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# Not All Traces on the Circle Come from Functions of Least Gradient in the Disk

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# *Not All Traces on the Circle Come from Functions of Least Gradient in the Disk*

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ABSTRACT. We provide an example of an $L^1$ function on the unit circle that cannot be the trace of a function of bounded variation of least gradient in the unit disk.

## 1. INTRODUCTION

Sternberg *et al.* in [3], and Sternberg and Ziemer in [4], consider the question of existence, uniqueness, and regularity for functions of least gradient and prescribed trace. More precisely, for $\Omega \subset \mathbb{R}^n$ a Lipchitz domain, and for a continuous map $g \in C(\partial\Omega)$, the authors formulate the problem

$$\min\left\{ \int_\Omega |\mathrm{D}u| : u \in \mathrm{BV}(\Omega),\ u|_{\partial\Omega} = g \right\}, \tag{1.1}$$

where $\mathrm{BV}(\Omega)$ denotes the space of functions of bounded varation, the integral is understood in the sense of the Radon measure $|\mathrm{D}u|$ of $\Omega$, and the trace at the boundary is in the sense of the trace of functions of bounded varation. Solutions to the minimization problem (1.1) are called functions of least gradient. For domains $\Omega$ with boundary of non-negative curvature, and that are not locally area minimizing, the authors prove existence, uniqueness, and regularity of the solution. Moreover, if the boundary of the domain fails either of the two assumptions, they provide counterexamples to existence.

It is known that traces of functions $f \in \mathrm{BV}(\Omega)$ of bounded variation are in $L^1(\partial\Omega)$, and that conversely, any function in $L^1(\partial\Omega)$ admits an extension (in the sense of trace) in $\mathrm{BV}(\Omega)$ (in fact, in $W^{1,1}(\Omega)$); see, for example, [1]. The question we address here is whether solutions of the problem (1.1) exist in the case of traces that are merely in $L^1(\partial\Omega)$ and not continuous. We answer this question in the



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negative by providing a counterexample for the unit disk, which has a boundary of positive curvature and which is not locally length minimizing.

Let $\mathbb{D}$ denote the unit disk in the plane, and $\mathbb{S}$ be its boundary. We prove the following:

***Theorem 1.1.*** *There exists $f \in L^1(\mathbb{S})$ such that the minimization problem*

$$\min\left\{\int_{\mathbb{D}} |Dw| : w \in \mathrm{BV}(\mathbb{D}),\ w|_{\mathbb{S}} = f\right\}$$

*has no solution.*

A renewed interest in functions of least gradient with variable weights appeared recently because of its applications to current density impedance imaging (see [2] and references therein). Our counterexample sets a limit on the roughness of the boundary data one can afford to use.

## 2. PROOF OF THEOREM 1.1

We will call the $L^1(\mathbb{S})$-function satisfying Theorem 1.1 "$f_\infty$". This function is the characteristic function of a fat Cantor set. Define $C_0 \supset C_1 \supset C_2 \supset \cdots$ inductively as follows:

$$C_0 = \left\{(\cos\theta, \sin\theta) \mid \frac{\pi}{2} - \frac{1}{2} \le \theta \le \frac{\pi}{2} + \frac{1}{2}\right\};$$

if $C_n$ consists of $2^n$ disjoint closed arcs, each with arc length

$$\theta_n = \frac{1}{2^n}\prod_{i=1}^{n}\left(1 - \frac{1}{2^i}\right) \tag{2.1}$$

(if $n = 0$, the "empty product" is interpreted as 1), then $C_{n+1}$ is obtained by removing an open arc of arc length $(1-1/2^{n+1})\theta_n$ from the center of each of those arcs. Then, $C_{n+1}$ consists of $2^{n+1}$ disjoint closed arcs, each with arc length $\theta_{n+1}$. For $n = 0, 1, 2, \ldots$, with $\mathcal{H}^1$ denoting one-dimensional Hausdorff measure,

$$\mathcal{H}^1(C_n) = 2^n\theta_n = \prod_{i=1}^{n}\left(1 - \frac{1}{2^i}\right) \equiv K_n. \tag{2.2}$$

Define $C_\infty = \bigcap_{n=0}^{\infty} C_n$. Then, $C_\infty$ is a compact and nowhere dense subset of $\mathbb{S}$, with

$$\mathcal{H}^1(C_\infty) = \prod_{i=1}^{\infty}\left(1 - \frac{1}{2^i}\right) = \lim_{n\to\infty} K_n \equiv K_\infty > 0.$$

Note that $K_\infty$ is well defined and positive, since all the terms in the infinite product are positive, and $\sum_{i=1}^{\infty} 1/2^i < \infty$.

We define $f_\infty \in L^1(\mathbb{S})$ to be the characteristic function of $C_\infty$:

$$f_\infty = \chi_{C_\infty} \in L^1(\mathbb{S}).$$

From [1, Theorem 2.16, Remark 2.17], we have that

$$\inf\left\{\int_{\mathbb{D}} |\mathrm{D}u| : u \in \mathrm{BV}(\mathbb{D}),\ u|_{\mathbb{S}} = f_\infty\right\} \le \|f_\infty\|_{L^1(\mathbb{S})} = K_\infty.$$

We will show that, for any $u \in \mathrm{BV}(\mathbb{D})$ with $u|_{\mathbb{S}} = f_\infty$,

$$\int_{\mathbb{D}} |\mathrm{D}u| > K_\infty,$$

proving Theorem 1.1.

The idea of the proof is as follows: we construct a compact subset $B_\infty$ of $\bar{\mathbb{D}}$, with empty interior, with the property that

(2.3.i) If $u \in \mathrm{BV}(\mathbb{D})$ with $u|_{\mathbb{S}} = f_\infty$ and $\displaystyle\int_{\mathbb{D}\setminus B_\infty} |u|\,\mathrm{d}x > 0$,

then $\displaystyle\int_{\mathbb{D}} |\mathrm{D}u| > K_\infty$,

and

(2.3.ii) If $u \in \mathrm{BV}(\mathbb{D})$ with $u|_{\mathbb{S}} = f_\infty$ and $\displaystyle\int_{\mathbb{D}\setminus B_\infty} |u|\,\mathrm{d}x = 0$,

then $\displaystyle\int_{\mathbb{D}} |\mathrm{D}u| > K_\infty$.

Theorem 1.1 obviously follows from this. $B_\infty$ has the form

$$B_\infty = \bigcap_{n=1}^{\infty} B_n, \tag{2.4}$$

where $B_1 \supset B_2 \supset B_3 \supset \cdots$, and for each $n \ge 1$, $B_n$ is a compact subset of $\bar{\mathbb{D}}$ with $2^n$ path components, with each path component the union of a polygon and two circular segments ("circular segment" is the standard term for the region between an arc and a chord connecting two points on a circle). That polygon will be defined precisely as the union of at least one triangle with at least one trapezoid. In Figure 2.2, $B_1$ is the union of the two shaded regions. In Figure 2.3, the two shaded regions constitute the upper portion of the right half of $B_2$. The four shaded regions in Figure 2.4 are indistinguishable from the top portion of $B_3$ (the set $S_1$ mentioned in the caption does not include eight tiny circular segments that hug $\mathbb{S}$ and are so small they are not visible in the figure). The entire set $B_3$ is formed by extending the shaded regions in Figure 2.4 downward to the bottom of

$\mathbb{S}$, similarly to $B_1$ (see Figure 2.2). In all four figures, the arclengths and lengths of arcs and line segments are not necessarily scaled consistently with (2.1), but were chosen to try to make the figures easy to read.

