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Multiple-Soliton Solutions for Extended Shallow Water Wave Equations

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Abstract: Four extended shallow water wave equations are introduced and studied for complete integrability. We show that the additional terms do not kill the integrability of the typical equations. The Hereman's simplified method and the Cole-Hopf transformation method are used to show this goal. Multiple soliton solutions will be derived for each model. The analysis highlights the effects of the extension terms on the structures of the obtained solutions.

Key Words: Shallow Water Wave Equations; Complete Integrability; Multiple-Soliton Solutions

1. INTRODUCTION

In [1–2], the (2+1)-dimensional shallow water wave equations

$$u_{yt} + u_{xxxy} - 3u_{xx}u_y - 3u_x u_{xy} = 0, \tag{1}$$

and

$$u_{xt} + u_{xxxy} - 2u_{xx}u_y - 4u_x u_{xy} = 0, \tag{2}$$

were studied. Both equations reduce to the potential KdV equation for $y = x$. The difference between the two models (1) and (2) is that x replaces y in the term u_{yt} and in the coefficients of the other terms.

In [1,3], the (3+1)-dimensional shallow water wave equations

$$u_{yzt} + u_{xxxyz} - 6u_x u_{xyz} - 6u_{xz}u_{xy} = 0, \tag{3}$$

and

$$u_{xzt} + u_{xxxyz} - 2(u_{xx}u_{yz} + u_y u_{xxz}) - 4(u_x u_{xyz} + u_{xz}u_{xy}) = 0, \tag{4}$$

were also studied. Both equations reduce to the potential KdV equation for $z = y = x$. The difference between the first terms of the two models is that x replaces y in the term u_{yzt}.

The focus of the studies on Eqs. (1)–(4) in [1–3], and some of the references therein was to show that each model is completely integrable and each one gives rise to multiple soliton solutions. For more details about the results obtained for these equations, read [1–3] and some of the references therein.

In this work, we will introduce four extended shallow water wave equations in (2+1) and (3+1) dimensions. We first introduce the first two extended equations

$$u_{yt} + u_{xxxy} - 3u_{xx}u_y - 3u_x u_{xy} + \alpha u_{xy} = 0, \tag{5}$$

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and

$$u_{xt} + u_{xxxy} - 2u_{xx}u_y - 4u_xu_{xy} + \alpha u_{xy} = 0. \quad (6)$$

We next introduce the two extended (3+1)-dimensional shallow water wave equations

$$u_{yzt} + u_{xxxyz} - 6u_xu_{xyz} - 6u_{xz}u_{xy} + \alpha u_{xyz} = 0, \quad (7)$$

and

$$u_{xzt} + u_{xxxyz} - 2(u_{xx}u_{yz} + u_yu_{xxz}) - 4(u_xu_{xyz} + u_{xz}u_{xy}) + \alpha u_{xyz} = 0. \quad (8)$$

The extended equations are established by adding the derivative of $u(x,t)$ with respect to the space variables x and y for the first two equations (1) and (2), and with respect to the space variables x, y, and z for the last two equations (3) and (4).

A variety of distinct methods are used for classification of integrable equations. The Painlevé analysis, the inverse scattering method, the Bäcklund transformation method, the conservation laws method, and the Hirota bilinear method [4–13] are mostly used in the literature for investigating complete integrability. The Hirota's bilinear method [1–22] is rather heuristic and possesses significant features that make it ideal for the determination of multiple soliton solutions for a wide class of nonlinear evolution equations.

Our aim from this work is two fold. We aim first to show that the additional terms αu_{xy} for the first two equations, and αu_{xyz} for the last two equations do not kill the integrability of the typical shallow water wave equations (1)–(4). We next aim to derive multiple soliton solutions for these extended forms (5)–(8) and to show the effect of these new terms on the structures of the obtained solutions. The Cole-Hopf transformation combined with the Hereman's method, that was established by Hereman et. al. in [13] will be used to achieve the goals set for this work. The Hereman's method can be found in [13–22], hence our main focus will be on applying this method.

