CSCanada
Studies in Mathematical Sciences
Vol. 6, No. 2, 2013, pp. [28–39]
DOI: 10.3968/j.sms.1923845220130602.3449

ISSN 1923-8444 [Print]
ISSN 1923-8452 [Online]
www.cscanada.net
www.cscanada.org

Predator Harvesting in Systems with One Predator and Two Prey Habitats

R. P. Jaju[a], D. R. Owen[b],* and B. S. Bhatt[b],*

[a] Department of Computer Science, Faculty of Science and Engineering, University of Swaziland, P/Bag 4, Kwaluseni, Swaziland.
[b] Department of Mathematics and Statistics, The University of The West Indies, St. Augustine, Trinidad (W. I.).

* Corresponding author.
Address: Department of Mathematics and Statistics, The University of The West Indies, St. Augustine, Trinidad (W. I.); E-Mail: bal.bhatt@sta.uwi.edu, drowentt@yahoo.com

Received: March 2, 2013/ Accepted: April 22, 2013/ Published: May 31, 2013

Abstract: Harvesting of the predator in systems consisting of one predator and two preys where the preys live in two different habitats is considered. It is assumed that the prey species have resources in abundance and the predator specie is able to switch to the most abundant prey specie. We consider two types of predator harvesting. One is when we harvest a fixed amount of predators (constant harvest quota) and the other is when we harvest a fixed percentage of the predators (constant harvest effort). Stability analysis is carried out and hypothetical cases are used to support our analysis.

Key words: Prey; Predator harvesting; Constant harvest quota; Constant harvest effort; Bifurcation points; Stability; Differential equations

Jaju, R. P., Owen, D. R., & Bhatt, B. S. (2013). Predator Harvesting in Systems with One Predator and Two Prey Habitats. *Studies in Mathematical Sciences, 6*(2), 28–39. Available from http://www.cscanada.net/index.php/sms/article/view/j.sms.1923845220130602.3449 DOI: 10.3968/j.sms.1923845220130602.3449

1. INTRODUCTION

The population models with two preys and one predator habitat have been studied for quite some time using general predation and harvesting functions and involving prey which were assumed to have abundant resources and which can move freely from one habitat to the other. The predator could switch its preference of feeding itself from one prey specie habitat to the other in different ways and due to different reasons. The present authors studied the predation and harvesting of preys in [1,2]. The authors also provided a good account of several systems with different predation and harvesting functions in [3]. The predation and harvesting functions could be made to be similar and exchangeable in forms.

Skalkski and Gillian [4] and Crawley and Martin [5] have used functional responses in their systems of population studies. Beddington [7] studied the mutual interference between predators, and applied the idea in a practical application on searching efficiency.

The current work extends the earlier work by the authors, by incorporating the harvesting of the predators. The harvesting of predators is considered in two ways. One in which the predator harvesting amount is fixed (constant harvest quota) and the other in which a proportion or percentage of predator (constant harvest effort) is harvested. The equilibrium states, their stability conditions and Hopf bifurcation points with multiple parameter sets are presented.

In Section 2 the equations defining the model are given. The stability of the general equilibrium states is examined in section 3, while in Section 4 we give a Hopf bifurcation theorem, numerical bifurcation results and figures depicting stable and unstable cases. Also bifurcation points for both types of harvesting are included.

2. THE MODEL

The model we used to describe the predator-prey systems, where the prey live in different habitats and which allows the prey to have finite resources and both prey and predator to be harvested, is represented by the equations:

$$\frac{dx_1}{dt} = (\alpha_1 - \epsilon_1)x_1 - \alpha_{11}x_1^2 + \epsilon_2 p_{21} x_2 - \beta_1 x_1 y k_1(x_1, x_2) - \delta_1 x_1 H_1(x_1, x_2) \quad (1)$$

$$\frac{dx_2}{dt} = (\alpha_2 - \epsilon_2)x_2 - \alpha_{22}x_2^2 + \epsilon_1 p_{12} x_1 - \beta_2 x_2 y k_2(x_1, x_2) - \delta_2 x_2 H_2(x_1, x_2) \quad (2)$$

$$\frac{dy}{dt} = \left[-\mu + c_1\beta_1 x_1 k_1(x_1, x_2) + c_2\beta_2 x_2 k_2(x_1, x_2)\right] y - \delta_0 \tilde{H}(y) \quad (3)$$

where

x_i: represents the prey population in the two different habitats,

y: represents the population of predator species,

β_i: the predator response rates towards the prey x_i,

δ_i: represents the harvesting rates of the prey,

δ_0: represents the harvesting rate of the predator,

c_i: the rate of conversion of prey to predator,

ϵ_i: inversion barrier strength in going out of the habitat,

p_{ij}: the probability of successful transition from the i^{th} habitat to the j^{th} habitat,

