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# Lattice-Valued T-Filters and Induced Structures

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# LATTICE-VALUED $\top$-FILTERS AND INDUCED STRUCTURES

by

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# ABSTRACT

A complete lattice is called a frame provided meets distribute over arbitrary joins. The implication operation in this context plays a central role. Intuitively, it measures the degree to which one element is less than or equal to another. In this setting, a category is defined by equipping each set with a T-convergence structure which is defined in terms of T-filters. This category is shown to be topological, strongly Cartesian closed, and extensional. It is well known that the category of topological spaces and continuous maps is neither Cartesian closed nor extensional.

Subcategories of compact and of complete spaces are investigated. It is shown that each T-convergence space has a compactification with the extension property provided the frame is a Boolean algebra. T-Cauchy spaces are defined and sufficient conditions for the existence of a completion are given. T-uniform limit spaces are also defined and their completions are given in terms of the T-Cauchy spaces they induce. Categorical properties of these subcategories are also investigated. Further, for a fixed T-convergence space, under suitable conditions, it is shown that there exists an order preserving bijection between the set of all strict, regular, Hausdorff compactifications and the set of all totally bounded T-Cauchy spaces which induce the fixed space.

To my wife Gabriela;

thank you for your love and support.

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| | |
|---|---|
| $\alpha \to \beta$ | $= \vee\{\delta \in L : \alpha \wedge \delta \leq \beta\}$ |
| $[a, b]$ | $= \bigwedge_{x \in X} \big(a(x) \to b(x)\big)$ |
| $\mathbf{1}_S(x)$ | $= \begin{cases} \top, & x \in S \\ \bot, & x \notin S \end{cases}$ |
| $\alpha \mathbf{1}_S(x)$ | $= \begin{cases} \alpha, & x \in S \\ \bot, & x \notin S \end{cases}$ |
| $f^{\to}(a)(y)$ | $= \vee\{a(x) : f(x) = y\}$ |
| $f^{\leftarrow}(b)$ | $= b \circ f$ |
| $f^{\Uparrow}(\nu)(b)$ | $= \nu(f^{\to}(b))$ |
| $f^{\Downarrow}(\nu)(a)$ | $= \vee\{\nu(b) : f^{\leftarrow}(b) \leq a\}$ |
| $\dot{x}(a)$ | $= a(x)$, Strat. filter gen. by $x$ |
| $\mathfrak{F}_L^S(X)$ | Strat.-$L$-filters on $X$ |

| | |
|---|---|
| $f^{\Rightarrow}\mathfrak{F}$ | $\top$-filter gen. by $\{f^{\to}(a) : a \in \mathfrak{F}\}$ |
| $f^{\Leftarrow}\mathfrak{G}$ | $\top$-filter gen. by $\{f^{\leftarrow}(b) : b \in \mathfrak{G}\}$ |
| $[x]$ | $= \{a \in L^X : a(x) = \top\}$, $\top$-filter gen. by $x$ |
| $\mathfrak{F}_L^{\top}(X)$ | $\top$-filters on $X$ |
| $\nu_{\mathfrak{F}}(a)$ | $= \bigvee_{b \in \mathfrak{F}} [b, a]$ |
| $\mathfrak{F}_\nu$ | $= \{a \in L^X : \nu(a) = \top\}$ |
| $\overline{a}(x)$ | $= \vee\{\nu_{\mathfrak{F}}(a) : \mathfrak{F} \xrightarrow{q} x\}$ |
| $e_{\mathfrak{G}}(b)$ | $= \nu_{\mathfrak{G}}(b)$ |
| $\kappa\sigma\mathfrak{H}$ | $= \{b \in L^X : e_b \circ \sigma \in \mathfrak{H}\}$ |
| $(a \times b)(x, y)$ | $= a(x) \wedge b(y)$ |
| $(a \circ b)(x, y)$ | $= \bigvee_{z \in x} \big(a(x, z) \wedge b(z, y)\big)$ |
| $a^{-1}(x, y)$ | $= a(y, x)$ |

# CHAPTER 1: INTRODUCTION

The use of filters to study various topological properties has been profoundly successful. It is natural, then, to use a many-valued version of a filter to adapt classical or crisp properties in topology to their many-valued or fuzzy counterparts. In this work, one such notion of a fuzzy or many-valued filter, the $\top$-filter, is used to establish several topological and categorical properties and structures in the fuzzy setting. The notion of a $\top$-filter is due to Höhle [10]. The particular version which follows here is due to Fang and Yu [29].

## Preliminaries

### *L-Sets*

A lattice $(L, \wedge, \vee)$ is called a **complete Heyting algebra** or **frame** provided it is complete and obeys $\alpha \wedge \left( \bigvee_{j \in J} \beta_j \right) = \bigvee_{j \in J} (\alpha \wedge \beta_j)$ for all $\alpha, \beta_j \in L$, $j \in J$. The implication operator $\rightarrow : L \times L \rightarrow L$ is defined by $\alpha \rightarrow \beta = \vee\{\delta \in L : \alpha \wedge \delta \leq \beta\}$. Let $\perp(\top)$ denote the bottom(top) member of the complete lattice $L$, respectively. In a bounded lattice, the **pseudo-complement** of an element $\alpha$ is an element $\neg\alpha$ such that $\alpha \wedge \neg\alpha = \perp$. If, in addition, $\alpha \vee \neg\alpha = \top$, then $\neg a$ is called a **complement**. If we let $\neg\alpha = \alpha \rightarrow \perp$, then $\alpha \wedge \neg\alpha = \perp$ but $\alpha \vee \neg\alpha$ does not always equal $\top$. Therefore, $\neg\alpha = \alpha \rightarrow \perp$ is a pseudo-complement and $\alpha \rightarrow \beta$ is sometimes referred to as the **relative pseudo-complement** of $\alpha$ with respect to $\beta$. If $\alpha \rightarrow \perp$ is a complement for each $\alpha \in L$, then the frame $L$ is called a **complete Boolean Algebra**.

Since meets distribute over arbitrary joins, $\alpha \rightarrow \beta = \max\{\delta \in L : \alpha \wedge \delta \leq \beta\}$. Given a set $X$, an $L$**-fuzzy subset of** $X$, or an $L$**-set** is map $a : X \longrightarrow L$. Intuitively, an $L$-set assigns each member of $X$ a degree of membership, indexed by $L$. We denote the set of all $L$-subsets of $X$ by $L^X$. Then we may identify classical subsets of $X$ with the characteristic functions. That is, if $S \subseteq X$ then we can identify $S$ with the $L$-set, $\mathbf{1}_S(x) = \begin{cases} \top, & x \in S \\ \bot, & x \notin S \end{cases}$. In this sense, $L$-sets are a natural generalization of the classical set. The lattice operations on $L$ can be extended point-wise to $L^X$ as follows: $\left(\bigvee_{j\in J} a_j\right)(x) = \bigvee_{j\in J} a_j(x)$, $\left(\bigwedge_{j\in J} a_j\right)(x) = \bigwedge_{j\in J} a_j(x)$, and $(a \to b)(x) = a(x) \to b(x)$ for each $x \in X$. Then $(L^X, \wedge, \vee)$ is also a frame, and $\mathbf{1}_\varnothing(\mathbf{1}_X)$ are bottom(top) members of $L^X$, respectively. Also, if $\alpha \in L$ we let $\alpha\mathbf{1}_S(x) = \begin{cases} \alpha, & x \in S \\ \bot, & x \notin S \end{cases}$.

Let $f : X \longrightarrow Y$ be a map. Then $f^{\to} : L^X \longrightarrow L^Y$ and $f^{\leftarrow} : L^Y \longrightarrow L^X$ are defined respectively by $f^{\to}(a)(y) = \vee\{a(x) : f(x) = y\}$ for each $a \in L^X$, $y \in Y$, and $f^{\leftarrow}(b) = b \circ f$ for all $b \in L^Y$.

If $L$ is a frame, $X$ a set, and $a, b \in L^X$, define $[a, b] = \bigwedge_{x\in X} \big(a(x) \to b(x)\big)$. Note that if $a \leq b$, then $a(x) \to b(x) = \top$ for each $x \in X$ and thus $[a, b] = \top$. It follows that we may think of $[a, b]$ as a measure of the degree to which $a \leq b$. If the underlying set $X$ is ever unclear, we may write $[a, b]_X$ for clarity.

The following lemma is a collection of properties of the implication operator and can be found in [29].

**Lemma 1.1.** *Let $L$ be a frame and $X$ a set. Then,*

*(i)* $\alpha \leq \beta$ *if and only if* $\alpha \to \beta = \top$,

*(ii)* $\alpha \wedge \beta = \alpha \wedge (\alpha \to \beta)$,

*(iii)* $(\alpha \to \gamma) \wedge (\beta \to \delta) \leq (\alpha \wedge \beta) \to (\gamma \wedge \delta)$,

*(iv)* $\bigwedge_{j\in J}(\alpha \to \beta_j) = \alpha \to \left(\bigwedge_{j\in J} \beta_j\right)$,

*(v)* $\bigwedge_{j\in J}(\alpha_j \to \beta) = \left(\bigvee_{j\in J}\alpha_j\right) \to \beta$,

*(vi)* $a \leq b$ *if and only if* $[a,b] = \top$,

*(vii)* $[a, b \wedge c] = [a,b] \wedge [a,c]$,

*(viii)* $[a,b] \leq [b,c] \to [a,c]$ *and* $[b,c] \leq [a,b] \to [a,c]$,

*(ix)* $[b \vee c, a] = [b,a] \wedge [c,a]$,

*(x)* $[c,a] \leq [b,a]$ *whenever* $b \leq c$, *and*

*(xi)* $[a,b] \leq [f^{\to}(a), f^{\to}(b)]$ *and* $[c,d] \leq [f^{\leftarrow}(c), f^{\leftarrow}(d)]$ *whenever* $f : X \to Y$ *is a map.*

$\top$*-Filters and Stratified* $L$*-Filters*

One way we can explore familiar notions of continuity of maps, convergence, compactness and other properties of interest is to define filters on these non-standard sets.

**Definition 1.1.** Let $L$ be a frame. A map $\nu : L^X \to L$ is called a **stratified $L$-filter** provided:

(F1) $\nu(\mathbf{1}_\varnothing) = \bot$, and $\nu(\alpha\mathbf{1}_X) \geq \alpha$, each $\alpha \in L$,

(F2) $a \leq b$ implies $\nu(a) \leq \nu(b)$, and

(F3) $\nu(a) \wedge \nu(b) \leq \nu(a \wedge b)$, for each $a, b \in L^X$.

Intuitively, each $L$-set is given a degree of membership in the stratified $L$-filter $\nu$. If $\nu(\alpha\mathbf{1}_X) = \alpha$ for each $\alpha \in L$, then $\nu$ is said to be **tight**. If $L$ is a complete Boolean algebra, then stratified $L$-filters are automatically tight.

Let $\mathfrak{F}_L^S(X)$ be the set of all stratified $L$-filters on $X$. If $\nu_1$ and $\nu_2$ are two stratified $L$-filters on $X$, denote $\nu_1 \leq \nu_2$ whenever $\nu_1(a) \leq \nu_2(a)$ for each $a \in L^X$. Moreover, for $x \in X$, define $\dot{x} \in \mathfrak{F}_L^S(X)$ by $\dot{x}(a) = a(x)$, for each $a \in L^X$. A Zorn's Lemma argument easily shows that each stratified $L$-filter on $X$ is contained in a maximal stratified $L$-filter, called a **stratified** $L$**-ultrafilter.** Höhle [11] proved the following fundamental results:

**Theorem 1.1.** [11] *Suppose that $L$ is a frame and $\nu$ is a stratified $L$-filter on $X$. Then for all $a \in L^X$,*

*(i) $\nu$ is a stratified $L$-ultrafilter on $X$ if and only if $\nu(a) = \nu(a \to \mathbf{1}_\varnothing) \to \bot$ for each $a \in L^X$*

*(ii) if $\nu$ is a stratified $L$-ultrafilter, then $\nu(a \to \mathbf{1}_\varnothing) = \nu(a) \to \bot$*

*(iii)* $\bigwedge\limits_{x\in X} a(x) \leq \nu(a) \leq \left( \bigvee\limits_{x\in X} a(x) \to \bot \right) \to \bot.$

Next, if $\mu \in \mathfrak{F}_L^S(X)$, then the **image stratified $L$-filter of** $\mu$ under $f$ is defined by $f^{\Uparrow}(\mu)(b) = \mu(f^{\to}(b))$, for each $b \in L^X$. Further, if $\nu \in \mathfrak{F}_L^S(Y)$, then the **inverse image stratified $L$-filter of** $\nu$ under $f$ is defined as $f^{\Downarrow}(\nu)(a) = \vee\{\nu(b) : f^{\leftarrow}(b) \leq a\}$ whenever it exists. It is straightforward to check that $\mathfrak{F}_L^S(X)$ has a smallest element $\nu_\bot$ defined by $\nu_\bot(a) = \bigwedge\limits_{x\in X} a(x)$.

Stratified $L$-filters have been well studied. Therefor our attention will turn to a different type of filter, the so called $\top$-filter. Still, it will often be useful to connect $\top$-filters to stratified $L$-filters. The notion of a $\top$-filter is due to Höhle [10]. A particular version which follows here is due to Fang and Yu [29].

**Definition 1.2.** [29] Suppose that $L$ is a frame and $X$ a set. A non-empty subset $\mathfrak{F} \subseteq L^X$ is called a $\top$**-filter** provided:

($\top$F1) $\bigvee\limits_{x\in X} b(x) = \top$ for each $b \in \mathfrak{F}$,

($\top$F2) if $a, b \in \mathfrak{F}$, then $a \wedge b \in \mathfrak{F}$,

($\top$F3) if $\bigvee_{b \in \mathfrak{F}} [b, d] = \top$, then $d \in \mathfrak{F}$.

One major difference between stratified $L$-filters and $\top$-filters is that in the $\top$-filter case, $L$-sets are either contained in the $\top$-filter or they are not, unlike in the stratified case were $L$-sets have degrees of membership. One way to think about $\top$-filters is to think of taking the $L$-sets from the top-level of a stratified $L$-filter and forming a filter. This notion will be studied in greater detail in a later chapter.

Let $\mathfrak{F}_L^{\top}(X)$ denote the set of all $\top$-filters on $X$. Let $x \in X$, define $[x] = \{a \in L^X : a(x) = \top\}$. If the underlining set is ever unclear, we may write $[x]_X$.

**Lemma 1.2.** *Let $x \in X$, then $[x]$ is a $\top$-filter on $X$.*

*Proof.* Let $a \in [x]$ then $a(x) = \top$ and $\bigvee_{y \in X} a(y) \geq a(x) = \top$. Hence ($\top$F1) is satisfied. If $a, b \in [x]$ then $(a \wedge b)(x) = a(x) \wedge b(x) = \top \wedge \top = \top$. Hence $a \wedge b \in [x]$ and ($\top$F2) is valid. Finally, suppose that $\bigvee_{a \in [x]} [a, b] = \top$. Then using Lemma 1.1 (x), $\top = \bigvee_{a \in [x]} [a, b] \leq [\mathbf{1}_{\{x\}}, b] = \bigwedge_{y \in X} \left(\mathbf{1}_{\{x\}}(y) \to b(y)\right) = \bigwedge_{y \in X} \begin{cases} \bot \to b(y), & y \neq x \\ \top \to b(x), & y = x \end{cases} = \bigwedge_{y \in X} \begin{cases} \top, & y \neq x \\ b(x), & y = x \end{cases} = b(x)$. Therefore $b(x) = \top$, $b \in [x]$, ($\top$F3) is satisfied and $[x]$ is a $\top$-filter on $X$. $\square$

It is often convenient to work with $\top$-filter bases as defined below.

**Definition 1.3.** A non-empty subset $\mathcal{B} \subseteq L^X$ is said to be a $\top$**-filter base** whenever:

($\top$B1) for each $b \in \mathcal{B}$, $\bigvee_{x \in X} b(x) = \top$, and

($\top$B2) if $a_1, a_2 \in \mathcal{B}$ then $\bigvee_{b \in \mathcal{B}} [b, a_1 \wedge a_2] = \top$.

According to [29], a $\top$-filter base $\mathcal{B}$ generates the $\top$-filter $\mathfrak{F} = \{a \in L^X : \bigvee_{b\in\mathcal{B}}[b,a] = \top\}$; that is, $\mathfrak{F}$ is the smallest $\top$-filter containing $\mathcal{B}$. Moreover, if $f : X \longrightarrow Y$ is a map, then the image $f^{\Rightarrow}(\mathcal{B}) = \{f^{\rightarrow}(b) : b \in \mathcal{B}\}$ is a $\top$-filter base, and the **image of a $\top$-filter** $\mathfrak{F}$, denoted by $f^{\Rightarrow}(\mathfrak{F})$, is defined to be the $\top$-filter on $Y$ having the $\top$-filter base $\{f^{\rightarrow}(a) : a \in \mathfrak{F}\}$. Further, if $\mathfrak{G}$ is a $\top$-filter on $Y$, then the **inverse image of** $\mathfrak{G}$, denoted by $f^{\Leftarrow}(\mathfrak{G})$, exists if and only if $\bigvee_{x\in X} a(f(x)) = \top$ for each $a \in \mathfrak{G}$. In this case, $f^{\Leftarrow}(\mathfrak{G})$ is defined to be the $\top$-filter on $X$ whose $\top$-filter base is $\{f^{\leftarrow}(a) : a \in \mathfrak{G}\}$.

**Lemma 1.3.** *Suppose that $L$ is a frame, $X$ a set and $\mathcal{B}$ a $\top$-filter base for the $\top$-filter $\mathfrak{F}$. Then for $d \in L^X$, $\bigvee_{b\in\mathcal{B}}[b,d] = \bigvee_{b_1,b_2\in\mathcal{B}}[b_1 \wedge b_2, d] = \bigvee_{a\in\mathfrak{F}}[a,d]$.*

*Proof.* Assume that $d \in L^X$, $c \in \mathcal{B}$, $a \in \mathfrak{F}$, then according to Lemma 1.1 (viii), $[b,a] \leq [a,d] \rightarrow [b,d]$. Since $a \in \mathfrak{F}$, $\top = \bigvee_{b\in\mathcal{B}}[b,a] \leq \bigvee_{b\in\mathcal{B}}\big([a,d] \rightarrow [b,d]\big) \leq [a,d] \rightarrow \bigvee_{b\in\mathcal{B}}[b,d]$. Then since $\top = [a,d] \rightarrow \bigvee_{b\in\mathcal{B}}[b,d]$, by Lemma 1.1 (i), $[a,d] \leq \bigvee_{b\in\mathcal{B}}[b,d]$. Consequently, $\bigvee_{a\in\mathfrak{F}}[a,d] \leq \bigvee_{b\in\mathcal{B}}[b,d]$, and since $\mathcal{B} \subseteq \mathfrak{F}$, $\bigvee_{a\in\mathfrak{F}}[a,d] = \bigvee_{b\in\mathcal{B}}[b,d]$.

Next, fix $b_1, b_2 \in \mathcal{B}$; then since $\mathcal{B}$ is a $\top$ filter base, $\bigvee_{b\in\mathcal{B}}[b, b_1 \wedge b_2] = \top$, and it follows from the definition of $\mathfrak{F}$ that $b_1 \wedge b_2 \in \mathfrak{F}$. Then $\mathcal{B} \subseteq \{b_1 \wedge b_2 : b_1, b_2 \in \mathcal{B}\} \subseteq \mathfrak{F}$, so that $\bigvee_{b\in\mathcal{B}}[b,d] \leq \bigvee_{b_1,b_2\in\mathcal{B}}[b_1 \wedge b_2, d] \leq \bigvee_{a\in\mathfrak{F}}[a,d]$ and by the previous part of this proof, we have equality throughout this last expression. □

**Lemma 1.4.** *Let $f : X \longrightarrow Y$ be a map, $\mathfrak{F}, \mathfrak{G} \in \mathfrak{F}_L^{\top}(X)$ and $x \in X$. The following hold:*

*(i)* $f^{\Rightarrow}(\mathfrak{F} \cap \mathfrak{G}) = f^{\Rightarrow}\mathfrak{F} \cap f^{\Rightarrow}\mathfrak{G}$,

*(ii)* $f^{\Rightarrow}[x]_X = [f(x)]_Y$

*(iii) if $\mathcal{B}$ is a base for $\mathfrak{F} \in \mathfrak{F}_L^{\top}(X)$ then $\hat{\mathcal{B}} = \{f^{\rightarrow}(b) : b \in \mathcal{B}\}$ is a base for $f^{\Rightarrow}\mathfrak{F}$.*

*Proof.* (i) Let $a \in \mathfrak{F} \cap \mathfrak{G}$. A typical base member for $f^{\Rightarrow}(\mathfrak{F} \cap \mathfrak{G})$ is given by $f^{\rightarrow}(a)$. But since $a \in \mathfrak{F}$ and $a \in \mathfrak{G}$, we have that $f^{\rightarrow}(a) \in f^{\Rightarrow}\mathfrak{F} \cap f^{\Rightarrow}\mathfrak{G}$. Hence $f^{\Rightarrow}(\mathfrak{F} \cap \mathfrak{G}) \subseteq f^{\Rightarrow}\mathfrak{F} \cap f^{\Rightarrow}\mathfrak{G}$

Next if we assume $a \in \{f^{\rightarrow}(c) : c \in \mathfrak{F}\} \cap \{f^{\rightarrow}(d) : d \in \mathfrak{G}\}$, then it follows that $a = f^{\rightarrow}(b)$ for some $b \in \mathfrak{F}$ and $b \in \mathfrak{G}$. Thus $b \in \mathfrak{F} \cap \mathfrak{G}$ and $a \in \{f^{\rightarrow}(b) : b \in \mathfrak{F} \cap \mathfrak{G}\}$. Hence $f^{\Rightarrow}\mathfrak{F} \cap f^{\Rightarrow}\mathfrak{G} \subseteq f^{\Rightarrow}(\mathfrak{F} \cap \mathfrak{G})$ and the result follows.

(ii) We first show that $f^{\Rightarrow}[x]_X \subseteq [f(x)]_Y$. To do this, it suffices to show that any member of a base for $f^{\Rightarrow}[x]_X$ is contained in $[f(x)]_Y$. Let $a \in \{f^{\rightarrow}(b) : b \in [x]_X\}$ which is a base for $f^{\Rightarrow}[x]_X$. Then $a = f^{\rightarrow}(b)$ for some $b \in [x]_X$ and thus $a(f(x)) = f^{\rightarrow}(b)(f(x)) = \bigvee_{f(z)=f(x)} b(z) \geq b(x) = \top$. Hence $a(f(x)) = \top$ and $a \in [f(x)]_Y$.

Next suppose that $a \in [f(x)]_Y$. In order to show that $[f(x)]_Y \subseteq f^{\Rightarrow}[x]_X$, it suffices to show that $a \in \{f^{\rightarrow}(b) : b \in [x]_X\}$. That is, it suffices to show that $a = f^{\rightarrow}(b)$ for some $b \in [x]$. Consider $f^{\leftarrow}(a) \in L^X$. Note that $f^{\leftarrow}(a)(x) = (a \circ f)(x) = a(f(x)) = \top$. Therefore $f^{\leftarrow}(a) \in [x]_X$. Further, $f^{\rightarrow}(f^{\leftarrow}(a))(y) = \bigvee_{f(z)=y} (a \circ f)(z) = \bigvee_{f(z)=y} a(f(z)) = \bigvee_{f(z)=y} a(y) = a(y)$. Hence $a = f^{\rightarrow}(f^{\leftarrow}(a)) \in \{f^{\rightarrow}(b) : b \in [x]_X\}$ and the result follows.

(iii) First we must show that $\hat{\mathcal{B}}$ is a $\top$-filter base. We have,

$$\bigvee_{y \in Y} f^{\rightarrow}(b)(y) = \bigvee_{y \in Y} \bigvee_{f(x)=y} b(x) = \left( \bigvee_{y \in f(X)} \bigvee_{f(x)=y} b(x) \right) \vee \left( \bigvee_{y \in Y \setminus f(X)} b(x) \right)$$
$$= \bigvee_{y \in f(X)} \bigvee_{f(x)=y} b(x) \vee \bot = \bigvee_{f(x)=y} b(x) = \bigvee_{x \in X} b(x) = \top.$$

Next if $a_1, a_2 \in \mathcal{B}$ then

$$\begin{aligned}
\bigvee_{b\in\mathcal{B}} [f^{\rightarrow}(b), f^{\rightarrow}(a_1) \wedge f^{\rightarrow}(a_2) &= \bigvee_{b\in\mathcal{B}} [f^{\rightarrow}(b), f^{\rightarrow}(a_1)] \wedge [f^{\rightarrow}(b), f^{\rightarrow}(a_2)] \qquad \text{Lemma 1.1 } (vii) \\
&\geq \bigvee_{b\in\mathcal{B}} [b, a_1] \wedge [b, a_2] \qquad \text{Lemma 1.1 } (xi) \\
&= \bigvee_{b\in\mathcal{B}} [b, a_1 \wedge a_2] = \top. \qquad \text{Lemma 1.1 } (vii)
\end{aligned}$$

Hence $\hat{\mathcal{B}}$ is a $\top$-filter base. Let $\mathfrak{G}$ denote the generated filter. Then clearly $\mathcal{B} \subseteq f^{\Rightarrow}\mathfrak{F}$. On the other hand if $a \in \mathfrak{F}$ then, again employing Lemma 1.1 $(xi)$, Lemma 1.3 and the fact that $\mathcal{B}$ is a base for $\mathfrak{F}$ we have, $\bigvee_{c\in\mathfrak{G}}[c, f^{\rightarrow}(a)] = \bigvee_{b\in\mathcal{B}}[f^{\rightarrow}(b), f^{\rightarrow}(a)] \geq \bigvee_{b\in\mathcal{B}}[b, a] = \top$. Hence the result follows.

□

**Definition 1.4.** A $\top$-filter $\mathfrak{F} \in \mathfrak{F}_L^{\top}(X)$ is called a **maximal** $\top$-filter on $X$ or a $\top$-**ultrafilter** on $X$ if for any $\top$-filter $\mathfrak{G}$ on $X$, $\mathfrak{F} \subseteq \mathfrak{G}$ implies $\mathfrak{F} = \mathfrak{G}$.

**Proposition 1.1.** *$\top$-filters generated by a point are maximal.*

*Proof.* Let $x \in X$ and suppose that $[x]$ is not maximal. Then there exists some $\top$-filter $\mathfrak{F}$ so that $[x] \subsetneq \mathfrak{F}$. Let $a \in \mathfrak{F} \setminus [x]$. Then $\mathbf{1}_{\{x\}} \wedge a \in \mathfrak{F}$. Since $a \notin [x]$, $a(x) < \top$, it follows that $(\mathbf{1}_{\{x\}} \wedge a)(t) = \begin{cases} a(x) & t = x \\ \bot & t \neq x \end{cases}$. But then $\bigvee_{t\in X}(\mathbf{1}_{\{x\}} \wedge a)(t) = a(x) < \top$, a contradiction. □

# CHAPTER 2: $\top$-CONVERGENCE SPACES

Now that we have a notion of a filter, we may define filter convergence.

## Definitions and Categorical Properties of $\top$-**Conv**

**Definition 2.1.** Assume that $L$ is a frame and $X$ a set. A function $q : \mathfrak{F}_L^{\top}(X) \longrightarrow 2^X$ is called a $\top$**-convergence structure on** $X$ provided:

($\top$CS1) $[x] \xrightarrow{q} x$ for all $x \in X$, and

($\top$CS2) if $\mathfrak{F} \xrightarrow{q} x$ and $\mathfrak{F} \subseteq \mathfrak{G}$, then $\mathfrak{G} \xrightarrow{q} x$.

Note that $\mathfrak{F} \xrightarrow{q} x$ is shorthand for $x \in q(\mathfrak{F})$. The pair $(X, q)$ is called a $\top$**-convergence space**.

A map $f : (X, q) \longrightarrow (Y, p)$ between two $\top$-convergence spaces is **continuous** if $f^{\Rightarrow}(\mathfrak{F}) \xrightarrow{p} f(x)$ whenever $\mathfrak{F} \xrightarrow{q} x$. Let $\top$-**Conv** denote the category whose objects are all the $\top$-convergence spaces and whose morphisms are all the continuous maps between objects. It has been shown by Fang and Yu that the category $\top$-**Conv** is a topological construct and is Cartesian closed.[1] Since $\top$-**Conv** is a topological construct we may say that $(X, q)$ is a **subspace** of $(Y, p)$ if $X \subseteq Y$ and $q$ is the initial structure on $X$ with respect to the natural injection $j : X \longrightarrow (Y, p)$.

The notion of convergence of a stratified $L$-filter has also been defined.

**Definition 2.2.** Suppose that $L$ is a frame and $X$ a set. The pair $(X, \overline{q})$, where $\overline{q} = (q_\alpha)_{\alpha \in L}$

[1] See Appendix for definitions of topological constructs, Cartesian closed categories and initial structures.

and $q_\alpha : \mathfrak{F}^S_L(X) \longrightarrow 2^X$, is called a **stratified $L$-convergence space** provided it satisfies:

(SL1) $\dot{x} \xrightarrow{q_\alpha} x$ and $\nu_\perp \xrightarrow{q_\perp} x$, for each $x \in X$ and $\alpha \in L$,

(SL2) $\mu \geq \nu \xrightarrow{q_\alpha} x$ implies $\mu \xrightarrow{q_\alpha} x$, and

(SL3) if $\mu \xrightarrow{q_\beta} x$ and $\alpha \leq \beta$, then $\mu \xrightarrow{q_\alpha} x$.

Again, note that $\mu \xrightarrow{q_\alpha} x$ is shorthand for $x \in q_\alpha(\mu)$. Intuitively, we may think of $\nu \xrightarrow{q_\alpha} x$ to mean that $\nu$ converges to $x$ with certainty $\alpha$. A map $f : (X, \overline{q}) \longrightarrow (Y, \overline{p})$ between two stratified $L$-convergence spaces is said to be **continuous** provided that $f^{\Uparrow}(\mu) \xrightarrow{p_\alpha} f(x)$ whenever $\mu \xrightarrow{q_\alpha} x$. Let **S$L$-CS** denote the category whose objects are all the stratified $L$-convergence spaces and whose morphisms are all the continuous maps between objects.

The following results due to Höhle ([11], [10]) provide a connection between $\top$-filters and stratified $L$-filters.

**Theorem 2.1.** *(See* [11], [10]*.)*

*(i) Assume that $L$ is a frame, $\mathfrak{F} \in \mathfrak{F}^\top_L(X)$, and define $\nu_{\mathfrak{F}}(a) = \bigvee_{b \in \mathfrak{F}} [b, a]$ for each $a \in L^X$. Then $\nu_{\mathfrak{F}} \in \mathfrak{F}^S_L(X)$ and $\mathfrak{F} = \{a \in L^X : \nu_{\mathfrak{F}}(a) = \top\}$.*

*(ii) Suppose that the frame $L$ is also a Boolean algebra, $\nu \in \mathfrak{F}^S_L(X)$, and define $\mathfrak{F}_\nu = \{a \in L^X : \nu(a) = \top\}$. Then the map $\nu \mapsto \mathfrak{F}_\nu$ is an order preserving bijection from $\mathfrak{F}^S_L(X)$ onto $\mathfrak{F}^\top_L(X)$. In particular, a $\top$-filter $\mathfrak{F}$ is maximal if and only if $\nu_{\mathfrak{F}}$ is maximal.*

Throughout this work, if **Cat** is a category, we let $|$**Cat**$|$ denote the objects of the category **Cat** and will write $A \in |$**Cat**$|$ to mean $A$ is an object of **Cat**.

**Theorem 2.2.** *The construct* $\top$-**Conv** *is extensional.* [2]

[2]See Appendix for definitions of constructs and extensional constructs.

*Proof.* Let $(X,q) \in |\top\text{-}\mathbf{Conv}|$. Define $(X^*,q^*)$ by $X^* = X \cup \{\infty_X\}$ where $\infty_X$ is not an element of $X$, and $q^*$ is given by the following:

(i) $\mathfrak{F} \xrightarrow{q^*} \infty_X$ for all $\mathfrak{F} \in \mathfrak{F}_L^\top(X^*)$

(ii) If $\mathfrak{F} \supseteq [\infty]_X$ then $\mathfrak{F} \xrightarrow{q^*} x$ for all $x \in X^*$

(iii) $\mathfrak{F} \xrightarrow{q^*} x$ provided $j^{\Leftarrow}\mathfrak{F} \xrightarrow{q} x$ or $j^{\Leftarrow}\mathfrak{F}$ fails to exist

where $j : X \longrightarrow X^*$ is the natural injection. We must show that $(X^*,q^*)$ is a $\top$ convergence space. Note that $[\infty_X] \xrightarrow{q^*} \infty_X$ by definition. On the other hand, if $x \in X$ then $j^{\Leftarrow}[x]_{X^*} = [x]_X$. Indeed, a base for $j^{\Leftarrow}[x]_{X^*}$ is $\{b \circ j : b \in [x]_{X^*}\}$. If $b \in [x]_{X^*}$, then $b(x) = \top$. Then $(b \circ j)(x) = b(j(x)) = b(x) = \top$ and $b \circ j \in [x]_X$. Thus $j^{\Leftarrow}[x]_{X^*} \subseteq [x]_X$. On the other hand, if $a \in [x]$ we may define $a^* \in L^{X^*}$ by $a^*(x) = x$, $x \in X$ and $a^*(\infty_X) = \top$. Then $a^* \in [x]_{X^*}$ and $a = a^* \circ j$. Thus $a \in j^{\Leftarrow}[x]_{X^*}$ and $j^{\Leftarrow}[x]_{X^*} = [x]_X$ as desired. Hence we have that $[x]_{X^*} \xrightarrow{q^*} x$ for any $x \in X^*$ and ($\top$CS1) is valid.

Next suppose that $\mathfrak{G} \supseteq \mathfrak{F} \xrightarrow{q^*} x$. If $x = \infty_X$ then $\mathfrak{G} \xrightarrow{q^*} \infty_X$; if $\mathfrak{F} \supseteq [\infty_X]$ then $\mathfrak{G} \supseteq [\infty_X]$ and thus $\mathfrak{G} \xrightarrow{q^*} x$; and if $j^{\Leftarrow}\mathfrak{F} \xrightarrow{q} x$, then $j^{\Leftarrow}\mathfrak{F} \subseteq j^{\Leftarrow}\mathfrak{G}$ and thus $j^{\Leftarrow}\mathfrak{G} \xrightarrow{q} x$ so that $\mathfrak{G} \xrightarrow{q^*} x$ also. Hence ($\top$CS2) is verified and we have that $(X^*,q^*) \in |\top\text{-}\mathbf{Conv}|$.

Next let $(Y,p)$, $(Z,r) \in |\top\text{-}\mathbf{Conv}|$ such that $(Y,p)$ is a subspace of $(Z,r)$. Also let $f : (Y,p) \longrightarrow (X,q)$ be continuous in $\top$-**Conv** and define $f^* : Z \longrightarrow X^*$ by

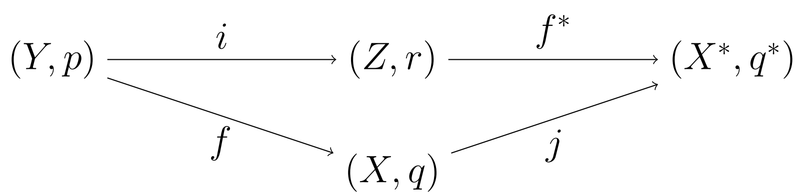
$$f^*(z) = \begin{cases} f(z), & z \in Y \\ \infty_X, & z \notin Y \end{cases}.$$

We must show that $f^* : (Z,r) \longrightarrow (X^*,q^*)$ is continuous in $\top$-**Conv**.

Suppose that $\mathfrak{F} \xrightarrow{r} z$. If $z \notin Y$ then $f^{*\Rightarrow}\mathfrak{F} \xrightarrow{q^*} f(z) = \infty_X$.

Next, suppose that $z \in Y$ and let $i : Y \longrightarrow Z$ be the natural injection. Further, suppose that $i^{\Leftarrow}\mathfrak{F}$ exists.

The following diagram is provided for convenience.




By virtue of $(Y,p)$ being a subspace of $(Z,r)$ we have that $Y \subseteq Z$ and $p : \mathfrak{F}_L^{\top}(Y) \longrightarrow 2^Y$ is given by $p(\mathfrak{G}) = \{y \in Y : i^{\Rightarrow}\mathfrak{G} \xrightarrow{r} i(y)\}$. Hence $i^{\Rightarrow}i^{\Leftarrow}\mathfrak{F} = \mathfrak{F} \xrightarrow{r} z = i(z)$ and thus $i^{\Leftarrow}\mathfrak{F} \xrightarrow{p} z$. By the continuity of $f$, this implies that $f^{\Rightarrow}i^{\Leftarrow}\mathfrak{F} \xrightarrow{q} f(z)$. We claim that $j^{\Rightarrow}j^{\Leftarrow}f^{\Rightarrow}i^{\Leftarrow}\mathfrak{F} = j^{\Leftarrow}f^{*\Rightarrow}\mathfrak{F}$. Indeed, if $a \in \mathfrak{F}$ and $x \in X$ then

$$\begin{aligned} j^{\leftarrow}(j^{\rightarrow}(f^{\rightarrow}(i^{\leftarrow}(a))))(x) &= \bigvee_{j(f(w))=j(x)} a(i(w)) = \bigvee_{f(w)=j(x)} a(w) = \bigvee_{f^*(w)=j(x)} a(w) \\ &= j^{\leftarrow}(f^{*\rightarrow}(a))(x). \end{aligned}$$

Since the left hand side and right hand side of the above are base members for $j^{\Rightarrow}j^{\Leftarrow}f^{\Rightarrow}i^{\Leftarrow}\mathfrak{F}$ and $j^{\Leftarrow}f^{*\Rightarrow}\mathfrak{F}$, respectively, our claim is justified. But $j^{\Rightarrow}j^{\Leftarrow}f^{\Rightarrow}i^{\Leftarrow}\mathfrak{F} = f^{\Rightarrow}i^{\Leftarrow}\mathfrak{F} \xrightarrow{q} f(z)$ and therefore $j^{\Leftarrow}f^{*\Rightarrow}\mathfrak{F} \xrightarrow{q} f(z)$. By definition of $q^*$ this implies that $f^{*\Rightarrow}\mathfrak{F} \xrightarrow{q} f(z)$.

Finally, if $i \Leftarrow \mathfrak{F}$ fails to exist, we will show that $j^{\Leftarrow} f^{*\Rightarrow}\mathfrak{F}$ also fails to exist. We compute,

$$\bigvee_{x\in X} j^{\leftarrow}\Big(f^{*\rightarrow}(a)\Big((x) = \bigvee_{x\in X} \bigvee_{\substack{f^*(t)=j(x)\\ t\in Z}} a(t)$$

$$= \bigvee_{x\in X} \bigvee_{\substack{f(i(s))=j(x)\\ s\in Y}} a(i(s)) = \bigvee_{x\in X} \bigvee_{\substack{f(i(s))=j(x)\\ s\in Y}} a(s)$$

$$\leq \bigvee_{s\in Y} a(i(s)).$$

Now since $i^{\Leftarrow}\mathfrak{F}$ fails to exist, there must exist some $a \in \mathfrak{F}$ such that $\bigvee_{s\in Y} \Big(i^{\leftarrow}a\Big)(s) = \bigvee_{s\in Y} a(i(s)) < \top$. Hence for this same $a \in \mathfrak{F}$, $\bigvee_{x\in X} j^{\leftarrow}\Big(f^{*\rightarrow}(a)\Big((x) < \top$ and $j^{\Leftarrow} f^{*\Rightarrow}\mathfrak{F}$ fails to exist. Therefore $f^{*\Rightarrow}\mathfrak{F} \xrightarrow{q^*} z = f^*(z)$. Hence $f^* : (Z, r) \longrightarrow (X^*, q^*)$ is continuous and we have that $\top$-**Conv** is extensional. □

Next we wish to show that the product of quotient maps is a quotient map. Before we can do so, we must explore the notion of an arbitrary product of $\top$-filters. Let $f_j : X_j \longrightarrow Y_j$, $j \in J$ be a family of maps. Denote $X = \prod_{j\in J} X_j$, $Y = \prod_{j\in J} Y_j$ and let $f : X \longrightarrow Y$ be the product map; that is, $f(x) = \Big(f_j(x_j)\Big)$, where $x = (x_j) \in X$. Let $\pi_{X_j} : X \longrightarrow X_j$ and $\pi_{Y_j} : Y \longrightarrow Y_j$ be the natural projections.

**Lemma 2.1.** *Assume that $a \in L^{X_j}$ and $b \in L^{X_k}$ for $j \neq k$. Then $f^{\rightarrow}\Big(\pi_{X_j}^{\leftarrow} a \wedge \pi_{X_k}^{\leftarrow} b\Big) = \pi_{Y_j}^{\leftarrow} f_j^{\rightarrow}(a) \wedge \pi_{Y_k}^{\leftarrow} f_k^{\rightarrow}(b)$.*

*Proof.* Fix $y = (y_i)$, $i \in J$. Then for $j \neq k$, $f^{\rightarrow}\Big(\pi_{X_j}^{\leftarrow} a \wedge \pi_{X_k}^{\leftarrow} b\Big)(y) = \bigvee_{f(x)=y} \pi_{X_j}^{\leftarrow} a(x) \wedge \pi_{X_k}^{\leftarrow} b(x) = \bigvee_{\substack{f_j(x_j)=y_j\\ f_k(x_k)=y_k}} a(x_j) \wedge b(x_k) = \Big(\bigvee_{f_j(s)=y_j} a(s)\Big) \wedge \Big(\bigvee_{f_k(t)=y_k} b(t)\Big) = \Big(f_j^{\rightarrow} a\Big)(y_j) \wedge \Big(f_k^{\rightarrow} b\Big)(y_k) = \Big(\pi_{Y_j}^{\leftarrow} f_j^{\rightarrow} a\Big)(y) \wedge \Big(\pi_{Y_k}^{\leftarrow} f_k^{\rightarrow} b\Big)(y) = \Big(\pi_{y_j}^{\leftarrow} f_j^{\rightarrow}(a) \wedge \pi_{Y_k}^{\leftarrow} f_k^{\rightarrow}(b)\Big)(y)$. Hence the result follows. □

**Corollary 2.1.** *Let $S$ be a finite collection of distinct members of $J$ and $a_j \in L^{X_j}$ for each*

$j \in S$. *Then* $f^{\rightarrow}\Big(\bigwedge_{j\in S} \pi_{X_j}^{\leftarrow} a_j\Big) = \bigwedge_{j\in S} \pi_{Y_j}^{\leftarrow} f_j^{\rightarrow}(a_j)$.

Suppose that $\mathfrak{F}_i \in \mathfrak{F}_L^{\top}(X_j)$, $j \in J$. Define $\mathcal{B} = \Big\{\bigwedge_{i\in S} \pi_{X_i}^{\leftarrow} a_i : a_i \in \mathfrak{F}_i, S \subseteq J, |S| < \infty\Big\}$. Note that if $a \in \mathfrak{F}_j$ and $b \in \mathfrak{F}_k$, $j \neq k$, then $\bigvee_{x\in X} \Big(\pi_{X_j}^{\leftarrow} a \wedge \pi_{X_k}^{\leftarrow} b\Big)(x) = \bigvee_{x=(x_i)\in X} a(x_j) \wedge b(x_k) = \bigvee_{\substack{s\in X_j \\ t\in X_k}} a(s) \wedge b(t) = \Big(\bigvee_{s\in X_j} a(s)\Big) \wedge \Big(\bigvee_{t\in X_k} b(t)\Big) = \top \wedge \top = \top$. Further, this is valid for any finite number of terms and so it follows that $\mathcal{B}$ is a $\top$-filter base.

**Definition 2.3.** If $\mathfrak{F}_j \in \mathfrak{F}_L^{\top}(X_j)$, $j \in J$, then the **product $\top$-filter** is defined to be the $\top$-filter on $X$ having base $\mathcal{B} = \Big\{\bigwedge_{i\in S} \pi_{X_i}^{\leftarrow} a_i : a_i \in \mathfrak{F}_i, S \subseteq J, |S| < \infty\Big\}$ and is denoted by $\prod_{j\in J} \mathfrak{F}_j$.

The following lemma justifies the above definition.

**Lemma 2.2.** *Let* $\mathfrak{F}_j \in \mathfrak{F}_L^{\top}(X_j)$, $j \in J$. *Then* $\prod_{j\in J} \mathfrak{F}_j$ *is the coarsest (smallest)* $\top$*-filter on* $X$ *containing* $\pi_{X_j}^{\Leftarrow} \mathfrak{F}_j$, *for each* $j \in J$.

*Proof.* It is clear that $\prod_{j\in J} \mathfrak{F}_j$ contains $\pi_{X_j}^{\Leftarrow} \mathfrak{F}_j$ for each $j \in J$. Indeed, a base member of $\pi_{X_j}^{\Leftarrow} \mathfrak{F}_j$ is $\pi_{X_j}^{\leftarrow} a_j$ where $a_j \in \mathfrak{F}_j$ and, by definition, $\pi_{X_j}^{\leftarrow} a_j$ is also in $\mathcal{B}$, a base for $\prod_{j\in J} \mathfrak{F}_j$. Next suppose that $\mathfrak{G} \in \mathfrak{F}_L^{\top}\Big(\prod_{j\in J} X_j\Big)$ such that $\pi_{X_j}^{\Leftarrow} \mathfrak{F}_j \subseteq \mathfrak{G}$ for each $j \in J$. A general base member of $\prod_{j\in J} \mathfrak{F}_j$ is given by $\bigwedge_{i\in S} \pi_{X_j}^{\leftarrow} a_i$ where $S \subseteq J$ is finite and $a_i \in \mathfrak{F}_i$, $i \in S$. But $\pi_{X_i}^{\leftarrow} a_i \in \pi_{X_i}^{\Leftarrow} \mathfrak{F}_i \subseteq \mathfrak{G}$ for each $i \in S$. Hence $\bigwedge_{i\in S} \pi_{X_j}^{\leftarrow} a_i \in \mathfrak{G}$ as $\top$-filters are closed under finite meets. Thus $\prod_{j\in J} \mathfrak{F}_j \subseteq \mathfrak{G}$. $\square$

**Lemma 2.3.** *Let* $f_j : (X_j, q_j) \longrightarrow (Y_j, p_j)$, $j \in J$ *be maps and let* $f$ *be the product map. Then* $f^{\Rightarrow}\left(\prod_{j\in J} \mathfrak{F}_j\right) = \prod_{j\in J} f_j^{\Rightarrow} \mathfrak{F}_j$.

*Proof.* It suffices to show that any base member of the left hand side is contained in the right hand side. Let $a_i \in \mathfrak{F}_i$ for each $i \in S \subseteq J$, $|S| < \infty$. Employing Lemma 1.4 (iii), a base member of the left hand side is given by $f^{\rightarrow}\left(\bigwedge_{i\in S}\pi_{X_i}^{\leftarrow}a_i\right)$. Employing Corollary 2.1 we have $f^{\rightarrow}\left(\bigwedge_{i\in S}\pi_{X_i}^{\leftarrow}a_i\right)=\bigwedge_{i\in S}\pi_{Y_i}^{\leftarrow}f_i^{\rightarrow}(a_i)\in\prod_{j\in J}f_j^{\Rightarrow}\mathfrak{F}_j$, and the result follows. □

**Lemma 2.4.** *If $\mathfrak{F}\in\mathfrak{F}_L^{\top}(X)$ with $X=\prod_{j\in J}X_j$ then $\mathfrak{F}\geq\prod_{j\in J}\pi_{X_j}^{\Rightarrow}\mathfrak{F}$.*

*Proof.* Note that if $a\in\mathfrak{F}$ and $x=(x_i)\in X$ then $\pi_{X_i}^{\leftarrow}\pi_{X_i}^{\rightarrow}a(x)=\bigvee_{\pi_{X_i}(z)=x_i}a(z)\geq a(x)$. Using Lemma 1.4 (iii), if $a\in\mathfrak{F}$ and $S$ is a finite subset of $J$, a base member of the right hand side is given by

$$\bigwedge_{i\in S}\pi_{X_i}^{\leftarrow}\pi_{X_i}^{\rightarrow}(a)\geq\bigwedge_{i\in S}a=a\in\mathfrak{F}.$$

The result follows. □

**Lemma 2.5.** *Let $(X,q)\in|\top\text{-}\mathbf{Conv}|$ and $f:(X,q)\longrightarrow Y$ be a surjection. Then the final structure with respect to this sink is given by $p(\mathfrak{G})=\{y\in Y:\exists\mathfrak{F}\in\mathfrak{F}_L^{\top}(X),\mathfrak{G}\geq f^{\Rightarrow}\mathfrak{F}$ and $\mathfrak{F}\xrightarrow{q}x\in f^{-1}(y)\}$.*

*Proof.* We must show that $p:\mathfrak{F}_L^{\top}(Y)\longrightarrow 2^Y$ is a $\top$-convergence structure on $Y$ and that a map $g:(Y,p)\longrightarrow(Z,r)$ is continuous if and only if $g\circ f:(X,q)\longrightarrow(Z,r)$ is continuous.

First to show that $p$ is a $\top$-convergence structure, note that if $\mathfrak{F}\subseteq\mathfrak{G}$ then clearly $p(\mathfrak{F})\subseteq p(\mathfrak{G})$ and so ($\top$CS2) is valid. Also, since $f^{\Rightarrow}[x]=[f(x)]=[y]$ for any $x\in f^{-1}(y)$, ($\top$CS1) is also valid and $(Y,p)\in|\top\text{-}\mathbf{Conv}|$.

Next let $g:(Y,p)\longrightarrow(Z,r)$ be a map. If $g$ is continuous then clearly $g\circ f$ is also continuous. On the other hand, if $g\circ f$ is continuous, let $\mathfrak{G}\xrightarrow{p}y$. Then there exists $\mathfrak{F}\in\mathfrak{F}_L^{\top}(X)$ such that $\mathfrak{G}\geq f^{\Rightarrow}\mathfrak{F}$ and $\mathfrak{F}\xrightarrow{q}x\in f^{-1}(y)$. Then $g^{\Rightarrow}\mathfrak{G}\geq g^{\Rightarrow}f^{\Rightarrow}\mathfrak{F}=(g\circ f)^{\Rightarrow}\mathfrak{F}\xrightarrow{r}(g\circ f)(x)=g(y)$. Hence $g$ is continuous. □

The following lemma is a direct result of Lemma 2.5.

