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An All-Against-One Game Approach for the Multi-Player Pursuit-Evasion Problem

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AN ALL-AGAINST-ONE GAME APPROACH FOR THE MULTI-PLAYER PURSUIT-EVASION PROBLEM

by

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ABSTRACT

The traditional pursuit-evasion game considers a situation where one pursuer tries to capture an evader, while the evader is trying to escape. A more general formulation of this problem is to consider multiple pursuers trying to capture one evader. This general multi-pursuer one-evader problem can also be used to model a system of systems in which one of the subsystems decides to dissent (evade) from the others while the others (the pursuer subsystems) try to pursue a strategy to prevent it from doing so. An important challenge in analyzing these types of problems is to develop strategies for the pursuers along with the advantages and disadvantages of each. In this thesis, we investigate three possible and conceptually different strategies for pursuers: (1) act non-cooperatively as independent pursuers, (2) act cooperatively as a unified team of pursuers, and (3) act individually as greedy pursuers. The evader, on the other hand, will consider strategies against all possible strategies by the pursuers. We assume complete uncertainty in the game i.e. no player knows which strategies the other players are implementing and none of them has information about any of the parameters in the objective functions of the other players. To treat the three pursuers strategies under one general framework, an all-against-one linear quadratic dynamic game is considered and the corresponding closed-loop Nash solution is discussed. Additionally, different necessary and sufficient conditions regarding the stability of the system, and existence and definiteness of the closed-loop Nash strategies under different strategy assumptions are derived. We deal with the uncertainties in the strategies by first developing the Nash strategies for each of the resulting games for all possible options available to both sides. Then we deal with the parameter uncertainties by performing a Monte Carlo analysis to determine probabilities of capture for the pursuers (or escape for the evader) for each resulting game. Results of the Monte Carlo simulation show that in general, pursuers do not always benefit from cooperating as a team and that acting as non-cooperating players may yield a higher probability of capturing of the evader.

Dedicated to my parents

and

beloved wife

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CHAPTER 1: INTRODUCTION

Many large enterprises consist of a large number of interacting subsystems. These system of systems (SoS) often operate optimally when all subsystems have a harmonious non-conflicting relationship among themselves. When one subsystem decides to operate in a manner that is not consistent with the others, the operation of the entire enterprise suffers resulting in an adversarial environment that affects not only the behavior of the dissenting subsystem but also possibly the behavior of the other conforming subsystems. The disruption caused by one dissenting subsystem may result in the entire system of systems disintegrating and behaving in a non-cooperative manner within itself. These types of complex system of systems are treated best using concepts from game theory. If the systems are dynamic in nature, then differential game theory is the appropriate framework to do the analysis. These types of problems are conceptually similar to multi-pursuer Pursuit-Evasion (PE) games. Traditional Pursuit-Evasion refers to a game in which a number of pursuers try to capture one or more evaders while the evaders try to escape. The solution of these types of games involves the development of movement strategies for both the pursuers and the evaders simultaneously that will conclude with the evader either escaping or being captured. Problems that include many evaders can be investigated as a combination of resource allocation problems involving multiple pursuers and one evader. That is, a wide range of problems can be covered by studying multi-pursuers one-evader games. Additionally, in real and practical problems there is barely certain information about the circumstances of the game. So, theoretical analysis only based on certain data may not be useful in practice and examination of the problem under uncertainties is vital. One of the most representative examples of the SoS approach to the pursuit evasion game is a system of entities such as UAVs or robots moving together in a coordinated fashion to perform a common task. If one entity decides to separate from the group and operate on its own, the remaining entities will face the problem of deciding how to proceed. For example, one UAV or one robot may decide to leave the group and proceed on its own with an objective that may be contrary or harmful to the group's common objective. The remaining entities may or may not agree on how to react. They may decide to pursue a group strategy in an attempt to prevent it from accomplishing its objective or they may disagree on the proper course of action resulting in a system-wide breakdown and producing a non-cooperative environment where each entity acts unilaterally on its own. In many ways, these problems are very similar in formulation to the traditional multi-pursuer PE problems. Another interesting examples of the SoS in presence of a dissenting subsystem is the European Union while they've been cooperating for years, suddenly the Britain stop to govern in coalition with other European countries and Brexit happens. Obviously, this action will affect not only the Britain, but also all countries around the world. As a consequence of Brexit, the question is what strategies should other countries in European Union use to minimize the effects of Brexit or possibly force the Britain to join the European Union again. They may choose to stay as an union or disintegrate and stop cooperating among themselves since the situation has changed. Of course that studying such a complicated topic is beyond the scope of this thesis and it was just a real example of situations where the concept of SoS in present of a dissenting subsystem may happen. So we restrict ourselves to studying multi-pursuer PE games under uncertainties in order to make the problem tractable and to simplify explanation of the main concept and rigorously carry the analysis further.

CHAPTER 2: LITERATURE REVIEW

In 1957, *Berge* [1] proposed a PE game in which the evader moves in a prescribed trajectory and the pursuers move with constant velocities but in directions that point directly towards the evader. In 1965, *Isaacs* [2] considered a zero-sum differential game with one pursuer and one evader. Following his work a variety of different formulations of PE games have been studied in the literature [3–9] including zero-sum [5], non-zero sum [10] and even Stackelberg [8, 9] games. The dynamics of the game can be formulated either in terms of velocity control [5, 11], or in terms of constant velocities but direction of movement control [12–14]. In recent years, multi-pursuers one-evader games have received considerable attention in the literature [3, 7, 15–18]. Compared to the original single-pursuer single-evader game, these games present numerous challenges in developing optimal strategies, including the structure, existence and uniqueness of the pursuers strategies [19–23]. The effectiveness of different strategies in presence of parameters uncertainty for a dynamic two-step look-ahead game has been studied in [24]. In [15], a multi-pursuers one-evader game has been studied in the presence of limited state information for the pursuers' side. The problem is formulated as a 2-player linear-quadratic game with the pursuers cooperating as one team (i.e. as one player) against the evader. This necessitated the introduction of so-called best achievable performance indices for the pursuers. In [25], a two-pursuer one-evader game is considered and decomposition for reduction of time complexity of the solution is introduced and comparing the run-time complexity of both the decomposed and the full game has shown that with increasing cardinality of each player's strategy space the decomposed game yields a relevant decrease of the run-time.

In this thesis, and before we investigate multi-pursuers one-evader games, we will consider a general framework that can be used to investigate several possible structural solutions for these types of PE games. More specifically, we consider an (N+1)-player linear quadratic game where N players are minimizing quadratic objective functions while the remaining player is maximizing a similar quadratic objective function. We refer to these games as all-against-one, reflecting the fact that one player's objective is directly opposed to all the other players. In chapter 3 we consider a general treatment of all-against-one Linear Quadratic (LQ) games including the derivation of new sufficient conditions for existence of closed-loop Nash solutions. In chapter 4 we formulate the N-pursuers one-evader game as a special case of the all-against-one game and investigate three different strategies for the pursuers. These strategies include Non-cooperating pursuers, Co-operating pursuers and Greedy pursuers. In chapter 5 we present several illustrative examples that describe different pursuit-evasion scenarios and show simulation results for all three pursuers' strategies when the evader is using a strategy that yields the Nash equilibrium in each case. In the same chapter, we present some preliminary results where neither the pursuers nor the evader have knowledge the objective functions of the other side and hence need to implement strategies that are secure against possible worst strategies by the other side. Results of Monte Carlo simulations of these incomplete information games are also reported. Concluding remarks are presented in chapter 6.

CHAPTER 3: All-AGAINST-ONE GAMES

In this chapter we consider an (N+1)-player game where N players are minimizing similar objective functions while the remaining player is minimizing another objective function which is completely opposing to the other players. We refer to these games as all-against-one, reflecting the fact that one player's objective is directly opposed to all the others. Let $M = N + 1$, where player 1 is the opposing player and players 2 through M are all against player 1. To make the problem tractable, let the game has linear dynamics and quadratic objective functions for players in the standard form [10] and [26]

$$\dot{z} = Az + B_1 u_1 + \sum_{j=2}^{M} B_j u_j, \quad z(t_0) = z_0; \tag{3.1}$$

where z is the state vector and u_1 through u_M are the player's control input vector. Matrix A is the open-loop system dynamics, matrices B_1 through B_M are player's input matrices, respectively. Let the objective function of player 1 be of the form

$$J_1 = \frac{1}{2} z(t_f)^\intercal S_{1f} z(t_f) + \frac{1}{2} \int_{t_0}^{t_f} [z(t)^\intercal Q_1 z(t) + u_1(t)^\intercal R_1 u_1(t)] dt, \tag{3.2}$$

where S_{1f} and Q_1 are symmetric negative definite matrices[1] and R_1 is symmetric positive definite matrix. Let the objective functions of players 2 through M be of the form

$$J_i = \frac{1}{2} z(t_f)^\intercal S_{if} z(t_f) + \frac{1}{2} \int_{t_0}^{t_f} [z(t)^\intercal Q_i z(t) + u_i(t)^\intercal R_i u_i(t)] dt; \tag{3.3}$$

for $i = 2, 3, ..., M$, where S_{if} and Q_i are symmetric positive semi-definite matrices and R_i is symmetric positive definite matrix. The main difference between this game and the standard LQ game considered in the literature is that player 1 has weight matrices S_{1f} and Q_1 that are not of the

[1] We require these matrices to be negative definite instead of negative semi-definite so that player 1 would not have the option to exclude any of the players who are against it.

same sign as similar matrices of the other players. In other words, player 1 is trying to unregulate (drive the state vector of the system away from the origin) while players 2 through M are trying to regulate the system (drive the state vector of the system to the origin). It is well known that necessary conditions for the closed-loop Nash strategies for this game are of the form [10]

$$u_i^* = -R_i^{-1} B_i S_i z \tag{3.4}$$

for $i = 1, 2, ..., M$, where S_i's satisfy the following M-coupled differential Riccati equations,

$$\dot{S}_i + S_i A + A^\intercal S_i + Q_i + S_i B_i R_i^{-1} B_i^\intercal S_i - \sum_{j=1}^{M} (S_i B_j R_j^{-1} B_j^\intercal S_j + S_j B_j R_j^{-1} B_j^\intercal S_i) = 0, \tag{3.5}$$

where $S_i(t_f) = S_{if}$. It is well-known that for the standard LQ non-zero-sum game that if $Q_i \geq 0$ and $S_{if} \geq 0$, then the solution for (3.5) will be $S_i(t) \geq 0$ for $i = 1, 2, ..., M$ [27]. The following lemma provides the equivalent result for the all-against-one game.