2. THE FIRST EXTENDED SHALLOW WATER WAVE EQUATION

In this section we will study the extended (2+1)-dimensional shallow water wave equation

$$u_{yt} + u_{xxxy} - 3u_{xx}u_y - 3u_xu_{xy} + \alpha u_{xy} = 0. \quad (9)$$

As stated before, the Hereman's method and the Cole-Hopf transformation method will be used for this analysis.

2.1 Multiple Soliton-Solutions

Substituting

$$u(x,y,t) = e^{\theta_i}, \theta_i = k_ix + r_iy - c_it, \quad (10)$$

into the linear terms of (9), and solving the resulting equation, the dispersion relation

$$c_i = k_i^3 + \alpha k_i, i = 1, 2, \cdots N, \quad (11)$$

and hence

$$\theta_i = k_ix + r_iy - (k_i^3 + \alpha k_i)t, \quad (12)$$

are readily obtained. Notice that the dispersion relation c_i is affected by the extension term αu_{xy}.

To determine R, we substitute the Cole-Hopf transformation

$$u(x,y,t) = R\,(\ln\ f(x,y,t))_x = R\frac{f_x(x,y,t)}{f(x,y,t)}, \tag{13}$$

where the auxiliary function $f(x,y,t)$ is given by

$$f(x,y,t) = 1 + e^{k_1x+r_1y-(k_1^3+\alpha k_1)t}, \tag{14}$$

into Eq. (9) and solve to find that $R = -2$. This means that the single soliton solution is given by

$$u(x,y,t) = -\frac{2k_1 e^{k_1x+r_1y-(k_1^3+\alpha k_1)t}}{1+e^{k_1x+r_1y-(k_1^3+\alpha k_1)t}}. \tag{15}$$

For the two-soliton solutions, we use the auxiliary function

$$f(x,y,t) = 1 + e^{\theta_1} + e^{\theta_2} + a_{12}e^{\theta_1+\theta_2}, \tag{16}$$

into (13), and substitute the result in Eq. (9) to find the phase shift

$$a_{12} = \frac{(k_1-k_2)(r_1-r_2)}{(k_1+k_2)(r_1+r_2)}, \tag{17}$$

and hence

$$a_{ij} = \frac{(k_i-k_j)(r_i-r_j)}{(k_i+k_j)(r_i+r_j)}, 1 \le i < j \le N. \tag{18}$$

Comparing the results for the phase shifts does not show any effect from the extension term αu_{xy}. The result (17) is the same as obtained for (1) in [2].

It is also clear that the phase shifts $a_{ij}, 1 \le i < j \le N$ depend on the coefficients k_m and r_m of the spatial variables x and y respectively. Moreover, we point out that the first extended shallow water wave equation does not show any resonant phenomenon [10] because the phase shift term a_{12} in (17) cannot be 0 or ∞ for $|k_1| \ne |k_2|$ and $|r_1| \ne |r_2|$.

This in turn gives

$$\begin{aligned} f(x,y,t) &= 1 + e^{k_1x+r_1y-(k_1^3+\alpha k_1)t} + e^{k_2x+r_2y-(k_2^3+\alpha k_2)t} \\ &+ \tfrac{(k_1-k_2)(r_1-r_2)}{(k_1+k_2)(r_1+r_2)}\, e^{(k_1+k_2)x+(r_1+r_2)y-(k_1^3+\alpha k_1+k_2^3+\alpha k_2)t}. \end{aligned} \tag{19}$$

To determine the two-soliton solutions explicitly, we substitute (19) into the formula $u = -2[\ln f(x,y,t)]_x$.