**Lemma 2.6.** *A surjection $f:(X,q)\longrightarrow(Y,p)$ is a quotient map iff for all $\mathfrak{G}\xrightarrow{p}y$, there exists $\mathfrak{F}\xrightarrow{q}x\in f^{-1}(y)$ such that $\mathfrak{G}\geq f^{\Rightarrow}\mathfrak{F}$.*

**Theorem 2.3.** *In $\top$-**Conv**, the product of quotient maps are quotient maps.*

*Proof.* Let $f_j:X_j\longrightarrow Y_j$, $j\in J$ be a family of surjections. Denote $X=\prod\limits_{j\in J}X_j$, $Y=\prod\limits_{j\in J}Y_j$ and let $f:X\longrightarrow Y$ be the product map; that is, $f(x)=\big(f_j(x_j)\big)$, where $x=(x_j)\in X$. Let $\pi_{Y_j}:Y\longrightarrow Y_j$ be the natural projection. Assume that $\mathfrak{G}\xrightarrow{p}y=(y_j)$; let $\mathfrak{G}_j=\pi_{Y_j}^{\Rightarrow}\mathfrak{G}$, $j\in J$. Then by Lemma 2.4, $\mathfrak{G}\geq\prod\limits_{j\in J}\mathfrak{G}_j$. Since $f_j:(X_j,q_j)\longrightarrow(Y_j,p_j)$ is a quotient map and $\mathfrak{G}_j\xrightarrow{p_j}y_j$, there exists $\mathfrak{F}_j\xrightarrow{q_j}f_j^{-1}(y_j)$ such that $f_j^{\Rightarrow}\mathfrak{F}_j\leq\mathfrak{G}_j$, $j\in J$. It follows from Corollary 2.1 and Lemma 2.3 that $f^{\Rightarrow}\Big(\prod\limits_{j\in J}\mathfrak{F}_j\Big)=\prod\limits_{j\in J}f_j^{\Rightarrow}\mathfrak{F}_j\leq\prod\limits_{j\in J}\mathfrak{G}_j\leq\mathfrak{G}$. Since $\prod\limits_{j\in J}\mathfrak{F}_j\xrightarrow{q}f^{-1}(y)=\big(f_j^{-1}(y_j)\big)$, it follows that $f:(X,q)\longrightarrow(Y,p)$ is a quotient map. □

**Theorem 2.4.** *The category $\top$-**Conv** is a strongly topological universe.* [3]

*Proof.* It was shown by Fang and Yu [29] that $\top$-**Conv** is both a topological construct and is Cartesian closed. Theorems 2.2 and 2.3 show $\top$-**Conv** is extensional and the product of quotient maps are quotient maps. □

## Embedding $\top$-**Conv** in **S*L*-CS**

It is shown in this section that the category $\top$-**Conv** can be embedded in **S*L*-CS**.

**Lemma 2.7.** *Let $L$ be a frame, $f:X\longrightarrow Y$ a map, and $\mathfrak{F}\in\mathfrak{F}_L^{\top}(X)$; then $f^{\Uparrow}(\nu_{\mathfrak{F}})=\nu_{f^{\Rightarrow}(\mathfrak{F})}$.*

*Proof.* Let $c\in L^X$. By definition, $f^{\Uparrow}(\nu_{\mathfrak{F}})(c)=\nu_{\mathfrak{F}}(f^{\leftarrow}(c))=\bigvee\limits_{d\in\mathfrak{F}}[d,f^{\leftarrow}(c)]$. According to Lemma 1.1 (xi), $[d,f^{\leftarrow}(c)]\leq[f^{\rightarrow}(d),f^{\rightarrow}(f^{\leftarrow}(c))]$. Since $f^{\rightarrow}(f^{\leftarrow}(c))\leq c$ and $[\bullet,\bullet]$ is in-

[3]See Appendix for the definition of a strong topological universe.

creasing in the second component, we have $f^{\Uparrow}(\nu_{\mathfrak{F}})(c) \leq \bigvee_{d \in \mathfrak{F}} [f^{\rightarrow}(d), c]$. Since $\{f^{\rightarrow}(d) : d \in \mathfrak{F}\}$ is a $\top$-filter base for $f^{\Rightarrow}(\mathfrak{F})$, it follows from Lemma 1.3 that $\bigvee_{d \in \mathfrak{F}} [f^{\rightarrow}(d), c] = \bigvee_{e \in f^{\Rightarrow}(\mathfrak{F})} [e, c] = \nu_{f^{\Rightarrow}(\mathfrak{F})}(c)$. Hence $f^{\Uparrow}(\nu_{\mathfrak{F}}) \leq \nu_{f^{\Rightarrow}(\mathfrak{F})}$.

Conversely, $\nu_{f^{\Rightarrow}(\mathfrak{F})}(c) = \bigvee_{b \in \mathfrak{F}} [f^{\rightarrow}(b), c]$ and by Lemma 1.1 (xi), $[f^{\rightarrow}(b), c] \leq [f^{\leftarrow}(f^{\rightarrow}(b)), f^{\leftarrow}(c)]$. Since $b \leq f^{\leftarrow}(f^{\rightarrow}(b))$, it follows that $\nu_{f^{\Rightarrow}(\mathfrak{F})}(c) \leq \bigvee_{b \in \mathfrak{F}} [b, f^{\leftarrow}(c)] = \nu_{\mathfrak{F}}(f^{\leftarrow}(c))$, and thus $\nu_{f^{\Rightarrow}(\mathfrak{F})} \leq f^{\Uparrow}(\nu_{\mathfrak{F}})$. Therefore $f^{\Uparrow}(\nu_{\mathfrak{F}}) = \nu_{f^{\Rightarrow}(\mathfrak{F})}$. □

**Definition 2.4.** Given $(X, q) \in |\top\text{-}\mathbf{Conv}|$, we define $(X, \overline{q}_*)$, $\overline{q}_* = (q_{*,\alpha})_{\alpha \in L}$ as follows:

(i) $\mu \xrightarrow{q_{*,\top}} x$ if and only if there exists $\mathfrak{F} \xrightarrow{q} x$ such that $\mu \geq \nu_{\mathfrak{F}}$, and

(ii) for $\alpha < \top$, $\mu \xrightarrow{q_{*,\alpha}} x$ if and only if $\mu \geq \nu_{\perp}$.

**Lemma 2.8.** *If* $(X, q) \in |\top\text{-}\mathbf{Conv}|$, *then* $(X, \overline{q}_*) \in |\boldsymbol{SL}\text{-}\boldsymbol{CS}|$.

*Proof.* Note that $\dot{x} = \nu_{[x]}$. Indeed, $\nu_{[x]}(a) = \bigvee_{b \in [x]} [b, a] = [\mathbf{1}_{\{x\}}, a] = \bigwedge_{z \in X} \Big(\mathbf{1}_{\{x\}}(z) \rightarrow a(z) = \bigwedge_{z \in X} \begin{cases} \top \rightarrow a(x), & z = x \\ \perp \rightarrow a(z), & z \neq x \end{cases} = \bigwedge_{z \in X} \begin{cases} a(x), & z = x \\ \top, & z \neq x \end{cases} = a(x) = \dot{x}(a)$. Since $[x] \xrightarrow{q} x$, we have that $\dot{x} \xrightarrow{q_{*,\alpha}} x$ for each $x \in X$ and $\alpha \in L$. Also, condition (ii) in Definition 2.4 assures that $\nu_{\perp} \xrightarrow{q_{*,\perp}} x$ for all $x \in X$ and thus (S$L$1) is valid.

Next assume $\mu \geq \nu \xrightarrow{q_{*,\alpha}} x$. Then if $\alpha < \top$ then there is nothing to prove. If $\alpha = \top$, then for some $\mathfrak{F} \xrightarrow{q} x$, $\nu \geq \nu_{\mathfrak{F}}$. But then $\mu \geq \nu \geq \nu_{\mathfrak{F}}$ implies $\mu \xrightarrow{q_{*,\top}} x$ also. Hence (S$L$2) is valid.

Finally suppose that $\mu \xrightarrow{q_{*,\beta}} x$ and $\alpha \leq \beta$. If $\alpha = \beta$, there is nothing to prove. If $\alpha < \beta \leq \top$, then condition (ii) in Definition 2.4 assures that $\mu \xrightarrow{q_{*,\alpha}} x$ and (S$L$3) is verified. Hence the result holds. □

**Definition 2.5.** Further, given $(X, q) \in |\top\text{-}\mathbf{Conv}|$, define $\overline{q}^* = (q^*_{\alpha})_{\alpha \in L}$ as follows:

(a) $\mu \xrightarrow{q^*_\perp} x$ if and only if $\mu \geq \nu_\perp$, and

(b) for $\alpha > \perp$, $\mu \xrightarrow{q^*_\alpha} x$ if and only if there exists $\mathfrak{F} \xrightarrow{q} x$ such that $\mu \geq \nu_{\mathfrak{F}}$.

**Lemma 2.9.** *If* $(X, q) \in |\top\text{-}\mathsf{Conv}|$, *then* $(X, \overline{q}^*) \in |\boldsymbol{SL\text{-}CS}|$.

*Proof.* Clearly $\nu_\perp \xrightarrow{q^*_\perp} x$ is satisfied for all $x \in X$. Also, as verified in the proof of Lemma 2.8, $\nu_{[x]} = \dot{x}$ and hence $\dot{x} \xrightarrow{q^*_\alpha} x$ for each $x \in X$ and $\alpha \in L$. Thus (S$L$1) is valid.

Next assume $\mu \geq \nu \xrightarrow{q^*_\alpha} x$. Then if $\alpha = \perp$ then there is nothing to prove. If $\alpha > \perp$, then for some $\mathfrak{F} \xrightarrow{q} x$, $\nu \geq \nu_{\mathfrak{F}}$. But then $\mu \geq \nu \geq \nu_{\mathfrak{F}}$ implies $\mu \xrightarrow{q^*_\alpha} x$ also. Hence (S$L$2) is valid.

Finally suppose that $\mu \xrightarrow{q^*_\beta} x$ and $\alpha \leq \beta$. If $\alpha = \perp$ there is nothing to prove. If $\alpha > \perp$ then $\beta > \perp$ and thus there exists some $\mathfrak{F} \xrightarrow{q} x$ such that $\mu \geq \nu_{\mathfrak{F}}$. Hence $\mu \xrightarrow{q^*_\alpha} x$ and (S$L$3) is valid. Hence the result holds. $\square$

Observe that for each $\alpha \in L$, $q^*_\alpha \geq q_{*,\alpha}$ and thus $\overline{q}^* \geq \overline{q}_*$. Let $E_*$ and $E^*$ denote the full subcategories of **S$L$-CS** whose objects are of the form $(X, \overline{q}_*)$ and $(X, \overline{q}^*)$, respectively, where $(X, q) \in |\top\text{-}\mathsf{Conv}|$.

**Lemma 2.10.** *The categories* $\top\text{-}\mathsf{Conv}$, $E_*$ *and* $E^*$ *are isomorphic.*

*Proof.* Let the functor $\theta : \top\text{-}\mathsf{Conv} \longrightarrow E_*$ be defined by $\theta(X, q) = (X, \overline{q}_*)$ and $\theta(f) = f$. Suppose that $f : (X, q) \longrightarrow (Y, p)$ is continuous in $\top\text{-}\mathsf{Conv}$; it is shown that $f : (X, \overline{q}_*) \longrightarrow (Y, \overline{p}_*)$ is continuous in $E_*$. If $\mu \xrightarrow{q_{*,\top}} x$, then by definition there exists $\mathfrak{F} \xrightarrow{q} x$ with $\mu \geq \nu_{\mathfrak{F}}$. By Lemma 2.7, $f^{\Uparrow}(\mu) \geq f^{\Uparrow}(\nu_{\mathfrak{F}}) = \nu_{f^{\Rightarrow}(\mathfrak{F})}$, and since $f^{\Rightarrow}(\mathfrak{F}) \xrightarrow{p} f(x)$, it follows that $f^{\Uparrow}(\nu_{\mathfrak{F}}) \xrightarrow{p_{*,\top}} f(x)$ and hence $f^{\Uparrow}(\mu) \xrightarrow{p_{*,\top}} f(x)$. Next, if $\mu \xrightarrow{q_{*,\alpha}} x$, $\alpha < \top$, then $\mu \geq \nu_\perp$ on $X$ and thus $f^{\Uparrow}(\mu) \geq f^{\Uparrow}(\nu_\perp) \xrightarrow{p_{*,\alpha}} f(x)$ and $f^{\Uparrow}(\mu) \xrightarrow{p_{*,\alpha}} f(x)$. Therefore $f : (X, \overline{q}_*) \longrightarrow (Y, \overline{p}_*)$ is continuous and $\theta$ is a functor.

By definition, $\theta$ is a surjection onto the objects of $E_*$; next we show $\theta$ is an injection. Assume that $\theta(X,q)=\theta(X,p)$ and $\mathfrak{F}\xrightarrow{q}x$. We must show that $\mathfrak{F}\xrightarrow{p}x$. We have that $\nu_{\mathfrak{F}}\xrightarrow{q_{*,\top}}x$ and thus $\nu_{\mathfrak{F}}\xrightarrow{p_{*,\top}}x$. Thus there exists $\mathfrak{G}\xrightarrow{p}x$ such that $\nu_{\mathfrak{F}}\geq\nu_{\mathfrak{G}}$. If $c\in\mathfrak{G}$, then $\nu_{\mathfrak{G}}(c)=\top$ and thus $\nu_{\mathfrak{F}}(c)=\top$. By Theorem 2.1 (i) this implies that $c\in\mathfrak{F}$. Hence $\mathfrak{F}\geq\mathfrak{G}$ and $\mathfrak{F}\xrightarrow{p}x$, and thus $p=q$.

Finally, suppose that $f:(X,\overline{q}_*)\longrightarrow(Y,\overline{p}_*)$ is continuous in $E_*$; it is shown that $f:(X,q)\longrightarrow(Y,p)$ is continuous in $\top$-**Conv**. Assume that $\mathfrak{F}\xrightarrow{q}x$; then $\nu_{\mathfrak{F}}\xrightarrow{q_{*,\top}}x$ and thus $f^{\Uparrow}(\nu_{\mathfrak{F}})=\nu_{f^{\Rightarrow}(\mathfrak{F})}\xrightarrow{p_{*,\top}}f(x)$ by the continuity of $f$. It follows that there exists $\mathfrak{G}\xrightarrow{p}f(x)$ such that $\nu_{f^{\Rightarrow}(\mathfrak{F})}\geq\nu_{\mathfrak{G}}$, and thus as before $f^{\Rightarrow}(\mathfrak{F})\geq\mathfrak{G}$. Then $f^{\Rightarrow}(\mathfrak{F})\xrightarrow{p}f(x)$ and thus $f:(X,q)\longrightarrow(Y,p)$ is continuous in $\top$-**Conv**. Therefore $\theta:\top$-**Conv** $\longrightarrow E_*$ is an isomorphism.

Next, we show in a similar fashion that $\top$-**Conv** and $E^*$ are isomorphic. Let $\phi:\top$-**Conv** $\longrightarrow E^*$ be defined by $\phi(X,q)=(X,\overline{q}^*)$ and $\phi(f)=f$. Suppose that $f:(X,q)\longrightarrow(Y,p)$ is continuous in $\top$-**Conv**; it is shown that $f:(X,\overline{q}^*)\longrightarrow(Y,\overline{p}^*)$ is continuous in $E^*$. If $\mu\xrightarrow{q^*_\alpha}x$, $\alpha>\bot$, then by definition there exists $\mathfrak{F}\xrightarrow{q}x$ such that $\mu\geq\nu_{\mathfrak{F}}$. Then by Lemma 2.7, $f^{\Uparrow}(\mu)\geq f^{\Uparrow}(\nu_{\mathfrak{F}})=\nu_{f^{\Rightarrow}(\mathfrak{F})}$, and since $f^{\Rightarrow}(\mathfrak{F})\xrightarrow{p}f(x)$, it follows that $f^{\Uparrow}(\mu)\xrightarrow{p^*_\alpha}f(x)$. Next, if $\mu\xrightarrow{q^*_\bot}x$ then $\mu\geq\nu_\bot$ on $X$ and thus $f^{\Uparrow}(\mu)\geq f^{\Uparrow}(\nu_\bot)\xrightarrow{p^*_\bot}f(x)$ and thus $f^{\Uparrow}(\mu)\xrightarrow{p^*_\bot}f(x)$. Therefore $f:(X,\overline{q}^*)\longrightarrow(Y,\overline{p}^*)$ is continuous and $\phi$ is a functor.

By definition $\phi$ is a surjection onto the objects of $E^*$; next we show it is an injection. Assume that $\phi(X,q)=\phi(X,p)$ and $\mathfrak{F}\xrightarrow{q}x$. We must show that $\mathfrak{F}\xrightarrow{p}x$. We have that for each $\alpha>\bot$, $\nu_{\mathfrak{F}}\xrightarrow{q^*_\alpha}x$ and thus for each $\alpha>\bot$, we also have $\nu_{\mathfrak{F}}\xrightarrow{p^*_\alpha}x$. Thus there exists $\mathfrak{G}\xrightarrow{p}x$ such that $\nu_{\mathfrak{F}}\geq\nu_{\mathfrak{G}}$. If $c\in\mathfrak{G}$, then $\nu_{\mathfrak{G}}(c)=\top$ and thus $\nu_{\mathfrak{F}}(c)=\top$. By Theorem 2.1 (i), this implies that $c\in\mathfrak{F}$. Hence $\mathfrak{F}\geq\mathfrak{G}$ and $\mathfrak{F}\xrightarrow{p}x$, and thus $p=q$.

Finally, suppose that $f:(X,\overline{q}^*)\longrightarrow(Y,\overline{p}^*)$ is continuous; it is shown that $f:(X,q)\longrightarrow(Y,p)$ is continuous. Assume that $\mathfrak{F}\xrightarrow{q}x$; then $\nu_{\mathfrak{F}}\xrightarrow{q^*_\alpha}x$ and thus $f^{\Uparrow}(\nu_{\mathfrak{F}})=\nu_{f^{\Rightarrow}(\mathfrak{F})}\xrightarrow{p^*_\alpha}f(x)$

by the continuity of $f$. Therefore, there exists $\mathfrak{G} \xrightarrow{p} f(x)$ with $\nu_{f^{\Rightarrow}(\mathfrak{F})} \geq \nu_{\mathfrak{G}}$, and thus as before $f^{\Rightarrow}(\mathfrak{F}) \geq \mathfrak{G}$. Then $f^{\Rightarrow}(\mathfrak{F}) \xrightarrow{p} f(x)$ and so $f : (X, q) \longrightarrow (Y, p)$ is continuous. Therefore $\phi : \top\text{-}\mathbf{Conv} \longrightarrow E^*$ is an isomorphism. □

**Lemma 2.11.** *Assume that the frame $L$ is also a Boolean algebra, and let $(X, \overline{q})$ be a stratified $L$-convergence space. Then there exists $(X, Q) \in |\top\text{-}\mathbf{Conv}|$ such that $(X, \overline{Q}_*) \in E_*$ with $q_\top = Q_{*,\top}$.*

*Proof.* Given $(X, \overline{q})$, where $\overline{q} = (q_\alpha)_{\alpha \in L}$, define $Q$ as follows: $\mathfrak{F} \xrightarrow{Q} x$ if and only if $\nu_{\mathfrak{F}} \xrightarrow{q_\top} x$. Then $[x] \xrightarrow{Q} x$ since $\nu_{[x]} = \dot{x}$ and if $\mathfrak{G} \geq \mathfrak{F} \xrightarrow{Q} x$, it follows that $\nu_{\mathfrak{G}} \geq \nu_{\mathfrak{F}}$ and thus $\nu_{\mathfrak{G}} \xrightarrow{q_\top} x$. Hence $\mathfrak{G} \xrightarrow{Q} x$ and $(X, Q)$ is a $\top$-convergence space. As defined above, $(X, \overline{Q}_*) \in E_*$, where $\mu \xrightarrow{Q_{*,\top}} x$ if and only if there exists $\mathfrak{F} \xrightarrow{Q} x$ such that $\mu \geq \nu_{\mathfrak{F}}$ and for $\alpha < \top$, $\mu \xrightarrow{Q_{*,\alpha}} x$ if and only if $\mu \geq \nu_\perp$.

It remains to show that $q_\top = Q_{*,\top}$. Assume that $\nu \xrightarrow{q_\top} x$. Since $L$ is a Boolean algebra, by Theorem 2.1 (ii), $\nu_{\mathfrak{F}_\nu} = \nu$. Since $\nu_{\mathfrak{F}_\nu} = \nu \xrightarrow{q_\top} x$, it follows that $\mathfrak{F}_\nu \xrightarrow{Q} x$ and thus $\nu = \nu_{\mathfrak{F}_\nu} \xrightarrow{Q_{*,\top}} x$. Hence $q_\top \geq Q_{*,\top}$.

Conversely, suppose that $\mu \xrightarrow{Q_{*,\top}} x$; then there exists $\mathfrak{F} \xrightarrow{Q} x$ such that $\mu \geq \nu_{\mathfrak{F}}$. It follows that $\nu_{\mathfrak{F}} \xrightarrow{q_\top} x$ and thus $\mu \xrightarrow{q_\top} x$. Hence $Q_{*,\top} \geq q_\top$ and $q_\top = Q_{*,\top}$. □

**Theorem 2.5.** *Assume that $L$ is a frame. Then,* [4]

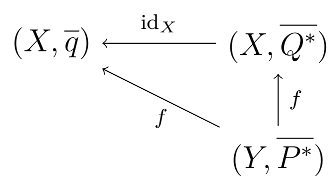
*(i)* $\top$-**Conv** *is embedded as a bicoreflective subcategory of* ***SL-CS****, and*

*(ii) provided that $L$ is also a Boolean algebra,* $\top$-**Conv** *is embedded as a bireflective subcategory of* ***SL-CS****.*

[4]See Appendix for definitions of bicoreflective and bireflective categories.

*Proof.* (i) Using Lemma 2.10, it suffices to show that $E^*$ is bicoreflective in $\mathbf{S}L\text{-}\mathbf{CS}$. Let $(X, \overline{q}) \in |\mathbf{S}L\text{-}\mathbf{CS}|$, where $\overline{q} = (q_\alpha)_{\alpha \in L}$. Define $Q$ as follows: $\mathfrak{F} \xrightarrow{Q} x$ if and only if $\nu_{\mathfrak{F}} \xrightarrow{q_\top} x$. Then $(X, Q) \in |\top\text{-}\mathsf{Conv}|$ and define $\overline{Q^*} = (Q^*_\alpha)_{\alpha \in L}$ as in Definition 2.5. Then $(X, \overline{Q^*}) \in |E^*|$. It is shown that $\mathrm{id}_X : (X, Q^*_\alpha) \longrightarrow (X, q_\alpha)$ is continuous, $\bot < \alpha$. Suppose $\mu \xrightarrow{Q^*_\alpha} x$; then there exists $\mathfrak{F} \xrightarrow{Q} x$ with $\mu \geq \nu_{\mathfrak{F}}$. Since $\nu_{\mathfrak{F}} \xrightarrow{q_\top} x$, $\nu_{\mathfrak{F}} \xrightarrow{q_\alpha} x$ and thus $\mu \xrightarrow{q_\alpha} x$. Hence $Q^*_\alpha \geq q_\alpha$, $\bot < \alpha$, and also $Q^*_\bot = q_\bot$. Then $\mathrm{id}_X : (X, \overline{Q^*}) \longrightarrow (X, \overline{q})$ is continuous.

Let $\phi : \top\text{-}\mathsf{Conv} \longrightarrow E^*$ be as in Lemma 2.10. Consider the diagram below; where $f : (Y, \overline{P^*}) \longrightarrow (X, \overline{q})$ is continuous.




It is shown that $f : (Y, \overline{P^*}) \longrightarrow (X, \overline{Q^*})$ is continuous in $E^*$. Since $\phi : \top\text{-}\mathsf{Conv} \longrightarrow E^*$ is an isomorphism, it is sufficient to show that $f : (Y, P) \longrightarrow (X, Q)$ is continuous in $\top\text{-}\mathsf{Conv}$.

Suppose that $\mathfrak{G} \xrightarrow{P} y$; then for $\bot < \alpha$, $\nu_{\mathfrak{G}} \xrightarrow{P^*_\alpha} y$ and thus by the continuity of $f : (Y, \overline{P^*}) \longrightarrow (X, \overline{q})$ in $\mathbf{S}L\text{-}\mathbf{CS}$, $\nu_{f^{\Rightarrow}(\mathfrak{G})} = f^{\Uparrow}(\nu_{\mathfrak{G}}) \xrightarrow{q_\alpha} f(y)$. It follows from the definition of $Q$ that $f^{\Rightarrow}(\mathfrak{G}) \xrightarrow{Q} f(y)$, and thus $f : (Y, P) \longrightarrow (X, Q)$ is continuous in $\top$-$\mathsf{Conv}$. Hence $f : (Y, \overline{P^*}) \longrightarrow (X, \overline{Q^*})$ is continuous in $E^*$, and thus $E^*$ is bicoreflective in $\mathbf{S}L\text{-}\mathbf{CS}$.

(ii) Assume that $(X, \overline{q}) \in |\mathbf{S}L\text{-}\mathbf{CS}|$ and define $Q$ as above; then $\mu \xrightarrow{Q_{*,\top}} x$ if and only if there exists $\mathfrak{F} \xrightarrow{Q} x$ such that $\mu \geq \nu_{\mathfrak{F}}$; otherwise $\mu \xrightarrow{Q_{*\alpha}} x$ if and only if $\mu \geq \nu_\bot$ for $\alpha < \top$. Then by Lemma 2.11, $q_\top = Q_{*,\top}$ and so $\mathrm{id}_X : (X, \overline{q}) \longrightarrow (X, \overline{Q_*})$ is continuous.

Suppose that $f : (X, \overline{q}) \longrightarrow (Y, \overline{P_*})$ is continuous and consider the diagram; where $f : (X, \overline{q}) \longrightarrow (Y, \overline{P_*})$ is continuous.

$$\begin{array}{ccc} (X, \overline{q}) & \xrightarrow{\mathrm{id}_X} & (X, \overline{Q_*}) \\ & \searrow^{f} & \downarrow f \\ & & (Y, \overline{P_*}) \end{array}$$

It remains to show that $f : (X, \overline{Q_*}) \longrightarrow (Y, \overline{P_*})$ is continuous in $E_*$. Let $\theta : \top$-**Conv** $\longrightarrow E_*$ be as in Lemma 2.10. Since $\theta$ is an isomorphism, it suffices to show that $f : (X, Q) \longrightarrow (Y, P)$ is continuous in $\top$-**Conv**.

Assume that $\mathfrak{F} \xrightarrow{Q} x$; then $\nu_{\mathfrak{F}} \xrightarrow{q_\top} x$ and thus $\nu_{f^{\Rightarrow}(\mathfrak{F})} = f^{\Uparrow}(\nu_{\mathfrak{F}}) \xrightarrow{P_{*,\top}} f(x)$. Hence there exists $\mathfrak{G} \xrightarrow{P} f(x)$ such that $\nu_{f^{\Rightarrow}(\mathfrak{F})} \geq \nu_{\mathfrak{G}}$, and it follows that $f^{\Rightarrow}(\mathfrak{F}) \geq \mathfrak{G}$. Then $f^{\Rightarrow}(\mathfrak{F}) \xrightarrow{P} f(x)$ and thus $f : (X, \overline{Q_*}) \longrightarrow (Y, \overline{P_*})$ is continuous. Therefore $E_*$ is bireflective in **S$L$-CS** whenever $L$ is a Boolean algebra. By Lemma 2.10, $E_*$ and $\top$-**Conv** are isomorphic and so $\top$-**Conv** is bireflective in **S$L$-CS**.

□

## Regularity in $\top$-**Conv**

Regularity for lattice-valued convergence spaces has been studied; for example, see Jäger [16] and Li and Jin [21]. Regularity has also been studied in the context of $\top$-convergence spaces by Fang and Yue [5], but only in terms of a diagonal condition. In this chapter the notion of closure of a $\top$-filter is defined and related to regularity as defined by [5].

The following connection between a $\top$-filter base and a $\top$-filter is useful. Using Lemma 1.3 and Lemma 1.1 (xi), it is straightforward to verify that if $f : X \longrightarrow Y$ is a map and $\mathcal{B}$ is a $\top$-filter base for $\mathfrak{F}$ on $X$, then $\{f^{\rightarrow}(b) : b \in \mathcal{B}\}$ is a $\top$-filter base for $f^{\Rightarrow}(\mathfrak{F})$ on $Y$.

**Definition 2.6.** Assume that $(X, q) \in |\top\text{-}\mathbf{Conv}|$ and $a \in L^X$. The **closure of** $a$ is defined by $\overline{a}(x) = \vee\{\nu_{\mathfrak{G}}(a) : \mathfrak{G} \xrightarrow{q} x\}$, for each $x \in X$.

Some basic properties of the closure operation are listed below.

**Lemma 2.12.** *Let $(X, q) \in |\top\text{-}\mathbf{Conv}|$, $a, b \in L^X$ and $\alpha \in L$. Then*

*(i)* $\overline{\mathbf{1}_{\varnothing}} = \mathbf{1}_{\varnothing}$,

*(ii)* $a \leq \overline{a}$,

*(iii)* $a \leq b$ *implies* $\overline{a} \leq \overline{b}$,

*(iv)* $\overline{a \wedge \alpha\mathbf{1}_X} = \overline{a} \wedge \alpha\mathbf{1}_X$,

*(v) if $L$ is a Boolean algebra, it follows that* $\overline{a \vee b} = \overline{a} \vee \overline{b}$.

*Proof.* (i)–(iv) These follow easily from the properties of stratified $L$-filters.

(v) Clearly $\overline{a} \vee \overline{b} \leq \overline{a \vee b}$. Employing Corollary 2.1.6 [10], $\mu(a \vee b) = \mu(a) \vee \mu(b)$ for each stratified $L$-ultrafilter $\mu$ on $X$. Since closures are determined by $\top$-ultrafilters, $\overline{a \vee b}(x) = \bigvee\{\nu_{\mathfrak{F}}(a \vee b) : \mathfrak{F} \xrightarrow{q} x, \mathfrak{F} \text{ a } \top\text{-ultrafilter}\} = \bigvee\{\nu_{\mathfrak{F}}(a) \vee \nu_{\mathfrak{F}}(b) : \mathfrak{F} \xrightarrow{q} x, \mathfrak{F} \text{ a } \top\text{-ultrafilter}\} \leq \overline{a}(x) \vee \overline{b}(x)$ and thus $\overline{a \vee b} = \overline{a} \vee \overline{b}$.

□

**Definition 2.7.** Given $(X, q) \in |\top\text{-}\mathbf{Conv}|$ and $\mathfrak{F} \in \mathfrak{F}_L^{\top}(X)$, the **closure of** $\mathfrak{F}$, denoted by $\overline{\mathfrak{F}}$, is defined to be the $\top$-filter whose $\top$-filter base is $\{\overline{a} : a \in \mathfrak{F}\}$. Further, if $\mathcal{B}$ is a $\top$-filter base, define $\overline{\mathcal{B}} = \{\overline{b} : b \in \mathcal{B}\}$.

**Lemma 2.13.** *Let* $(X, q) \in |\top\text{-}\mathbf{Conv}|$ *and* $b, c \in L^X$. *Then* $[b, c] \leq [\overline{b}, \overline{c}]$.

*Proof.* Since $[\overline{b}, \overline{c}] = \bigwedge_{x \in X} (\overline{b}(x) \to \overline{c}(x))$, it suffices to show that for fixed $x \in X$, $[b, c] \leq \overline{b}(x) \to \overline{c}(x)$. According to Lemma 1.1 (viii), $[b, c] \leq [a, b] \to [a, c]$ for any $a \in L^X$. Recall that $\nu_{\mathfrak{G}}(c) = \bigvee_{h \in \mathfrak{G}} [h, c]$. Further fix $\mathfrak{G} \xrightarrow{q} x$ and let $g \in \mathfrak{G}$. Then

$$[g, c] \leq \bigvee_{h \in \mathfrak{G}} [h, c] = \nu_{\mathfrak{G}}(c) \leq \bigvee \{\nu_{\mathfrak{H}}(c) : \mathfrak{H} \xrightarrow{q} x\} = \overline{c}(x).$$

Now since the implication operation is increasing in the second component, we have $[b, c] \leq [g, b] \to [g, c] \leq [g, b] \to \overline{c}(x)$. It follows from the distributive property in Lemma 1.1 (v) that

$$[b, c] \leq \bigwedge \{[g, b] \to \overline{c}(x) : g \in \mathfrak{G}\} = \left( \bigvee_{g \in \mathfrak{G}} [g, b] \right) \to \overline{c}(x) = \nu_{\mathfrak{G}}(b) \to \overline{c}(x).$$

Thus we have,

$$[b, c] \leq \bigwedge_{\mathfrak{G} \xrightarrow{q} x} \big(\nu_{\mathfrak{G}}(b) \to \overline{c}(x)\big) = \left( \bigvee_{\mathfrak{G} \xrightarrow{q} x} \nu_{\mathfrak{G}}(b) \right) \to \overline{c}(x) = \overline{b}(x) \to \overline{c}(x).$$

As this holds for any $x \in X$, $[b, c] \leq \bigwedge_{x \in X} \big(\overline{b}(x) \to \overline{c}(x)\big) = [\overline{b}, \overline{c}]$. $\square$

**Lemma 2.14.** *Let* $\mathcal{B}$ *be a* $\top$*-filter base for the* $\top$*-filter* $\mathfrak{F}$ *on* $(X, q)$. *Then* $\overline{\mathcal{B}}$ *is a base for* $\overline{\mathfrak{F}}$.

*Proof.* Note that by Lemma 2.12 (iii), Lemma 2.13 and the fact that $[\bullet, \bullet]$ is increasing in the second component, if $b_1, b_2 \in \mathcal{B}$, then $\bigvee_{b \in \mathcal{B}} [\overline{b}, \overline{b_1} \wedge \overline{b_2}] \geq \bigvee_{b \in \mathcal{B}} [\overline{b}, \overline{b_1 \wedge b_2}] \geq \bigvee_{b \in \mathcal{B}} [b, b_1 \wedge b_2] = \top$, as $\mathcal{B}$ is a $\top$-filter base for $\mathfrak{F}$. Also $\bigvee_{x \in X} \overline{b}(x) \geq \bigvee_{x \in X} b(x) = \top$ and thus $\overline{\mathcal{B}}$ is a $\top$-filter base.

Let $c \in \mathfrak{F}$; then by Lemma 2.13, as $\mathcal{B}$ is a base for $\mathfrak{F}$, $\bigvee_{b \in \mathcal{B}} [\bar{b}, \bar{c}] \geq \bigvee_{b \in \mathcal{B}} [b, c] = \top$. Thus $\bar{c}$ belongs to the $\top$-filter generated by $\overline{\mathcal{B}}$; that is, the $\top$-filter generated by $\overline{\mathcal{B}}$ includes $\{\bar{c} : c \in \mathfrak{F}\}$. Therefore $\overline{\mathcal{B}}$ generates $\overline{\mathfrak{F}}$. □

Kowalsky [18] introduced a diagonal axiom which characterizes when a convergence space is topological. The dual of the diagonal axiom was shown by Cook and Fischer [2] to characterize when a convergence space, or topological space, is regular. An appropriate diagonal axiom is used by Fang and Yue [5] to define regularity in $\top$-**Conv**.

Let $(X, q)$ be a $\top$-convergence space, $J$ a non-empty set and let $\psi : J \longrightarrow X$ and $\sigma : J \longrightarrow \mathfrak{F}_L^{\top}(X)$ be maps such that $\sigma(j) \xrightarrow{q} \psi(j)$ for each $j \in J$. For each $b \in L^X$, define $e_b : \mathfrak{F}_L^{\top}(X) \longrightarrow L$ by $e_b(\mathfrak{G}) = \nu_{\mathfrak{G}}(b)$. Then, given $\mathfrak{H} \in \mathfrak{F}_L^{\top}(J)$ we define $\kappa\sigma\mathfrak{H} = \{b \in L^X : e_b \circ \sigma \in \mathfrak{H}\}$. The definition of $\kappa\sigma\mathfrak{H}$ is due to Fang and Yue [5]. It is shown in Lemma 3.6 of [5] that $\kappa\sigma\mathfrak{H}$ is a $\top$-filter on $X$, and it is referred to as the $\top$-diagonal filter of $\mathfrak{H}$. They use the diagonal filter to define regularity.

The following definition is given by Fang and Yue [5]. The notation "(TR)" is used in [5] to denote the diagonal condition.

**Definition 2.8.** Suppose that $L$ is a frame and $(X, q)$ is a $\top$-convergence space. We say that $(X, q)$ is **regular** in $\top$-**Conv**, provided that for any non-empty set $J$ and maps $\psi : J \longrightarrow X$ and $\sigma : J \longrightarrow \mathfrak{F}_L^{\top}(X)$ such that $\sigma(j) \xrightarrow{q} \psi(j)$ for all $j \in J$, $\psi^{\Rightarrow}(\mathfrak{H}) \xrightarrow{q} x$ whenever $\mathfrak{H} \in \mathfrak{F}_L^{\top}(J)$ and $\kappa\sigma\mathfrak{H} \xrightarrow{q} x$.

**Lemma 2.15.** *Assume that $L$ is a frame and $(X, q)$ is a $\top$-convergence space. Denote $J = \{(\mathfrak{G}, y) \in \mathfrak{F}_L^{\top}(X) \times X : \mathfrak{G} \xrightarrow{q} y\}$ and define $\psi : J \longrightarrow X$ by $\psi(\mathfrak{G}, y) = y$ and define $\sigma : J \longrightarrow \mathfrak{F}_L^{\top}(X)$ by $\sigma(\mathfrak{G}, y) = \mathfrak{G}$. Then for each $\mathfrak{F} \in \mathfrak{F}_L^{\top}(X)$ there exists $\hat{\mathfrak{F}} \in \mathfrak{F}_L^{\top}(J)$ such that $\mathfrak{F} \subseteq \kappa\sigma\hat{\mathfrak{F}}$.*

*Proof.* Suppose that $a \in L^X$; define $\hat{a} : J \longrightarrow L$ by $\hat{a}(\mathfrak{G}, y) = \nu_{\mathfrak{G}}(a)$. Then $\hat{a} \in L^J$ and note that if $a, b \in L^X$, $\widehat{(a \wedge b)}(\mathfrak{G}, y) = \nu_{\mathfrak{G}}(a \wedge b) = \nu_{\mathfrak{G}}(a) \wedge \nu_{\mathfrak{G}}(b) = \hat{a}(\mathfrak{G}, y) \wedge \hat{b}(\mathfrak{G}, y)$ and thus $\widehat{(a \wedge b)} = \hat{a} \wedge \hat{b}$. Observe that if $a \in \mathfrak{F}$, then $\bigvee_{x \in X} a(x) = \top$ and thus

$$\begin{aligned}\bigvee\{\hat{a}(\mathfrak{G}, y) : \mathfrak{G} \xrightarrow{q} y\} &\geq \bigvee_{x \in X} \hat{a}([x], x) = \bigvee_{x \in X} \nu_{[x]}(a) \\ &= \bigvee_{x \in X} \bigvee_{b \in [x]} [b, a] = \bigvee_{x \in X} [\mathbf{1}_{\{x\}}, a] \\ &= \bigvee_{x \in X} a(x) = \top.\end{aligned}$$

Thus $\bigvee\{\hat{a}(\mathfrak{G}, y) : \mathfrak{G} \xrightarrow{q} y\} = \top$. It follows that $\mathcal{D} = \{\hat{a} : a \in \mathfrak{F}\}$ is a $\top$-filter base on $J$ which is closed under finite infima. Let $\hat{\mathfrak{F}}$ be the $\top$-filter with base $\mathcal{D}$.

Next it is shown that $\mathfrak{F} \subseteq \kappa\sigma\hat{\mathfrak{F}}$. Assume that $a \in \mathfrak{F}$; then $\hat{a} \in \mathcal{D}$ and it remains to show that $e_a \circ \sigma \in \hat{\mathfrak{F}}$. Indeed, $(e_a \circ \sigma)(\mathfrak{G}, y) = e_a(\mathfrak{G}) = \nu_{\mathfrak{G}}(a) = \hat{a}(\mathfrak{G}, y)$ and so $e_a \circ \sigma = \hat{a} \in \mathcal{D} \subseteq \hat{\mathfrak{F}}$. Thus according to the definition, $a \in \kappa\sigma\hat{\mathfrak{F}}$ and $\mathfrak{F} \subseteq \kappa\sigma\hat{\mathfrak{F}}$. □

**Theorem 2.6.** *Suppose that $L$ is a frame and $(X, q)$ is a $\top$-convergence space. Then $(X, q)$ is regular if and only if $\overline{\mathfrak{F}} \xrightarrow{q} x$ whenever $\mathfrak{F} \xrightarrow{q} x$.*

*Proof.* Assume that $(X, q)$ is such that $\overline{\mathfrak{F}} \xrightarrow{q} x$ whenever $\mathfrak{F} \xrightarrow{q} x$. Suppose that $J \neq \varnothing$ is a set, $\psi : J \longrightarrow X$ and $\sigma : J \longrightarrow \mathfrak{F}_L^\top(X)$ are such that $\sigma(j) \xrightarrow{q} \psi(j)$ for each $j \in J$. Let $\mathfrak{H} \in \mathfrak{F}_L^\top(J)$ such that $\kappa\sigma\mathfrak{H} \xrightarrow{q} x$. It suffices to show that $\overline{\kappa\sigma\mathfrak{H}} \subseteq \psi^\Rightarrow(\mathfrak{H})$.

Recall that $\mathcal{B}_{\mathfrak{H}} = \{b \in L^X : e_b \circ \sigma \in \mathfrak{H}\}$ is a $\top$-filter base for $\kappa\sigma\mathfrak{H}$ and $\mathcal{B}_{\mathfrak{H}}$ is closed under finite infima. It follows from Lemma 2.14 that $\overline{\mathcal{B}_{\mathfrak{H}}} = \{\bar{b} : b \in \mathcal{B}_{\mathfrak{H}}\}$ is a $\top$-filter base for $\overline{\kappa\sigma\mathfrak{H}}$ on $X$ and it suffices to show that $\overline{\mathcal{B}_{\mathfrak{H}}} \subseteq \psi^\Rightarrow(\mathfrak{H})$.

Let $b \in \mathcal{B}_{\mathfrak{H}}$; then $e_b \circ \sigma \in \mathfrak{H}$ and $\psi^\rightarrow(e_b \circ \sigma)(y) = \bigvee\{(e_b \circ \sigma)(j) : \psi(j) = y\} = \bigvee\{\nu_{\sigma(j)}(b) :$

$\psi(j) = y\}$. Since $\sigma(j) \xrightarrow{q} \psi(j)$, it follows that $\psi^{\rightarrow}(e_b \circ \sigma)(y) = \bigvee\{\nu_{\sigma(j)}(b) : \psi(j) = y\} \leq \bigvee\{\nu_{\mathfrak{G}}(b) : \mathfrak{G} \xrightarrow{q} y\} = \overline{b}(y)$ and thus $\psi^{\rightarrow}(e_b \circ \sigma) \leq \overline{b}$. It follows that $\overline{b} \in \psi^{\Rightarrow}(\mathfrak{H})$ and thus $\overline{\mathcal{B}_{\mathfrak{H}}} \subseteq \psi^{\Rightarrow}(\mathfrak{H})$. Hence $(X, q)$ is regular.

Conversely, suppose that $(X, q)$ is regular and assume that $\mathfrak{F} \xrightarrow{q} x$. It must be shown that $\overline{\mathfrak{F}} \xrightarrow{q} x$. Let $J$, $\psi$, $\sigma$ and $\hat{\mathfrak{F}} \in \mathfrak{F}_L^{\top}(J)$ be as in Lemma 2.15. According to Lemma 2.15, $\mathfrak{F} \subseteq \kappa\sigma\hat{\mathfrak{F}}$ and thus $\kappa\sigma\hat{\mathfrak{F}} \xrightarrow{q} x$ and since $(X, q)$ is regular, $\psi^{\Rightarrow}(\hat{\mathfrak{F}}) \xrightarrow{q} x$. It remains to show that $\psi^{\Rightarrow}(\hat{\mathfrak{F}}) \subseteq \overline{\mathfrak{F}}$.

Recall that $\mathcal{D} = \{\hat{a} : a \in \mathfrak{F}\}$ is a $\top$-filter base for $\hat{\mathfrak{F}}$ which is closed under finite infima. Then $\psi^{\rightarrow}(\hat{a})(y) = \bigvee\{\hat{a}(\mathfrak{K}, z) : \psi(\mathfrak{K}, z) = y\} = \bigvee\{\nu_{\mathfrak{G}}(a) : \mathfrak{G} \xrightarrow{q} y\} = \overline{a}(y)$, and thus $\psi^{\rightarrow}(\hat{a}) = \overline{a}$. Hence $\psi^{\Rightarrow}(\mathcal{D}) \subseteq \overline{\mathfrak{F}}$ and it follows that $\overline{\mathfrak{F}} \xrightarrow{q} x$. □

*Regular Subcategory of* $\top$-**Conv**

Let $f : (X, q) \longrightarrow (Y, p)$ be a continuous map in $\top$-**Conv**. If $\mathfrak{F} \xrightarrow{q} x$, it easily follows that $\overline{\mathfrak{F}} \subseteq [x]$ and $\overline{f^{\Rightarrow}(\mathfrak{F})} \subseteq f^{\Rightarrow}(\overline{\mathfrak{F}})$.

Let $\top$-**RConv** denote the full subcategory of $\top$-**Conv** consisting of all of the regular objects in $\top$-**Conv**. Fang and Yu [29] have shown that $\top$-**Conv** is a topological construct that is also Cartesian-closed. The proof of the next result uses a standard argument.

**Theorem 2.7.** *The category* $\top$-**RConv** *is a concretely bireflective subcategory of* $\top$-**Conv**.[5]

*Proof.* Note that the indiscrete $\top$-convergence structure $\rho$ on $X$ is regular. Since initial structures exist in $\top$-**Conv**, let $\sigma$ denote the initial structure on $X$ determined by $f_j$ :

[5]See Appendix for definitions concrete and bireflective.

$X \longrightarrow (Y_j, p_j)$, $j \in J$, where each $(Y_j, p_j) \in |\top\text{-}\mathbf{RConv}|$. Then $\mathfrak{F} \xrightarrow{\sigma} x$ if and only if $f_j^{\Rightarrow}(\mathfrak{F}) \xrightarrow{p_j} f_j(x)$, for each $j \in J$, and thus $f_j : (X, \sigma) \longrightarrow (Y_j, p_j)$ is continuous for each $j \in J$. Then for $a \in \mathfrak{F}$, $f_j^{\rightarrow}(\overline{a}^{\sigma}) \subseteq \overline{f_j^{\rightarrow}(a)}^{p_j}$ and thus $f_j^{\Rightarrow}(\overline{\mathfrak{F}}^{\sigma}) \supseteq \overline{f_j^{\Rightarrow}(\mathfrak{F})}^{p_j} \xrightarrow{p_j} f_j(x)$, for each $j \in J$. Hence $\overline{\mathfrak{F}}^{\sigma} \xrightarrow{\sigma} x$ and thus $(X, \sigma)$ is regular.

Let $rq$ denote the largest regular $\top$-convergence structure on $X$ such that $rq \leq q$. Then $\mathrm{id}_X : (X, q) \longrightarrow (X, rq)$ is a continuous map.

Suppose that $f : (X, q) \longrightarrow (Y, p)$ is any continuous map and $(Y, p) \in |\top\text{-}\mathbf{RConv}|$. Let $\delta$ denote the initial $\top$-convergence structure on $X$ defined by $f : X \longrightarrow (Y, p)$. Then $f : (X, \delta) \longrightarrow (Y, p)$ is continuous, $\delta \leq q$, and $(X, \delta) \in |\top\text{-}\mathbf{RConv}|$. It follows that $rq \geq \delta$ and thus $f : (X, rq) \longrightarrow (Y, p)$ is continuous. The following diagram commutes:

$$\begin{array}{ccc} (X, q) & \xrightarrow{\mathrm{id}_X} & (X, rq) \\ & \searrow^{f} & \downarrow f \\ & & (Y, p) \end{array}$$

and thus $\top\text{-}\mathbf{RConv}$ is concretely bireflective in $\top\text{-}\mathbf{Conv}$. □

*Regularity in* $\boldsymbol{SL\text{-}CS}$

Let $(X, \overline{q}) \in |\mathbf{S}L\text{-}\mathbf{CS}|$, $\alpha \in L$, $J$ an non-empty set and let $\psi : J \longrightarrow X$ and $\Sigma : J \longrightarrow \mathfrak{F}_L^S(X)$ be maps such that $\Sigma(j) \xrightarrow{q_\alpha} \psi(j)$ for each $j \in J$. Fix $b \in L^X$ and define $E_b : \mathfrak{F}_L^S(X) \longrightarrow L$ by $E_b(\nu) = \nu(b)$, for each $\nu \in \mathfrak{F}_L^S(X)$. Let $\mu \in \mathfrak{F}_L^S(J)$ and let $K\Sigma\mu \in \mathfrak{F}_L^S(X)$ be defined by $K\Sigma\mu(b) = \mu(E_b \circ \Sigma)$, for $b \in L^X$.

**Definition 2.9.** Assume that $L$ is a frame and $(X, \overline{q}) \in |\mathbf{S}L\text{-}\mathbf{CS}|$. Then $(X, \overline{q})$ is said to be **regular** in $\mathbf{S}L\text{-}\mathbf{CS}$ provided that for each $\alpha \in L$, $\psi : J \longrightarrow X$, $\Sigma : J \longrightarrow \mathfrak{F}_L^S(X)$ such that

$\Sigma(j) \xrightarrow{q_\alpha} \psi(j)$ and for $\mu \in \mathfrak{F}_L^S(J)$, we have that $\psi^{\Uparrow}\mu \xrightarrow{q_\alpha} x$ whenever $K\Sigma\mu \xrightarrow{q_\alpha} x$.

**Lemma 2.16.** *Let $L$ be a frame, $X$ a set, $\mathfrak{G} \in \mathfrak{F}_L^\top(X)$ and $\mu \in \mathfrak{F}_L^S(X)$. Then*

*(i)* $\mathfrak{F}_{\nu_\mathfrak{G}} = \mathfrak{G}$, *and*

*(ii)* $\mu = \nu_{\mathfrak{F}_\mu}$ *whenever $L$ is a Boolean algebra.*

*Proof.* Parts (i) and (ii) follow from Theorem 2.1 (i) and (ii), respectively. □

**Theorem 2.8.** *Assume that the frame $L$ is a Boolean algebra, $(X, q) \in |\top\text{-}\mathbf{Conv}|$, and let $(X, \overline{q}_*) \in |\boldsymbol{SL}\text{-}\boldsymbol{CS}|$ be as given in Definition 2.4. Then $(X, \overline{q}_*)$ is regular in $\boldsymbol{SL}\text{-}\boldsymbol{CS}$ if and only if $(X, q)$ is regular in $\top\text{-}\mathbf{Conv}$.*

*Proof.* Suppose that $(X, \overline{q}_*)$ is regular in $\mathbf{S}L\text{-}\mathbf{CS}$ and assume that $\psi : J \longrightarrow X$, $\sigma : J \longrightarrow \mathfrak{F}_L^\top(X)$ is such that $\sigma(j) \xrightarrow{q} \psi(j)$ for each $j \in J$. Let $\mathfrak{G} \in \mathfrak{F}_L^\top(J)$ be such that $\kappa\sigma\mathfrak{G} \xrightarrow{q} x$; it is shown that $\psi^{\Rightarrow}\mathfrak{G} \xrightarrow{q} x$. Define $\Sigma(j) = \nu_{\sigma(j)}$ and since $\sigma(j) \xrightarrow{q} \psi(j)$, it follows that $\Sigma(j) \xrightarrow{q_{*,\top}} \psi(j)$ for each $j \in J$.

First it is shown that $\kappa\sigma\mathfrak{G} = \mathfrak{F}_{K\Sigma\nu_\mathfrak{G}}$. Assume that $b \in \mathfrak{F}_{K\Sigma\nu_\mathfrak{G}}$; observe that $(E_b \circ \Sigma)(j) = E_b(\nu_{\sigma(j)}) = \nu_{\sigma(j)}(b) = (e_b \circ \sigma)(j)$ and thus $E_b \circ \Sigma = e_b \circ \sigma$. Moreover, using Theorem 2.1 we have $\top = K\Sigma\nu_\mathfrak{G}(b) = \nu_\mathfrak{G}(E_b \circ \Sigma) = \nu_\mathfrak{G}(e_b \circ \sigma)$ and hence $e_b \circ \sigma \in \mathfrak{G}$. It follows that $b \in \kappa\sigma\mathfrak{G}$ and thus $\mathfrak{F}_{K\Sigma\nu_\mathfrak{G}} \subseteq \kappa\sigma\mathfrak{G}$.