Lemma 1. *For the all-against-one game let $S_i(t)$ for $i = 1, 2, ..., M$ be solutions of (3.5) for $t \in [t_0, t_f]$, then $S_1(t) < 0$ and $S_i(t) \geq 0$ for all $t \in [t_0, t_f]$, for $i = 2, 3, ..., M$.*

Proof. Assume $\lambda_{S_1}^{max}(t)$ and $v(t)$ are maximal eigenvalue and corresponding unit eigenvector of $S_1(t)$, respectively. $S_1(t)$ is piecewise continuously differentiable and symmetric for $t \in [t_0, t_f]$ but not necessarily analytic. So based on Theorem 3.6.1 in [27], $\lambda_{S_1}^{max}(t)$ is continuous in time and at any point that it is differentiable, its derivative satisfies

$$\begin{aligned} \frac{d}{dt}\big(\lambda_{S_1}^{max}(t)\big) =& v^\intercal(t) \dot{S}_1 v(t) \qquad (3.6) \\ =& - v^\intercal(t)(S_1 A + A^\intercal S_1 + S_1 B_1 R_1^{-1} B_1^\intercal S_1) v(t) + \sum_{j=1}^{M} \big(v^\intercal(t) S_1 B_j R_j^{-1} B_j^\intercal S_j v(t) \\ &+ v^\intercal(t) S_j B_j R_j^{-1} B_j^\intercal S_1 v(t)\big) - v^\intercal(t) Q_1 v(t) \qquad (3.7) \end{aligned}$$

$$
\begin{aligned}
=&\lambda_{S_1}^{max}(t)v^{\intercal}(t)\Big[-2A-\lambda_{S_1}^{max}(t)B_1R_1^{-1}B_1^{\intercal}+\sum_{j=1}^{M}\big(B_jR_j^{-1}B_j^{\intercal}S_j+S_jB_jR_j^{-1}B_j^{\intercal}\big)\Big]v(t)\\
&-v^{\intercal}(t)Q_1v(t).
\end{aligned}
\tag{3.8}
$$

Now for sufficiently small $\lambda_{S_1}^{max}(t)$ we have

$$
\frac{d}{dt}\big(\lambda_{S_1}^{max}(t)\big)\approx -v^{\intercal}(t)Q_1v(t) \tag{3.9}
$$

Since $Q_1<0$, $\frac{d}{dt}\big(\lambda_{S_1}^{max}(t)\big)>0$ holds for sufficiently small $\lambda_{S_1}^{max}(t)$. Given $S_1(t_f)<0$, we know $\lambda(t_f)<0$ and, hence by (3.9) and the continuity of $\lambda_{S_1}^{max}(t)$, the maximal eigenvalue $\lambda_{S_1}^{max}(t)<0$ for all $t\in[t_0,t_f]$. This proves that $S_1(t)<0$ for all $t\in[t_0,t_f]$.

Additionally, Let

$$
\bar{A}=A-\sum_{j=1}^{M}B_jR_j^{-1}B_j^{\intercal}S_j, \tag{3.10}
$$

following (3.5), $S_i(t)$ should satisfy

$$
\dot{S}_i+S_i\bar{A}+\bar{A}^{\intercal}S_i+Q_i+S_iB_iR_i^{-1}B_i^{\intercal}S_i=0; \tag{3.11}
$$

for $i=1,2,3,...,M$, and $S_i(t_f)=S_{if}$. Let $\Phi(t,\tau)$ be the state transition matrix of $-\bar{A}^{\intercal}(t)$. It is known that for $t,\tau\in[t_0,t_f]$,

$$
\frac{d\Phi(t,\tau)}{dt}=-\bar{A}^{\intercal}(t)\Phi(t,\tau),\quad \Phi(\tau,\tau)=I; \tag{3.12}
$$

Then according to [27], $S_i(t)$ will satisfy

$$
S_i(t)=\Phi(t,t_f)S_i(t_f)\Phi^{\intercal}(t,t_f)+\int_t^{t_f}\Phi(t,\tau)[Q_i+S_iB_iR_i^{-1}B_i^{\intercal}S_i]\Phi^{\intercal}(t,\tau)d\tau. \tag{3.13}
$$

Now, since $S_i(t_f) \geq 0,\ Q_i \geq 0$ and $R_i > 0$ for $i = 2, 3..., M$, it follows from (3.13) that $S_i(t) \geq 0$ for all $t \in [t_0, t_f]$ for $i = 2, 3, ..., M$. □

Next, we present a theorem that gives sufficient conditions for uniform exponential stability and assures that all states of the system are within a defined ball at the end of the game [28]. Later in section 5, this result is used to guarantee capture of the evader.

Theorem 2. *For the all-against-one game let $S_i(t)$ for $i = 1, 2, ..., M$ be solution of (3.5) for $t \in [t_0, t_f]$, and let*

$$C(t) \triangleq \sum_{j=1}^{M} (Q_j + S_j B_j R_j^{-1} B_j^{\intercal} S_j) \quad \textit{and} \quad P(t) \triangleq \sum_{j=1}^{M} S_j(t).$$

If $C(t) > 0$ for all $t \in [t_0, t_f]$ and $P(t_f) \geq 0$, then the system will be uniformly exponentially stable and $P(t) > 0$ for all $t \in [t_0, t_f)$. Furthermore, for any positive scalar $\delta > 0$, if

$$t_f \geq t_0 + \frac{\bar{\lambda}_P}{\underline{\lambda}_C} \ln \left(\frac{\bar{\lambda}_P . \left\| z_0 \right\|^2}{\underline{\lambda}_P . \delta^2} \right) \tag{3.14}$$

then it follows that $\|z(t_f)\| \leq \delta$.

In the above theorem, $\bar{\lambda}_{(.)} = \max\limits_{t \in [t_0, t_f]} \lambda_{(.)}^{max}$, $\underline{\lambda}_{(.)} = \min\limits_{t \in [t_0, t_f]} \lambda_{(.)}^{min}$ and $\|.\|$ refers to the Euclidean norm.

***Proof*.** The matrix $P(t)$ satisfies the differential equation

$$\dot{P} + \bar{A}^{\intercal} P + P \bar{A} + C(t) = 0, \quad P(t_f) = P_{t_f}; \tag{3.15}$$

where $P_{t_f} = \sum_{j=1}^{M} S_{jf}$ and $\bar{A}(t)$ is as defined in (3.10). Consequently, following the definition of

$\Phi(t, t_f)$ in (3.12), $P(t)$ also satisfies

$$P(t) = \Phi(t, t_f)P(t_f)\Phi^{\intercal}(t, t_f) + \int_t^{t_f} \Phi(t, \tau)C(\tau)\Phi^{\intercal}(t, \tau)d\tau.$$

Since $P(t_f) \geq 0$ and $C(t) > 0$, it follows that $P(t) > 0$, for all $t \in [t_0, t_f)$. Now we can define the Lyapunov function

$$V(t) = z^{\intercal}P(t)z, \tag{3.16}$$

then

$$\begin{aligned} \dot{V}(t) &= \dot{z}^{\intercal}P(t)z + z^{\intercal}\dot{P}(t)z + z^{\intercal}P(t)\dot{z} && (3.17) \\ &= z^{\intercal}(\bar{A}^{\intercal}P + \dot{P} + P\bar{A})z && (3.18) \\ &= -z^{\intercal}C(t)z && (3.19) \end{aligned}$$

So $C(t) > 0$ results in the system to be uniformly exponentially stable. Additionally, since

$$\dot{V} \leq -\frac{\underline{\lambda}_C}{\bar{\lambda}_P} z^{\intercal}Pz \leq -\frac{\underline{\lambda}_C}{\bar{\lambda}_P} V \tag{3.20}$$

then one can conclude that

$$V(t) \leq V(t_0)\exp[-\frac{\underline{\lambda}_C}{\bar{\lambda}_P}(t - t_0)]. \tag{3.21}$$

Moreover,

$$\begin{aligned} \|z(t)\|^2 &\leq \frac{1}{\underline{\lambda}_P}V(t) && (3.22) \\ &\leq \frac{V(t_0)}{\underline{\lambda}_P}\exp[-\frac{\underline{\lambda}_C}{\bar{\lambda}_P}(t - t_0)] && (3.23) \\ &\leq \frac{\bar{\lambda}_P}{\underline{\lambda}_P}\|z_0\|^2\exp[-\frac{\underline{\lambda}_C}{\bar{\lambda}_P}(t - t_0)]. && (3.24) \end{aligned}$$

Finally, in order to achieve $\|z(t_f)\| \leq \delta$, t_f should satisfy

$$\frac{\bar{\lambda}_P}{\underline{\lambda}_P} \|z_0\|^2 \exp[-\frac{\lambda_C}{\bar{\lambda}_P}(t_f - t_0)] \leq \delta^2 \tag{3.25}$$

which yields (3.14) and this ends the proof. □

Remark 1. The above theorem has been proved for M players with different quadratic cost functions in a linear game while they are implementing their Nash strategies. Later in chapter 4, different Nash solutions will be presented in which players either utilizes their Nash strategies according to an M-player (all-against-one) game or 2-player (greedy or one-against-one) games or cooperative (one-team-against-one) game. Thus, the above theorem can be applied directly to each of those scenarios.