Unfortunately, defining each $B_n$ precisely requires a slew of definitions. For $n \geq 0$, $C_n$ is the disjoint union of $2^n$ closed arcs. Call this collection of arcs $\mathcal{A}_n$. For example, $\mathcal{A}_0 = \{C_0\}$. For each $A \in \mathcal{A}_n$, we will define a set $B_A \subset \bar{\mathbb{D}}$, then define $B_n$ as the disjoint union

$$B_n = \bigsqcup_{A \in \mathcal{A}_n} B_A. \tag{2.5}$$

Each such $B_A$ is the connected union of the following: a closed circular segment, $n$ closed polygons (all of which are triangles or trapezoids), and a "bottom" piece that is the connected union of a trapezoid and a (different) circular segment. (In Figure 2.2, the arc in $\mathcal{A}_1$ in the right half of the $x_1$-$x_2$ plane is called "$A$", and $B_A$ is the shaded region in the right half of $\bar{\mathbb{D}}$. If we call the other arc in $\mathcal{A}_1$ "$A'$", then the shaded region in the left half of $\bar{\mathbb{D}}$ is $B_{A'}$. In Figure 2.3, the shaded region on the left is the top of $B_\alpha$, and the shaded region on the right is the top of $B_\beta$, where $\alpha$ and $\beta$ are the two arcs in $\mathcal{A}_2$ in the right half of the $x_1$-$x_2$ plane. The other notation used in Figures 2.2 and 2.3 will be defined momentarily). For an arc $A$, let $\mathrm{Cho}(A)$ denote the chord connecting the endpoints of $A$, and $W(A)$ the closed circular segment enclosed by $A$ and $\mathrm{Cho}(A)$. For $A \in \mathcal{A}_n$ with $n \geq 1$, define $\mathrm{Par}(A)$ (the "parent" of $A$) to be the arc in $\mathcal{A}_{n-1}$ containing $A$:

$$\mathrm{Par}(A) = A' : A' \in \mathcal{A}_{n-1},\ A \subset A'.$$

More generally, for $A \in \mathcal{A}_n$ ($n \geq 0$), define

$$\begin{aligned}
&\mathrm{Par}^0(A) = A,\ \mathrm{Par}^1(A) = \mathrm{Par}(A),\ \mathrm{Par}^2(A) = \mathrm{Par}(\mathrm{Par}^1(A)),\\
&\mathrm{Par}^3(A) = \mathrm{Par}(\mathrm{Par}^2(A)), \dots,\ \mathrm{Par}^n(A) = C_0 \in \mathcal{A}_0.
\end{aligned}$$

For $A \in \mathcal{A}_n$ ($n \geq 0$), define the two "children" of $A$, $\mathrm{Chi}_L(A)$ and $\mathrm{Chi}_R(A)$, by

$$\begin{aligned}
&\mathrm{Chi}_L(A),\ \mathrm{Chi}_R(A) \in \mathcal{A}_{n+1},\ \mathrm{Chi}_L(A), \mathrm{Chi}_R(A) \subset A,\\
&\mathrm{Chi}_L(A) \cap \mathrm{Chi}_R(A) = \emptyset,\\
&\mathrm{Chi}_L(A) \text{ is "to the left" or counterclockwise from } \mathrm{Chi}_R(A).
\end{aligned}$$

For an arc $A$ of $\mathbb{S}$ of arc length less than $\pi$, let $\mathbf{v}(A)$ denote the unit vector perpendicular to $\mathrm{Cho}(A)$ and pointing from $\mathrm{Cho}(A)$ toward $\mathbf{0} \in \mathbb{R}^2$. For $A \in \mathcal{A}_n$ with $n \geq 1$, let $T(A)$ denote the unique closed right triangle whose longer leg is $\mathrm{Cho}(A)$ and whose hypotenuse is a subset of $\mathrm{Cho}(\mathrm{Par}(A))$.

Figure 2.1 shows an arc $A$ belonging to $\mathcal{A}_n$ for some $n \geq 1$, along with $\operatorname{Par}(A)$, $\operatorname{Cho}(A)$, $\operatorname{Cho}(\operatorname{Par}(A))$, $T(A)$, and $\mathbf{v}(A)$. The lengths of the segments and arcs are not necessarily to scale, and the length of $\mathbf{v}(A)$ is definitely not to scale, since $\mathbf{v}(A)$ is a unit vector and $\operatorname{Par}(A)$ is an arc of $\mathbb{S}$.

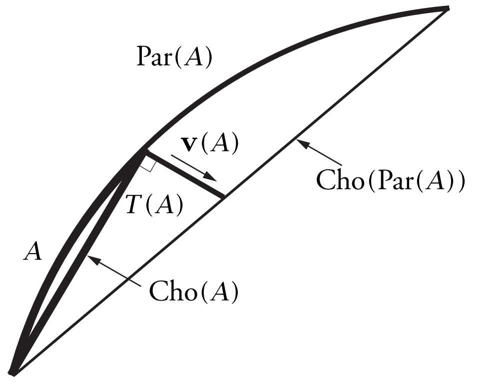


FIGURE 2.1.

We are finally ready to define $B_A$ (for $A \in \mathcal{A}_n$ with $n \geq 1$). We will do the $n = 1$ and $n = 2$ cases first, then the general case. Suppose $A \in \mathcal{A}_1$ (so $A = \operatorname{Chi}_L(C_0)$ or $\operatorname{Chi}_R(C_0)$). Note that $B_A$ is the union of $W(A)$, $T(A)$, and a "bottom" piece that is the union of a trapezoid and a circular segment. To help establish the pattern for general $n$, we introduce some notation that is not needed here but will be necessary later. Define $L_1(A)$ to be $\operatorname{Cho}(A)$ (a line segment), $T_0(A)$ to be $T(A)$ (a triangle), and $L_2(A)$ (a line segment, of course) to be the hypotenuse of $T_0(A)$, which can also be defined by

$$L_2(A) = \left\{ \mathbf{x} \in \partial T_0(A) \mid x_2 = \sin\left(\frac{\pi}{2} - \frac{1}{2}\right) \right\}. \tag{2.6}$$

Define the "bottom" part of $B_A$, $\operatorname{Bot}(A)$, to be the set of all points in $\bar{\mathbb{D}}$ on or directly "below" $L_2(A)$; that is,

$$\operatorname{Bot}(A) = \left\{ \mathbf{x} \in \bar{\mathbb{D}} \mid x_2 \leq \sin\left(\frac{\pi}{2} - \frac{1}{2}\right),\ x_1 = y_1 \text{ for some } \mathbf{y} \in L_2(A) \right\}. \tag{2.7}$$

$\operatorname{Bot}(A)$ is the union of a closed trapezoid and a closed circular segment. Finally, define

$$B_A = W(A) \cup T_0(A) \cup \operatorname{Bot}(A). \tag{2.8}$$

In Lemma A.1 in Appendix A, it is proven that, for any $A \in \mathcal{A}_n$ (for $n \geq 0$), $T(\operatorname{Chi}_L(A))$ and $T(\operatorname{Chi}_R(A))$ are disjoint (use $\theta = \theta_n$ and $\alpha = \theta_n/2^{n+1} \geq \theta_n^2/2$,