α_i: specific growth rate of the prey in the absence of predation,

α_{ii}: measure of the maximum sustainable yield,

μ: per capita death rate of the predator,

and $k_i(x_1, x_2)$, $H_i(x_1, x_2)$ for $i = 1, 2$ and $\tilde{H}(y)$ are positive functions which satisfy one general condition to be given later. The coefficients $\alpha_i,\ \alpha_{ii},\ \epsilon_i,\ p_{ij},\ \beta_i, \mu, \delta_i$ and δ_0 are all positive quantities.

In the following we shall take $\tilde{H}(y) = yg(y)$ where $g(y)$ is a positive function.

3. EQUILIBRIUM AND STABILITY

Let $E = (X_1, X_2, Y)$ be an equilibrium point of Eqs. (1)–(3) then X_1, X_2 and Y satisfy

$$(\alpha_1 - \epsilon_1)X_1 - \alpha_{11}X_1^2 + \epsilon_2 p_{21}X_2 - \beta_1 X_1 Y k_1\left(X_1, X_2\right) - \delta_1 X_1 H_1(X_1, X_2) = 0 \quad (4)$$
$$(\alpha_2 - \epsilon_2)X_2 - \alpha_{22}X_2^2 + \epsilon_1 p_{12}X_1 - \beta_2 X_2 Y k_2\left(X_1, X_2\right) - \delta_2 X_2 H_2(X_1, X_2) = 0 \quad (5)$$
$$\left[-\mu + c_1\beta_1 X_1 k_1\left(X_1, X_2\right) + c_2\beta_2 X_2 k_2\left(X_1, X_2\right)\right] Y - \delta_0 \tilde{H}(Y) = 0 \quad (6)$$

where $Y \neq 0$ and $\tilde{H}(y) = yg(y)$.

To solve these equations for X_1, X_2 and Y let us set $X_1 = \bar{X}X_2$, where $\bar{X} > 0$, and on substituting into Eq. (4) we get, after dividing by X_2 and solving for Y,

$$Y(\bar{X}, X_2) = \frac{(\alpha_1 - \epsilon_1)\bar{X} - \alpha_{11}\bar{X}^2 X_2 + \epsilon_2 p_{21} - \delta_1 \bar{X} H_1(\bar{X}, X_2)}{\beta_1 \bar{X} k_1(\bar{X}, X_2)} \quad (7)$$

and similarly from Eq. (5)

$$Y(\bar{X}, X_2) = \frac{(\alpha_2 - \epsilon_2) - \alpha_{22}X_2 + \epsilon_1 p_{12}\bar{X} - \delta_2 H_2(\bar{X}, X_2)}{\beta_2 k_2(\bar{X}, X_2)}. \quad (8)$$

We point out that after substitution of $X_1 = \bar{X}X_2$ in $H_i(X_1, X_2)$ and $k_i(X_1, X_2)$ we have written $H_i(\bar{X}X_2, X_2)$ as $H_i(\bar{X}, X_2)$ and $k_i(\bar{X}X_2, X_2)$ as $k_i(\bar{X}, X_2)$.

In order that the solution Y exist we must equate both expressions for Y. This gives

$$\frac{(\alpha_1 - \epsilon_1)\bar{X} - \alpha_{11}\bar{X}^2 X_2 + \epsilon_2 p_{21} - \delta_1 \bar{X} H_1(\bar{X}, X_2)}{\beta_1 \bar{X} k_1(\bar{X}, X_2)} = \frac{(\alpha_2 - \epsilon_2) - \alpha_{22}X_2 + \epsilon_1 p_{12}\bar{X} - \delta_2 H_2(\bar{X}, X_2)}{\beta_2 k_2(\bar{X}, X_2)} \quad (9)$$

which provides us with an equation which must be satisfied by $\bar{X}, X_2$.

