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A Class of Nondifferentiable Multiobjective Control Problems

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Abstract: Optimality conditions are derived for a class of nondifferentiable multiobjective control problems having a nondifferentiable term in each component of vector-valued integrand of objective functional. Using Karush-Kuhn-Tucker type optimality conditions, we formulate Mond-Weir type dual to the nondifferentiable control problem and derive duality results extensively under generalized invexity. Finally, it is indicated that our duality results can be considered as dynamic generalizations of those of nondifferentiable nonlinear programming problems recently obtained.

Key words: Nondifferentiable multiobjective control; Efficient solution; Generalized invexity; Duality; Nondifferentiable multiobjective nonlinear programming problem

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1. INTRODUCTION

The problem of optimal control was first formulated by Mond and Hanson [1] as mathematical programming problems with equality and inequality constraints in infinite dimensional space. Subsequently, a number of authors, notably, and Chandra *et al.* [2], Mond and Smart [3], Nohak and Nanda [4] etc. most of them considered the Wolfe and Mond-Weir type for a single objective control problem.

In the recent past, some researchers studied duality for multiobjective control problems motivated with Bector and Husain [5]. Bhatia and Kumar [6] discussed multiobjective control problems with ρ-pseudoinvexity, ρ-strict pseudoinvexity, ρ-qausi-invexity or ρ-strict quasi-invexity. Nahak and Nanda [4] discussed efficiency and duality for multiobjective variational control problems with (F, ρ)-convexity. The objective functionals and constraints functionals in both references [4] and [6] were differentiable. In the present research expositions, we study duality and optimality for a class of nondifferentiable multiobjective control problems in which nondifferentiability enter due having a term of square root a quadratic form in each component of the vector-valued integrand of objective functional. The relationship of our results with those of a class of nondifferentiable nonlinear programming problems is briefly indicated.

2. RELATED PRE-REQUISITES AND NONDIFFERENTIABLE MULTIOBJECTIVE CONTROL PROBLEMS

Let $I = [a, b]$ be a real interval, and let $f^i : I \times R^n \times R^m \to R$, $i = 1, 2, ..., p$, $g^j : I \times R^n \times R^m \to R^l$, and $h : I \times R^n \times R^m \to R^n$ be continuously differentiable functions. Denote by X the space of piecewise smooth functions $x : I \to R^n$, with the norm $\|x\| = \|x\|_\infty + \|Dx\|_\infty$ and by U the space of piecewise continuous control functions $u : I \to R^m$ with the norm $\|u\|_\infty$, where the differentiation operator D is given by

$$u = D\,x \Leftrightarrow x(a) + \int_a^t u(s)\,ds,$$

where $x(a)$ is a given boundary value. Denote the partial derivatives of f_i with respect to t, x, and u, respectively, by f_t^i, f_x^i, and f_u^i such that

$$f_x^i = \left(\frac{\partial f^i}{\partial x^1}, \frac{\partial f^i}{\partial x^2}, \dots, \frac{\partial f^i}{\partial x^n}\right)^T, \quad f_u^i = \left(\frac{\partial f^i}{\partial u^1}, \frac{\partial f^i}{\partial u^2}, \dots, \frac{\partial f^i}{\partial u^n}\right)^T,$$

$i = 1, 2, ...p$, where T denotes the transpose operator. The partial derivatives of the vector functions g and h are similarly defined, using $m \times n$ matrix and $m \times n$ matrix respectively.

Consider the following multiobjective control problem:

(VCP): Minimize

$$\left(\int_I \left(f^1(t,x,u) + \left(u(t)^T B^1(t)\,u(t)\right)^{1/2}\right) dt,\right.$$
$$\left. ..., \int_I \left(f^p(t,x,u) + \left(u(t)^T B^p(t)\,u(t)\right)^{1/2}\right) dt\right)$$

subject to

$$x(a) = \alpha, \; x(b) = \beta \tag{1}$$

$$\dot{x} = h(t, x, u), \; t \in I \tag{2}$$

$$g(t,x,u) \leqq 0, \ \ t \in I \tag{3}$$

The following convention for equality and inequality will be used. If $\alpha, \beta \in R^n$, then

$$\begin{aligned}
&\alpha = \beta \ \Leftrightarrow \alpha^i = \beta^i && i = 1,2,...,n \\
&\alpha \geqq \beta \ \Leftrightarrow \alpha^i \geqq \beta^i && i = 1,2,...,n \\
&\alpha \geqslant \beta \ \Leftrightarrow \alpha \geqq \beta \ \text{ and } && \alpha \neq \beta \\
&\alpha > \beta \ \Leftrightarrow \alpha^i > \beta^i && i = 1,2,...,n
\end{aligned}$$