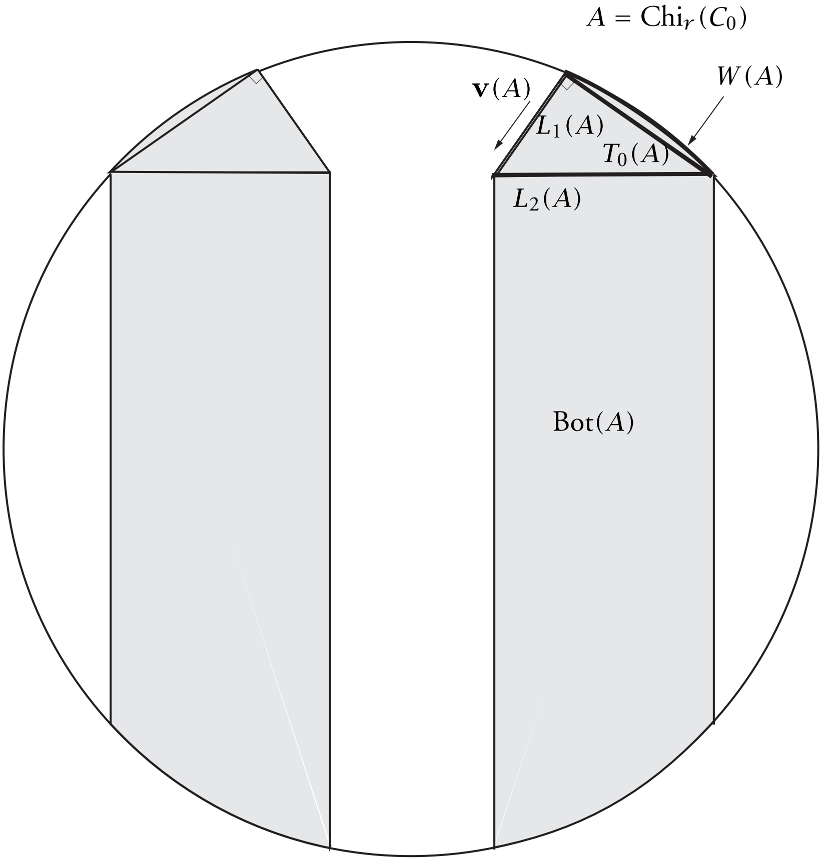


FIGURE 2.2. $B_1$

with $\theta_n$ as in (2.1)). It follows that $B_{\mathrm{Chi}_L(A)}$ and $B_{\mathrm{Chi}_R(A)}$ are disjoint. Now, $B_1$ is defined as in (2.5). The two shaded regions in Figure 2.2 compose $B_1$. The arc $\mathrm{Chi}_R(C_0)$ is called $A$, and the parts of $B_A$ (which is the right half of $B_1$) are labeled, along with the vector $\mathbf{v}(A)$, which is perpendicular to $\mathrm{Cho}(A)$. The lengths of the segments and arcs are not truly scaled, and the unit vector $\mathbf{v}(A)$ is drawn with shorter than unit length in order to fit in the picture.

Next, suppose $A \in \mathcal{A}_2$. $B_A$ is the union of a chain of four sets: a closed circular segment, followed by a closed triangle, then a closed triangle or trapezoid, then finally a closed "bottom" piece which is the union of a trapezoid and a circular segment, as in the $n = 1$ case. The intersection of any two consecutive sets in the chain is a line segment.

Figure 2.3 shows the top of the right half of $B_2$, so it shows the top portions of the two rightmost of the four components of $B_2$. As before, the lengths of segments and arcs are not necessarily scaled truly, and $\mathbf{v}(\mathrm{Chi}_R(C_0))$ is actually a unit vector, contrary to the picture. The arc $aj$ is $\mathrm{Chi}_R(C_0)$. For brevity in notation, we have defined $\alpha = ae = \mathrm{Chi}_L(\mathrm{Chi}_R(C_0))$ and $\beta = fj = \mathrm{Chi}_R(\mathrm{Chi}_R(C_0))$. The two connected gray regions are the upper portions of $B_\alpha$ and of $B_\beta$. Also, $B_\alpha$ is the union of $W(\alpha)$ (a very thin circular segment in the figure), the triangle $\triangle ade$, the trapezoid $abcd$, and a "bottom" piece $\mathrm{Bot}(\alpha)$ consisting of all the points in $\bar{\mathbb{D}}$ on or directly below the line segment $\overline{bc}$. Finally, $B_\beta$ is the union of $W(\beta)$ (a very thin circular segment in the figure), the triangle $\triangle fgj$, the triangle $\triangle ghj$, and a "bottom" piece $\mathrm{Bot}(\beta)$ consisting of all the points in $\bar{\mathbb{D}}$ on or directly below the line segment $\overline{hj}$.

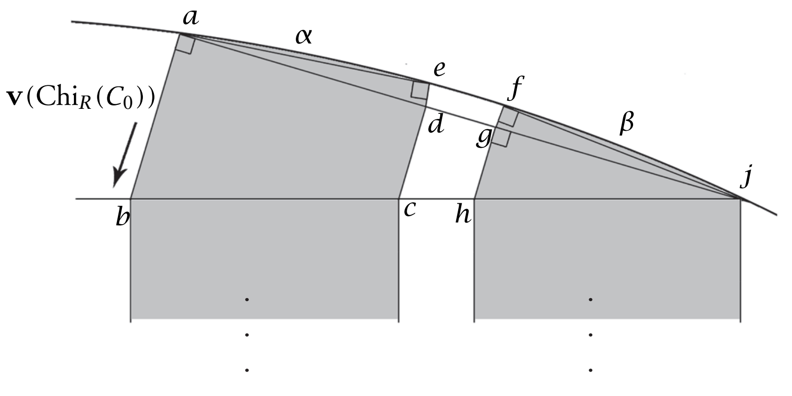


FIGURE 2.3. The upper part of the right half of $B_2$

Generally, for $A \in \mathcal{A}_2$, define the line segment $L_1(A) = \mathrm{Cho}(A)$ (thus, $L_1(\alpha) = \overline{ae}$ and $L_1(\beta) = \overline{fj}$), and define the triangle $T_0(A) = T(A)$ (thus, $T_0(\alpha) = \triangle ade$ and $T_0(\beta) = \triangle fgj$). Define the line segment

$$L_2(A) = \partial T_0(A) \cap \mathrm{Cho}(\mathrm{Par}(A)).$$

Then, $L_2(A)$ can also be described as the hypotenuse of $T_0(A)$. In Figure 2.3, $L_2(\alpha) = \overline{ad}$ and $L_2(\beta) = \overline{gj}$. Define $T_1(A) \subset T(\mathrm{Par}^1(A))$ to be the set of all points $\mathbf{x}$ in the triangle $T(\mathrm{Par}^1(A))$ with the property that, for some point $\mathbf{y} \in L_2(A)$, the vector $\mathbf{x} - \mathbf{y}$ is parallel to $\mathbf{v}(\mathrm{Par}^1(A)) \equiv \mathbf{v}(\mathrm{Par}(A))$. Note that $T_1(A)$ is either a triangle (this occurs if $A = \mathrm{Chi}_L(\mathrm{Chi}_L(C_0))$ or $\mathrm{Chi}_R(\mathrm{Chi}_R(C_0))$) or a trapezoid (this occurs if $A = \mathrm{Chi}_L(\mathrm{Chi}_R(C_0))$ or $\mathrm{Chi}_R(\mathrm{Chi}_L(C_0))$). In Figure 2.3, $T_1(\alpha)$ is the trapezoid $abcd$, with $\alpha = \mathrm{Chi}_L(\mathrm{Chi}_R(C_0))$, and $T_1(\beta)$ is the triangle $\triangle ghj$, with $\beta = \mathrm{Chi}_R(\mathrm{Chi}_R(C_0))$. $T_1(A)$ can be defined succinctly by

$$T_1(A) = \{\mathbf{x} \in T(\mathrm{Par}^1(A)) : \mathbf{x} - \mathbf{y} \parallel \mathbf{v}(\mathrm{Par}^1(A)) \text{ for some } \mathbf{y} \in L_2(A)\}.$$

In a way similar to what was done in (2.6), define the horizontal line segment $L_3(A)$ to be the set of all points in $\partial T_1(A)$ with $x_2$-coordinate $\sin(\pi/2 - \frac{1}{2})$:

$$L_3(A) = \left\{ \mathbf{x} \in \partial T_1(A) \mid x_2 = \sin\left(\frac{\pi}{2} - \frac{1}{2}\right) \right\}.$$

In other words, $L_3(A)$ is the side of the polygon $\partial T_1(A)$ that is a subset of the horizontal line $\{\mathbf{x} \mid x_2 = \sin(\pi/2 - \frac{1}{2})\}$. In Figure 2.3, we have $L_3(\alpha) = \overline{bc}$ and $L_3(\beta) = \overline{hj}$. As in (2.7), define the "bottom" part of $B_A$, $\mathrm{Bot}(A)$, to be the set of all points in $\bar{\mathbb{D}}$ on or directly below $L_3(A)$; that is,

$$\mathrm{Bot}(A) = \left\{ \mathbf{x} \in \bar{\mathbb{D}} \mid x_2 \le \sin\left(\frac{\pi}{2} - \frac{1}{2}\right),\ x_1 = y_1 \text{ for some } \mathbf{y} \in L_3(A) \right\}.$$

As before, $\mathrm{Bot}(A)$ is the union of a trapezoid and a circular segment. In a way similar to what was done in (2.8), define

$$B_A = W(A) \cup T_0(A) \cup T_1(A) \cup \mathrm{Bot}(A).$$

As in the $n = 1$ case, by Lemma A.1 in Appendix A, the sets $B_A$ for the four elements of $\mathcal{A}_2$ are disjoint. $B_2$ is defined by (2.5). Clearly, $B_2 \subset B_1$.