From Eqs. (7)–(8), since Y must be positive, in order to represent a real population, then $\bar{X}, X_2$ are chosen to satisfy the inequalities:

$$(\alpha_1 - \epsilon_1)\bar{X} - \alpha_{11}\bar{X}^2 X_2 + \epsilon_2 p_{21} - \delta_1 \bar{X} H_1(\bar{X}, X_2) > 0 \text{ and}$$
$$(\alpha_2 - \epsilon_2) - \alpha_{22}X_2 + \epsilon_1 p_{12}\bar{X} - \delta_2 H_2(\bar{X}, X_2) > 0. \quad (10)$$

To get another equation involving $\bar{X}, X_2$ we can substitute for Y into Eq. (6) and after simplifying, we get

$$-\mu + c_1\beta_1 \bar{X} X_2 k_1(\bar{X}, X_2) + c_2\beta_2 X_2 k_2(\bar{X}, X_2) - \delta_0 g(Y(\bar{X}, X_2)) = 0 \quad (11)$$

Solving Eqs. (9) and (11), for $\bar{X} > 0,\ X_2 > 0$, subject to both the conditions in Eq. (10), we obtain $\bar{X}, X_2$ and using $X_1 = \bar{X}X_2$ we get X_1 and from either Eqs. (7) or (8) we get Y, thus giving the equilibrium point $E = (X_1, X_2, Y)$. We may write these results as the following:

Lemma 1. *The solution (X_1, X_2, Y) of Eqs. (1)–(3) exists and represents real populations if $\bar{X}(> 0)$ and X_2 are solutions of Eqs. (9) and (11) and satisfy both the inequalities in Eq. (10).*

3.1. General Assumptions

In this investigation we make four assumptions. The first is a general assumption and is used to examine the stability of the equilibrium state (X_1, X_2, Y) where X_1, X_2, Y are all positive, while the others are related to the particular methods employed by the predators in attacking the preys and the harvester in harvesting the predator. They are as follows:

Assumption I: Each of the predatory and harvesting functions, $k_i(x_1, x_2)$ and $H_i(x_1, x_2)$ for $i = 1, 2$, is a function of the ratio of the prey populations $\dfrac{x_1}{x_2}$, is positive, smooth and has a Taylor expansion about the point (X_1, X_2). In the case of predator harvesting function $\tilde{H}(y)$, it has a Taylor expansion about $y = Y$, where (X_1, X_2, Y) is an equilibrium point of Eqs. (1)–(3).

Assumption II: The predator functions $k_1\left(\dfrac{x_2}{x_1}\right), k_2\left(\dfrac{x_1}{x_2}\right)$ are such that

$$k_1^{'}\left(\frac{x_2}{x_1}\right) < 0,\ k_2^{'}\left(\frac{x_1}{x_2}\right) < 0$$

where x_1, x_2 are positive and $k_1^{'}(z_1),\ z_1 = \dfrac{x_2}{x_1}, k_2^{'}(z), z = \dfrac{x_2}{x_1}$ denote the derivatives of k_1, k_2 with respect to z_1, z respectively.

Assumption III: This is a switching assumption and becomes effective only when we apply our theory to a particular system. We use it in the construction of the functions and describes the type of feeding mechanism adapted by the predator. The present feeding mechanism is a switching mechanism where the predators are allowed to switch to the most abundant prey population. The assumption is

1. for $\dfrac{x_1}{x_2} >> 1$, i.e. as $\dfrac{x_1}{x_2} \to \infty,\ k_1\left(\dfrac{x_2}{x_1}\right) \to 1,\ k_2\left(\dfrac{x_1}{x_2}\right) \to 0$,
2. for $\dfrac{x_2}{x_1} >> 1$, i.e. as $\dfrac{x_2}{x_1} \to \infty,\ k_1\left(\dfrac{x_2}{x_1}\right) \to 0,\ k_2\left(\dfrac{x_1}{x_2}\right) \to 1$.

Assumption IV: This is an assumption which affects the harvesting of the predator and is controlled by the sign of the derivative of $g(y)$ with respect to y. We have considered $\dfrac{dg(y)}{dy} \leq 0$.

3.2. Stability analysis

We now examine the stability of an equilibrium point $E = (X_1, X_2, Y)$ by linearizing Eqs. (1)–(3) about E by introducing a small perturbation, that is, we substitute

