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Wu Jing
University of Central Florida

Shijie Lu

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GENERALIZED JORDAN DERIVATIONS ON PRIME RINGS AND STANDARD OPERATOR ALGEBRAS

Wu Jing and Shijie Lu

Abstract. In this paper we initiate the study of generalized Jordan derivations and generalized Jordan triple derivations on prime rings and standard operator algebras.

1. Introduction

Motivated by the concept of generalized derivations (cf.[8]) we initiate the concepts of generalized Jordan derivations and generalized Jordan triple derivations as follows:

Definition 1.1. *Let R be a ring and $\delta : R \to R$ an additive map, if there is a Jordan derivation $\tau : R \to R$ such that*

$$\delta(a^2) = \delta(a)a + a\tau(a)$$

for each $a \in R$, then δ is called a generalized Jordan derivation, and τ the relating Jordan derivation.

Definition 1.2. *Let R be a ring and $\delta : R \to R$ an additive map, if there is a Jordan triple derivation $\tau : R \to R$ such that*

$$\delta(aba) = \delta(a)ba + a\tau(b)a + ab\tau(a)$$

for every $a, b \in R$, then δ is called a generalized Jordan triple derivation, and τ the relating Jordan triple derivation.

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We refer the readers to [1] for a good account of the theory of Jordan derivations and Jordan triple derivations. And we should mention that the generalized Jordan derivation defined in [10] is a special case of our definition. In this present paper we extend some results concerning Jordan derivations and Jordan triple derivations on prime rings to generalized Jordan derivations and generalized Jordan triple derivations. Namely, we proved that every generalized Jordan derivation and generalized Jordan triple derivation on 2-torsion free prime ring is a generalized derivation.

In section 4, we study the generalized Jordan derivations on standard operator algebras in Banach space X. We will prove that every weak operator topology continuous generalized Jordan linear derivation on standard operator algebra is a generalized inner derivation.

By $B(X)$ we denote the algebra of all bounded linear operators on X, and $F(X)$ denotes the algebra of all finite rank operators in $B(X)$. Recall that a standard operator algebra is any subalgebra of $B(X)$ which contains $F(X)$. The dual of X will be denoted by X^*. For any $x \in X$ and $f \in X^*$ we denote by $x \otimes f$ the bounded linear operator on X defined by $(x \otimes f)y = f(y)x$ for $y \in X$. Note that every operator of rank one can be written in this form and every operator of finite rank can be represented as a sum of operators of rank one.

In this paper R will represent an associative ring with center $Z(R)$. Recall that R is prime if $aRb = \{0\}$ implies that either $a = 0$ or $b = 0$, and is semiprime if $aRa = \{0\}$ implies $a = 0$.

2. Generalized Jordan Derivations on Prime Rings

Throughout this section, R will be a 2-torsion free ring. Let $\delta : R \to R$ be a generalized Jordan derivation and $\tau : R \to R$ the relating Jordan derivation.

Lemma 2.1. *For all $a, b, c \in R$, the following statements hold*:
(1) $\delta(ab + ba) = \delta(a)b + a\tau(b) + \delta(b)a + b\tau(a)$;
(2) $\delta(aba) = \delta(a)ba + a\tau(b)a + ab\tau(a)$;
(3) $\delta(abc + cba) = \delta(a)bc + a\tau(b)c + ab\tau(c) + \delta(c)ba + c\tau(b)a + cb\tau(a)$.

Proof. (1) $\delta((a+b)^2) = \delta(a+b)(a+b)+(a+b)\tau(a+b)$, however, $\delta((a+b)^2) = \delta(a^2 + ab + ba + b^2) = \delta(a)a + a\tau(a) + \delta(ab + ba) + \delta(b)b + b\tau(b)$.
Comparing these two expressions we obtain

$$\delta(ab + ba) = \delta(a)b + a\tau(b) + \delta(b)a + b\tau(a).$$

(2) Let $w = \delta(a(ab + ba) + (ab + ba)a)$.
On the one hand, using (1) we have

$$w = \delta(a)(ab + ba) + a\tau(ab + ba) + \delta(ab + ba)a + (ab + ba)\tau(a).$$

On the other hand,

$$w = \delta(a^2b + ba^2 + 2aba) = \delta(a^2)b + a^2\tau(b) + \delta(b)a^2 + b\tau(a^2) + 2\delta(aba).$$

