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Spectral Radius of Nonnegative Centrosymmetric Matrices

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Abstract: In this paper, we present some results about the spectral radius of a kind of structured matrices, nonnegative centrosymmetric matrices. Furthermore, we constructrue a algorithm to compute the spectral radius of nonnegative centrosymmetric matrices.

Keywords: Centrosymmetric matrices; Spectral radius; Nonnegative matrices

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INTRODUCTION

The classical theory of nonnegative matrices has proved that there exists a nonnegative eigenvalue $\rho(A)$ for a nonnegative square matrix A; where $\rho(A)$ is the spectral radius of A.

Our interest is focused on nonnegative matrices with central symmetric structure. Recall that a matrix A is said to be centrosymmetric if $A = JAJ$ where J is the exchange matrix with ones on the cross diagonal (bottom left to top right) and zeros elsewhere. Centrosymmetric matrices appear in the numerical solution of certain differential equations[2], in the study of Markov processes[6] and in various physics and engineering problems[3], we will review some basic notations frequently used.

Definition 0.1[1] A matrix $A = (a_{ij})_{n\times n} \in R^{n,n}$is called a *centrosymmetric* matrix, if the elements of A satisfy the relation

$$J_nAJ_n = A \tag{1}$$

where$J_n = (e_n, e_{n-1}, \cdots, e_1)$, $e_i$denotes the standard unit vector with the ith entry 1.

For simplicity, we restrict our attention to the case of even, $n = 2m$.

For $n = 2m$, a centrosymmetric matrix can be written as the form[1,9]:

$$A = \begin{bmatrix} B & J_mCJ_m \\ C & J_mBJ_m \end{bmatrix} \quad \text{with} \quad B, C \in R^{m,m}.$$

We have known the following results, see[1,2]

Lemma 0.1[1]. *Let $A \in R^{n,n}$ be a centrosymmetric matrix, for $n = 2m$, let $P = \frac{\sqrt{2}}{2}\begin{bmatrix} I_m & I_m \\ -J_m & J_m \end{bmatrix}$, then*
$P^{-1}AP = \begin{bmatrix} B - J_mC & \\ & B + J_mC \end{bmatrix}$.

We shall use the concept of nonnegative matrices[4,11].

Definition 0.2 Let $B = (b_{ij})_{n\times m} \in R^{n,m}$ and $A = (a_{ij})_{n\times m} \in R^{n,m}$. We write $B \geq 0 \quad (> 0)$ if all $b_{ij} \geq 0 \quad (> 0)$; $A \geq B \quad (A > B)$ if $A - B \geq 0 (A - B > 0)$.
If $A \geq 0$, we say A is a *nonnegative*, and if $A > 0$, we say A is *positive*.

1. THE SPECTRAL RADIUS OF NONNEGATIVE CENTROSYMMETRIC MATRICES

Lemma 1.1[4] *Let $A = (a_{ij})_{n\times n}, B = (b_{ij})_{n\times n} \in R^{n,n}$, if $|A| \leq B$,then*

$$\rho(A) \leq \rho(|A|) \leq \rho(B),$$

where $|A| = (|a_{ij}|)_{n\times n}$, and $\rho(A)$ is the spectral radius of A.

According to the Lemma 2.1 and Lemma 1.1, we have the following result.

Theorem 1.1 *Let $A \in R^{n,n}$ be a nonnegative centrosymmetric matrix,*

$$A = \begin{bmatrix} B & J_mCJ_m \\ C & J_mBJ_m \end{bmatrix}, \quad \text{then} \rho(A) = \rho(B + J_mC).$$

Proof. From the hypothesis, we have that A is nonnegative. Then, according to the definition, B and C are both nonnegative. By Lemma 1.1,

$$P^{-1}AP = \begin{bmatrix} B - J_mC & \\ & B + J_mC \end{bmatrix}.$$

Note that B and C are both nonnegative, which implies

$$-B - J_mC \leq B - J_mC \leq B + J_mC.$$

That is, $|B - J_mC| \leq B + J_mC$. From Lemma 2.1, we can deduce that $\rho(B - J_mC) \leq \rho(B + J_mC)$. It is obvious that

$$\rho(A) = \rho(P^{-1}AP) = \max\{\rho(B - J_mC), \rho(B + J_mC)\}$$

we get $\rho(A) = \rho(B + J_mB)$.