Finally, we consider the $n > 2$ case. Let $A \in \mathcal{A}_n$. Note that $B_A$ is the union of a chain of $n+2$ closed sets: a closed circular segment, followed by $n$ closed polygons that are all triangles or trapezoids, and finally a bottom piece called $\mathrm{Bot}(A)$ (as before) that is the union of a closed trapezoid and a closed circular segment. Either all $n$ of the polygons are triangles, or (more likely), the first $k$ of them are triangles for some $1 \le k \le n-1$, while the remaining $n-k$ polygons are trapezoids. The intersection of any two consecutive sets in the chain is a line segment. Also, $B_A$ has the form

$$B_A = W(A) \cup \bigcup_{k=0}^{n-1} T_k(A) \cup \mathrm{Bot}(A),$$

where $T_k(A)$ and $\mathrm{Bot}(A)$ will be defined precisely in a moment. To do so, we must also name the intersections of consecutive sets in the chain, which are line segments, and which we will call $L_1(A), \dots, L_{n+1}(A)$. We will also need to use $L_1(A), \dots, L_{n+1}(A)$ to prove (2.3).

Define

(2.9.i) $$L_1(A) = \mathrm{Cho}(A),$$

(2.9.ii) $$T_0(A) = T(A),$$

(2.9.iii) $$L_2(A) = \partial T_0(A) \cap \mathrm{Cho}(\mathrm{Par}^1(A)),$$

(2.9.iv) $$T_1(A) = \Big\{ \mathbf{x} \in T(\mathrm{Par}^1(A)) : \mathbf{x} - \mathbf{y} \parallel \mathbf{v}(\mathrm{Par}^1(A)) \text{ for some } \mathbf{y} \in L_2(A) \Big\},$$

(2.9.v) $$L_3(A) = \partial T_1(A) \cap \operatorname{Cho}(\operatorname{Par}^2(A)),$$

(2.9.vi) $$T_2(A) = \Big\{\mathbf{x} \in T(\operatorname{Par}^2(A)) : \mathbf{x} - \mathbf{y} \parallel \mathbf{v}(\operatorname{Par}^2(A)) \\ \text{for some } \mathbf{y} \in L_3(A)\Big\},$$

$$\vdots$$

(2.9.vii) $$T_{n-1}(A) = \Big\{\mathbf{x} \in T(\operatorname{Par}^{n-1}(A)) : \mathbf{x} - \mathbf{y} \parallel \mathbf{v}(\operatorname{Par}^{n-1}(A)) \\ \text{for some } \mathbf{y} \in L_n(A)\Big\},$$

(2.9.viii) $$L_{n+1}(A) = \left\{\mathbf{x} \in \partial T_{n-1}(A) : x_2 = \sin\left(\frac{\pi}{2} - \frac{1}{2}\right)\right\},$$

(2.9.ix) $$\operatorname{Bot}(A) = \Big\{\mathbf{x} \in \bar{\mathbb{D}} : x_2 \le \sin\left(\frac{\pi}{2} - \frac{1}{2}\right),\ x_1 = y_1 \\ \text{for some } \mathbf{y} \in L_{n+1}(A)\Big\}.$$

As before, the $B_A$ are disjoint for all the $2^n$ arcs $A$ in $\mathcal{A}_n$, and $B_n$ is defined by (2.5). Clearly, $B_1 \supset B_2 \supset B_3 \supset \cdots$. We define $B_\infty$ by (2.4).

Having defined $B_\infty$, we show that it has property (2.3), from which Theorem 1.1 follows. This requires three lemmas, followed by an easy proof of (2.3.i), then a more involved proof of (2.3.ii).

***Lemma 2.1.*** *Let $u \in C^\infty(\mathbb{D}) \cap \mathrm{BV}(\mathbb{D})$ with $u|_{\mathbb{S}} = f_\infty$, $n \ge 1$, and $A \in \mathcal{A}_n$. Let $T_0(A), T_1(A), \dots, T_{n-1}(A)$ and* $\operatorname{Bot}(A)$ *be as in* (2.9). *Then,*

(2.10) $$\int_{W(A)} |\nabla u \cdot \mathbf{v}(A)|\,\mathrm{d}x + \sum_{k=0}^{n-1} \int_{T_k(A)} |\nabla u \cdot \mathbf{v}(\operatorname{Par}^k(A))|\,\mathrm{d}x \\ + \int_{\operatorname{Bot}(A)} |\nabla u \cdot \mathbf{j}|\,\mathrm{d}x \ge \cos\left(\frac{K_n}{2^{n+1}}\right)\frac{K_\infty}{2^n}.$$

Here, $\mathbf{j} = \langle 0, 1\rangle$, as usual, and $K_n$ is from (2.2). There is a slight abuse of notation in (2.10): the domain of $u$ is $\mathbb{D}$, not $\bar{\mathbb{D}}$, but $W(A)$ and $\operatorname{Bot}(A)$ intersect $\mathbb{S} \equiv \partial\mathbb{D}$, and $T_k(A)$ may intersect $\mathbb{S}$. In all cases, the intersection has $\mathcal{H}^2$-measure zero. It would be better formally to replace "$W(A)$", "$T_k(A)$", and "$\operatorname{Bot}(A)$" in (2.10) with their interiors, or with their intersections with $\mathbb{D}$. However, this might make the proof of Lemma 2.1 less readable, so we will keep the notation of (2.10) in the proof of the lemma, and in the remainder of this section.

*Proof.* Define $s_n = 2\sin(K_n/2^{n+1})$, which is the length of $\operatorname{Cho}(A)$. Let $L_1(A), \dots, L_{n+1}(A)$ be as in (2.9). For $k = 1, 2, \dots, n+1$, let $\varphi_k : [0, s_n] \to \bar{\mathbb{D}}$ be the linear map with $\varphi_k(0)$ the left endpoint of $L_k(A)$ and $\varphi_k(s_n)$ the right endpoint of $L_k(A)$ ($L_k(A)$ is not vertical). Define $\varphi_0 : (0, s_n) \to A$ so that $\varphi_0(t)$ is the projection of $\varphi_1(t)$ onto $A$ in the direction $-\mathbf{v}(A)$ (the explicit formula for

$\varphi_0(t)$ is fairly complicated; since we do not use it, we omit it here). Now, define $g_0, g_1, \dots, g_{n+1} \in L^1((0, s_n))$ by

$$g_0(t) = f_\infty(\varphi_0(t)), \; g_k(t) = u(\varphi_k(t)) \quad \text{for } 1 \le k \le n+1.$$

Now, $\mathcal{H}^1(C_\infty \cap A) = K_\infty/2^n$, so $\int_A f_\infty \, \mathrm{d}\mathcal{H}^1 = K_\infty/2^n$. Recall $\theta_n$ from (2.5). Since the angle between $A$ and $\mathrm{Cho}(A)$ is at most $\theta_n/2 = K_n/2^{n+1}$, we have

$$\|g_0\|_{L^1((0,s_n))} \ge \cos\left(\frac{K_n}{2^{n+1}}\right) \int_A f_\infty \, \mathrm{d}\mathcal{H}^1 = \cos\left(\frac{K_n}{2^{n+1}}\right) \frac{K_\infty}{2^n}. \tag{2.11}$$

Obviously,

$$g_0 = (g_0 - g_1) + (g_1 - g_2) + (g_2 - g_3) + \cdots + (g_n - g_{n+1}) + g_{n+1},$$

so by the triangle inequality,

$$\begin{aligned} \|g_0\|_{L^1((0,s_n))} \le \|g_1 - g_0\|_{L^1((0,s_n))} &+ \sum_{k=1}^{n} \|g_{k+1} - g_k\|_{L^1((0,s_n))} \\ &+ \|g_{n+1}\|_{L^1((0,s_n))}. \end{aligned} \tag{2.12}$$