Since R is 2-torsion free, we get

$$\delta(aba) = \delta(a)ba + a\tau(b)a + ab\tau(a).$$

(3) Linearizing (2) by replacing a by $a + c$ and we obtain

$$\delta(abc + cba) = \delta(a)bc + a\tau(b)c + ab\tau(c) + \delta(c)ba + c\tau(b)a + cb\tau(a).$$

Lemma 2.2. ([1]) *Let R be a semiprime ring, if $axb = 0$ for all $x \in R$, then $bxa = 0$.*

It will be convenient to denote $a^b = \delta(ab) - \delta(a)b - a\tau(b)$ and $[a, b] = ab - ba$. We can easily have
(1) $a^b = -b^a$;
(2) $a^{(b+c)} = a^b + a^c$;
(3) $(a + b)^c = a^c + b^c$.

Theorem 2.3. *If R is semiprime, then $a^b x[a, b] = 0$ for arbitrary $a, b, x \in R$.*

Proof. Let $w = \delta(abxba + baxab)$, then we get

$$w = \delta(ab)xba + ab\tau(x)ba + abx\tau(ba) + \delta(ba)xab + ba\tau(x)ab + bax\tau(ab).$$

On the other hand,

$$w = \delta(a)bxba + a\tau(bxb)a + abxb\tau(a) + \delta(b)axab + b\tau(axa)b + baxa\tau(b).$$

Since R is a semiprime ring, by [2], τ is a derivation. Comparing the above two expressions we have $a^b xba + b^a xab = 0$ and hence $a^b x[a, b] = 0$.

Corollary 2.4. *If R is semiprime, then $a^b \in Z(R)$.*

Proof. For arbitrary $a, b, c, x \in R$, by Theorem 2.3, we have $a^{(b+c)}x[a, b+c] = 0$, and so $a^b x[a, c] + a^c x[a, b] = 0$. By Theorem 2.3 $a^c y[a, c] = 0$ for every $y \in R$, by Lemma 2.2, we have $[a, c]ya^c = 0$, and so

$$(a^b x[a, c])y(a^b x[a, c]) = -a^b x[a, c]ya^c x[a, b] = 0.$$

This implies that $a^b x[a, c] = 0$.

Similarly, we have $a^b x[d,c]=0$ for all $d\in R$.

In particular,

$$[a^b,c]x[a^b,c]=(a^bc-ca^b)x[a^b,c]=a^b(cx)[a^b,c]-ca^bx[a^b,c]=0.$$

This yields that $[a^b,c]=0$, i. e., $a^b\in Z(R)$.

Now we are in a position to prove

Theorem 2.5. *Let R be a 2-torsion free prime ring, then every generalized Jordan derivation on R is a generalized derivation.*

Proof. Suppose that $\delta: R\to R$ is a generalized Jordan derivation and τ is the relating Jordan derivation on R. By the proof of Corollary 2.4, we have that $a^bx[c,d]=0$ for all $a,b,c,d,x\in R$. We have two cases:

Case 1. R is not commutative.

Then there exist $c,d\in R$ such that $[c,d]\neq 0$, by the primeness of R, we conclude that $a^b=0$, i. e. δ is a generalized derivation.

Case 2. R is commutative.

Let $w=\delta(a^2b+ba^2)$, then

$$w=\delta(a)ab+a\tau(ab)+\delta(ab)a+ab\tau(a)=\delta(a)ab+a\tau(a)b+a^2\tau(b)+\delta(ab)a+ab\tau(a).$$

On the other hand,

$$\begin{aligned}w&=\delta(a^2)b+a^2\tau(b)+\delta(b)a^2+b\tau(a^2)\\&=\delta(a)ab+a\tau(a)b+a^2\tau(b)+\delta(b)a^2+b\tau(a)a+ba\tau(a).\end{aligned}$$

These two equations yield that

$$(\delta(ab)-\delta(b)a-b\tau(a))a=b^aa=0.$$

A linearization of the above expression with repect to a gives

$$b^ca+b^ac=0.$$

Then $(b^ac)x(b^ac)=-b^caxb^ac=-(b^cc)x(b^aa)=0$, hence $b^ac=0$. Furthermore, we have $b^acb^a=0$, and so $b^a=0$.