Definition 1. A feasible solution $(\bar{x}, \bar{u})$ for (VCP) is said to be an efficient solution for (VCP) if there is no other solution (x, u), such that

$$\int_a^b f(t,x,u)dt < \int_a^b f(t,\bar{x},\bar{u})dt, \quad \text{for some} \ \ i \in \{1,2,...,p\}$$

$$\int_a^b f^j(t,x,u)dt \leqq \int_a^b f^j(t,\bar{x},\bar{u})dt, \quad \text{for all} \ \ j \in \{1,2,...,p\}$$

Definition 2 (i). If there exist vector functions $\eta(t,x,\bar{x}) \in R^n$, with $\eta = 0$ at t if $x(t) = \bar{x}(t)$, and $\zeta\,(t,u,\bar{u}) \in R^m$ such that for the scalar function $h(t,x,u)$ the functional $H(x,u) = \int_a^b h(t,x,u)dt$ satisfies

$$H\,(x,u) - H\,(\bar{x},\bar{u}) \geqslant \int_a^b \left[\eta^T h_x\,(t,\bar{x},\bar{u}) + \frac{d\eta^T}{dt} h_{\dot{x}}\,(t,\bar{x},\bar{u}) + \zeta^T h_u\,(t,\bar{x},\bar{u})\right] dt,$$

then H is said to be invex in $\bar{x}$ and $\bar{u}$ on $[a,b]$ with respect to η and ζ.

(ii). If for all $x \in X$ and $u \in U$,

$$\int_a^b \left[\eta^T h_x\,(t,\bar{x},\bar{u}) + \frac{d\eta^T}{dt} h_{\dot{x}}\,(t,\bar{x},\bar{u}) + \zeta^T h_u\,(t,\bar{x},\bar{u})\right] dt \geqslant 0$$
$$\Rightarrow H\,(x,u) \geqslant H\,(\bar{x},\bar{u}),$$

then H is said to be pseudoinvex in $\bar{x}$ and $\bar{u}$ on $[a,b]$ with respect to η and ζ.

(iii). If for all $x \in X$ and $u \in U$,

$$\int_a^b \left[\eta^T h_x\,(t,\bar{x},\bar{u}) + \frac{d\eta^T}{dt} h_{\dot{x}}\,(t,\bar{x},\bar{u}) + \zeta^T h_u\,(t,\bar{x},\bar{u})\right] dt \geqslant 0$$
$$\Rightarrow H\,(x,u) > H\,(\bar{x},\bar{u}),$$

then H is said to be strictly pseudoinvex in $\bar{x}$ and $\bar{u}$ on $[a,b]$ with respect to η and ζ.

(iv). If for all $x \in X$ and $u \in U$,

$$H(x,u) \leqq H(\bar{x},\bar{u})$$
$$\Rightarrow \int_a^b \left[\eta^T h_x(t,\bar{x},\bar{u}) + \frac{d\eta^T}{dt} h_{\dot{x}}(t,\bar{x},\bar{u}) + \zeta^T h_u(t,\bar{x},\bar{u})\right] dt \leqq 0,$$

then H is said to be quasi-invex in $\bar{x}$ and $\bar{u}$ on $[a,b]$ with respect to η and ζ.

(v). If for all $x \in X$ and $u \in U$,

$$H(x,u) < H(\bar{x},\bar{u})$$
$$\Rightarrow \int_a^b \left[\eta^T h_x(t,\bar{x},\bar{u}) + \frac{d\eta^T}{dt} h_{\dot{x}}(t,\bar{x},\bar{u}) + \zeta^T h_u(t,\bar{x},\bar{u})\right] dt < 0,$$

then H is said to be quasi-invex in $\bar{x}$ and $\bar{u}$ on $[a,b]$ with respect to η and ζ.

The generalized Schwartz inequality [2] which will be invoked in the forthcoming analysis, states that

$$x(t)^T B(t) w(t) \leqq \left(x(t)^T B(t) x(t)\right)^{1/2} \left(w(t)^T B(t) w(t)\right)^{1/2}$$

with equality in the above if (and only if)

$$B(t)x(t) = q(t)B(t)z(t), \quad \text{for some} \quad q(t) \in R.$$

3. NECESSARY OPTIMALITY CONDITIONS

In this section, we obtain necessary optimality conditions for the nondifferentiable multiobjective control problems (VCP), using the relationship between efficient solution of the problem (VCP) and the optimal solution of the associated nondifferentiable scalar control problem.