$x_1 = X_1 + u, x_2 = X_2 + v, y = Y + w$ into all terms in Eqs. (1)–(3) and using **Assumption I** expand each term by its Taylor expansion neglecting second and higher order terms in u, v and w. We end up with the general linearized equation

$$\frac{dV}{dt} = JV$$

where $V = \begin{pmatrix} u \\ v \\ w \end{pmatrix}$ and the matrix J is given by

$$J = \begin{pmatrix} \frac{\partial F}{\partial X_1} & \frac{\partial F}{\partial X_2} & \frac{\partial F}{\partial Y} \\ \\ \frac{\partial G}{\partial X_1} & \frac{\partial G}{\partial X_2} & \frac{\partial G}{\partial Y} \\ \\ \frac{\partial H}{\partial X_1} & \frac{\partial H}{\partial X_2} & \frac{\partial H}{\partial Y} \end{pmatrix}$$

and F, G and H represent the right hand sides of Eqs. (1)–(3) respectively. Now writing

$$\begin{aligned} AY &= \frac{\partial H}{\partial X_1}, \quad BY = \frac{\partial H}{\partial X_2}, \quad \tilde{E} = \frac{\partial H}{\partial Y}, \\ C &= \frac{\partial G}{\partial X_1}, \quad \tilde{C} = \frac{\partial G}{\partial X_2}, \quad -w_2 = \frac{\partial G}{\partial Y}, \\ \tilde{D} &= \frac{\partial F}{\partial X_1}, \quad D = \frac{\partial F}{\partial X_2}, \quad -w_1 = \frac{\partial F}{\partial Y} \end{aligned}$$

then we can write J as

$$J = \begin{pmatrix} \tilde{D} & D & -w_1 \\ C & \tilde{C} & -w_2 \\ AY & BY & \tilde{E} \end{pmatrix}$$

where $A, B, C, \tilde{C}, D, \tilde{D}, \tilde{E}, w_1, w_2$ are given by

$$\begin{aligned} A &= c_1\beta_1\Big(k_1(x_1,x_2) + X_1\frac{\partial k_1}{\partial X_1}\Big) + c_2\beta_2 X_2\frac{\partial k_2}{\partial X_1}, \\ B &= c_1\beta_1 X_1\frac{\partial k_1}{\partial X_2} + c_2\beta_2\Big(k_2(x_1,x_2) + X_2\frac{\partial k_2}{\partial X_2}\Big), \\ C &= \epsilon_1 p_{12} - \beta_2 X_2 Y\frac{\partial k_2}{\partial X_1} - \delta_2 X_2\frac{\partial H_2}{\partial X_1}, \\ \tilde{C} &= -\bar{X}C - \alpha_{22}X_2 - \beta_2 Y\Big(X_1\frac{\partial k_2}{\partial X_1} + X_2\frac{\partial k_2}{\partial X_2}\Big) - \delta_2\Big(X_1\frac{\partial H_2}{\partial X_1} + X_2\frac{\partial H_2}{\partial X_2}\Big), \\ D &= \epsilon_2 p_{21} - \beta_1 Y X_1\frac{\partial k_1}{\partial X_2} - \delta_1 X_1\frac{\partial H_1}{\partial X_2}, \\ \tilde{D} &= -\frac{D}{\bar{X}} - \alpha_{11}X_1 - \beta_1 Y\Big(X_1\frac{\partial k_1}{\partial X_1} + X_2\frac{\partial k_1}{\partial X_2}\Big) - \delta_1\Big(X_1\frac{\partial H_1}{\partial X_1} + X_2\frac{\partial H_1}{\partial X_2}\Big), \\ \tilde{E} &= -\delta_0 Y\frac{dg}{dY}, \end{aligned}$$

$$\begin{aligned} w_1 &= \beta_1 X_1 k_1(x_1, x_2), \\ w_2 &= \beta_2 X_2 k_2(x_1, x_2). \end{aligned} \tag{12}$$

The characteristic equation

$$|J - \lambda I| = 0$$

where I is the unit matrix, may be written as

$$\lambda^3 + b_1\lambda^2 + b_2\lambda + b_3 = 0 \tag{13}$$

where

$$\begin{aligned} b_1 &= -(\tilde{C} + \tilde{D} + \tilde{E}), \\ b_2 &= \tilde{C}\tilde{D} - CD + \tilde{D}\tilde{E} + \tilde{C}\tilde{E} + AYw_1 + BYw_2, \\ b_3 &= -\tilde{C}\tilde{D}\tilde{E} + CD\tilde{E} + DAYw_2 - B\tilde{D}Yw_2 + CBYw_1 - A\tilde{C}Yw_1. \end{aligned} \tag{14}$$

Now in order that the equilibrium point $E = (X_1, X_2, Y)$ be stable, the eigenvalue solutions λ of Eq. (13) must have negative real parts and the Routh–Hurwitz criteria provide us with the conditions to be satisfied. By the Routh–Hurwitz criteria we will have a stable equilibrium if and only if

$$b_1 > 0,\ b_3 > 0,\ b_1b_2 - b_3 > 0.$$

From the above we may write the following theorem:

