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The Optimal Portfolio Model Based on Multivariate T Distribution with Fuzzy Mathematics Method

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Abstract
This paper proposed the optimal portfolio model maximizing returns and minimizing the risk expressed as CvaR under the assumption that the portfolio yield subject to multivariate t distribution. With fuzzy mathematics method, we solve the multi-objectives model, and compare the model results to the case under the assumption of normal distribution yield, based on the portfolio VAR through empirical research. It is showed that our returns and risk are higher than M-V model.

Key words
Multivariate t distribution; The optimal portfolio; VAR; CVAR; Multi-objectives programming; Fuzzy mathematics

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INTRODUCTION

Markowitz[1] used the mean-variance and quadratic programming method to solve the optimal portfolio problem, which is considered as the cornerstone of modern financial theory. The model supposes the return of investment has normal distribution, and the investor's utility function is determined by mean and variance, where the variance reflects the investment risk. Undeniably, Markowitz's portfolio theory pioneered the quantitative measurement of financial risk and management, which is followed up by many other theories. With the deepening of financial theory and practice of financial measurement and modeling technology development, the inadequacy of the theory gradually emerged. Firstly, it is the suppose of normal distribution return, with further research and practical testing, some researchers found that assets were heavy-tailed and skewed distribution ,so literatures have improved the suppose. Bollerslev (1987)[2] described the foreign exchange return with t-distribution firstly. But they did not considered skewed distribution, so income distribution also caused changes in portfolio risk change with the characterization. Hansen (1994)[3] proposed skewed-t-distribution firstly, and considered both capital gains and fat-tail of the skewed nature of consideration. In recently research, some studies extended the single-variable distribution to multi-variate distribution, such as introducing Copula function. Multi-t distribution also can describe the heavy tail of return.Bao etc supposed hedge portfolio returns subject to multi-t-distribution, whose objective function is to minimize the VAR. They only consider the case of a sub-bit value, not take into account the limits of part of the loss. Rockafellar and S.Uryasev[5]proposed CVaR (Conditional Value at Risk), which is a new model for researching credit risk.. CVaR means the conditional mean of the loss of Var, which is better to satisfy the additive need, and showing a monotonic.

This paper supposes the portfolio return subject to multi-t-distribution, using CVAR to describe risk, and proposes a model maximizing mean and minimizing CVAR. At last, with Fuzzy Mathematics, we solve the multi-objectives model, and compare the model results to the case under the assumption of normal distribution yield, based on the portfolio VAR through empirical research. It is showed that our returns and risk are higher than M-V model.

1. RELATED CONCEPTS AND PROPERTIES

1.1 Multi-t-distribution and Related Properties[6]

Suppose $X = (T_1(\nu), T_2(\nu) \cdots T_n(\nu))$subject to n dimension with νdegree£according to Johnson, N. J. and Kotz, S. In the literature [5], which gives the definition of multi-t-distribution ,whose p.d.f as:

$$\begin{aligned} p(y) &= \frac{\Gamma\left(\frac{\nu+n}{2}\right)}{\Gamma\left(\frac{\nu}{2}\right)\sqrt{|V|}\,(\pi\nu)^{\frac{n}{2}}} \cdot \left(1 + \frac{1}{\nu}(y-\mu)^T V^{-1}(y-\mu)\right)^{-\frac{\nu+n}{2}} \\ &= |V|^{-\frac{1}{2}} f\left((y-\mu)^T \frac{V^{-1}}{\nu}(y-\mu)\right) \end{aligned}$$

That is:

$$f(u) = \frac{\Gamma\left(\frac{\nu+n}{2}\right)}{\Gamma\left(\frac{\nu}{2}\right)(\pi\nu)^{\frac{n}{2}}} \cdot \left(1 + \frac{u}{\nu}\right)^{-\frac{\nu+n}{2}}$$

let

$$A = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 1 \end{pmatrix},$$

$\mu = (\mu_1, \mu_2 \cdots \mu_n)$, $Y = AX + \mu$, $V = \frac{\nu}{\nu-2} AA^T$, suppose portfolio return is $R = w^T Y$, then: $E(R) = w^T \mu$, where $\mu^T = (\mu_1, \mu_2 \cdots \mu_n)$ is the n asset returns, $w^T = (w_1, w_2 \cdots w_n)$ are their proportion, and $\sum_{i=1}^{n} w_i = 1, 0 \leq w_i \leq 1$.

1.2 The Definition of CVAR

For each w, the loss f(w, y) is a random variable having a distribution induced by that of y. The underlying probability distribution of y will be assumed for convenience to have density, which we denote by p(y) as (1).