Similarly, to determine the three-soliton solutions, we set

$$\begin{aligned} f(x,y,t) &= 1 + e^{k_1x+r_1y-(k_1^3+\alpha k_1)t} + e^{k_2x+r_2y-(k_2^3+\alpha k_2)t} + e^{k_3x+r_3y-(k_3^3+\alpha k_3)t} \\ &+ \tfrac{(k_1-k_2)(r_1-r_2)}{(k_1+k_2)(r_1+r_2)}\, e^{(k_1+k_2)x+(r_1+r_2)y-(k_1^3+\alpha k_1+k_2^3+\alpha k_2)t} \\ &+ \tfrac{(k_1-k_3)(r_1-r_3)}{(k_1+k_3)(r_1+r_3)}\, e^{(k_1+k_3)x+(r_1+r_3)y-(k_1^3+\alpha k_1+k_3^3+\alpha k_3)t} \\ &+ \tfrac{(k_2-k_3)(r_2-r_3)}{(k_2+k_3)(r_2+r_3)}\, e^{(k_2+k_3)x+(r_2+r_3)y-(k_2^3+\alpha k_2+k_3^3+\alpha k_3)t} \\ &+ b_{123}\, e^{(k_1+k_2+k_3)x+(r_1+r_2+r_3)y-(k_1^3+\alpha k_1+k_2^3+\alpha k_2+k_3^3+\alpha k_3)t}, \end{aligned} \tag{20}$$

into (13) and substitute it into the Eq. (9) to find that

$$b_{123} = a_{12}a_{13}a_{23}. \tag{21}$$

To determine the three-soliton solutions explicitly, we substitute the last result for $f(x,y,t)$ in the formula $u(x,y,t) = -2(\ln f(x,y,t))_x$. The higher level soliton solutions, for $n \ge 4$ can be obtained in a parallel manner. This shows that the first extended (2+1)-dimensional shallow water wave equation (9) is completely integrable and gives rise to multiple-soliton solutions of any order.

3. THE SECOND EXTENDED SHALLOW WATER WAVE EQUATION

In this section we will study the second extended (2+1)-dimensional shallow water wave equation

$$u_{xt} + u_{xxxy} - 2u_{xx}u_y - 4u_xu_{xy} + \alpha u_{xy} = 0. \tag{22}$$

We will follow a manner parallel to the approach employed before.

3.1 Multiple Soliton-Solutions

Substituting

$$u(x, y, t) = e^{\theta_i}, \theta_i = k_i x + r_i y - c_i t, \tag{23}$$

into the linear terms of (22), and solving the resulting equation we obtain the dispersion relation

$$c_i = k_i^2 r_i + \alpha r_i, i = 1, 2, \cdots N, \tag{24}$$

and hence θ_i becomes

$$\theta_i = k_i x + r_i y - (k_i^2 r_i + \alpha r_i)t. \tag{25}$$

Notice that the dispersion relation c_i depends on the coefficients k_i and r_i of the spatial variables x and y respectively, and is affected by the extension term αu_{xy}.

To determine R, we substitute

$$u(x, y, t) = R\,(\ln\ f(x, y, t))_x = R\frac{f_x(x, y, t)}{f(x, y, t)}, \tag{26}$$

where the auxiliary function

$$f(x, y, t) = 1 + e^{k_1 x + r_1 y - (k_1^2 r_1 + \alpha r_1)t}, \tag{27}$$

into Eq. (22) and solve to find that $R = -2$. This means that the single soliton solution is given by

$$u(x, y, t) = -\frac{2k_1 e^{k_1 x + k_1 y - (k_1^2 r_1 + \alpha r_1)t}}{1 + e^{k_1 x + k_1 y - (k_1^2 r_1 + \alpha r_1)t}}. \tag{28}$$