To conclude this section, we conjecture that every generalized Jordan derivation on 2-torsion free semiprime ring is a generalized derivation since every Jordan derivation on 2-torsion free semiprime ring is a derivation. But to our knowledgement we can not prove it.

3. Generalized Jordan Triple Derivations on Prime Rings

In this section, R will be a 2-torsion free ring. Let $\delta : R \to R$ be a generalized Jordan triple derivation and $\tau : R \to R$ the relating Jordan triple derivation.

Lemma 3.1. *For arbitrary $a, b, c \in R$ we have*

$$\delta(abc + cba) = \delta(a)bc + a\tau(b)c + ab\tau(c) + \delta(c)ba + c\tau(b)a + cb\tau(a).$$

Proof. In a similar argument as in the proof of Lemma 2.1 we compute $w = \delta((a+c)b(a+c))$. On the one hand we have $w = \delta(a+c)b(a+c)+(a+c)\tau(b)(a+c) + (a+c)b\tau(a+c)$ and on the other hand $w = \delta(aba) + \delta(cbc) + \delta(abc + cba)$. Comparing two expressions we obtain the assertion of the lemma.

For the purpose of this section we shall write $A(abc) = \delta(abc) - \delta(a)bc - a\tau(b)c - ab\tau(c)$ and $B(abc) = abc - cba$. We list a few elementary properties of A and B:

(1) $A(abc) + A(cba) = 0$;

(2) $A((a+b)cd) = A(acd) + A(bcd)$ and $B((a+b)cd) = B(acd) + B(bcd)$;

(3) $A(a(b+c)d) = A(abd) + A(acd)$ and $B(a(b+c)d) = B(abd) + B(acd)$;

(4) $A(ab(c+d)) = A(abc) + A(abd)$ and $B(ab(c+d)) = B(abc) + B(abd)$.

Lemma 3.2. *Let R be a semiprime ring, then for arbitrary $a, b, c, x \in R$ we have*

$$A(abc)xB(abc) = 0.$$

Proof. Since R is a semiprime ring, it follows from Theorem 4.3 of [1] that τ is a derivation. Let $w = \delta(abcxcba + cbaxabc)$, and compute w in two ways. Then the lemma follows easily.

Theorem 3.3. *Let R be a semiprime ring, then $A(abc)xB(rst) = 0$ holds for all $a, b, c, x, r, s, t \in R$.*

Proof. By Lemma 3.2 we have $A((a+r)bc)xB((a+r)bc) = 0$, which implies that $A(abc)xB(rbc) + A(rbc)xB(abc) = 0$, and so

$$A(abc)xB(rbc)yA(abc)xB(rbc) = -A(abc)xB(rbc)yA(rbc)xB(abc) = 0, \forall y \in R.$$

By the semiprimeness of R, we have that $A(abc)xB(rbc) = 0$, similarly we can get that $A(abc)xB(rsc) = 0$ and furthermore $A(abc)xB(rst) = 0$.

Corollary 3.4. *If R is a prime ring and $Z(R) \neq \{0\}$, then $A(abc) \in Z(R)$ for all $a, b, c \in R$.*

Proof. For arbitrary $a, b, c, x, r, s \in R$ we have

$$\begin{aligned} B(A(abc)rs)xB(A(abc)rs) &= (A(abc)rs - srA(abc))xB(A(abc)rs) \\ &= A(abc)rsxB(A(abc)rs) - srA(abc)xB(A(abc)rs) \\ &= 0. \end{aligned}$$

The primeness of R yields that $B(A(abc)rs) = 0$, i. e. $A(abc)rs = srA(abc)$.

Choose nonzero $r_0 \in Z(R)$, then $A(abc)r_0s = sr_0A(abc) = sA(abc)r_0$, which implies that $A(abc)r_0 \in Z(R)$. Since R is prime, we can easily conclude that $A(abc) \in Z(R)$.

Now we are ready to prove

Theorem 3.5. *Let R be a 2-torsion free prime ring, then every generalized Jordan triple derivation on R is a generalized derivation.*

Proof. Suppose that $\delta : R \to R$ is a generalized Jordan triple derivation and τ is the relating Jordan triple derivation on R. Let a, b, c, x be arbitrary element of R. We have two cases:

Case 1. There exist $r, s, t \in R$ such that $B(rst) \neq 0$.