Conversely, if $b \in \mathcal{B}$, where $\mathcal{B}$ is a base for $\kappa\sigma\mathfrak{G}$, then $e_b \circ \sigma \in \mathfrak{G}$. If follows that $K\Sigma\nu_\mathfrak{G}(b) = \nu_\mathfrak{G}(E_b \circ \Sigma) = \nu_\mathfrak{G}(e_b \circ \sigma) = \top$ since $e_b \circ \sigma \in \mathfrak{G}$. Then using Theorem 2.1, $b \in \mathfrak{F}_{K\Sigma\nu_\mathfrak{G}}$ and thus $\kappa\sigma\mathfrak{G} = \mathfrak{F}_{K\Sigma\nu_\mathfrak{G}}$.

According to Lemma 2.16, since $\kappa\sigma\mathfrak{G} \xrightarrow{q} x$, $K\Sigma\nu_\mathfrak{G} = \nu_{\mathfrak{F}_{K\Sigma\nu_\mathfrak{G}}} \xrightarrow{q_{*,\top}} x$. However, $(X, \overline{q}_*)$ is regular in $\mathbf{S}L\text{-}\mathbf{CS}$ and it follows that $\psi^{\Uparrow}\nu_\mathfrak{G} \xrightarrow{q_{*,\top}} x$ and by Lemma 2.7, $\nu_{\psi^{\Rightarrow}\mathfrak{G}} = \psi^{\Uparrow}(\nu_\mathfrak{G}) \xrightarrow{q_{*,\top}}$

$x$. Then $\psi^{\Rightarrow}\mathfrak{G} \xrightarrow{q} x$ and hence $(X, q)$ is regular in $\top$-**Conv**.

Conversely, assume that $(X, q)$ is regular in $\top$-**Conv**; it is shown that $(X, \overline{q}_*)$ is regular in **S**$L$-**CS**. Suppose that $\psi : J \longrightarrow X$ and $\Sigma : J \longrightarrow \mathfrak{F}^S_L(X)$ are such that $\Sigma(j) \xrightarrow{q_*, \top} \psi(j)$ for each $j \in J$, and $\mu \in \mathfrak{F}^S_L(X)$ for which $K\Sigma\mu \xrightarrow{q_*, \top} x$. Define $\sigma(j) = \mathfrak{F}_{\Sigma(j)}$; then $\Sigma(j) \xrightarrow{q_*, \top} \psi(j)$ implies there exists $\mathfrak{G} \xrightarrow{q} \psi(j)$ such that $\Sigma(j) \geq \nu_{\mathfrak{G}}$ and thus by Lemma 2.16 (i), $\mathfrak{F}_{\Sigma(j)} \geq \mathfrak{F}_{\nu_{\mathfrak{G}}} = \mathfrak{G}$. Hence $\sigma(j) \xrightarrow{q} \psi(j)$ for each $j \in J$. It is shown that $\kappa\sigma\mathfrak{F}_\mu = \mathfrak{F}_{K\Sigma\mu}$.

Suppose that $b \in \mathcal{B}$, where $\mathcal{B}$ is the base for $\kappa\sigma\mathfrak{F}_\mu$; then $e_b \circ \sigma \in \mathfrak{F}_\mu$. Hence $K\Sigma\mu(b) = \mu(E_b \circ \Sigma) = \mu(e_b \circ \sigma) = \top$, and thus $b \in \mathfrak{F}_{K\Sigma\mu}$ implies that $\kappa\sigma\mathfrak{F}_\mu \subseteq \mathfrak{F}_{K\Sigma\mu}$. Conversely, if $b \in \mathfrak{F}_{K\Sigma\mu}$, then $\top = K\Sigma\mu(b) = \mu(e_b \circ \sigma)$ and thus $e_b \circ \sigma \in \mathfrak{F}_\mu$. Therefore $b \in \kappa\sigma\mathfrak{F}_\mu$ and hence $\kappa\sigma\mathfrak{F}_\mu = \mathfrak{F}_{K\Sigma\mu}$. Since $K\Sigma\mu \xrightarrow{q_*, \top} x$, it follows $\kappa\sigma\mathfrak{F}_\mu \xrightarrow{q} x$ and thus $\psi^{\Rightarrow}\mathfrak{F}_\mu \xrightarrow{q} x$. However, $\mu = \nu_{\mathfrak{F}_\mu}$, implying $\psi^{\Uparrow}\mu = \psi^{\Uparrow}(\nu_{\mathfrak{F}_\mu}) = \nu_{\psi^{\Rightarrow}\mathfrak{F}_\mu} \xrightarrow{q_*, \top} x$, and thus $(X, \overline{q}_*)$ is regular in **S**$L$-**CS**. $\square$

*The Dual of Regularity: Topological*

The next definition is the dual of Definition 2.8 and is given in [5]. The notation "(TF)" is used in [5] to denote the diagonal condition.

**Definition 2.10.** Suppose that $L$ is a frame. Then $(X, q) \in |\top$-**Conv**$|$ is called **topological** in $\top$-**Conv** provided that for each $\psi : J \longrightarrow X$, $\sigma : J \longrightarrow \mathfrak{F}^T_L(X)$ such that $\sigma(j) \xrightarrow{q} \psi(j)$ for each $j \in J$, we have $\kappa\sigma\mathfrak{H} \xrightarrow{q} x$ whenever $\psi^{\Rightarrow}\mathfrak{H} \xrightarrow{q} x$, $\mathfrak{H} \in \mathfrak{F}^\top_L(J)$.

The definition of a strong $L$-topological space used here can be found in Fang and Yue [29].

**Definition 2.11.** Let $L$ be a frame and $\tau \subseteq L^X$. Then the pair $(X, \tau)$ is called a **strong $L$-topological space** provided it satisfies:

(ST1) $\alpha \mathbf{1}_X \in \tau$ for each $\alpha \in L$,

(ST2) $a, b \in \tau$ implies $a \wedge b \in \tau$,

(ST3) $a_j \in \tau$ for each $j \in J$ implies $\bigvee_{j \in J} a_j \in \tau$, and

(ST4) $a \in \tau$ implies $\alpha \mathbf{1}_X \to a \in \tau$ for each $\alpha \in L$.

The following result appears as Theorem 3.11 in [5].

**Theorem 2.9.** [5] *Suppose that $(X, q) \in |\top\text{-}\mathbf{Conv}|$. Then $(X, q)$ is topological if and only if it is a strong $L$-topological space.*

**Theorem 2.10.** *Assume that the frame $L$ is a Boolean algebra, $(X, q) \in |\top\text{-}\mathbf{Conv}|$, and let $(X, \overline{q}_*) \in |\boldsymbol{SL\text{-}CS}|$ be as given in Definition 2.4. Then $(X, q)$ is topological in $\top$-**Conv** if and only if $(X, \overline{q}_*)$ is topological in $\boldsymbol{SL\text{-}CS}$.*

*Proof.* Suppose that $(X, \overline{q}_*)$ is topological in $\mathbf{S}L\text{-}\mathbf{CS}$. Assume that $\psi : J \longrightarrow X$ and $\sigma : J \longrightarrow \mathfrak{F}_L^{\top}(X)$ are such that $\sigma(j) \xrightarrow{q} \psi(j)$ for each $j \in J$, and $\mathfrak{G} \in \mathfrak{F}_L^{\top}(J)$ is such that $\psi^{\Rightarrow}\mathfrak{G} \xrightarrow{q} x$. Define $\Sigma(j) = \nu_{\sigma(j)}$ for each $j \in J$, and note that $\Sigma(j) \xrightarrow{q_*, \top} \psi(j)$. According to Lemma 2.7, $\psi^{\Uparrow}(\nu_{\mathfrak{G}}) = \nu_{\psi^{\Rightarrow}\mathfrak{G}} \xrightarrow{q_*, \top} x$ and it follows that $K\Sigma\nu_{\mathfrak{G}} \xrightarrow{q_*, \top} x$. It is shown in the proof of Theorem 2.8 that $\kappa\sigma\mathfrak{G} = \mathfrak{F}_{K\Sigma\nu_{\mathfrak{G}}}$. Since $K\Sigma\nu_{\mathfrak{G}} \xrightarrow{q_*, \top} x$, according to Definition 2.4 we have that $K\Sigma\nu_{\mathfrak{G}} \geq \nu_{\mathfrak{H}}$ for some $\mathfrak{H} \xrightarrow{q} x$, and thus $\mathfrak{F}_{K\Sigma\nu_{\mathfrak{G}}} \geq \mathfrak{F}_{\nu_{\mathfrak{H}}} = \mathfrak{H}$. Then $\mathfrak{F}_{K\Sigma\nu_{\mathfrak{G}}} \xrightarrow{q} x$ and thus $\kappa\sigma\mathfrak{G} \xrightarrow{q} x$. It follows that $(X, q)$ is topological in $\top$-**Conv**.

Conversely, assume that $(X, q)$ is topological in $\top$-**Conv**, $\psi : J \longrightarrow X$ and $\Sigma : X \longrightarrow \mathfrak{F}_L^S(X)$ are such that $\Sigma(j) \xrightarrow{q_*, \top} \psi(j)$ for each $j \in J$, and that $\mu \in \mathfrak{F}_L^S(J)$ is such that $\psi^{\Uparrow}\mu \xrightarrow{q_*, \top} x$.

Define $\sigma(j) = \mathfrak{F}_{\Sigma(j)}$; then $\sigma(j) \xrightarrow{q} \psi(j)$ for each $j \in J$. It is straightforward to show that $\psi^{\Rightarrow}\mathfrak{F}_{\mu} = \mathfrak{F}_{\psi^{\Uparrow}\mu}$ and hence $\psi^{\Rightarrow}\mathfrak{F}_{\mu} \xrightarrow{q} x$. Since $(X, q)$ is topological in $\top$-**Conv**, $\kappa\sigma\mathfrak{F}_{\mu} \xrightarrow{q} x$. It is shown in Theorem 2.8 that $\kappa\sigma\mathfrak{F}_{\mu} = \mathfrak{F}_{K\Sigma\mu}$. It follows from Lemma 2.16 that $K\Sigma\mu = \nu_{\mathfrak{F}_{K\Sigma\mu}}$ and thus $K\Sigma\mu \xrightarrow{q_{*},\top} x$. Hence $(X, \overline{q}_{*})$ is topological in **S$L$-CS**. □

**Remark 2.1.** Theorems 2.8 and 2.10 remain valid whenever $(X, \overline{q_{*}})$ is replaced by $(X, \overline{q^{*}})$.

Compactifications in $\top$-**Conv**

Whenever $L$ is a frame, Jäger [14] showed that every lattice-valued convergence space possesses a compactification. The same ideas used by Jäger are employed in our construction. In order to show that our extension space is compact, the assumption that $L$ is a Boolean algebra is needed. Hence the bijection between the stratified $L$-ultrafilters and $\top$-ultrafilters is used to show compactness of our extension space. The object $(X, q) \in |\top\text{-}\mathbf{Conv}|$ is said to be **compact** if every maximal $\top$-filter, or $\top$-ultrafilter, converges.

**Definition 2.12.** Assume that $(X, q) \in |\top\text{-}\mathbf{Conv}|$ is not compact. Then $\big((Y, p), f\big)$ is called a **compactification** of $(X, q)$ provided:

(i) $(Y, p)$ is compact,

(ii) $f : (X, q) \longleftarrow \Big(f(X), p\,\Big|_{f(X)}\Big)$ and $f^{-1}$ are continuous, and

(iii) for each $y \in Y$, there exists $\mathfrak{F} \in \mathfrak{F}_{L}^{\top}(X)$ such that $f^{\Rightarrow}\mathfrak{F} \xrightarrow{p} y$.

Whenever $L$ is a complete Boolean algebra, a compactification of each non-compact $(X, q) \in |\top\text{-}\mathbf{Conv}|$ is constructed. Further, each continuous map from $(X, q)$ into a compact regular object in $\top$-**Conv** has a continuous extension to the compactification.

Assume that $L$ is a complete Boolean algebra and let $(X, q) \in |\top\text{-}\mathbf{Conv}|$ which is not compact. Let $\eta$ denote the set of all $\top$-ultrafilters on $X$ which fail to converge. Define $X^* = X \cup \{\langle \mathfrak{G} \rangle : \mathfrak{G} \in \eta\}$ and let $j : X \longrightarrow X^*$ denote the natural injection $j(x) = x$, $x \in X$. Recall from Theorem 2.1, that $\mathfrak{F} \mapsto \nu_{\mathfrak{F}}$ defines an order preserving bijection from the set of all $\top-$filters on $X$ onto the set of all stratified $L$-filters on $X$, where $\nu_{\mathfrak{F}}(a) = \bigvee_{f \in \mathfrak{F}} [f, a]$, $a \in L^X$, and $\mathfrak{F} = \{b \in L^X : \nu_{\mathfrak{F}}(b) = \top\}$.

Given $a \in L^X$, define $a^* \in L^{X^*}$ as $a^*(z) = \begin{cases} a(x), & z = j(x) \\ \nu_{\mathfrak{G}}(a), & z = \langle \mathfrak{G} \rangle \end{cases}$. Observe that $(\bot \mathbf{1}_X)^* = \bot \mathbf{1}_{X^*}$ and $(\alpha \mathbf{1}_X)^* \geq \alpha \mathbf{1}_{X^*}$ since $(\alpha \mathbf{1}_X)^*(\langle \mathfrak{G} \rangle) = \nu_{\mathfrak{G}}(\alpha \mathbf{1}_X) \geq \alpha = (\alpha \mathbf{1}_{X^*})(\langle \mathfrak{G} \rangle)$. Moreover, $(a \wedge b)^*(\langle \mathfrak{G} \rangle) = \nu_{\mathfrak{G}}(a \wedge b) = \nu_{\mathfrak{G}}(a) \wedge \nu_{\mathfrak{G}}(b) = a^*(\langle \mathfrak{G} \rangle) \wedge b^*(\langle \mathfrak{G} \rangle) = (a^* \wedge b^*)(\langle \mathfrak{G} \rangle)$ and thus $(a \wedge b)^* = a^* \wedge b^*$. Observe that if $\mathcal{B}$ is a $\top$-filter base on $X$ that is closed under finite infima, then $\mathcal{B}^* = \{b^* : b \in \mathcal{B}\}$ is a $\top$-filter base on $X^*$ that is also closed under finite infima. Note that if $b \in \mathcal{B}$, then $\bigvee_{z \in X^*} b^*(z) \geq \bigvee_{z \in X} b(x) = \top$. In particular, if $\mathfrak{F}$ is a $\top$-filter on $X$, then $\{f^* : f \in \mathfrak{F}\}$ is a $\top$-filter base on $X^*$; let $\mathfrak{F}^*$ denote the $\top$-filter on $X^*$ that it generates.

Using the notation above, define a structure $q^*$ on $X^*$ as follows:

$\mathfrak{H} \xrightarrow{q^*} j(x)$ if and only if $\mathfrak{H} \geq \mathfrak{F}^*$ for some $\mathfrak{F} \xrightarrow{q} x$,

$\mathfrak{H} \xrightarrow{q^*} \langle \mathfrak{G} \rangle$ if and only if $\mathfrak{H} \geq \mathfrak{G}^*$.

Note that $[j(x)] \geq [x]^* \xrightarrow{q^*} x$ and thus $[j(x)] \xrightarrow{q^*} j(x)$ for each $x \in X$. Also, observe that $[\langle \mathfrak{G} \rangle] \geq \mathfrak{G}^*$. Indeed, if $g \in \mathfrak{G}$, then $g^*(\langle \mathfrak{G} \rangle) = \nu_{\mathfrak{G}}(g) = \top$ and thus $g^* \in [\langle \mathfrak{G} \rangle]$. It follows that the $\top$-filter base $\{g^* : g \in \mathfrak{G}\} \subseteq [\langle \mathfrak{G} \rangle]$ and thus the $\top$-filter $\mathfrak{G}^* \subseteq [\langle \mathfrak{G} \rangle]$. Clearly, if $\mathfrak{H} \xrightarrow{q^*} z$ and $\mathfrak{K} \geq \mathfrak{H}$, then $\mathfrak{K} \xrightarrow{q^*} z$ and hence $(X^*, q^*) \in |\top\text{-}\mathbf{Conv}|$.

**Theorem 2.11.** *Assume that the frame $L$ is a Boolean algebra and suppose that $(X, q) \in |\top$-$\mathbf{Conv}|$ is not compact. Then $\big((X^*, q^*), j\big)$, as defined above, is a compactification of $(X, q)$ in*

$\top$-**Conv**. *Moreover, if* $\theta : (X, q) \longrightarrow (Y, p)$ *is continuous and* $(Y, p)$ *is compact and regular, then* $\theta$ *has a continuous extension* $\theta^* : (X^*, q^*) \longrightarrow (Y, p)$ *such that* $\theta^* \circ j = \theta$.

*Proof.* It was shown in the above that $(X^*, q^*) \in |\top\text{-}\mathbf{Conv}|$. We show that $j$ is continuous. Observe that if $\mathfrak{F} \xrightarrow{q} x$, then $j^{\Rightarrow}(\mathfrak{F}) \supseteq \mathfrak{F}^*$. Indeed, if $f \in \mathfrak{F}$, then $j^{\rightarrow}(f)(z) = \begin{cases} f(x), & z = j(x) \\ \bot, & z = \langle \mathfrak{G} \rangle \end{cases}$ and thus $j^{\rightarrow}(f) \leq f^*$. Then $f^* \in j^{\Rightarrow}(\mathfrak{F})$ for each $f \in \mathfrak{F}$ and thus $j^{\Rightarrow}(\mathfrak{F}) \supseteq \mathfrak{F}^*$, and this implies that $j^{\Rightarrow}(\mathfrak{F}) \xrightarrow{q^*} j(x)$. Hence $j$ is continuous.

Conversely, if $\mathfrak{F}$ is any $\top$-filter on $X$ such that $j^{\Rightarrow}(\mathfrak{F}) \xrightarrow{q^*} j(x)$, then $j^{\Rightarrow}(\mathfrak{F}) \geq \mathfrak{K}^*$ for some $\mathfrak{K} \xrightarrow{q} x$. Hence $\mathfrak{F} = j^{\Leftarrow}(j^{\Rightarrow}(\mathfrak{F})) \geq j^{\Leftarrow}(\mathfrak{K}^*) = \mathfrak{K}$ and thus $\mathfrak{F} \xrightarrow{q} x$. It follows that $j : (X, q) \longrightarrow (X^*, q^*)$ is an embedding. Further, if $\mathfrak{G} \in \mathcal{N}$, then $j^{\Rightarrow}(\mathfrak{G}) \supseteq \mathfrak{G}^*$ implies that $j^{\Rightarrow}(\mathfrak{G}) \xrightarrow{q^*} \langle \mathfrak{G} \rangle$ and thus $j : (X, q) \longrightarrow (X^*, q^*)$ is a dense embedding.

It is shown that $(X^*, q^*)$ is compact. Assume that $\mathfrak{H}$ is a $\top$-ultrafilter on $X^*$. According to Theorem 2.1, $\nu_{\mathfrak{H}}$ is a stratified $L$-ultrafilter on $X^*$ and, moreover, $d \in \mathfrak{H}$ if and only if $\nu_{\mathfrak{H}}(d) = \top$. Define $\mu_{\mathfrak{H}} : L^X \to L$ by $\mu_{\mathfrak{H}}(a) = \nu_{\mathfrak{H}}(a^*)$ for each $a \in L^X$.

Observe that $\mu_{\mathfrak{H}}(\bot \mathbf{1}_X) = \nu_{\mathfrak{H}}(\bot \mathbf{1}_{X^*}) = \bot$, $\mu_{\mathfrak{H}}(\alpha \mathbf{1}_X) = \nu_{\mathfrak{H}}((\alpha \mathbf{1}_X)^*) \geq \nu_{\mathfrak{H}}(\alpha \mathbf{1}_{X^*}) \geq \alpha$ and $\mu_{\mathfrak{H}}(a \wedge b) = \nu_{\mathfrak{H}}((a \wedge b)^*) = \nu_{\mathfrak{H}}(a^*) \wedge \nu_{\mathfrak{H}}(b^*) = \mu_{\mathfrak{H}}(a) \wedge \mu_{\mathfrak{H}}(b)$, for $a, b \in L^X$ and $\alpha \in L$. It follows that $\mu_{\mathfrak{H}}$ is a stratified $L$-filter on $X$. According to Theorem 1.1 (i), $\mu_{\mathfrak{H}}$ is a stratified $L$-ultrafilter if and only if for each $a \in L^X$, $\mu_{\mathfrak{H}}(a) = \mu_{\mathfrak{H}}(a \to \mathbf{1}_{\varnothing}) \to \bot$. Employing Theorem 1.1 (ii), for any $\mathfrak{G} \in \eta$, $(a \to \mathbf{1}_{\varnothing})^*(\langle \mathfrak{G} \rangle) = \nu_{\mathfrak{G}}(a \to \mathbf{1}_{\varnothing}) = \nu_{\mathfrak{G}}(a) \to \bot = a^*(\langle \mathfrak{G} \rangle) \to \bot = (a^* \to \mathbf{1}_{\varnothing})(\langle \mathfrak{G} \rangle)$. Hence $(a \to \mathbf{1}_{\varnothing})^* = a^* \to \mathbf{1}_{\varnothing}$.

Then $\mu_{\mathfrak{H}}(a) = \nu_{\mathfrak{H}}(a^*) = \nu_{\mathfrak{H}}(a^* \to \mathbf{1}_{\varnothing}) \to \bot = \nu_{\mathfrak{H}}((a \to \mathbf{1}_{\varnothing})^*) \to \bot = \mu_{\mathfrak{H}}(a \to \mathbf{1}_{\varnothing}) \to \bot$, and thus $\mu_{\mathfrak{H}}$ is a stratified $L$-ultrafilter on $X$. It follows from Theorem 2.1 (ii) that $\mathfrak{F}_{\mathfrak{H}} = \{a \in L^X : \mu_{\mathfrak{H}}(a) = \top\}$ is a $\top$-ultrafilter on $X$. Moreover, $a \in \mathfrak{F}_{\mathfrak{H}}$ if and only if $\nu_{\mathfrak{H}}(a^*) = \top$ if and only if $a^* \in \mathfrak{H}$. That is, $a \in \mathfrak{F}_{\mathfrak{H}}$ if and only if $a^* \in \mathfrak{H}$. Observe that $\mathcal{B} = \{a^* : a \in \mathfrak{F}_{\mathfrak{H}}\}$ is a $\top$-filter base on $X^*$ which is closed under finite infima. Let $\mathfrak{F}^*_{\mathfrak{H}}$ denote the $\top$-filter on $X^*$ that it generates; then $\mathfrak{F}^*_{\mathfrak{H}} \subseteq \mathfrak{H}$.

Assume that $\mathfrak{F}_{\mathfrak{H}} \xrightarrow{q} x$; then $\mathfrak{H} \xrightarrow{q^*} j(x)$. If $\mathfrak{F}_{\mathfrak{H}} \in \mathcal{N}$, then $\mathfrak{H} \xrightarrow{q^*} \langle \mathfrak{F}_{\mathfrak{H}} \rangle$ and it follows that $(X^*, q^*)$ is compact and therefore $\big((X^*, q^*), j\big)$ is a compactification of $(X, q)$ in $\top$-**Conv**.

Next, suppose that $\theta : (X, q) \longrightarrow (Y, p)$ is a continuous map. Define $\theta^* : (X^*, q^*) \longrightarrow (Y, p)$ by $\theta^*(j(x)) = \theta(x)$ for $x \in X$, and $\theta^*(\langle \mathfrak{G} \rangle) = y$ where $y$ is one of the limits of $\theta^{\Rightarrow}(\mathfrak{G})$ in $(Y, p)$. First, for $a \in L^X$, $x \in X$ and $\mathfrak{G} \in \eta$, it is shown that $\nu_{[\theta(x)]}(\theta^{\rightarrow}(a)) \geq a(x)$ and $\nu_{\theta^{\Rightarrow}(\mathfrak{G})}(\theta^{\rightarrow}(a)) \geq a^*(\langle \mathfrak{G} \rangle)$.

Note that

$$\nu_{[\theta(x)]}(\theta^{\rightarrow}(a)) = \bigvee_{b \in [\theta(x)]} [b, \theta^{\rightarrow}(a)] \geq [\mathbf{1}_{\{\theta(x)\}}, \theta^{\rightarrow}(a)] \geq \top \rightarrow a(x) = a(x),$$

and thus $\nu_{[\theta(x)]}(\theta^{\rightarrow}(a)) \geq a(x)$. Further, if $\mathfrak{G} \in \mathcal{N}$, then using Lemma 1.1 (xi),

$$\nu_{\theta^{\Rightarrow}(\mathfrak{G})}(\theta^{\rightarrow}(a)) \geq \bigvee_{g \in \mathfrak{G}} [\theta^{\rightarrow}(g), \theta^{\rightarrow}(a)] \geq \bigvee_{g \in \mathfrak{G}} [g, a] = \nu_{\mathfrak{G}}(a) = a^*(\langle \mathfrak{G} \rangle),$$

and hence $\nu_{\theta^{\Rightarrow}(\mathfrak{G})}(\theta^{\rightarrow}(a)) \geq a^*(\langle \mathfrak{G} \rangle)$.

If $a \in L^X$, it is shown that $\theta^{*\rightarrow}(a^*) \leq \overline{\theta^{\rightarrow}(a)}$. First, assume that $y = \theta^*(j(z)) = \theta(z) \in Y$. Then $\overline{\theta^{\rightarrow}(a)}(y) = \bigvee\{\nu_{\mathfrak{K}}(\theta^{\rightarrow}(a) : \mathfrak{K} \xrightarrow{p} y\} \geq \nu_{[\theta(z)]}(\theta^{\rightarrow}(a)) \geq a(z) = a^*(j(z))$. Next, suppose that $\theta^*(\langle \mathfrak{G} \rangle) = y$, where $\theta^{\Rightarrow}(\mathfrak{G}) \xrightarrow{p} y$. Then $\overline{\theta^{\rightarrow}(a)}(\theta^*(\langle \mathfrak{G} \rangle) = \bigvee\{\nu_{\mathfrak{K}}(\theta^{\rightarrow}(a)) : \mathfrak{K} \xrightarrow{p} y\} \geq \nu_{\theta^{\Rightarrow}(\mathfrak{G})}(\theta^{\rightarrow}(a)) \geq a^*(\langle \mathfrak{G} \rangle)$. Combining these two results, $\overline{\theta^{\rightarrow}(a)}(y) \geq \bigvee\{a^*(z) : \theta^*(z) = y\} = \theta^{*\rightarrow}(a^*)(y)$ and thus $\theta^{*\rightarrow}(a^*) \leq \overline{\theta^{\rightarrow}(a)}$.

Assume $\mathfrak{F}$ is a $\top$-filter on $X$; then $\mathcal{B}_1 = \{\overline{\theta^{\rightarrow}(a)} : a \in \mathfrak{F}\}$ and $\mathcal{B}_2 = \{\theta^{*\rightarrow}(a^*) : a \in \mathfrak{F}\}$ are

$\top$-filter bases on $Y$. Then $\overline{\theta^{\Rightarrow}(\mathfrak{F})}$ and $\theta^{*\Rightarrow}(\mathfrak{F}^*)$ denote the $\top$-filters generated by $\mathcal{B}_1$ and $\mathcal{B}_2$, respectively. Since for each $a \in \mathfrak{F}$, $\theta^{*\rightarrow}(a^*) \leq \overline{\theta^{\rightarrow}(a)}$, it follows that $\theta^{*\Rightarrow}(\mathfrak{F}^*) \geq \overline{\theta^{\Rightarrow}(\mathfrak{F})}$.

Finally, suppose that $\mathfrak{H} \xrightarrow{q^*} j(x)$. Then $\mathfrak{H} \geq \mathfrak{F}^*$ for some $\mathfrak{F} \xrightarrow{q} x$, and thus $\theta^{*\Rightarrow}(\mathfrak{H}) \geq \theta^{*\Rightarrow}(\mathfrak{F}^*) \geq \overline{\theta^{\Rightarrow}(\mathfrak{F})}$. Since $(Y, q)$ is regular, it follows that $\theta^{*\Rightarrow}(\mathfrak{H}) \xrightarrow{p} \theta(x) = \theta^*(j(x))$. Similarly, if $\mathfrak{H} \xrightarrow{q^*} \langle \mathfrak{G} \rangle$, then $\mathfrak{H} \geq \mathfrak{G}^*$ and thus $\theta^{*\Rightarrow}(\mathfrak{H}) \geq \theta^{*\Rightarrow}(\mathfrak{G}^*) \geq \overline{\theta^{\Rightarrow}(\mathfrak{G})}$ and $\theta^{*\Rightarrow}(\mathfrak{H}) \xrightarrow{p} y = \theta^*(\langle \mathfrak{G} \rangle)$, where $\theta^{\Rightarrow}(\mathfrak{G}) \xrightarrow{p} y$. Hence $\theta^* : (X^*, q^*) \longrightarrow (Y, p)$ is continuous and $j \circ \theta^* = \theta$. $\square$

Connections between the compactification constructed in Theorem 2.11 and that given by Jäger [14] are made below. Assume that $(X, q) \in |\top\text{-}\mathbf{Conv}|$ is not compact. In order to simplify the notation, let $\big((X^*, s), j\big)$ denote the compactification of $(X, q)$ given in Theorem 2.11. According to Theorem 4.1 [6], there is an isomorphism between the full subcategory S$L$-LC-CS of "left-continuous" objects in S$L$-CS and the category S$L$-GCS of stratified $L$-generalized convergence spaces.

Let $(X, \overline{q}^*) \in |\mathbf{S}L\text{-}\mathbf{CS}|$ denote the object given in Definition 2.5; it easily follows that $(X, \overline{q}^*) \in |\mathrm{S}L\text{-LC-CS}|$ but, in general, $(X, \overline{q}_*)$ is not left-continuous.

Jäger's [14] compactification $\big((X^*, \overline{p}), j\big)$ of $(X, \overline{q}^*)$ in S$L$-CS is described below. If $\mu \in \mathfrak{F}^S_L(X^*)$, define $\widetilde{\mu} \in \mathfrak{F}^S_L(X)$ by $\widetilde{\mu}(a) = \mu(a^*)$, for each $a \in L^X$. Then $\overline{p} = (p_\alpha)_{\alpha \in L}$ is defined as follows: for $\alpha > \perp$

$$\mu \xrightarrow{p_\alpha} j(x) \iff \widetilde{\mu} \xrightarrow{q^*_\alpha} x$$

$$\mu \xrightarrow{p_\alpha} \langle \mathfrak{G} \rangle \iff \widetilde{\mu} = \nu_{\mathfrak{G}}$$

$$\mu \xrightarrow{p_\perp} z \iff \mu \geq \nu_\perp,\ z \in X^*.$$

It is shown in Theorem 2.12 below that $\overline{p} = \overline{s}$.

**Lemma 2.17.** *Suppose that $(X, q) \in |\top\text{-}\mathbf{Conv}|$ is not compact and let $X^* = X \cup \{\langle \mathfrak{G} \rangle : \mathfrak{G} \in \eta\}$. Then for $a, b \in L^X$, $\mathfrak{K} \in \mathfrak{F}_L^\top(X)$, and $\mathfrak{J} \in \mathfrak{F}_L^\top(X^*)$:*

*(i)* $[b^*, a^*] = [b, a]$

*(ii)* $\widetilde{\nu}_{\mathfrak{K}^*} = \nu_{\mathfrak{K}}$

*(iii)* $\widetilde{\nu_{\mathfrak{J}}} \geq \nu_{\mathfrak{K}}$ *implies* $\mathfrak{J} \geq \mathfrak{K}^*$.

*Proof.* (i) Observe that $[b^*, a^*] = \bigwedge\limits_{x \in X} \Big(b^*(j(x)) \to a^*(j(x))\Big) \wedge \bigwedge\limits_{\mathfrak{G} \in \mathcal{N}} \Big(b^*(\langle \mathfrak{G} \rangle) \to a^*(\langle \mathfrak{G} \rangle)\Big) = [b, a] \wedge \bigwedge\limits_{\mathfrak{G} \in \mathcal{N}} \Big(\nu_{\mathfrak{G}}(b) \to \nu_{\mathfrak{G}}(a)\Big)$. According to Corollary 3.3 [4], $\nu_{\mathfrak{G}}(b) \to \nu_{\mathfrak{G}}(a) \geq [b, a]$ and it follows that $[b^*, a^*] = [b, a]$.

(ii) Fix $a \in L^X$; then using (i), $\widetilde{\nu}_{\mathfrak{K}^*}(a) = \nu_{\mathfrak{K}^*}(a^*) = \bigvee\limits_{b \in \mathfrak{K}} [b^* a^*] = \bigvee\limits_{b \in \mathfrak{K}} [b, a] = \nu_{\mathfrak{K}}(a)$. Hence $\widetilde{\nu}_{\mathfrak{K}^*} = \nu_{\mathfrak{K}}$.

(iii) Assume that $a \in \mathfrak{K}$; then $\top = \nu_{\mathfrak{K}}(a) \leq \widetilde{\nu_{\mathfrak{J}}}(a) = \nu_{\mathfrak{J}}(a^*)$ and thus $a^* \in \mathfrak{J}$. Hence $\mathfrak{K}^* \subseteq \mathfrak{J}$. □

**Theorem 2.12.** *Assume that $L$ is a complete Boolean algebra, $(X, q) \in |\top\text{-}\mathbf{Conv}|$ is not compact, $\big((X^*, s), j\big)$ is the compactification of $(X, q)$ given in Theorem 2.11, $(X, \overline{q}^*)$ and $(X^*, s^*)$ are as defined in Definition 2.5. If $\big((X^*, \overline{p}, j\big)$ denotes the compactification of $(X, \overline{q}^*)$ given by Jäger [14], then $\overline{s}^* = \overline{p}$.*

*Proof.* Fix $\alpha > \bot$. First, suppose that $\mu \xrightarrow{p_\alpha} j(x)$; then $\widetilde{\mu} \xrightarrow{q^*_\alpha} x$ and thus $\widetilde{\mu} \geq \nu_{\mathfrak{F}}$ for some $\mathfrak{F} \xrightarrow{q} x$. Since $\mu = \nu_{\mathfrak{H}}$ for some $\mathfrak{H} \in \mathfrak{F}_L^\top(X^*)$, $\widetilde{\nu}_{\mathfrak{H}} \geq \nu_{\mathfrak{F}}$, and by Lemma 2.17 (iii), $\mathfrak{H} \geq \mathfrak{F}^*$. Then $\mu = \nu_{\mathfrak{H}} \geq \nu_{\mathfrak{F}^*}$ and $\mathfrak{F}^* \xrightarrow{s} j(x)$ implies that $\mu \xrightarrow{s^*_\alpha} j(x)$. Next, assume that $\mu \xrightarrow{p_\alpha} \langle \mathfrak{G} \rangle$;

then $\mu = \nu_{\mathfrak{H}}$ and $\widetilde{\mu} = \nu_{\mathfrak{G}}$. Since $\widetilde{\nu}_{\mathfrak{H}} = \nu_{\mathfrak{G}}$, it follows by Lemma 2.17 (iii) that $\mathfrak{H} \geq \mathfrak{G}^*$ and thus $\mu = \nu_{\mathfrak{H}} \geq \nu_{\mathfrak{G}^*}$. Hence $\mu \xrightarrow{s^*_\alpha} \langle \mathfrak{G} \rangle$ and thus $\overline{p} \geq \overline{s}^*$.

Conversely, suppose that $\alpha > \perp$ and $\mu \xrightarrow{s^*_\alpha} j(x)$; then $\mu \geq \nu_{\mathfrak{F}^*}$ for some $\mathfrak{F} \xrightarrow{q} x$. It follows from Lemma 2.17 (ii) that $\widetilde{\mu} \geq \widetilde{\nu}_{\mathfrak{F}^*} = \nu_{\mathfrak{F}}$ and thus $\widetilde{\mu} \xrightarrow{q^*_\alpha} x$. Hence $\mu \xrightarrow{p_\alpha} j(x)$. Next, assume that $\mu \xrightarrow{s^*_\alpha} \langle \mathfrak{G} \rangle$; then $\mu \geq \nu_{\mathfrak{G}^*}$ implies that $\widetilde{\mu} \geq \nu_{\mathfrak{G}}$. Since $\nu_{\mathfrak{G}}$ is a stratified $L$-ultrafilter on $X$, $\widetilde{\mu} = \nu_{\mathfrak{G}}$ and thus $\mu \xrightarrow{p_\alpha} \langle \mathfrak{G} \rangle$. Then $\overline{s}^* \geq \overline{p}$ and thus $\overline{s}^* = \overline{p}$. $\square$

# CHAPTER 3: $\top$-CAUCHY SPACES

The study of completions using Cauchy filters dated back to Kowalsky [18]. Later Cook and Fischer [3] introduced uniform convergence spaces which also gave a framework for the study of completions in terms of Cauchy filters. Keller [17] gave a set of axioms which characterize the Cauchy filters of a uniform convergence spaces. Spaces satisfying these axioms are now called Cauchy spaces, and has led to the study of completions from the context of Cauchy spaces. Fundamental results in this area can be found in the works of Reed [25], Lowen [20], and Preuss [24]. Jäger [14] defined a Cauchy space in the lattice-valued setting and developed a completion theory in this context.

## Definitions and Categorical Properties of $\top$-**Chy**

If $\mathfrak{F}, \mathfrak{G} \in \mathfrak{F}_L^{\top}(X)$, then $\mathfrak{F} \vee \mathfrak{G}$ denotes the smallest $\top$-filter on $X$ containing both $\mathfrak{F}$ and $\mathfrak{G}$, provided it exists.

**Lemma 3.1.** *If $\mathfrak{F}, \mathfrak{G} \in \mathfrak{F}_L^{\top}(X)$ then $\mathfrak{F} \vee \mathfrak{G}$ exists if and only if for each $f \in \mathfrak{F}$ and $g \in \mathfrak{G}$, $\bigvee_{x \in X}(f(x) \wedge g(x)) = \top$. In particular, for any $\mathfrak{F}, \mathfrak{G}, \mathfrak{H} \in \mathfrak{F}_L^{\top}(X)$, $(\mathfrak{F} \cap \mathfrak{H}) \vee (\mathfrak{G} \cap \mathfrak{H})$ exists.*

*Proof.* Assume that $\mathfrak{F} \vee \mathfrak{G}$ exists, $f \in \mathfrak{F}$ and $g \in \mathfrak{G}$. It follows that $f \wedge g \in \mathfrak{F} \vee \mathfrak{G}$ and thus $\bigvee_{z \in X}\Big(f(z) \wedge g(z)\Big) = \top$. Conversely, suppose that for each $f \in \mathfrak{F}, g \in \mathfrak{G}$, $\bigvee_{z \in X}\Big(f(z) \wedge g(z)\Big) = \top$. Define $\mathcal{B} = \{f \wedge g : f \in \mathfrak{F}, g \in \mathfrak{G}\}$. Then $\mathcal{B}$ is closed under finite infima and thus $\bigvee_{c \in \mathcal{B}}[c, f \wedge g] = \top$. Since $\bigvee_{z \in \mathcal{B}}\Big(f(z) \wedge g(z)\Big) = \top$, it follows that $\mathcal{B}$ is a base for $\mathfrak{F} \vee \mathfrak{G}$. Further, if $a \in \mathfrak{F} \cap \mathfrak{H}$ and $b \in \mathfrak{G} \cap \mathfrak{H}$, then $a \wedge b \in \mathfrak{H}$ implies that $\bigvee_{x \in X}\Big(a(x) \wedge b(x)\Big) = \top$ and thus $(\mathfrak{F} \cap \mathfrak{H}) \vee (\mathfrak{G} \cap \mathfrak{H})$ exists. $\square$

**Definition 3.1.** The pair $(X, \mathcal{C})$ is called a **$\top$-Cauchy space** and $\mathcal{C}$ is called a **$\top$-Cauchy structure on** $X$ provided that $\mathcal{C} \subseteq \mathfrak{F}_L^{\top}(X)$ obeys:

($\top$C1) $[x] \in \mathcal{C}$ for each $x \in X$,

($\top$C2) $\mathfrak{F} \geq \mathfrak{G} \in \mathcal{C}$ implies $\mathfrak{F} \in \mathcal{C}$, and

($\top$C3) $\mathfrak{F}, \mathfrak{G} \in \mathcal{C}$ and $\mathfrak{F} \vee \mathfrak{G}$ exists implies that $\mathfrak{F} \cap \mathfrak{G} \in \mathcal{C}$.

The axioms of a $\top$-Cauchy space coincide with those in the classical case provided $\top$-filters on $X$ replace set filters. A map $\theta : (X, \mathcal{C}) \longrightarrow (Y, \mathcal{D})$ between two $\top$-Cauchy spaces is said to be **Cauchy-continuous** provided that $\theta^{\Rightarrow}\mathfrak{F} \in \mathcal{D}$ whenever $\mathfrak{F} \in \mathcal{C}$. Let **$\top$-Chy** denote the category whose objects consist of all $\top$-Cauchy spaces and whose morphisms are all the Cauchy-continuous maps.

**Definition 3.2.** The pair $(X, q)$ is called a **$\top$-limit space** provided that $(X, q) \in |\top\text{-}\mathbf{Conv}|$ and, additionally, $\mathfrak{F}, \mathfrak{G} \xrightarrow{q} x$ implies that $\mathfrak{F} \cap \mathfrak{G} \xrightarrow{q} x$. Let **$\top$-Lim** denote the full subcategory of $\top$-**Conv** consisting of all the $\top$-limit spaces

For each $(X, \mathcal{C}) \in |\top\text{-}\mathbf{Chy}|$, define $(X, q_{\mathcal{C}})$ as follows: $\mathfrak{F} \xrightarrow{q_{\mathcal{C}}} x$ iff $\mathfrak{F} \cap [x] \in \mathcal{C}$.

**Lemma 3.2.** *If* $(X, \mathcal{C}) \in |\top\text{-}\mathbf{Chy}|$, *then* $(X, q_{\mathcal{C}}) \in |\top\text{-}\mathbf{Lim}|$.

*Proof.* First observe that if $x \in X$, then by ($\top$C1), $[x] \in \mathcal{C}$ and thus $[x] \cap [x] \in \mathcal{C}$ and $[x] \xrightarrow{q_{\mathcal{C}}} x$. Next if $\mathfrak{F} \xrightarrow{q_{\mathcal{C}}} x$ and $\mathfrak{F} \subseteq \mathfrak{G}$, then $\mathfrak{G} \cap [x] \supseteq \mathfrak{F} \cap [x] \in \mathcal{C}$. By ($\top$C2) this implies $\mathfrak{G} \cap [x] \in \mathcal{C}$ and thus $\mathfrak{G} \xrightarrow{q_{\mathcal{C}}} x$. Finally, if $\mathfrak{F}, \mathfrak{G} \xrightarrow{q_{\mathcal{C}}} x$ then $\mathfrak{F} \cap [x], \mathfrak{G} \cap [x] \in \mathcal{C}$. Further, by Lemma 3.1, $(\mathfrak{F} \cap [x]) \vee (\mathfrak{G} \cap [x])$ exists and thus by ($\top$C3) we have that $(\mathfrak{F} \cap [x]) \cap (\mathfrak{G} \cap [x]) = (\mathfrak{F} \cap \mathfrak{G}) \cap [x] \in \mathcal{C}$. Hence $\mathfrak{F} \cap \mathfrak{G} \xrightarrow{q_{\mathcal{C}}} x$. Hence $(X, q_{\mathcal{C}}) \in |\top\text{-}\mathbf{Lim}|$. $\square$

Next, using Keller's [17] argument, we characterize precisely when a $\top$-limit structure is induced by a $\top$-Cauchy structure.

**Lemma 3.3.** *An object* $(X, q) \in |\top\text{-}\mathbf{Lim}|$ *is induced by some* $(X, \mathcal{C}) \in |\top\text{-}\mathbf{Chy}|$ *if and only if for each* $x \neq y$ *in* $X$*, either* $q$*-convergence to* $x$ *and* $y$ *coincides or* $x$ *and* $y$ *have no common* $q$*-convergent filters.*

*Proof.* First suppose that $(X, q) \in |\top\text{-}\mathbf{Lim}|$ is induced by some $(X, \mathcal{C}) \in |\top\text{-}\mathbf{Chy}|$. That is $q = q_{\mathcal{C}}$. Let $x, y \in X$ be distinct. To show that either $q$-convergence to $x$ and $y$ coincides or have no common $q$-convergent filters, we suppose that there is a $\mathfrak{F} \in \mathfrak{F}_L^\top(X)$ which converges to $x$ and not to $y$ and then show that have no common $q$-convergent filters. Suppose by way of contradiction that $x$ and $y$ do have a convergent filter in common, say $\mathfrak{G} \xrightarrow{q} x, y$. Then since $\mathfrak{F}, \mathfrak{G} \xrightarrow{q} x$, it follows from Definition 3.2 of a $\top$-limit space that $\mathfrak{F} \cap \mathfrak{G} \xrightarrow{q} x$. Hence $\mathfrak{F} \cap \mathfrak{G} \in \mathcal{C}$. Also, since $\mathfrak{G} \xrightarrow{q} y$ and $q = q_{\mathcal{C}}$ it follows that $\mathfrak{G} \cap [y] \in \mathcal{C}$. Employing Lemma 3.1 we have that $(\mathfrak{F} \cap \mathfrak{G}) \vee (\mathfrak{G} \cap [y])$ exists and thus $(\mathfrak{F} \cap \mathfrak{G}) \cap (\mathfrak{G} \cap [y]) = (\mathfrak{F} \cap \mathfrak{G}) \cap [y] \in \mathcal{C}$. Hence $\mathfrak{F} \cap \mathfrak{G} \xrightarrow{q_{\mathcal{C}}} y$ and thus $\mathfrak{F} \xrightarrow{q} y$, a contradiction.

Next, suppose that $(X, q) \in |\top\text{-}\mathbf{Lim}|$ is such that for each $x \neq y$ in $X$, either $q$-convergence to $x$ and $y$ coincides or have no common $q$-convergent filters. Define $\mathcal{C}^q = \{\mathfrak{F} \in \mathfrak{F}_L^\top(X) : \mathfrak{F}\ q\text{-converges}\}$. Then by Definition 2.1 ($\top$CS1), since $[x] \xrightarrow{q} x$ for each $x \in X$, we have that $[x] \in \mathcal{C}^q$. Also if $\mathfrak{G} \supseteq \mathfrak{F} \xrightarrow{q} x$ then by Definition 2.1 ($\top$CS2) it follows that $\mathfrak{G} \xrightarrow{q} x$ and hence that $\mathfrak{G} \in \mathcal{C}^q$. Assume that $\mathfrak{F}, \mathfrak{G} \in \mathcal{C}^q$ such that $\mathfrak{F} \vee \mathfrak{G}$ exists. Then $\mathfrak{F} \xrightarrow{q} x$, $\mathfrak{G} \xrightarrow{q} y$ for some $x, y \in X$. By assumption, convergence to $x$ and $y$ either coincides or share no filters. Since $\mathfrak{F}, \mathfrak{G} \subseteq \mathfrak{F} \vee \mathfrak{G}$, it follows that $\mathfrak{F} \vee \mathfrak{G} \xrightarrow{q} x, y$. Hence $x$ and $y$ must have the same $q$-convergent filters. In particular $\mathfrak{F}, \mathfrak{G} \xrightarrow{q} x$ and thus by Definition 3.2 of a $\top$-limit space, it follows that $\mathfrak{F} \cap \mathfrak{G} \xrightarrow{q} x$. Thus $\mathfrak{F} \cap \mathfrak{G} \in \mathcal{C}^q$. Therefore $\mathcal{C}^q$ is a $\top$-Cauchy structure on $X$. It follows that $q = q_{\mathcal{C}^q}$. □

An object $(X, \mathcal{C}) \in |\top\text{-}\mathbf{Chy}|$ called **Hausdorff**, or $T_2$ if no two distinct points share convergent $\top$-filters in $(X, q_{\mathcal{C}})$. It follows that each $T_2$ object $(X, q) \in |\top\text{-}\mathbf{Lim}|$ is induced by some $(X, \mathcal{C}) \in |\top\text{-}\mathbf{Chy}|$.

**Lemma 3.4.** *If $f : (X, \mathcal{C}) \longrightarrow (Y, \mathcal{D})$ is Cauchy-continuous in $\top$-**Chy**, then $f : (X, q_{\mathcal{C}}) \longrightarrow (Y, q_{\mathcal{D}})$ is continuous in $\top$-**Conv**.*

*Proof.* Suppose that $f : (X, \mathcal{C}) \longrightarrow (Y, \mathcal{D})$ is Cauchy-continuous in $\top$-**Chy** and assume that $\mathfrak{F} \xrightarrow{q_{\mathcal{C}}} x$. We need to show that $f^{\Rightarrow}\mathfrak{F} \xrightarrow{q_{\mathcal{D}}} f(x)$. That is, we need to show that $f^{\Rightarrow}\mathfrak{F} \cap [f(x)] \in \mathcal{D}$. Since $\mathfrak{F} \xrightarrow{q_{\mathcal{C}}} x$ this implies that $\mathfrak{F} \cap [x] \in \mathcal{C}$. The Cauchy-continuity of $f$ implies that $f^{\Rightarrow}(\mathfrak{F} \cap [x]) = f^{\Rightarrow}\mathfrak{F} \cap f^{\Rightarrow}[x] = f^{\Rightarrow}\mathfrak{F} \cap [f(x)] \in \mathcal{D}$, as desired. □

It is shown by Fang and Yu in [29] that $\top$-**Conv** is a topological construct which is also Cartesian closed. Making only minor modifications, the theorems below shows that $\top$-**Chy** is also a topological construct and is Cartesian-closed. We begin with the following lemma.

**Lemma 3.5.** *Let $f : (X, q) \longrightarrow (Y, p)$ be a map between two $\top$-Cauchy spaces and let $\mathfrak{F}, \mathfrak{G} \in \mathfrak{F}_L^{\top}(X)$. If $\mathfrak{F} \vee \mathfrak{G}$ exists then $f^{\Rightarrow}\mathfrak{F} \vee f^{\Rightarrow}\mathfrak{G}$ also exists.*

*Proof.* By Lemma 3.1, we must show that $\bigvee_{y \in Y} \left(f^{\rightarrow}a\right)(y) \wedge \left(f^{\rightarrow}b\right)(y) = \top$ for each $a \in \mathfrak{F}$ and $b \in \mathfrak{G}$. We compute,

$$\bigvee_{y \in Y} \left(f^{\rightarrow}a\right)(y) \wedge \left(f^{\rightarrow}b\right)(y) = \bigvee_{y \in Y} \left( \bigvee_{f(x)=y} a(x) \wedge \bigvee_{f(t)=y} b(t) \right)$$
$$\geq \bigvee_{y \in Y} \bigvee_{f(x)=y} a(x) \wedge b(x) = \bigvee_{x \in X} a(x) \wedge b(x).$$

Since $\mathfrak{F} \vee \mathfrak{G}$ exists, Lemma 3.1 implies that $\bigvee_{x \in X} a(x) \wedge b(x) = \top$ and the result follows. □

**Theorem 3.1.** *The category* $\top$-**Chy** *is a topological construct.*[1]

*Proof.* Consider the source $f_j : X \longrightarrow (Y_j, \mathcal{D}_j)$, $j \in J$. Define $\mathcal{C} = \{\mathfrak{F} \in \mathfrak{F}_L^\top(X) : f_j^\Rightarrow \mathfrak{F} \in \mathcal{D}_j,\ \forall\, j \in J\}$. Since for each $x \in X$, $f_j^\Rightarrow[x] = [f_j(x)] \in \mathcal{D}_j$ for all $j \in J$, $[x] \in \mathcal{C}$. Next if $\mathfrak{G} \supseteq \mathfrak{F} \in \mathcal{C}$, then for each $j \in J$, $f_j^\Rightarrow \mathfrak{G} \supseteq f_j^\Rightarrow \mathfrak{F} \in \mathcal{D}_j$ and hence $f_j^\Rightarrow \mathfrak{G} \in \mathcal{D}_j$ and $\mathfrak{G} \in \mathcal{C}$. Finally if $\mathfrak{F}, \mathfrak{G} \in \mathcal{C}$ and $\mathfrak{F} \vee \mathfrak{G}$ exists, then by Lemma 3.5, $f_j^\Rightarrow \mathfrak{F} \vee f_j^\Rightarrow \mathfrak{G}$ exists for each $j \in J$ and using Lemma 1.4 (i), $f_j^\Rightarrow \mathfrak{F} \cap f_j^\Rightarrow \mathfrak{G} = f_j(\mathfrak{F} \cap \mathfrak{G}) \in \mathcal{D}_j$ for each $j \in J$. Thus $(X, \mathcal{C}) \in |\top\text{-}\mathbf{Chy}|$.