For a portfolio$w$, the loss is defined $f(w, y) = -w^T y$, given a believe degree$\beta\,(0 < \beta < 1)$, $VaR_\beta(w)$, $CVaR_\beta(w)$ are defined as [7]:

$$VaR_\beta(w) = \min\left\{\alpha \left| \int_{f(w,y)\leq\alpha} p(y)dy \geq \beta \right.\right\}$$

$$CVaR_{\beta}(w) = (1-\beta)^{-1} \int_{f(w,y)\geq VaR_{\beta}(x)} f(w,y)\, p(y) dy$$

Lemma: under the suppose of multivariate t distribution, with believe degree$\beta(0 < \beta < 1)$:

$$VaR(w) = -w^T\mu + S\left|w^T V w\right|^{\frac{1}{2}}$$
$$CVaR(w) = -w^T\mu + H\left|w^T V w\right|^{\frac{1}{2}}$$

where:

$$H = \frac{\Gamma\left(\frac{\nu-1}{2}\right)}{2(1-\alpha)\Gamma\left(\frac{\nu}{2}\right)\sqrt{\pi}} \nu^{\frac{\nu}{2}} \left(\left(\frac{w^T\mu + VaR}{\left|w^T V w\right|^{\frac{1}{2}}}\right)^2 + \nu\right)^{-\frac{\nu-1}{2}}$$

and S is solved by the equation:

$$\alpha = \frac{\pi^{\frac{n-1}{2}}}{\Gamma\left(\frac{n-1}{2}\right)} \int_S^{+\infty} \int_{z^2}^{+\infty} \left(u - z^2\right)^{\frac{n-3}{2}} f(u)\, du dz$$

2. THE OPTIMAL INVESTMENT MODEL

For investors, the aim is to seek the maximize returns while controlling risk as minimal risk.Suppose that the returns of portfolio follows multivariate t distribution.We propose the optimal portfolio under multi-t-distribution:

$$(P1)\begin{cases} \max \quad E(R) = w^T\mu & (1) \\ \min \quad CVaR(w) & (2) \\ s.t \quad \sum_{i=1}^{n} w_i = 1 & (3) \\ 0 \leq w_i \leq 1 & (4) \end{cases}$$

Model($P1$)is a multi-objective problem. We will use fuzzy mathematic method to solve it, the steps are follows:

(1) to solve objective(1)(2) to get the max and min value under constrains (3)(4).

(2) let stretching targets are equals to the difference between maximum and minimum values, transform the model to a single objective optimization problem, using of membership functions.

3. EMPIRICAL RESEARCH

3.1 The Optimal Portfolio under Multi-t-distribution

We choose two stocks (Handan Iron & Steel and Baidu internet), date begins 2010.1.7.3 to 2011.8.29, with 1174 closed days, calculate each day yield: $y_{i,j} = \frac{p_{i,j}-p_{i,j-1}}{p_{i,j-1}}$, where$p_{i,j}$ is previous day's closing price and $p_{i,j-1}$is the day after. And we also can get the average yield of Handan Iron & Steel is $\mu_1 = 0.0004181$, kurtosis is 15.775; the average yield of Baidu internet is $\mu_2 = 0.0019742$, kurtosis is 122.535, compare to normal return's kurtosis 3, the two stocks returns's is larger ,that is, the return is heavy tails.

let m=1174, $\alpha = 0.01$, using LINGO to solve model ($P1$) and ($P1'$):

$$(P2)\begin{cases} \max \quad E(R) = w^T\mu \\ s.t \quad \sum_{i=1}^{n} w_i = 1 \\ 0 \leq w_i \leq 1 \end{cases} \qquad (P2')\begin{cases} \min \quad E(R) = w^T\mu \\ s.t \quad \sum_{i=1}^{n} w_i = 1 \\ 0 \leq w_i \leq 1 \end{cases}$$

We get $R^+ = \max \quad E(R) = 0.0019742, R^- = \min \quad E(R) = 0.0006898$

Then to solve$(P2)$and$(P2')$

$$(P2)\begin{cases} \min \quad CVaR(w) \\ s.t \quad \sum_{i=1}^{n} w_i = 1 \\ 0 \le w_i \le 1 \end{cases} \qquad (P2')\begin{cases} \max \quad CVaR(w) \\ s.t \quad \sum_{i=1}^{n} w_i = 1 \\ 0 \le w_i \le 1 \end{cases}$$

And get: $C^+ = \max \quad CVaR(w) = 0.020338 \quad C^- = \min \quad CVaR(w) = \text{-}0.0019742$

At last, to solve model$(P0)$:

$$\max \quad \lambda$$
$$s.t \quad \begin{cases} E(R) - (R^+ - R^-)\lambda \ge R^- \\ CVaR(w) - (C^+ - C^-)\lambda \le C^+ \\ \sum_{i=1}^{n} w_i = 1 \\ 0 \le w_i \le 1 \\ \lambda \ge 0 \end{cases}$$

we get the optimal portfolio is$w = [0.25, 0.75]$, and $\max E(w) = 0.0016531$, $\min CVaR(w) = 0.014971$

3.2 Mean-VaR Model Under Normal Distribution Mean-VaR Model under Normal Distribution is $(P0')$:

$$(P0')\begin{cases} \max \ \mathrm{E}(R(w)) = w^T R = \sum_{i=1}^{n} w_i R_i \\ \min \ V(w) = w^T V w = \sum_{i=1}^{n}\sum_{j=1}^{n} x_i x_j \sigma_{ij} \\ s.t \quad \sum_{i=1}^{n} w_i = 1 \\ 0 \le w_i \le 1 \end{cases}$$

we get the optimal portfolio is $w = [0.35, 0.65]$, andmax $\mathrm{E}(R(w)) = 0.0015247$, min $V(w) = 0.013999$

It is showed that our max return is equal to and risk is higher than M-V model. so the CVaR predicts the potential risk of the portfolio, which help investors cautious investment.Although the empirical research only include two stocks, but it is easy to be extended to many stocks case.

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