Theorem 1. *If $E = (X_1, X_2, Y)$ is an equilibrium point of Eqs. (1)–(3), then provided* **Lemma 1** *holds together with* **Assumption I** *and $A, B, C, \tilde{C}$, $D, \tilde{D}, \tilde{E}, w_1$, and w_2 are defined by Eq. (12) then the equilibrium point $E = (X_1, X_2, Y)$ exists, represents real populations, and is stable if and only if*

$$b_1 > 0,\ b_3 > 0 \text{ and } \ b_1b_2 - b_3 > 0$$

where b_1, b_2, b_3 are given by Eq. (14).

4. APPLICATIONS

We are extending our earlier works [1–3] by allowing the predators to be harvested. In our examples we also allow the prey to have access to limited as well as unlimited resources, and the predator can switch to the most abundant prey population.

The predatory functions k_1, k_2 are chosen to be functions of the ratio $\dfrac{x_1}{x_2}$ so that we can easily represent switching, that is, we choose $k_1(x_1, x_2) = k_1\left(\dfrac{x_2}{x_1}\right)$ and $k_2(x_1, x_2) = k_2\left(\dfrac{x_1}{x_2}\right)$.

We note that such functions satisfy

$$X_1\frac{\partial k_1}{\partial X_1} + X_2\frac{\partial k_1}{\partial X_2} = 0$$

and

$$X_1\frac{\partial k_2}{\partial X_1} + X_2\frac{\partial k_2}{\partial X_2} = 0.$$

The harvesting of the predator is described by the function $\tilde{H}(y) = yg(y)$. Considering $\alpha_{11} = \alpha_{22} = \delta_1 = \delta_2 = 0$, that is, the prey having unlimited resources and are not harvested, the expressions for Y are

$$Y = \frac{(\alpha_1 - \epsilon_1)\bar{X} + \epsilon_2 p_{21}}{\beta_1 \bar{X} k_1\left(\frac{1}{\bar{X}}\right)},$$

and

$$Y = \frac{\alpha_2 - \epsilon_2 + \epsilon_1 p_{12}\bar{X}}{\beta_2 k_2(\bar{X})}.$$

The equation for $\bar{X} > 0$ becomes

$$\left[(\alpha_1 - \epsilon_1) + \frac{\epsilon_2 p_{21}}{\bar{X}}\right]\beta_2 k_2(\bar{X}) = \beta_1 k_1\left(\frac{1}{\bar{X}}\right)[\alpha_2 - \epsilon_2 + \epsilon_1 p_{12}\bar{X}] \tag{15}$$

where $\bar{X}$ satisfies the inequalities

$$(\alpha_1 - \epsilon_1)\bar{X} + \epsilon_2 p_{21} > 0$$
$$\alpha_2 - \epsilon_2 + \epsilon_1 p_{12}\bar{X} > 0,$$

since Y must be positive. Now Eq. (10) gives

$$X_2 = \frac{\mu + \delta_0 g(Y(\bar{X}))}{c_1\beta_1\bar{X}k_1\left(\frac{1}{\bar{X}}\right) + c_2\beta_2 k_2(\bar{X})} > 0$$

and X_1 is again given by $X_1 = \bar{X}X_2$. We can write these results in the form of the following Lemma:

Lemma 2. *The equilibrium point, (X_1, X_2, Y), of the Eqs. (1)–(3) with $\alpha_{11} = \alpha_{22} = \delta_1 = \delta_2 = 0$, exists and represents real populations if $\bar{X}(> 0), X_2$ satisfy the equation*

$$\left[(\alpha_1 - \epsilon_1) + \frac{\epsilon_2 p_{21}}{\bar{X}}\right]\beta_2 k_2(\bar{X}) = \beta_1 k_1\left(\frac{1}{\bar{X}}\right)[\alpha_2 - \epsilon_2 + \epsilon_1 p_{12}\bar{X}]$$

and the inequalities:

$$(\alpha_1 - \epsilon_1)\bar{X} + \epsilon_2 p_{21} > 0$$
$$\alpha_2 - \epsilon_2 + \epsilon_1 p_{12}\bar{X} > 0$$

and X_2 is given by

$$X_2 = \frac{\mu + \delta_0 g(Y(\bar{X}))}{c_1\beta_1\bar{X}k_1\left(\frac{1}{\bar{X}}\right) + c_2\beta_2 k_2(\bar{X})}(> 0).$$

Eq. (12) gives an equation for $\bar{X}$ only, and with the inequalities provide the existence of Y. From Eq. (11) and the equation immediately above this paragraph, X_2 exists and represents a real population. The inequalities for $\bar{X}$ ensure that Y represents a real population.