Now,

$$\|g_1 - g_0\|_{L^1((0,s_n))} = \int_0^{s_n} |g_1(t) - g_0(t)| \, \mathrm{d}t \le \int_{W(A)} |\nabla u(x) \cdot \mathbf{v}(A)| \, \mathrm{d}x. \tag{2.13}$$

For $1 \le k \le n$, the Fundamental Theorem of Calculus yields

$$\begin{aligned} \|g_{k+1} - g_k\|_{L^1((0,s_n))} &= \int_0^{s_n} |g_{k+1}(t) - g_k(t)| \, \mathrm{d}t \\ &\le \int_{T_{k-1}(A)} |\nabla u(x) \cdot \mathbf{v}(\mathrm{Par}^{k-1}(A))| \, \mathrm{d}x. \end{aligned} \tag{2.14}$$

Since $f_\infty = 0$ on the bottom half of $\mathbb{S}$, $u|_{\mathbb{S}} = f_\infty$, and $\mathrm{Bot}(A) \cap \mathbb{S}$ is a subset of the bottom half of $\mathbb{S}$, it follows that

$$\|g_{k+1}\|_{L^1((0,s_n))} = \int_0^{s_n} |g_{k+1}(t)| \, \mathrm{d}t \le \int_{\mathrm{Bot}(A)} |\nabla u \cdot \mathbf{j}| \, \mathrm{d}x. \tag{2.15}$$

Putting (2.11) and (2.12)–(2.15) together yields (2.10). □

Define $\mathbb{D}_- \subset \mathbb{D}$, the "lower part" of $\mathbb{D}$, by

$$\mathbb{D}_- = \left\{ \mathbf{x} \in \mathbb{D} \mid x_2 < \sin\left(\frac{\pi}{2} - \frac{1}{2}\right) \right\}. \tag{2.16}$$

From Lemma 2.1, we have the following result.

***Lemma 2.2.*** *Let $u$ be as in Lemma 2.1: $u \in C^\infty(\mathbb{D}) \cap \mathrm{BV}(\mathbb{D})$ with $u|_{\mathbb{S}} = f_\infty$. Let $n \geq 1$. Then,*

$$\sum_{A \in \mathcal{A}_n} \int_{W(A)} |\nabla u \cdot \mathbf{v}(A)|\, \mathrm{d}x + \sum_{m=1}^{n} \sum_{A \in \mathcal{A}_m} \int_{T(A) \cap B_n} |\nabla u \cdot \mathbf{v}(A)|\, \mathrm{d}x + \int_{B_n \cap \mathbb{D}_-} |\nabla u \cdot \mathbf{j}|\, \mathrm{d}x \geq \cos\left(\frac{K_n}{2^{n+1}}\right) K_\infty. \tag{2.17}$$

*Proof.* By Lemma 2.1,

$$\sum_{A \in \mathcal{A}_n} \int_{W(A)} |\nabla u \cdot \mathbf{v}(A)|\, \mathrm{d}x + \sum_{A \in \mathcal{A}_n} \sum_{k=0}^{n-1} \int_{T_k(A)} |\nabla u \cdot \mathbf{v}(A)|\, \mathrm{d}x + \sum_{A \in \mathcal{A}_n} \int_{\mathrm{Bot}(A)} |\nabla u \cdot \mathbf{j}|\, \mathrm{d}x \geq \cos\left(\frac{K_n}{2^{n+1}}\right) K_\infty. \tag{2.18}$$

We must prove that inequalities (2.17) and (2.18) are equivalent. The right-hand sides and first terms of the left-hand sides are exactly the same. The third summands in the left-hand sides are equal because $B_n \cap \bar{\mathbb{D}}_-$ is the disjoint union of the sets $\mathrm{Bot}(A)$ for the $2^n$ arcs $A$ in $\mathcal{A}_n$. We must show that the second summands on the left-hand sides of (2.17) and (2.18) are equal. Call the common integrand of the integrals "$g(x)$". Generally, any two distinct sets of the form $T_k(A)$, for $\ell \geq 1$, $A \in \mathcal{A}_\ell$, and $k \in \{0, \dots, \ell - 1\}$, have intersection of $\mathcal{H}^2$-measure zero. This includes the case of $k = 0$, $T_k(A) = T_0(A) \equiv T(A)$. Therefore, the second summands on the left-hand sides of (2.17) and (2.18) have the form $\int_{S_1} g\, \mathrm{d}x$ and $\int_{S_2} g\, \mathrm{d}x$, where

$$S_1 = \bigcup_{m=1}^{n} \bigcup_{A \in \mathcal{A}_m} (T(A) \cap B_n) = B_n \cap \Big( \bigcup_{m=1}^{n} \bigcup_{A \in \mathcal{A}_m} T(A) \Big),$$

$$S_2 = \bigcup_{A \in \mathcal{A}_n} \bigcup_{k=0}^{n-1} T_k(A).$$

We must show $S_1 = S_2$. This is easy to see if one uses a picture, but unfortunately is difficult to explain in words. We will do both.

In Figure 2.4, the shaded region (comprised of eight components) is $S_1$ (which equals $S_2$) in the case $n = 3$.

On one hand, the shaded region is $S_2$: $\mathcal{A}_3$ contains eight disjoint closed arcs $A$. For each such $A$, the union of the polygons $T_0(A)$, $T_1(A)$, and $T_2(A)$ equals one of the eight components of the shaded region: on top, $T_0(A)$ is a tiny triangle that is barely visible; just below, $T_1(A)$ is a larger triangle or trapezoid; and on

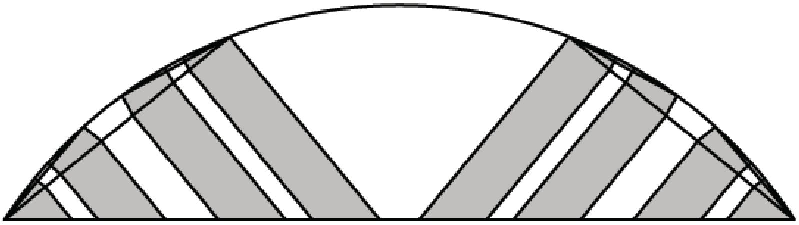

FIGURE 2.4. The set $S_1$ (which equals $S_2$) from the proof of Lemma 2.2

the bottom, $T_2(A)$ is a triangle or trapezoid, one of whose sides is a subset of the horizontal chord $\mathrm{Cho}(C_0)$ at the bottom of Figure 2.4. On the other hand, the shaded region is $S_1$: each component of the shaded region is the union of three polygons. The union of the polygons on the bottom of the components (there are eight such polygons) is $B_3 \cap \bigcup_{A\in\mathcal{A}_1} T(A)$. The union of the middle polygons of the components (there are eight such polygons) is $B_3 \cap \bigcup_{A\in\mathcal{A}_2} T(A)$. Finally, the union of the top polygons of each component (there are eight such polygons, and they are all tiny triangles) is $B_3 \cap \bigcup_{A\in\mathcal{A}_3} T(A)$.