The following remark represents usefulness of the above theorem by providing a practical explicit sufficient condition resulting in exponential stability of the system.

Remark 2. For all-against-one games, let the implemented strategies for player $1, 2, \ldots, M$ be as (3.4). Noting that S_{1f} and Q_1 are negative semi-definite matrices, if $S_{1f} + \sum_{j=2}^{M} S_{jf} \geq 0$ and $Q_1 + \sum_{j=2}^{M} Q_j \geq 0$ for all $t \in [t_0, t_f]$ then the system is exponentially stable regardless of initial state vector z_0. In other words, if the collective weights of players 2 through M overpower the weights of player 1 (opposing player) in the objective functions then the exponential stability of the whole system is assured.

Remark 3. Note that not only these results are applied to Pursuit-Evasion Games, but also they can be utilized in any system of systems (or Multi-agent systems) in presence of a malicious subsystem (or agent) [29]. The above theorem provides the exponential stability of the whole system only in Linear Quadratic cases, however, the rest of paper presents a general idea which can be applied to any kind of system as long as the aforementioned strategies can be calculated.

Remark 4. As it is shown in the second part of theorem, note that if the hypothesis of theorem is satisfied and if termination time t_f is large enough, then the state of system is guaranteed to end up in a small ball. In next section, this means that the capturing of the evader in a PE game is guaranteed no matter how far the pursuers are at the beginning.

A sufficient condition for existence of solutions to (3.5) have been discussed in [27], where all Q's and S's matrices are positive semi definite. The following theorem provides conditions for existence of solutions to M-coupled differential Riccati equations which arise in the all-against-one games.

Theorem 3. ***(Sufficient Condition for Existence of Nash Solution)*** *Let $Q \in \mathbb{R}^{nN\times nN}$ be symmetric and $W(t) = -S_1 + \sum_{i=2}^{M} S_i$ while S_i are the solutions of (3.5). Then $S_i, \quad \forall i \in \{1, 2, ..., M\}$ exist for $\forall t \leq t_f$ with*

$$0 \leq W(t) \leq L_Q(t) \tag{3.26}$$

while

$$R(S_1, S_2, ..., S_M, Q) \geq 0; \tag{3.27}$$

With

$$\begin{aligned} R(S_1, ..., S_M, Q) = Q + W(\sum_{j=1}^{M} B_j R_j^{-1} B_j^\intercal S_j) + (\sum_{j=1}^{M} S_j B_j R_j^{-1} B_j^\intercal)W \\ + \Big(S_1 B_1 R_1^{-1} B_1^\intercal S_1 - \sum_{i=2}^{M}(S_i B_i R_i^{-1} B_i^\intercal S_i)\Big) \end{aligned} \tag{3.28}$$

and $L_Q(t)$ is the unique solution of following linear terminal value problem (Lyapunov differential *equation) while* $L_Q(t_f) = -S_{1f} + \sum_{i=2}^{M} S_{if}$

$$\dot{L}_Q(t) = -L_Q(t)A - A^{\intercal}L_Q(t) - (Q - Q_1 + \sum_{i=2}^{M} Q_i). \tag{3.29}$$

Since $R(S_1, S_2, ..., S_M, Q)$ is not explicitly a second order term in W, the Comparison Theorem in [27] is not applicable to proof of this theorem. So we will prove the following theorem directly.

***Proof*.** The first inequality of (3.26) is a consequence of Theorem 1. If A, Q and Q_i are bounded then $L_Q(t)$ exists for $t \leq t_f$ since it follows a Lyapunov differential equation. Also, $W(t_f) = -S_{1f} + \sum_{i=2}^{M} S_{if}$ and

$$\begin{aligned}
\dot{W} =& - WA - A^{\intercal}W - (Q - Q_1 + \sum_{i=2}^{M} Q_i) + \\
& Q + W(\sum_{j=1}^{M} B_j R_j^{-1} B_j^{\intercal} S_j) + (\sum_{j=1}^{M} S_j B_j R_j^{-1} B_j^{\intercal})W \\
& + \Big(S_1 B_1 R_1^{-1} B_1^{\intercal} S_1 - \sum_{i=2}^{M} (S_i B_i R_i^{-1} B_i^{\intercal} S_i)\Big) \\
=& - WA - A^{\intercal}W - (Q - Q_1 + \sum_{i=2}^{M} Q_i) \\
& + R(S_1, S_2, ..., S_M, Q),
\end{aligned} \tag{3.30}$$

Now define $Y = L_Q - W$, then $Y(t_f) = \mathbf{0}$ and

$$\begin{aligned}
\dot{Y} =& - (L_Q - W)A - A^{\intercal}(L_Q - W) \\
& - R(S_1, S_2, ..., S_M, Q) \\
=& - YA - A^{\intercal}Y - R(S_1, S_2, ..., S_M, Q)
\end{aligned} \tag{3.31}$$

Since $R(S_1, S_2, ..., S_M, Q) \geq 0$ then

$$\dot{Y} \leq -YA - A^{\intercal}Y. \tag{3.32}$$

Let's define $Z(t,\tau) = \Phi_1^{\intercal}(t,\tau)Y(t)\Phi_1(t,\tau)$ while $\Phi_1(t,\tau)$ for all $t, \tau \leq t_f$ is defined as follows

$$\dot{\Phi}_1(t,\tau) = A\Phi_1(t,\tau), \quad \Phi_1(\tau,\tau) = I. \tag{3.33}$$

Since $Y(t_f) = \mathbf{0}$ and $\det \Phi_1(t,\tau) \neq 0$ then $Z(t_f,\tau) = \mathbf{0}$ for all $\tau \leq t_f$. Consequently,

$$\frac{\partial Z(t,\tau)}{\partial t} = \Phi_1^{\intercal}(t,\tau)\big[A^{\intercal}Y + \dot{Y} + YA\big]\Phi_1(t,\tau) \leq 0 \tag{3.34}$$

if Y is a solution of (3.31). Now if we define $g(t,\tau,v) = v^{\intercal}Z(t,\tau)v$ for all $t, \tau \leq t_f$ and $v \in \mathbb{R}^{nN}$ then $\frac{\partial}{\partial t}g(t,\tau,v) \leq 0$ for all $t \leq t_f$. Therefore the mean value theorem yields

$$0 \geq g(t_1,\tau,v) - g(t_2,\tau,v) = v^{\intercal}[Z(t_1,\tau) - Z(t_2,\tau)]v \tag{3.35}$$

for all $t_1 \leq t_2 \leq t_f$ and as a result $Z(t_1,\tau) \geq Z(t_2,\tau)$. So we conclude that $Z(t,\tau) \geq Z(t_f,\tau)$ for all $t, \tau \leq t_f$. Finally choosing $\tau = t$ yields $Y(t) \geq 0$ for all $t \leq t_f$. This ends the proof. □

Remark 5. If we define

$$\bar{S}_1 = -S_1, \qquad H_1 = -B_1^{\intercal}R_1^{-1}B_1; \tag{3.36}$$

$$\bar{S}_i = S_i, \qquad H_i = B_i^{\intercal}R_i^{-1}B_i, \qquad \forall i \in \{2, 3, ..., M\} \tag{3.37}$$

then the sufficient condition becomes

$$R(S_1,...,S_M,Q) = Q + W(\sum_{j=1}^{M} H_j)W - \sum_{i=1}^{M}\sum_{\substack{j=1\\j\neq i}}^{M}\sum_{\substack{k=1\\k\neq j}}^{M} \bar{S}_i H_j \bar{S}_k \tag{3.38}$$

Proof. Notice that $\bar{S}_i > 0 \quad \forall i \in \{1,2,...,M\}$, on the other hand $H_1 < 0$ and $H_i > 0 \quad \forall i \in \{2,3,...,M\}$ while $W = \sum_{i=1}^{M} \bar{S}_i$. So the equation follows since

$$\begin{aligned} W(\sum_{j=1}^{M} H_j)W =& (\sum_{j=1}^{M} \bar{S}_j H_j)W + \sum_{i=1}^{M}\sum_{\substack{j=1\\j\neq i}}^{M}\sum_{k=1}^{M} \bar{S}_i H_j \bar{S}_k \\ =& (\sum_{j=1}^{M} \bar{S}_j H_j)W + W(\sum_{j=1}^{M} H_j \bar{S}_j) \\ & - \sum_{i=1}^{M} \bar{S}_i H_i \bar{S}_i + \sum_{i=1}^{M}\sum_{\substack{j=1\\j\neq i}}^{M}\sum_{\substack{k=1\\k\neq j}}^{M} \bar{S}_i H_j \bar{S}_k. \end{aligned} \tag{3.39}$$

So we obtain

$$\begin{aligned} W(\sum_{j=1}^{M} H_j)W - \sum_{i=1}^{M}\sum_{\substack{j=1\\j\neq i}}^{M}\sum_{\substack{k=1\\k\neq j}}^{M} \bar{S}_i H_j \bar{S}_k = & (\sum_{j=1}^{M} S_j B_j^{\intercal} R_j^{-1} B_j)W + W(\sum_{j=1}^{M} B_j^{\intercal} R_j^{-1} B_j S_j) \\ & + S_1 B_1^{\intercal} R_1^{-1} B_1 S_1 - \sum_{i=2}^{M} S_i B_i^{\intercal} R_i^{-1} B_i S_i \end{aligned} \tag{3.40}$$

So we can rewrite the sufficient condition of the Theorem 3 as follow

$$R(S_1,S_2,...,S_M,Q) = R(-\bar{S}_1,\bar{S}_2,...,\bar{S}_M,Q) = Q + W(\sum_{j=1}^{M} H_j)W - \sum_{i=1}^{M}\sum_{\substack{j=1\\j\neq i}}^{M}\sum_{\substack{k=1\\k\neq j}}^{M} \bar{S}_i H_j \bar{S}_k \tag{3.41}$$

□

CHAPTER 4: METHODOLOGY

Following the all-against-one game formulation in chapter 3, and for the sake of tractability of the problem, we study a linear quadratic Pursuit-Evasion game as an all-against-one game. This simplifies the explanation of the general concepts of the All-against-one games and its application to a practical problem. In a multi-player pursuit-evasion game consisting of many pursuers and one evader, each pursuer's objective function reflects a desire to minimize the distance between itself and the evader while the evader's objective function reflects a need to escape by maximizing a weighted measure of the distances between itself and the pursuers.