4.1. Stability of the Equilibrium Point

The characteristic equation of this system is

$$\lambda^3 + b_1\lambda^2 + b_2\lambda + b_3 = 0$$

where $A, B, C, \tilde{C}, D, \tilde{D}, \tilde{E}, w_1$ and w_2 and b_1, b_2, b_3 are given by Eqs. (12) and (14), respectively, with $\alpha_{11} = \alpha_{22} = \delta_1 = \delta_2 = 0$.

For Stability, we note that since k_1 and k_2 are functions of $\dfrac{x_2}{x_1}$ and $\dfrac{x_1}{x_2}$ respectively, then $\tilde{C}, \tilde{D}$ are negative and $\tilde{C}\tilde{D} = CD$, hence we see that if **Assumptions I-III** hold, if $A > 0, B > 0$, then $b_3 > 0$ and thus the conditions for stability reduce to two, namely

$$b_1 > 0,\ b_1b_2 - b_3 > 0.$$

We can write the following theorem:

Theorem 2. *If (X_1, X_2, Y) is an equilibrium point of Eqs. (1)–(3) then providing Lemma 2 and Assumptions I-III hold and $A, B, C, \tilde{C}, D, \tilde{D}, \tilde{E}$, w_1, w_2, and b_1, b_2, b_3 are defined by Eqs. (12) and (14) with $A > 0, B > 0$ then the equilibrium point exists, represents real populations and is stable if and only if*

$$b_1 > 0,\ b_1b_2 - b_3 > 0.$$

4.2. Hopf Bifurcation

Following the analysis of [2] we can show that if **Assumptions I-III** are satisfied and $A > 0, B > 0$ then C, D, b_1, b_2, b_3 are all positive. Also we see that $\bar{X}, Y, C, D, b_1, w_1$ and w_2 are all independent of c_1, c_2 and again from the analysis of [2] we can write the following Hopf bifurcation theorem:

Theorem 3. *Consider the predator functions k_i, $i = 1, 2$ as satisfying Assumptions I-III and let Lemma 2 hold. Let $A, B, C, \tilde{C}, D, \tilde{D}, \tilde{E}, w_1$ and w_2 and b_1, b_2, b_3 be defined by Eqs. (12) and (14) respectively and let $A > 0, B > 0$, then if $\bar{c}_1$ is a positive root of $b_1b_2 = b_3$ we have a Hopf bifurcation as c_1 passes through $\bar{c}_1$ provided $\beta_1 k_1\left(\dfrac{X_2}{X_1}\right) \neq \beta_2 k_2(\bar{X})$.*

A similar analysis with c_2, as the varying parameter, will give a similar result.

4.3. Application 1

The first system we investigated was defined by the functions:

$$k_1(x_1, x_2) = \frac{1}{1 + \left(\dfrac{x_2}{x_1}\right)^n},\ k_2(x_1, x_2) = \frac{1}{1 + \left(\dfrac{x_1}{x_2}\right)^n},$$

with $n = 1, 2, 3, ...$ and $g(y) = 1$. This predator harvesting function, $g(y)$, implies that we are harvesting a fixed fraction of the predator population $\delta_0 H(y) = \delta_0 y g(y) = \delta_0 y$ i.e. constant harvest effort.

We note that in this case $\dfrac{dg}{dy} = 0$ hence from Eq. (12), $\tilde{E} = 0$.

Stability is assured if and only if

$$b_1 > 0, \; b_3 > 0 \text{ and } b_1 b_2 - b_3 > 0.$$

The following set of parameters: $\mu = .01, \alpha_1 = .015, \alpha_2 = .025, p_{12} = .3, p_{21} = .2$, $\epsilon_1 = .02, \epsilon_2 = .03, \beta_1 = .02, \beta_2 = .01, n = 2, \alpha_{11} = 0, \alpha_{22} = 0$, and $\delta_0 = .00001, \delta_1 = 0, \delta_2 = 0$ were used to numerically solve the Eqs. (1)–(3), for a system where the prey have unlimited resources. Two stable and unstable cases are graphically depicted in Figure 1 and Figure 2.

