CSCanada
Studies in Mathematical Sciences
Vol. 6, No. 1, 2013, pp. [18–25]
DOI: 10.3968/j.sms.1923845220130601.7625

ISSN 1923-8444 [Print]
ISSN 1923-8452 [Online]
www.cscanada.net
www.cscanada.org

Homogenization of Stokes Equation by Multiple Scale Expansion Method

E. K. Essel[a],*

[a] Department of Mathematics and Statistics, University of Cape Coast, Cape Coast, Ghana.

* Corresponding author.
Address: Department of Mathematics and Statistics, University of Cape Coast, Cape Coast, Ghana; E-Mail: ekessel04@yahoo.co.uk

Received: December 7, 2012/ Accepted: February 12, 2013/ Published: February 28, 2013

Abstract: In this paper we prove the homogenization of the Stokes equation by the method of multiple scale expansion. In particular the cell problems are clearly defined and an algorithm for obtaining the homogenized solution is well stated in the concluding part.

Key words: Homogenization; Stokes equation; Multiple scale method

Essel, E. K. (2013). Homogenization of Stokes Equation by Multiple Scale Expansion Method. *Studies in Mathematical Sciences, 6*(1), 18–25. Available from http://www.cscanada.net/index.php/sms/article/view/j.sms.1923845220130601.7625 DOI: 10.3968/j.sms.1923845220130601.7625

1. INTRODUCTION

Homogenization methods are used for studying the limit behaviour of solution of boundary value problem with rapidly oscillating coefficients. This theory facilitates the analysis of partial differential equations with rapidly oscillating coefficients, see e. g. Jikov *et al.* [25]. Homogenization was recently applied to different problems connected to fluid flow through machine elements with much success, see e. g. [11–15,17–24,27]. One technique within the homogenization theory is the formal method of multiple scale expansion, see e. g. [1] or [16]. The Stokes equation is a simplification of the Navier-Stokes equation especially in the incompressible Newtonian case. It is used in modelling flow of fluid through porous media. A stokes flow has no dependence on time. This means that, given the boundary conditions of a Stokes flow, the flow can be found without knowledge of the flow at any other time. The homogenization of Stokes equation and Navier Stokes equations in perforated domains have been studied by different authors see e. g. [6] and [7]. In this paper we consider the homogenization of Stokes equation using the multiple scale expansion method.

2. THE HOMOGENIZATION PROCEDURE

In this section we consider Stokes equation that governs incompressible Newtonian flow. Let Ω be an open bounded subset of $\mathbb{R}^2$, $Y = (0,1)^2$. We introduce the auxiliary matrix $\mathbf{A} = (a_{ij})$, where $a_{ij} = a_{ij}(x,y)$, and $i = 1,2$, and $j = 1,2$ are smooth functions that are Y periodic in y. It is also assumed that a constant $\alpha > 0$ exists such that

$$\sum_{i,j=1}^{2} a_{ij}(x,y)\xi_i\xi_j \geq \alpha \left|\xi\right|^2 \qquad \text{for every } \xi \in \mathbb{R}^2.$$

We now define the matrix $\mathbf{A}_\varepsilon$ as

$$\mathbf{A}_\varepsilon(x) = \begin{pmatrix} a^\varepsilon_{11}(x) & a^\varepsilon_{12}(x) \\ a^\varepsilon_{21}(x) & a^\varepsilon_{22}(x) \end{pmatrix} = \mathbf{A}(x, x/\varepsilon),$$

and we consider the homogenization of the following boundary value problem

$$\begin{aligned} A^\varepsilon u_\varepsilon &= f - \nabla p_\varepsilon \quad \text{in } \Omega \\ \text{div } u_\varepsilon &= 0 \ \text{ in } \Omega \\ u_\varepsilon &= 0 \text{ on } \Gamma \end{aligned} \tag{1}$$

where

$$A^\varepsilon = -\nabla_x \cdot \left(\mathbf{A}_\varepsilon(x)\nabla_x\right).$$