2. AN ALGORITHM ON THE SPECTRAL RADIUS OF IRREDUCIBLE NONNEGATIVE MATRICES

Lemma 2.1[11] *Let $B \in R^{n,n}$ be a positive (or irreducible nonnegative) matrix, and $z = (z_1, \cdots, z_n)^T, y = (y_1, \cdots, y_n)^T$.*

(1) If $z \geq 0, z \neq 0$ and $Bz = \lambda z$, then $z > 0, \lambda = \rho(B)$.

(2) If$Bz = \rho(B)z, By = \rho(B)y, z > 0, y > 0$ then $y = kz, k > 0$

Lemma 2.2[4] *Let $A \in R^{n,n}$ be an irreducible nonnegative matrix, then*

$$(I + A)^{n-1} > 0$$

where I is the identity matrix of order n. And for any nonnegative nonzero vector x, we have $(I+A)^{n-1}x > 0$.

Definition 2.2 (*C-W function*)[7] Let $A = (a_{ij})_{n\times n}$ be an irreducible nonnegative matrix. For any vector $x = (x_1, \cdots, x_n)^T > 0$, $F_A(x)$ and $G_A(x)$ are defined as

$$F_A(x) = \min_{1\le i\le n} \frac{(Ax)_i}{x_i}; \quad G_A(x) = \max_{1\le i\le n} \frac{(Ax)_i}{x_i}.$$

Lemma 2.3[9] *Let $A = (a_{ij})_{n\times n} \in R^{n,n}$ be an irreducible nonnegative matrix, $F_A(x)$ and $G_A(x)$ are the C-W functions of A. Then*

(1) $F_A(tx) = F_A(x), G_A(tx) = G_A(x)$ *for* $t > 0$.

(2) $Ax - kx \ge 0$ $(x > 0)$ implies $F_A(x) \ge k$, *and*

$Ax - mx \le 0$ $(x > 0)$ implies $G_A(x) \le m$.

(3) If $x > 0$ *and* $y = (I + A)^{n-1}x$, *then* $F_A(x) \le F_A(y), G_A(x) \ge G_A(y)$.

Let $A \in R^{n,n}$ be an irreducible nonnegative matrix of, and $B = (I + A)^{n-1}$. Let the initial vector $x^{(0)} = (x_1^{(0)}, \cdots, x_n^{(0)})^T > 0$. Define the iteration as follows:

$$y^{(k)} = Bx^{(k-1)} = (I + A)^{n-1}x^{(k-1)}, \quad x^{(k)} = [1/\left\|y^{(k)}\right\|_1]y^{(k)}, \quad k = 1, 2, \cdots \tag{2}$$

where $\|x\|_1 = \sum_{i=1}^{n} |x_i|$. It is obviously that $\left\|x^{(k)}\right\|_1 = 1(k = 1, 2, \cdots)$.

Theorem 2.1 (Convergent Theorem) *Let $A \in R^{n,n}$ be an irreducible nonnegative matrix, $B = (I + A)^{n-1}$, $\left\{x^{(k)} : k = 1, 2, \cdots\right\}$ is a vector sequence defined in (2). Then*

$$\lim_{n\to\infty} F_A(x^{(k)}) = \lim_{n\to\infty} G_A(x^{(k)}) = \rho(A)$$

and $\lim_{n\to\infty} x^{(k)} = z$, *where z satisfies* $z > 0, Az = \rho(A)z$, *and* $\|z\|_1 = 1$.

Proof. According to Definition 2.2, we can see

$$Ax - F_A(x)x \ge 0, \quad \text{for} \quad x > 0. \tag{3}$$

By Lemma 2.2(1),(3) and the fact that $x^{(k)} = [1/\left\|y^{(k)}\right\|_1]y^{(k)}$, we know

$$F_A(x^{(k)}) \le F_A(x^{(k+1)}), k = 1, 2, \cdots$$

This means $\{F_A(x^{(k)})\}$ is a monotonic sequence bounded above (from Lemma 2.3 (4)). Therefore, $\{F_A(x^{(k)})\}$ is a convergent sequence. Let $\lim_{n\to\infty} F_A(x^{(k)}) = l$.