4.1 Pursuit-Evasion Game Formulation

Consider a PE game with N pursuers against one evader. The game is formulated as a finite time LQ game in which the goal of the i-th pursuer is to minimize its distance with respect to the evader while the evader's goal is to maximize a weighted sum of the distances between itself and the pursuers. In this section we are concerned with developing only strategies for the pursuers. In doing so, the pursuers will assume that the evader will be using the corresponding Nash strategy but will not know whether the evader will implement that strategy or not. The first strategy is characterized by pursuers who cooperate as a team in their effort to catch the evader. The resulting game is referred to as a *Cooperating Pursuers Game*. The second strategy is characterized by non-cooperating pursuers who act in a non-cooperative manner among themselves and the evader. The resulting game is referred to as a *Non-Cooperating Pursuers Game*. The third strategy is characterized by greedy pursuers who act independently and selfishly each on its own in an attempt to catch the evader. The resulting game is referred to as *Greedy Pursuers Game*.

Let $x_1 \in \mathbb{R}^n$ be the evader's position vector and $x_{i+1} \in \mathbb{R}^n$ be i-th pursuer's position vector and let

$$z_i = x_1 - x_{i+1}, \tag{4.1}$$

for $i = 1, 2, ..., N$, be the difference between position vector of the i-th pursuer and the evader as illustrated in Figure 4.1. The system dynamics follows (3.1), where the state vector z is defined as $z = \begin{bmatrix} z_1^\intercal & z_2^\intercal & \dots & z_N^\intercal \end{bmatrix}^\intercal$ and $u_i \in \mathbb{R}^{n\times 1}$. Since the pursuers' dynamics are uncoupled, the system dynamics matrix $A = blkdiag\{A_1, ..., A_N\} \in \mathbb{R}^{nN\times nN}$ is block diagonal where $A_i \in \mathbb{R}^{n\times n}$ for $i = 1, 2, ..., N$. The input matrix of the evader is $B_e = B_1 = \mathbf{1}_N \otimes \mathbf{I}_n \in \mathbb{R}^{nN\times n}$ and the input matrices of the pursuers are $B_{pi} = B_{i+1} = -\mathbf{e}_i \otimes \mathbf{I}_n \in \mathbb{R}^{nN\times n}$ for $i = 1, 2, ..., N$. $\mathbf{I}_n \in \mathbb{R}^{n\times n}$ is the identity matrix, $\mathbf{e}_i \in \mathbb{R}^N$ denotes the vector with a 1 in the i-th coordinate and 0's elsewhere, $\mathbf{1}_N \in \mathbb{R}^{N\times 1}$ is a vector with all the entries equal to 1 and $\otimes$ is the Kronecker product. Dimension n can be any positive integer, for example 2 or 3 which represents a PE game with 2 or 3 dimensions, respectively. Each player should minimize the objective function as defined in (3.2) and (3.3), where $S_{if} \in \mathbb{R}^{nN\times nN}, Q_i \in \mathbb{R}^{nN\times nN}$ and $R_i \in \mathbb{R}^{n\times n}$ are the dimensions of weight matrices of the i-th player. For the player 1, the evader, $Q_e = Q_1 < 0$, $S_{e_f} = S_{1f} < 0$ and $R_e = R_1 > 0$. For players 2 thorough M, the pursuers, $Q_{p_i} = Q_{(i+1)} \geq 0$, $S_{p_if} = S_{(i+1)f} \geq 0$ and $R_{p_i} = R_{(i+1)} > 0$ for $i = 1, 2, ..., N$.

In this thesis, we consider three different strategies for the pursuers which have different Nash attributes. These strategies are discussed in the following subsections. The weights in the players' objective functions (3.2) and (3.3) are chosen as follow

$$\begin{aligned} S_{p_{fi}} =& f_{p_i}\mathbf{e}_i \otimes I_n, \quad S_{e_f} = -\,diag\{f_{e_1}, ..., f_{e_N}\} \otimes I_n, \\ Q_{p_i} =& q_{p_i}\mathbf{e}_i \otimes I_n, \quad Q_e = -\,diag\{q_{e_1}, ..., q_{e_N}\} \otimes I_n, \\ R_{p_i} =& r_{p_i} \otimes I_n, \qquad R_e = r_e \otimes I_n, \end{aligned} \tag{4.2}$$

where f_{e_i}, q_{e_i}, r_e and $f_{p_i}, q_{p_i}, r_{p_i}$ for $i = 1, 2, ..., N$ are positive scalar weights for the evader and corresponding i-th pursuer, respectively. Furthermore, for simplicity, we will assume that $t_0 = 0$.

We should also mention that in developing the pursuers strategies we will assume that all players have full measurements of the state vectors they need to implement their closed-loop strategies. In practice, however, full state measurements may not be available. Problems with partial state measurement and problems with information network topology among the players will be considered in a follow-up paper.

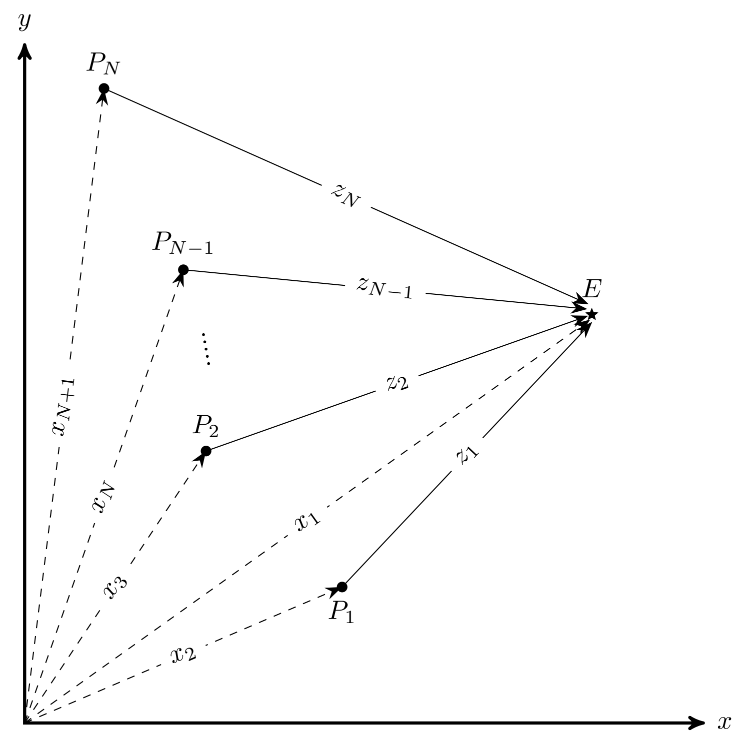


Figure 4.1: Pursuers and the Evader on an x-y Coordinate System

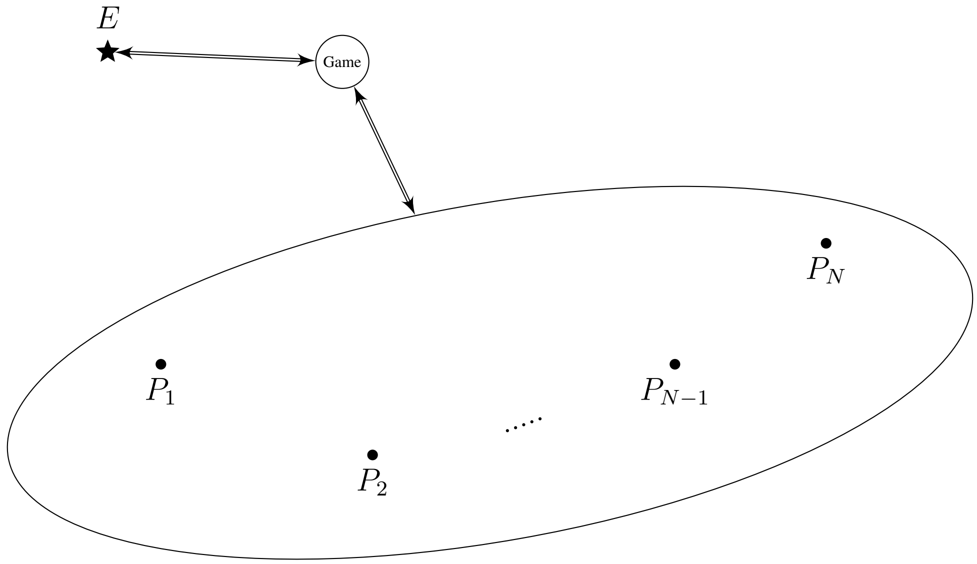


Figure 4.2: PE Game Between Cooperative Pursuers (as a Team) and the Evader.