According to the formal method of multiple scale expansion it is assumed that p_ε and u_ε are of the form

$$\begin{aligned} u_\varepsilon(x) &= \sum_{i=0}^{\infty} \varepsilon^i u_i(x, x/\varepsilon) \\ p_\varepsilon(x) &= \sum_{i=0}^{\infty} \varepsilon^i p_i(x, x/\varepsilon), \end{aligned} \tag{2}$$

where $u_i = u_i(x,y)$, $p_i = p_i(x,y)$ are vectors which are both Y periodic. Applying the chain rule to a smooth function ψ defined by

$$\psi_\varepsilon(x) = \psi(x,y),$$

where $y = \frac{x}{\varepsilon}$, we see that

$$\nabla_x \to \nabla_x + \frac{1}{\varepsilon}\nabla_y. \tag{3}$$

Pluging into (1) we have,

$$-\left[\left(\nabla_x + \varepsilon^{-1}\nabla_y\right) \cdot \left(\mathbf{A}\left(\nabla_x + \varepsilon^{-1}\nabla_y\right)\right)\right] u_\varepsilon = f - \left(\nabla_x + \varepsilon^{-1}\nabla_y\right) p_\varepsilon,$$

or

$$\begin{aligned} & -\left[\varepsilon^{-2}\left[\nabla_y \cdot (\mathbf{A}\nabla_y)\right] + \varepsilon^{-1}\left[\nabla_x \cdot (\mathbf{A}\nabla_y) + \nabla_y \cdot (\mathbf{A}\nabla_x)\right] + \nabla_x \cdot (\mathbf{A}\nabla_x)\right] u_\varepsilon \\ = \; & f - \left(\nabla_x + \varepsilon^{-1}\nabla_y\right) p_\varepsilon \end{aligned}$$

or

$$\left[\varepsilon^{-2} A_1 + \varepsilon^{-1} A_2 + \varepsilon^0 A_3\right] u_\varepsilon = f - \left(\nabla_x + \varepsilon^{-1}\nabla_y\right) p_\varepsilon, \tag{4}$$

where the differential operators A_1, A_2 and A_3 are defined as

$$\begin{aligned} A_1 &= -\nabla_y \cdot (\mathbf{A}\nabla_y) \\ A_2 &= -\left(\nabla_x \cdot (\mathbf{A}\nabla_y)\right) - \nabla_y \cdot (\mathbf{A}\nabla_x) \\ A_3 &= -\nabla_x \cdot (\mathbf{A}\nabla_x). \end{aligned}$$

Also

$$\text{div } u_\varepsilon = \left(\text{div}_x + \varepsilon^{-1}\text{div}_y\right) u_\varepsilon.$$

Using (4) and (2) we obtain the full expansion of (1) as,

$$\begin{aligned} & \left[\varepsilon^{-2} A_1 + \varepsilon^{-1} A_2 + \varepsilon^0 A_3\right] \left(u_0 + \varepsilon u_1 + \varepsilon^2 u_2 + ...\right) \\ = \; & f - \left(\nabla_x + \varepsilon^{-1}\nabla_y\right) \left(p_0 + \varepsilon p_1 + \varepsilon^2 p_2 + ...\right) \end{aligned}$$

$$\begin{aligned} \left(\text{div}_x + \varepsilon^{-1}\text{div}_y\right) \left(u_0 + \varepsilon u_1 + \varepsilon^2 u_2 + ...\right) &= 0 \\ u_\varepsilon &= 0. \end{aligned}$$

Re-arranging we obtain

$$\begin{aligned} & \varepsilon^{-2} A_1 u_0 + \varepsilon^{-1}\left(A_1 u_1 + A_2 u_0\right) + \varepsilon^0 \left(A_3 u_0 + A_1 u_2 + A_2 u_1\right) + ... \\ = \; & \varepsilon^{-1}\left(\nabla_y p_0\right) f - + \varepsilon^0 \left(\nabla_x p_0 + \nabla_y p_1\right) + \varepsilon^1 \left(\nabla_x p_1 + \nabla_y p_2\right) + \varepsilon^2 \left(\nabla_x p_2\right). \end{aligned}$$

and

$$\varepsilon^0 \left(\text{div}_x u_0 + \text{div}_y u_1\right) + \varepsilon^{-1}\left(\text{div}_y u_0\right) + \varepsilon^1 \left(\text{div}_x u_1 + \text{div}_y u_2\right) = 0.$$