For the two-soliton solutions, we us the auxiliary function

$$f(x, y, t) = 1 + e^{\theta_1} + e^{\theta_2} + a_{12}e^{\theta_1 + \theta_2}, \tag{29}$$

into (26), with $R = -2$, and we use the outcome into Eq. (22) to obtain

$$a_{12} = \frac{(k_1 - k_2)^2}{(k_1 + k_2)^2}, \tag{30}$$

and hence

$$a_{ij} = \frac{(k_i - k_j)^2}{(k_i + k_j)^2}, 1 \le i < j \le N. \tag{31}$$

It is obvious that the phase shifts $a_{ij}, 1 \le i < j \le N$ do not depend on the coefficients r_i of the spatial variable y. Moreover, we point out that the second shallow water wave equation does not show any resonant phenomenon [10] because the phase shift term a_{12} in (30) cannot be 0 or ∞ for $|k_1| \neq |k_2|$. Moreover, the phase shift was not affected by the extension term αu_{xy}.

Consequently, we obtain

$$\begin{aligned} f(x,y,t) &= 1 + e^{k_1x+r_1y-(k_1^2r_1+\alpha r_1)t} + e^{k_2x+r_2y-(k_2^2r_2+\alpha r_2)t} \\ &+ \frac{(k_1-k_2)^2}{(k_1+k_2)^2}\, e^{(k_1+k_2)x+(r_1+r_2)y-((k_1^2r_1+\alpha r_1)+(k_2^2r_2+\alpha r_2))t}. \end{aligned} \tag{32}$$

To determine the two-soliton solutions explicitly, we substitute (32) into the formula $u = -2[\ln f(x,y,t)]_x$.

Similarly, to determine the three-soliton solutions, we set

$$\begin{aligned} f(x,y,t) &= 1 + e^{k_1x+r_1y-(k_1^2r_1+\alpha r_1)t} + e^{k_2x+r_2y-(k_2^2r_2+\alpha r_2)t} + e^{k_3x+r_3y-(k_3^2r_3+\alpha r_3)t} \\ &+ \frac{(k_1-k_2)^2}{(k_1+k_2)^2}\, e^{(k_1+k_2)x+(r_1+r_2)y-((k_1^2r_1+\alpha r_1)+(k_2^2r_2+\alpha r_2))t} \\ &+ \frac{(k_1-k_3)^2}{(k_1+k_3)^2}\, e^{(k_1+k_3)x+(r_1+r_3)y-((k_1^2r_1+\alpha r_1)+(k_3^2r_3+\alpha r_3))t} \\ &+ \frac{(k_2-k_3)^2}{(k_2+k_3)^2}\, e^{(k_2+k_3)x+(r_2+r_3)y-((k_2^2r_2+\alpha r_2)+(k_3^2r_3+\alpha r_3))t} \\ &+ b_{123}\, e^{(k_1+k_2+k_3)x+(r_1+r_2+r_3)y-(k_1^2r_1+\alpha r_1+k_2^2r_2+\alpha r_2+k_3^2r_3+\alpha r_3)t}, \end{aligned} \tag{33}$$

into (26) and substitute it into Eq. (22) to find that

$$b_{123} = a_{12}a_{13}a_{23}. \tag{34}$$

To determine the three-soliton solutions explicitly, we substitute the last result for $f(x,y,t)$ in the formula $u(x,y,t) = -2(\ln f(x,y,t))_x$. The higher level soliton solutions, for $n \geq 4$ can be obtained in a parallel manner. This shows that the second extended (2+1)-dimensional shallow water wave equation (22) is completely integrable and gives rise to multiple-soliton solutions of any order.

4. THE THIRD EXTENDED SHALLOW WATER WAVE EQUATION

In this section we will study the extended (3+1)-dimensional shallow water wave equation

$$u_{yzt} + u_{xxxyz} - 6u_xu_{xyz} - 6u_{xz}u_{xy} + \alpha u_{xyz} = 0. \tag{35}$$

We will apply the approach used before, hence we will skip details.