Theorem 3.3 and the primeness of R yield that $A(abc) = 0$.

Case 2. $B(rst) = 0$ holds for all $r, s, t \in R$, i. e. $rst = tsr$.

Let Q be the central closure or the Martindale right quotient ring (see [9] for the definitions) of R, then Q is a prime ring with identity and contains R. By [5], Q satisfies the same generalized polynomial identities as R. In particular $rst = tsr$ for all r, s, t in Q. Taking $s = 1$ yields the commutativity of Q and R.

Now let $w = \delta(a^3bc + cba^3)$, then $w = \delta(a^3)bc + a^3\tau(b)c + a^3b\tau(c) + \delta(c)ba^3 + c\tau(b)a^3 + cb\tau(a^3)$ and $w = \delta(abcaa + aaabc) = \delta(abc)a^2 + abc\tau(a)a + abca\tau(a) + \delta(a)a^2bc + a\tau(a)abc + a^2\tau(abc)$. These two expressions yield that $[\delta(abc) - \delta(c)ba - c\tau(b)a - cb\tau(a)]a^2 = 0$, i. e.

$$A(abc)a^2 = 0.$$

Then we have $(A(abc)a)x(A(abc)a) = A(abc)a^2xA(abc) = 0$, hence $A(abc)a = 0$, furthermore, with the same approach as in the proof of Theorem 2.5, we obtain $A(abc)x = 0$ and so $A(abc)xA(abc) = 0$, which implies that $A(abc) = 0$.

It remains to prove that δ is a generalized derivation.

Again, let a, b, c, d, x, y be arbitrary elements of R and $w = \delta(abxab)$. On one hand we have $w = \delta(ab)xab + ab\tau(x)ab + abx\tau(ab)$, on the other hand $w = \delta(a)bxab + a\tau(bxa)b + abxa\tau(b)$. Then we have $(\delta(ab) - \delta(a)b - a\tau(b))xab = 0$. By the notation $a^b = \delta(ab) - \delta(a)b - a\tau(b)$, we get $a^bxab = 0$. Furthermore

$(a+c)^b x(a+c)b = 0$, this shows that $a^b xcb = -c^b xab$, and $(a^b xcb)y(a^b xcb) = -a^b x(cbyc^b)xab = 0$, hence $a^b xcb = 0$. Similarly we have $a^b xcd = 0$, in particular, $a^b cxa^b c = 0$, and so $a^b c = 0$, which leads to $a^b = 0$.

As in section 2, we also conjecture that every generalized Jordan triple derivation on 2-torsion free semiprime ring is a generalized derivation.

4. Generalized Jordan Derivations on Standard Operator Algebras

Theorem 4.1. *Let $M_n(\mathbf{C})$ denote the algebra of all $n \times n$ complex matrices and $\mathcal{B}$ be an arbitrary algebra over the complex field $\mathbf{C}$. Suppose that $\delta : M_n(\mathbf{C}) \to \mathcal{B}$ is a linear mapping such that $\delta(P) = \delta(P)P + P\tau(P)$ holds for all idempotent P in $M_n(\mathbf{C})$, where $\tau : M_n(\mathbf{C}) \to \mathcal{B}$ is a linear mapping satisfying $\tau(P) = \tau(P)P + P\tau(P)$ for any idempotent P in $M_n(\mathbf{C})$, then δ is a generalized Jordan derivation. Moreover, δ is a generalized derivation.*

Proof. By Theorem 2.3 of [3], τ is a derivation.

Pick a Hermitian matrix $H \in M_n(\mathbf{C})$. Then $H = \sum_{i=1}^n t_i P_i$ where $t_i \in \mathbf{R}$ and P_i are idempotents such that $P_i P_j = P_j P_i = 0$ for $i \neq j$. Since $P_i + P_j$ is an idempotent if $i \neq j$, we have $\delta(P_i+P_j) = \delta(P_i+P_j)(P_i+P_j)+(P_i+P_j)\tau(P_i+P_j)$. This yields that $\delta(P_i)P_j + P_i\tau(P_j) + \delta(P_j)P_i + P_j\tau(P_i) = 0$. Using this relation we see that $\delta(H^2) = \delta(H)H + H\tau(H)$.