4.1.1 Cooperative Pursuers Strategy

This game describes a situation where the pursuers act as one team against the evader as illustrated in Figure 4.2. Essentially, it is a 2-player game where all the pursuers' controls are combined as one control vector. The cooperation among the pursuers can be formulated as a convex combination of their cost functions as $J_P = \sum_{i=1}^{N} \alpha_i J_{p_i}$, where $\alpha_i > 0$, $\sum_{i=1}^{N} \alpha_i = 1$ and $J_{p_i} = J_{i+1}$ for $i = 1, 2, ..., N$ as they are defined in (3.2) and (3.3). The weight α_i can be interpreted as the fraction of the total effort that the i-th pursuer is contributing towards the team. Consequently, the cost function for the pursuers as a team becomes

$$J_P = \frac{1}{2} z(t_f)^\intercal S_{P_f} z(t_f) + \frac{1}{2} \int_0^{t_f} [z(t)^\intercal Q_P z(t) + u_i(t)^\intercal R_P u_i(t)] dt \tag{4.3}$$

where

$$S_{P_f} = \sum_{i=1}^{N} \alpha_i S_{p_{fi}}, \quad Q_P = \sum_{i=1}^{N} \alpha_i Q_{p_i}, \quad R_P = \sum_{i=1}^{N} \alpha_i R_{p_i} \tag{4.4}$$

The evader's cost function as defined in (3.2) becomes $J_e = J_1$. The system dynamics of (3.1) becomes

$$\dot{z} = Az + B_e u_e + B_P u_P, \tag{4.5}$$

where $u_e = u_1 \in \mathbb{R}^{n\times 1}$ is the evader's control vector and $u_P = [u_2^\intercal, u_3^\intercal, ..., u_M^\intercal]^\intercal \in \mathbb{R}^{Nn\times 1}$ is the combined control vector of the pursuers as a team. The matrix $B_e = \mathbf{1}_N \otimes \mathbf{I}_n$ is the input matrix of the evader and $B_P = -\mathbf{I}_N \otimes \mathbf{I}_n$ is the combined input matrix of pursuers as a team.

The system dynamics (4.5), and objective functions (4.3) and J_e form 2-player LQ game. A Pareto Front solution of cooperation among the pursuers can be obtained by using different values of α_i which impacts the performance of the pursuers as will be illustrated in Section 5.3. The closed-loop Nash strategies for the pursuers can be obtained from (3.4) as follow

$$u_P^* = -R_P^{-1} B_P S_P z, \tag{4.6}$$

$$u_e^* = -R_e^{-1} B_e S_e z; \tag{4.7}$$

where the matrices S_P and S_e are the solutions to the following coupled differential Riccati equations

$$\dot{S}_P + S_P A + A^\intercal S_P + Q_P - S_P B_P R_P^{-1} B_P^\intercal S_P - S_P B_e R_e^{-1} B_e^\intercal S_e - S_e B_e R_e^{-1} B_e^\intercal S_P = 0, \tag{4.8}$$

$$\dot{S}_e + S_e A + A^\intercal S_e + Q_e - S_e B_e R_e^{-1} B_e^\intercal S_e - S_e B_P R_P^{-1} B_P^\intercal S_P - S_P B_P R_P^{-1} B_P^\intercal S_e = 0, \tag{4.9}$$

where $S_P(t_f) = S_{P_f}$ and $S_e(t_f) = S_{e_f}$. We note that the evader's strategy in (4.6) is only calculated in order to determine the pursuers' strategy, but there is no guarantee that the evader will actually use that strategy.

4.1.2 Non-Cooperative Pursuers Strategies

This game represent a situation where the pursuers are competing among themselves in their effort to capture the evader as illustrated in Figure 4.3.The Nash equilibrium in this case will involve all pursuers and evader simultaneously, and the corresponding non-cooperative pursuers strategies can be determined using (3.4) and (3.5).

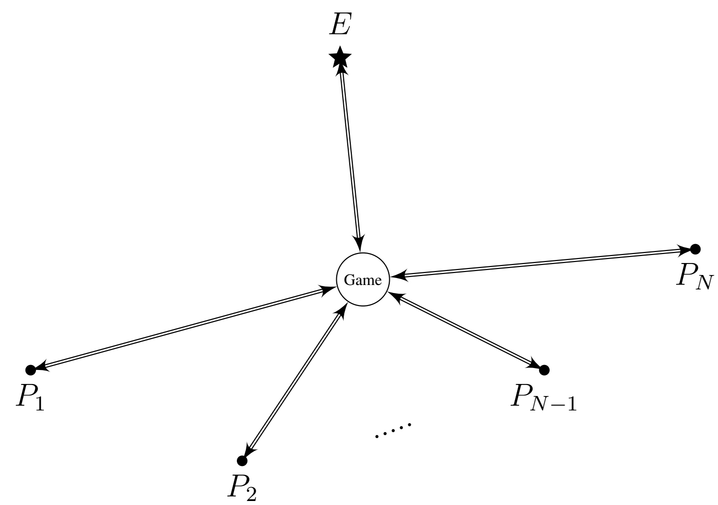


Figure 4.3: PE Game among Non-Cooperative Pursuers and the Evader.

4.1.3 Greedy Pursuers Strategies

This game represent a situation where the pursuers are greedy. That is, each pursuer decides to ignore all other pursuers and pursue the evader on its own as illustrated in Figure 4.4. Essentially, in this case we have N separate PE games between each pursuer and the evader. The pursuers' strategies are similar to the ones proposed in [1] where each pursuer moves exactly toward the evader at each instant of time, ignoring what the others are doing.

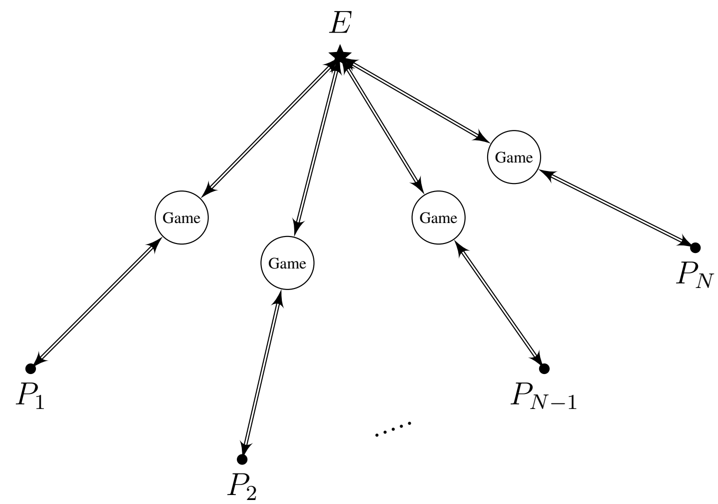


Figure 4.4: Separate PE Games Between Each Greedy Pursuer and the Evader.

According to the problem formulation (4.2) and (4.1), dynamics of the games can be represented by N decoupled systems

$$\dot{z}_i = A_i z_i + B_{e_i} u_{e_i} + B_{p_i} u_{p_i}, \tag{4.10}$$

for $i = 1, 2, ..., N$, where $B_{e_i} = B_e \circ (\mathbf{e}_i \otimes \mathbf{I}_n)$ is the corresponding part of the original input matrix B_e as defined in (3.3) with respect to i-th player, with input matrix $B_{p_i} \in \mathbb{R}^{n\times n}$, and $\circ$ is the Hadamard product. If each pursuer ignores the effect of the other pursuers, its game with the evader assumes that the evader is playing only versus that pursuer and ignoring the others. In another word, the evader's cost function against i-th pursuer is as follows

$$J_{e_i} = \frac{1}{2} z(t_f)^\intercal S_{e_i f} z(t_f) + \frac{1}{2}\int_0^{t_f} [z(t)^\intercal Q_{e_i} z(t) + u_{e_i}(t)^\intercal R_e u_{e_i}(t)]dt, \tag{4.11}$$

where $S_{e_i f} = S_{e_f} \circ blkdiag\{\mathbf{e}_i \otimes \mathbf{I}_n\}$ and $Q_{e_i} = Q_e \circ blkdiag\{\mathbf{e}_i \otimes \mathbf{I}_n\}$ are the evader weights instead of S_{e_f}, Q_e in (4.2), respectively. According to (3.4) and (3.5), the closed-loop Nash greedy strategies can be obtained for each pursuer as follows

$$u_{pig}^{*} = -R_{p_i}^{-1} B_{p_i} S_{p_{fi}} z, \tag{4.12}$$

$$u_{eig}^{*} = -R_{e}^{-1} B_{e_i} S_{e_i} z; \tag{4.13}$$

where $S_{p_{fi}}$ and S_{e_i} are the solutions to the following coupled differential Riccati equations

$$\dot{S}_{p_i} + S_{p_i}\bar{A}_i + \bar{A}_i^{\intercal} S_{p_i} + Q_{p_i} - S_{p_i} B_{p_i} R_{p_i}^{-1} B_{p_i}^{\intercal} S_{p_i} - S_{p_i} B_{e_i} R_{e}^{-1} B_{e_i}^{\intercal} S_{e_i} - S_{e_i} B_{e_i} R_{e}^{-1} B_{e_i}^{\intercal} S_{p_i} = 0 \tag{4.14}$$

$$\dot{S}_{e_i} + S_{e_i}\bar{A}_i + \bar{A}_i^{\intercal} S_{e_i} + Q_{e_i} - S_{e_i} B_{e_i} R_{e}^{-1} B_{e_i}^{\intercal} S_{e_i} - S_{e_i} B_{p_i} R_{p_i}^{-1} B_{p_i}^{\intercal} S_{p_i} - S_{p_i} B_{p_i} R_{p_i}^{-1} B_{p_i}^{\intercal} S_{e_i} = 0 \tag{4.15}$$

for $i = 1, 2, ..., N$, where $S_{p_i}(t_f) = S_{pif}$, $S_{e_i}(t_f) = S_{eif}$ and matrix $\bar{A}_i = blkdiag\{\mathbf{e}_i \otimes A_i\}$. Note that equation (4.12) is in form of full state closed-loop. However, $S_{p_{fi}}$ contains zeros in all entries except the ones related to i-th pursuer, indicating that the i-th pursuer needs only measurement of the distance between itself and the evader to implement its control.