Assume that $g : (Z, \mathcal{E}) \longrightarrow (X, \mathcal{C})$ is a map such that $f_j \circ g : (Z, \mathcal{E}) \longrightarrow (Y_j, \mathcal{D}_j)$ is Cauchy-continuous for each $j \in J$. If $\mathfrak{G} \in \mathcal{E}$, then $f_j^\Rightarrow(g^\Rightarrow \mathfrak{G}) = (f_j \circ g)^\Rightarrow \mathfrak{G} \in \mathcal{D}_j$ for each $j \in J$ and thus $g^\Rightarrow \mathfrak{G} \in \mathcal{C}$. It follows that $g : (Z, \mathcal{E}) \longrightarrow (X, \mathcal{C})$ is also Cauchy-continuous. Conversely, if $g$ is Cauchy-continuous, then clearly the composition $f_j \circ g$ is Cauchy-continuous for each $j \in J$. Thus $\top$-**Chy** possesses initial structures.

Suppose that $X$ is any fixed set. Then the class of all $\top$-Cauchy structures on $X$ is a subset of $2^{\mathfrak{F}_L^\top(X)}$ and is thus a set. Next, assume that $|X| = 1$, that is, $X = \{x\}$. Note that if $a \in L^X$, then $a = \alpha \mathbf{1}_X$ for some $\alpha \in L$. Since $X = \{x\}$, $[x] = \mathbf{1}_X$ for each $\mathfrak{F} \in \mathfrak{F}_L^\top(X)$ and thus $\mathcal{C} = \{\mathfrak{F} \in \mathfrak{F}_L^\top(X) : \mathfrak{F} = [x]\}$ is the only structure on $X$ such that $(X, \mathcal{C}) \in |\top\text{-}\mathbf{Chy}|$. If $X = \varnothing$, then $\mathcal{C} = \varnothing$ and thus there is exactly one object in $\top$-Chy whenever $X = \varnothing$ or $X = \{x\}$. Hence $\top$-**Chy** is a topological construct. □

Let $X_1, X_2$ be two sets. If $a_i \in L^{X_i}$, $i = 1, 2$, then we define $a_1 \times a_2 \in L^{X_1 \times X_2}$ by $(a_1 \times a_2)(x_1, x_2) = a_1(x_1) \wedge a_2(x_2)$. If $\pi_i$ is the $i^{\text{th}}$ projection, note that $\pi_1^\rightarrow(a_1 \times a_2)(x_1) = \bigvee_{\pi_1(s,t)=x_1} (a_1 \times a_2)(s, t) = \bigvee_{x_2 \in X_2} (a_1(x_1) \wedge a_2(x_2)) = a_1(x) \wedge \Big(\bigvee_{x_2 \in X_2} a_2(x_2)\Big) \leq a_1(x_1)$. Hence $\pi_1^\rightarrow(a_1 \times a_2) \leq a_1$. Similarly, $\pi_2^\rightarrow(a_1 \times a_2) \leq a_2$.

[1]See Appendix for the definition of topological constructs.

Let $\mathfrak{F}_i \in \mathfrak{F}_L^\top(X_i)$, $i = 1, 2$; then $\mathcal{B} = \{a_1 \times a_2 : a_i \in \mathfrak{F}_i\}$ is shown in [29] to be a $\top$-filter base, and the generated $\top$-filter is denoted by $\mathfrak{F}_\mathbf{1} \times \mathfrak{F}_\mathbf{2}$. Further, assume that $\mathcal{B}_i$ is a $\top$-filter base for $\mathfrak{F}_i$, $i = 1, 2$. It is shown in [29] that $\{b_1 \times b_2 : b_i \in \mathcal{B}_i, i = 1, 2\}$ is a $\top$-filter base which generates the filter $\mathfrak{F}_1 \times \mathfrak{F}_2$. The following lemma is also found in [29].

**Lemma 3.6.** *Let $\theta_i : X_i \longrightarrow Y_i$ and let $\mathfrak{F}_i \in \mathfrak{F}_L^\top(X_i)$, $i = 1, 2$. Also let $\pi_i : X_1 \times X_2 \longrightarrow X_i$ denote the $i^{th}$ projection map. The following hold:*

*(i)* $(\theta_1 \times \theta_2)^\Rightarrow(\mathfrak{F}_i \times \mathfrak{F}_2) = \theta_1^\Rightarrow(\mathfrak{F}_1) \times \theta_2^\Rightarrow(\mathfrak{F}_2)$,

*(ii)* $\pi_i^\Rightarrow(\mathfrak{F}_1 \times \mathfrak{F}_2) = \mathfrak{F}_i$, $i = 1, 2$.

Let $(X, \mathcal{C}), (Y, \mathcal{D}) \in |\top\text{-}\mathbf{Chy}|$, and let $C(X, Y)$ denote the set of all Cauchy-continuous maps. Define $\mathrm{ev} : C(X, Y) \times X \to Y$ by $\mathrm{ev}(f, x) = f(x)$. Note that since $\top$-**Chy** possesses initial structures, it has product structures. In particular, if $\mathfrak{H} \in \mathfrak{F}_L^\top(X \times Y)$, then $\mathfrak{H} \in \mathcal{C} \times \mathcal{D}$ (product structure) iff $\pi_1^\Rightarrow\mathfrak{H} \in \mathcal{C}$ and $\pi_2^\Rightarrow\mathfrak{H} \in \mathcal{D}$. Define $\Sigma \subseteq \mathfrak{F}_L^\top(C(X, Y))$ as follows: $\Phi \in \Sigma$ iff for each $\mathfrak{F} \in \mathcal{C}$, $\mathrm{ev}^\Rightarrow(\Phi \times \mathfrak{F}) \in \mathcal{D}$.

**Theorem 3.2.** *The category $\top$-**Chy** is Cartesian closed.*

*Proof.* First, we show that $\Sigma$ as defined above is a $\top$-Cauchy structure on $C(X, Y)$. Fix $\theta \in C(X, Y)$. It is shown that if $\mathfrak{F} \in \mathcal{C}$, then $\mathrm{ev}^\Rightarrow([\theta] \times \mathfrak{F}) \in \mathcal{D}$. Since $\mathcal{B}_1 = \{\mathbf{1}_{\{\theta\}}\}$ is a $\top$-filter base for $[\theta]$, $\mathcal{B} = \{\mathbf{1}_{\{\theta\}} \times a : a \in \mathfrak{F}\}$ is a $\top$-filter base for $[\theta] \times \mathfrak{F}$ and thus $\hat{\mathcal{B}} = \{\mathrm{ev}^\rightarrow(\mathbf{1}_{\{\theta\}} \times a) : a \in \mathfrak{F}\}$ is a $\top$-filter base for $\mathrm{ev}^\Rightarrow([\theta] \times \mathfrak{F})$. Observe that for $y \in Y$,

$$\begin{aligned}\mathrm{ev}^\rightarrow(\mathbf{1}_{\{\theta\}} \times a)(y) &= \bigvee_{\mathrm{ev}(\psi,z)=y} (\mathbf{1}_{\{\theta\}} \times a)(\psi, z)\\ &= \bigvee_{\mathrm{ev}(\psi,z)=y} \mathbf{1}_{\{\theta\}}(\psi) \wedge a(z)\\ &= \bigvee_{\theta(z)=y} a(z) = \theta^\rightarrow(a)(y).\end{aligned}$$

Hence $\mathrm{ev}^{\rightarrow}(\mathbf{1}_{\{\theta\}} \times a) = \theta^{\rightarrow}(a)$ for each $a \in \mathfrak{F}$. Since $\hat{\mathcal{B}}$ is a $\top$-filter base for $\mathrm{ev}^{\Rightarrow}([\theta] \times \mathfrak{F})$ and $\{\theta^{\rightarrow}(a) : a \in \mathfrak{F}\}$ is a $\top$-filter base for $\theta^{\Rightarrow}\mathfrak{F}$, we have $\mathrm{ev}^{\Rightarrow}([\theta] \times \mathfrak{F}) = \theta^{\Rightarrow}\mathfrak{F} \in \mathcal{D}$. Hence $[\theta] \in \Sigma$.

Clearly, if $\Psi \geq \Phi \in \Sigma$, then $\Psi \in \Sigma$.

Next, assume that $\Psi, \Phi \in \Sigma$ such that $\Psi \vee \Phi$ exists. If $\mathfrak{F} \in \mathcal{C}$, then $\Phi \times \mathfrak{F} \subseteq (\Psi \vee \Phi) \times \mathfrak{F}$ and $\Psi \times \mathfrak{F} \subseteq (\Psi \vee \Phi) \times \mathfrak{F}$ and hence $(\Phi \times \mathfrak{F}) \vee (\Psi \times \mathfrak{F})$ exists. It follows from Lemma 3.5 that $\mathrm{ev}^{\Rightarrow}(\Phi \times \mathfrak{F}) \vee \mathrm{ev}^{\Rightarrow}(\Psi \times \mathfrak{F})$ exists. Since $\mathrm{ev}^{\Rightarrow}(\Phi \times \mathfrak{F}), \mathrm{ev}^{\Rightarrow}(\Psi \times \mathfrak{F}) \in \mathcal{D}$, we have that $\mathrm{ev}^{\Rightarrow}(\Phi \times \mathfrak{F}) \cap \mathrm{ev}^{\Rightarrow}(\Psi \times \mathfrak{F}) \in \mathcal{D}$. Observe that $\mathrm{ev}^{\Rightarrow}((\Phi \cap \Psi) \times \mathfrak{F}) = \mathrm{ev}^{\Rightarrow}(\Phi \times \mathfrak{F}) \cap \mathrm{ev}^{\Rightarrow}(\Psi \times \mathfrak{F})$, and thus $\Sigma$ is a $\top$-Cauchy structure and $\big(C(X,Y), \Sigma\big) \in |\top\text{-}\mathbf{Chy}|$.

Note that if $\Gamma \in \Sigma \times \mathcal{C}$, then $\Gamma \geq \pi_1^{\Rightarrow}(\Gamma) \times \pi_2^{\Rightarrow}(\Gamma)$, where $\pi_1^{\Rightarrow}(\Gamma) \in \Sigma$ and $\pi_2^{\Rightarrow}(\Gamma) \in \mathcal{C}$. Since $\mathrm{ev}^{\Rightarrow}\Gamma \geq \mathrm{ev}^{\Rightarrow}(\pi_1^{\Rightarrow}(\Gamma) \times \pi_2^{\Rightarrow}(\Gamma)) \in \mathcal{D}$, it follows that $\mathrm{ev} : \big(C(X,Y), \Sigma\big) \times (X, \mathcal{C}) \longrightarrow (Y, \mathcal{D})$ is Cauchy-continuous.

Next, assume that $f : (Z, \mathcal{E}) \times (X, \mathcal{C}) \longrightarrow (Y, \mathcal{D})$ is Cauchy-continuous. Fix $z \in Z$ and define $f_z : X \longrightarrow Y$ by $f_z(x) = f(z,x)$. It is shown that $f_z \in C(X,Y)$. Indeed, let $\mathfrak{F} \in \mathcal{C}$; it is shown that $f_z^{\Rightarrow}\mathfrak{F} = f^{\Rightarrow}([z] \times \mathfrak{F})$. A $\top$-filter base for $[z] \times \mathfrak{F}$ is $\{\mathbf{1}_{\{z\}} \times a : a \in \mathfrak{F}\}$. Observe that if $y \in Y$ and $a \in \mathfrak{F}$, $f^{\rightarrow}(\mathbf{1}_{\{z\}} \times a)(y) = \bigvee_{f(s,t)=y} (\mathbf{1}_{\{z\}} \times a)(s,t) = \bigvee_{f(z,t)=y} a(t) = \bigvee_{f_z(t)=y} a(t) = f_z^{\rightarrow}(a)(y)$. Hence $f^{\rightarrow}(\mathbf{1}_{\{z\}} \times a) = f_z^{\rightarrow}(a)$ and $f_z^{\Rightarrow}\mathfrak{F} = f^{\Rightarrow}([z] \times \mathfrak{F})$ since their $\top$-filter bases coincide. Since $f^{\Rightarrow}([z] \times \mathfrak{F}) \in \mathcal{D}$, $f_z : (X, \mathcal{C}) \longrightarrow (Y, \mathcal{D})$ is Cauchy-continuous and thus $f_z \in C(X,Y)$. Define $f^* : Z \longrightarrow C(X,Y)$ by $f^*(z) = f_z$ for $z \in X$. It is shown that $f^* : (Z, \mathcal{E}) \longrightarrow \big(C(X,Y), \Sigma\big)$ is Cauchy-continuous.

In [29] it was shown that $\mathrm{ev} \circ (f^* \times \mathrm{id}_X) = f$. Indeed, if $(s,t) \in Z \times X$, then $\big(\mathrm{ev} \circ (f^* \times \mathrm{id}_X)\big)(s,t) = \mathrm{ev}(f^*(s), t) = \mathrm{ev}(f_s, t) = f_s(t) = f(s,t)$. Observe that if $\mathfrak{G} \in \mathcal{E}$ and $\mathfrak{F} \in \mathcal{C}$, then $f^{\Rightarrow}(\mathfrak{G} \times \mathfrak{F}) \in \mathcal{D}$ since $f$ is Cauchy-continuous. Then $f^{\Rightarrow}(\mathfrak{G} \times \mathfrak{F}) = \big(\mathrm{ev}^{\rightarrow}(f^* \times \mathrm{id}_X)\big)^{\Rightarrow}(\mathfrak{G} \times \mathfrak{F}) = \mathrm{ev}^{\Rightarrow}(f^{*\Rightarrow}(\mathfrak{G}) \times \mathfrak{F}) \in \mathcal{D}$ for each $\mathfrak{F} \in \mathcal{D}$. It follows that $f^{*\Rightarrow}\mathfrak{G} \in \Sigma$ and thus

$f^*:(Z,\mathcal{E})\longrightarrow\Big(C(X,Y),\Sigma\Big)$ is Cauchy-continuous. That is, if $f:(Z,\mathcal{E})\times(X,\mathcal{C})\to(Y,\mathcal{D})$ is Cauchy-continuous, then $f^*:(Z,\mathcal{E})\longrightarrow\Big(C(X,Y),\Sigma\Big)$ is Cauchy-continuous. Hence $\top$-**Chy** is Cartesian closed. □

## Cauchy Completions

An object $(X,\mathcal{C})\in|\top\text{-}\mathbf{Chy}|$ is called **complete** provided each $\mathfrak{F}\in\mathcal{C}$ converges in $(X,q_{\mathcal{C}})$. Moreover, $\Big((Y,\mathcal{D}),\phi\Big)$ is called a **completion** of $(X,\mathcal{C})$ in $\top$-**Chy** provided that $\phi:(X,\mathcal{D})\longrightarrow(Y,\mathcal{D})$ is a dense $\top$-Cauchy embedding and $(Y,\mathcal{D})$ is complete. Here denseness means that for each $y\in Y$, there exists $\mathfrak{F}\in\mathfrak{F}_L^{\top}(X)$ such that $\phi^{\Rightarrow}\mathfrak{F}\xrightarrow{q_{\mathcal{D}}}y$. It is shown below that each $\top$-Cauchy space has a finest completion in $\top$-**Chy**, and also each Cauchy-continuous map into a complete $\top$-Cauchy space can be extended to a Cauchy-continuous map on the completion.

Let $(X,\mathcal{C})\in|\top\text{-}\mathbf{Chy}|$; then $\mathfrak{F}\sim\mathfrak{G}$ iff $\mathfrak{F}\cap\mathfrak{G}\in\mathcal{C}$ defines an equivalence relation on $\mathcal{C}$. Denote $\mathcal{N}_{\mathcal{C}}=\{\mathfrak{F}\in\mathcal{C}:\mathfrak{F}$ fails to $q_{\mathcal{C}}$-converge$\}$ and let $\langle\mathfrak{G}\rangle_{\mathcal{C}}=\{\mathfrak{F}\in\mathcal{N}_{\mathcal{C}}:\mathfrak{F}\sim\mathfrak{G}\}$. When the structure is clear, we will write $\mathcal{N}$ ($\langle\mathfrak{G}\rangle$) instead of $\mathcal{N}_{\mathcal{C}}$ ($\langle\mathfrak{G}\rangle_{\mathcal{C}}$), respectively. Define $X^*=X\cup\{\langle\mathfrak{G}\rangle:\mathfrak{G}\in\mathcal{N}\}$ and let $j:X\longrightarrow X^*$ be the natural injection.

**Definition 3.3.** A completion $\Big((Y,\mathcal{D}),\phi\Big)$ of $(X,\mathcal{C})$ in $\top$-**Chy** is said to be in **standard form** provided that $Y=X^*$, $\phi=j$, and $j^{\Rightarrow}\mathfrak{H}\xrightarrow{q_{\mathcal{D}}}\langle\mathfrak{G}\rangle$ in $(Y,q_{\mathcal{D}})$ whenever $\mathfrak{H}\sim\mathfrak{G}$.

**Definition 3.4.** Assume that $\Big((Y,\mathcal{D}),\phi\Big)$ and $\Big((Z,\mathcal{E}),\psi\Big)$ are two completions of $(X,\mathcal{C})$ in $\top$-**Chy**. Then $\Big((Y,\mathcal{D}),\phi\Big)\geq\Big((Z,\mathcal{E}),\psi\Big)$ is defined to mean that there exists a Cauchy-continuous map $h:(Y,\mathcal{D})\longrightarrow(Z,\mathcal{E})$ such that $h\circ\phi=\psi$. That is, the diagram below commutes:

$$\begin{array}{ccc} & \xrightarrow{\phi} & (Y,\mathcal{D}) \\ (X,\mathcal{C}) & & \big\downarrow \exists h \\ & \xrightarrow[\psi]{} & (Z,\mathcal{E}) \end{array}$$

As in the classical setting, $\geq$ is a partial order on the set of all completions of $(X,\mathcal{C})$. Moreover, if $\big((Y,\mathcal{D}),\phi\big) \geq \big((Z,\mathcal{E}),\psi\big)$ and vice versa, then the two completions are said to be **equivalent** and in this case $h$ is a $\top$-Cauchy isomorphism. Verification of the following lemma follows the proof in the classical setting of Theorem 5 given by Reed in [25].

**Lemma 3.7.** *Every $T_2$ completion of $(X,\mathcal{C})$ in $\top$-$\mathbf{Chy}$ is equivalent to one in standard form.*

Assume that $(X,\mathcal{C}) \in |\top\text{-}\mathbf{Chy}|$ and let $X^* = X \cup \{\langle\mathfrak{G}\rangle : \mathfrak{G} \in \mathcal{N}\}$ and $j : X \longrightarrow X^*$ be the natural injection. Define

$$\mathcal{C}^* = \{\mathfrak{H} \in \mathfrak{F}_L^{\top}(X^*) : \text{either } \mathfrak{H} \geq j^{\Rightarrow}\mathfrak{F} \text{ for some } q_{\mathcal{C}}\text{-convergent } \mathfrak{F} \text{ or}$$
$$\mathfrak{H} \geq j^{\Rightarrow}\mathfrak{G} \cap [\langle\mathfrak{G}\rangle] \text{ for some } \mathfrak{G} \in \mathcal{N}\}.$$

**Theorem 3.3.** *Suppose that $(X,\mathcal{C}) \in |\top\text{-}\mathbf{Chy}|$. Then*

*(i)* $\big((X^*,\mathcal{C}^*),j\big)$ *is the finest completion of $(X,\mathcal{C})$ in $\top$-$\mathbf{Chy}$ which is in standard form,*

*(ii) if $f : (X,\mathcal{C}) \longrightarrow (Y,\mathcal{D})$ is a Cauchy-continuous map and $(Y,\mathcal{D})$ is complete, then $f$ has a Cauchy-continuous extension $f^* : (X^*,\mathcal{C}^*) \longrightarrow (Y,\mathcal{D})$ such that $f^* \circ j = f$, and*

*(iii) $(X^*,\mathcal{C}^*)$ is $T_2$ iff $(X,\mathcal{C})$ is $T_2$.*

*Proof.* (i) First, it is shown that $\mathcal{C}^*$ is a $\top$-Cauchy structure on $X^*$. Since $j^{\Rightarrow}([x]) = [j(x)]$, it follows that $[j(x)]$ and $[\langle\mathfrak{G}\rangle]$ are in $\mathcal{C}^*$. Clearly $\mathfrak{K} \geq \mathfrak{H} \in \mathcal{C}^*$ implies that $\mathfrak{K} \in \mathcal{C}^*$. Suppose that $j^{\Rightarrow}\mathfrak{F}_1 \vee j^{\Rightarrow}\mathfrak{F}_2$ exists where $\mathfrak{F}_1, \mathfrak{F}_2$ are $q_{\mathcal{C}}$-convergent. Fix $f_i \in \mathfrak{F}_i$, $i = 1, 2$; then

$$\begin{aligned}
\top &= \bigvee_{z \in X^*} j^{\rightarrow} f_1(z) \wedge j^{\rightarrow} f_2(z) \\
&= \left( \bigvee_{x \in X} j^{\rightarrow} f_1(j(x)) \wedge j^{\rightarrow} f_2(j(x)) \right) \vee \left( \bigvee_{\mathfrak{G} \in \mathcal{N}} j^{\rightarrow} f_1(\langle \mathfrak{G} \rangle) \wedge j^{\rightarrow} f_2(\langle \mathfrak{G} \rangle) \right) \\
&= \left( \bigvee_{x \in X} \Big( \vee \{ f_1(z) : j(z) = x \} \Big) \wedge \Big( \vee \{ f_2(z) : j(z) = x \} \Big) \right) \\
&\qquad \vee \left( \bigvee_{\mathfrak{G} \in \mathcal{N}} \Big( \vee \{ f_1(z) : j(z) = \langle \mathfrak{G} \rangle \} \Big) \wedge \Big( \vee \{ f_2(z) : j(z) = \langle \mathfrak{G} \rangle \} \Big) \right) \\
&= \left( \bigvee_{x \in X} f_1(x) \wedge f_2(x) \right) \vee \left( \bigvee_{\mathfrak{G} \in \mathcal{N}} (\vee \varnothing) \wedge (\vee \varnothing) \right) \\
&= \left( \bigvee_{x \in X} f_1(x) \wedge f_2(x) \right) \vee \left( \bigvee_{\mathfrak{G} \in \mathcal{N}} \bot \wedge \bot \right) \\
&= \bigvee_{x \in X} f_1(x) \wedge f_2(x),
\end{aligned}$$

and thus $\mathfrak{F}_1 \vee \mathfrak{F}_2$ exists. Since $\mathfrak{F}_1$ and $\mathfrak{F}_2$ are $q_{\mathcal{C}}$-convergent, it is simple to show that $\mathfrak{F}_1 \cap \mathfrak{F}_2$ is also $q_{\mathcal{C}}$-convergent. Since $\mathfrak{F}_1 \cap \mathfrak{F}_2$ is $q_{\mathcal{C}}$-convergent and $j^{\Rightarrow}\mathfrak{F}_1 \cap j^{\Rightarrow}\mathfrak{F}_2 = j^{\Rightarrow}(\mathfrak{F}_1 \cap \mathfrak{F}_2) \in \mathcal{C}^*$, we have that $j^{\Rightarrow}\mathfrak{F}_1 \cap j^{\Rightarrow}\mathfrak{F}_2 \in \mathcal{C}^*$.

Observe that if $\mathfrak{F}$ $q_{\mathcal{C}}$-converges and $\mathfrak{G} \in \mathcal{N}$, then $j^{\Rightarrow}\mathfrak{F} \vee \Big( j^{\Rightarrow}\mathfrak{G} \cap [\langle \mathfrak{G} \rangle] \Big)$ fails to exist. Indeed, $\mathfrak{F} \vee \mathfrak{G}$ fails to exist since $\mathfrak{G}$ fails to $q_{\mathcal{C}}$-converge. Therefore there exists $a \in \mathfrak{F}$ and $b \in \mathfrak{G}$ such that $\bigvee_{x \in X} (a(x) \wedge b(x)) \neq \top$. A base member for $j^{\Rightarrow}\mathfrak{G} \cap [\langle \mathfrak{G} \rangle]$ is $c = j^{\rightarrow}(b) \vee \mathbf{1}_{\{\langle \mathfrak{G} \rangle\}}$. It follows that $\bigvee_{z \in X^*} \Big( j^{\rightarrow}(a)(z) \wedge c(z) \Big) = \bigvee_{x \in X} a(x) \wedge b(x) \neq \top$, and thus $j^{\Rightarrow}\mathfrak{F} \vee (j^{\Rightarrow}\mathfrak{G} \cap [\langle \mathfrak{G} \rangle])$ fails to exist. This argument also shows that $j^{\Rightarrow}\mathfrak{G}_1 \cap [\langle \mathfrak{G}_1 \rangle] \vee j^{\Rightarrow}\mathfrak{G}_2 \cap [\langle \mathfrak{G}_2 \rangle]$ exists iff $\langle \mathfrak{G}_1 \rangle = \langle \mathfrak{G}_2 \rangle$, and it follows that $\mathcal{C}^*$ is a $\top$-Cauchy structure on $X^*$.

The definition of $\mathcal{C}^*$ implies that $j : (X, \mathcal{C}) \longrightarrow (X^*, \mathcal{C}^*)$ is Cauchy-continuous. Conversely, assume that $\mathfrak{L} \in \mathfrak{F}_L^{\top}(X)$ such that $j^{\Rightarrow}\mathfrak{L} \in \mathcal{C}^*$. If $j^{\Rightarrow}\mathfrak{L} \geq j^{\Rightarrow}\mathfrak{F}$ for some $q_{\mathcal{C}}$-convergent $\mathfrak{F}$, then since $j$ is one-to-one, $\mathfrak{L} = j^{\Leftarrow}j^{\Rightarrow}\mathfrak{L} \geq \mathfrak{F}$ and thus $\mathfrak{L} \in \mathcal{C}$.

Next, suppose that $j^{\Rightarrow}\mathfrak{L} \geq j^{\Rightarrow}\mathfrak{G} \cap [\langle \mathfrak{G} \rangle]$ for some $\mathfrak{G} \in \mathcal{N}$. It is shown that $\mathfrak{L} \geq \mathfrak{G}$. Indeed, if $g \in \mathfrak{G}$; then $b = j^{\rightarrow}g \vee \mathbf{1}_{\{\langle \mathfrak{G} \rangle\}}$ is a base member of $j^{\Rightarrow}\mathfrak{G} \cap [\langle \mathfrak{G} \rangle]$ and $b \in j^{\Rightarrow}\mathfrak{L}$. Hence $j^{\leftarrow}(b) \in j^{\Leftarrow}j^{\Rightarrow}\mathfrak{L} = \mathfrak{L}$. However, for each $x \in X$, $j^{\leftarrow}(b)(x) = (b \circ j)(x) = g(x)$. Then $j^{\leftarrow}(b) = g \in \mathfrak{L}$ and thus $\mathfrak{L} \geq \mathfrak{G}$. It follows that $j : (X, \mathcal{C}) \longrightarrow (X^*, \mathcal{C}^*)$ is a Cauchy embedding. Since $j^{\Rightarrow}\mathfrak{G} \xrightarrow{q_{\mathcal{C}^*}} \langle \mathfrak{G} \rangle$, $j(X)$ is dense in $X^*$. It follows from the definition of $\mathcal{C}^*$ that $(X^*, \mathcal{C}^*)$ is complete. Hence $\big((X^*, \mathcal{C}^*), j\big)$ is a completion of $(X, \mathcal{C})$ in $\top$-**Chy**.

Finally, assume that $\big((X^*, \mathcal{D}), j\big)$ is another completion of $(X, \mathcal{C})$ in standard form. If $\mathfrak{F} \in \mathcal{C}$, then $j^{\Rightarrow}\mathfrak{F} \in \mathcal{D}$. Moreover, if $\mathfrak{G} \in \mathcal{N}$, then since $\big((X^*, \mathcal{D}), j\big)$ is in standard form, $j^{\Rightarrow}\mathfrak{G} \xrightarrow{q_{\mathcal{D}}} \langle \mathfrak{G} \rangle$. Hence $j^{\Rightarrow}\mathfrak{G} \cap [\langle \mathfrak{G} \rangle] \in \mathcal{D}$ and thus $\mathcal{C}^* \subseteq \mathcal{D}$. It follows that $\big((X^*, \mathcal{C}^*), j\big)$ is the finest completion in $\top$-**Chy** which is in standard form.

(ii) Suppose that $f : (X, \mathcal{C}) \longrightarrow (Y, \mathcal{D})$ is Cauchy-continuous. Define $f^*(j(x)) = f(x)$ and $f^*(\langle \mathfrak{G} \rangle) = y$, where $y$ is one of the limits of $f^{\Rightarrow}\mathfrak{G}$ in $(Y, \mathcal{D})$. Then $f^* \circ j = f$. If $\mathfrak{F} \in \mathfrak{F}_L^{\top}(X)$ is $q_{\mathcal{C}}$-convergent, then $f^{*\Rightarrow}(j^{\Leftarrow}\mathfrak{F}) = f^{\Rightarrow}\mathfrak{F} \in \mathcal{D}$.
Next, suppose that $\mathfrak{G} \in \mathcal{N}$; then $j^{\Rightarrow}\mathfrak{G} \cap [\langle \mathfrak{G} \rangle] \in \mathcal{C}^*$ and $f^{*\Rightarrow}(j^{\Rightarrow}\mathfrak{G} \cap [\langle \mathfrak{G} \rangle]) = f^{\Rightarrow}\mathfrak{G} \cap f^{*\Rightarrow}([\langle \mathfrak{G} \rangle]) = f^{\Rightarrow}\mathfrak{G} \cap [f^*(\langle \mathfrak{G} \rangle)] = f^{\Rightarrow}\mathfrak{G} \cap [y]$ where $f^{\Rightarrow}\mathfrak{F} \xrightarrow{q_{\mathcal{D}}} y$. It follows that $f^{\Rightarrow}\mathfrak{G} \cap [y] \in \mathcal{D}$ and thus $f^* : (X^*, \mathcal{C}^*) \longrightarrow (Y, \mathcal{D})$ is a Cauchy-continuous extension of $f$.

(iii) Since $j$ is a Cauchy-embedding, if $(X^*, \mathcal{C}^*)$ is $T_2$, then $(X, \mathcal{C})$ is $T_2$. Next, suppose that $(X, \mathcal{C})$ is $T_2$ and $\mathfrak{H} \xrightarrow{q_{\mathcal{C}^*}} z_1, z_2$. If $z_i = j(x_i)$, then $\mathfrak{H} \geq j^{\Rightarrow}\mathfrak{F}_i$ for some $\mathfrak{F}_i \xrightarrow{q_{\mathcal{C}}} x_i$ and $x_1 = x_2$. If $z_1 = j(x_1)$ and $z_2 \in X^* \setminus j(X)$, then $\mathfrak{H} \geq j^{\Rightarrow}\mathfrak{F}$ for some $\mathfrak{F} \xrightarrow{q_{\mathcal{C}}} x_1$ and $\mathfrak{H} \geq j^{\Rightarrow}\mathfrak{G} \cap [\langle \mathfrak{G} \rangle]$ for some $\mathfrak{G} \in \mathcal{N}$ with $z_2 = \langle \mathfrak{G} \rangle$. It follows that $j^{\Rightarrow}\mathfrak{F} \vee j^{\Rightarrow}\mathfrak{G}$ exists and thus $\mathfrak{F} \vee \mathfrak{G}$ exists, which implies that $\mathfrak{G}$ is $q_{\mathcal{C}}$-convergent. Hence this case is impossible. Moreover, if $z_i = \langle \mathfrak{G}_i \rangle$, then $\mathfrak{H} \geq j^{\Rightarrow}\mathfrak{K}_i \cap [\langle \mathfrak{G}_i \rangle]$ for some $\mathfrak{K}_i \in \langle \mathfrak{G}_i \rangle$, $i = 1, 2$. It follows that $\langle \mathfrak{G}_1 \rangle = \langle \mathfrak{G}_2 \rangle$; otherwise, $j^{\Rightarrow}\mathfrak{K}_1 \vee j^{\Rightarrow}\mathfrak{K}_2$ exists and thus $\mathfrak{K}_1 \vee \mathfrak{K}_2$ exists, which implies that $\langle \mathfrak{G}_1 \rangle = \langle \mathfrak{G}_2 \rangle$. Hence $(X^*, \mathcal{C}^*)$ is $T_2$. □

## Selection Maps and Completions

In this section we give a general approach for obtaining a completion using selection maps.

**Definition 3.5.** Let $(X, \mathcal{C}) \in |\top\text{-}\mathbf{Chy}|$. A map $\alpha : X^* \longrightarrow \mathcal{C}$ is called a **selection map** for $(X, \mathcal{C}) \in |\top\text{-}\mathbf{Chy}|$, or simply a selection map if the context is clear, if $\alpha(x) = [x]$ whenever $x \in X$ and $\alpha(\langle \mathfrak{G} \rangle) \in \langle \mathfrak{G} \rangle$ whenever $\mathfrak{G} \in \mathcal{N}$. Given $a \in L^X$ we define $a^\alpha \in L^{X^*}$ by

$$a^\alpha(x) = \nu_{\alpha(x)}(a) = \begin{cases} a(x), & x \in X \\ \nu_{\alpha(x)}(a), & x \in X^* \setminus X \end{cases}$$

Notationally, instead of writing $\alpha(\langle \mathfrak{G} \rangle)$, which is quite cumbersome, we write $\mathfrak{G}_\alpha$. It is easily shown that $(a \wedge b)^\alpha = a^\alpha \wedge b^\alpha$ and hence if $\mathfrak{F} \in \mathfrak{F}_L^\top(X)$ then $\{a^\alpha : a \in \mathfrak{F}\}$ is a $\top$-filter base for some $\top$-filter on $X^*$, denoted by $\mathfrak{F}^\alpha$. The following properties are needed.

**Lemma 3.8.** *Assume that $(X, \mathcal{C})$ is a $\top$-Cauchy space, $a, b \in L^X$ and let $\mathcal{B}$ denote a $\top$-filter base for $\mathfrak{F} \in \mathfrak{F}_L^\top(X)$. Then*

*(i)* $\nu_{\mathfrak{F}}(a \wedge b) \leq \bigvee_{x \in X} (a(x) \wedge b(x))$,

*(ii)* $[a^\alpha, b^\alpha] = [a, b]$,

*(iii)* $j^{\rightarrow}(a) \leq a^\alpha$ *and* $j^{\leftarrow}(a^\alpha) = a$,

*(iv)* $\mathcal{B}^\alpha = \{b^\alpha : b \in \mathcal{B}\}$ *is a* $\top$*-filter base for* $\mathfrak{F}^\alpha$,

*(v) $\mathfrak{K} \vee \mathfrak{L}$ exists on $X$ iff $\mathfrak{K}^\alpha \vee \mathfrak{L}^\alpha$ exists on $X^*$.*

*Proof.* (i) First, observe that if $\beta \in L$ and $\mathfrak{F} \in \mathfrak{F}_L^\top(X)$, then $\nu_\mathfrak{F}(\beta \mathbf{1}_X) = \beta$. Indeed, if $a \in \mathfrak{F}$, then it follows from Lemma 1.1 (v) that $[a, \beta \mathbf{1}_X] = \bigwedge\limits_{x \in X} (a(x) \to \beta) = \left( \bigvee\limits_{x \in X} a(x) \right) \to \beta = \top \to \beta = \beta$ and thus $\nu_\mathfrak{F}(\beta \mathbf{1}_X) = \bigvee\limits_{a \in \mathfrak{F}} [a, \beta \mathbf{1}_X] = \bigvee\limits_{a \in \mathfrak{F}} \beta = \beta$. Let $\beta = \bigvee\limits_{x \in X} (a(x) \wedge b(x))$; then $\nu_\mathfrak{F}(a \wedge b) \leq \nu_\mathfrak{F}(\beta \mathbf{1}_X) = \beta$ and thus the result follows.

(ii) According to Corollary 3.3 [4], $\nu_\mathfrak{F}(a) \to \nu_\mathfrak{F}(b) \geq [a, b]$ is valid for each $\mathfrak{F} \in \mathfrak{F}_L^\top(X)$. Then $[a^\alpha, b^\alpha] = [a, b] \wedge \bigwedge\limits_{\mathfrak{G} \in \mathcal{N}} \Big( a^\alpha(\langle \mathfrak{G} \rangle) \to b^\alpha(\langle \mathfrak{G} \rangle) \Big) = [a, b] \wedge \bigwedge\limits_{\mathfrak{G} \in \mathcal{N}} \Big( \nu_{\mathfrak{G}_\alpha}(a) \to \nu_{\mathfrak{G}_\alpha}(b) \Big) \geq [a, b]$, and thus $[a^\alpha, b^\alpha] = [a, b]$.

(iii) $j^\to(a)(x) = a(x)$ if $x \in X$ and $j^\to(a)(x) = \perp$ if $x \notin X$. Hence $j^\to(a) \leq a^\alpha$. Next $j^\leftarrow(a^\alpha)(x) = a^\alpha(j(x)) = a^\alpha(x) = a(x)$ as needed.

(iv) If $b \in \mathcal{B}$, then $\bigvee\limits_{z \in X^*} b^\alpha(z) \geq \bigvee\limits_{x \in X} b(x) = \top$ and thus $\bigvee\limits_{z \in X^*} b^\alpha(z) = \top$. Next, suppose that $b_1, b_2 \in \mathcal{B}$; then by (ii) above, $\bigvee\limits_{b \in \mathcal{B}} [b^\alpha, b_1^\alpha \wedge b_2^\alpha] = \bigvee\limits_{b \in \mathcal{B}} [b^\alpha, (b_1 \wedge b_2)^\alpha] = \bigvee\limits_{b \in \mathcal{B}} [b, b_1 \wedge b_2] = \top$. Hence $\mathcal{B}^\alpha$ is a $\top$-filter base for some $\top$-filter $\mathfrak{H} \subseteq \mathfrak{F}^\alpha$. Moreover, assume that $a \in \mathfrak{F}$; then $a^\alpha$ belongs to the $\top$-filter base $\{f^\alpha : f \in \mathfrak{F}\}$ for the $\top$-filter denoted by $\mathfrak{F}^\alpha$. It suffices to show that $a^\alpha \in \mathfrak{H}$. According to (ii) above, $\bigvee\limits_{b \in \mathcal{B}} [b^\alpha, a^\alpha] = \bigvee\limits_{b \in \mathcal{B}} [b, a] = \top$ and thus $\mathcal{B}^\alpha$ is a $\top$-filter base for $\mathfrak{F}^\alpha$.

(v) Fix $a \in \mathfrak{K}$, $b \in \mathfrak{L}$ and suppose that $\mathfrak{K}^\alpha \vee \mathfrak{L}^\alpha$ exists. Then,

$$
\begin{aligned}
\top &= \bigvee_{z\in X^*}\Big(a^\alpha(z)\wedge b^\alpha(z)\Big)\\
&= \bigvee_{x\in X}\Big(a(x)\wedge b(x)\Big)\vee\bigvee_{\mathfrak{G}\in\mathcal{N}}\Big(a^\alpha(\langle\mathfrak{G}\rangle)\wedge b^\alpha(\langle\mathfrak{G}\rangle)\Big)\\
&= \bigvee_{x\in X}\Big(a(x)\wedge b(x)\Big)\vee\bigvee_{\mathfrak{G}\in\mathcal{N}}\Big(\nu_{\alpha(\langle\mathfrak{G}\rangle)}(a)\wedge \nu_{\alpha(\langle\mathfrak{G}\rangle)}(b)\Big)\\
&= \bigvee_{x\in X}\Big(a(x)\wedge b(x)\Big)\vee\bigvee_{\mathfrak{G}\in\mathcal{N}}\nu_{\alpha(\langle\mathfrak{G}\rangle)}(a\wedge b)\\
&= \bigvee_{x\in X}\Big(a(x)\wedge b(x)\Big) \qquad\qquad \text{by (i).}
\end{aligned}
$$

Hence $\bigvee_{x\in X}\Big(a(x)\wedge b(x)\Big)=\top$ and thus $\mathfrak{K}\vee\mathfrak{L}$ exists. The other direction is clear. ☐

Now given any $(X,\mathcal{C})\in|\top\text{-}\mathbf{Chy}|$ and a selection map $\alpha$, define $\mathcal{C}^\alpha=\{\mathfrak{H}\in\mathfrak{F}_L^\top(X^*):\mathfrak{H}\geq\mathfrak{F}^\alpha$ for some $\mathfrak{F}\in\mathcal{C}\}$.

**Theorem 3.4.** *Let $(X,\mathcal{C})$ be a $\top$-Cauchy space which is not complete and $\alpha$ a selection map. Then $\Big((X^*,\mathcal{C}^\alpha),j\Big)$ is a completion in $\top$-**Chy** which is in standard form.*

*Proof.* First we must show that $\mathcal{C}^\alpha$ is a $\top$-Cauchy structure on $X^*$. Let $x\in X$ then $[j(x)]\supseteq[x]^\alpha$. Indeed, if $a\in[x]$ then a base member of $[x]^\alpha$ is $a^\alpha$ and $a^\alpha(j(x))=a(x)=\top$. Hence $a^\alpha\in[j(x)]$ and $[j(x)]\supseteq[x]^\alpha$. Therefore $[j(x)]\in\mathcal{C}^\alpha$ for each $x\in X$. Next we show that $[\langle\mathfrak{G}\rangle]\geq(\mathfrak{G}_\alpha)^\alpha$. Let $g\in\mathfrak{G}_\alpha$ then $g^\alpha(\langle\mathfrak{G}\rangle)=\nu_{\mathfrak{G}_\alpha}(g)=\top$ and so $g^\alpha\in[\langle\mathfrak{G}\rangle]$ and $[\langle\mathfrak{G}\rangle]\supseteq(\mathfrak{G}_\alpha)^\alpha$. Since $\mathfrak{G}_\alpha\in\mathcal{C}$, it follows that $[\langle\mathfrak{G}\rangle]\in\mathcal{C}^\alpha$.

Clearly $\mathfrak{K}\geq\mathfrak{H}\in\mathcal{C}^\alpha$ implies $\mathfrak{K}\in\mathcal{C}^\alpha$. Next, assume that $\mathfrak{H},\mathfrak{K}\in\mathcal{C}^\alpha$ such that $\mathfrak{H}\vee\mathfrak{K}$ exists, where $\mathfrak{H}\geq\mathfrak{F}^\alpha$ and $\mathfrak{K}\geq\mathfrak{G}^\alpha$ for some $\mathfrak{F},\mathfrak{G}\in\mathcal{C}$. Then $\mathfrak{F}^\alpha\vee\mathfrak{G}^\alpha$ exists, and by Lemma 3.8 (v), $\mathfrak{F}\vee\mathfrak{G}$ exists also. Thus $\mathfrak{F}\cap\mathfrak{G}\in\mathcal{C}$ and since $\mathfrak{F}^\alpha\cap\mathfrak{G}^\alpha\geq(\mathfrak{F}\cap\mathfrak{G})^\alpha$, $\mathfrak{H}\cap\mathfrak{K}\in\mathcal{C}^\alpha$. Therefore $\mathcal{C}^\alpha$ is a $\top$-Cauchy structure on $X^*$ as desired.

Suppose that $\mathfrak{F} \in \mathcal{C}$. According to Lemma 3.8 (iii), $j^{\rightarrow}(a) \leq a^{\alpha}$ for each $a \in \mathfrak{F}$ and thus $j^{\Rightarrow}\mathfrak{F} \geq \mathfrak{F}^{\alpha}$. Hence $j : (X, \mathcal{C}) \longrightarrow (X^*, \mathcal{C}^{\alpha})$ is Cauchy-continuous. Next, assume that $\mathfrak{H} \in \mathfrak{F}_L^{\top}(X)$ such that $j^{\Rightarrow}\mathfrak{H} \geq \mathfrak{F}^{\alpha}$ for some $\mathfrak{F} \in \mathcal{C}$. Since $j^{\Leftarrow}(\mathfrak{F}^{\alpha})$ exists, it follows from Lemma 3.8 (iii) that $\mathfrak{H} = j^{\Leftarrow}(j^{\Rightarrow}\mathfrak{H}) \geq j^{\Leftarrow}(\mathfrak{F}^{\alpha}) = \mathfrak{F} \in \mathcal{C}$. Then $\mathfrak{H} \in \mathcal{C}$ and $j : (X, \mathcal{C}) \longrightarrow (X^*, \mathcal{C}^{\alpha})$ is a $\top$-Cauchy embedding.

Fix $\mathfrak{G} \in \mathcal{N}$; it is shown that $j^{\Rightarrow}(\mathfrak{G}_{\alpha}) \cap [\langle \mathfrak{G} \rangle] \geq (\mathfrak{G}_{\alpha})^{\alpha}$. A base member of $(\mathfrak{G}_{\alpha})^{\alpha}$ is $g^{\alpha}$ where $g \in \mathfrak{G}_{\alpha}$. Since $g^{\alpha} \geq j^{\rightarrow}g$, we have $g^{\alpha} \in j^{\Rightarrow}\mathfrak{G}_{\alpha}$. Also, $g^{\alpha}(\langle \mathfrak{G} \rangle) = \nu_{\mathfrak{G}_{\alpha}}(g) = \top$ so that $g^{\alpha} \in [\langle \mathfrak{G} \rangle]$. Hence $g^{\alpha} \in j^{\Rightarrow}(\mathfrak{G}_{\alpha}) \cap [\langle \mathfrak{G} \rangle]$ and $j$ is a dense embedding.

Finally, we must show that $(X^*, \mathcal{C}^{\alpha})$ is complete. Assume that $\mathfrak{H} \in \mathcal{C}^{\alpha}$. Then $\mathfrak{H} \geq \mathfrak{F}^{\alpha}$ with $\mathfrak{F} \in \mathcal{C}$. There are two possibilities: $\mathfrak{F} \xrightarrow{q_{\mathcal{C}}} x$ for some $x \in X$ or $\mathfrak{F} \in \mathcal{N}_{\mathcal{C}}$. If $\mathfrak{F} \xrightarrow{q_{\mathcal{C}}} x$, then $\mathfrak{F} \cap [x] \in \mathcal{C}$ implies that $\mathfrak{H} \cap [j(x)] \geq \mathfrak{F}^{\alpha} \cap [j(x)] \geq \mathfrak{F}^{\alpha} \cap [x]^{\alpha} \geq (\mathfrak{F} \cap [x])^{\alpha}$. Hence $\mathfrak{H} \cap [j(x)] \in \mathcal{C}^{\alpha}$ and $\mathfrak{H} \xrightarrow{q_{\mathcal{C}^{\alpha}}} j(x)$. Next, if $\mathfrak{F} \in \mathcal{N}_{\mathcal{C}}$, note that $\mathfrak{F}^{\alpha} \cap (\mathfrak{F}_{\alpha})^{\alpha} \leq [\langle \mathfrak{F} \rangle]$. Indeed if $a \in \mathfrak{F}, b \in \mathfrak{F}_{\alpha}$ then $(a^{\alpha} \vee b^{\alpha})(\langle \mathfrak{F} \rangle) = \nu_{\mathfrak{F}_{\alpha}}(a) \vee \nu_{\mathfrak{F}_{\alpha}}(b) \geq \nu_{\mathfrak{F}_{\alpha}}(b) = \top$. Hence $\mathfrak{H} \cap [\langle \mathfrak{F} \rangle] \geq \mathfrak{F}^{\alpha} \cap (\mathfrak{F}_{\alpha})^{\alpha} \cap [\langle \mathfrak{F} \rangle] = \mathfrak{F}^{\alpha} \cap (\mathfrak{F}_{\alpha})^{\alpha} \geq (\mathfrak{F} \cap \mathfrak{F}_{\alpha})^{\alpha}$. Since $\mathfrak{F} \sim \mathfrak{F}_{\alpha}$, we have that $\mathfrak{F} \cap \mathfrak{F}_{\alpha} \in \mathcal{C}$ and thus $\mathfrak{H} \cap [\langle \mathfrak{F} \rangle] \in \mathcal{C}^{\alpha}$ and $\mathfrak{H} \xrightarrow{q_{\mathcal{C}^{\alpha}}} \langle \mathfrak{F} \rangle$. Hence $(X^*, \mathcal{C}^{\alpha})$ is complete and the result follows. □

Given any $(X, \mathcal{C}) \in |\top\text{-}\mathbf{Chy}|$ and selection map $\alpha$, Theorem 3.3 (i) implies $\mathcal{C}^* \subseteq \mathcal{C}^{\alpha}$. However, even in the classical case where $L = \{0, 1\}$, examples exist where $\mathcal{C}^* \neq \mathcal{C}^{\alpha}$ for any selection map $\alpha$.

*Pretopological Completions*

In this section we look at a particular selection map. Assume that $(X, \mathcal{C}) \in |\top\text{-}\mathbf{Chy}|$ is not complete. Then $(X, \mathcal{C})$ is called **relatively full** if for each $\mathfrak{G} \in \mathcal{N}$, $\langle \mathfrak{G} \rangle$ contains a smallest member, denoted as $\mathfrak{G}_{\min}$. If in addition, each $x \in X$ has a coarsest $q_{\mathcal{C}}$-convergent $\top$-filter, denoted by $\mathcal{U}_{\mathcal{C}}(x)$, then $(X,\mathcal{C})$ is said to be **full**.

**Definition 3.6.** A completion $\big((Y,\mathcal{D}),\phi\big)$ of $(X,\mathcal{C})$ in $\top$-**Chy** is said to be **remainder-pretopological** if each $y \in Y \setminus \phi(X)$ has a coarsest $q_{\mathcal{D}}$-convergent $\top$-filter. The completion is called **pretopological** if the above holds for each $y \in Y$.

Suppose that $(X,\mathcal{C})$ is relatively full, then we may choose the selection map $\alpha$ which sends $x \mapsto [x]$, $x \in X$ and $\langle \mathfrak{G} \rangle \mapsto \mathfrak{G}_{\min}$, $\mathfrak{G} \in \mathcal{N}$. For this special selection map we will denote $a^{\alpha}, \mathfrak{F}^{\alpha}$ and $\mathcal{C}^{\alpha}$, respectively, by $\widetilde{a}, \widetilde{\mathfrak{F}}$ and $\widetilde{\mathcal{C}}$.