Comparing terms with the same order of ε from -2 to 0, we obtain the following system of equations

$$A_1 u_0 = 0 \tag{5}$$

$$A_1 u_1 + A_2 u_0 = -\nabla_y p_0 \tag{6}$$

$$A_3 u_0 + A_1 u_2 + A_2 u_1 = f - \nabla_x p_0 - \nabla_y p_1 \tag{7}$$

in addition to the following

$$\text{div}_x u_0 + \text{div}_y u_1 = 0 \tag{8}$$

$$\text{div}_y u_0 = 0. \tag{9}$$

Lemma 1. *Consider the boundary value problem*

$$A_0 \Phi = F \text{ in the unit cell } Y, \tag{10}$$

where $F \in L^2(Y)$ and $\Phi(y)$ is Y-periodic. Then the following holds true:

(i) There exists a weak Y- periodic solution Φ of (10) if and only if

$$\frac{1}{|Y|}\int_Y F dy = 0.$$

(ii) If there exists a weak Y- periodic solution of (10), then it is unique up to a constant, that is, if we find one solution $\Phi_0(y)$, every solution is of the form $\Phi(y) = \Phi_0(y) + c$, where c is a constant.

Proof. See [1, p. 39]. □

From (5)

$$A_1 u_0 = 0$$

implies that

$$u_0(x, y) = u_0(x),$$

since A_1 is a differential operator in y.

From (8)

$$\text{div}_x u_0 + \text{div}_y u_1 = 0$$

Integrating over the period Y we have

$$\int_Y (\text{div}_x u_0(x))\, dy + \int_Y \text{div}_y u_1(x, y)\, dy = 0.$$

By periodicity $\int_Y \text{div}_y u_1(x, y)\, dy = 0$ and $\int_Y (\text{div}_x u_0(x))\, dy = |Y| \text{div}_x u_0(x)$. Hence, the last equation reduces to

$$|Y|\, \text{div}_x u_0(x) = 0$$

or

$$\text{div}_x u_0(x) = 0. \tag{11}$$

Substituting (11) into (8) we find that

$$\text{div}_y u_1 = 0. \tag{12}$$

From (6)

$$\begin{aligned} A_1 u_1 + A_2 u_0 &= -\nabla_y p_0 \\ A_1 u_1 &= [\nabla_x \cdot (\mathbf{A}\nabla_y) + \nabla_y \cdot (\mathbf{A}\nabla_x)]\, u_0(x) - \nabla_y p_0 \\ &= \nabla_y \cdot (\mathbf{A}\nabla_x)\, u_0(x) - \nabla_y p_0. \end{aligned} \tag{13}$$

But u_0 being a function of x only, implies that $\nabla_x \cdot (\mathbf{A}\nabla_y) u_0(x) = 0$.

From (12) and (13) we have

$$\begin{aligned} A_1 u_1 &= (\nabla_y \cdot (\mathbf{A}\nabla_x))\, u_0(x) - \nabla_y p_0 \\ \text{div}_y u_1 &= 0. \end{aligned}$$

To obtain the cell problems we simplify (13) as follows

$$A_1 u_1 = (\nabla_y \cdot (\mathbf{A}\nabla_x))\, u_0(x) - \nabla_y p_0.$$

By linearity u_1 must be of the form

$$u_1 = v_1(x,y)\frac{\partial u_0}{\partial x_1} + v_2(x,y)\frac{\partial u_0}{\partial x_2} + v_0\,(x)$$

and so

$$A_1\left(v_1(x,y)\frac{\partial u_0}{\partial x_1} + v_2(x,y)\frac{\partial u_0}{\partial x_2} + v_0\,(x)\right) = \nabla_y\cdot(\mathbf{A}\nabla_x)\,u_0\,(x) - \nabla_y p_0$$

$$\begin{aligned} & -\nabla_y\cdot(\mathbf{A}\nabla_y)\left(v_1(x,y)\frac{\partial u_0}{\partial x_1} + v_2(x,y)\frac{\partial u_0}{\partial x_2} + v_0\,(x)\right) \\ = \quad & \nabla_y\cdot\left(\mathbf{A}e_1\frac{\partial u_0}{\partial x_1}\right) + \nabla_y\cdot\left(\mathbf{A}e_2\frac{\partial u_0}{\partial x_1}\right) - \nabla_y p_0. \end{aligned}$$