4.1 Multiple Soliton-Solutions

To determine multiple-soliton solutions for Eq. (35), we first substitute

$$u(x,y,z,t) = e^{\theta_i}, \theta_i = k_ix + r_iy + s_iz - c_it, \tag{36}$$

into the linear terms of (35), and solving the result to obtain the dispersion relation

$$c_i = k_i^3 + \alpha k_i, i = 1, 2, \cdots N, \tag{37}$$

and hence θ_i becomes

$$\theta_i = k_ix + r_iy + s_iz - (k_i^3 + \alpha k_i)t. \tag{38}$$

Notice that the dispersion relation c_i depends only on the coefficients k_i of x and does not depend on the coefficients r_i and s_i of the spatial variables y and z respectively. Moreover, the extension term αu_{xyz} affected the dispersion relation.

To determine R, we substitute the Cole-Hopf transformation

$$u(x,y,z,t) = R\,(\ln\ f(x,y,z,t))_x = R\frac{f_x(x,y,z,t)}{f(x,y,z,t)}, \tag{39}$$

where $f(x,y,z,t) = 1 + e^{k_1x+r_1y+s_1z-(k_1^3+\alpha k_1)t}$ into Eq. (35) and solve to find that $R = -2$. This gives the single soliton solution by

$$u(x,y,z,t) = -\frac{2k_1e^{k_1x+k_1y+s_1z-(k_1^3+\alpha k_1)t}}{1+e^{k_1x+k_1y+s_1z-(k_1^3+\alpha k_1)t}}. \tag{40}$$

For the two-soliton solutions, we substitute

$$f(x,y,z,t) = 1 + e^{\theta_1} + e^{\theta_2} + a_{12}e^{\theta_1+\theta_2}, \tag{41}$$

into Eq. (35) to obtain the phase shift

$$a_{12} = \frac{(k_1-k_2)^2}{(k_1+k_2)^2}, \tag{42}$$

and hence

$$a_{ij} = \frac{(k_i-k_j)^2}{(k_i+k_j)^2}, 1 \le i < j \le N. \tag{43}$$

It is clear that the phase shifts $a_{ij}, 1 \le i < j \le N$ depend only on the coefficients k_m of the spatial variable x. Moreover, the extension term αu_{xyz} has no effect on the phase shift. We point out that the third extended shallow water wave equation does not show any resonant phenomenon [10] because the phase shift term a_{12} in (17) cannot be 0 or ∞ for $|k_1| \neq |k_2|$.

This in turn gives

$$\begin{aligned} f(x,y,z,t) \;=\; & 1 + e^{k_1x+r_1y+s_1z-(k_1^3+\alpha k_1)t} + e^{k_2x+r_2y+s_2z-(k_2^3+\alpha k_2)t} \\ & + \tfrac{(k_1-k_2)^2}{(k_1+k_2)^2}\, e^{(k_1+k_2)x+(r_1+r_2)y+(s_1+s_2)z-((k_1^3+\alpha k_1)+(k_2^3+\alpha k_2))t}. \end{aligned} \tag{44}$$

To determine the two-soliton solutions explicitly, we substitute (44) into the (39) and using $R = -2$.

Similarly, to determine the three-soliton solutions, we set

$$\begin{aligned} f(x,y,z,t) \;=\; & 1 + e^{k_1x+r_1y+s_1z-(k_1^3+\alpha k_1)t} + e^{k_2x+r_2y+s_2z-(k_2^3+\alpha k_2)t} + e^{k_3x+r_3y+s_3z-(k_3^3+\alpha k_3)t} \\ & + \tfrac{(k_1-k_2)^2}{(k_1+k_2)^2}\, e^{(k_1+k_2)x+(r_1+r_2)y+(s_1+s_2)z-((k_1^3+\alpha k_1)+(k_2^3+\alpha k_2))t} \\ & + \tfrac{(k_1-k_3)^2}{(k_1+k_3)^2}\, e^{(k_1+k_3)x+(r_1+r_3)y+(s_1+s_3)z-((k_1^3+\alpha k_1)+(k_3^3+\alpha k_3))t} \\ & + \tfrac{(k_2-k_3)^2}{(k_2+k_3)^2}\, e^{(k_2+k_3)x+(r_2+r_3)y+(s_2+s_3)z-((k_2^3+\alpha k_2)+(k_3^3+\alpha k_3))t} \\ & + b_{123}\, e^{(k_1+k_2+k_3)x+(r_1+r_2+r_3)y+(s_1+s_2+s_3)z-(k_1^3+\alpha k_1+k_2^3+\alpha k_2+k_3^3+\alpha k_3)t}, \end{aligned} \tag{45}$$