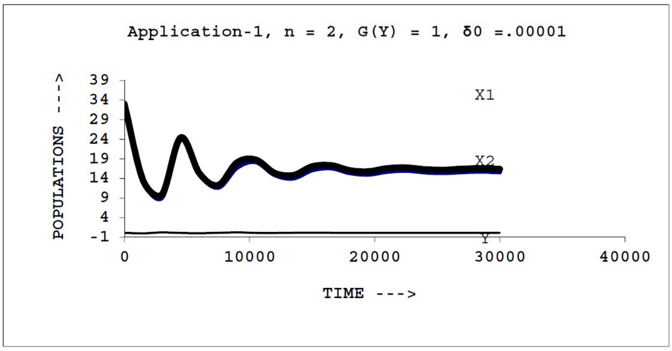

Figure 1
A Stable Case Which Corresponds to $c_1 = 0.05$ and $c_2 = 0.03$

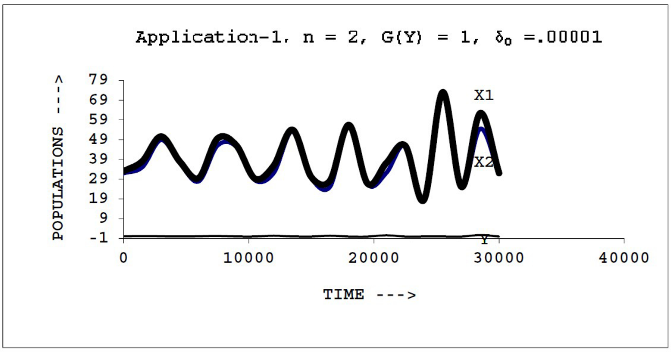

Figure 2
An Unstable Case Which Corresponds to $c_1 = 0.01$ and $c_2 = 0.03$

4.4. Application 2

For the second application we considered harvesting a constant amount of the predator, i.e. constant harvest quota. The actual functions used were:

$$k_1(x_1, x_2) = \frac{1}{1 + \left(\dfrac{x_2}{x_1}\right)^n}, \; k_2(x_1, x_2) = \frac{1}{1 + \left(\dfrac{x_1}{x_2}\right)^n},$$

with $n = 1, 2, 3, ...$ and $g(y) = \frac{1}{y}$, so that $\delta_0 H(y) = \delta_0 y g(y) = \delta_0$.

We see that $\frac{dg(y)}{dy} < 0$ and from Eq. (12), $\tilde{E} > 0$. The equation and inequalities satisfied by $\bar{X}$ and X_2 are those required by Lemma 1.

In this system stability is determined by the conditions: $b_1 > 0$, $b_3 > 0$ and $b_1 b_2 - b_3 > 0$.

The parameters used are the same as used in the first application where the prey have unlimited resources.

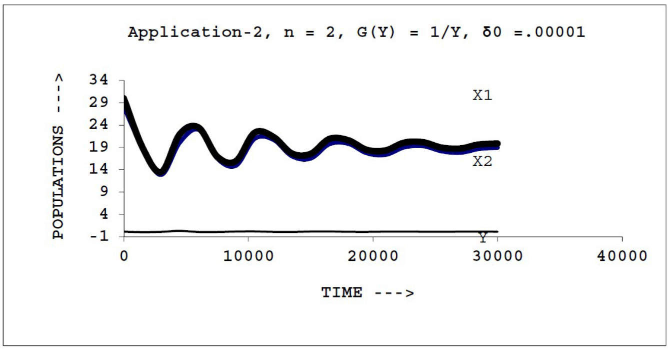

Figure 3
A Stable Case Which Corresponds to $c_1 = 0.04$ and $c_2 = 0.03$

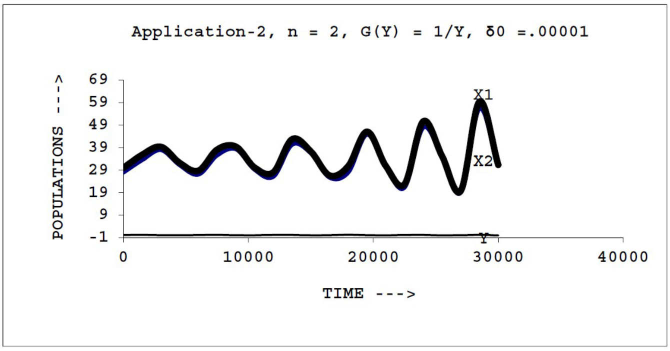

Figure 4
An Unstable Case Which Corresponds to $c_1 = 0.015$ and $c_2 = 0.03$

All the four graphs Figure 1 to Figure 4 also confirm our analysis. In both the applications, the system of Eqs. (1)–(3) remains stable when c_1 is in the interval of stability, otherwise it becomes unstable.