**Theorem 3.5.** *Suppose that $(X,\mathcal{C})$ is relatively full (full). Then $\big((X^*,\widetilde{\mathcal{C}}),j\big)$ is a remainder-pretopological (pretopological) completion in $\top$-**Chy**, respectively.*

*Proof.* Assume that $(X,\mathcal{C})$ is relatively full. Then by Theorem 3.4 $\big((X^*,\widetilde{\mathcal{C}}),j\big)$ is a completion. It must be shown that $\big((X^*,\widetilde{\mathcal{C}}),j\big)$ is remainder-pretopological. Fix $\mathfrak{G} \in \mathcal{N}$ and assume that $\mathfrak{H} \longrightarrow \langle \mathfrak{G} \rangle$ in $(X^*, q_{\widetilde{\mathcal{C}}})$. It is shown that $\mathfrak{H} \geq \widetilde{\mathfrak{G}_{\min}} \longrightarrow \langle \mathfrak{G} \rangle$ in $(X^*, q_{\widetilde{\mathcal{C}}})$. Since $\mathfrak{H} \longrightarrow \langle \mathfrak{G} \rangle$ in $(X^*, q_{\widetilde{\mathcal{C}}})$, $\mathfrak{H} \cap [\langle \mathfrak{G} \rangle] \geq \widetilde{\mathfrak{K}}$ for some $\mathfrak{K} \in \mathcal{C}$, and it follows that $\widetilde{\mathfrak{K}} \cap [\langle \mathfrak{G} \rangle] \in \widetilde{\mathcal{C}}$. Hence $j^{\Rightarrow}\mathfrak{G} \longrightarrow \langle \mathfrak{G} \rangle$ in $(X^*, q_{\widetilde{\mathcal{C}}})$ implies that $j^{\Rightarrow}(\mathfrak{K} \cap \mathfrak{G}) \longrightarrow \langle \mathfrak{G} \rangle$ in $(X^*,\widetilde{\mathcal{C}})$, and hence $\mathfrak{K} \cap \mathfrak{G} \in \mathcal{C}$. Therefore $\langle \mathfrak{K} \rangle = \langle \mathfrak{G} \rangle$ and thus $\mathfrak{H} \geq \widetilde{\mathfrak{K}} \geq \widetilde{\mathfrak{G}_{\min}} \longrightarrow \langle \mathfrak{G} \rangle$ in $(X^*,\widetilde{\mathcal{C}})$. It follows that $\big((X^*,\widetilde{\mathcal{C}}),j\big)$ is a remainder-pretopological completion of $(X,\mathcal{C})$ in $\top$-**Chy**.

Next, assume that $(X,\mathcal{C})$ is full and fix $x \in X$. Let $\mathcal{U}_{\mathcal{C}}(x)$ denote the coarsest $\top$-filter which $q_{\mathcal{C}}$-converges to $[x]$. It is shown that $\widetilde{\mathcal{U}_{\mathcal{C}}(x)}$ is the coarsest $q_{\widetilde{\mathcal{C}}}$-convergent $\top$-filter to converge to $j(x)$. Suppose that $\mathfrak{H} \cap [j(x)] \in \widetilde{\mathcal{C}}$; then $\mathfrak{H} \cap [j(x)] \geq \widetilde{\mathfrak{F}}$ for some $\mathfrak{F} \in \mathcal{C}$. Note that $\widetilde{\mathfrak{F}} \vee (\mathfrak{H} \cap [j(x)])$ exists and thus $\widetilde{\mathfrak{F}} \cap [j(x)] \in \widetilde{\mathcal{C}}$. Since $j^{\Rightarrow}(\mathfrak{F} \cap [x]) \geq \widetilde{\mathfrak{F}} \cap [j(x)]$, it follows that $\mathfrak{F} \cap [x] \in \mathcal{C}$ and thus $\mathfrak{F} \geq \mathcal{U}_{\mathcal{C}}(x)$. Hence $\mathfrak{H} \geq \widetilde{\mathfrak{F}} \geq \widetilde{\mathcal{U}_{\mathcal{C}}(x)}$ and thus $(X^*,\widetilde{\mathcal{C}})$ is pretopological. ☐

**Corollary 3.1.** *Assume that $(X,\mathcal{C}) \in |\top$-**Chy**$|$. Then $(X,\mathcal{C})$ has a remainder-pretopological (pretopological) completion in $\top$-**Chy** which is in standard form if and only if $(X,\mathcal{C})$ is* *relatively full (full), respectively.*

*Proof.* Suppose that $(X, \mathcal{C})$ is relatively full (full). Then by Theorem 3.5 we have that $(X, \mathcal{C})$ has a remainder-pretopological (pretopological) completion in standard form, respectively.

Conversely, suppose that $\big((X^*, \mathcal{D}), j\big)$ is a remainder-pretopological completion of $(X, \mathcal{C})$ in standard form. Then if $y \in X^* \setminus j(X)$, then by definition of $X^*$, $y = \langle \mathfrak{G} \rangle$ for some $\mathfrak{G} \in \mathcal{N}$. It follows that $j^{\Rightarrow}\left(\bigcap_{\mathfrak{H} \in \langle \mathfrak{G} \rangle} \mathfrak{H}\right) = \bigcap_{\mathfrak{H} \in \langle \mathfrak{G} \rangle} j^{\Rightarrow}\mathfrak{H} \in \mathcal{D}$ and since $j$ is a $\top$-Cauchy embedding, $\langle \mathfrak{G} \rangle$ contains a smallest member. Hence $(X, \mathcal{C})$ is relatively full. A similar argument holds whenever $\big((X^*, \mathcal{D}), j\big)$ is a pretopological completion in standard form. $\square$

**Corollary 3.2.** *Suppose that $(X, \mathcal{C})$ is relatively full. Then $(X^*, \widetilde{\mathcal{C}})$ is $T_2$ if and only if $(X, \mathcal{C})$ is $T_2$.*

## Topological Completion

Fang and Yu [29] defined when a object in $\top$-**Conv** is topological. Further, Fang and Yue [5] showed that this definition characterizes when a $\top$-convergence space is strong $L$-topological as defined in [29]. The following definition of topological is a version suited for the category $\top$-**Chy**.

**Definition 3.7.** An object $(X, \mathcal{C}) \in |\top\text{-}\mathbf{Chy}|$ is said to be **topological** in $\top$-**Chy** provided the following conditions are satisfied. Let $J$ be any set, $\psi : J \longrightarrow X^*$, $\sigma : J \longrightarrow \mathcal{C}$ such that if $\psi(y) \in X$ then $\sigma(y) \xrightarrow{q_{\mathcal{C}}} \psi(y)$, and otherwise if $\psi(y) = \langle \mathfrak{G} \rangle$, then $\sigma(y) \in \langle \mathfrak{G} \rangle$. If $\mathfrak{H} \in \mathfrak{F}_L^{\top}(J)$ and $\psi^{\Rightarrow}\mathfrak{H} \geq j^{\Rightarrow}\mathfrak{F}$ for some $q_{\mathcal{C}}$-convergent $\mathfrak{F}$ or $\psi^{\Rightarrow}\mathfrak{H} \geq j^{\Rightarrow}\mathfrak{G} \cap [\langle \mathfrak{F} \rangle]$ for some $\mathfrak{G} \in \mathcal{N}_{\mathcal{C}}$, then $\kappa\sigma\mathfrak{H} \in \mathcal{C}$. Here $\kappa\sigma\mathfrak{H} = \{b \in L^X : e_b \circ \sigma \in \mathfrak{H}\}$ and $e_b : \mathfrak{F}_L^{\top}(X) \longrightarrow L$ is defined as $e_b(\mathfrak{G}) = \nu_{\mathfrak{G}}(b)$, for each $\mathfrak{G} \in \mathfrak{F}_L^{\top}(X)$. It was shown by Fang and Yue in [5] that $\kappa\sigma\mathfrak{H}$ is a $\top$-filter on $X$.

The notion of being topological has been defined in previous chapters in the context of $\top$-convergence structures. The following two results connect the definition in the $\top$-convergence space setting to our discussion here.

**Lemma 3.9.** *Suppose that $(X, \mathcal{C}) \in |\top\text{-}\mathbf{Chy}|$ is complete. Then $(X, \mathcal{C})$ is topological in $\top\text{-}\mathbf{Chy}$ if and only if $(X, q_{\mathcal{C}})$ is topological in $\top\text{-}\mathbf{Conv}$.*

*Proof.* Let $J$ be any set, $\psi : J \longrightarrow X$, $\sigma : J \longrightarrow \mathfrak{F}_L^{\top}(X)$ such that $\sigma(y) \xrightarrow{q_{\mathcal{C}}} \psi(y)$ for each $y \in J$. Suppose that $\mathfrak{H} \in \mathfrak{F}_L^{\top}(J)$ and $\psi^{\Rightarrow}\mathfrak{H} \xrightarrow{q_{\mathcal{C}}} x$. Using Theorem 3.11 in [5] it must be shown that $\kappa\sigma\mathfrak{H} \xrightarrow{q_{\mathcal{C}}} x$; equivalently; $\kappa\sigma\mathfrak{H} \cap [x] \in \mathcal{C}$. Since $\sigma(y) \xrightarrow{q_{\mathcal{C}}} \psi(y)$ for each $y \in J$, it suffices to assume that $\sigma(y) \leq [\psi(y)]$ for each $y \in J$. Otherwise, $\sigma$ can be replaced by $\sigma^*$, where $\sigma^*(y) = \sigma(y) \cap [\psi(y)]$, $y \in J$. Hence assume that $\sigma(y) \leq [\psi(y)]$ for each $y \in J$.

It is shown that $\psi^{\Rightarrow}\mathfrak{H} \geq \kappa\sigma\mathfrak{H}$. Fix $b \in \kappa\sigma\mathfrak{H}$. Since $\sigma(y) \leq [\psi(y)]$ for each $y \in J$, for $z \in X$, $\psi^{\Rightarrow}(e_b \circ \sigma)(z) = \vee\{(e_b \circ \sigma)(y) : \psi(y) = z\} = \vee\{\nu_{\sigma(y)}(b) : \psi(y) = z\} \leq \vee\{\nu_{[\psi(y)]}(b) : \psi(y) = z\} = \nu_{[z]}(b) = b(z)$. Then $\psi^{\rightarrow}(e_b \circ \sigma) \leq b$ and $e_b \circ \sigma \in \mathfrak{H}$ implies that $b \in \psi^{\Rightarrow}\mathfrak{H}$. Therefore $\kappa\sigma\mathfrak{H} \leq \psi^{\Rightarrow}\mathfrak{H}$. Since $(X, \mathcal{C})$ is topological, $\kappa\sigma\mathfrak{H} \in \mathcal{C}$, and hence $\psi^{\Rightarrow}\mathfrak{H} \cap [x] \in \mathcal{C}$ implies that $\kappa\sigma\mathfrak{H} \vee (\psi^{\Rightarrow}\mathfrak{H} \cap [x])$ exists. Then $\kappa\sigma\mathfrak{H} \cap [x] \in \mathcal{C}$ and $\kappa\sigma\mathfrak{H} \xrightarrow{q_{\mathcal{C}}} x$. Therefore $(X, q_{\mathcal{C}})$ is topological in $\top$-**Conv**.

Conversely, assume that $(X, \mathcal{C})$ is complete and $(X, q_{\mathcal{C}})$ is topological in $\top$-**Conv**. Given $\psi : J \longrightarrow X$, $\sigma : J \longrightarrow \mathcal{C}$ such that $\sigma(y) \in \mathcal{C}$ satisfies $[\psi(y)] \geq \sigma(y)$ for each $y \in J$. It follows that $\sigma(y) \cap [\psi(y)] \in \mathcal{C}$ and thus $\sigma(y) \xrightarrow{q_{\mathcal{C}}} \psi(y)$. Suppose that $\mathfrak{H} \in \mathfrak{F}_L^{\top}(J)$ and $\psi^{\Rightarrow}\mathfrak{H} \in \mathcal{C}$; Then since $(X, \mathcal{C})$ is complete, $\psi^{\Rightarrow}\mathfrak{H} \xrightarrow{q_{\mathcal{C}}} x$ for some $x \in X$. It follows that $\kappa\sigma\mathfrak{H} \xrightarrow{q_{\mathcal{C}}} x$ as $(X, q_{\mathcal{C}})$ is topological, and thus $\kappa\sigma\mathfrak{H} \in \mathcal{C}$. Hence $(X, \mathcal{C})$ is topological in $\top$-**Chy**. □

**Porism 3.1.** *Assume that $(X, \mathcal{C}) \in |\top\text{-}\mathbf{Chy}|$ is topological in $\top\text{-}\mathbf{Chy}$; then $(X, q_{\mathcal{C}})$ is topological in $\top\text{-}\mathbf{Conv}$.*

*Proof.* Note that the first half of the proof of Lemma 3.9 did not use completeness. □

**Lemma 3.10.** *If* $(X, \mathcal{C})$ *is topological in* $\top$-**Chy** *then it is full.*

*Proof.* Let $\mathfrak{G} \in \mathcal{N}$ and let $J = \{y : \mathfrak{G}_y \in \langle \mathfrak{G} \rangle\}$ be an index for the elements in $\langle \mathfrak{G} \rangle$. Define $\psi : J \longrightarrow X^*$ by $\psi(y) = \langle \mathfrak{G} \rangle$ and $\sigma(y) = \mathfrak{G}_y \in \mathcal{C}$ for each $y \in J$. Let $\mathfrak{H} = \{\mathbf{1}_J\} \in \mathfrak{F}_L^\top(J)$; then $\psi^\Rightarrow \mathfrak{H} = [\langle \mathfrak{G} \rangle]$. Indeed, $\psi^\rightarrow(\mathbf{1}_J)(z) = \vee\{\mathbf{1}_J(y) : \psi(y) = z\} = \begin{cases} \bot, & z \neq \langle \mathfrak{G} \rangle \\ \top, & z = \langle \mathfrak{G} \rangle \end{cases} = \mathbf{1}_{\{\langle \mathfrak{G} \rangle\}}(z)$, and thus $\psi^\rightarrow(\mathbf{1}_J) = \mathbf{1}_{\{\langle \mathfrak{G} \rangle\}}$. It follows that $\psi^\Rightarrow \mathfrak{H} = [\langle \mathfrak{G} \rangle] \geq j^\Rightarrow \mathfrak{G} \cap [\langle \mathfrak{G} \rangle]$ and since $(X, \mathcal{C})$ is topological in $\top$-**Chy**, $\kappa\sigma\mathfrak{H} \in \mathcal{C}$. It remains to show that $\kappa\sigma\mathfrak{H} = \bigcap_{y \in J} \mathfrak{G}_y$. Recall that $\kappa\sigma\mathfrak{H} = \{b \in L^X : e_b \circ \sigma \in \mathfrak{H}\}$. Observe that $b \in \kappa\sigma\mathfrak{H}$ iff $e_b \circ \sigma = \mathbf{1}_J$. Equivalently, $b \in \kappa\sigma\mathfrak{H}$ iff for each $y \in J$, $\nu_{\sigma(y)}(b) = \top$, or iff for each $y \in J$, $b \in \sigma(y) = \mathfrak{G}_y$. Hence $b \in \kappa\sigma\mathfrak{H}$ iff $b \in \bigcap_{y \in J} \mathfrak{G}_y$. It follows that $\kappa\sigma\mathfrak{H} = \bigcap_{y \in J} \mathfrak{G}_y$. Since $\kappa\sigma\mathfrak{H} \in \mathcal{C}$, $\langle \mathfrak{G} \rangle$ contains a minimum member.

Likewise, fix $x \in X$ and let $J = \{y : \mathfrak{F}_y \xrightarrow{q_\mathcal{C}} x\}$ be an index set for all of the $\top$-filters which $q_\mathcal{C}$-converge to $x$. Define $\psi : J \longrightarrow X^*$ as $\psi(y) = x$ and $\sigma(y) = \mathfrak{F}_y \in \mathcal{C}$, $y \in J$. The argument used above shows that $\mathcal{U}_\mathcal{C}(x) = \bigcap_{y \in J} \mathfrak{F}_y \in \mathcal{C}$. Since $[x] \geq \mathcal{U}_\mathcal{C}(x)$, it follows that $\mathcal{U}_\mathcal{C}(x) \xrightarrow{q_\mathcal{C}} x$. Hence $\mathcal{U}_\mathcal{C}(x)$ is the coarsest $\top$-filter which $q_\mathcal{C}$-converges to $x$. □

The next result appears within the proof of Theorem 3.5, where $\mathcal{U}_\mathcal{C}(x) = \cap\{\mathfrak{F} : \mathfrak{F} \xrightarrow{q_\mathcal{C}} x\}$, $x \in X$.

**Lemma 3.11.** *Assume that* $(X, \mathcal{C})$ *is full and* $\mathfrak{H} \in \mathfrak{F}_L^\top(X^*)$. *Then*

*(i)* $\mathfrak{H} \longrightarrow j(x)$ *in* $(X^*, \widetilde{\mathcal{C}})$ *iff* $\mathfrak{H} \geq \widetilde{\mathcal{U}_\mathcal{C}}(x)$, $x \in X$, *and*

*(ii)* $\mathfrak{H} \longrightarrow \langle \mathfrak{G} \rangle$ *in* $(X^*, \widetilde{\mathcal{C}})$ *iff* $\mathfrak{H} \geq \widetilde{\mathfrak{G}_{min}}$, $\mathfrak{G} \in \mathcal{N}$.

Note that Lemma 3.11 shows that if $(X, \mathcal{C})$ is topological in $\top$-**Chy**, then the completion $\big((X^*, \widetilde{\mathcal{C}}), j\big)$ is pretopological, that is, $(X^*, q_{\widetilde{\mathcal{C}}})$ is a pretopological space.

**Theorem 3.6.** *Suppose that* $(X, \mathcal{C})$ *is topological in* $\top$-**Chy**. *Then* $(X^*, \widetilde{\mathcal{C}})$ *is also topological in* $\top$-**Chy**.

*Proof.* Define $\psi : X \longrightarrow X^*$ by $\psi(x) = j(x)$ and $\sigma : X \longrightarrow \mathcal{C}$ by $\sigma(x) = \mathcal{U}_{\mathcal{C}}(x)$, $x \in X$. Since $(X, \mathcal{C})$ is topological in $\top$-**Chy**, it follows that $\kappa\sigma\mathcal{U}_{\mathcal{C}}(x) = \mathcal{U}_{\mathcal{C}}(x)$ and $\kappa\sigma\mathfrak{G}_{\min} = \mathfrak{G}_{\min}$, for each $x \in X$ and $\mathfrak{G} \in \mathcal{N}$. Define $\delta : X^* \longrightarrow \widetilde{\mathcal{C}}$ by

$$\delta(z) = \begin{cases} \widetilde{\mathcal{U}_{\mathcal{C}}(x)}, & z = j(x) \\ \widetilde{\mathfrak{G}_{\min}}, & z = \langle \mathfrak{G} \rangle \end{cases}.$$

Since $(X^*, q_{\widetilde{\mathcal{C}}})$ is pretopological, it suffices to show that $\kappa\delta\widetilde{\mathcal{U}_{\mathcal{C}}(x)} \geq \widetilde{\mathcal{U}_{\mathcal{C}}(x)}$ and $\kappa\delta\widetilde{\mathfrak{G}_{\min}} \geq \widetilde{\mathfrak{G}_{\min}}$ for each $x \in X$ and $\mathfrak{G}_{\min} \in \mathcal{N}$.

First, it is shown that $\kappa\delta\widetilde{\mathcal{U}_{\mathcal{C}}(x)} \geq \widetilde{\mathcal{U}_{\mathcal{C}}(x)}$, $x \in X$. Recall that $\kappa\sigma\mathcal{U}_{\mathcal{C}}(x) = \{b \in L^X : e_b \circ \sigma\} = \mathcal{U}_{\mathcal{C}}(x)$. Assume that $b \in \kappa\sigma\mathcal{U}_{\mathcal{C}}(x)$; it is shown that $\widetilde{b} \in \kappa\delta\widetilde{\mathcal{U}_{\mathcal{C}}(x)}$. Observe that

$$\begin{aligned}(e_{\widetilde{b}} \circ \delta)(j(x)) &= e_{\widetilde{b}}\Big(\widetilde{\mathcal{U}_{\mathcal{C}}(x)}\Big) = \nu_{\widetilde{\mathcal{U}_{\mathcal{C}}(x)}}(\widetilde{b}) = \bigvee_{a \in \mathcal{U}_{\mathcal{C}}(x)} [\widetilde{a}, \widetilde{b}] \\ &= \bigvee_{a \in \mathcal{U}_{\mathcal{C}}(x)} [a, b] = \nu_{\mathcal{U}_{\mathcal{C}}(x)}(b) = (e_b \circ \sigma)(x) = \widetilde{e_b \circ \sigma}(j(x)),\end{aligned}$$

where Lemma 3.8 (ii) is used from the first to second line.

Further,

$$(e_{\widetilde{b}} \circ \delta)(\langle \mathfrak{G} \rangle) = e_{\widetilde{b}}(\widetilde{\mathfrak{G}_{\min}}) = \bigvee_{c \in \mathfrak{G}_{\min}} [\widetilde{c}, \widetilde{b}]$$
$$= \bigvee_{c \in \mathfrak{G}_{\min}} [c, b] = \nu_{\mathfrak{G}_{\min}}(b) \geq \nu_{\mathfrak{G}_{\min}}(e_b \circ \sigma) = \widetilde{e_b \circ \sigma}(\langle \mathfrak{G} \rangle),$$

where again Lemma 3.8 (ii) was used.

Hence $e_{\widetilde{b}} \circ \delta \geq \widetilde{e_b \circ \sigma}$ and since $\widetilde{e_b \circ \sigma} \in \widetilde{\mathcal{U}_{\mathcal{C}}(x)}$, $e_{\widetilde{b}} \circ \delta \in \widetilde{\mathcal{U}_{\mathcal{C}}(x)}$. Therefore $\widetilde{b} \in \kappa\delta\widetilde{\mathcal{U}_{\mathcal{C}}(x)}$ and $\kappa\delta\widetilde{\mathcal{U}_{\mathcal{C}}(x)} \geq \widetilde{\mathcal{U}_{\mathcal{C}}(x)}$; hence $\kappa\delta\widetilde{\mathcal{U}_{\mathcal{C}}(x)} \xrightarrow{q_{\widetilde{\mathcal{C}}}} j(x)$, $x \in X$.

It remains to verify that $\kappa\delta\widetilde{\mathfrak{G}_{\min}} \geq \widetilde{\mathfrak{G}_{\min}}$ whenever $\mathfrak{G}_{\min} \in \mathcal{N}$. As above $\kappa\sigma\mathfrak{G}_{\min} = \{b \in L^X : e_b \circ \sigma \in \mathfrak{G}_{\min}\}$. Let $b \in \kappa\sigma\mathfrak{G}_{\min}$; it follows as shown above that $e_{\widetilde{b}} \circ \delta \geq \widetilde{e_b \circ \sigma} \in \widetilde{\mathfrak{G}_{\min}}$ and thus $e_{\widetilde{b}} \circ \delta \in \widetilde{\mathfrak{G}_{\min}}$. Thence $\widetilde{b} \in \kappa\delta\widetilde{\mathfrak{G}_{\min}}$, $\kappa\delta\widetilde{\mathfrak{G}_{\min}} \geq \widetilde{\mathfrak{G}_{\min}}$, and thus $\kappa\delta\widetilde{\mathfrak{G}_{\min}} \xrightarrow{q_{\widetilde{\mathcal{C}}}} \langle \mathfrak{G} \rangle$. Therefore $(X^*, \widetilde{\mathcal{C}})$ is topological in $\top$-**Chy**. □

Let $\top$-**TopChy** denote the full subcategory of $\top$-**Chy** consisting of all the objects that are topological in the sense of Definition 3.7.

**Lemma 3.12.** *The subcategory* $\top$-**TopChy** *of* $\top$-**Chy** *possesses initial structures. In particular,* $\top$-**TopChy** *is a concretely bireflective subcategory of* $\top$-**Chy**. [2]

*Proof.* Let $X$ be any set and consider any indexed family $f_i : X \longrightarrow (Y_i, \mathcal{D}_i)$, $i \in I$, where $(Y_i, \mathcal{D}_i) \in |\top\text{-}\mathbf{TopChy}|$. Then $\mathcal{C} = \{\mathfrak{F} \in \mathfrak{F}_L^{\top}(X) : f_i^{\Rightarrow}\mathfrak{F} \in \mathcal{D}_i \text{ for each } i \in I\}$ is the initial structure in $\top$-**Chy**. It is shown that $(X, \mathcal{C}) \in |\top\text{-}\mathbf{TopChy}|$.

Suppose that $\psi : J \longrightarrow X^*$, $\sigma : J \longrightarrow \mathcal{C}$ such that $\sigma(y) \xrightarrow{q_{\mathcal{C}}} \psi(y)$ whenever $\psi(y) \in X$ and $\sigma(y) \in \langle \mathfrak{G} \rangle$ provided $\psi(y) = \langle \mathfrak{G} \rangle$. For each $i \in I$ define $\theta_i : X^* \longrightarrow Y_i^*$ by $\theta_i(t) = f_i(t)$

[2]See Appendix for definitions concrete and bireflective.

whenever $t \in X$, and $\theta_i\Big(\langle\mathfrak{G}\rangle_{\mathcal{C}}\Big) = \begin{cases} z & \text{if } f_i^{\Rightarrow}\mathfrak{G} \xrightarrow{q_{\mathcal{D}_i}} z, \\ \langle f_i^{\Rightarrow}\mathfrak{G}\rangle_{\mathcal{D}_i}, & \text{if } f_i^{\Rightarrow}\mathfrak{G} \in \mathcal{N}_{\mathcal{D}_i} \end{cases}$, where $z$ is a selected limit of $f_i^{\Rightarrow}\mathfrak{G}$. Let $\psi_i^* = \theta_i \circ \psi : J \longrightarrow Y_i^*$ and define $\sigma_i^*$ by $\sigma_i^*(y) = f_i^{\Rightarrow}(\sigma(y))$. Since $f_i : (X, \mathcal{C}) \longrightarrow (Y_i, \mathcal{D}_i)$ is Cauchy-continuous, $\sigma_i^* : J \longrightarrow \mathcal{D}_i$.

Fix $y \in J$. If $\psi(y) = t \in X$, then $\sigma(y) \xrightarrow{q_{\mathcal{C}}} t$ and $\sigma_i^*(y) = f_i^{\Rightarrow}\Big(\sigma(y)\Big) \xrightarrow{q_{\mathcal{D}_i}} f_i(t) = \psi_i^*(y)$. Next, suppose that $\psi(y) = \langle\mathfrak{G}\rangle_{\mathcal{C}}$, $\mathfrak{G} \in \mathcal{N}_{\mathcal{C}}$. If $\theta_i\Big(\langle\mathfrak{G}\rangle_{\mathcal{C}}\Big) = z$, then $\psi_i^*(y) = z \in Y_i$. Note that $\sigma(y) \in \langle\mathfrak{G}\rangle_{\mathcal{C}}$ and thus $\sigma_i^*(y) \xrightarrow{q_{\mathcal{D}_i}} z = \psi_i^*(y)$. Further, assume that $\theta_i\Big(\langle\mathfrak{G}\rangle_{\mathcal{C}}\Big) = \langle f_i^{\Rightarrow}\mathfrak{G}\rangle_{\mathcal{D}_i}$, where $f_i^{\Rightarrow}\mathfrak{G} \in \mathcal{N}_{\mathcal{D}_i}$, then $\psi_i^*(y) = \langle f_i^{\Rightarrow}\mathfrak{G}\rangle_{\mathcal{D}_i}$ and $\sigma_i^*(y) = f_i^{\Rightarrow}(\sigma(y)) \in \langle f_i^{\Rightarrow}\mathfrak{G}\rangle_{\mathcal{D}_i} = \psi_i^*(y)$.

Let $\mathfrak{H} \in \mathfrak{F}_L^{\top}(J)$; first assume that $\psi^{\Rightarrow}\mathfrak{H} \geq j_X^{\Rightarrow}\mathfrak{F}$ for some $\mathfrak{F} \xrightarrow{q_{\mathcal{C}}} x$. It must be shown that $\kappa\sigma\mathfrak{H} \in \mathcal{C}$. Since $(Y_i, \mathcal{D}_i) \in |\top\text{-}\textsf{\textbf{TopChy}}|$ and $\psi_i^{*\Rightarrow}\mathfrak{H} = \theta_i^{\Rightarrow}(\psi^{\Rightarrow}\mathfrak{H}) \geq \theta_i^{\Rightarrow}\Big(j_X^{\Rightarrow}\mathfrak{F}\Big) = j_{Y_i}^{\Rightarrow} f_i^{\Rightarrow}\mathfrak{F} \xrightarrow{q_{\mathcal{D}_i}} j_{Y_i}(f_i(x))$, it follows that $\kappa\sigma_i^*\mathfrak{H} \in \mathcal{D}_i$. Recall that $\kappa\sigma_i^*\mathfrak{H} = \{b^* \in L^{Y_i} : e_{b^*} \circ \sigma^* \in \mathfrak{H}\}$. Fix $b^* \in \kappa\sigma_i^*\mathfrak{H}$; it is shown that $b^* \in f_i^{\Rightarrow}(\kappa\sigma\mathfrak{H})$. Define $b = f_i^{\leftarrow}(b^*)$. Observe that if $y \in J$, $\Big(e_{b^*} \circ \sigma_i^*\Big)(y) = e_{b^*}\Big(\sigma_i^*(y)\Big) = \nu_{\sigma_i^*(y)}(b^*) = \nu_{f_i^{\Rightarrow}(\sigma(y))}(b^*) = \bigvee_{c \in \sigma(y)} [f_i^{\rightarrow}(c), b^*] \leq \bigvee_{c \in \sigma(y)} [f_i^{\leftarrow}(f_i^{\rightarrow}(c)), f_i^{\leftarrow}(b^*)] \leq \bigvee_{c \in \sigma(y)} [c, b] = \nu_{\sigma(y)}(b) = (e_b \circ \sigma)(y)$, and thus $e_{b^*} \circ \sigma_i^* \leq e_b \circ \sigma$. Since $b^* \in \kappa\sigma_i^*\mathfrak{H}$, $e_{b^*} \circ \sigma_i^* \in \mathfrak{H}$ and thus $e_b \circ \sigma \in \mathfrak{H}$. Then $b = f_i^{\leftarrow}(b^*) \in \kappa\sigma\mathfrak{H}$ and hence $b^* \in f_i^{\Rightarrow}(\kappa\sigma\mathfrak{H})$ implies that $\kappa\sigma_i^*\mathfrak{H} \subseteq f_i^{\Rightarrow}(\kappa\sigma\mathfrak{H})$. In this case, $f_i^{\Rightarrow}(\kappa\sigma\mathfrak{H}) \in \mathcal{D}_i$.

Next, suppose that $\psi^{\Rightarrow}\mathfrak{H} \geq j_X^{\Rightarrow}\mathfrak{H} \cap [\langle\mathfrak{G}\rangle_{\mathcal{C}}]$ for some $\mathfrak{G} \in \mathcal{N}_{\mathcal{C}}$. Then $\psi_i^{*\Rightarrow}\mathfrak{H} \geq \theta_i(j_X^{\Rightarrow}\mathfrak{G}) \cap \theta_i^{\Rightarrow}\Big([\langle\mathfrak{G}\rangle_{\mathcal{C}}]\Big) = j_{Y_i}^{\Rightarrow}(f_i^{\Rightarrow}\mathfrak{G}) \cap \theta_i^{\Rightarrow}\Big([\langle\mathfrak{G}\rangle_{\mathcal{C}}]\Big)$. First, assume that $f_i^{\Rightarrow}\mathfrak{G} \xrightarrow{q_{\mathcal{D}_i}} z$; then $\psi_i^{*\Rightarrow}\mathfrak{H} \geq j_{Y_i}^{\Rightarrow}(f_i^{\Rightarrow}\mathfrak{G}) \cap [j_{Y_i}(z)]$. Then as in the case above, $f_i^{\Rightarrow}(\kappa\sigma\mathfrak{H}) \in \mathcal{D}_i$. Finally, suppose that $f_i^{\Rightarrow}\mathfrak{G} \in \mathcal{N}_{\mathcal{D}_i}$; then $\psi_i^{*\Rightarrow}\mathfrak{H} \geq j_{Y_i}^{\Rightarrow}(f_i^{\Rightarrow}\mathfrak{G}) \cap [\langle f_i^{\Rightarrow}\mathfrak{G}\rangle_{\mathcal{D}_i}]$. It follows that $\kappa\sigma_i^*\mathfrak{H} \in \mathcal{D}_i$ and thus $f_i^{\Rightarrow}(\kappa\sigma\mathfrak{H}) \in \mathcal{D}_i$. Since $f_i^{\Rightarrow}(\kappa\sigma\mathfrak{H}) \in \mathcal{D}_i$ for each $i \in I$, it follows that $\kappa\sigma\mathfrak{H} \in \mathcal{C}$ and thus $(X, \mathcal{C}) \in |\top\text{-}\textsf{\textbf{TopChy}}|$. Hence $\top$-**TopChy** possesses initial structures, and it follows from Corollary 2.2.6 [24] that $\top$-**TopChy** is concretely bireflective in $\top$-**Chy**. □

Lemma 3.12 implies, in particular, that if $(X, \mathcal{C}) \in |\top\text{-}\mathbf{TopChy}|$, then each subspace $(A, \mathcal{C}_A)$ of $(X, \mathcal{C})$ formed in $\top$-**Chy** also belongs to $\top$-**TopChy**. Combining Lemma 3.12 and Theorem 3.6 gives the next result.

**Corollary 3.3.** *An object $(X, \mathcal{C}) \in |\top\text{-}\mathbf{Chy}|$ has a topological completion in standard form in $\top$-**Chy** iff it is topological.*

Next we explore the question of whether the completion which takes $(X, \mathcal{C})$ to $\big((X^*, \mathcal{C}^*), j\big)$ preserves the property of being topological.

**Lemma 3.13.** *Let $(X, \mathcal{C}) \in |\top\text{-}\mathbf{Chy}|$ be full and not be complete and let $\big((X^*, \mathcal{C}^*), j\big)$ denote the completion given in Theorem 3.3. Define $\sigma : X \longrightarrow \mathcal{C}$ by $\sigma(x) = \mathcal{U}_{\mathcal{C}}(x)$ and $\delta : X^* \longrightarrow \mathcal{C}^*$ by $\delta(j(x)) = j^{\Rightarrow}\mathcal{U}_{\mathcal{C}}(x)$ and $\delta(\langle\mathfrak{G}\rangle) = j^{\Rightarrow}\mathfrak{G}_{min} \cap [\langle\mathfrak{G}\rangle]$, whenever $\mathfrak{G} \in \mathcal{N}_{\mathcal{C}}$. If $a \in L^X$ and $b \in L^{X^*}$, then*

*(i)* $j^{\rightarrow}(e_a \circ \sigma) = e_{j^{\rightarrow}a} \circ \delta$

*(ii)* $j^{\leftarrow}(e_b \circ \delta) = e_{j^{\leftarrow}b} \circ \sigma$.

*Proof.* (i) Fix $x \in X$. Then $(e_{j^{\rightarrow}a} \circ \delta)(j(x)) = e_{j^{\rightarrow}a}(j^{\Rightarrow}\mathcal{U}_{\mathcal{C}}(x)) = \nu_{j^{\Rightarrow}\mathcal{U}_{\mathcal{C}}(x)}(j^{\rightarrow}a) = \nu_{\mathcal{U}_{\mathcal{C}}(x)}(a) = (e_a \circ \sigma)(x) = j^{\rightarrow}(e_a \circ \sigma)(j(x))$. Moreover $j^{\rightarrow}(e_a \circ \sigma)(\langle\mathfrak{G}\rangle) = \bot$ and $(e_{j^{\rightarrow}a} \circ \delta)(\langle\mathfrak{G}\rangle) = e_{j^{\rightarrow}a}(j^{\Rightarrow}\mathfrak{G}_{\min} \cap [\langle\mathfrak{G}\rangle]) = \nu_{j^{\Rightarrow}\mathfrak{G}_{\min} \cap [\langle\mathfrak{G}\rangle]}(j^{\rightarrow}a) = \nu_{j^{\Rightarrow}\mathfrak{G}_{\min}}(j^{\rightarrow}a) \wedge \nu_{[\langle\mathfrak{G}\rangle]}(j^{\rightarrow}a)$. Observe that $\nu_{[\langle\mathfrak{G}\rangle]}(j^{\rightarrow}a) = \bigvee_{c \in [\langle\mathfrak{G}\rangle]} [c, j^{\rightarrow}a] = [\mathbf{1}_{\{\langle\mathfrak{G}\rangle\}}, j^{\rightarrow}a] = \top \rightarrow j^{\rightarrow}a(\langle\mathfrak{G}\rangle) = \top \rightarrow \bot = \bot$. Then $(e_{j^{\rightarrow}a} \circ \delta)(\langle\mathfrak{G}\rangle) = \bot$. and hence (i) is valid.

(ii) Let $x \in X$ and denote $a = j^{\leftarrow}b$. Then $j^{\leftarrow}(e_b \circ \delta)(x) = (e_b \circ \delta)(j(x)) = e_b(j^{\Rightarrow}\mathcal{U}_{\mathcal{C}}(x)) = \nu_{j^{\Rightarrow}\mathcal{U}_{\mathcal{C}}(x)}(b) = \bigvee_{c \in \mathcal{U}_{\mathcal{C}}(x)} [j^{\rightarrow}c, b] = \bigvee_{c \in \mathcal{U}_{\mathcal{C}}(x)} [c, a] = \nu_{\mathcal{U}_{\mathcal{C}}(x)}(a) = (e_a \circ \sigma)(x)$. Therefore (ii) is satisfied. □

**Theorem 3.7.** *Assume that* $(X, \mathcal{C})$ *is topological in but not complete in* $\top$-**Chy** *and let* $\big((X^*, \mathcal{C}^*), j\big)$ *denote the completion given in Theorem 3.3. Then* $(X^*, \mathcal{C}^*)$ *is also topological in* $\top$-**Chy**.

*Proof.* Again, $\mathcal{U}_{\mathcal{C}}(x)$ denotes the $\top$-neighborhood filter at $x$. Let $\sigma$ and $\delta$ be as defined in Lemma 3.13. First, it is shown that $\kappa\delta j^{\Rightarrow}\mathcal{U}_{\mathcal{C}}(x) = j^{\Rightarrow}\mathcal{U}_{\mathcal{C}}(x)$, $x \in X$. Since $(X, \mathcal{C})$ is topological, $\mathcal{U}_{\mathcal{C}}(x) = \kappa\sigma\mathcal{U}_{\mathcal{C}}(x) = \{a \in L^X : e_a \circ \sigma \in \mathcal{U}_{\mathcal{C}}(x)\}$. Assume that $a \in \mathcal{U}_{\mathcal{C}}(x)$, then $e_a \circ \sigma \in \mathcal{U}_{\mathcal{C}}(x)$ and thus $j^{\rightarrow}(e_a \circ \sigma) \in j^{\Rightarrow}\mathcal{U}_{\mathcal{C}}(x)$. Since $(X, \mathcal{C})$ is topological, it is also full and hence by Lemma 3.13 (i) it follows that $e_{j^{\rightarrow}a} \circ \delta \in j^{\Rightarrow}\mathcal{U}_{\mathcal{C}}(x)$ and therefore $j^{\rightarrow}a \in \kappa\delta j^{\Rightarrow}\mathcal{U}_{\mathcal{C}}(x)$. Then $j^{\Rightarrow}\mathcal{U}_{\mathcal{C}}(x) \subseteq \kappa\delta j^{\Rightarrow}\mathcal{U}_{\mathcal{C}}(x)$. Conversely, if $b \in \kappa\delta j^{\Rightarrow}\mathcal{U}_{\mathcal{C}}(x)$, then $e_b \circ \delta \in j^{\Rightarrow}\mathcal{U}_{\mathcal{C}}(x)$. According to Lemma 3.13 (ii), $e_{j^{\leftarrow}b} \circ \sigma = j^{\leftarrow}(e_b \circ \delta) = j^{\Leftarrow}j^{\Rightarrow}\mathcal{U}_{\mathcal{C}}(x) = \mathcal{U}_{\mathcal{C}}(x)$ and thus $j^{\leftarrow}b \in \kappa\sigma\mathcal{U}_{\mathcal{C}}(x)$ implies that $\kappa\delta f^{\Rightarrow}\mathcal{U}_{\mathcal{C}}(x) = j^{\Rightarrow}\mathcal{U}_{\mathcal{C}}(x)$ for each $x \in X$.

It remains to show that for $\mathfrak{G} \in \mathcal{N}$, $\kappa\delta(j^{\Rightarrow}\mathfrak{G}_{\min} \cap [\langle\mathfrak{G}\rangle]) = j^{\Rightarrow}\mathfrak{G}_{\min} \cap [\langle\mathfrak{G}\rangle]$. Suppose that $b \in j^{\Rightarrow}\mathfrak{G}_{\min} \cap [\langle\mathfrak{G}\rangle]$. Then $j^{\leftarrow}b \in j^{\Leftarrow}j^{\Rightarrow}\mathfrak{G}_{\min} = \mathfrak{G}_{\min} = \kappa\sigma\mathfrak{G}_{\min}$. It follows that $e_{j^{\leftarrow}b} \circ \sigma \in \mathfrak{G}_{\min}$ and thus by Lemma 3.13 (ii), $j^{\leftarrow}(e_b \circ \delta) \in \mathfrak{G}_{\min}$. Then $e_b \circ \delta \geq j^{\rightarrow}j^{\leftarrow}(e_b \circ \delta) \in j^{\Rightarrow}\mathfrak{G}_{\min}$ and hence $e_b \circ \delta \in j^{\Rightarrow}\mathfrak{G}_{\min}$. Moreover, $e_b \circ \delta \in j^{\Rightarrow}\mathfrak{G}_{\min}$ and $b \in [\langle\mathfrak{G}\rangle]$ implies that $(e_b \circ \delta)(\langle\mathfrak{G}\rangle) = e_b(j^{\Rightarrow}\mathfrak{G}_{\min} \cap [\langle\mathfrak{G}\rangle]) = \nu_{j^{\Rightarrow}\mathfrak{G}_{\min}}(b) \wedge \nu_{[\langle\mathfrak{G}\rangle]}(b) = \top$ and thus $e_b \circ \delta \in [\langle\mathfrak{G}\rangle]$. Then $e_b \circ \delta \in j^{\Rightarrow}\mathfrak{G}_{\min} \cap [\langle\mathfrak{G}\rangle]$ and hence $b \in \kappa\delta(j^{\Rightarrow}\mathfrak{G}_{\min} \cap [\langle\mathfrak{G}\rangle])$. Conversely, assume that $b \in \kappa\delta(j^{\Rightarrow}\mathfrak{G}_{\min} \cap [\langle\mathfrak{G}\rangle])$; then $e_b \circ \delta \in j^{\Rightarrow}\mathfrak{G}_{\min} \cap [\langle\mathfrak{G}\rangle]$. Denote $a = j^{\leftarrow}b$ and by Lemma 3.13 (ii), $e_a \circ \sigma = j^{\leftarrow}(e_b \circ \delta) \in j^{\Leftarrow}j^{\Rightarrow}\mathfrak{G}_{\min} = \mathfrak{G}_{\min} = \kappa\sigma\mathfrak{G}_{\min}$. It follows that $a \in \mathfrak{G}_{\min}$ and thus $b \geq j^{\rightarrow}j^{\leftarrow}b \in j^{\Rightarrow}\mathfrak{G}_{\min}$ implies that $b \in j^{\Rightarrow}\mathfrak{G}_{\min}$. Further, $e_b \circ \delta \in [\langle\mathfrak{G}\rangle]$ gives $\top = (e_b \circ \delta)(\langle\mathfrak{G}\rangle) = e_b(j^{\Rightarrow}\mathfrak{G}_{\min} \cap [\langle\mathfrak{G}\rangle]) = \nu_{j^{\Rightarrow}\mathfrak{G}_{\min}}(b) \wedge \nu_{[\langle\mathfrak{G}\rangle]}(b) \leq \nu_{[\langle\mathfrak{G}\rangle]}(b)$. It follows that $b \in [\langle\mathfrak{G}\rangle]$ and hence $b \in j^{\Rightarrow}\mathfrak{G}_{\min} \cap [\langle\mathfrak{G}\rangle]$. Then $\kappa\delta(j^{\Rightarrow}\mathfrak{G}_{\min} \cap [\langle\mathfrak{G}\rangle]) = j^{\Rightarrow}\mathfrak{G}_{\min} \cap [\langle\mathfrak{G}\rangle]$ and thus $(X^*, \mathcal{C}^*)$ is topological in $\top$-**Chy**. □

# CHAPTER 4: $\top$-UNIFORM LIMIT SPACES

In terms of set filters, Cook and Fischer [3] introduced the notion of a uniform convergence space in order to study completions and various convergences in function spaces. Several authors have extended these concepts to the lattice context. A $\top$-uniform limit space is introduced and investigated here; our main thrust is toward completions, but first we establish some categorical properties.

## Definitions and Categorical Properties of $\top$-**ULS**

If $a_i \in L^{X_i}$, $i = 1, 2$, then the product $a_1 \times a_2 \in L^{X_1 \times X_2}$ is defined by $(a_1 \times a_2)(x, y) = a_1(x) \wedge a_2(y)$, $(x, y) \in X_1 \times X_2$. If $\mathfrak{F}_i \in \mathfrak{F}_L^{\top}(X_i)$, $i = 1, 2$, then the product $\top$-filter on $X_1 \times X_2$, denoted by $\mathfrak{F}_1 \times \mathfrak{F}_2$, is defined to be the $\top$-filter whose base is $\{a_1 \times a_2 : a_1 \in \mathfrak{F}_1, a_2 \in \mathfrak{F}_2\}$. It is shown in [29] that if $\mathcal{B}_i$ is any $\top$-filter base for $\mathfrak{F}_i$, $i = 1, 2$, then $\mathcal{B}_1 \times \mathcal{B}_2$ is a $\top$-filter base for $\mathfrak{F}_1 \times \mathfrak{F}_2$.

Let $X^2 = X \times X$ and let $a, b \in L^{X^2}$. The composition of $a$ and $b$ is defined by $(a \circ b)(x, y) = \bigvee_{z \in X} \Big(a(x, z) \wedge b(z, y)\Big)$, where $(x, y) \in X^2$. If $\Phi_1, \Phi_2 \in \mathfrak{F}_L^{\top}(X^2)$, let $\mathcal{B} = \{a_1 \circ a_2 : a_i \in \Phi_i,\ i = 1, 2\}$ and observe that for $a_1 \circ a_2, b_1 \circ b_2 \in \mathcal{B}$, $(a_1 \circ a_2) \wedge (b_1 \circ b_2) \geq (a_1 \wedge b_1) \circ (a_2 \wedge b_2) \in \mathcal{B}$. It follows that $\mathcal{B}$ is a base for a $\top$-filter on $X^2$ iff for each $a_1 \circ a_2 \in \mathcal{B}$, $\bigvee_{(x,y) \in X^2} (a_1 \circ a_2)(x, y) = \top$. Whenever $\mathcal{B}$ is a base, we say that $\Phi_1 \circ \Phi_2$ exists and define $\Phi_1 \circ \Phi_2$ to be the $\top$-filter it generated by $\mathcal{B}$.

**Lemma 4.1.** *Let $a_i, b_i \in L^{X^2}$ and let $\mathfrak{D}_i$ be any $\top$-filter base for $\Phi_i \in \mathfrak{F}_L^{\top}(X^2)$, $i = 1, 2$. Assume that $\Phi_1 \circ \Phi_2$ exists. Then*

*(i)* $[a_1, b_1] \wedge [a_2, b_2] \leq [a_1 \circ a_2, b_1 \circ b_2]$

*(ii)* $\mathfrak{D} = \{b_1 \circ b_2 : b_i \in \mathfrak{D}_i,\ i = 1, 2\}$ *is a* $\top$*-filter base for* $\Phi_1 \circ \Phi_2$.

*Proof.*

(i) Employing Lemma 1.1 (iii) and (v), $[a_1, b_1] \wedge [a_2, b_2] = \bigwedge_{x,y\in X} \big(a_1(x, y) \to b_1(x, y)\big) \wedge \bigwedge_{s,t\in X} \big(a_2(s, t) \to b_2(s, t)\big) = \bigwedge_{\substack{x,y\\ s,t}} \Big[\big(a_1(x, y) \to b_1(x, y)\big) \wedge \big(a_2(s, t) \to b_2(s, t)\big)\Big] \leq \bigwedge_{\substack{x,y\\ s,t}} \big(a_1(x, y) \wedge a_2(s, t)\big) \to \big(b_1(x, y) \wedge b_2(s, t)\big) \leq \bigwedge_{x,t,z\in X} \big(a_1(x, z) \wedge a_2(z, t) \to b_1(x, z) \wedge b_2(z, t)\big] \leq \bigwedge_{x,t\in X} \bigwedge_{z\in X} \big(a_1(x, z)\wedge a_2(z, t) \to (b_1\circ b_2)(x, t)\big) = \bigwedge_{x,t\in X} \Bigg[\bigg(\bigvee_{z\in X} \big(a_1(x, z) \wedge a_2(z, t)\big)\bigg) \to (b_1 \circ b_2)(x, t)\bigg)\Bigg] = \bigwedge_{x,t\in X} \big((a_1 \circ a_2)(x, t) \to (b_1 \circ b_2)(x, t)\big) = [a_1 \circ a_2, b_1 \circ b_2]$.

(ii) First, $\mathfrak{D}$ is a $\top$-filter base on $X^2$. Indeed, if $b_i \in \mathfrak{D}_i$, then $b_1 \circ b_2 \in \Phi_1 \circ \Phi_2$ and thus $\bigvee_{x,y\in X} (b_1\circ b_2)(x, y) = \top$. Next, if $c_1\circ c_2, d_1\circ d_2 \in \mathfrak{D}$, it is shown that $\bigvee_{b_i\in\mathfrak{D}_i} [b_1\circ b_2, (c_1\circ c_2)\wedge (d_1\circ d_2)] = \top$. Since $\mathfrak{D}_i$ are $\top$-filter bases, $\top = \bigvee_{b_i\in\mathfrak{D}_i} [b_i, c_i\wedge d_i]$, $i = 1, 2$. According to (i) above, $\top = \bigvee_{b_i\in\mathfrak{D}_i} [b_1, c_1\wedge d_1]\wedge [b_2, c_2\wedge d_2] \leq \bigvee_{b_i\in\mathfrak{D}_i} [b_1\circ b_2, (c_1\wedge d_1)\circ (c_2\wedge d_2)]$. Observe that $(c_1 \wedge d_1) \circ (c_2 \wedge d_2) \leq (c_1 \circ c_2) \wedge (d_1 \circ d_2)$ and hence $\top = \bigvee_{b_i\in\mathfrak{D}_i} [b_1 \circ b_2, (c_1 \circ c_2) \wedge (d_1 \circ d_2)]$. Therefore $\mathfrak{D}$ is a $\top$-filter base on $X^2$.

In order to show that $\mathfrak{D}$ generates $\Phi_1 \circ \Phi_2$, it suffices to show that for any $a_i \in \Phi_i$, $\bigvee_{b_i\in\mathfrak{D}_i} [b_1 \circ b_2, a_1 \circ a_2] = \top$, $i = 1, 2$. Since $\mathfrak{D}_i$ is a $\top$-filter base for $\Phi_i$, $\top = \bigvee_{b_i\in\mathfrak{D}_i} [b_i, a_i]$, $i = 1, 2$. Applying (i) above, $\top = \bigvee_{b_i\in\mathfrak{D}_i} \big([b_1, a_1] \wedge [b_2, a_2]\big) \leq \bigvee_{b_i\in\mathfrak{D}_i} [b_1 \circ b_2, a_1 \circ a_2]$, and hence $\bigvee_{b_i\in\mathfrak{D}_i} [b_1 \circ b_2, a_1 \circ a_2] = \top$, $i = 1, 2$. Therefore $\mathfrak{D}$ is a $\top$-filter base for $\Phi_1 \circ \Phi_2$.