Grouping like terms we obtain

$$\begin{aligned} -\nabla_y\cdot\left(\mathbf{A}e_1\frac{\partial u_0}{\partial x_1}\right) - \nabla_y\cdot(\mathbf{A}\nabla_y)\,v_1(x,y)\frac{\partial u_0}{\partial x_1} &= 0 \\ -\nabla_y\cdot\left(\mathbf{A}e_2\frac{\partial u_0}{\partial x_1}\right) - \nabla_y\cdot(\mathbf{A}\nabla_y)\,v_2(x,y)\frac{\partial u_0}{\partial x_2} &= 0 \\ -\nabla_y\cdot(\mathbf{A}\nabla_y)\,v_0\,(x) + \nabla_y p_0 &= 0. \end{aligned}$$

This simplifies to the following cell problems

$$\begin{aligned} \nabla_y\cdot(\mathbf{A}\,(\nabla_y v_1 + e_1)) &= 0 \\ \nabla_y\cdot(\mathbf{A}\,(\nabla_y v_2 + e_2)) &= 0 \\ -\nabla_y\cdot(\mathbf{A}\nabla_y v_0) + \nabla_y p_0 &= 0. \end{aligned} \tag{14}$$

From (7) we have

$$A_3 u_0 + A_1 u_2 + A_2 u_1 = f - \nabla_x p_0 - \nabla_y p_1.$$

Averaging over Y we have

$$\int_Y (A_3 u_0 + A_1 u_2 + A_2 u_1)\,dy = \int_Y (f - \nabla_x p_0 - \nabla_y p_1)\,dy.$$

But $\int_Y A_1 u_2 dy = 0$ and $\int \nabla_y p_1 dy = 0$ by Y-periodicity (since u_2 and p_1 are periodic in Y). This reduces the last equation to

$$\int_Y (A_2 u_1 + A_3 u_0)\,dy = \int_Y (f - \nabla_x p_0)\,dy$$

or

$$\begin{aligned} & \int_Y -\left[\nabla_x\cdot(\mathbf{A}\nabla_y u_1) + \nabla_y\cdot(\mathbf{A}\nabla_x u_1)\right]dy - \int_Y \nabla_x\cdot(\mathbf{A}\nabla_x u_0)\,dy \\ = \quad & \int_Y (f - \nabla_x p_0)\,dy. \end{aligned}$$

Now $\int_Y \nabla_y \cdot (\mathbf{A}\nabla_x u_1)\, dy = 0$ since $A\nabla_x u_1$ is periodic in Y. We thus, have

$$\int_Y -\nabla_x \cdot (\mathbf{A}\nabla_y u_1)\, dy - \int_Y \nabla_x \cdot (\mathbf{A}\nabla_x u_0)\, dy = |Y|\, f - \nabla_x \int_Y p_0\,(x,y)\, dy,$$

i. e.,

$$\begin{aligned} & \int_Y -\nabla_x \cdot \left[\mathbf{A}\nabla_y \left(v_1(x,y)\frac{\partial u_0}{\partial v_1} + v_2(x,y)\frac{\partial u_0}{\partial v_2} + v_0\,(x)\right)\right] dy \\ & -\int_Y \nabla_x \cdot \left(\mathbf{A}\left(\frac{\partial u_0}{\partial x_1} e_1 + \frac{\partial u_0}{\partial x_1} e_2\right)\right) dy \\ = \;& |Y|\, f - \nabla_x \int_Y p_0\,(x,y)\, dy, \end{aligned}$$

or

$$\begin{aligned} & \nabla_x \cdot \left\{-\frac{\partial u_0}{\partial x_1}\int_Y \mathbf{A}\,(\nabla_y v_1 + e_1)\, dy - \frac{\partial u_0}{\partial x_2}\int_Y \mathbf{A}\,(\nabla_y v_2 + e_2)\, dy\right. \\ = \;& |Y|\, f - \nabla_x \int_Y p_0\,(x,y)\, dy + \nabla_x \cdot \int_Y \mathbf{A}\nabla_y v_0\,(x)\, dy. \end{aligned}$$