into (39) and substitute it into Eq. (35) to find that

$$b_{123} = a_{12}a_{13}a_{23}. \tag{46}$$

To determine the three-soliton solutions explicitly, we substitute the last result for $f(x,y,z,t)$ in the formula $u(x,y,z,t) = -2(\ln f(x,y,z,t))_x$. The higher level soliton solutions, for $n \ge 4$ can be obtained in a parallel manner. This shows that the extended (3+1)-dimensional shallow water wave equation (35) is completely integrable and gives rise to multiple-soliton solutions of any order.

5. THE FOURTH EXTENDED SHALLOW WATER WAVE EQUATION

We close our analysis by studying the fourth extended (3+1)-dimensional shallow water wave equation

$$u_{xzt} + u_{xxxyz} - 2(u_{xx}u_{yz} + u_y u_{xxz}) - 4(u_x u_{xyz} + u_{xz}u_{xy}) + \alpha u_{xyz} = 0. \quad (47)$$

The Hereman's method and the Cole-Hopf transformation will be used to conduct this analysis.

5.1 Multiple Soliton-Solutions

To determine multiple-soliton solutions for Eq. (47), we follow the steps presented above. We first substitute

$$u(x, y, z, t) = e^{\theta_i}, \theta_i = k_i x + r_i y + s_i z - c_i t, \quad (48)$$

into the linear terms of the (47), and solving the resulting equation we obtain the dispersion relation

$$c_i = k_i^2 r_i + \alpha r_i, i = 1, 2, \cdots N, \quad (49)$$

and hence we set

$$\theta_i = k_i x + r_i y + s_i z - (k_i^2 r_i + \alpha r_i)t. \quad (50)$$

Notice that the dispersion relation c_i depends only on the coefficients k_i and r_i of the spatial variables x and y respectively, and on the extension term αu_{xyz}.

We next use the Cole-Hopf transformation

$$u(x, y, z, t) = R\frac{f_x(x, y, z, t)}{f(x, y, z, t)}, \quad (51)$$

where $f(x, y, z, t) = 1 + e^{k_1 x + r_1 y + s_1 z - (k_1^2 r_1 + \alpha r_1)t}$ into Eq. (47) and solve to find that $R = -2$. This gives the single soliton solution

$$u(x, y, z, t) = -\frac{2k_1 e^{k_1 x + k_1 y + s_1 z - (k_1^2 r_1 + \alpha r_1)t}}{1 + e^{k_1 x + k_1 y + s_1 z - (k_1^2 r_1 + \alpha r_1)t}}. \quad (52)$$

For the two-soliton solutions, we use

$$f(x, y, z, t) = 1 + e^{\theta_1} + e^{\theta_2} + a_{12}e^{\theta_1 + \theta_2}, \quad (53)$$

into Eq. (47) to obtain

$$a_{12} = \frac{(k_1 - k_2)^2}{(k_1 + k_2)^2}, \quad (54)$$

and hence

$$a_{ij} = \frac{(k_i - k_j)^2}{(k_i + k_j)^2}, 1 \leq i < j \leq N. \quad (55)$$

It is clear that the phase shifts $a_{ij}, 1 \leq i < j \leq N$ depend only on the coefficients k_m of the spatial variable x. The other coefficients r_m, s_m and α has no effect on the phase shifts. Moreover, we point out that the first shallow water wave equation does not show any resonant phenomenon [10] because the phase shift term a_{12} in (54) cannot be 0 or ∞ for $|k_1| \neq |k_2|$.