It is obvious that

$$x^{(k)} > 0, \left\|x^{(k)}\right\| = 1(k = 1, 2, \cdots) \tag{4}$$

So $\{x^{(k)}\}$ is a bounded vector sequence. Let $\{v^{(k)}\}(k = 1, 2, \cdots)$ be a arbitrary convergent subsequence of $\{x^{(k)}\}$, and $z = \lim_{k\to\infty} v^{(k)}$. From (2),(3),(4), we obtain

$$\|z\|_1 = 1, \quad z \ge 0, \quad Bz = \lambda z, \quad Az - lz > 0. \tag{5}$$

By Lemma 2.1, $Bz = \lambda z = \rho(B)z, z > 0$. Besides, Lemma 2.2 and (3.4) imply that $\lambda > 0$. Next we will show $Az - lz = 0$. If $Az - lz \neq 0$, then

$$Az - lz = A(\frac{1}{\lambda}Bz) - l(\frac{1}{\lambda}Bz) = \frac{1}{\lambda}B(Az - lz) > 0$$

from Lemma 2.2. By Definition 3.2 and Lemma 2.3, we know that

$$l < F_A(z) = \lim_{k\to\infty} F_A(v^k) = l$$

which contradicts. Thus, $Az - lz = 0$, or $Az = lz$. From Lemma 2.1, we get

$$l = \rho(A), Az = \rho(A)z \tag{6}$$

Assume that $\{u^k\}(k = 1, 2, \cdots)$ is another convergent subsequence of $\{x^k\}$ and $\lim_{k\to\infty} u^{(k)} = y$, then we can also prove

$$\|y\|_1 = 1, y > 0, By = \rho(B)y.$$

However, by Lemma 2.1, we have $y = z$. That is to say, any convergent subsequence of $\{x^k\}(k = 1, 2, \cdots)$ converges to the same vector z. Thus, $\{x^k\}$ itself is convergent and $\lim_{k\to\infty} x^k = z$. From (6), we know that $\lim_{n\to\infty} F_A(x^{(k)}) = l = \rho(A)$.

Similarly, we can prove the following results

$$\lim_{k\to\infty} G_A(x^{(k)}) = h, Az - hz \le 0, \lim_{k\to\infty} x^{(k)} = z > 0.$$

Likewise, we get $Az = hz, h = \rho(A), \lim_{k\to\infty} G_A(x^{(k)}) = \rho(A)$,

Corollary 2.1 From the proof above, we have
$0 < F_A(x^{(0)}) \le F_A(x^{(1)}) \le \cdots \le F_A(x^{(k)}) \le \cdots \le \rho(A) \le \cdots \le G_A(x^{(k)}) \le \cdots \le G_A(x^{(1)}) \le G_A(x^{(0)})$.

Based on this theorem, we present a algorithm to compute the spectral radius of nonnegative square matrices:

Algorithm 1.

Step1. Let $x^{(0)} = (1, 1, \cdots, 1)^T$(or any other positive vector), give precision $\varepsilon > 0$.

Step2. Compute $x^{(k)}$ from $x^{(k-1)}$, $k = 1, 2, \cdots$

$$y^{(k)} = (I + A)^{n-1}x^{(k-1)}, \quad x^{(k)} = [1/\sum_{i=1}^{n} y_i^{(k)}]y^{(k)}$$

Step3. Compute $F_A(x^{(k)})$, $G_A(x^{(k)})$:

$$F_A(x) = \min_{1\le i\le n} \frac{(Ax)_i}{x_i}; \quad G_A(x) = \max_{1\le x\le n} \frac{(Ax)_i}{x_i}.$$

Step4. If $G_A(x^{(k)}) - F_A(x^{(k)}) < \varepsilon$, goto Step5; otherwise go back to Step 2.

Step5. Let $\lambda = \frac{1}{2}(G_A(x^{(k)}) + F_A(x^{(k)}))$, and λ is the approximation of the spectral radius of A.

We have the following result which shows **Algorithm 1** is convergent.