Since $\nabla_x \cdot \int_Y \mathbf{A}\nabla_y v_0\,(x)\, dy = 0$, If we let $\int_Y p_0\,(x,y)\, dy = \tilde{p}_0\,(x)$ and $|Y| = 1$ then we have the following homogenized equation

$$\begin{aligned} & \nabla_x \cdot \left\{\frac{\partial u_0}{\partial x_1}\begin{pmatrix} b_{11}\,(x) \\ b_{21}\,(x) \end{pmatrix} + \frac{\partial u_0}{\partial x_2}\begin{pmatrix} b_{12}\,(x) \\ b_{22}\,(x) \end{pmatrix}\right\} \\ = \;& f - \nabla_x \tilde{p}_0\,(x) \end{aligned}$$

$$\begin{aligned} & \nabla_x \cdot \left\{\begin{pmatrix} b_{11}\,(x) & b_{12}\,(x) \\ b_{21}\,(x) & b_{22}\,(x) \end{pmatrix}\begin{pmatrix} \frac{\partial u_0}{\partial x_1} \\ \frac{\partial u_0}{\partial x_2} \end{pmatrix}\right\} \\ = \;& f - \nabla_x \tilde{p}_0\,(x)\,. \end{aligned}$$

In a more compact form the homogenized equation can be written as

$$\nabla_x \cdot \{B\,(x)\,\nabla u_0\} = f - \nabla_x \tilde{p}_0\,(x)\,, \tag{15}$$

where the matrix $B\,(x)$ is a matrix function defined by $B\,(x) = b_{ij}\,(x)$ in terms of v_1 and v_2 as

$$\begin{aligned} \begin{pmatrix} b_{11}\,(x) \\ b_{21}\,(x) \end{pmatrix} &= -\int_Y [\mathbf{A}\,(\nabla_y v_1 + e_1)]\, dy, \\ \begin{pmatrix} b_{12}\,(x) \\ b_{22}\,(x) \end{pmatrix} &= -\int_Y [\mathbf{A}\,(\nabla_y v_2 + e_2)]\, dy, \end{aligned} \tag{16}$$

If we let

$$A^0 = \nabla_x \cdot \{B\,(x)\,\nabla\}\,,$$

(15) can be written as

$$A^0 u_0 = f - \nabla_x \tilde{p}_0\,(x) \;\text{ in } \Omega. \tag{17}$$

Combining (11) and (17) the homogenized boundary value problem of (1) is given by

$$\begin{aligned} A^0 u_0 &= f - \nabla_x \tilde{p}_0(x) \text{ in } \Omega \\ \text{div}_x u_0(x) &= 0 \text{ in } \Omega \\ u_0(x) &= 0 \text{ on } \Gamma. \end{aligned}$$

3. CONCLUSION

We have proved that an approximate solution of equation (1) can be obtained by following the algorithm below.

1. Solve the three local problems (14).
2. Insert the solutions of the local problems into (16) and compute the homogenized coefficient $B(x)$.
3. Insert $B(x)$ into the homogenized equation (15), which corresponds to the approximate solution for (1), and solve for u_0.

REFERENCES

[1] Persson, L. E., Persson, L., Svanstedt, N., & Wyller, J. (1993). *The homogenization method: an introduction*. Studentlitteratur, Lund.

[2] Almqvist, A., Essel, E. K., Persson, L. E., & Wall, P. (2007). Homogenization of the unstationary incompressible Reynolds equation. *Tribol. Int., 40*, 1344-1350.

[3] Bensoussan, A., Lions, J. L., & Papanicolaou, G. (1978). *Asymptotic analysis for periodic structures*. Amsterdam: North-Holland.

[4] Lions, L., Lukkassen, D., Persson, L. E., & Wall, P. (2000). Reiterated homogenization of monotone operators. *C. R. Acad. Sci. Paris, t. 330, Serie I*, 675-680.

[5] Lions, J. L., Lukkassen, D., Persson, L. E., & Wall, P. (2001). Reiterated homogenization of nonlinear monotone operators. *Chin. Ann. of Math., 22*(B), 1-12.

[6] Svandedt, N., & Wellander, N. (2005). *Multiscale homogenization of the Navier-Stokes equation*. Lecture notes in Computational Science and Engineering 44, Springer Verlag, 265-276.