This in turn gives

$$\begin{aligned} f(x, y, z, t) \quad = \quad & 1 + e^{k_1 x + r_1 y + s_1 z - (k_1^2 r_1 + \alpha r_1)t} + e^{k_2 x + r_2 y + s_2 z - (k_2^2 r_2 + \alpha r_2)t} \\ & + \frac{(k_1 - k_2)^2}{(k_1 + k_2)^2} e^{(k_1 + k_2)x + (r_1 + r_2)y + (s_1 + s_2)z - ((k_1^2 r_1 + \alpha r_1) + (k_2^2 r_2 + \alpha r_2))t}. \end{aligned} \quad (56)$$

To determine the two-soliton solutions explicitly, we substitute (56) into (51) with $R = -2$.

Similarly, to determine the three-soliton solutions, we set

$$\begin{aligned} f(x,y,z,t) \;=\;& 1 + e^{k_1x+r_1y+s_1z-(k_1^2r_1+\alpha r_1)t} + e^{k_2x+r_2y+s_2z-(k_2^2r_2+\alpha r_2)t} \\ &+ e^{k_3x+r_3y+s_3z-(k_3^2r_3+\alpha r_3)t} \\ &+ \frac{(k_1-k_2)^2}{(k_1+k_2)^2}\, e^{(k_1+k_2)x+(r_1+r_2)y+(s_1+s_2)z-((k_1^2r_1+\alpha r_1)+(k_2^2r_2+\alpha r_2))t} \\ &+ \frac{(k_1-k_3)^2}{(k_1+k_3)^2}\, e^{(k_1+k_3)x+(r_1+r_3)y+(s_1+s_3)z-((k_1^2r_1+\alpha r_1)+(k_3^2r_3+\alpha r_3))t} \\ &+ \frac{(k_2-k_3)^2}{(k_2+k_3)^2}\, e^{(k_2+k_3)x+(r_2+r_3)y+(s_2+s_3)z-((k_2^2r_2+\alpha r_2)+(k_3^2r_3+\alpha r_3))t} \\ &+ b_{123}\, e^{(k_1+k_2+k_3)x+(r_1+r_2+r_3)y+(s_1+s_2+s_3)z-(k_1^2r_1+\alpha r_1+k_2^2r_2+\alpha r_2+k_3^2r_3+\alpha r_3)t}, \end{aligned} \tag{57}$$

and proceed as before we obtain

$$b_{123} = a_{12}a_{13}a_{23}. \tag{58}$$

To determine the three-soliton solutions explicitly, we substitute the last result for $f(x,y,t)$ in the formula $u(x,y,z,t) = -2(\ln f(x,y,z,t))_x$. The higher level soliton solutions, for $n \geq 4$ can be obtained in a parallel manner. This shows that the fourth extended (3+1)-dimensional shallow water wave equation (47) is completely integrable and gives rise to multiple-soliton solutions of any order.

6. CONCLUSIONS AND DISCUSSIONS

From the results obtained above, we can make the following conclusions:

1. The extension terms αu_{xy} and αu_{xyz} that were added to the first two models and the last two models in (1)–(4) did not kill the integrability of these four models. The extended models were proved to retain the integrability and multiple soliton solutions were formally derived for each extended model.

2. The only effect of the extension terms was on the dispersion relation as shown above.

3. The phase shifts of the typical shallow water waves equations (1)–(4) and for the extended models (5)–(8) were the same without any change and not affected by the extension terms.

In this work we have examined four extended shallow water waves equations in higher dimensions. We have showed that the extended terms do not kill the integrability of the extended equations. Multiple soliton solutions have been formally derived for these equations. The only effect caused by the extended terms is on the dispersion relations, whereas the phase shifts remain unchanged. The Hereman's method and the Cole-Hopf transformation method show effectiveness and reliability in handling nonlinear evolution equations.

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