CHAPTER 5: RESULTS

In this chapter, we demonstrate applications of the strategies (developed in chapter 4) in various situations by illustrating examples of games with and without uncertainties in parameters and information about players. Not only the strategies are significantly different, but also the number of pursuers chasing the evader as well as their initial position and characteristic are crucial factors that lead to various outcome in different situations. Additionally, it's significantly useful if one could predict the outcome of a game given the information about players and their characteristics in an All-against-one game. In order to make it feasible, we have developed a user-friendly software package that provides the opportunity of simulating any planar multi-player Pursuit-Evasion game by only inputting the information about the players. So if this data were available precisely, then one could forecast the outcome of a game thoroughly by using that software. However, in most of the real problems, this information may not be known by other players and only uncertain estimations of parameters might be feasible. We discuss these kind of problems in section 5.3 and we study similar problems to the examples in section 5.2 under different scenarios where there are uncertainties about the strategies and also parameters in the cost function of each player.

5.1 PE Game Software

In this section, we explain the software that we have developed to solve planar PE games with any number of pursuers chasing an evader. The software gives the opportunity to choose different strategies for each side of the game as shown in Figure 5.1 . For example, the pursuers can choose to play non-cooperatively as independent players or they can decide o play cooperatively as a team. Also, possibility of different team formation has been augmented in the software, i.e. the portion of contribution of each pursuer towards the team can be chosen in each team formation as shown in Figures 5.2a and 5.2b. In the next section, a number of different examples shows the output of software for simulation of some three-pursuers one-evader games.

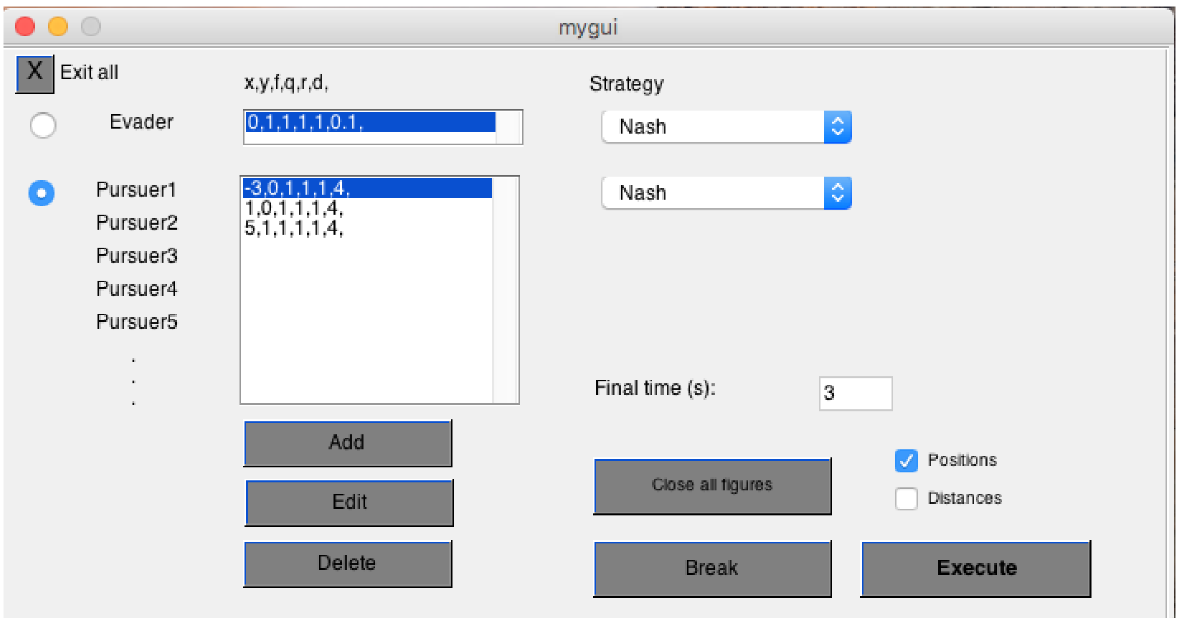
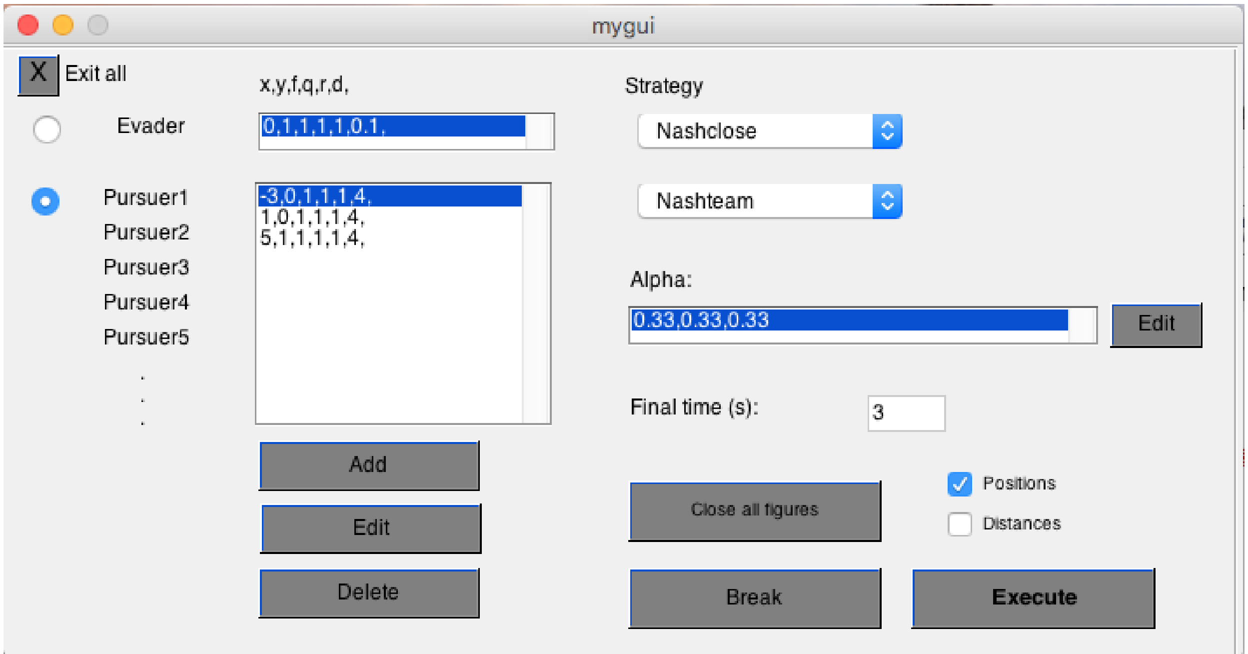
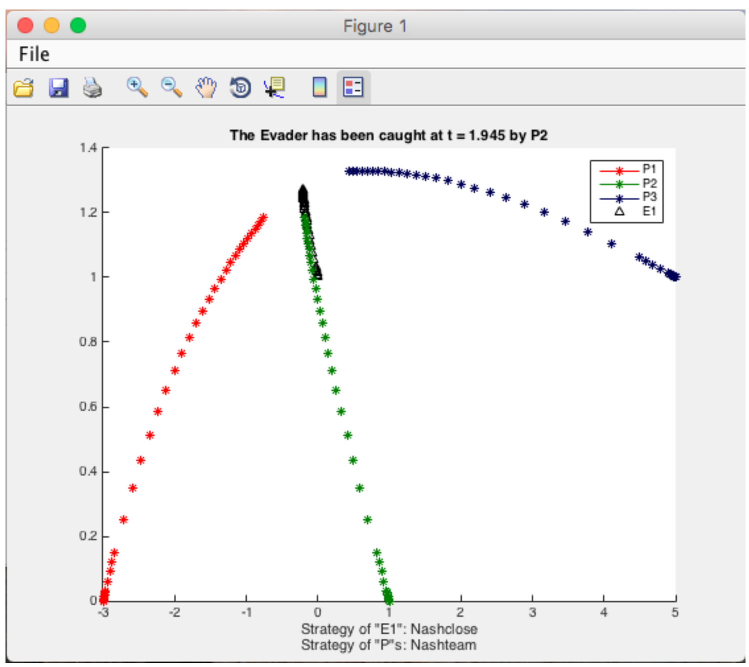


Figure 5.1: Software - Compatible with Mac OS X and Windows



(a) Main Page



(b) Output

Figure 5.2: Software - An example of Cooperative pursuers versus the evader escaping from the closest pursuer

5.2 Illustrative Examples

Consider a three-pursuer one-evader game on the plane (so $n = 2$) with parameters as described in Table 5.1. The entries under the position column are the x-y coordinates of each player, under the f column represent the weights in S_{if}, under the r column represent the weights in R_i. The parameters in the evader row are the same against each of three pursuers. For simplicity, the control input of each player is assumed to be velocity which leads to a system dynamics matrix of $A = \mathbf{0}$. The terminal time is assumed to be $t_f = 3$ and the evader is assumed to be captured if it enters a capture circle of radius $\delta = 0.1$ from any one pursuer. The three pursuers' strategies are determined as described in the previous section and the evader in each case is implementing the corresponding Nash strategy in each case.