Additionally, we also computed stable / unstable intervals and bifurcation points for several data sets for both the applications. These are given in Table 1. The other parameter values are the same as used for Figure 1 earlier.

Table 1
Stable/Unstable Intervals and Bifurcation Points for Several Data Sets for Both Applications

β_1	β_2	n	STABLE	UNSTABLE	BIF POINT
			Application 1 c_1 varies, $c_2 = .03$		
.01	.02	1	$0 \leq c_1 \leq .059374$	$c_1 \geq .059375$	.059374
.01	.02	2	$0 \leq c_1 \leq .059518$	$c_1 \geq .059519$	.059518
.02	.01	1	$c_1 \geq .015072$	$0 \leq c_1 \leq .015071$	.015072
.02	.01	2	$c_1 \geq .015390$	$0 \leq c_1 \leq .015389$	.015390
			Application 1 c_2 varies, $c_1 = .03$		
.01	.02	1	$c_2 \geq .015158$	$0 \leq c_2 \leq .015157$	.015158
.01	.02	2	$c_2 \geq .015122$	$0 \leq c_2 \leq .015121$	.015122
.02	.01	1	$0 \leq c_2 \leq .059716$	$c_2 \geq .059717$	.059716
.02	.01	2	$0 \leq c_2 \leq .058482$	$c_2 \geq .058483$	.058482
			Application 2 c_1 varies, $c_2 = .03$		
.01	.02	1	$0 \leq c_1 \leq .040600$	$c_1 \geq .040601$	.040600
.01	.02	2	$0 \leq c_1 \leq .059518$	$c_1 \geq .059519$	.059518
.02	.01	1	$c_1 \geq .020764$	$0 \leq c_1 \leq .020763$	.020764
.02	.01	2	$c_1 \geq .019514$	$0 \leq c_1 \leq .019513$	.019514
			Application 2 c_2 varies, $c_1 = .03$		
.01	.02	1	$c_2 \geq .022168$	$0 \leq c_2 \leq .022167$	.022168
.01	.02	2	$c_2 \geq .018542$	$0 \leq c_2 \leq .018541$	.018542
.02	.01	1	$0 \leq c_2 \leq .043340$	$c_2 \geq .043341$	.043340
.02	.01	2	$0 \leq c_2 \leq .046122$	$c_2 \geq .046123$	.046122

4.5. Some Interesting Observations

It is interesting to note that in all the data sets, which we have considered, the harvesting of the predator decreased the interval of stability.

Also for the same data set, the interval of stability in the first application, where we use constant harvest effort, is larger than that in the second application where we use constant harvest quota.

Another interesting observation is as follows:

From Eq. (15) we note that $\bar{X}$ is independent of δ_0 and hence so is Y, however, from case 1, X_1 and X_2 both depend linearly on δ_0. This means that as we vary δ_0, for example if we increase the predator harvesting, the value of the prey populations at equilibrium both increase while the predator population remains constant.

A similar observation occurs when we use c_1 or c_2 (rate of conversion of preys to predator) as the varying parameter.

REFERENCES

[1] Bhatt, B. S., Owen, D. R., & Jaju, R. P. (2002). Switching effect of predation on prey species living in different habitats with arbitrary predatory rates. *J. Phys. Soc. Japan, 71*(9), 2304-2309.

[2] Bhatt, B. S., Owen, D. R., & Jaju, R. P. (2008). Switching of predation on prey species in the presence of Predator interference. *Int. J. Pure & Appl. Maths., 45*(4), 584-598.

[3] Bhatt, B. S., Owen, D. R., & Jaju, R. P. (2010). Switching of predation on prey species in the presence of Predator interference-II. *Int. J. Pure & Appl. Maths., 61*(3), 281-295.

[4] Skalkski, G. T., & Gillian, J. F. (2001). Functional responses with predator interference: viable alternatives to the Holling type II model. *Ecology, 82*, 3082-3092.

[5] Crowley, P. H., & Martin, E. K. (1989). Functional response and interference within and between year classes of dragonfly population. *Journal of the North American Benthological Society, 8*, 211-221.

[6] Marsden, J. E., & McCracken, M. (1976). *The Hopf Bifurcation and its application.* New York: Springer-Verlag.

[7] Beddington, J. R. (1975). Mutual interference between parasites or predators and its effect on searching efficiency. *Journal of Animal Biology, 51*, 331-340.