To formally prove $S_1 = S_2$, we show that the two sets are subsets of each other. First, we show $S_1 \subset S_2$. Let $m' \in \{1,\dots,n\}$ and $A' \in \mathcal{A}_{m'}$. We will show $B_n \cap T(A') \subset S_2$. Let $A \in \mathcal{A}_n$. Since $T(A') \subset \mathbb{D} \setminus \mathbb{D}_-$ and $\mathrm{Bot}(A) \subset \bar{\mathbb{D}}_-$, $x_2 = \sin(\pi/2 - \frac{1}{2})$ along $T(A') \cap \mathrm{Bot}(A)$. Now,

$$\mathrm{Bot}(A) \cap \left\{\mathbf{x} \mid x_2 = \sin\left(\frac{\pi}{2} - \frac{1}{2}\right)\right\} = T_{n-1}(A) \cap \left\{\mathbf{x} \mid x_2 = \sin\left(\frac{\pi}{2} - \frac{1}{2}\right)\right\}.$$

Thus, we have $T(A') \cap \mathrm{Bot}(A) \subset T_{n-1}(A)$. Also, since $m' \le n$, we have that $T(A') \cap W(A) \subset T(A) \equiv T_0(A)$. Therefore,

$$\begin{aligned}
B_n \cap T(A') &\equiv \Big( \bigcup_{A\in\mathcal{A}_n} \Big(W(A) \cup \mathrm{Bot}(A) \cup \bigcup_{k=0}^{n-1} T_k(A)\Big)\Big) \cap T(A') \\
&= \bigcup_{A\in\mathcal{A}_n} \Big(\Big(W(A) \cup \mathrm{Bot}(A) \cup \bigcup_{k=0}^{n-1} T_k(A)\Big) \cap T(A')\Big) \\
&= \bigcup_{A\in\mathcal{A}_n} \bigcup_{k=0}^{n-1} (T_k(A) \cap T(A')) \subset \bigcup_{A\in\mathcal{A}_n} \bigcup_{k=0}^{n-1} T_k(A) = S_2,
\end{aligned}$$

and $S_1 \subset S_2$. Next, we prove $S_2 \subset S_1$. Let $A' \in \mathcal{A}_n$ and $k' \in \{0, \dots, n-1\}$. We will show $T_{k'}(A') \subset S_1$. First,

$$T_{k'}(A') \subset \bigcup_{k=0}^{n-1} T_k(A') \subset B_{A'} \subset B_n. \tag{2.19}$$

Next, let $m' = n - k' \in \{1, \dots, n\}$. Since we have $T_{k'}(A') \subset T(\mathrm{Par}^{k'}(A'))$ and $\mathrm{Par}^{k'}(A') \in \mathcal{A}_{n-k'} = \mathcal{A}_{m'}$, it follows that

$$T_{k'}(A') \subset T(\mathrm{Par}^{k'}(A')) \subset \bigcup_{m=1}^{n} \bigcup_{A \in \mathcal{A}_m} T(A). \tag{2.20}$$

By (2.19), (2.20), and the definition of $S_1$, we have $T_{k'}(A') \subset S_1$. Therefore, $S_2 \subset S_1$. Lemma 2.2 is proven. ◻

Now, as $n \to \infty$, $\mathcal{H}^2(\bigcup_{A \in \mathcal{A}_n} W(A)) \to 0$. Also, $K_n \to K_\infty$ as $n \to \infty$. Thus, taking limits of both sides of (2.17) as $n \to \infty$ yields the second inequality in the lemma below.

***Lemma 2.3.*** *Let $u \in C^\infty(\mathbb{D}) \cap \mathrm{BV}(\mathbb{D})$ with $u|_{\mathbb{S}} = f_\infty$. Then,*

$$\int_{B_\infty \cap \mathbb{D}} |\nabla u| \, \mathrm{d}x \geq \sum_{n=1}^{\infty} \sum_{A \in \mathcal{A}_n} \int_{T(A) \cap B_\infty} |\nabla u \cdot \mathbf{v}(A)| \, \mathrm{d}x + \int_{\mathbb{D}_- \cap B_\infty} |\nabla u \cdot \mathbf{j}| \, \mathrm{d}x \geq K_\infty.$$

The first inequality is obvious because any two different triangles in the collection $\{T(A) \mid A \in \mathcal{A}_\ell, \ \ell \geq 1\}$ have intersection with zero $\mathcal{H}^2$-measure, and all such triangles are disjoint with $\mathbb{D}_-$.

Now, let us prove (2.3.i). Suppose

$$u \in \mathrm{BV}(\mathbb{D}), \quad \text{with } u|_{\mathbb{S}} = f_\infty \text{ and } \int_{\mathbb{D} \setminus B_\infty} |u| \, \mathrm{d}x > 0.$$

$\mathbb{D} \setminus B_\infty$ consists of countably many open components. For at least one such component $\Omega$, $\int_\Omega |u| \, \mathrm{d}x > 0$. Note that $\partial\Omega$ contains an arc of $\mathbb{S}$ of positive arc length along which $f_\infty$ equals zero. Therefore,

$$\int_\Omega |\mathrm{D}u| > 0.$$

By [1, Theorem 1.17, Remark 1.18, Remark 2.12], there then exists a sequence $(u_m) \subset C^\infty(\mathbb{D}) \cap \mathrm{BV}(\mathbb{D})$ with $u_m|_{\mathbb{S}} = f_\infty$ for all $m$, $u_m \to u$ in $L^1(\mathbb{D})$, and $\int_{\mathbb{D}} |\nabla u_m| \, \mathrm{d}x \to \int_{\mathbb{D}} |\mathrm{D}u|$ as $m \to \infty$. By [1, Theorem 1.19],

$$\liminf_{m \to \infty} \int_\Omega |\nabla u_m| \, \mathrm{d}x \geq \int_\Omega |\mathrm{D}u| > 0.$$

Therefore, using Lemma 2.3,

$$\int_{\mathbb{D}} |Du| = \lim_{m\to\infty} \int_{\mathbb{D}} |\nabla u_m| \,\mathrm{d}x \geq \liminf_{m\to\infty} \left( \int_{\mathbb{D}\cap B_\infty} |\nabla u_m| \,\mathrm{d}x + \int_{\Omega} |\nabla u_m| \,\mathrm{d}x \right)$$
$$\geq B_\infty + \int_{\Omega} |Du| > B_\infty.$$

Next, we prove (2.3.ii), which will complete the proof of Theorem 1.1. Suppose $u \in \mathrm{BV}(\mathbb{D})$ with $u|_{\mathbb{S}} = f_\infty$ and $\int_{\mathbb{D}\setminus B_\infty} |u| \,\mathrm{d}x = 0$. Recall $\mathbb{D}_-$, defined in (2.16). Since $u \neq 0$,

$$\int_{\mathbb{D}_-} |u| \,\mathrm{d}x > 0 \qquad \text{or} \qquad \int_{\mathbb{D}\setminus\mathbb{D}_-} |u| \,\mathrm{d}x > 0.$$

We examine the former case first. Assume $\int_{\mathbb{D}_-} |u| \,\mathrm{d}x > 0$. Then, there exists a closed rectangle $[a,b] \times [c,d] \subset \mathbb{D}_-$ and $\delta > 0$ with

$$\int_{[a,b]\times[c,d]} |u| \,\mathrm{d}x > \delta.$$

$B_\infty$ is a compact subset of $\bar{\mathbb{D}}$ with empty interior. The restriction of $\chi_{B_\infty}$ to $\mathbb{D}_-$ is constant on vertical line segments. Therefore, there exists an open, dense subset $U$ of $[a,b]$ with

$$(U \times [c,d]) \cap B_\infty = \emptyset.$$

Let $a < a_1 < b_1 < b$ with $a_1, b_1 \in U$ and

$$\int_{[a_1,b_1]\times[c,d]} |u| \,\mathrm{d}x > \frac{\delta}{2}.$$

Let $(u_m) \subset C^\infty(\mathbb{D}) \cap \mathrm{BV}(\mathbb{D})$ be given by the construction in [1, Theorem 1.17]: $u_m|_{\mathbb{S}} = f_\infty$ for all $m$, $u_m \to u$ in $L^1(\mathbb{D})$, and $\int_{\mathbb{D}} |\nabla u| \,\mathrm{d}x \to \int_{\mathbb{D}} |Du| > 0$ as $m \to \infty$. Furthermore, the $u_m$ are obtained by convolving $u$ with $C^\infty$ mollifier functions, supported on discs, with the radii of the discs approaching 0 as $m \to \infty$ uniformly on the rectangle $[a,b] \times [c,d]$. Thus, for large enough $m$, we have $u_m = 0$ on the vertical line segments $\{a_1\} \times [c,d]$ and $\{b_1\} \times [c,d]$. By Lemma A.2 in Appendix A,

$$\int_{[a_1,b_1]\times[c,d]} \left| \frac{\partial u_m}{\partial x_1} \right| \,\mathrm{d}x \geq \frac{2}{b_1 - a_1} \int_{[a_1,b_1]\times[c,d]} |u_m| \,\mathrm{d}x > \frac{\delta}{b_1 - a_1} \equiv \delta_2$$

for large enough $m$. Clearly, for large enough $m$,

$$\int_{[a_1,b_1]\times[c,d]} \left| \frac{\partial u_m}{\partial x_2} \right| \,\mathrm{d}x \leq \int_{[a_1,b_1]\times[c,d]} |\nabla u_m| \,\mathrm{d}x \leq \int_{\mathbb{D}} |\nabla u_m| \,\mathrm{d}x < 2\int_{\mathbb{D}} |Du|.$$