□

Let $a \in L^{X^2}$; then $a^{-1} \in L^{X^2}$ is defined by $a^{-1}(x, y) = a(y, x)$ for $(x, y) \in X^2$. Further, if $\Phi \in \mathfrak{F}^{\top}_L(X^2)$, then $\Phi^{-1}$ denotes the $\top$-filter $\Phi^{-1} = \{a^{-1} : a \in \Phi\}$.

The axioms listed below are similar to those used by Jäger and Burton [13] in the definition of a stratified $L$-uniform convergence space.

**Definition 4.1.** Given a pair $(X, \Lambda)$, where $\Lambda \subseteq \mathfrak{F}_L^\top(X^2)$, is called a $\top$**-uniform limit space** provided it satisfies:

(UL1) $[(x, x)] \in \Lambda$, for each $x \in X$,

(UL2) $\Psi \geq \Phi \in \Lambda$ implies $\Psi \in \Lambda$,

(UL3) $\Phi \in \Lambda$ implies $\Phi^{-1} \in \Lambda$,

(UL4) $\Phi \circ \Psi \in \Lambda$ whenever $\Phi, \Psi \in \Lambda$ and $\Phi \circ \Psi$ exists, and

(UL5) $\Phi, \Psi \in \Lambda$ implies $\Phi \cap \Psi \in \Lambda$.

Moreover, $\Lambda$ above is said to be a $\top$**-uniform limit structure** on $X$.

A map $k : (X, \Lambda) \longrightarrow (Y, \Sigma)$ between two $\top$-uniform limit spaces is called **uniformly continuous** if $(k \times k)^{\Rightarrow}\Phi \in \Sigma$ whenever $\Phi \in \Lambda$. Let $\top$-**ULS** denote the category of all $\top$-uniform limit spaces and uniformly continuous maps between them.

**Lemma 4.2.** *Let* $\mathfrak{F}, \mathfrak{G}, \mathfrak{H}, \mathfrak{K} \in \mathfrak{F}_L^\top(X)$.

*(i)* $(\mathfrak{F} \cap \mathfrak{G}) \times (\mathfrak{F} \cap \mathfrak{G}) = (\mathfrak{F} \times \mathfrak{F}) \cap (\mathfrak{G} \times \mathfrak{F}) \cap (\mathfrak{F} \times \mathfrak{G}) \cap (\mathfrak{G} \times \mathfrak{G})$

*(ii) If* $\mathfrak{G} \vee \mathfrak{H}$ *exists, then* $\mathfrak{F} \times \mathfrak{K} = (\mathfrak{F} \times \mathfrak{G}) \circ (\mathfrak{H} \times \mathfrak{K})$.

*Proof.* (i) A base member for $(\mathfrak{F} \times \mathfrak{F}) \cap (\mathfrak{G} \times \mathfrak{F}) \cap (\mathfrak{F} \times \mathfrak{G}) \cap (\mathfrak{G} \times \mathfrak{G})$ is $a = (f \times f) \vee (g \times f) \vee (f \times g) \vee (g \times g)$ where $f \in \mathfrak{F}$, $g \in \mathfrak{G}$. It is shown that $a \geq (f \vee g) \times (f \vee g)$. According to Jäger and Burton [16] page 14, $a_1 \times c_2 \leq c$ and $b_1 \times c_2 \leq c$ is equivalent to

$(a_1 \vee b_1) \times c_2 \leq c$. Note that $f \times (f \vee g) \leq a$. Indeed, $(f \times (f \vee g))(x, y) = f(x) \wedge (f(y) \vee g(y)) = (f(x) \wedge f(y)) \vee (f(x) \wedge g(y)) \leq a(x, y)$. Similarly, it is clear that $g \times (f \vee g) \leq a$. Therefore it follows that $(f \vee g) \times (f \vee g) \leq a$. Hence $a \in (\mathfrak{F} \cap \mathfrak{G}) \times (\mathfrak{F} \cap \mathfrak{G})$ and it follows that $(\mathfrak{F} \times \mathfrak{F}) \cap (\mathfrak{G} \times \mathfrak{F}) \cap (\mathfrak{F} \times \mathfrak{G}) \cap (\mathfrak{G} \times \mathfrak{G}) \subseteq (\mathfrak{F} \cap \mathfrak{G}) \times (\mathfrak{F} \cap \mathfrak{G})$. The other direction is clear.

(ii) Let $f \in \mathfrak{F}$, $g \in \mathfrak{G}$, $h \in \mathfrak{H}$ and $k \in \mathfrak{K}$. We have $(f \times g) \circ (h \times k)(x, y) = \bigvee_{z \in X} \Big(f(x) \wedge g(z) \wedge h(z) \wedge k(y)\Big) = f(x) \wedge k(y) \wedge \bigvee_{z \in X} \Big(g(z) \wedge h(z)\Big)$. But $\bigvee_{z \in X} g(z) \wedge h(z) = \top$ because $\mathfrak{G} \vee \mathfrak{H}$ exists. Hence $\Big((f \times g) \circ (h \times k)\Big)(x, y) = f(x) \wedge k(y) = (f \times k)(x, y)$, and the result follows. □

**Lemma 4.3.** *Let* $(X, \Lambda) \in |\top\text{-}\mathbf{ULS}|$ *and* $\Phi \in \Lambda$, *then* $\Phi^{-1} \circ \Phi$ *and* $\Phi \circ (\Phi^{-1} \circ \Phi)$ *exist.*

*Proof.* First to show that $\Phi^{-1} \circ \Phi$ exists we let $f, g \in \Phi$. Then $\bigvee_{x,y \in X} (f^{-1} \circ g)(x, y) = \bigvee_{x,y,z \in X} f^{-1}(x, z) \wedge g(z, y) = \bigvee_{x,y,z]inX} f(z, x) \wedge g(z, y) \geq \bigvee_{x,z} f(z, x) \wedge g(z, x) = \top$ since $f \wedge g \in \Phi$. Thus $\Phi^{-1} \circ \Phi$ exists.

Next, if $f, g, h \in \Phi$, then $\bigvee_{x,y} \Big(f \circ (g^{-1} \circ h)\Big)(x, y) = \bigvee_{x,y,z,w \in X} f(x, z) \wedge g(w, z) \wedge h(w, y) \geq \bigvee_{y,z,w \in X} f(w, z) \wedge g(w, z) \wedge h(z, y) = \bigvee_{y,z,w \in X} (f \wedge g)^{-1}(w, z) \wedge h(z, y) = \bigvee_{w,y \in X} \Big((f \wedge g) \circ h\Big)(w, y) = \top$ since $(f \wedge g)^{-1} \circ h \in \Phi^{-1} \circ \Phi$.

□

**Lemma 4.4.** *Suppose that* $\Phi, \Psi \in \mathfrak{F}_L^{\top}(X^2)$, $\Phi \circ \Psi$ *exists and* $k : X \longrightarrow Y$ *is any map, then* $(k \times k)^{\Rightarrow}\Phi \circ (k \times k)^{\Rightarrow}\Psi$ *exists and* $(k \times k)^{\Rightarrow}(\Phi \circ \Psi) \geq (k \times k)^{\Rightarrow}\Phi \circ (k \times k)^{\Rightarrow}\Psi$.

*Proof.* Assume that $\Phi \circ \Psi$ exists, $a \in \Phi$, $b \in \Psi$, and $(y_1, y_2) \in Y^2$. Then

$$\Big((k \times k)^{\rightarrow} a \circ (k \times k)^{\rightarrow} b\Big)(y_1, y_2) = \bigvee_{z \in Y} \Big(\Big((k \times k)^{\rightarrow} a\Big)(y_1, z) \wedge \Big((k \times k)^{\rightarrow} b\Big)(z, y_2)\Big)$$

$$= \bigvee_{z \in Y} \Big(\Big( \vee \{a(x_1, s) : (k \times k)(x_1, s) = (y_1, z)\}\Big) \wedge \Big( \vee \{b(t, x_2) : (k \times k)(t, x_2) = (z, y_2)\}\Big)\Big)$$

$$= \bigvee_{z \in Y} \Big( \vee \{a(x_1, s) \wedge b(t, x_2) : (k \times k)(x_1, s) = (y_1, z), (k \times k)(t, x_2) = (z, y_2)\}\Big)$$

$$\geq \bigvee_{z \in Y} \Big( \vee \{a(x_1, s) \wedge b(s, x_2) : (k \times k)(x_1, s) = (y_1, z), (k \times k)(s, x_2) = (z, y_2)\}\Big)$$

$$= \bigvee_{s \in X} \{a(x_1, s) \wedge b(s, x_2) : (k \times k)(x_1, x_2) = (y_1, y_2)\}$$

$$= (k \times k)^{\rightarrow}(a \circ b)(y_1, y_2).$$

Hence $(k \times k)^{\rightarrow} a \circ (k \times k)^{\rightarrow} b \geq (k \times k)^{\rightarrow}(a \circ b)$. Observe that $\bigvee_{(y_1, y_2) \in Y^2} (k \times k)^{\rightarrow}(a \circ b)(y_1, y_2) = \bigvee_{(x_1, x_2) \in X^2} (a \circ b)(x_1, x_2) = \top$ since $\Phi \circ \Psi$ exists.

Then $\bigvee_{(y_1, y_2) \in Y^2} \Big((k \times k)^{\rightarrow} a \circ (k \times k)^{\rightarrow} b\Big)(y_1, y_2) = \top$ and thus $(k \times k)^{\Rightarrow} \Phi \circ (k \times k)^{\Rightarrow} \Psi$ exists. The above inequality shows that $(k \times k)^{\rightarrow} a \circ (k \times k)^{\rightarrow} b \in (k \times k)^{\Rightarrow}(\Phi \circ \Psi)$, where $a \in \Phi$ and $b \in \Psi$.

The desired result follows from Lemma 4.1 (ii). $\square$

**Theorem 4.1.** *The category* $\top$-**ULS** *is a topological construct.*[1]

*Proof.* Consider the source $k_j : X \longrightarrow (Y_j, \Lambda_j)$, $j \in J$. Define $\Lambda = \{\Phi \in \mathfrak{F}_L^{\top}(X^2) : (k_j \times k_j)^{\Rightarrow} \Phi \in \Lambda_j, \forall j \in J\}$. Since $(k_j \times k_j)^{\Rightarrow}([x, x]) = [(k_j \times k_j)(x, x))] = \Big[\Big(k_j(x), k_j(x)\Big)\Big] \in \Lambda_j$ for each $j \in J$, we have that $[(x, x)] \in \Lambda$ for each $x \in X$. Thus (UL1) is satisfied. Clearly if $\Psi \geq \Phi \in \Lambda$ then $(k_j \times k_j)^{\Rightarrow} \Psi \geq (k_j \times k_j)^{\Rightarrow} \Phi \in \Lambda_j$ for each $j \in J$ and hence $\Psi \in \Lambda$ and (UL2) is valid.

[1]See Appendix for the definition of a topological construct.

Next, assume that $\Phi \in \Lambda$. Let $y_1, y_2 \in Y_j$ and $a \in \Phi$; then $\Big((k_j \times k_j)^{\rightarrow}(a^{-1})\Big)(y_1, y_2) = \bigvee\limits_{\substack{k_j(x_1)=y_1 \\ k_j(x_2)=y_2}} a^{-1}(x_1, x_2) = \bigvee\limits_{\substack{k_j(x_1)=y_1 \\ k_j(x_2)=y_2}} a(x_2, x_1) = \Big((k_j \times k_j)^{\rightarrow}(a)\Big)(y_2, y_1) = \Big((k_j \times k_j)^{\rightarrow}(a)\Big)^{-1}(y_1, y_2)$. Hence $(k_j \times k_j)^{\rightarrow}(a^{-1}) = (k_j \times k_j)^{\rightarrow}(a)$ for each $a \in \Phi$ and therefore $(k_j \times k_j)^{\Rightarrow}\Phi = (k_j \times k_j)^{\Rightarrow}(\Phi^{-1}) \in \Lambda_j$. This implies $\Phi^{-1} \in \Lambda$ and (UL3) is valid.

To show (UL4), suppose that $\Phi, \Psi \in \Lambda$ and $\Phi \circ \Psi$ exists. According to Lemma 4.4, $(k_j \times k_j)^{\Rightarrow}(\Phi \circ \Psi) \geq (k_j \times k_j)^{\Rightarrow}\Phi \circ (k_j \times k_j)^{\Rightarrow}\Psi \in \Lambda_j$ for each $j \in J$. Hence $\Phi \circ \Psi \in \Lambda$ and (UL4) is satisfied.

Finally, suppose that $\Phi, \Psi \in \Lambda$. Then employing Lemma 1.4 $(i)$, $(k_j \times k_j)^{\Rightarrow}(\Phi \cap \Psi) = (k_j \times k_j)^{\Rightarrow}\Phi \cap (k_j \times k_j)^{\Rightarrow}\Psi \in \Lambda_j$ for all $j \in J$. Hence $\Phi \cap \Psi \in \Lambda$ and (UL5) is valid. Hence $(X, \Lambda) \in |\top\text{-}\mathbf{ULS}|$.

Let $\ell : (Z, \Sigma) \longrightarrow (X, \Lambda)$ be a map and assume that $k_j \circ \ell : (Z, \Sigma) \longrightarrow (Y_j, \Lambda_j)$ is uniformly continuous for each $j \in J$. If $\Phi \in \Sigma$, then $(k_j \times k_j)^{\Rightarrow}\Big((\ell \times \ell)^{\Rightarrow}\Phi\Big) = \Big((k_j \times k_j) \circ (\ell \times \ell)\Big)^{\Rightarrow}\Phi \in \Lambda_j$ for each $j \in J$. Then by definition of $\Lambda$, $(\ell \times \ell)^{\Rightarrow}\Phi \in \Lambda$ and thus $\ell : (Z, \Sigma) \longrightarrow (X, \Lambda)$ is uniformly continuous. Conversely, if $\ell$ is uniformly continuous, then clearly the composition $k_j \circ \ell$ is also uniformly continuous for each $j \in J$. Moreover, $\Lambda$ is the unique such structure having this property and hence $\top$-**ULS** contains initial structures.

For any set $X$, the class of all $\top$-uniform limit structures on $X$ is a subset of $2^{\mathfrak{F}_L^{\top}(X^2)}$ and hence is also a set. Further if $X = \{x\}$ is a singleton, then $\mathfrak{F}_L^{\top}(X \times X) = \{(x, x)\}$ and $\Lambda = \{[(x, x)]\}$ is the only $\top$-uniform limit structure on $X$; if $X = \varnothing$ then $\Lambda = \varnothing$. Hence $\top$-**ULS** is a topological construct. $\square$

Let $(X, \Lambda), (Y, \Gamma) \in |\top\text{-}\mathbf{ULS}|$, and let $UC(X, Y)$ denote the set of all uniformly continuous maps in $\top$-**ULS** from $X$ to $Y$. Define $\text{ev} : UC(X, Y) \times X \to Y$ by $\text{ev}(f, x) = f(x)$. Note that since $\top$-**ULS** possesses initial structures, it has product structures. In particular, if $\Phi \in \mathfrak{F}_L^\top\Big((X\times Y)^2\Big)$, then $\Phi \in \Lambda \times \Gamma$ (product structure) iff $(\pi_1\times\pi_1)^{\Rightarrow}\Phi \in \Lambda$ and $(\pi_2\times\pi_2)^{\Rightarrow}\Phi \in \Gamma$. Let $\eta : \Big(UC(X,Y)\times UC(X,Y)\Big)\times(X\times X) \longrightarrow \Big(UC(X,Y)\times X\Big)\times\Big(UC(X,Y)\times X\Big)$ be given by $\eta\Big((\phi,\psi),(x_1,x_2)\Big) = \Big((\phi,x_1),(\psi,x_2)\Big)$. Define $\Sigma \subseteq \mathfrak{F}_L^\top(UC(X,Y)^2)$ as follows: $\Psi \in \Sigma$ iff for each $\Phi \in \Lambda$, $(\mathrm{ev}\times\mathrm{ev})^{\Rightarrow}\Big(\eta^{\Rightarrow}(\Psi\times\Phi)\Big) \in \Gamma$.

**Theorem 4.2.** *The category* $\top$-**ULS** *is Cartesian closed.*

*Proof.* First we show that $\Sigma$ as defined above is a $\top$-uniform limit structure on $UC(X,Y)$. Fix $\theta \in UC(X,Y)$, it is shown that if $\Phi \in \Lambda$, then $(\mathrm{ev}\times\mathrm{ev})^{\Rightarrow}\Big(\eta^{\Rightarrow}([(\theta,\theta)]\times\Phi)\Big) \in \Gamma$. Since $\{\mathbf{1}_{\{(\theta,\theta)\}}\}$ is a $\top$-filter base for $[(\theta,\theta)]$, $\{(\mathbf{1}_{\{(\theta,\theta)\}}\times a : a \in \Phi\}$ is a $\top$-filter base for $[(\theta,\theta)]\times\Phi$ and thus $\mathcal{B} = \{(\mathrm{ev}\times\mathrm{ev})^{\rightarrow}\Big(\eta^{\rightarrow}(\mathbf{1}_{\{(\theta,\theta)\}}\times a)\Big) : a \in \Phi\}$ is a $\top$-filter base for $(\mathrm{ev}\times\mathrm{ev})^{\Rightarrow}\Big(\eta^{\Rightarrow}([(\theta,\theta)]\times\Phi)\Big)$. Observe that

$$
\begin{aligned}
&(\mathrm{ev}\times\mathrm{ev})^{\rightarrow}\Big(\eta^{\rightarrow}(\mathbf{1}_{\{(\theta,\theta)\}}\times a)\Big)(y_1,y_2)\\
&= \bigvee_{(\mathrm{ev}\times\mathrm{ev})\Big((\phi,x_1),(\psi,x_2)\Big)=(y_1,y_2)} \ \bigvee_{\eta\Big((\xi,\gamma),(z_1,z_2)\Big)=\Big((\phi,x_1),(\psi,x_2)\Big)} \Big(\mathbf{1}_{(\theta,\theta)}\times a\Big)\Big((\xi,\gamma),(z_1,z_2)\Big)\\
&= \bigvee_{\substack{\phi(x_1)=y_1\\ \psi(x_2)=y_2}} \ \bigvee_{\substack{(\xi,z_1)=(\phi,x_1)\\ (\psi,x_2)=(\gamma,z_2)}} \Big(\mathbf{1}_{(\theta,\theta)}\times a\Big)\Big((\xi,\gamma),(z_1,z_2)\Big)\\
&= \bigvee_{\substack{\phi(x_1)=y_1\\ \psi(x_2)=y_2}} \Big(\mathbf{1}_{(\theta,\theta)}\times a\Big)\Big((\phi,\psi),(x_1,x_2)\Big)\\
&= \bigvee_{\substack{\theta(x_1)=y_1\\ \theta(x_2)=y_2}} a(x_1,x_2)\\
&= (\theta\times\theta)^{\rightarrow}(a)(y_1,y_2)
\end{aligned}
$$

Hence $(\mathrm{ev}\times\mathrm{ev})^{\rightarrow}\Big(\eta^{\rightarrow}(\mathbf{1}_{\{(\theta,\theta)\}}\times a)\Big) = (\theta\times\theta)^{\rightarrow}(a)$ for each $a \in \Phi$. Since $\mathcal{B}$ is a $\top$-filter base for $(\mathrm{ev}\times\mathrm{ev})^{\Rightarrow}\Big(\eta^{\Rightarrow}([(\theta,\theta)]\times\Phi)\Big)$ and $\{(\theta\times\theta)^{\rightarrow}(a) : a \in \Phi\}$ is a $\top$-filter base for $(\theta\times\theta)^{\Rightarrow}\Phi$,

$(\mathrm{ev} \times \mathrm{ev})^{\Rightarrow}\Big(\eta^{\Rightarrow}([(\theta,\theta)] \times \Phi)\Big) = (\theta \times \theta)^{\Rightarrow}\Phi \in \Gamma$. Hence $[(\theta,\theta)] \in \Sigma$ and (UL1) is satisfied by $\Sigma$.

Clearly, if $\Psi \geq \Phi \in \Sigma$, then $\Psi \in \Sigma$ and therefore (UL2) is satisfied by $\Sigma$.

A straightforward computation confirms that if $\phi \in \Phi \in \Sigma$, $f \in \mathfrak{F} \in \Lambda$ and $y_1, y_2 \in Y$, then

$$(\mathrm{ev} \times \mathrm{ev})^{\rightarrow}(\eta^{\rightarrow}(\phi^{-1} \times f^{-1}))(y_1, y_2) = \Big((\mathrm{ev} \times \mathrm{ev})^{\rightarrow}(\eta^{\rightarrow}(\phi \times f))\Big)^{-1}(y_1, y_2).$$

Hence if $\Phi \in \Sigma$, then $\Phi^{-1} \in \Sigma$ and (UL3) is valid.

Next assume that $\Phi, \Psi \in \Sigma$ and $\Phi \circ \Psi$ exists. We must show that $\Phi \circ \Psi \in \Sigma$. Let $\mathfrak{F} \in \Lambda$. Lemma C from [13] implies that

$$(\mathrm{ev} \times \mathrm{ev})^{\Rightarrow}\Big(\eta^{\Rightarrow}((\Phi \circ \Psi) \times \mathfrak{F})\Big) \geq (\mathrm{ev} \times \mathrm{ev})^{\Rightarrow}\Big(\eta^{\Rightarrow}(\Phi \times \mathfrak{F})\Big) \circ (\mathrm{ev} \times \mathrm{ev})^{\Rightarrow}\Big(\eta^{\Rightarrow}(\Psi \times (\mathfrak{F}^{-1} \circ \mathfrak{F}))\Big).$$

Note that by Lemma 4.3, if $\mathfrak{F} \in \Lambda$ then $\mathfrak{F}^{-1} \circ \mathfrak{F}$ exists. Hence it suffices to show that if $\Phi \circ \Psi$ exists and $\mathfrak{F} \in \Lambda$, then $(\mathrm{ev} \times \mathrm{ev})^{\Rightarrow}\Big(\eta^{\Rightarrow}(\Phi \times \mathfrak{F})\Big) \circ (\mathrm{ev} \times \mathrm{ev})^{\Rightarrow}\Big(\eta^{\Rightarrow}(\Psi \times (\mathfrak{F}^{-1} \circ \mathfrak{F}))\Big)$ exists. Let $\phi \in \Phi, \psi \in \Psi$, $f_1 \in \mathfrak{F}$ and $f_2 \in \mathfrak{F}^{-1} \circ \mathfrak{F}$. We have,

$$\begin{aligned} &\bigvee_{y_1, y_2 \in Y} \Big((\mathrm{ev} \times \mathrm{ev})^{\rightarrow}(\eta^{\rightarrow}(\phi \times f_1)) \circ (\mathrm{ev} \times \mathrm{ev})^{\rightarrow}(\eta^{\rightarrow}(\psi \times f_2))\Big)(y_1, y_2) \\ &= \bigvee_{z, y_1, y_2 \in Y} (\mathrm{ev} \times \mathrm{ev})^{\rightarrow}(\eta^{\rightarrow}(\phi \times f_1))(y_1, z) \wedge (\mathrm{ev} \times \mathrm{ev})^{\rightarrow}(\eta^{\rightarrow}(\psi \times f_2))(z, y_2) \qquad (4.1) \end{aligned}$$

and,

$$(\mathrm{ev}\times\mathrm{ev})^{\to}(\eta^{\to}(\phi\times f_1))(y_1,z) = \bigvee_{\substack{(\mathrm{ev}\times\mathrm{ev})\big((\theta_1,x_1),(\theta_2,x_2)\big)\\ =(y_1,z)}} \bigvee_{\substack{\eta\big(\xi_1,\xi_2),(w_1,w_2)\big)\\ =\big((\theta_1,x_1),(\theta_2,x_2)\big)}} (\phi\times f_1)\Big((\xi_1,\xi_2),(w_1,w_2)\Big)$$

$$= \bigvee_{\substack{\theta_1(x_1)=y_1\\ \theta_2(x_2)=z}} (\phi\times f_1)\Big((\theta_1,\theta_2),(x_1,x_2)\Big) = \bigvee_{\substack{\theta_1(x_1)=y_1\\ \theta_2(x_2)=z}} \phi(\theta_1,\theta_2)\wedge f_1(x_1,x_2).$$

Similarly, $(\mathrm{ev}\times\mathrm{ev})^{\to}(\eta^{\to}(\psi\times f_2))(z,y_2) = \bigvee_{\substack{\theta_1'(x_1')=z\\ \theta_2'(x_2')=y_2}} \psi(\theta_1',\theta_2')\wedge f_2(x_1',x_2')$. Therefore, picking up from Equation (4.1),

$$\begin{aligned}
(4.1) &= \bigvee_{z,y_1,y_2\in Y} \bigvee_{\substack{\theta_1(x_1)=y_1\\ \theta_2(x_2)=z\\ \theta_1'(x_1')=z\\ \theta_2'(x_2')=y_2}} \phi(\theta_1,\theta_2)\wedge f_1(x_1,x_2)\wedge\psi(\theta_1',\theta_2')\wedge f_2(x_1',x_2')\\
&\geq \bigvee_{z,y_1,y_2\in Y} \bigvee_{\substack{\theta_1(x_1)=y_1\\ \theta(x)=z\\ \theta_2'(x_2')=y_2}} \phi(\theta_1,\theta)\wedge\psi(\theta,\theta_2')\wedge f_1(x_1,x)\wedge f_2(x,x_2')\\
&= \bigvee_{\substack{z,y_1,y_2\in Y\\ \theta_1(x_1)=y_1\\ \theta(x)=z\\ \theta_2'(x_2')=y_2}} \phi(\theta_1,\theta)\wedge\psi(\theta,\theta_2')\wedge f_1(x_1,x)\wedge f_2(x,x_2')\\
&= \bigvee_{\substack{\theta_1,\theta_2',\theta\in UC(X,Y)\\ x_1,x_2',x\in X}} \phi(\theta_1,\theta)\wedge\psi(\theta,\theta_2')\wedge f_1(x_1,x)\wedge f_2(x,x_2')\\
&= \bigvee_{\theta_1,\theta_2'\in UC(X,Y)} \left(\bigvee_{\theta\in UC(X,Y)} \phi(\theta_1,\theta)\wedge\psi(\theta,\theta_2')\right)\wedge \bigvee_{x_1,x_2'\in X}\left(\bigvee_{x\in X} f(x_1,x)\wedge f_2(x,x_2')\right)\\
&= \bigvee_{\theta_1,\theta_2'\in UC(X,Y)} (\phi\circ\psi)(\theta_1,\theta_2')\wedge \bigvee_{x_1,x_2'\in X} (f_1\circ f_2)(x_1,x_2')\\
&= \top\wedge\top = \top. \qquad\qquad \text{(Lemma 4.3)}
\end{aligned}$$

Where on the penultimate line above, Lemma 4.3 is used on $\mathfrak{F} \circ (\mathfrak{F}^{-1} \circ \mathfrak{F})$. Therefore $(\mathrm{ev} \times \mathrm{ev})^{\Rightarrow}\Big(\eta^{\Rightarrow}(\Phi \times \mathfrak{F})\Big) \circ (\mathrm{ev} \times \mathrm{ev})^{\Rightarrow}\Big(\eta^{\Rightarrow}(\Psi \times (\mathfrak{F}^{-1} \circ \mathfrak{F}))\Big)$ exists, $\Phi \circ \Psi \in \Sigma$ and (UL4) verified.

To prove (UL5), assume that $\Phi, \Psi \in \Sigma$ and let $\mathfrak{F} \in \Lambda$. We must show $\Phi \cap \Psi \in \Sigma$. For this, we must show $(\mathrm{ev} \times \mathrm{ev})^{\Rightarrow}(\eta^{\Rightarrow}((\Phi \cap \Psi) \times \mathfrak{F})) \in \Gamma$. Note that $(\Phi \cap \Psi) \times \mathfrak{F} = (\Phi \times \mathfrak{F}) \cap (\Psi \times \mathfrak{F})$ and therefore, using Lemma 1.4 (i), $(\mathrm{ev} \times \mathrm{ev})^{\Rightarrow}(\eta^{\Rightarrow}((\Phi \cap \Psi) \times \mathfrak{F})) = (\mathrm{ev} \times \mathrm{ev})^{\Rightarrow}(\eta^{\Rightarrow}(\Phi \times \mathfrak{F})) \cap (\mathrm{ev} \times \mathrm{ev})^{\Rightarrow}(\eta^{\Rightarrow}(\Psi \times \mathfrak{F})) \in \Gamma$ as desired. Hence (UL5) holds.

Since (UL1) – (UL5) have been verified, we have that $\Big(UC(X,Y), \Sigma\Big) \in |\top\text{-}\mathbf{ULS}|$.

Next, we show that if $\Xi \in \Sigma \times \Lambda$ then $\Xi \geq \eta^{\Rightarrow}\Big((\pi_1 \times \pi_1)^{\Rightarrow}\Xi \times (\pi_2 \times \pi_2)^{\Rightarrow}\Xi\Big)$. Let $\phi, \psi \in \Xi$. For convienience we will abreviate $UC(X,Y)$ with $UC$; it is also helpful to recall that $\Xi \subseteq L^{(UC \times X)^2}$. Also let $f, g \in UC$ and $x, y \in X$. We compute,

$$\begin{aligned}
&\Big(\eta^{\rightarrow}\Big[(\pi_1 \times \pi_1)^{\rightarrow}\phi \times (\pi_2 \times \pi_2)^{\rightarrow}\psi\Big]\Big)\Big((f,x),(g,y)\Big) \\
&= \bigvee_{\substack{((h,k),(z,w)) \in UC^2 \times X^2 \\ \eta((h,k),(z,w)) = ((f,x),(g,y))}} \Big[(\pi_1 \times \pi_1)^{\rightarrow}\phi \times (\pi_2 \times \pi_2)^{\rightarrow}\psi\Big]\Big((h,k),(z,w)\Big) \\
&= \bigvee_{\substack{((h,k),(z,w)) \in UC^2 \times X^2 \\ \eta((h,k),(z,w)) = ((f,x),(g,y))}} \left( \bigvee_{\substack{((\ell,v),(m,u)) \in (UC \times X)^2 \\ (\pi_1 \times \pi_1)((\ell,v),(m,u)) = (h,k)}} \phi\Big((\ell,v),(m,u)\Big) \wedge \bigvee_{\substack{((\ell,v),(m,u)) \in (UC \times X)^2 \\ (\pi_2 \times \pi_2)((q,s),(q,t)) = (z,w)}} \psi\Big((p,s),(q,t)\Big) \right) \\
&\geq \bigvee_{\substack{((\ell,v),(m,u)) \in (UC \times X)^2 \\ (\pi_1 \times \pi_1)((\ell,v),(m,u)) = (f,g)}} \phi\Big((\ell,v),(m,u)\Big) \wedge \bigvee_{\substack{((\ell,v),(m,u)) \in (UC \times X)^2 \\ (\pi_2 \times \pi_2)((q,s),(q,t)) = (x,y)}} \psi\Big((p,s),(q,t)\Big) \\
&= \bigvee_{u,v \in X} \phi\Big((f,v),(g,u)\Big) \wedge \bigvee_{p,q \in UC} \psi\Big((p,x),(q,y)\Big) \\
&\geq \phi\Big((f,x),(g,y)\Big) \wedge \psi\Big((f,x),(g,y)\Big) \\
&= (\phi \wedge \psi)\Big((f,x),(g,y)\Big)
\end{aligned}$$

Hence $\eta^{\rightarrow}\Big[(\pi_1 \times \pi_1)^{\rightarrow}\phi \times (\pi_2 \times \pi_2)^{\rightarrow}\psi\Big] \geq \phi \wedge \psi \in \Xi$. Therefore it follows that $\Xi \geq \eta^{\Rightarrow}\Big((\pi_1 \times$

$\pi_1)^{\Rightarrow}\Xi \times (\pi_2 \times \pi_2)^{\Rightarrow}\Xi\Big)$. Next, since $(\pi_1 \times \pi_1)^{\Rightarrow}\Xi \in \Sigma$ and $(\pi_2 \times \pi_2)^{\Rightarrow}\Xi \in \Lambda$, by definition of $\Sigma$ this implies $(\text{ev} \times \text{ev})^{\Rightarrow}\Xi \geq (\text{ev} \times \text{ev})^{\Rightarrow}\Big(\eta^{\Rightarrow}\Big((\pi_1 \times \pi_1)^{\Rightarrow}\Xi \times (\pi_2 \times \pi_2)^{\Rightarrow}\Xi\Big)\Big) \in \Gamma$. It follows that $\text{ev} : \Big(UC(X,Y), \Sigma\Big) \times (X, \Lambda) \longrightarrow (Y, \Gamma)$ is uniformly continuous in $\top$-**ULS**.

Next, assume that $f : (Z, \Upsilon) \times (X, \Lambda) \longrightarrow (Y, \Gamma)$ is uniformly continuous. Fix $z \in Z$ and define $f_z : X \longrightarrow Y$ by $f_z(x) = f(z,x)$. It is shown that $f_z \in UC(X,Y)$. To do this, it is shown that if $\Phi \in \Lambda$, then $(f_z \times f_z)^{\Rightarrow}\Phi = (f \times f)^{\Rightarrow}(\zeta^{\Rightarrow}([(z,z)] \times \Phi))$, where $\zeta : (Z \times Z) \times (X \times X) \longrightarrow (Z \times X) \times (Z \times X)$ maps $\Big((z_1, z_2), (x_1, x_2)\Big) \mapsto \Big((z_1, x_1), (z_2, x_2)\Big)$. Let $\phi \in \Phi$. Bases for $(f_z \times f_z)^{\Rightarrow}\Phi$, and $(f \times f)^{\Rightarrow}(\zeta^{\Rightarrow}([(z,z)] \times \Phi))$ are given by $\{(f_z \times f_z)^{\rightarrow}\phi : \phi \in \Phi\}$ and $\{(f \times f)^{\rightarrow}(\zeta^{\rightarrow}(\mathbf{1}_{\{(z,z)\}} \times \phi)) : \phi \in \Phi\}$, respectively. Let $\phi \in \Phi$ and $y_1, y_2 \in Y$; then,

$$(f \times f)^{\rightarrow}(\zeta^{\rightarrow}(\mathbf{1}_{\{(z,z)\}} \times \phi))(y_1, y_2) = \bigvee_{\substack{f(z_1,x_1)=y_1 \\ f(z_2,x_2)=y_2}} \mathbf{1}_{\{(z,z)\}}(z_1, z_2) \wedge \phi(x_1, x_2)$$
$$= \bigvee_{\substack{f(z,x_1)=y_1 \\ f(z,x_2)=y_2}} \phi(x_1, x_2) = \bigvee_{(f_z \times f_z)(x_1,x_2)=(y_1,y_2)} \phi(x_1, x_2)$$
$$= (f_z \times f_z)^{\rightarrow}\phi(y_1, y_2).$$

Hence $(f_z \times f_z)^{\Rightarrow}\Phi = (f \times f)^{\Rightarrow}(\zeta^{\Rightarrow}([(z,z)] \times \Phi))$. Now it must be shown that $(f \times f)^{\Rightarrow}(\zeta^{\Rightarrow}([(z,z)] \times \Phi)) \in \Gamma$. Since $f$ is uniformly continuous, it suffices to show that $\zeta^{\Rightarrow}([(z,z)] \times \Phi) \in \Upsilon \times \Lambda$. That is, we must show that $(\pi_i \times \pi_i)^{\Rightarrow}(\zeta^{\Rightarrow}([(z,z)] \times \Phi)) \in \Upsilon(\Lambda)$ if $i = 1(2)$, respectively. Let $\phi \in \Phi$ and $y_1, y_2 \in Z(X)$ when $i = 1(2)$ respectively. Then,

$$(\pi_i \times \pi_i)^{\rightarrow}(\zeta^{\rightarrow}(\mathbf{1}_{\{(z,z)\}} \times \phi))(y_1, y_2) = \bigvee_{\substack{\pi_i(z_1,x_1)=y_1 \\ \pi_i(z_2,x_2)=y_2}} \mathbf{1}_{\{(z,z)\}}(z_1, z_2) \wedge \phi(x_1, x_2)$$
$$= \begin{cases} \bigvee_{x_1,x_2 \in X} \mathbf{1}_{\{(z,z)\}}(y_1, y_2) \wedge \phi(x_1, x_2), & i = 1 \\ \bigvee_{z_1,z_2 \in Z} \mathbf{1}_{\{(z,z)\}}(z_1, z_2) \wedge \phi(y_1, y_2), & i = 2 \end{cases} = \begin{cases} \mathbf{1}_{\{(z,z)\}}(y_1, y_2) \wedge \top, i = 1 \\ \top \wedge \phi(y_1, y_2), i = 2 \end{cases} .$$

Since $\mathbf{1}_{\{(z,z)\}} \in [(z,z)]$ and $\phi \in \Phi \in \Lambda$, it follows that $\zeta^{\Rightarrow}([(z,z)] \times \Phi) \in \Upsilon \times \Lambda$. Therefore $(f_z \times f_z)^{\Rightarrow}\Phi = (f \times f)^{\Rightarrow}(\zeta^{\Rightarrow}([(z,z)] \times \Phi)) \in \Gamma$ and $f_z$ is uniformly continuous.

Define $f^* : Z \longrightarrow UC(X,Y)$ by $f^*(z) = f_z$ for $z \in X$. It is shown that $f^* : (Z, \Upsilon) \longrightarrow \Big(UC(X,Y), \Sigma\Big)$ is uniformly continuous. It was shown in [29] that $\mathrm{ev} \circ (f^* \times \mathrm{id}_X) = f$ (Here, $\circ$ is traditional function composition). Indeed, if $(s,t) \in Z \times X$, then $\Big(\mathrm{ev} \circ (f^* \times \mathrm{id}_X)\Big)(s,t) = \mathrm{ev}(f^*(s), t) = \mathrm{ev}(f_s, t) = f_s(t) = f(s,t)$. Let $\Phi \in \Upsilon$, we must show that $(f^* \times f^*)^{\Rightarrow}\Phi \in \Sigma$. It suffices to show that for any $\mathfrak{G} \in \Lambda$, we have $(\mathrm{ev} \times \mathrm{ev})^{\Rightarrow}\Big(\eta^{\Rightarrow}\Big([(f^* \times f^*)^{\Rightarrow}\Phi] \times \mathfrak{G}\Big)\Big) \in \Gamma$. Let $\phi \in \Phi$, $g \in \mathfrak{G}$ and $y_1, y_2 \in Y$. We have,

$$(\mathrm{ev} \times \mathrm{ev})^{\rightarrow}\Big(\eta^{\rightarrow}\Big((f^* \times f^*)^{\rightarrow}\phi \times g\Big)\Big)(y_1, y_2) = \bigvee_{\substack{\theta_1(x_1)=y_1 \\ \theta_2(x_2)=y_2}} \left( \bigvee_{\substack{(f^* \times f^*)(z_1,z_2) \\ =(\theta_1,\theta_2)}} \phi(z_1, z_2) \right) \wedge g(x_1, x_2)$$

$$= \bigvee_{\substack{\theta_1(x_1)=y_1 \\ \theta_2(x_2)=y_2}} \left( \bigvee_{\substack{f_{z_1}=\theta_1 \\ f_{z_2}=\theta_2}} \phi(z_1, z_2) \right) \wedge g(x_1, x_2) = \bigvee_{\substack{\theta_1(x_1)=y_1 \\ \theta_2(x_2)=y_2}} \left( \bigvee_{\substack{f_{z_1}=\theta_1 \\ f_{z_2}=\theta_2}} \phi(z_1, z_2) \right) \wedge g(x_1, x_2)$$

$$= \bigvee_{\substack{f_{z_1}(x_1)=y_1 \\ f_{z_2}(x_2)=y_2}} \phi(z_1, z_2) \wedge g(x_1, x_2) = \bigvee_{\substack{f(z_1,x_1)=y_1 \\ f(z_2,x_2)=y_2}} \phi(z_1, z_2) \wedge g(x_1, x_2)$$

$$= (f \times f)^{\rightarrow}\Big(\zeta^{\rightarrow}(\phi \times g)\Big)(y_1, y_2)$$

Thus $(\mathrm{ev} \times \mathrm{ev})^{\Rightarrow}\Big(\eta^{\Rightarrow}\Big([(f^* \times f^*)^{\Rightarrow}\Phi] \times \mathfrak{G}\Big)\Big) = (f \times f)^{\Rightarrow}\Big(\zeta^{\Rightarrow}(\Phi \times \mathfrak{G})\Big)$. Now we must show that $(f \times f)^{\Rightarrow}\Big(\zeta^{\Rightarrow}(\Phi \times \mathfrak{G})\Big) \in \Gamma$. Since $f$ is uniformly continuous, it suffices to show that $\zeta^{\Rightarrow}(\Phi \times \mathfrak{G}) \in \Upsilon \times \Lambda$. That is, we must show that $(\pi_i \times \pi_i)^{\Rightarrow}(\zeta^{\Rightarrow}(\Phi \times \mathfrak{G})) \in \Upsilon\,(\Lambda)$ if $i = 1\,(2)$, respectively. Let $\phi \in \Phi$, $g \in \mathfrak{G}$ and $y_i \in Z(X)$ when $i = 1\,(2)$, respectively. Then,

$$(\pi_i \times \pi_i)^{\rightarrow}(\zeta^{\rightarrow}(\phi \times g))(y_1, y_2) = \bigvee_{\substack{\pi_i(z_1,x_1)=y_1 \\ \pi_i(z_2,x_2)=y_2}} \phi(z_1, z_2) \wedge g(x_1, x_2)$$

$$= \begin{cases} \bigvee\limits_{x_1,x_2 \in X} \phi(y_1, y_2) \wedge g(x_1, x_2), & i = 1 \\ \bigvee\limits_{z_1,z_2 \in Z} \phi(z_1, z_2) \wedge g(y_1, y_2), & i = 2 \end{cases} = \begin{cases} \phi(y_1, y_2) \wedge \top, & i = 1 \\ \top \wedge g(y_1, y_2), & i = 2 \end{cases}$$

Since $\phi \in \Phi \in \Upsilon$ and $g \in \mathfrak{G} \in \Lambda$, it follows that $\zeta^{\Rightarrow}(\Phi \times \mathfrak{G}) \in \Upsilon \times \Lambda$. Therefore $(f \times f)^{\Rightarrow}(\zeta^{\Rightarrow}(\Phi \times \mathfrak{G})) \in \Gamma$. Hence $f^* : (Z, \Upsilon) \longrightarrow \big(UC(X, Y), \Sigma\big)$ is uniformly continuous and $\top$-**ULS** is a Cartesian closed category. $\square$

## Selection Maps and Completions

Suppose that $(X, \Lambda) \in |\top\text{-}\mathbf{ULS}|$; define $\mathcal{C}_\Lambda = \{\mathfrak{F} \in \mathfrak{F}_L^{\top}(X) : \mathfrak{F} \times \mathfrak{F} \in \Lambda\}$. If $\mathfrak{F}, \mathfrak{G} \in \mathcal{C}_\Lambda$ such that $\mathfrak{F} \vee \mathfrak{G}$ exists, then using Lemma 4.2 (i) and (ii), $(\mathfrak{F} \cap \mathfrak{G}) \times (\mathfrak{F} \cap \mathfrak{G}) = (\mathfrak{F} \times \mathfrak{F}) \cap [(\mathfrak{G} \times \mathfrak{G}) \circ (\mathfrak{F} \times \mathfrak{F})] \cap [(\mathfrak{F} \times \mathfrak{F}) \circ (\mathfrak{G} \times \mathfrak{G})] \cap (\mathfrak{G} \times \mathfrak{G}) \in \Lambda$, and thus it follows that $\mathfrak{F} \cap \mathfrak{G} \in \mathcal{C}_\Lambda$. Hence $(X, \mathcal{C}_\Lambda) \in |\top\text{-}\mathbf{Chy}|$.

An object $(X, \Lambda) \in |\top\text{-}\mathbf{ULS}|$ is called **complete** if $(X, \mathcal{C}_\Lambda)$ is complete in $\top$-**Chy**. Moreover, $\big((Y, \Sigma), \phi\big)$ is called a **completion** of $(X, \mathcal{C})$ in $\top$-**ULS** provided that $\phi : (X, \Lambda) \longrightarrow (Y, \Sigma)$ is a dense $\top$-uniform embedding and $(Y, \Sigma)$ is complete.

Much as was done in the $\top$-**Chy** setting, we take advantage of selection maps to achieve completions. Given $(X, \Lambda) \in |\top\text{-}\mathbf{ULS}|$, let $X^* = X \cup \{\langle \mathfrak{G} \rangle : \mathfrak{G} \in \mathcal{N}_{\mathcal{C}_\Lambda}\}$ and let $\alpha : X^* \longrightarrow \mathcal{C}_\Lambda$

be a selection map. For each $a \in L^{X^2}$, define $a^\alpha \in L^{(X^*)^2}$ as follows:

$$a^\alpha(x,y) = \nu_{\alpha(x)\times\alpha(y)}(a).$$

If $\Phi \in \mathfrak{F}_L^\top(X^2)$, then let $\Phi^\alpha$ be the $\top$-filter on $(X^*)^2$ generated by the $\top$-filter base $\{a^\alpha : \alpha \in \Phi\}$. Indeed, this is a $\top$-filter base as it can easily be shown that $(a \wedge b)^\alpha = a^\alpha \wedge b^\alpha$ for each $a, b \in L^{X^2}$. Further, if $a \in \Phi$ then $\bigvee\limits_{(x,y)\in(X^*)^2} a^\alpha(x,y) \geq \bigvee\limits_{(x,y)\in X^2} a(x,y) = \top$.

**Lemma 4.5.** *Assume that $a, b \in L^{X^2}$, $\mathfrak{F}, \mathfrak{G} \in \mathfrak{F}_L^\top(X)$, $\Phi, \Psi \in \mathfrak{F}_L^\top(X^2)$ and $\alpha$ is a selection map for $(X, \mathcal{C}) \in |\top\text{-}\mathbf{Chy}|$. Then*

*(i)* $[a, b] = [a^{-1}, b^{-1}]$

*(ii)* $\nu_\Phi(a) = \nu_{\Phi^{-1}}(a^{-1})$

*(iii)* $(a^\alpha)^{-1} = (a^{-1})^\alpha$ *and therefore* $(\Phi^\alpha)^{-1} = (\Phi^{-1})^\alpha$

*(iv) if* $\Phi \circ \Psi$ *exists then* $\nu_\Phi(a) \wedge \nu_\Psi(b) \leq \nu_{\Phi\circ\Psi}(a \circ b)$

*(v)* $a^\alpha \circ b^\alpha \leq (a \circ b)^\alpha$

*(vi)* $a^\alpha \times b^\alpha \leq (a \times b)^\alpha$ *and therefore* $(\mathfrak{F} \times \mathfrak{G})^\alpha \subseteq \mathfrak{F}^\alpha \times \mathfrak{G}^\alpha$

*(vii) if* $\Phi^\alpha \circ \Psi^\alpha$ *exists, then* $\Phi \circ \Psi$ *exists and* $\Phi^\alpha \circ \Psi^\alpha \geq (\Phi \circ \Psi)^\alpha$

*(viii)* $\Phi^\alpha \cap \Psi^\alpha \geq (\Phi \cap \Psi)^\alpha$

*Proof.* (i) Note that $[a, b] = \bigwedge\limits_{(x,y)\in X^2} \Big(a(x,y) \to b(x,y)\Big) = \bigwedge\limits_{(x,y)\in X^2} \Big(a^{-1}(y,x) \to b^{-1}(y,x)\Big) = \bigwedge\limits_{(s,t)\in X^2} \Big(a^{-1}(s,t) \to b^{-1}(s,t)\Big) = [a^{-1}, b^{-1}]$.

(ii) Using (i), $[b,a] = [b^{-1}, a^{-1}]$, and thus $\nu_\Phi(a) = \bigvee_{b\in\Phi}[b,a] = \bigvee_{b\in\Phi}[b^{-1}, a^{-1}] = \bigvee_{c\in\Phi^{-1}}[c, a^{-1}] = \nu_{\Phi^{-1}}(a^{-1})$.

(iii) Fix $z_1, z_2 \in X^*$; then using (ii), $(a^{-1})^\alpha(z_1, z_2) = \nu_{\alpha(z_1)\times\alpha(z_2)}(a^{-1}) = \nu_{\alpha(z_2)\times\alpha(z_1)}(a) = a^\alpha(z_2, z_1) = (a^\alpha)^{-1}(z_1, z_2)$. Hence $(a^{-1})^\alpha = (a^\alpha)^{-1}$.

(iv) Applying Lemma 4.1 (i), since $\Phi \circ \Psi$ exists, $\nu_\Phi(a) \wedge \nu_\Psi(b) = \bigvee_{\substack{c\in\Phi\\ d\in\Psi}} \Big([c,a] \wedge [d,b]\Big) \leq \bigvee_{\substack{c\in\Phi\\ d\in\Psi}}[c\circ d, a\circ b] = \nu_{\Phi\circ\Psi}(a\circ b)$. Hence the result follows.

(v) Fix $z_1, z_2 \in X^*$; it follows from $(iv)$ that $(a^\alpha \circ b^\alpha)(z_1, z_2) = \bigvee_{w\in X^*} \Big(a^\alpha(z_1, w) \wedge b^\alpha(w, z_2)\Big) = \bigvee_{w\in X^*} \Big(\nu_{\alpha(z_1)\times\alpha(w)}(a) \wedge \nu_{\alpha(w)\times\alpha(z_2)}(b)\Big) \leq \nu_{\alpha(z_1)\times\alpha(z_2)}(a\circ b)$ since $(\alpha(z_1)\times\alpha(w))\circ(\alpha(w)\times\alpha(z_2)) = \alpha(z_1)\times\alpha(z_2)$ by Lemma 4.2 (ii). Thus $a^\alpha \circ b^\alpha \leq (a\circ b)^\alpha$.

(vi) Let $x, y \in X^*$. Then employing Lemma 1.1 (iii), $(a\times b)^\alpha(x,y) = \nu_{\alpha(x)\times\alpha(y)}(a\times b) = \bigvee_{\substack{c\in\alpha(x)\\ d\in\alpha(y)}}[c\times d, a\times b] = \bigvee_{\substack{c\in\alpha(x)\\ d\in\alpha(y)}} \bigwedge_{w,z\in X^*} \Big(\big(c(w)\wedge d(z)\big) \to \big(a(w)\wedge b(z)\big)\Big) \geq$
$\bigvee_{\substack{c\in\alpha(x)\\ d\in\alpha(y)}} \bigwedge_{w,z\in X^*} \Big(\big(c(w)\to a(w)\big) \wedge \big(d(z)\to b(z)\big)\Big) =$
$\bigvee_{c\in\alpha(x)}[c,a] \wedge \bigvee_{d\in\alpha(y)}[d,b] = \nu_{\alpha(x)}(a)\wedge\nu_{\alpha(y)}(b) = (a^\alpha\times b^\alpha)(x,y)$.