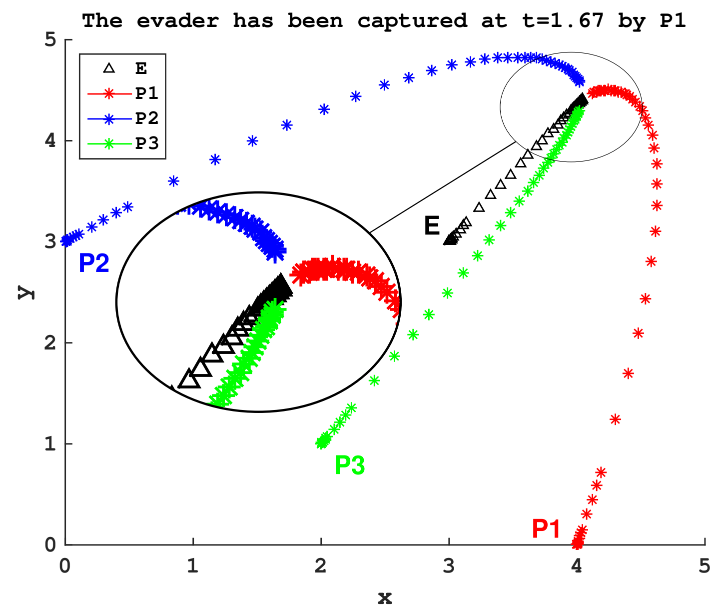
Table 5.1: Positions and Parameters of Player in the Example

	Position	Parameters[1]		
		f	q	r
Evader	$[3, 3]$	-10	-10	1
Pursuer 1	$[4, 0]$	3	3	0.4
Pursuer 2	$[0, 3]$	3	3	0.8
Pursuer 3	$[2, 1]$	4	4	2.8

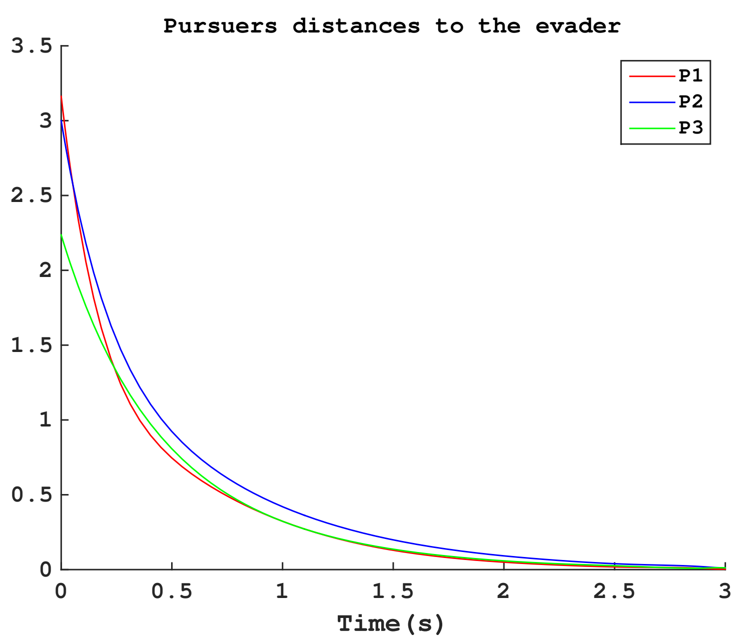
1. Parameters are the diagonal entries in (4.2).

The resulting trajectories of pursuers are illustrated in Figure 5.3a, 5.4a and 5.5a, and the corresponding distances of pursuers to the evader are illustrated in Figure 5.3b, 5.4b and 5.5b. In the cooperative pursuers game, $\alpha_i = 0.\bar{3}$ are used in the team objective functions, representing equal contributions by all pursuers.Note that the evader is able to escape when each pursuer uses the greedy strategy but it gets captured when the pursuers implement either the cooperative or the non-cooperative strategies. However, the method by which the evader gets captured in these two games are noticeably different. When the pursuers cooperate as a team they seem to encircle and entrap the evader before capturing (Figure 5.3a) while when they do not cooperate they seem to target the evader directly (Figure 5.4a). In both of these games the conditions of Theorem 2 are satisfied yielding trajectories that are exponentially stable (Figures 5.4b and 5.3b).

In the greedy pursuers strategy, the fact that each pursuers are moving exactly towards the evader causes the evader to escape (Figure 5.5a). In this case the condition of Theorem 2 are not satisfied. We should note that in this example we have assumed that all players have full information about the parameters of other side, which also may not be feasible in realistic situations.

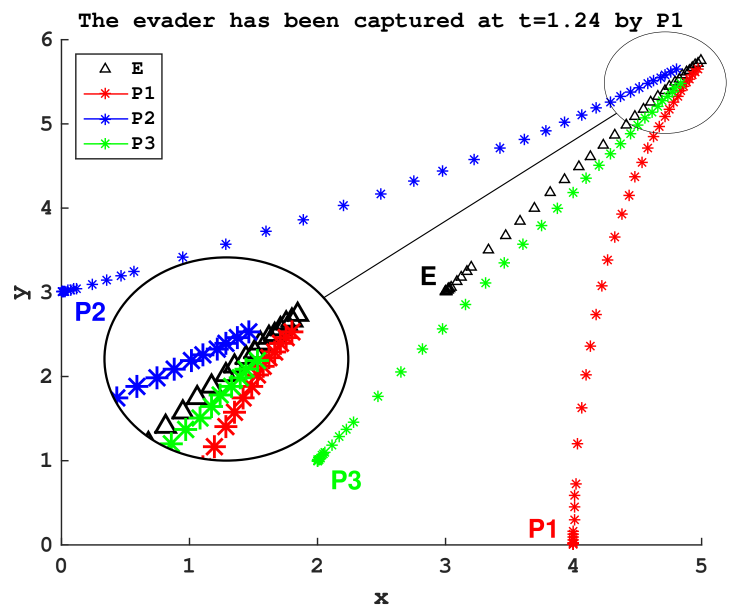


(a) Trajectories

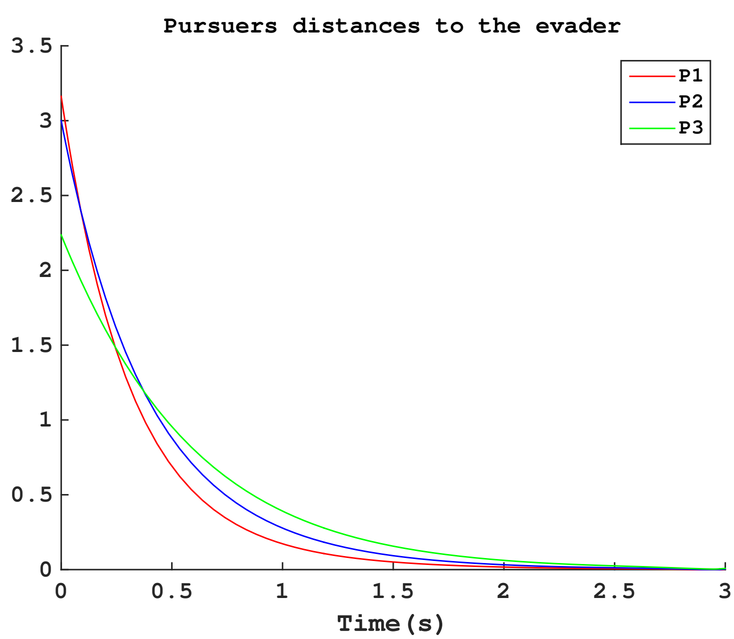


(b) Distances

Figure 5.3: Example - PE Game with Cooperative Pursuers.

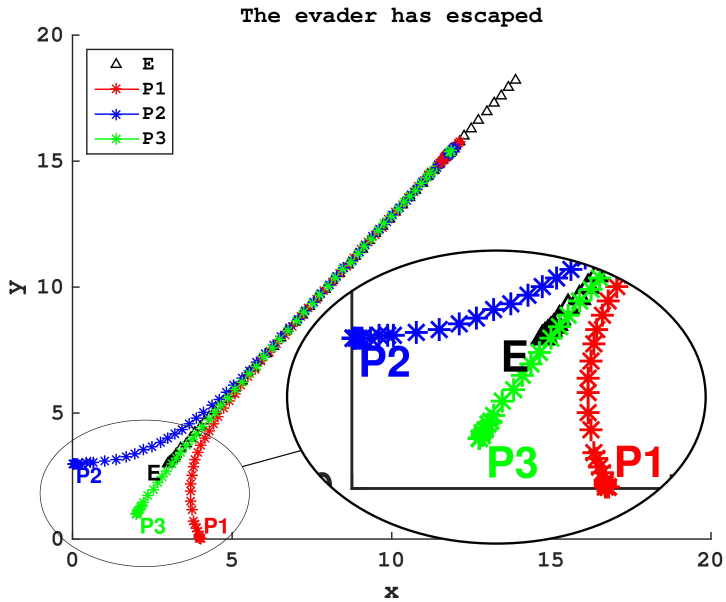


(a) Trajectories

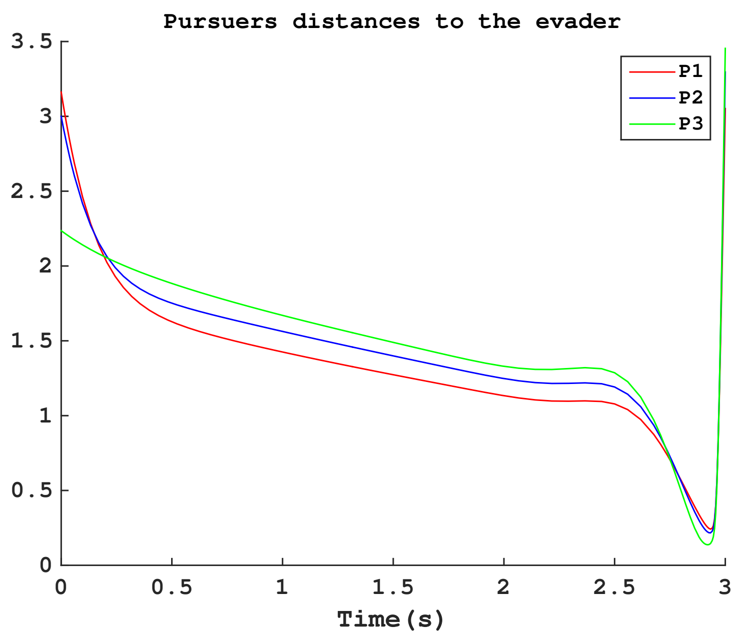


(b) Distances

Figure 5.4: Example - PE Game with Non-Cooperative Pursuers.



(a) Trajectories



(b) Distances

Figure 5.5: Example - PE Game with Greedy Pursuers.