[7] Namlyeyeva, Y. V., Skrypnik, I. I., & Necasova, S. (2008). *Homogenization of the Dirichlet problem for steady Navier-Stokes equations in domains with channels*. Institute of Mathematics, AS, CR, Prague.

[8] Allaire, G. (1992). Homogenization and two-scale convergence. *SIAM J. Math. Anal., 23*, 1482-1518.

[9] Allaire, G., Braides, A., Buttazzo, G., Defranceschi, A., & Gibiansky, L. (1993). *School on homogenization*. Lecture Notes of the Cources held at ICTP, Trieste, 4-17 September 1993. Preprint SISSA, Trieste.

[10] Almqvist, A. (2006). *On the effects of surface roughness in lubrication*. Ph.D. thesis 2006: 31, Luleå University of Technology, Luleå.

[11] Almqvist, A., Larsson, R., & Wall, P. (2006). The homogenization process of the time dependent Reynolds equation describing compressible liquid flow. Research Report, No. 4, ISSN 1400-4003, Department of Mathematics, Luleå University of Technology.

[12] Almqvist, A., & Dasht, J. (2013). The homogenization process of the Reynolds equation describing compressible liquid flow. To appear in *Trib. Int.*.

[13] Bayada, G., & Faure, J. B. (1989). A Double scale analysis approach of the Reynolds roughness comments and application to the journal bearing. *J. of Tribology, 111*, 323-330.

[14] Bayada, G., & Chambat, M. (1989). Homogenization of the Stokes system in a thin film flow with rapidly varying thickness. *Model. Math. Anal. Numer., 23*(2), 205-234.

[15] Bayada, G., Ciuperca, S., & Jai, M. (2000). Homogenization of variational equations and inequalities with small oscillating parameters. Application to the study of thin film unstationary lubrication flow. *C. R. Acad. Sci. Paris, t. 328, Serie II b*, 819-824.

[16] Bensoussan, A., Lions, J. L., & Papanicolaou, G. (1978). *Asymptotic analysis for periodic structures.* Amsterdam: North-Holland.

[17] Buscaglia, G., & Jai, M. (2000). Sensitivity analysis and Taylor expansions in numerical homogenization problems. *Numer. Math., 85*, 49-75.

[18] Buscaglia, G., & Jai, M. (2001). A new numerical scheme for non uniform homogenized problems: application to the non linear Reynolds compressible equation. *Mathematical Problems in Engineering, 7*, 355-378.

[19] Buscaglia, G., Ciuperca, I., & Jai, M. (2002). Homogenization of the transient Reynolds equation. *Asymptotic Anal., 32*, 131-152.

[20] Buscaglia, G., & Jai, M. (2004). Homogenization of the generalized Reynolds equation for ultra-thin gas films and its resolution by FEM. *J. of Tribology, 126*, 547-552.

[21] Jai, M. (1995). Homogenization and two-scale convergence of the compressible Reynolds lubrication equation modelling the flying characteristics of a rough magnetic head over a rough rigid-disk surface. *Math. Modelling and Numer. Anal., 29*(2), 199-233.

[22] Jai, M., & Bou-Said, B. (2002). A comparison of homogenization and averaging techniques for the treatment of roughness in slip-flow-modified Reynolds equation. *J. of Tribology, 124*, 327-335.

[23] Kane, M., & Bou-Said, B. (2004). Comparison of homogenization and direct techniques for the treatment of roughness in incompressible lubrication. *J. of Tribology, 126*, 733-737.

[24] Kane, M., & Bou-Said, B. (2005). A study of roughness and non-Newtonian effects in lubricated contacts. *J. of Tribology, 127*, 575-581.

[25] Jikov, V. V., Kozlov, S. M., & Oleinik, O. A. (1994). *Homogenization of differential operators and integral functionals.* New York: Springer-Verlag.

[26] Lions, J. L., Lukkassen, D., Persson, L. E., & Wall, P. (2001). Reiterated homogenization of nonlinear monotone operators. *Chin. Ann. of Math., 22*(B), 1-12.

[27] Lukkassen, D., Meidell, A., & Wall, P. (2006). Bounds on the effective behavior of a homogenized Reynold-type equation. Research Report, No. 3, ISSN 1400-4003, Department of Mathematics, Luleå University of Technology, 16 pages.