(vii) Recall that $\Phi\circ\Psi$ exists iff for each $a\in\Phi$, $b\in\Psi$, $\bigvee_{(x,y)\in X^2}(a\circ b)(x,y) = \top$. Employing Lemma 4.5 (v), since $\Phi^\alpha\circ\Psi^\alpha$ exists, $\top = \bigvee_{z_1,z_2\in X^*}(a^\alpha\circ b^\alpha)(z_1,z_2) \leq \bigvee_{z_1,z_2\in X^*}(a\circ b)^\alpha(z_1,z_2) =$

$\bigvee_{x,y\in X}(a\circ b)(x,y) \vee \bigvee_{\substack{z_1,z_2\in X^* \\ z_1 \text{ or } z_2\in(X^*\smallsetminus X)}}(a\circ b)^{\alpha}(z_1,z_2)$. Recall that in the proof of Lemma 3.8 (i), it was established that for any $\beta\in L$ and $\mathfrak{F}\in\mathfrak{F}_L^{\top}(X)$ that $\nu_{\mathfrak{F}}(\beta\mathbf{1}_X)=\beta$. Hence taking $\mathfrak{F}=\alpha(z_1)\times\alpha(z_2)$ and $\beta=\bigvee_{x,y\in X}(a\circ b)(x,y)$, we have $(a\circ b)^{\alpha}(z_1,z_2)=\nu_{\mathfrak{H}_{z_1}\times\mathfrak{H}_{z_2}}(a\circ b)\leq \bigvee_{x,y\in X}(a\circ b)(x,y)$. It follows that $\bigvee_{x,y\in X}(a\circ b)(x,y)=\top$ and thus $\Phi\circ\Psi$ exists. According to Lemma 4.5 (v), $\Phi^{\alpha}\circ\Psi^{\alpha}\geq(\Phi\circ\Psi)^{\alpha}$.

(viii) The verification is clear. □

Assume that $(X,\Lambda)\in|\top\text{-}\mathbf{ULS}|$ and $\alpha$ is a selection map for $(X,\mathcal{C}_{\Lambda})$; define

$$\Lambda^{\alpha}=\{\Gamma\in\mathfrak{F}_L^{\top}((X^*)^2):\Gamma\geq\Phi^{\alpha}\text{ for some }\Phi\in\Lambda\}.$$

**Lemma 4.6.** *Given* $(X,\Lambda)\in|\top\text{-}\mathsf{ULS}|$. *Then*

*(i)* $(X^*,\Lambda^{\alpha})\in|\top\text{-}\mathsf{ULS}|$,

*(ii)* $j:(X,\Lambda)\longrightarrow(X^*,\Lambda^{\alpha})$ *is a dense embedding in* $\top\text{-}\mathsf{ULS}$, *and*

*(iii)* $\mathfrak{H}\in\mathfrak{F}_L^{\top}(X^*)$ *implies that* $\kappa\alpha\mathfrak{H}=\{b\in L^X:b^{\alpha}\in\mathfrak{H}\}\in\mathfrak{F}_L^{\top}(X)$.

*Proof.* (i) If $a\in[(x,x)]$, then $a(x,x)=\top$ and thus $a^{\alpha}(j(x),j(x))=a(x,x)=\top$. Hence $[(x,x)]^{\alpha}\subseteq[j(x),j(x)]$ and thus $[(j(x),j(x))]\in\Lambda^{\alpha}$. Suppose that $\mathfrak{G}\in\mathcal{N}_{\mathcal{C}_{\Lambda}}$; it is shown that $(\mathfrak{G}_{\alpha}\times\mathfrak{G}_{\alpha})^{\alpha}\subseteq[(\langle\mathfrak{G}\rangle,\langle\mathfrak{G}\rangle)]$, where $\alpha(\langle\mathfrak{G}\rangle)=\mathfrak{G}_{\alpha}$. Let $a,b\in\mathfrak{G}_{\alpha}$; then $a\times b$ is a $\top$-filter base member of $\mathfrak{G}_{\alpha}\times\mathfrak{G}_{\alpha}$. Then $(a\times b)^{\alpha}(\langle\mathfrak{G}\rangle,\langle\mathfrak{G}\rangle)=\nu_{\mathfrak{G}_{\alpha}\times\mathfrak{G}_{\alpha}}(a\times b)=\top$ and hence $(a\times b)^{\alpha}\in[(\langle\mathfrak{G}\rangle,\langle\mathfrak{G}\rangle)]$. Therefore $(\mathfrak{G}_{\alpha}\times\mathfrak{G}_{\alpha})^{\alpha}\leq[(\langle\mathfrak{G}\rangle,\langle\mathfrak{G}\rangle)]$ implies that

$[(\langle\mathfrak{G}\rangle,\langle\mathfrak{G}\rangle)] \in \Lambda^\alpha$ and (UL1) is satisfied. Clearly (UL2) is valid. It follows from Lemma 4.5 (iii) that (UL3) holds. Lemma 4.5 (vii) implies that (UL4) is true. Further, (UL5) follows from Lemma 4.5 (viii).

(ii) Recall that if $a \in L^{X^2}$, then $(j\times j)^{\rightarrow}(a) \le a^\alpha$ and $(j\times j)^{\leftarrow}(a^\alpha) = a$. If $\Phi \in \Lambda$, then $(j\times j)^{\Rightarrow}\Phi \ge \Phi^\alpha$ and thus $j : (X,\Lambda) \longrightarrow (X^*,\Lambda^\alpha)$ is uniformly continuous. Moreover, if $\Psi \in \mathfrak{F}_L^{\top}(X^2)$ such that $(j\times j)^{\Rightarrow}\Psi \in \Lambda^\alpha$, then $(j\times j)^{\Rightarrow}\Psi \ge \Phi^\alpha$ for some $\Phi \in \Lambda$. Hence $\Psi \ge (j\times j)^{\Leftarrow}\Phi^\alpha = \Phi$ and $\Psi \in \Lambda$. Therefore $j : (X,\Lambda) \longrightarrow (X^*,\Lambda^\alpha)$ is a uniform embedding. In order to show that the embedding is dense, it suffices to verify that if $\mathfrak{G} \in \mathcal{N}_{\mathcal{C}_\Lambda}$, then $j^{\Rightarrow}(\mathfrak{G}_\alpha) \times [\langle\mathfrak{G}\rangle] \ge (\mathfrak{G}_\alpha \times \mathfrak{G}_\alpha)^\alpha$. Let us show that if $a, b \in \mathfrak{G}_\alpha$, then $(a\times b)^\alpha \ge j^{\rightarrow}a \times \mathbf{1}_{\{\langle\mathfrak{G}\rangle\}}$. Fix $z_1, z_2 \in X^*$ and note that $(j^{\rightarrow}a \times \mathbf{1}_{\{\langle\mathfrak{G}\rangle\}})(z_1,z_2) = \bot$ whenever either $z_1 \notin j(X)$ or $z_2 \ne \langle\mathfrak{G}\rangle$. Assume that $z_1 = j(x)$ and $z_2 = \langle\mathfrak{G}\rangle$; then $(j^{\rightarrow}a \times \mathbf{1}_{\{\langle\mathfrak{G}\rangle\}})(j(x),\langle\mathfrak{G}\rangle) = a(x)$. Also, $(a\times b)^\alpha(j(x),\langle\mathfrak{G}\rangle) = \nu_{[x]\times\mathfrak{G}_\alpha}(a\times b) = \bigvee_{c\in\mathfrak{G}_\alpha}[\mathbf{1}_{\{x\}}\times c, a\times b] \ge [\mathbf{1}_{\{x\}}\times b, a\times b] = \bigwedge_{s,t\in X}\Big(\big(\mathbf{1}_{\{x\}}\times b\big)(s,t) \to (a\times b)(s,t)\Big) = \bigwedge_{t\in X}\Big(b(t) \to a(x)\wedge b(t)\Big) \ge a(x)$. It follows that $(a\times b)^\alpha(j(x),\langle\mathfrak{G}\rangle) \ge (j^{\rightarrow}a \times \mathbf{1}_{\{\langle\mathfrak{G}\rangle\}})(j(x),\langle\mathfrak{G}\rangle)$ and hence $j^{\Rightarrow}(\mathfrak{G}_\alpha) \times [\langle\mathfrak{G}\rangle] \ge (\mathfrak{G}_\alpha\times\mathfrak{G}_\alpha)^\alpha$ and thus $j^{\Rightarrow}\mathfrak{G}_\alpha \xrightarrow{q_{\mathcal{C}_{\Lambda^\alpha}}} \langle\mathfrak{G}\rangle$. Then $(X,\Lambda)$ is uniformly embedded in $(X^*,\Lambda^\alpha)$ as a dense subspace.

(iii) Notice that $b^\alpha = e_b \circ \alpha$ and hence $\{b \in L^X : b^\alpha \in \mathfrak{H}\} = \kappa\alpha\mathfrak{H}$. If $b \in \kappa\alpha\mathfrak{H}$, then $b^\alpha \in \mathfrak{H}$ implies that $\top = \bigvee_{z\in X^*} b^\alpha(z) = \bigvee_{x\in X} b(x) \vee \bigvee_{\mathfrak{G}\in\mathcal{N}} \nu_{\mathfrak{G}_\alpha}(b)$. Since $\nu_{\mathfrak{G}_\alpha}(b) \le \bigvee_{x\in X} b(x)$ (Lemma 3.8 (i) with $a = b$), it follows that $\bigvee_{x\in X} b(x) = \top$. Also, if $b_1, b_2 \in \kappa\alpha\mathfrak{H}$, then $(b_1\wedge b_2)^\alpha = {b_1}^\alpha \wedge {b_2}^\alpha \in \mathfrak{H}$ and thus $b_1 \wedge b_2 \in \kappa\alpha\mathfrak{H}$. Finally, assume that $c \in L^{X^2}$ such that $\bigvee_{b\in\kappa\alpha\mathfrak{H}}[b,c] = \top$. According to Lemma 3.8 (ii), $\top = \bigvee_{b\in\kappa\alpha\mathfrak{H}}[b,c] = \bigvee_{b\in\kappa\alpha\mathfrak{H}}[b^\alpha,c^\alpha] \le \bigvee_{d\in\mathfrak{H}}[d,c^\alpha]$ and hence $c^\alpha \in \mathfrak{H}$ implies that $c \in \kappa\alpha\mathfrak{H}$. Therefore $\kappa\alpha\mathfrak{H} \in \mathfrak{F}_L^{\top}(X)$.

□

**Theorem 4.3.** *Assume that* $(X,\Lambda) \in |\top\text{-}\mathbf{ULS}|$ *and let* $(X,\mathcal{C}_\Lambda)$ *denote the induced* $\top$*-Cauchy* *space. If $\mathcal{C}_{\Lambda^\alpha}$ and $(\mathcal{C}_\Lambda)^\alpha$ possess the same $\top$-ultrafilters on $X^*$ for some selection map $\alpha$ on $(X, \mathcal{C}_\Lambda)$, then $\big((X^*, \Lambda^\alpha), j\big)$ is a completion of $(X, \Lambda)$ in $\top$-**ULS**. Moreover, $\mathcal{C}_{\Lambda^\alpha} = (\mathcal{C}_\Lambda)^\alpha$ iff $\mathfrak{H} \in \mathcal{C}_{\Lambda^\alpha}$ implies that $\kappa\alpha\mathfrak{H} \in \mathcal{C}_\Lambda$.*

*Proof.* According to Lemma 4.6 (i, ii), $(X^*, \Lambda^\alpha) \in |\top\text{-}\mathbf{ULS}|$ and $j : (X, \Lambda) \longrightarrow (X^*, \Lambda^\alpha)$ is a dense embedding in $\top$-**ULS**. It must be shown that $(X^*, \Lambda^\alpha)$ is complete in $\top$-**ULS**. By Lemma 4.5 (vi) it follows that if $\mathfrak{K} \in \mathcal{C}_\Lambda$, then $\mathfrak{K}^\alpha \times \mathfrak{K}^\alpha \geq (\mathfrak{K} \times \mathfrak{K})^\alpha \in \Lambda^\alpha$ and thus $\mathfrak{K}^\alpha \in \mathcal{C}_{\Lambda^\alpha}$. Hence $(\mathcal{C}_\Lambda)^\alpha \subseteq \mathcal{C}_{\Lambda^\alpha}$ always holds.

Assume that $\mathfrak{H} \in \mathcal{C}_{\Lambda^\alpha}$ and let $\mathfrak{L} \geq \mathfrak{H}$ be a $\top$-ultrafilter on $X^*$. Then $\mathfrak{L} \in \mathcal{C}_{\Lambda^\alpha}$, and by hypothesis, $\mathfrak{L} \in (\mathcal{C}_\Lambda)^\alpha$. Since $(X^*, (\mathcal{C}_\Lambda)^\alpha)$ is complete, $\mathfrak{L} \cap [z] \in (\mathcal{C}_\Lambda)^\alpha \subseteq \mathcal{C}_{\Lambda^\alpha}$ for some $z \in X^*$. It follows that $(\mathfrak{L} \cap [z]) \vee \mathfrak{H}$ exists and hence $\mathfrak{H} \cap [z] \in \mathcal{C}_{\Lambda^\alpha}$. Therefore $(X^*, \Lambda^\alpha)$ is complete, and thus $\big((X^*, \Lambda^\alpha), j\big)$ is a completion of $(X, \mathcal{C})$ in $\top$-**ULS**.

Finally, if $\mathcal{C}_{\Lambda^\alpha} = (\mathcal{C}_\Lambda)^\alpha$ and $\mathfrak{H} \in \mathcal{C}_{\Lambda^\alpha} = (\mathcal{C}_\Lambda)^\alpha$, then $\mathfrak{H} \geq \mathfrak{K}^\alpha$ for some $\mathfrak{K} \in \mathcal{C}_\Lambda$. Hence $\kappa\alpha\mathfrak{H} \geq \mathfrak{K}$ and thus $\kappa\alpha\mathfrak{H} \in \mathcal{C}_\Lambda$. Conversely, suppose that $\mathfrak{H} \in \mathcal{C}_{\Lambda^\alpha}$ and $\kappa\alpha\mathfrak{H} = \mathfrak{K} \in \mathcal{C}_\Lambda$; then $\mathfrak{H} \geq \mathfrak{K}^\alpha \in (\mathcal{C}_\Lambda)^\alpha$. Hence $(\mathcal{C}_\Lambda)^\alpha = \mathcal{C}_{\Lambda^\alpha}$. □

**Definition 4.2.** Assume that $(X, \Lambda) \in |\top\text{-}\mathbf{ULS}|$; then $(X, \Lambda)$ is said to be **relatively full** in $\top$-**ULS** provided that $(X, \mathcal{C}_\Lambda)$ is relatively full in $\top$-**Chy**.

Whenever $(X, \Lambda)$ is relatively full we may choose the selection map $\alpha$ which sends $x \mapsto [x]$, $x \in X$ and $\langle \mathfrak{G} \rangle \mapsto \mathfrak{G}_{\min}$, $\mathfrak{G} \in \mathcal{N}_{\mathcal{C}_\Lambda}$. For this special selection map we will denote $a^\alpha, \mathfrak{F}^\alpha$, $\kappa\alpha\mathfrak{H}$ and $\Lambda^\alpha$, respectively, by $\widetilde{a}, \widetilde{\mathfrak{F}}$, $\check{\mathfrak{H}}$ and $\widetilde{\Lambda}$. The next result follows from Theorem 4.3.

**Corollary 4.1.** *Assume that $(X, \Lambda) \in |\top\text{-}\mathbf{ULS}|$ is relatively full, and let $(X, \mathcal{C}_\Lambda)$ denote the induced $\top$-Cauchy space. Let $\big((X^*, \widetilde{\mathcal{C}_\Lambda}), j\big)$ be the completion of $(X, \mathcal{C}_\Lambda)$ in $\top$-**Chy**. If $\mathcal{C}_{\widetilde{\Lambda}}$ and $\widetilde{\mathcal{C}_\Lambda}$ possess the same $\top$-ultrafilters on $X^*$, then $\big((X^*, \widetilde{\Lambda}), j\big)$ is a completion of $(X, \Lambda)$ in $\top$-**ULS**. Moreover, $\mathcal{C}_{\widetilde{\Lambda}} = \widetilde{\mathcal{C}_\Lambda}$ iff $\mathfrak{H} \in \mathcal{C}_{\widetilde{\Lambda}}$ implies that $\check{\mathfrak{H}} \in \mathcal{C}_\Lambda$.*

An elementary example of a completion is given below. First, a lemma which may be of independent interest is presented.

**Lemma 4.7.** *Assume that $L$ is a complete Boolean algebra. Suppose that $(X, \mathcal{C}) \in |\top\text{-}\mathsf{Chy}|$ is not complete, and let $\alpha$ be a selection map which chooses a $\top$-ultrafilter $\mathfrak{G}_\alpha$ from each $\langle \mathfrak{G} \rangle$, $\mathfrak{G} \in \mathcal{N}$. If $\mathfrak{H}$ is a $\top$-ultrafilter on $X^*$, then there exists a $\top$-ultrafilter $\mathfrak{F}$ on $X$ such that $\mathfrak{F}^\alpha \subseteq \mathfrak{H}$.*

*Proof.* Since $\mathfrak{H}$ is a $\top$-ultrafilter, it follows from results due to Höhle ([10], [11]) that $\nu_\mathfrak{H}$ is a stratified $L$-ultrafilter on $X^*$. Define for each $a \in L^X$, $\mu_\mathfrak{H}(a) = \nu_\mathfrak{H}(a^\alpha)$. Note that $\mu_\mathfrak{H}(\bot \mathbf{1}_X) = \nu_\mathfrak{H}(\bot \mathbf{1}_{X^*}) = \bot$, $\mu_\mathfrak{H}(\beta \mathbf{1}_X) = \nu_\mathfrak{H}\Big((\beta \mathbf{1}_X)^\alpha\Big) \geq \nu_\mathfrak{H}(\beta \mathbf{1}_{X^*}) \geq \beta$ and $\mu_\mathfrak{H}(a \wedge b) = \nu_\mathfrak{H}((a \wedge b)^\alpha) = \nu_\mathfrak{H}(a^\alpha \wedge b^\alpha) = \nu_\mathfrak{H}(a^\alpha) \wedge \nu_\mathfrak{H}(b^\alpha) = \mu_\mathfrak{H}(a) \wedge \mu_\mathfrak{H}(b)$, for each $a, b \in L^X$ and $\beta \in L$. Hence $\mu_\mathfrak{H}$ is a stratified $L$-filter on $X$.

According to Höhle ([10]), $\mu_\mathfrak{H}$ is a stratified $L$-ultrafilter on $X$ iff for each $a \in L^X$, $\mu_\mathfrak{H}(a) = \mu_\mathfrak{H}(a \to \mathbf{1}_\varnothing) \to \bot$. He also shows that $\nu_\mathfrak{G}(a \to \mathbf{1}_\varnothing) = \nu_\mathfrak{G}(a) \to \bot$ whenever $\mathfrak{G}$ is a $\top$-ultrafilter on $X$. As before, we denote $\alpha(\langle \mathfrak{G} \rangle) = \mathfrak{G}_\alpha$ for $\mathfrak{G} \in \mathcal{N}$. Then $(a \to \mathbf{1}_\varnothing)^\alpha(\langle \mathfrak{G} \rangle) = \nu_{\mathfrak{G}_\alpha}(a \to \mathbf{1}_\varnothing) = \nu_{\mathfrak{G}_\alpha}(a) \to \bot = a^\alpha(\langle \mathfrak{G} \rangle) \to \bot = (a^\alpha \to \mathbf{1}_\varnothing)(\langle \mathfrak{G} \rangle)$ and thus $(a \to \mathbf{1}_\varnothing)^\alpha = a^\alpha \to \mathbf{1}_\varnothing$. Then $\mu_\mathfrak{H}(a) = \nu_\mathfrak{H}(a^\alpha) = \nu_\mathfrak{H}(a^\alpha \to \mathbf{1}_\varnothing) \to \bot = \nu_\mathfrak{H}\Big((a \to \mathbf{1}_\varnothing)^\alpha\Big) \to \bot = \mu_\mathfrak{H}(a \to \mathbf{1}_\varnothing) \to \bot$, and hence $\mu_\mathfrak{H}$ is a stratified $L$-ultrafilter on $X$.

Since $L$ is a complete Boolean algebra, it follows again from Höhle ([10], [11]) that $\mathfrak{F} \mapsto \nu_\mathfrak{F}$ defines a bijection between the $\top$-ultrafilters and the stratified $L$-ultrafilters on $X$. Then $\mathfrak{F} = \{a \in L^X : \mu_\mathfrak{H}(a) = \top\}$ is a $\top$-ultrafilter on $X$. Further, $a \in \mathfrak{F}$ iff $\nu_\mathfrak{H}(a^\alpha) = \top$ iff $a^\alpha \in \mathfrak{H}$, and it follows that $\mathfrak{F}^\alpha \subseteq \mathfrak{H}$. $\square$

A $\top$-uniform limit space $(X,\Lambda)$ is said to be **totally bounded** whenever each $\top$-ultrafilter on $X$ is $\Lambda$-Cauchy, that is, each $\top$-ultrafilter $\mathfrak{F} \in \mathcal{C}_\Lambda$. Also, $(X,\Lambda)$ is said to be **compact** if every $\top$-ultrafilter on $X$ converges in $(X, q_{\mathcal{C}_\Lambda})$. Let us conclude this subsection with the following restricted example.

**Example 4.1.** Suppose that $L$ is a complete Boolean algebra and $(X,\Lambda) \in |\top\text{-}\mathbf{ULS}|$ is totally bounded but not complete. Assume that $\alpha$ is a selection map such that $\alpha(\langle\mathfrak{G}\rangle) = \mathfrak{G}_\alpha$ is a $\top$-ultrafilter in $\langle\mathfrak{G}\rangle$, for each $\mathfrak{G} \in \mathcal{N}_{\mathcal{C}_\Lambda}$. Let $\big((X^*, (\mathcal{C}_\Lambda)^\alpha), j\big)$ denote the corresponding completion of $(X, \mathcal{C}_\Lambda)$ in $\top$-**Chy**. Then $\big((X^*, \Lambda^\alpha), j\big)$ is a completion of $(X,\Lambda)$ in $\top$-**ULS** which is also compact.

*Proof.* According to Lemma 4.6 (ii), $j : (X,\Lambda) \longrightarrow (X^*, \Lambda^\alpha)$ is a dense embedding in $\top$-**ULS**. Since compactness of $(X^*, \Lambda^\alpha)$ implies completeness, it suffices to show $(X^*, \Lambda^\alpha)$ is compact. Let $\mathfrak{H}$ be a $\top$-ultrafilter on $X^*$; then by Lemma 4.7 there exists a $\top$-ultrafilter $\mathfrak{F}$ on $X$ such that $\mathfrak{F}^\alpha \subseteq \mathfrak{H}$. Since $(X,\Lambda)$ is totally bounded, $\mathfrak{F} \in \mathcal{C}_\Lambda$ and thus $\mathfrak{F}^\alpha \in (\mathcal{C}_\Lambda)^\alpha$. It follows that $\mathfrak{H} \in (\mathcal{C}_\Lambda)^\alpha$ and since $(X^*, (\mathcal{C}_\Lambda)^\alpha)$ is complete, $\mathfrak{H} \cap [z] \in (\mathcal{C}_\Lambda)^\alpha$ for some $z \in X^*$. As shown in the proof of Theorem 4.3 $(\mathcal{C}_\Lambda)^\alpha \subseteq \mathcal{C}_{\Lambda^\alpha}$ is always valid. It follows that $\mathfrak{H}$ converges in $(X^*, q_{\mathcal{C}_{\Lambda^\alpha}})$, and hence $(X^*, \Lambda^\alpha)$ is both compact and complete. □

The authors are unsure as to whether or not $(\mathcal{C}_\Lambda)^\alpha = \mathcal{C}_{\Lambda^\alpha}$ in Example 4.1.

## An Alternate Approach to Completions

In the classical case, if $(X, \mathcal{V})$ is a uniform space and $\mathcal{F} \times \mathcal{F} \geq \mathcal{V}$, then $\mathcal{F}$ is a Cauchy filter and $\mathcal{V}(\mathcal{F}) = \{A \subseteq X : V(F) \subseteq A \text{ for some } F \in \mathcal{F} \text{ and } V \in \mathcal{V}\}$, where $V(F) = \{y \in X : (x,y) \in V \text{ for some } x \in F\}$, is the smallest Cauchy filter on $X$ contained in $\mathcal{F}$. Our aim in this subsection is to outline an extension of this technique to the lattice context.

Fix $(X, \Lambda) \in |\top\text{-}\mathbf{ULS}|$ and assume that $\mathfrak{F} \times \mathfrak{F} \geq \Phi \in \Lambda$. If $a \in \Phi$ and $b \in \mathfrak{F}$, define $a(b) \in L^X$ as follows:

$$a(b)(y) = \bigvee_{x \in X} a(x, y) \wedge b(x), \quad y \in X.$$

Denote $\mathcal{B} = \{a(b) : a \in \Phi, b \in \mathfrak{F}\}$; then $\mathcal{B}$ is a $\top$-filter base and let $\Phi(\mathfrak{F})$ denote the generated $\top$-filter. The following lemma lists some extensions of well-known classical results to the lattice setting. The proof is omitted.

**Lemma 4.8.** *Assume that $(X, \Lambda) \in |\top\text{-}\mathbf{ULS}|$ and $\mathfrak{F} \times \mathfrak{F} \geq \Phi = \Phi^{-1} \in \Lambda$. Then,*

*(i) $\mathcal{B}$ is a $\top$-filter base,*

*(ii) $\Phi(\mathfrak{F}) \times \Phi(\mathfrak{F}) = \Phi \circ (\mathfrak{F} \times \mathfrak{F}) \circ \Phi$, and*

*(iii) $\Phi(\mathfrak{F}) \vee \mathfrak{F}$ exists.*

According to Lemma 4.8 (ii, iii), $\Phi(\mathfrak{F})$ and $\Phi(\mathfrak{F}) \cap \mathfrak{F}$ belong to $\mathcal{C}_\Lambda$. Our final completion result listed below is not entirely satisfactory since the characterization is not given completely in terms of the underlying $\top$-uniform limit space. Here $(X^*, \Lambda^\alpha) \in |\top\text{-}\mathbf{ULS}|$ denotes the space given in Lemma 4.6.

**Theorem 4.4.** *Suppose that $(X, \Lambda) \in |\top\text{-}\mathbf{ULS}|$ and $\alpha$ is a selection map for $(X, \mathcal{C}_\Lambda)$. Then $\big((X^*, \Lambda^\alpha), j\big)$ is a completion of $(X, \Lambda)$ in $\top$-$\mathbf{ULS}$ iff for each $\mathfrak{H} \times \mathfrak{H} \geq \Phi^\alpha$, for some $\Phi = \Phi^{-1} \in \Lambda$, there exists an $\mathfrak{L} \in \mathcal{C}_{\Lambda^\alpha}$ such that $\mathfrak{L} \leq \Phi^\alpha(\mathfrak{H}) \cap \mathfrak{H}$ and $j^{\Leftarrow}\mathfrak{L}$ exists.*

*Proof.* Assume that $\big((X^*, \Lambda^\alpha), j\big)$ is a completion of $(X, \Lambda)$ in $\top$-$\mathbf{ULS}$ and $\mathfrak{H} \times \mathfrak{H} \geq \Phi^\alpha$ for some $\Phi = \Phi^{-1} \in \Lambda$. Then $\mathfrak{H} \xrightarrow{p} z$, for some $z \in X^*$, where $p = q_{\mathcal{C}_{\Lambda^\alpha}}$. Since $j(X)$ is dense in $X^*$, choose $\mathfrak{K} \in \mathfrak{F}_L^\top(X)$ such that $j^{\Rightarrow}\mathfrak{K} \xrightarrow{p} z$. According to Lemma 4.8 (iii), $\Phi^\alpha(\mathfrak{H}) \cap \mathfrak{H} \cap [z] \in \mathcal{C}_{\Lambda^\alpha}$, and it follows that $\mathfrak{L} = j^{\Rightarrow}\mathfrak{K} \cap \Phi^\alpha(\mathfrak{H}) \cap \mathfrak{H} \cap [z] \in \mathcal{C}_{\Lambda^\alpha}$. Then $\mathfrak{L} \leq \Phi^\alpha(\mathfrak{H}) \cap \mathfrak{H}$ and $\mathfrak{L} \leq j^{\Rightarrow}\mathfrak{K}$ implies that $j^{\Leftarrow}\mathfrak{L}$ exits.

Conversely, by Lemma 4.6 (ii), $j : (X, \Lambda) \longrightarrow (X^*, \Lambda^\alpha)$ is a dense embedding in $\top$-**ULS**. It remains to show that $(X^*, \Lambda^\alpha)$ is complete. Suppose that $\mathfrak{H} \times \mathfrak{H} \geq \Phi^\alpha$ for some $\Phi = \Phi^{-1} \in \Lambda$. Then there exists $\mathfrak{L} \in \mathcal{C}_{\Lambda^\alpha}$ such that $\mathfrak{L} \leq \Phi^\alpha(\mathfrak{H}) \cap \mathfrak{H}$ and $\mathfrak{M} = j^{\Leftarrow}\mathfrak{L}$ exists. Note that $\mathfrak{M} \in \mathcal{C}_\Lambda$. If $\mathfrak{M} \xrightarrow{q_\Lambda} x$, then $j^{\Rightarrow}\mathfrak{M} \xrightarrow{p} j(x)$ and $j^{\Rightarrow}\mathfrak{M} \cap [j(x)] \cap \Phi^\alpha(\mathfrak{H}) \cap \mathfrak{H} \in \mathcal{C}_{\Lambda^\alpha}$. Then $\mathfrak{H} \xrightarrow{p} j(x)$. A similar argument shows that if $\mathfrak{M}$ fails to $q_\Lambda$-converge, then $\langle \mathfrak{M} \rangle \in X^*$ and $\mathfrak{H} \xrightarrow{p} \langle \mathfrak{M} \rangle$. Hence $(X^*, \Lambda^\alpha)$ is complete. □

**Corollary 4.2.** *Suppose that $(X, \Lambda) \in |\top$-**ULS**$|$ is relatively full. Then $\left((X^*, \widetilde{\Lambda}), j\right)$ is a completion of $(X, \Lambda)$ in $\top$-**ULS** iff for each $\mathfrak{H} \times \mathfrak{H} \geq \widetilde{\Phi}$, for some $\Phi = \Phi^{-1} \in \Lambda$, there exists an $\mathfrak{L} \in \mathcal{C}_{\widetilde{\Lambda}}$ such that $\mathfrak{L} \leq \widetilde{\Phi}(\mathfrak{H}) \cap \mathfrak{H}$ and $j^{\Leftarrow}\mathfrak{L}$ exists.*

# CHAPTER 5: STRICT $\top$-EMBEDDINGS

For a fixed $\top$-limit space, under suitable conditions, an order preserving injection between the set of all equivalence classes of all strict $T_3$-compactifications of the $\top$-limit space and all the totally bounded $\top$-Cauchy spaces which induce the $\top$-limit space and have a strict $T_3$-completion is given. Unfortunately, the author was unable to determine whether or not the injection is a bijection. In the case that the underlying lattice is a complete Boolean algebra, the injection is in fact a bijection. Further, a characterization as to when a totally bounded $\top$-Cauchy space has a $T_3$ (strict $T_3$)-completion is an open problem.

## $T_3$-Embeddings

Suppose that $(X, q) \in |\top\text{-}\mathbf{Lim}|$ and $a \in L^X$; recall that the closure of $a$ is defined by $\overline{a}(x) = \bigvee\{\nu_{\mathfrak{F}}(a) : \mathfrak{F} \xrightarrow{q} x\}$, $x \in X$. If $\mathfrak{H} \in \mathfrak{F}_L^{\top}(X)$, then $\overline{\mathfrak{H}}$ denotes the $\top$-filter on $X$ whose $\top$-filter base is $\{\overline{c} : c \in \mathfrak{H}\}$. It is shown in Lemma 2.14 that if $\mathcal{B}$ is a $\top$-filter base for $\mathfrak{H}$ then $\overline{\mathcal{B}} = \{\overline{b} : b \in \mathcal{B}\}$ is also a $\top$-filter base for $\overline{\mathfrak{H}}$. Fang and Yue [5] defined regularity of $(X, q) \in |\top\text{-}\mathbf{Lim}|$ in terms of a diagonal axiom. This definition is shown in Theorem 2.6 to be equivalent to $\overline{\mathfrak{F}} \xrightarrow{q} x$ whenever $\mathfrak{F} \xrightarrow{q} x$. Further, define $(X, \mathcal{C}) \in |\top\text{-}\mathbf{Chy}|$ to be regular provided that $\overline{\mathfrak{F}} \in \mathcal{C}$ whenever $\mathfrak{F} \in \mathcal{C}$. Moreover, $(X, \mathcal{C}) \in |\top\text{-}\mathbf{Chy}|$ is said to be $T_3$ provided it is $T_2$ and regular. A similar definition holds for objects of $\top\text{-}\mathbf{Lim}$ . Suppose that $\theta : X \longrightarrow (Y, p) \in |\top\text{-}\mathbf{Lim}|$ is a dense injection. Since $\theta(X)$ is dense in $Y$, for each $y \in Y \smallsetminus \theta(X)$, choose a $\top$-ultrafilter $\mathfrak{G}_y$ on $X$ such that $\theta^{\Rightarrow}\mathfrak{G}_y \xrightarrow{p} y$. Define for each $a \in L^X$, $\hat{a} \in L^Y$ by

$$\hat{a}(y) = \begin{cases} a(x), & y = \theta(x) \\ \nu_{\mathfrak{G}_y}(a), & y \in Y \smallsetminus \theta(X) \end{cases}.$$

Observe that $\hat{a} \wedge \hat{b} = \widehat{a \wedge b}$ for each $a, b \in L^X$, and if $\mathfrak{F} \in \mathfrak{F}_L^\top(X)$ we let $\hat{\mathfrak{F}}$ denote the $\top$-filter on $Y$ whose $\top$-filter base is $\{\hat{b} : b \in \mathfrak{F}\}$.

**Lemma 5.1.** *Assume that $L$ is a complete Boolean algebra, $\theta : X \longrightarrow (Y, p) \in |\top\text{-}\mathbf{Lim}|$ is a dense injection and $\mathfrak{H}$ is a $\top$-ultrafilter on $Y$. Then there exists a $\top$-ultrafilter $\mathfrak{F}$ on $X$ such that $\overline{\theta^{\Rightarrow}\mathfrak{F}} \subseteq \hat{\mathfrak{F}} \subseteq \mathfrak{H}$ and for each $b \in L^X$, $\nu_{\mathfrak{F}}(b) = \nu_{\mathfrak{H}}(\hat{b})$.*

*Proof.* The proof is a slight modification of an argument used in the proof of Theorem 2.11. For each $y \in Y \smallsetminus \theta(X)$, choose a $\top$-ultrafilter $\mathfrak{G}_y \xrightarrow{p} y$ and for each $b \in L^X$ define $\hat{b}$ as above. Define $\mu(b) = \nu_{\mathfrak{H}}(\hat{b})$ for each $b \in L^X$; the argument given in the proof of Theorem 2.11 shows that $\mu$ is a stratified $L$-ultrafilter on $X$. Since $L$ is a complete Boolean algebra, it follows from Theorem 2.1 that $\mu = \nu_{\mathfrak{F}}$, where $\mathfrak{F} = \{b \in L^X : \mu(b) = \top\}$ is a $\top$-ultrafilter on $X$. Note that $b \in \mathfrak{F}$ iff $\hat{b} \in \mathfrak{H}$ and thus $\hat{\mathfrak{F}} \subseteq \mathfrak{H}$; further, $\nu_{\mathfrak{F}}(b) = \mu(b) = \nu_{\mathfrak{H}}(\hat{b})$. Since $\hat{b} \leq \overline{\theta^{\rightarrow} b}$ for each $b \in L^X$, $\overline{\theta^{\Rightarrow}\mathfrak{F}} \subseteq \hat{\mathfrak{F}} \subseteq \mathfrak{H}$. □

Assume that $\theta : X \longrightarrow (Y, p) \in |\top\text{-}\mathbf{Lim}|$ is a dense injection. Define for each $a \in L^X$, $a^\dagger \in L^Y$ as follows:

$$a^\dagger(y) = \bigvee\{\nu_{\mathfrak{F}}(a) : \theta^{\Rightarrow}\mathfrak{F} \xrightarrow{p} y\},\ y \in Y.$$

Observe that $a^\dagger \leq \overline{\theta^{\rightarrow} a}$. Indeed, if $\theta^{\Rightarrow}\mathfrak{F} \xrightarrow{p} y$, then $\nu_{\mathfrak{F}}(a) = \bigvee_{b \in \mathfrak{F}} [b, a] \leq \bigvee_{b \in \mathfrak{F}} [\theta^{\rightarrow} b, \theta^{\rightarrow} a] = \nu_{\theta^{\Rightarrow}\mathfrak{F}}(\theta^{\rightarrow} a) \leq \bigvee\{\nu_{\mathfrak{H}}(\theta^{\rightarrow} a) : \mathfrak{H} \xrightarrow{p} y\} = \overline{\theta^{\rightarrow} a}(y)$. Hence $a^\dagger(y) = \bigvee\{\nu_{\mathfrak{F}}(a) : \theta^{\Rightarrow}\mathfrak{F} \xrightarrow{p} y\} \leq \overline{\theta^{\rightarrow} a}(y)$ and thus $a^\dagger \leq \overline{\theta^{\rightarrow} a}$.

**Definition 5.1.** Suppose that $\theta : X \longrightarrow (Y, p) \in |\top\text{-}\mathbf{Lim}|$ is a dense injection. Consider the following axioms:

(S1) $a^\dagger = \overline{\theta^{\rightarrow} a}$ for each $a \in L^X$

(S2) for each $\top$-filter $\mathfrak{H} \stackrel{p}{\longrightarrow} y$, there exists a $\top$-filter $\mathfrak{F}$ on $X$ such that $\theta^{\Rightarrow}\mathfrak{F} \stackrel{p}{\longrightarrow} y$ and $\overline{\theta^{\Rightarrow}\mathfrak{F}} \subseteq \mathfrak{H}$

(S3) Same as (S2) with $\mathfrak{H}$ and $\mathfrak{F}$ being $\top$-ultrafilters.

The map $\theta$ is said to be **strict** whenever (S1) and (S2) are satisfied.

**Lemma 5.2.** *Suppose that $L$ is a complete Boolean algebra and $\theta : X \longrightarrow (Y, p) \in |\top\text{-}\mathbf{Lim}|$ is a dense injection and $(Y, p)$ is compact $T_3$. Then $\theta$ obeys (S1) and (S3).*

*Proof.* Assume that $\mathfrak{H}$ is a $\top$-ultrafilter on $Y$ such that $\mathfrak{H} \stackrel{p}{\longrightarrow} y$. Employing Lemma 5.1, there exists a $\top$-ultrafilter $\mathfrak{F}$ on $X$ which satisfies $\overline{\theta^{\Rightarrow}\mathfrak{F}} \subseteq \hat{\mathfrak{F}} \subseteq \mathfrak{H}$ and $\nu_{\mathfrak{F}}(a) = \nu_{\mathfrak{H}}(\hat{a})$ for each $a \in L^X$. Since $(Y, p)$ is compact $T_3$, it follows that $\theta^{\Rightarrow}\mathfrak{F} \stackrel{p}{\longrightarrow} y$ and thus (S3) is valid. Moreover, $a^{\dagger}(y) = \bigvee\{\nu_{\mathfrak{K}}(a) : \theta^{\Rightarrow}\mathfrak{K} \stackrel{p}{\longrightarrow} y\} \geq \nu_{\mathfrak{F}}(a) = \nu_{\mathfrak{H}}(\hat{a}) \geq \nu_{\mathfrak{H}}(\theta^{\rightarrow}a)$. Therefore $a^{\dagger}(y) \geq \bigvee\{\nu_{\mathfrak{H}}(\theta^{\rightarrow}a) : \mathfrak{H} \stackrel{p}{\longrightarrow} y\} = \overline{\theta^{\rightarrow}a}(y)$. Since $a^{\dagger}(y) \leq \overline{\theta^{\rightarrow}a}(y)$ always holds, it follows that (S1) is satisfied. □

Assume that $(X, \mathcal{C}) \in |\top\text{-}\mathbf{Chy}|$; let $X^* = X \cup \{\langle\mathfrak{G}\rangle : \mathfrak{G} \in \mathcal{N}\}$ and let $j : X \longrightarrow X^*$ denote the natural injection. Define the following $\top$-limit structure $\sigma$ on $X^*$:

$$\mathfrak{H} \stackrel{\sigma}{\longrightarrow} j(x) \quad \text{iff} \quad \mathfrak{H} \geq j^{\Rightarrow}\mathfrak{F} \text{ for some } \mathfrak{F} \stackrel{qc}{\longrightarrow} x$$

$$\mathfrak{H} \stackrel{\sigma}{\longrightarrow} \langle\mathfrak{G}\rangle \quad \text{iff} \quad \mathfrak{H} \geq j^{\Rightarrow}\mathfrak{G} \cap [\langle\mathfrak{G}\rangle] \text{ for some } \mathfrak{G} \in \mathcal{N}.$$

Then $(X^*, \sigma) \in |\top\text{-}\mathbf{Lim}|$ and $j : X \longrightarrow (X^*, \sigma)$ is a dense injection. If $a \in L^X$, then define $a^{\dagger}(y) = \bigvee\{\nu_{\mathfrak{F}}(a) : j^{\Rightarrow}\mathfrak{F} \stackrel{\sigma}{\longrightarrow} y\}$, $y \in X^*$. Moreover, suppose that $\mathfrak{F} \in \mathfrak{F}_L^{\top}(X)$ and denote $\mathcal{B} = \{a^{\dagger} : a \in \mathfrak{F}\}$. It is shown that $\mathcal{B}$ is a $\top$-filter base on $X^*$. Indeed, if $a \in \mathfrak{F}$, then $\bigvee_{y \in X^*} a^{\dagger}(y) \geq \bigvee_{x \in X} a^{\dagger}(j(x)) \geq \bigvee_{x \in X} a(x) = \top$ and thus ($\top$B1) is satisfied. Next, assume that $b_1, b_2 \in \mathfrak{F}$; then $(b_1 \wedge b_2)^{\dagger}(y) = \bigvee\{\nu_{\mathfrak{F}}(b_1 \wedge b_2) : j^{\Rightarrow}\mathfrak{F} \stackrel{\sigma}{\longrightarrow} y\} = \bigvee\{\nu_{\mathfrak{F}}(b_1) \wedge \nu_{\mathfrak{F}}(b_2) : j^{\Rightarrow}\mathfrak{F} \stackrel{\sigma}{\longrightarrow}$

$y\} \leq \bigvee\{\nu_{\mathfrak{K}}(b_1) \wedge \nu_{\mathfrak{L}}(b_2) : j^{\Rightarrow}\mathfrak{K}, j^{\Rightarrow}\mathfrak{L} \xrightarrow{\sigma} y\} = \bigvee\{\nu_{\mathfrak{K}}(b_1) : j^{\Rightarrow}\mathfrak{K} \xrightarrow{\sigma} y\} \wedge \bigvee\{\nu_{\mathfrak{L}}(b_2) : j^{\Rightarrow}\mathfrak{L} \xrightarrow{\sigma} y\} = b_1^{\dagger}(y) \wedge b_2^{\dagger}(y) = (b_1^{\dagger} \wedge b_2^{\dagger})(y)$ and thus $(b_1 \wedge b_2)^{\dagger} \leq b_1^{\dagger} \wedge b_2^{\dagger}$. Hence, if $b_1, b_2 \in \mathfrak{F}$, $\bigvee\limits_{a\in\mathfrak{F}}[a^{\dagger}, b_1^{\dagger} \wedge b_2^{\dagger}] \geq \bigvee\limits_{a\in\mathfrak{F}}[a^{\dagger}, (b_1 \wedge b_2)^{\dagger}] \geq [(b_1 \wedge b_2)^{\dagger}, (b_1 \wedge b_2)^{\dagger}] = \top$ and thus ($\top$B2) is valid. Thus $\mathcal{B}$ is a $\top$-filter base for the $\top$-filter on $X^*$ denoted by $\mathfrak{F}^{\dagger}$. Define

$$\mathcal{C}^{\dagger} = \{\mathfrak{H} \in \mathfrak{F}_L^{\top}(X^*) : \mathfrak{H} \geq \mathfrak{F}^{\dagger},\ \text{for some } \mathfrak{F} \in \mathcal{C}\}$$

and note that $[j(x)] \geq \mathfrak{F}^{\dagger}$ whenever $\mathfrak{F} \xrightarrow{q\mathcal{C}} x$. Also, $[\langle\mathfrak{G}\rangle] \geq \mathfrak{G}^{\dagger}$ and $\mathfrak{H} \geq \mathfrak{K} \in \mathcal{C}^{\dagger}$ implies that $\mathfrak{H} \in \mathcal{C}^{\dagger}$. However, if $\mathfrak{F}_1, \mathfrak{F}_2 \in \mathcal{C}$ such that $\mathfrak{F}_1^{\dagger} \vee \mathfrak{F}_2^{\dagger}$ exists, $\mathfrak{F}_1^{\dagger} \cap \mathfrak{F}_2^{\dagger}$ may fail to belong to $\mathcal{C}^{\dagger}$. Hence $\mathcal{C}^{\dagger}$ may fail to be a $\top$-Cauchy structure on $X^*$. A necessary condition for $\big((X^*, \mathcal{C}^{\dagger}), j\big)$ to be a $T_2$-completion of $(X, \mathcal{C})$ is given below.

**Lemma 5.3.** *Suppose that $\mathfrak{F} \in \mathfrak{F}_L^{\top}(X)$, $(X, \mathcal{C}) \in |\top\text{-}\mathbf{Chy}|$. Then*

*(i) $j^{\Leftarrow}(\mathfrak{F}^{\dagger})$ exists and equals $\overline{\mathfrak{F}}$*

*(ii) $(X, \mathcal{C})$ is regular whenever $\big((X^*, \mathcal{C}^{\dagger}), j\big)$ is a $T_2$-completion of $(X, \mathcal{C})$.*

*Proof.* (i) Since $j^{\Leftarrow}j^{\Rightarrow}\mathfrak{F}$ exists and $\mathfrak{F}^{\dagger} \subseteq j^{\Rightarrow}\mathfrak{F}$, it follows that $j^{\Leftarrow}(\mathfrak{F}^{\dagger})$ exits. Next, it is shown that $j^{\Leftarrow}(\mathfrak{F}^{\dagger}) = \overline{\mathfrak{F}}$. Assume that $a \in \mathfrak{F}$ and thus $j^{\leftarrow}(a^{\dagger})$ is a $\top$-filter base member for $j^{\Leftarrow}(\mathfrak{F}^{\dagger})$. Note that $j^{\leftarrow}(a^{\dagger})(x) = a^{\dagger}(j(x)) = \bigvee\{\nu_{\mathfrak{G}}(a) : j^{\Rightarrow}\mathfrak{G} \xrightarrow{\sigma} j(x)\} = \bigvee\{\nu_{\mathfrak{G}}(a) : \mathfrak{G} \xrightarrow{q\mathcal{C}} x\} = \overline{a}(x)$. Since $j^{\leftarrow}(a^{\dagger}) = \overline{a} \in \overline{\mathfrak{F}}$, it follows that $j^{\Leftarrow}(\mathfrak{F}^{\dagger}) \subseteq \overline{\mathfrak{F}}$. Since $\{\overline{a} : a \in \mathfrak{F}\}$ is a $\top$-filter base for $\overline{\mathfrak{F}}$, $j^{\leftarrow}(a^{\dagger}) = \overline{a} \in j^{\Leftarrow}(\mathfrak{F}^{\dagger})$ whenever $a \in \mathfrak{F}$, and thus $\overline{\mathfrak{F}} \subseteq j^{\Leftarrow}(\mathfrak{F}^{\dagger})$. Then $j^{\Leftarrow}(\mathfrak{F}^{\dagger}) = \overline{\mathfrak{F}}$.
(ii) Verification here follows directly from (i). □

**Lemma 5.4.** *Assume that $\big((X^*, \mathcal{D}), j\big)$ is a $T_3$-completion of $(X, \mathcal{C})$ in standard form. Then*

*(i) $\big((X^*, \mathcal{C}^{\dagger}), j\big)$ is a $T_2$-completion of $(X, \mathcal{C})$ and $\mathcal{C}^{\dagger} \subseteq \mathcal{D}$*

*(ii)* $j : (X, \mathcal{C}) \longrightarrow (X^*, \mathcal{C}^\dagger)$ *satisfies (S2)*

*(iii)* $\big((X^*, \mathcal{C}^\dagger), j\big)$ *is the only possible strict* $T_3$*-completion of* $(X, \mathcal{C})$ *in standard form.*

*Proof.* (i) Suppose that $\mathfrak{H}_1, \mathfrak{H}_2 \in \mathcal{C}^\dagger$ such that $\mathfrak{H}_1 \vee \mathfrak{H}_2$ exists. Then there exists $\mathfrak{F}_i \in \mathcal{C}$ such that $\mathfrak{F}_i^\dagger \subseteq \mathfrak{H}_i$, $i = 1, 2$. Let $p = q_{\mathcal{D}}$; then $\overline{j^{\Rightarrow}\mathfrak{F}_i}^p \subseteq \mathfrak{F}_i^\dagger \subseteq \mathfrak{H}_i$ and since $(X^*, \mathcal{D})$ is regular, $\overline{j^{\Rightarrow}\mathfrak{F}_i}^p \in \mathcal{D}$, $i = 1, 2$. Hence $\overline{j^{\Rightarrow}\mathfrak{F}_i}^p \vee \overline{j^{\Rightarrow}\mathfrak{F}_2}^p$ exists and thus $\overline{j^{\Rightarrow}\mathfrak{F}_i}^p \cap \overline{j^{\Rightarrow}\mathfrak{F}_2}^p \in \mathcal{D}$ implies that $\mathfrak{F}_1 \cap \mathfrak{F}_2 \in \mathcal{C}$. Therefore $\mathfrak{F}_1^\dagger \cap \mathfrak{F}_2^\dagger \geq (\mathfrak{F}_1 \cap \mathfrak{F}_2)^\dagger \in \mathcal{C}^\dagger$ and hence $(X^*, \mathcal{C}^\dagger) \in |\top\text{-}\mathbf{Chy}|$ . Moreover, if $\mathfrak{F} \in \mathcal{C}$, then $\overline{j^{\Rightarrow}\mathfrak{F}}^p \subseteq \mathfrak{F}^\dagger$ implies that $\mathfrak{F}^\dagger \in \mathcal{D}$ and thus $\mathcal{C}^\dagger \subseteq \mathcal{D}$. Since $\mathcal{C}^\dagger \subseteq \mathcal{D}$, $\big((X^*, \mathcal{C}^\dagger), j\big)$ is a $T_2$-completion of $(X, \mathcal{C})$.

(ii) Denote $r = q_{\mathcal{C}^\dagger}$ and suppose that $\mathfrak{H} \xrightarrow{r} y$. Then $\mathfrak{H} \in \mathcal{C}^\dagger$ and thus $\mathfrak{H} \geq \mathfrak{F}^\dagger$ for some $\mathfrak{F} \in \mathcal{C}$. Hence $\mathfrak{F}^\dagger \cap [y] \in \mathcal{C}^\dagger$ and $j^{\Rightarrow}\mathfrak{F} \xrightarrow{r} y$. It follows that $\overline{j^{\Rightarrow}\mathfrak{F}}^r \subseteq \mathfrak{F}^\dagger \subseteq \mathfrak{H}$ and thus $j : (X, \mathcal{C}) \longrightarrow (X^*, \mathcal{C}^\dagger)$ obeys (S2).