Thus, for large enough $m$, by Lemma A.3 in Appendix A (using $g = |\partial u_m/\partial x_1|$ and $h = |\partial u_m/\partial x_2| = |\nabla u_m \cdot \mathbf{j}|$), we have

$$\begin{aligned}(2.21)\quad \int_{[a_1,b_1]\times[c,d]} |\nabla u_m|\,\mathrm{d}x &\geq \int_{[a_1,b_1]\times[c,d]} |\nabla u_m \cdot \mathbf{j}|\,\mathrm{d}x + \frac{\delta_2^2}{4\int_{\mathbb{D}} |\mathrm{D}u| + \delta_2} \\ &\equiv \int_{[a_1,b_1]\times[c,d]} |\nabla u_m \cdot \mathbf{j}|\,\mathrm{d}x + \delta_3.\end{aligned}$$

The collection of triangles $\{T(A) \mid A \in \mathcal{A}_\ell,\ \ell \geq 1\}$ is a countable family of sets, for which the intersection of any distinct pair has zero $\mathcal{H}^2$-measure. All these triangles are subsets of $\bar{\mathbb{D}} \setminus \mathbb{D}_-$. Therefore, applying (2.21) and Lemma 2.3, it follows that, for large enough $m$,

$$\begin{aligned}\int_{\mathbb{D}} |\nabla u_m| &= \int_{\mathbb{D}_-} |\nabla u_m|\,\mathrm{d}x + \int_{\mathbb{D}\setminus\mathbb{D}_-} |\nabla u_m|\,\mathrm{d}x \\ &\geq \int_{\mathbb{D}_-} |\nabla u_m \cdot \mathbf{j}|\,\mathrm{d}x + \delta_3 + \sum_{n=1}^{\infty}\sum_{A\in\mathcal{A}_n}\int_{T(A)} |\nabla u_m|\,\mathrm{d}x \\ &\geq \int_{\mathbb{D}_-\cap B_\infty} |\nabla u_m \cdot \mathbf{j}|\,\mathrm{d}x + \delta_3 + \sum_{n=1}^{\infty}\sum_{A\in\mathcal{A}_n}\int_{T(A)\cap B_\infty} |\nabla u_m \cdot \mathbf{v}(A)|\,\mathrm{d}x \\ &\geq K_\infty + \delta_3.\end{aligned}$$

Since $\int_{\mathbb{D}} |\nabla u_m|\,\mathrm{d}x \to \int_{\mathbb{D}} |\mathrm{D}u|$ as $m \to \infty$, it follows that $\int_{\mathbb{D}} |\mathrm{D}u| \geq K_\infty + \delta_3 > K_\infty$.

Next, suppose $\int_{\mathbb{D}\setminus\mathbb{D}_-} |u|\,\mathrm{d}x > 0$ (and $\int_{\mathbb{D}\setminus B_\infty} |u|\,\mathrm{d}x = 0$). Since

$$((\mathbb{D} \setminus \mathbb{D}_-) \cap B_\infty) \subset \bigcup_{n=1}^{\infty}\bigcup_{A\in\mathcal{A}_n} T(A),$$

there exists $n' \geq 1$ and $A \in \mathcal{A}_{n'}$ with

$$\int T(A)|u|\,\mathrm{d}x > 0.$$

There then exists a closed rectangle $R \subset T(A) \cap \mathbb{D}$ with sides parallel and perpendicular to $\mathbf{v}(A)$ and $\int_R |u|\,\mathrm{d}x > 0$.

Let $(u_m)$ be given by the construction in [1, Theorem 1.17], as before. Arguing as before, let the line segment $L$ be one of the two sides of $R$ perpendicular to $\mathbf{v}(A)$. Note that $L$ has an open and dense (with respect to the subspace topology on $L$) subset $X$ with $X \cap B_\infty = \emptyset$. From the way $B_\infty$ is constructed, if $\mathbf{x} \in R$ and the vector $\mathbf{x} - \mathbf{y}$ is parallel to $\mathbf{v}(A)$ for some $\mathbf{y} \in X$, then $\mathbf{x} \notin B_\infty$. Arguing as in the $\int_{\mathbb{D}_-} |u|\,\mathrm{d}x > 0$ case, there exists $\delta_3 > 0$ with

$$\int_R |\nabla u_m|\,\mathrm{d}x \geq \int_R |\nabla u_m \cdot \mathbf{v}(A)|\,\mathrm{d}x + \delta_3 \tag{2.22}$$

for large enough $m$. Using Lemma 2.3 and (2.22), for large enough $m$, we have

$$\begin{aligned}
\int_{\mathbb{D}} |\nabla u_m|\,\mathrm{d}x &= \int_{\mathbb{D}_-} |\nabla u_m|\,\mathrm{d}x + \int_{\mathbb{D}\setminus\mathbb{D}_-} |\nabla u_m|\,\mathrm{d}x \\
&\geq \int_{\mathbb{D}_-\cap B_\infty} |\nabla u_m|\,\mathrm{d}x + \sum_{n=1}^{\infty} \sum_{A\in\mathcal{A}_n} \int_{T(A)} |\nabla u_m|\,\mathrm{d}x \\
&\geq \int_{\mathbb{D}_-\cap B_\infty} |\nabla u_m \cdot \mathbf{j}|\,\mathrm{d}x + \delta_3 + \sum_{n=1}^{\infty} \sum_{A\in\mathcal{A}_n} \int_{T(A)} |\nabla u_m \cdot \mathbf{v}(A)|\,\mathrm{d}x \\
&\geq \int_{\mathbb{D}_-\cap B_\infty} |\nabla u_m \cdot \mathbf{j}|\,\mathrm{d}x + \delta_3 + \sum_{n=1}^{\infty} \sum_{A\in\mathcal{A}_n} \int_{T(A)\cap B_\infty} |\nabla u_m \cdot \mathbf{v}(A)|\,\mathrm{d}x \\
&\geq K_\infty + \delta_3.
\end{aligned}$$

As before, since $\int_{\mathbb{D}} |\nabla u_m|\,\mathrm{d}x \to \int_{\mathbb{D}} |\mathrm{D}u|$ as $m \to \infty$, it follows that we have $\int_{\mathbb{D}} |\mathrm{D}u| \geq K_\infty + \delta_3 > K_\infty$. The proof of Theorem 1.1 is thus complete. □

## APPENDIX A. THREE LEMMAS

This section contains three easy, self-contained lemmas, moved to the end of the paper in order not to interrupt the flow of the main proof.