5.3 Monte Carlo Simulation

In the illustrative example in the previous section, we have assumed that each player has full knowledge of the parameters in the objective functions of all other players (i.e. Table 5.1). In a practical PE game, however, this may not the case. We have also assumed that in each of the three pursuers strategies considered, the evader is implementing the strategy that yields a Nash equilibrium in each case. In a practical PE game, this also may not be the case.

With such lack of knowledge of the game parameters and strategy used by the evader, the pursuers might be interested in determining the probabilities of capturing the evader under some statistical distribution of these parameters and for several possible strategies used by the evader.

To accomplish this, we performed Monte Carlo simulations on a 3-pursuers one-evader game according to three different scenarios with parameter distributions and initial positions as shown in Table 5.2. The only difference between these scenarios is the initial position of players. Basically, this indicates the impact of initial positions of players on the output of the game.

Table 5.2: Positions and Parameters of Pursuers in MC Simulation

	Positions			Parameters		
	Scenario 1	Scenario 2	Scenario 3	f	q	r
Evader	$[6,6]$	$[6,6]$	$[6,6]$	$^{\star}U[-3,-1]$	$U[-3,-1]$	$U[1,3]$
Pursuer 1	$[7,3]$	$[4,0]$	$[4,0]$	$U[0,2]$	$U[0,2]$	$U[1,10]$
Pursuer 2	$[3,6]$	$[3,6]$	$[0,3]$	$U[0,2]$	$U[0,2]$	$U[1,10]$
Pursuer 3	$[5,4]$	$[5,4]$	$[5,4]$	$U[0,2]$	$U[0,2]$	$U[1,10]$

$\star$ $U[a,b]$ indicates the Uniform distribution between a and b.

In the cooperative pursuers strategies we used four different team configurations with different choices of α_i parameters as shown in Table 5.3. In Team 1, 2 and 3 we have given one pursuer a higher share of the total effort (0.8) than the other two and in Team 4 we have assigned equal share of the total effort (0.3) to the three pursuers.

Table 5.3: Value of α_i for Each Pursuer in Different Team Formations

	Pursuer 1 α_1	Pursuer 2 α_2	Pursuer 3 α_3
Team 1	0.8	0.1	0.1
Team 2	0.1	0.8	0.1
Team 3	0.1	0.1	0.8
Team 4	$0.\bar{3}$	$0.\bar{3}$	$0.\bar{3}$

Each sample of the MC simulation represents all different combinations of strategies for the pursuers and evader. The outcome of each simulation is defined to be a Bernoulli random variable of 1 if the pursuers capture the evader with probability of (ρ) and of 0 if the evader escapes with probability of ($1-\rho$). Assuming that the error distribution is normal as in [30, 31], the confidence interval of the simulated mean value would be

$$[e^-, e^+] = [\rho - z_{\gamma/2}.s,\ \rho + z_{\gamma/2}.s] \tag{5.1}$$

where $s = \sqrt{\frac{\rho(1-\rho)}{n}}$ is the standard deviation of n samples and $z_{\gamma/2}$ is the critical value of the Normal distribution for a given error level γ. The desired margin of error ε is half of the confidence interval, so the number of required samples can be calculated as follows:

$$\varepsilon = z_{\gamma/2}.s \tag{5.2}$$

$$n = \Big[\frac{z_{\gamma/2}\sqrt{\rho(1-\rho)}}{\varepsilon}\Big]^2 \tag{5.3}$$

Also in terms of desired percentage of margin of error, it can be calculated as:

$$E = 100 \times \frac{z_{\gamma/2}.s}{\rho} \tag{5.4}$$

$$n = (\frac{1}{\rho} - 1)\Big[\frac{100 z_{\gamma/2}}{E}\Big]^2 \tag{5.5}$$

For a 95% confidence level, the error level is $\gamma = 0.05$ and the critical value would be $z_{\gamma/2} = 1.96$. For a desired margin of error $\varepsilon = 0.01$, the resulting minimum number of required samples becomes $9,604$ which happens when $\rho = 0.5$. Consequently, in the worst case, with a confidence level of 95% we can assert that the percentages of capturing probability will be accurate with a maximum error of $\pm 1\%$. In our Monte Carlo simulation we used $10,000$ runs, each consisting of all possible combinations of strategies by pursuers and evader. This required 15 hours of CPU time on a super computer with 10 Intel Xeon 64-bit CPUs. The results of the simulation are shown in Tables 5.4, 5.5 and 5.6.

For each possible strategy of the evader, the pursuers strategy that yields the highest probability of capturing the evader is indicated by ‡ on Table 5.4. The highest such probability occurs when the pursuers are using a non-cooperative strategy and the evader uses a Nash Strategy against Team 3. The lowest capturing probability occurs when the pursuers use also the non-cooperative strategy and the evader uses a Nash strategy against the closest pursuer. In between these two extremes, the pursuers could run the risk of attaining very low probabilities of capturing the evader. For example, if the pursuers cooperate according to Team 3 to achieve a capturing probability of 67%, they run the risk of attaining 2% probability of capture (or 98% of probability of escape) if the evader uses a Nash strategy against the closest pursuer. Thus, the safest strategy for the pursuers is to use the non-cooperative strategy.

On the other side, the evader may perform a similar Monte Carlo simulation to arrive at Table 5.4. From the evader's perspective, for every strategy that the pursuers use, the strategy that yields the lowest probability of capture is indicated with a †. Consequently the safest strategy for the evader is to use the Nash strategy against the closest pursuer. If both sides use the safest strategy, this eventually would result in a 46% probability of capture (or 54% probability of escape) of the evader.

Table 5.4: Scenario 1: Monte Carlo Simulation (Capturing Probability in Percent)

			Evader's Nash Strategies versus:					
			Team 1	Team 2	Team 3	Team 4	Non-Cooperative Pursuers	Closest Pursuer
Pursuers Nash Strategies vs Evader	Cooperative Pursuers	Team 1	64	20†	29	61⋆	20†	24
		Team 2	60	32	14†	56	17	20
		Team 3	66⋆	9	7	7	32	2†
		Team 4	23	28	36	15†	25	17
	Non-Cooperative Pursuers		59	69⋆	74⋆	53	48⋆	**46⋆†**
	Greedy Pursuers		18	10	11	11	7†	31

⋆ Optimal reaction set of pursuers

† Optimal reaction set of evader

In scenarios 2 and 3, initial position of Pursuer 1 and 2 have been moved further out of the evader, respectively. As it is shown in Tables 5.5 and 5.6, the probabilities of captures are much lower in comparison to scenario 1. Also in Scenario 2, if both side of the game choose to play with security strategy the outcome of the game would be 40% probability of capture of evader. Similarly in Scenario 3, the secure strategy for the pursuers is Non-cooperative and for the evader is versus the Closet Pursuer, and if both chooses their security strategy the outcome of the game would be 20% probability of capture of the evader or 80% probability of escape.

Table 5.5: Scenario 2: Monte Carlo Simulation (Capturing Probability in Percent)

			Evader's Nash Strategies versus:					
			Team 1	Team 2	Team 3	Team 4	Non-Cooperative Pursuers	Closest Pursuer
Pursuers Nash Strategies vs Evader	Cooperative Pursuers	Team 1	$68^{\star}$	23	$10^{\dagger}$	60	27	14
		Team 2	39	10	5	32	11	$3^{\dagger}$
		Team 3	53	13	$8^{\dagger}$	36	21	35
		Team 4	28	$5^{\dagger}$	19	20	26	16
	Non-Cooperative Pursuers		61	$46^{\star}$	$41^{\star}$	$63^{\star}$	$59^{\star}$	$\mathbf{40}^{\star\dagger}$
	Greedy Pursuers		9	4	7	8	$3^{\dagger}$	28

⋆ Optimal reaction set of pursuers

† Optimal reaction set of evader

Table 5.6: Scenario 3: Monte Carlo Simulation (Capturing Probability in Percent)

			Evader's Nash Strategies versus:					
			Team 1	Team 2	Team 3	Team 4	Non-Cooperative Pursuers	Closest Pursuer
Pursuers Nash Strategies vs Evader	Cooperative Pursuers	Team 1	51	23	$6^{\dagger}$	44	19	14
		Team 2	39	10	$1^{\dagger}$	30	10	2
		Team 3	35	11	$6^{\dagger}$	42	12	$23^{\star}$
		Team 4	19	$2^{\dagger}$	16	12	20	8
	Non-Cooperative Pursuers		$54^{\star}$	$25^{\star}$	$36^{\star}$	$47^{\star}$	$56^{\star}$	$20^{\dagger}$
	Greedy Pursuers		6	3	3	6	$1^{\dagger}$	$23^{\star}$

⋆ Optimal reaction set of pursuers

† Optimal reaction set of evader

CHAPTER 6: CONCLUSION

In this thesis we considered an all-against-one LQ game and provided necessary conditions for the closed-loop Nash strategies as well as sufficient conditions for exponential stability of the resulting state trajectory. This all-against-one game was used as a framework to study three different pursuers strategies in a multi-pursuers one evader PE game. The strategies considered include cooperating, non-cooperating and greedy pursuers. In each case, these strategies are derived assuming that the evader is also using in each case the corresponding Nash strategy. An example is used to illustrate these strategies and show several different approaches that may be used by the pursuers to capture the evader. Finally, in the absence of complete information on the parameters in the objective functions, we performed a Monte Carlo simulation with 10,000 runs to produce probabilities of capture of the evader under uniform distribution of the parameter spaces and for several different strategies that could be used by the evader against the three proposed strategies for the pursuers.

The results indicate that in all-against-one games under parameter uncertainties, the Cooperative Nash strategy for the pursuers is not always the best choice. Furthermore, the Non-cooperative Nash strategy turns out to be the safest one in most of the cases. This is due to the existence of an adversary agent in the system and parameter uncertainties between players, so that it's not always safe for the pursuers to rely on the cooperation among themselves. Also, the Greedy Nash strategy is the worst one in almost all the cases unless the evader also chooses its Nash strategy versus only one of the pursuers (e.g. the closest one).

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