(iii) Assume that $\big((X^*, \mathcal{D}), j\big)$ is any strict $T_3$-completion of $(X, \mathcal{C})$ in standard form. It remains to show that $\mathcal{C}^\dagger = \mathcal{D}$. According to (i), $\mathcal{C}^\dagger \subseteq \mathcal{D}$. Let $\mathfrak{H} \in \mathcal{D}$ and $\mathfrak{H} \xrightarrow{p} y$, where $p = q_{\mathcal{D}}$. Since $j : (X, \mathcal{C}) \longrightarrow (X^*, \mathcal{D})$ obeys (S2), there exists a $\top$-filter $\mathfrak{F}$ on $X$ such that $j^{\Rightarrow}\mathfrak{F} \xrightarrow{p} y$ and $\overline{j^{\Rightarrow}\mathfrak{F}}^p \subseteq \mathfrak{H}$. Note that $\mathfrak{F} \in \mathcal{C}$. Applying (S1), $\mathfrak{F}^\dagger = \overline{j^{\Rightarrow}\mathfrak{F}}^p \subseteq \mathfrak{H}$ and thus $\mathfrak{H} \in \mathcal{C}^\dagger$. Hence $\mathcal{C}^\dagger = \mathcal{D}$ and $\big((X^*, \mathcal{C}^\dagger), j\big)$ is the only possible strict $T_3$-completion of $(X, \mathcal{C})$ in standard form. $\square$

**Lemma 5.5.** *Suppose that* $(X, \mathcal{C})$ *has a strict* $T_3$*-completion in* $\top\text{-}\mathbf{Chy}$ *. Assume that* $\psi : (X, \mathcal{C}) \longrightarrow (Y, \mathcal{D})$ *is Cauchy-continuous and* $(Y, \mathcal{D})$ *is* $T_3$ *and complete. Then there exists a Cauchy-continuous map* $\theta : (X^*, \mathcal{C}^\dagger) \longrightarrow (Y, \mathcal{D})$ *such that* $\theta \circ j = \psi$*. In particular, under these assumptions,* $\big((X^*, \mathcal{C}^\dagger), j\big)$ *is the largest* $T_3$*-completion of* $(X, \mathcal{C})$ *in* $\top\text{-}\mathbf{Chy}$ *.*

*Proof.* Denote $p = q_{\mathcal{C}^\dagger}$ and $r = q_{\mathcal{D}}$. Define $\theta : X^* \longrightarrow Y$ by $\theta(s) = t$, where $j^{\Rightarrow}\mathfrak{G} \xrightarrow{p} s$ and $\psi^{\Rightarrow}\mathfrak{G} \xrightarrow{r} t$. Since $(X^*, \mathcal{C})$ and $(Y, \mathcal{D})$ are $T_3$ and complete, $\theta$ is a well-defined map and $\theta \circ j = \psi$. To show that $\theta$ is Cauchy-continuous, let $\mathfrak{F} \in \mathcal{C}$; it suffices to show that $\overline{\psi^{\Rightarrow}\mathfrak{F}}^{r} \subseteq \theta^{\Rightarrow}(\mathfrak{F}^{\dagger})$. Choose $a \in \mathfrak{F}$ and $y \in Y$; it is shown that $\theta^{\rightarrow}(a^{\dagger})(y) \leq \overline{\psi^{\rightarrow}a}^{r}(y)$. Recall that $\theta^{\rightarrow}(a^{\dagger})(y) = \bigvee\{a^{\dagger}(z) : \theta(z) = y\}$ and observe that $\nu_{\mathfrak{G}}(a) = \bigvee_{b \in \mathfrak{G}}[b, a] \leq \bigvee_{b \in \mathfrak{G}}[\psi^{\rightarrow}b, \psi^{\rightarrow}a] = \nu_{\psi^{\rightarrow}\mathfrak{G}}(\psi^{\rightarrow}a)$. Fix $z \in \theta^{-1}(y)$; then $a^{\dagger}(z) = \bigvee\{\nu_{\mathfrak{G}}(a) : j^{\Rightarrow}\mathfrak{G} \xrightarrow{p} z\} \leq \bigvee\{\nu_{\psi^{\Rightarrow}\mathfrak{G}}(\psi^{\rightarrow}a) : j^{\Rightarrow}\mathfrak{G} \xrightarrow{p} z\} \leq \bigvee\{\nu_{\mathfrak{H}}(\psi^{\rightarrow}a) : \mathfrak{H} \xrightarrow{r} y\} = \overline{\psi^{\rightarrow}a}^{r}(y)$. Hence $\theta^{\rightarrow}(a^{\dagger})(y) = \bigvee\{a^{\dagger}(z) : \theta(z) = y\} \leq \overline{\psi^{\rightarrow}a}^{r}(y)$ and thus $\theta$ is Cauchy-continuous. $\square$

Let $(X, q) \in |\top\text{-}\mathbf{Lim}|$. Then $(X, q)$ is said to satisfy **property Q** provided: $\mathfrak{F} \xrightarrow{q} z$ and $[z] \xrightarrow{q} x$ implies that $\mathfrak{F} \xrightarrow{q} x$. Moreover, $(X, q) \in |\top\text{-}\mathbf{Lim}|$ is called **symmetric** provided it is regular and obeys property Q. Since $\top$-**Lim** possesses initial structures, it easily follows that if $(X, q) \in |\top\text{-}\mathbf{Lim}|$, then there exists a finest symmetric $\top$-limit structure which is coarser that $q$. Let $sq \leq q$ denote this structure. Observe that if $(X, \mathcal{C}) \in |\top\text{-}\mathbf{Chy}|$, then $(X, q_{\mathcal{C}})$ satisfies property Q. Verification of the following lemma is straightforward.

**Lemma 5.6.** *Assume that $(X, q) \in |\top\text{-}\mathbf{Lim}|$ is regular. Then there exists a (complete) Cauchy structure $\mathcal{C}$ such that $q_{\mathcal{C}} = q$ iff $(X, q)$ is symmetric.*

Let $(X, \mathcal{C}) \in |\top\text{-}\mathbf{Chy}|$ and let $r\mathcal{C} \leq \mathcal{C}$ denote the finest regular Cauchy structure on $X$ which is coarser than $\mathcal{C}$. According to Lemma 5.6, $q_{r\mathcal{C}}$ is symmetric.

**Lemma 5.7.** *Suppose that $(X, \mathcal{C}) \in |\top\text{-}\mathbf{Chy}|$ is complete and denote $q = q_{\mathcal{C}}$. Define $\mathcal{C}_{sq} = \{\mathfrak{F} \in \mathfrak{F}_L^{\top}(X) : \mathfrak{F}$ sq-converges$\}$; then $r\mathcal{C} = \mathcal{C}_{sq}$ and, moreover, $(X, r\mathcal{C})$ is complete.*

*Proof.* Since $sq$ is symmetric, it follows from Lemma 5.6 that $(X, \mathcal{C}_{sq}) \in |\top\text{-}\mathbf{Chy}|$. Also, $(X, \mathcal{C}_{sq})$ is complete and induces $(X, sq)$. Since $(X, \mathcal{C}_{sq})$ is regular, it follows that $\mathcal{C}_{sq} \leq r\mathcal{C} \leq \mathcal{C}$ or $\mathcal{C} \subseteq r\mathcal{C} \subseteq \mathcal{C}_{sq}$. Let $\mathfrak{F} \in \mathcal{C}_{sq}$; then $\mathfrak{F} \xrightarrow{sq} x$ for some $x \in X$. Since $q_{r\mathcal{C}}$ is symmetric and $q_{r\mathcal{C}} \leq q$, it follows that $q_{r\mathcal{C}} \leq sq \leq q$. Hence $\mathfrak{F} \xrightarrow{q_{r\mathcal{C}}} x$ and thus $\mathfrak{F} \in r\mathcal{C}$. Therefore $r\mathcal{C} = \mathcal{C}_{sq}$ and $(X, r\mathcal{C})$ is complete. $\square$

As used above, since $\top$-**Chy** possesses initial structures, it follows that for each $(Y,\mathcal{D}) \in|\top$-**Chy**$|$ there exists a finest regular $\top$-Cauchy structure on $Y$, which is coarser than $\mathcal{D}$, denoted by $(Y, r\mathcal{D})$. Moreover, if $f : (X,\mathcal{C}) \longrightarrow (Y, r\mathcal{D})$ is Cauchy-continuous in $\top$-**Chy** , then $f : (X, r\mathcal{C}) \longrightarrow (Y, r\mathcal{D})$ is also Cauchy-continuous. Assume that $(X,\mathcal{C}) \in|\top$-**Chy**$|$ is $T_2$ and define $\widetilde{\mathcal{C}}$ on $X^*$ as follows:

$$\widetilde{\mathcal{C}} = \{\mathfrak{H} \in \mathfrak{F}_L^{\top}(X^*) : \text{either } \mathfrak{H} \geq j^{\Rightarrow}\mathfrak{F} \text{ for some } q_{\mathcal{C}} \text{ convergent } \mathfrak{F}, \text{ or}$$
$$\mathfrak{H} \geq j^{\Rightarrow}\mathfrak{G} \cap [\langle[\mathfrak{G}]\rangle \text{ for some } \mathfrak{G} \in \mathcal{N}\}.$$

The following lemma appears above as Theorem 3.3. It is listed here for convenience.

**Lemma 5.8.** *Suppose that* $(X,\mathcal{C}) \in|\top$-**Chy**$|$ *is* $T_2$. *Then*

*(i)* $\big((X^*,\widetilde{\mathcal{C}}), j\big)$ *is the finest* $T_2$*-completion of* $(X,\mathcal{C})$ *in* $\top$-**Chy** *which is in standard form*

*(ii) If* $f : (X,\mathcal{C}) \longrightarrow (Y,\mathcal{D})$ *is a Cauchy-continuous map and* $(Y,\mathcal{D})$ *is complete,* $f$ *has a Cauchy-continuous extension* $\widetilde{f} : (X^*,\widetilde{\mathcal{C}}) \longrightarrow (Y,\mathcal{D})$ *such that* $\widetilde{f} \circ j = f$.

An object $(X,\mathcal{C}) \in|\top$-**Chy**$|$ is said to obey **property P** provided that for each $\mathfrak{F} \notin \mathcal{C}$ there exists a $T_3$-complete $(Y,\mathcal{D}) \in|\top$-**Chy**$|$ and a Cauchy-continuous map $f : (X,\mathcal{C}) \longrightarrow (Y,\mathcal{D})$ such that $f^{\Rightarrow}\mathfrak{F} \notin \mathcal{D}$.

**Lemma 5.9.** *Assume that* $(X,\mathcal{C}) \in|\top$-**Chy**$|$ *is* $T_3$. *Then* $(X,\mathcal{C})$ *has a* $T_3$*-completion in* $\top$-**Chy** *iff it satisfies property P.*

*Proof.* Suppose that $(X,\mathcal{C})$ possesses a $T_3$-completion in $\top$-**Chy**. Then clearly $(X,\mathcal{C})$ satisfies property P. Conversely, assume that $(X,\mathcal{C})$ obeys property P. It is shown that $\big((X^*, r\widetilde{\mathcal{C}}), j\big)$

is a $T_3$-completion of $(X,\mathcal{C})$. Denote $\delta = q_{r\widetilde{\mathcal{C}}}$; then $j : (X,\mathcal{C}) \longrightarrow (X^*, r\widetilde{\mathcal{C}})$ is Cauchy-continuous and $\mathrm{cl}_\delta(j(X)) = X^*$. That is, $\overline{\mathbf{1}_{j(X)}}^{\delta} = \mathbf{1}_{X^*}$. Indeed, if $x \in X$ then clearly $\overline{\mathbf{1}_{j(X)}}^{\delta}(x) = \top$ and if $\mathfrak{G} \in \mathcal{N}$ then $\overline{\mathbf{1}_{j(X)}}^{\delta}(\langle\mathfrak{G}\rangle) = \vee\{\nu_{\mathfrak{H}}(\mathbf{1}_{j(X)} : \mathfrak{H} \xrightarrow{\delta} \langle\mathfrak{G}\rangle\} = \vee\{\nu_{\mathfrak{H}}(\mathbf{1}_{j(X)} : \mathfrak{H} \geq j^{\Rightarrow}\mathfrak{G} \cap \langle\mathfrak{G}\rangle]\} \geq \nu_{j^{\Rightarrow}\mathfrak{G}(\mathbf{1}_{j(x)}} = \top$. Suppose that $\mathfrak{F} \in \mathfrak{F}_L^{\top}(X)$ such that $j^{\Rightarrow}\mathfrak{F} \in r\widetilde{\mathcal{C}}$ but $\mathfrak{F} \notin \mathcal{C}$. Then there exists a $T_3$-complete $(Y,\mathcal{D}) \in |\top\text{-}\mathbf{Chy}|$ and a Cauchy-continuous map $f : (X,\mathcal{C}) \longrightarrow (Y,\mathcal{D})$ such that $f^{\Rightarrow}\mathfrak{F} \notin \mathcal{D}$. Since by Lemma 5.8 $f$ has a Cauchy-continuous extension $\widetilde{f} : (X^*, r\widetilde{\mathcal{C}}) \longrightarrow (Y,\mathcal{D})$ such that $\widetilde{f} \circ j = f$, $f^{\Rightarrow}\mathfrak{F} = \widetilde{f}^{\Rightarrow}(j^{\Rightarrow}\mathfrak{F}) \in \mathcal{D}$, contrary to our assumption. Hence $j : (X,\mathcal{C}) \longrightarrow (X^*, r\widetilde{\mathcal{C}})$ is a dense embedding. Further, $(X^*, r\widetilde{\mathcal{C}})$ is $T_2$; otherwise, $\mathfrak{H} \xrightarrow{\delta} y_1, y_2$ for some $\mathfrak{H}$ and $y_1 \neq y_2$ and hence $y_1$ and $y_2$ have the same $\delta$-convergent $\top$-filters. In particular, $[y_1] \cap [y_2] \in r\widetilde{\mathcal{C}}$. If $y_i = j(x_i)$, $i = 1,2$, then $[x_1] \cap [x_2] \in \mathcal{C}$, which contradicts $(X,\mathcal{C})$ being $T_2$. Next if $y_1 = j(x_2)$ and $y_2 = \langle\mathfrak{G}\rangle$, $\mathfrak{G} \in \mathcal{N}$, then $j^{\Rightarrow}\mathfrak{G} \xrightarrow{\delta} j(x_1)$ implies that $\mathfrak{G} \xrightarrow{q_{\mathcal{C}}} x_1$, which violates $\mathfrak{G} \in \mathcal{N}$. Finally, suppose that $y_i = \langle\mathfrak{G}_i\rangle$, where $\mathfrak{G}_i \in \mathcal{N}$, $i = 1,2$ and $\langle\mathfrak{G}_1\rangle \neq \langle\mathfrak{G}_2\rangle$. This is impossible since $\langle\mathfrak{G}_1\rangle$ and $\langle\mathfrak{G}_2\rangle$ must have the same $\delta$-convergent $\top$-filters. Therefore $(X^*, r\widetilde{\mathcal{C}})$ is $T_3$. Moreover, since $(X^*, \widetilde{\mathcal{C}})$ is complete, it follows from Lemma 5.7 that $(X^*, r\widetilde{\mathcal{C}})$ is also complete. Hence $\big((X^*, r\widetilde{\mathcal{C}}), j\big)$ is a $T_3$-completion in $\top$-**Chy**. □

**Lemma 5.10.** *Assume that $(X,\mathcal{C}) \in |\top\text{-}\mathbf{Chy}|$ is $T_3$, obeys property P, and let $\big((X^*, r\widetilde{\mathcal{C}}), j\big)$ denote its $T_3$-completion in standard form. Then,*

*(i) $\big((X^*, \mathcal{C}^\dagger), j\big)$ is a $T_2$-completion of $(X,\mathcal{C})$ and $j$ obeys (S2)*

*(ii) if L is a complete Boolean algebra and $(X,\mathcal{C})$ is totally bounded, $\big((X^*, \mathcal{C}^\dagger), j\big)$ is a strict $T_3$-completion iff $\mathcal{C}^\dagger = r\widetilde{\mathcal{C}}$.*

*Proof.* (i) The result follows from Lemma 5.4 (i) and (ii).
(ii) Suppose that $\big((X^*, \mathcal{C}^\dagger), j\big)$ is a strict $T_3$-completion of $(X,\mathcal{C})$. Since $r\widetilde{\mathcal{C}} \leq \mathcal{C}^\dagger \leq \widetilde{\mathcal{C}}$ and $r\widetilde{\mathcal{C}}$

is the finest regular Cauchy structure which is coarser than $\widetilde{\mathcal{C}}$, $r\widetilde{\mathcal{C}} = \mathcal{C}^\dagger$. Conversely, assume that $\mathcal{C}^\dagger = r\widetilde{\mathcal{C}}$. According to (i), $j : (X, \mathcal{C}) \longrightarrow (X^*, \mathcal{C}^\dagger)$ obeys (S2). Since $L$ is a complete Boolean algebra and $(X, \mathcal{C})$ is totally bounded, it follows from Lemma 5.1 that $(X^*, \mathcal{C}^\dagger)$ is also totally bounded and thus compact. Then according to Lemma 5.2, $(X^*, \mathcal{C}^\dagger)$ obeys (S1) and hence $\Big((X^*, \mathcal{C}^\dagger), j\Big)$ is a strict $T_3$-completion of $(X, \mathcal{C})$. □

**Corollary 5.1.** *Under the assumptions of Lemma 5.10 (ii), both* $(X^*, \mathcal{C}^\dagger)$ *and* $(X^*, r\widetilde{\mathcal{C}})$ *are compact and hence* $\mathcal{C}^\dagger$ *and* $r\widetilde{\mathcal{C}}$ *possess the same* $\top$*-ultrafilters. Moreover,* $\Big((X^*, \mathcal{C}^\dagger), j\Big)$ *is a strict* $T_2$*-completion of* $(X, \mathcal{C})$ *in this case.*

Connecting $T_3$-Completions and $T_3$-Compactifications

Given $(X, q) \in |\top\text{-}\mathbf{Lim}|$ , assume that $(X, q)$ possesses a strict $T_3$-compactification. Let $\mathcal{A}$ denote the set of all equivalence classes of strict $T_3$-compactifications of $(X, q)$ in $\top$-**Lim** and let $\mathcal{B}$ denote the set of all totally bounded $\top$-Cauchy spaces $(X, \mathcal{C})$ such that $q_{\mathcal{C}} = q$ and which have a strict $T_3$-completion in $\top$-**Chy**. Define $\Theta : \mathcal{A} \longrightarrow \mathcal{B}$ by $\Theta\Big(\Big\langle\Big((Y, p), \psi\Big)\Big\rangle\Big) = (X, \mathcal{C}_p)$, where $\mathfrak{F} \in \mathcal{C}_p$ iff $\psi^{\Rightarrow}\mathfrak{F}$ $p$-converges.

**Theorem 5.1.** *The map* $\Theta : \mathcal{A} \longrightarrow \mathcal{B}$ *is an order preserving injection. Moreover,* $\Theta$ *is a bijection whenever* $L$ *is a complete Boolean algebra.*

*Proof.* Let $\Theta\Big(\Big\langle\Big((Y, p), \psi\Big)\Big\rangle\Big) = (X, \mathcal{C}_p)$ and note that $\mathcal{C}_p$ is a $\top$-Cauchy structure. Indeed $\psi^{\Rightarrow}([x]) = [\psi(x)] \xrightarrow{p} \psi(x)$ and thus $[x] \in \mathcal{C}_p$. If $\mathfrak{G} \geq \mathfrak{F} \in \mathcal{C}_p$, then $\mathfrak{G} \in \mathcal{C}_p$. Assume that $\mathfrak{F}_1, \mathfrak{F}_2 \in \mathcal{C}_p$ such that $\mathfrak{F}_1 \vee \mathfrak{F}_2$ exists. Then $\psi^{\Rightarrow}\mathfrak{F}_1 \vee \psi^{\Rightarrow}\mathfrak{F}_2$ exists and since $(Y, p)$ is $T_2$, $\psi^{\Rightarrow}(\mathfrak{F}_1 \cap \mathfrak{F}_2) = \psi^{\Rightarrow}\mathfrak{F}_1 \cap \psi^{\Rightarrow}\mathfrak{F}_2$ $p$-converges. Hence $\mathfrak{F}_1 \cap \mathfrak{F}_2 \in \mathcal{C}_p$ and thus $(X, \mathcal{C}_p)$ is a $\top$-Cauchy space. Moreover, $(X, \mathcal{C}_p)$ is $T_3$ since if $\mathfrak{F} \in \mathcal{C}_p$, then $\psi^{\Rightarrow}\overline{\mathfrak{F}}^q \geq \overline{\psi^{\Rightarrow}\mathfrak{F}}^p$ $p$-converges since $(Y, p)$ is $T_3$. Therefore $\overline{\mathfrak{F}}^q \in \mathcal{C}_p$ and $(X, \mathcal{C}_p)$ is $T_3$. Further, $\psi^{\Rightarrow}\mathfrak{F}$ $p$-converges for each $\top$-ultrafilter $\mathfrak{F}$ on $X$ and thus $\mathfrak{F} \in \mathcal{C}_p$. Then $(X, \mathcal{C}_p)$ is also totally bounded. Denote $\mathcal{D} = \{\mathfrak{H} \in \mathfrak{F}_L^{\top}(Y) : \mathfrak{H}\ p\text{-converges}\}$, and it follows that $\big((Y, \mathcal{D}), \psi\big)$ is a strict $T_3$-completion of the totally bounded $T_3$ space $(X, \mathcal{C}_p)$ and thus $(X, \mathcal{C}_p) \in \mathcal{B}$. We next show that $\Theta$ is an injection. Assume that $\Theta\Big(\Big\langle\big((Y_i, p_i), \psi_i\big)\Big\rangle\Big) = (X, \mathcal{C}_p)$, $i = 1, 2$. Define $h : (Y_1, p_1) \longrightarrow (Y_2, p_2)$ by $h(s) = t$, where $\psi_2^{\Rightarrow}\mathfrak{F} \xrightarrow{p_2} t$ whenever $\psi_1^{\Rightarrow}\mathfrak{F} \xrightarrow{p_1} s$. Suppose $\psi_1^{\Rightarrow}\mathfrak{F}_k \xrightarrow{p_1} s$, $k = 1, 2$; then $\psi_1^{\Rightarrow}(\mathfrak{F}_1 \cap \mathfrak{F}_2) \xrightarrow{p_1} s$ and thus $\mathfrak{F}_1 \cap \mathfrak{F}_2 \in \mathcal{C}_p$. It follows that $\psi_2^{\Rightarrow}(\mathfrak{F}_1 \cap \mathfrak{F}_2) \xrightarrow{p_2} t$ and thus $h$ is well-defined. Moreover, $\psi_1^{\Rightarrow}[x] = [\psi_1(x)] \xrightarrow{p_1} \psi_1(x)$ and $\psi_2^{\Rightarrow}[x] = [\psi_2(x)] \xrightarrow{p_2} \psi_2(x)$ implies that $h \circ \psi_1 = \psi_2$.

It remains to show that $h$ is an isomorphism. Since

$$\Theta\Big(\Big\langle\big((Y_1, p_1), \psi_1\big)\Big\rangle\Big) = \Theta\Big(\Big\langle\big((Y_2, p_2), \psi_2\big)\Big\rangle\Big) = (X, \mathcal{C}_p),$$

$\mathcal{C} = \{\mathfrak{F} \in \mathfrak{F}_L^{\top}(X) : \psi_1^{\Rightarrow}\mathfrak{F}\ p_1\text{-converges}\} = \{\mathfrak{F} \in \mathfrak{G}_L^{\top}(X) : \psi_2^{\Rightarrow}\mathfrak{G}\ p_2\text{-converges}\}$. Then for each $a \in L^X$, $\overline{\psi_1^{\rightarrow}a}^{p_1}(s) = a^{\dagger}(s) = \bigvee\{\nu_{\mathfrak{F}}(a) : \psi_1^{\Rightarrow}\mathfrak{F} \xrightarrow{p_1} s\} = \bigvee\{\nu_{\mathfrak{G}}(a) : \psi_2^{\Rightarrow}\mathfrak{G} \xrightarrow{p_2} t\} = a^{\dagger}(t) = \overline{\psi_2^{\rightarrow}a}^{p_2}(t)$, according to (S1). Next, assume that $\mathfrak{H} \xrightarrow{p_1} s$; then employing (S2), there exits an $\mathfrak{F}$ such that $\psi_1^{\Rightarrow}\mathfrak{F} \xrightarrow{p_1} x$ and $\overline{\psi_1^{\Rightarrow}\mathfrak{F}}^{p_1} \subseteq \mathfrak{H}$. Observe that $h^{\rightarrow}\Big(\overline{\psi_1^{\rightarrow}a}^{p_1}\Big)(t) = \bigvee\{\overline{\psi_1^{\rightarrow}(a)}^{p_1}(z) : h(z) = t\} = \overline{\psi_1^{\rightarrow}a}^{p_1}(s) = \overline{\psi_2^{\rightarrow}a}^{p_2}(t)$. Further, if $a \in \mathfrak{F}$, then $h^{\rightarrow}\Big(\overline{\psi_1^{\rightarrow}a}^{p_1}\Big)(t) = \overline{\psi_2^{\rightarrow}a}^{p_2}(t)$ implies that $h^{\Rightarrow}\mathfrak{H} \geq h^{\Rightarrow}\Big(\overline{\psi_1^{\Rightarrow}\mathfrak{F}}^{p_1}\Big) = \overline{\psi_2^{\Rightarrow}\mathfrak{F}}^{p_2}$. Then $h$ is continuous and by symmetry, $h$ is an isomorphism. Therefore $\Big\langle\big((Y_1, p_1), \psi_1\big)\Big\rangle = \Big\langle\big((Y_2, p_2), \psi_2\big)\Big\rangle$ and $\Theta$ is an injection.

Finally, we must show that the injection $\Theta$ is order preserving. Assume that $\Big\langle\big((Y_1, p_1), \psi_1\big)\Big\rangle \geq \Big\langle\big((Y_2, p_2), \psi_2\big)\Big\rangle$ and let $k : (Y_1, p_1) \longrightarrow (Y_2, p_2)$ be a continuous map such that $k \circ \psi_1 = \psi_2$. Denote $\mathcal{C}_{p_i} = \{\mathfrak{F} \in \mathfrak{F}_L^{\top}(X) : \psi_i^{\Rightarrow}\mathfrak{F}\ p_i\text{-converges}\}$, $i = 1, 2$. Suppose that $\mathfrak{F} \in \mathcal{C}_{p_1}$; then $\psi_2^{\Rightarrow}\mathfrak{F} = (k \circ \psi_1)^{\Rightarrow}\mathfrak{F}\ p_2$-converges and hence $\mathfrak{F} \in \mathcal{C}_{p_2}$. Therefore $\mathcal{C}_{p_1} \geq \mathcal{C}_{p_2}$. Conversely, suppose that $\mathcal{C}_1 \geq \mathcal{C}_2$ and $\Theta\left(\Big\langle\big((Y_i, p_i), \psi_i\big)\Big\rangle\right) = (X, \mathcal{C}_i)$, $i = 1, 2$. Define $h : (Y_1, p_1) \longrightarrow (Y_2, p_2)$ as follows: $h(s) = t$, where $\psi_1^{\Rightarrow}\mathfrak{F} \xrightarrow{p_1} s$ which implies that $\psi_2^{\Rightarrow}\mathfrak{F} \xrightarrow{p_2} t$. Since $\mathcal{C}_1 \subseteq \mathcal{C}_2$,

$h$ is well-defined and $h \circ \psi_1 = \psi_2$. If $a \in L^X$ and $s \in Y_1$, then $\overline{\psi_1^{\rightarrow} a}^{p_1}(s) = a^{\dagger}(s) = \bigvee\{\nu_{\mathfrak{F}}(a) : \psi_1^{\Rightarrow}\mathfrak{F} \xrightarrow{p_1} s\} \leq \bigvee\{\nu_{\mathfrak{G}}(a) : \psi_2^{\Rightarrow}\mathfrak{G} \xrightarrow{p_2} t\} = a^{\dagger}(t) = \overline{\psi_2^{\rightarrow} a}^{p_2}(t)$. It follows that $h^{\Rightarrow}\left(\overline{\psi_1^{\Rightarrow}\mathfrak{F}}^{p_1}\right) \geq \overline{\psi_2^{\Rightarrow}\mathfrak{F}}^{p_2} \xrightarrow{p_2} t$ whenever $\psi_1^{\Rightarrow}\mathfrak{F} \xrightarrow{p_1} s$ and thus $h$ is continuous. Therefore $\left\langle\Big((Y_1, p_1), \psi_1\Big)\right\rangle \geq \left\langle\Big((Y_2, p_2), \psi_2\Big)\right\rangle$ and $\Theta$ is order preserving.

Next, assume that $L$ is a complete Boolean algebra and $(X, \mathcal{C}) \in \mathcal{B}$. Let $\Big((Y, \mathcal{D}), \psi\Big)$ denote the strict $T_3$-completion of $(X, \mathcal{C})$ in $\top$-**Chy** . Define $p = q_{\mathcal{D}}$ and let $\mathcal{C}_p = \{\mathfrak{F} \in \mathfrak{F}_L^{\top}(X) : \psi^{\Rightarrow}\mathfrak{F}$ $p$-converges$\}$. It follows from Lemma 5.1 that $(Y, p)$ is compact and thus $\Big((Y, p), \psi\Big)$ is a strict $T_3$-compactification of $(X, q)$. Moreover, note that $\mathcal{C} = \mathcal{C}_p$, $\Theta\left(\left\langle\Big((Y, p), \psi\Big)\right\rangle\right) = (X, \mathcal{C})$ and hence $\Theta$ is a bijection in this case. □

Whenever $L$ is a complete Boolean algebra, the following example establishes the existence of a totally bounded $\top$-Cauchy space which has a strict $T_3$-completion.

**Example 5.1.** Suppose that $L$ is a complete Boolean algebra and $X \neq \varnothing$. Recall that by Proposition 1.1, $[x]$ is a $\top$-ultrafilter on $X$. Let $\eta$ denote the set of all $\top$-ultrafilters on $X$ which are not of the form $[x]$, for some $x \in X$, and define $\mathcal{C} = \{[x], \mathfrak{G} : x \in X, \mathfrak{G} \in \eta\}$. Then $(X, \mathcal{C}) \in |\top$-**Chy**$|$ is totally bounded and $\langle\mathfrak{G}\rangle$ is a singleton set, for each $\mathfrak{G} \in \eta$. Let $q = q_{\mathcal{C}}$ and note that $\mathfrak{F} \xrightarrow{q} x$ iff $\mathfrak{F} = [x]$. Fix $a \in L^X$ and observe that $\overline{a}^q(x) = \bigvee\{\nu_{\mathfrak{F}}(a) : \mathfrak{F} \xrightarrow{q} x\} = \nu_{[x]}(a) = a(x)$, $x \in X$. Hence $\overline{a}^q = a$ and thus $(X, \mathcal{C})$ is regular. Moreover, $a^{\dagger}(j(x)) = \bigvee\{\nu_{\mathfrak{F}}(a) : j^{\Rightarrow}\mathfrak{F} \xrightarrow{\sigma} j(x)\} = \nu_{[x]}(a) = a(x)$ and $a^{\dagger}(\langle\mathfrak{G}\rangle) = \bigvee\{\nu_{\mathfrak{K}}(a) : j^{\Rightarrow}\mathfrak{K} \xrightarrow{\sigma} \langle\mathfrak{G}\rangle\} = \nu_{\mathfrak{G}}(a)$. As usual $\mathcal{C}^{\dagger} = \{\mathfrak{H} \in \mathfrak{F}_L^{\top}(X^*) : \mathfrak{H} \geq \mathfrak{F}^{\dagger}$ for some $\mathfrak{F} \in \mathcal{C}\}$. Then $\mathcal{C}^{\dagger}$ obeys ($\top$C1) and ($\top$C2) but we must prove ($\top$C3). Suppose that $\mathfrak{F}_1 \vee \mathfrak{F}_2$ fails to exist, where $\mathfrak{F}_1, \mathfrak{F}_2 \in \mathcal{C}$. Then there exists $a_i \in \mathfrak{F}_i$ such that $\bigvee_{x \in X} a_1(x) \wedge a_2(x) = \alpha < \top$. Recall that for each $\top$-filter $\mathfrak{K}$ and $b \in L^X$, $\nu_{\mathfrak{K}}(b) \leq \bigvee_{x \in X} b(x)$. It follows that $\bigvee_{y \in X^*} a_1^{\dagger}(y) \wedge a_2^{\dagger}(y) = \bigvee_{x \in X}(a_1(x) \wedge a_2(x)) \wedge \bigvee_{\mathfrak{H} \in \eta}(\nu_{\mathfrak{G}}(a_1) \wedge \nu_{\mathfrak{G}}(a_2)) = \alpha \wedge \bigvee_{\mathfrak{H} \in \eta}(\nu_{\mathfrak{G}}(a_1) \wedge \nu_{\mathfrak{G}}(a_2)) = \alpha < \top$. It follows that $\mathfrak{F}_1^{\dagger} \vee \mathfrak{F}_2^{\dagger}$ fails to exist and thus $(X^*, \mathcal{C}^{\dagger}) \in |\top$-**Chy**$|$ . Let $r = q_{\mathcal{C}^{\dagger}}$. It is shown that $(X^*, \mathcal{C}^{\dagger})$ is regular. Using the notation prior to Lemma 5.1, since $\langle\mathfrak{G}\rangle$ is a singleton, $\hat{a} = a^\dagger$ for each $a \in L^X$ and thus $\hat{\mathfrak{F}} = \mathfrak{F}^\dagger$. Assume that $\mathfrak{H} \xrightarrow{r} y$, where $\mathfrak{H}$ is a $\top$-ultrafilter on $X^*$. According to Lemma 5.1, there exists a $\top$-ultrafilter $\mathfrak{F}$ on $X$ such that $\overline{j^{\Rightarrow}\mathfrak{F}}^r \subseteq \mathfrak{F}^\dagger = \hat{\mathfrak{F}} \subseteq \mathfrak{H}$ and $\nu_{\mathfrak{F}}(a) = \nu_{\mathfrak{H}}(\hat{a})$ for each $a \in L^X$. Since $(X, \mathcal{C})$ is totally bounded, $\mathfrak{F} \in \mathcal{C}$ and thus $\mathfrak{F}^\dagger \xrightarrow{r} y$ and, moreover, $\nu_{\mathfrak{F}}(a) = \nu_{\mathfrak{H}}(\hat{a}) = \nu_{\mathfrak{H}}(a^\dagger)$. Hence $\overline{a^\dagger}^r(y) = \bigvee\{\nu_{\mathfrak{H}}(a^\dagger) : \mathfrak{H} \xrightarrow{r} y\} = \bigvee\{\nu_{\mathfrak{F}}(a) : j^{\Rightarrow}\mathfrak{F} \xrightarrow{r} y\} = a^\dagger(y)$ and thus $\overline{a^\dagger}^r = a^\dagger$. Therefore $\overline{\mathfrak{F}^\dagger}^r = \mathfrak{F}^\dagger$ and $\big((X^*, \mathcal{C}^\dagger), j\big)$ is a $T_3$-completion of $(X, \mathcal{C})$ in standard form. Further, since $(X, \mathcal{C})$ is totally bounded, Lemma 5.1 implies that $(X^*, \mathcal{C}^\dagger)$ is also totally bounded and hence compact. According to Lemma 5.2 and Lemma 5.4, $j : (X, \mathcal{C}) \longrightarrow (X^*, \mathcal{C}^\dagger)$ obeys (S1) and (S2). Therefore $\big((X^*, \mathcal{C}^\dagger), j\big)$ is a strict $T_3$-completion of $(X, \mathcal{C})$ in standard form. $\square$

Continuing with the notation used in Theorem 5.1, denote $\mathcal{B} = \{(X, \mathcal{C}_k) : k \in J\}$ and define $\mathcal{C} = \bigcap_{k \in J} \mathcal{C}_k$. Since each $(X, \mathcal{C}_k)$ induces $(X, q)$, it follows that $q_{\mathcal{C}} = q$. As before, let $X^* = X \cup \{\langle\mathfrak{G}\rangle : \mathfrak{G} \in \mathcal{N}_{\mathcal{C}}\}$ and $j : X \longrightarrow X^*$ be the natural injection. Further, let $\mathcal{N}_k = \{\mathfrak{G} \in \mathcal{C}_k : \mathfrak{G}$ fails to $q$-converge$\}$. Since each $(X, \mathcal{C}_k)$, $k \in J$, is totally bounded, it follows that $(X, \mathcal{C})$ is also totally bounded. Denote $X_k^* = X \cup \{\langle\mathfrak{G}\rangle_k : \mathfrak{G} \in \mathcal{N}_k\}$ and let $\big((X_k^*, \mathcal{C}_k^\dagger), j_k\big)$ denote the strict $T_3$-completion of $(X, \mathcal{C}_k)$ in standard form. Since $\mathcal{C}$ and $\mathcal{C}_k$ have the same $\top$-ultrafilters and $\mathcal{C} \subseteq \mathcal{C}_k$, it follows that $\langle\mathfrak{G}\rangle \subseteq \langle\mathfrak{G}\rangle_k$, for each $\top$-ultrafilter $\mathfrak{G} \in \mathcal{N}_{\mathcal{C}}$ and $k \in J$. Let $\delta_k : X^* \longrightarrow X_k^*$ denote the bijection defined by $\delta_k(j(x)) = j_k(x)$ and $\delta_k(\langle\mathfrak{G}\rangle) = \langle\mathfrak{G}\rangle_k$, $x \in X$ and $\top$-ultrafilter $\mathfrak{G} \in \mathcal{N}_{\mathcal{C}}$. Since $\top$-**Chy** is a topological category, it possesses initial structures. Let $\mathcal{D}$ denote the initial $\top$-Cauchy structure on $X^*$ determined by the maps $\delta_k : X^* \longrightarrow (X_k^*, \mathcal{C}_k^\dagger)$, $k \in J$. Define $p = q_{\mathcal{D}}$ and $p_k = q_{\mathcal{C}_k^\dagger}$, $k \in J$. For sake of convenience, consider the commutative diagram:

$$\begin{array}{ccc} (X,\mathcal{C}) & \xrightarrow{\quad j \quad} & (X^*,\mathcal{D}) \\ \Big\downarrow \mathrm{id}_X & & \Big\downarrow \delta_k \\ (X,\mathcal{C}_k) & \xrightarrow{\quad j_k \quad} & (X_k^*,\mathcal{C}_k^\dagger) \end{array}$$

**Proposition 5.1.** *Assume that the frame $L$ is a Boolean algebra for parts (ii) and (iv); fix $(X,q) \in |\top\text{-}\mathbf{Lim}|$ . Using the notation given above, suppose that the assumptions made in Theorem 5.1 are valid and define $\mathcal{C} = \bigcap\limits_{k\in J} \mathcal{C}_k$. Then*

*(i) $\big((X^*,\mathcal{D}), j\big)$ is a $T_3$-completion of $(X,\mathcal{C})$ in standard form*

*(ii) $\big((X^*,p), j\big)$ is a $T_3$-compactification of $(X,q)$ and $j$ obeys (S1) and (S3), where $p = q_{\mathcal{D}}$*

*(iii) $\big((X^*,\mathcal{C}^\dagger), j\big)$ is a $T_2$-completion of $(X,\mathcal{C})$ and $j$ satisfies (S2)*

*(iv) $\big((X^*,r), j\big)$ is a $T_2$ compactification of $(X,q)$ and $j$ obeys (S2), where $r = q_{\mathcal{C}^\dagger}$.*

*Proof.* (i): Since each $(X_k^*,\mathcal{C}_k^\dagger)$ is a regular $\top$-Cauchy space and $\mathcal{D}$ is the initial $\top$-Cauchy structure on $X^*$ determined by $\delta_k : X^* \longrightarrow (X_k^*,\mathcal{C}_k^\dagger)$, it follows that $(X^*,\mathcal{D})$ is a regular $\top$-Cauchy space and $\delta_k : (X^*,\mathcal{D}) \longrightarrow (X_k^*,\mathcal{C}_k^\dagger)$ is Cauchy-continuous, $k \in J$. If $\mathfrak{H} \xrightarrow{p} z_1, z_2$, then $\delta_k^{\Rightarrow}\mathfrak{H} \xrightarrow{p_k} \delta_k(z_1), \delta_k(z_2)$ and thus $\delta_k(z_1) = \delta_k(z_2)$. Then $z_1 = z_2$ and $(X^*,\mathcal{D})$ is $T_3$. If $\mathfrak{F} \in \mathcal{C}$, then using the commutative diagram above, $\delta_k^{\Rightarrow}(j^{\Rightarrow}\mathfrak{F}) = j_k^{\Rightarrow}\mathfrak{F} \in \mathcal{C}_k^\dagger$ for each $k \in J$, and thus $j^{\Rightarrow}\mathfrak{F} \in \mathcal{D}$. Therefore $j$ is Cauchy-continuous. Next, assume that $\mathfrak{F} \in \mathfrak{F}_L^\top(X)$ and $j^{\Rightarrow}\mathfrak{F} \in \mathcal{D}$. Then $j_k^{\Rightarrow}\mathfrak{F} = (\delta_k \circ j)^{\Rightarrow}\mathfrak{F} \in \mathcal{C}_k^\dagger$ and hence $\mathfrak{F} \in \mathcal{C}_k$ for each $k \in J$. It follows that $\mathfrak{F} \in \mathcal{C}$ and thus $\big((X^*,\mathcal{D}), j\big)$ is a $T_3$-completion of $(X,\mathcal{C})$ in standard form.

(ii): Suppose that $\mathfrak{H}$ is a $\top$-ultrafilter on $X^*$. According to Lemma 5.1 there exists a $\top$-ultrafilter $\mathfrak{F}$ on $X$ such that $\mathfrak{H} \geq \overline{j^{\Rightarrow}\mathfrak{F}}^{p}$. If $\mathfrak{F} \xrightarrow{q} x$, then $\mathfrak{H} \xrightarrow{p} j(x)$. Assume that $\mathfrak{F} \in \mathcal{N}_{\mathcal{C}}$.

Since $j^{\Rightarrow}\mathfrak{F} \in \mathcal{D}$, it follows that $\delta_k^{\Rightarrow}(j^{\Rightarrow}\mathfrak{F}) = j_k^{\Rightarrow}\mathfrak{F} \xrightarrow{q_{\mathcal{C}_k^\dagger}} \delta_k(\langle\mathfrak{F}\rangle)$ for each $k \in J$. Then $j^{\Rightarrow}\mathfrak{F} \xrightarrow{p} \langle\mathfrak{F}\rangle$ and hence $(X^*, p)$ is compact. It follows that $\big((X^*, p), j\big)$ is a $T_3$-compactification of $(X, q)$ and by Lemma 5.2, $j$ obeys (S1) and (S3).

(iii): Employing Lemma 5.4, $\big((X, \mathcal{C}^\dagger), j\big)$ is a $T_2$-completion of $(X, \mathcal{C})$ which satisfies (S2).

(iv): Let $\mathfrak{H}$ be a $\top$-ultrafilter on $X^*$, and by Lemma 5.1 there exists a $\top$-ultrafilter $\mathfrak{F}$ on $X$ such that $\mathfrak{F}^\dagger \subseteq \mathfrak{H}$. Since $(X, \mathcal{C})$ is totally bounded and $(X^*, \mathcal{C}^\dagger)$ is complete, it follows that $\mathfrak{H}$ $r$-converges. Then $(X^*, r)$ is compact and $\big((X^*, r), j\big)$ is a $T_2$-compactification of $(X, q)$ which obeys (S2). □

The assumption that the frame $L$ is a Boolean algebra is used in the proof of Theorem 5.1 to show that the strict $T_3$-completion $\big((Y, \mathcal{D}), \psi\big)$ of the totally bounded $\top$-Cauchy space is compact. The key step being that each stratified $L$-ultrafilter on $X$ is the image of a $\top$-ultrafilter according to the mapping $\mathfrak{F} \mapsto \nu_{\mathfrak{F}}$ listed in Theorem 2.1. This property has been extended from the requirement that $L$ is a Boolean algebra to more general algebraic structures; for example, see Proposition 4.4.4 [12] and Proposition 9 [9]. These and related references may prove to be profitable in extending the results of this chapter as well as the compactification given in Chapter 2 and [26].

# APPENDIX: CATEGORICAL CONSIDERATIONS

The following definitions and theorems can be found in [24] and [1].

**Definition A.1** [24] A category **Cat** is said to be a **construct** if its objects are structured sets, i.e. pairs $(X, \xi)$ where $X$ is a set and $\xi$ a **Cat**-structure on $X$, its morphisms $f : (X, \xi) \longrightarrow (Y, \eta)$ are suitable maps between $X$ and $Y$ and its composition law is the usual composition of maps. A construct **Cat** is said to be **topological** if the following hold:

(T1) *Existence of initial structures:* For any set $X$, any family $\Big((X_i, \xi_i)\Big)_{i \in I}$ of **Cat**-objects indexed by a class $I$ and any family $(f : X \longrightarrow X_i)_{i \in I}$ of maps indexed by $I$ there exists a unique **Cat**-structure $\xi$ on $X$ such that for any **Cat**-object $(Y, \eta)$ a map $g : (Y, \eta) \longrightarrow (X, \xi)$ is a **Cat**-morphism iff for every $i \in I$ the composite map $f_i \circ g : (Y, \eta) \longrightarrow (X_i, \xi)$ is a **Cat**-morphism.

(T2) For any set $X$, the class $\{(Y, \eta) \in |\mathbf{Cat}| : X = Y\}$ of all **Cat**-objects with underlying set $X$ is a set.

(T3) For any set $X$ with cardinality at most one, there exists exactly one **Cat**-object with underlying set $X$.

The property of being a topological category is quite useful. For example, suppose the category **Cat** is topological, $(X, \xi) \in \mathbf{Cat}$ and $A \subseteq X$. The initial structure with respect to the natural injection $j : A \longrightarrow X$ defines a **Cat**-structure on $A$, say $\xi_A$. The **Cat**-object $(A, \xi_A)$ is often called a sub-structure of $(X, \xi)$.

Another example of the use of a topological category is the existence of product structures. Suppose that **Cat** is a topological category and $\Big((X_i, \xi_i)\Big)_{i \in I}$ are **Cat**-objects indexed by a class $I$. Let $\prod_{i \in I} X_i$ be the product set, and let $\prod_{i \in I} \xi_i$ be the initial structure on $\prod_{i \in I} X_i$ defined by the family of maps $\left(\pi_i : \prod_{i \in I} X_i \longrightarrow X_i\right)_{i \in I}$ where $\pi_i$ is the $i^{\text{th}}$ projection map. In this way we may obtain products in the topological category **Cat**.

Perhaps the greatest benefit to topological constructs is the existence of final structures. The following theorem appears in [24] as Theorem 1.2.1.1.

**Theorem A.1.** [24] *Let* **Cat** *be a construct. Then the following are equivalent:*

*(a)* **Cat** *satisfies (T1) in Definition 5.*

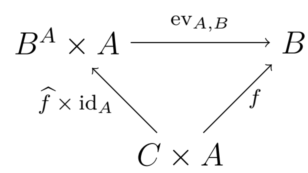
*(b) For any set $X$, any family $\big((X_i, \xi_i)\big)_{i \in I}$ of* **Cat***-objects indexed by some class $I$ and any family $(f_i : X_i \longrightarrow X)_{i \in I}$ of maps indexed by $I$ there exists a unique* **Cat***-structure $\xi$ on $X$ which is final with respect to $\big((X_i, \xi_i), f_i, X, I\big)$, i.e. such that for any* **Cat***-object $(Y, \eta)$ a map $g : (X, \xi) \longrightarrow (Y, \eta)$ is a* **Cat***-morphism iff for every $i \in I$ the composite map $g \circ f_i : (X_i, \xi_i) \longrightarrow (Y, \eta)$ is a* **Cat***-morphism.*

In Definition A.1 (T1) above, the structure $(f_i : X \longrightarrow X_i)_{i \in I}$ is often called a source and in Theorem A.1 (b) above, the structure $(f_i : X_i \longrightarrow X)_{i \in I}$ is often called a sink.

**Definition A.2** [24] A category **Cat** is called **Cartesian closed** provided the following conditions are satisfied:

(CC1) For each pair $(A, B)$ of **Cat**-objects, there exists a product $A \times B$ in **Cat**.

(CC2) For each **Cat**-object $A$, the following holds: For each **Cat**-object $B$, there exists some **Cat**-object $B^A$ (called *power object*) and some **Cat**-morphism $\mathrm{ev}_{A,B} : B^A \times A \longrightarrow B$ (called *evaluation morphism*) such that for each **Cat**-object $C$ and each **Cat**-morphism $f : C \times A \longrightarrow B$, there exists a unique **Cat**-morphism $\widehat{f} : C \longrightarrow B^A$ such that the diagram below commutes:




**Definition A.3** [24] (i) Let $(X, \xi), (Y, \eta)$ be objects of the topological construct **Cat**. Then the **Cat**-morphism $f : (X, \xi) \longrightarrow (Y, \eta)$ is said to be a **quotient map** if $f : X \longrightarrow Y$ is surjective and $\eta$ is the final **Cat**-structure with respect to the sink $f : (X, \xi) \longrightarrow Y$.

(ii) A topological construct is called **strongly Cartesian closed** provided it is Cartesian closed and the product of quotient maps in **Cat** are quotient maps in **Cat**.

**Definition A.4** [24] (i) In a topological construct **Cat**, a **partial morphism** from $A$ to $B$ is a **Cat**-morphism $f : C \longrightarrow B$ whose domain is a subobject of $A$.

(ii) A topological construct **Cat** is called **extensional** provided that every **Cat**-object $B$ has a **one-point extension** $B^* \in |\mathbf{Cat}|$, i.e. every $B \in |\mathbf{Cat}|$ can be embedded via the addition of a single point $\infty_B$ into a **Cat**-object $B^*$ such that, for every partial morphism $f : C \longrightarrow B$ from $A$ to $B$, the map $f^* : A \longrightarrow B^*$, defined by

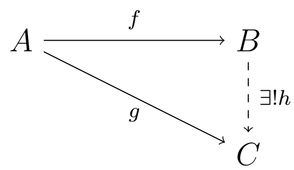
$$f^*(a) = \begin{cases} f(a), & a \in C \\ \infty_B, & a \notin C \end{cases},$$

is a **Cat**-morphism.

(iii) A topological construct **Cat** is called a **topological universe** if it is Cartesian closed and extensional. It is called a **strong topological universe** if it is strongly Cartesian closed and extensional.

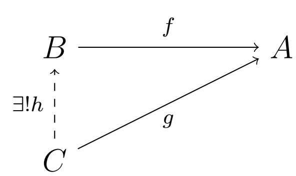
**Definition A.5** [24] Let **Cat** be a category and **Subcat** be a full subcategory of **Cat**.

We say that **Subcat** is **reflective** in **Cat** if for each $A \in |\mathbf{Cat}|$ there exists an object $B \in |\mathbf{Subcat}|$ and morphism $f : A \longrightarrow B$ of **Cat** such that for each morphism $g : A \longrightarrow C$ of **Cat** with $C \in |\mathbf{Subcat}|$, there exists a unique **Subcat** morphism $h : B \longrightarrow C$ so that the diagram below commutes:




Further, if $f : A \longrightarrow B$ can be chosen to be a bijection, then **Subcat** is said to be a **bireflective** subcategory of **Cat**.

**Definition A.6** [24] Let **Cat** be a category and **Subcat** be a full subcategory of **Cat**. We say that **Subcat** is **coreflective** in **Cat** if for each $A \in |\mathbf{Cat}|$ there exists an object $B \in |\mathbf{Subcat}|$ and morphism $f : B \longrightarrow A$ of **Cat** such that for each morphism $g : C \longrightarrow A$ of **Cat** with $C \in |\mathbf{Subcat}|$, there exists a unique **Subcat** morphism $h : C \longrightarrow B$ so that the diagram below commutes:




Further, if $f : B \longrightarrow A$ can be chosen to be a bijection, then **Subcat** is said to be a **bicoreflective** subcategory of **Cat**.

**Definition A.7** [1] Let **CAT** be a category. A **concrete** category over **CAT** is a pair $(\mathbf{Cat}, U)$ such that **Cat** is a category and $U : \mathbf{Cat} \longrightarrow \mathbf{CAT}$ is a faithful functor, often times the forgetful functor.

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