Now replacing H by $H + K$ where H and K are both Hermitian, we get $\delta(HK+KH) = \delta(H)K + H\tau(K) + \delta(K)H + K\tau(H)$. Since an arbitrary matrix $A \in M_n(\mathbf{C})$ can be written in the form $A = H + iK$ with H, K Hermitian, the last relations imply that $\delta(A^2) = \delta(A)A + A\tau(A)$, by Theorem 2.5, δ is a generalized derivation.

Theorem 4.2. *Let $\mathcal{B}$ be a $F(X)-$bimodule and $\delta : F(X) \to \mathcal{B}$ be a linear mapping such that $\delta(P) = \delta(P)P + P\tau(P)$ for any idempotent $P \in F(X)$, where $\tau : F(X) \to \mathcal{B}$ is a linear mapping satisfying $\tau(P) = \tau(P)P + P\tau(P)$ for any idempotent $P \in F(X)$, then δ is a generalized derivation. Furthermore, δ is a generalized inner derivation, i.e. there exist $S, T \in B(X)$ such that $\delta(A) = SA - AT$ for every $A \in F(X)$.*

Proof. By Theorem 3.6 in [3], τ is a derivation. Pick $A, B \in F(X)$, then there exists an idempotent $Q \in F(X)$ such that $QAQ = A$ and $QBQ = B$. Let $\{x_1, x_2, \cdots, x_n\}$ be a basis of the range of Q. Define linear functionals $f_1, f_2, \cdots, f_n$ on X by

$$f_i(x_j) = \delta_{ij},$$

$$f_i(z) = 0 \text{ for all } z \in KerQ.$$

Let $\mathcal{C} \subseteq F(X)$ be the algebra of all operators of the form $C = \sum_{i,j=1}^{n} t_{ij} x_i \otimes f_j$, $t_{ij} \in \mathbf{C}$ and note that $\mathcal{C}$ is isomorphic to $M_n(\mathbf{C})$ via the isomorpmism $C \longmapsto (t_{ij})$. Thus, for the restriction of δ to $\mathcal{C}$, Theorem 4.1 can be applied, hence, $\delta(AB) = \delta(A)B + A\tau(B)$.

To complete the proof, it remains to prove that δ is a generalized inner derivation.

Let $x_0 \in X$ and $f_0 \in X^*$ be chosen such that $f_0(x_0) = 1$. Define two operators $S : X \to X$ and $T : X \to X$ by $Sx = \delta(x \otimes f_0)x_0$ and $Tx = \tau(x \otimes f_0)x_0$ respectively.

For arbitrary $A \in F(X)$, we have

$$\delta(Ax \otimes f_0) = \delta(A)x \otimes f_0 + A\tau(x \otimes f_0).$$

Applying both operators in this equation to x_0, we get

$$\delta(A)x = SAx - ATx, \forall x \in X.$$

Hence $\delta(A) = SA - AT$.

It is easy to see that both S and T have closed graphs, and hence both are bounded. This completes the proof.

Since $F(X)$ is dense in the weak operator topology in every standard operator algebra, we have the following

Theorem 4.3. *Let $\mathcal{A}$ be a standard operator algebra. Suppose that linear mappings $\delta : \mathcal{A} \to B(X)$ and $\tau : \mathcal{A} \to B(X)$ satisfying $\delta(P) = \delta(P)P + P\tau(P)$ and $\tau(P) = \tau(P)P + P\tau(P)$ for every idempotent P in $\mathcal{A}$. If δ and τ are continuous in the weak operator topology, then there exist $S, T \in B(X)$ such that $\delta(A) = SA - AT, \quad \forall A \in \mathcal{A}$.*

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E-mail: wu511944@pegasus.cc.ucf.edu
Department of Mathematics,
Yuquan Campus, Zhejiang University,
Hangzhou 310027,
P. R. China
E-mail: jingwu@math.zju.edu.cn
Wu Jing
Department of Mathematics,
University of Central Florida,
Orlando, FL 32816-1364,
U.S.A.

Shijie Lu
Department of Mathematics,
Yuquan Campus, Zhejiang University,
Hangzhou 310027,
P. R. China
E-mail: lusj@zucc.edu.cn