***Lemma A.1.*** *Let $\theta \in (0,1]$ and $\alpha \in [\theta^2/2, \theta)$. Let $P$ and $S$ be points on $\mathbb{S}$ separated by arc length $\theta$. Let $Q$ and $R$ lie on the arc $PS$, with $QR$ having arc length $\alpha$, with $PQ$ and $RS$ having equal arc length, and with $Q$ between $P$ and $R$. Let $T$ and $U$ lie on the chord $\overline{PS}$, chosen such that $\triangle PQT$ and $\triangle RSU$ are right triangles. Then, $\triangle PQT$ and $\triangle RSU$ have disjoint closures.*

*Proof.* Clearly, it suffices to consider $\alpha = \theta^2/2$. By rotating the arc $PS$, we may assume

$$P = \left(\cos\frac{\theta}{2}, \sin\frac{\theta}{2}\right), \qquad S = \left(\cos\frac{\theta}{2}, -\sin\frac{\theta}{2}\right),$$
$$Q = \left(\cos\frac{\theta^2}{4}, \sin\frac{\theta^2}{4}\right), \quad R = \left(\cos\frac{\theta^2}{4}, -\sin\frac{\theta^2}{4}\right).$$

Define $V = (\cos(\theta/2), 0)$. It suffices to show the angle $\angle PQV$ is obtuse, using a dot product. We will show $\overrightarrow{QP} \cdot \overrightarrow{QV} < 0$. Using familiar trigonometric identities,

$$\begin{aligned}
\overrightarrow{QP} &= \left\langle \cos\left(\frac{1}{2}\theta\right) - \cos\left(\frac{1}{4}\theta^2\right), \sin\left(\frac{1}{2}\theta\right) - \sin\left(\frac{1}{4}\theta^2\right) \right\rangle, \\
\overrightarrow{QV} &= \left\langle \cos\left(\frac{1}{2}\theta\right) - \cos\left(\frac{1}{4}\theta^2\right), -\sin\left(\frac{1}{4}\theta^2\right) \right\rangle, \\
\overrightarrow{QP} \cdot \overrightarrow{QV} &= \left(\cos\left(\frac{1}{4}\theta^2\right) - \cos\left(\frac{1}{2}\theta\right)\right)^2 - \left(\sin\left(\frac{1}{2}\theta\right) - \sin\left(\frac{1}{4}\theta^2\right)\right)\sin\left(\frac{1}{4}\theta^2\right) \\
&= \cos^2\left(\frac{1}{4}\theta^2\right) + \cos^2\left(\frac{1}{2}\theta\right) - 2\cos\left(\frac{1}{4}\theta^2\right)\cos\left(\frac{1}{2}\theta\right) \\
&\qquad - \sin\left(\frac{1}{2}\theta\right)\sin\left(\frac{1}{4}\theta^2\right) + \sin^2\left(\frac{1}{4}\theta^2\right) \\
&= 1 + \left(\frac{1}{2} + \frac{1}{2}\cos(\theta)\right) - \left(\cos\left(\frac{1}{2}\theta + \frac{1}{4}\theta^2\right) + \cos\left(\frac{1}{2}\theta - \frac{1}{4}\theta^2\right)\right) \\
&\qquad - \frac{1}{2}\left(\cos\left(\frac{1}{2}\theta - \frac{1}{4}\theta^2\right) - \cos\left(\frac{1}{2}\theta + \frac{1}{4}\theta^2\right)\right) \\
&= \frac{3}{2} + \frac{1}{2}\cos\theta - \frac{1}{2}\cos\left(\frac{1}{2}\theta + \frac{1}{4}\theta^2\right) - \frac{3}{2}\cos\left(\frac{1}{2}\theta - \frac{1}{4}\theta^2\right).
\end{aligned}$$

By the Maclaurin series for cos and properties of alternating series,

$$1 - x^2/2 < \cos x < 1 - x^2/2 + x^4/24 \quad \text{for } 0 < x < 1.$$

Both $\theta/2 + \theta^2/4$ and $\theta/2 - \theta^2/4$ are between 0 and 1. Therefore,

$$\begin{aligned}
\overrightarrow{QP} \cdot \overrightarrow{QV} &< \frac{3}{2} + \frac{1}{2}\left(1 - \frac{\theta^2}{2} + \frac{\theta^4}{24}\right) - \frac{1}{2}\left(1 - \frac{1}{2}\left(\frac{\theta}{2} + \frac{\theta^2}{4}\right)^2\right) \\
&\qquad - \frac{3}{2}\left(1 - \frac{1}{2}\left(\frac{\theta}{2} - \frac{\theta^2}{4}\right)^2\right) = -\frac{1}{8}\theta^3 + \frac{1}{24}\theta^4 < 0.
\end{aligned}$$ ◻

***Lemma A.2.*** *Let $a < b$, $c < d$, and $u \in C^1([a,b] \times [c,d])$ with $u = 0$ on $\{a,b\} \times [c,d]$. Then,*

$$\int_{y=c}^{d} \int_{x=a}^{b} \left| \frac{\partial u}{\partial x} \right| \mathrm{d}x\,\mathrm{d}y \geq \frac{2}{b-a} \int_{y=c}^{d} \int_{x=a}^{b} |u(x,y)|\,\mathrm{d}x\,\mathrm{d}y. \tag{A.1}$$

*Proof.* Fix $y \in [c,d]$. Let $x_0 \in (a,b)$ with

$$|u(x_0, y)| = \max_{[a,b]\times\{y\}} |u|.$$

Then,

$$\text{(A.2)} \quad \int_{x=a}^{b} |u(x,y)|\,\mathrm{d}x \le (b-a)|u(x_0,y)|$$
$$= \left(\frac{b-a}{2}\right)(|u(x_0,y)-u(a,y)| + |u(b,y)-u(x_0,y)|)$$
$$= \left(\frac{b-a}{2}\right)\left(\left|\int_a^{x_0} \frac{\partial u}{\partial x}\,\mathrm{d}x\right| + \left|\int_{x^0}^{b} \frac{\partial u}{\partial x}\,\mathrm{d}x\right|\right)$$
$$\le \left(\frac{b-a}{2}\right)\left(\int_a^{x_0} \left|\frac{\partial u}{\partial x}\right| \mathrm{d}x + \int_{x_0}^{b} \left|\frac{\partial u}{\partial x}\right| \mathrm{d}x\right) = \left(\frac{b-a}{2}\right)\int_a^b \left|\frac{\partial u}{\partial x}\right| \mathrm{d}x.$$

Multiplying both sides of (A.2) by $2/(b-a)$ and integrating from $y = c$ to $y = d$ yields (A.1). ☐

***Lemma A.3.*** *Let $M, \delta > 0$, let $\Omega$ be an open subset of $\mathbb{R}^n$ $(n \ge 1)$, let $g, h \in L^1(\Omega)$ with $g, h \ge 0$ Lebesgue-almost everywhere, and assume $\int_\Omega g\,\mathrm{d}x \ge \delta$, $\int_\Omega h\,\mathrm{d}x \le M$. Then,*

$$\text{(A.3)} \qquad \int_\Omega \sqrt{g^2+h^2}\,\mathrm{d}x \ge \int_\Omega h\,\mathrm{d}x + \frac{\delta^2}{2M+\delta}.$$

*Proof.* This proof is courtesy of Oleksiy Klurman of the University of Manitoba.

Since $x^2/(\sqrt{x^2+y^2}+|y|) \to 0$ as $(x,y) \to (0,0)$, we will interpret the expression "$g^2/(\sqrt{g^2+h^2}+h)$" as zero when $g = h = 0$ below. By the Cauchy-Schwarz inequality,

$$\left(\int_\Omega g\,\mathrm{d}x\right)^2 = \left(\int_\Omega \frac{g}{\sqrt{\sqrt{g^2+h^2}+h}} \cdot \sqrt{\sqrt{g^2+h^2}+h}\,\mathrm{d}x\right)^2$$
$$\le \left(\int_\Omega \frac{g^2}{\sqrt{g^2+h^2}+h}\,\mathrm{d}x\right)\left(\int_\Omega \sqrt{g^2+h^2}+h\,\mathrm{d}x\right)$$
$$= \left(\int_\Omega \sqrt{g^2+h^2}-h\,\mathrm{d}x\right)\left(\int_\Omega \sqrt{g^2+h^2}+h\,\mathrm{d}x\right)$$
$$\le \left(\int_\Omega \sqrt{g^2+h^2}-h\,\mathrm{d}x\right)\left(2\int_\Omega h\,\mathrm{d}x + \int_\Omega g\,\mathrm{d}x\right)$$
$$\le \left(\int_\Omega \sqrt{g^2+h^2}-h\,\mathrm{d}x\right)\left(2M + \int_\Omega g\,\mathrm{d}x\right).$$

Therefore,

$$\int_\Omega \sqrt{g^2+h^2} - h \,\mathrm{d}x \geq \frac{\left(\int_\Omega g \,\mathrm{d}x\right)^2}{2M + \int_\Omega g \,\mathrm{d}x} \geq \frac{\delta^2}{2M+\delta}, \tag{A.4}$$

because the map $x \mapsto x^2/(2M+x)$ is an increasing function of $x$ for $x \geq 0$. Rearranging (A.4) yields (A.3). $\square$

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