Theorem 2.2 *Given a precision* $\varepsilon > 0$, *if*
$G_A(x^{(k)}) - F_A(x^{(k)}) < \varepsilon$, *then* $\left|\rho(A) - \lambda^{(k)}\right| < \frac{\varepsilon}{2}$,*where*$\lambda^{(k)} = \frac{1}{2}(F_A(x^k) + G_A(x^k))$.

3. COMPUTATION OF SPECTRAL RADIUS OF NONNEGATIVE CENTROSYMMETRIC MATRICES

As a application of Theorem 2.1 and **Algorithm 1,** we present **Algorithm 2** for computing the spectral radius of a nonnegative centrosymmetric matrix

For simplicity, we assume B is irreducible. We have the following result.

Lemma 3.1 *Let $B, C \in R^{n,n}$ be nonnegative matrices. If B is irreducible, then $B + C$ is irreducible.*

From the lemma above, we know that $D = B + J_m C$ is irreducible.

Algorithm 2

Step1. Compute D: $D = B + J_m C$.

Step2. Let $x^{(0)} = (1, 1, \cdots, 1)^T$, give precision $\varepsilon > 0$.

Step3. Compute $x^{(k)}$ from $x^{(k-1)}$ $\quad, k = 1, 2, \cdots$

$$y^{(k)} = (I + D)^{n-1} x^{(k-1)}, \quad x^{(k)} = [1/\sum_{i=1}^{n} y_i^{(k)}] y^{(k)}.$$

Step4. Compute $F_A(x^{(k)})$, $G_A(x^{(k)})$.

Step5. If $G_D(x^{(k)}) - F_D(x^{(k)}) < \varepsilon$, go to Step6; otherwise go back to Step 2.

Step6. Compute λ:$\lambda = \frac{1}{2}(G_D(x^{(k)}) + F_D(x^{(k)}))$.

Here λ is the approximation of $\rho(A)$ with the precision ε.

Example 1. Given a 8×8 nonnegative centrosymmetric matrix

$$A = \begin{bmatrix} 0.4326 & 0.8671 & 0.9441 & 0.9989 & 1.2025 & 1.5937 & 0.5928 & 0.7633 \\ 0.6656 & 0.7258 & 1.3362 & 0.6900 & 1.1908 & 1.2540 & 1.0668 & 1.1892 \\ 1.2533 & 0.5883 & 0.7143 & 0.8156 & 0.6686 & 0.8580 & 1.1393 & 1.1909 \\ 0.8768 & 1.1832 & 1.6236 & 0.7119 & 1.2902 & 0.6918 & 1.3645 & 1.1465 \\ 1.1465 & 1.3645 & 0.6918 & 1.2902 & 0.7119 & 1.6236 & 1.1832 & 0.8768 \\ 1.1909 & 1.1393 & 0.8580 & 0.6686 & 0.8156 & 0.7143 & 0.5883 & 1.2533 \\ 1.1892 & 1.0668 & 1.2540 & 1.1908 & 0.6900 & 1.3362 & 0.7258 & 0.6656 \\ 0.7633 & 0.5928 & 1.5937 & 1.2025 & 0.9989 & 0.9441 & 0.8671 & 0.4326 \end{bmatrix}.$$

Then we have

$$B = \begin{bmatrix} 0.4326 & 0.8671 & 0.9441 & 0.9989 \\ 0.6656 & 0.7258 & 1.3362 & 0.6900 \\ 1.2533 & 0.5883 & 0.7143 & 0.8156 \\ 0.8768 & 1.1832 & 1.6236 & 0.7119 \end{bmatrix}, C = \begin{bmatrix} 1.1465 & 1.3645 & 0.6918 & 1.2902 \\ 1.1909 & 1.1393 & 0.8580 & 0.6686 \\ 1.1892 & 1.0668 & 1.2540 & 1.1908 \\ 0.7633 & 0.5928 & 1.5937 & 1.2025 \end{bmatrix}$$

Imput A and $\varepsilon = 1 \times 10^{-6}$, and use the algorithm 2. The result comes out as $\lambda = 7.875600$. We recompute the spectral radius of A by MATLAB 7.1, and get $\rho(A) = 7.875600$. This example shows that Algorithm 2 is an efficient methods to compute the spectral radius of a nonnegative centrosymmetric matrix.

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