The following lemma will be used to obtain the Fritz John type optimality conditions for (VCP):

Lemma 1 (Chankong and Haimes [7]). If $(\bar{x},\bar{u})$ is an efficient solution of the (VCP) if and only if $(\bar{x},\bar{u})$ is the optimal solutions of the scalar control problems $P_k(\bar{x},\bar{u})$ for $k = \{1,2,...,p\}$ where $P_k(\bar{x},\bar{u})$ is defined as

$$P_k(\bar{x},\bar{u}): \quad \text{Minimize} \quad \int_I \left(f^k(t,x,u) + \left(u(t)^T B^k(t) u(t)\right)^{1/2}\right) dt$$

subject to

$$x(a) = \alpha, \ x(b) = \beta$$
$$\dot{x} = h(t,x,u), \ t \in I$$
$$g(t,x,u) \leqq 0, \ t \in I$$
$$f^j(t,x,u) + \left(u(t)^T B^j(t) u(t)\right)^{1/2} \leqq f^j(t,\bar{x},\bar{u}) + \left(\bar{u}(t)^T B^j(t) \bar{u}(t)\right)^{1/2}$$

for all $j \in \{1, 2, ..., p\}$, $j \neq k$

Chandra *et al.* [2] considered the following nondifferentiable single objective Control problem to determine the necessary optimality conditions:

(CP): Minimize $\int_I \left(f(t, x, u) + \left(u(t)^T B(t) u(t) \right)^{1/2} \right) dt$

subject to

$$x(a) = \alpha, \ x(b) = \beta$$
$$\dot{x} = h(t, x, u), \ t \in I$$
$$g(t, x, u) \leqq 0, \ t \in I$$

where f, g, h are the same as defined earlier. Following Craven [8], the differential equation $\dot{x} = h(t, x, u)$ with initial condition can be expressed as

$$x(t) = x(a) + \int_a^t h(s, x(s), u(s))ds, \quad t \in I$$

may be written as $Dx = H(x, u)$ where the map $H : X \times U \to C(I, R^n)$ is defined by

$$H(x, u)(t) = h(t, x(t), u(t)) \ , t \in I$$

In the following Fritz-John type optimality conditions, some constraint qualification to make the equality constraint locally solvable [8] is needed. For this, the Frechet derivative of

$$Dx - H(x, u) = Q(x, u), \quad \text{(say)}$$

with respect to (x, u),

$$Q' = Q'(\bar{x}, \bar{u}) = [D - H_x(\bar{x}, \bar{u}), -H_u(\bar{x}, \bar{u})] \quad \text{must be surjective.}$$

Theorem 3.1 (Fritz-John condition): If $(\bar{x}, \bar{u})$ is an optimal solution of (CP) and the Frechet derivative $Q' = [D - H_x(\bar{x}, \bar{u}), -H_u(\bar{x}, \bar{u})]$ is surjective, then there exist Lagrange multipliers $\tau_0 \in I$ piecewise smooth functions $y : I \to R^m$, $z : I \to R^n$ and $w : I \to R^n$ satisfying for all $t \in I$,

$$\tau_o f_x(t, \bar{x}, \bar{u}) + y(t)^T g_x(t, \bar{x}, \bar{u}) + z(t)^T h_x(t, \bar{x}, \bar{u}) + \dot{z}(t) = 0, t \in I$$

$$\tau_o \left(f_u(t, \bar{x}, \bar{u}) + B(t)w(t) \right) + y(t)^T g_u(t, \bar{x}, \bar{u}) + z(t)^T h_u(t, \bar{x}, \bar{u}) = 0, t \in I$$

$$u(t)^T B(t)w(t) = (u(t)^T B(t)u(t))^{1/2}, t \in I$$

$$y(t)^T g(t, \bar{x}, \bar{u}) = 0 \, , t \in I$$

$$w(t)^T B(t)w(t) \leqq 1, \ t \in I$$

$$(\tau_0, y(t)) \geqq 0, t \in I$$

$$(\tau_0, y(t), z(t)) \ \neq \ 0, t \in I$$

The above theorem gives the Karush-Kuhn-Tucker type optimality conditions if $\tau_0 = 1$, then $(\bar{x}, \bar{u})$ will be called d normal. For this, it sufficient to assume the Zowe's [9] form of the Slater condition is assumed.

Theorem 3.2 (Karush-Kuhn-Tucker type optimality conditions): If $(\bar{x},\bar{u})$ is an optimal and normal solution of (CP), and Frechet derivative $Q' = [D - H_x(\bar{x},\bar{u}), -H_u(\bar{x},\bar{u})]$ is surjective, then there exist piecewise smooth $y : I \to R^m$, $z : I \to R^n$ and $w : I \to R^n$, $i \in K$,

$$f_x(t,\bar{x},\bar{u}) + y(t)^T g_x(t,\bar{x},\bar{u}) + z(t)^T h_x(t,\bar{x},\bar{u}) + \dot{z}(t) = 0, t \in I$$

$$(f_u(t,\bar{x},\bar{u}) + B(t)w(t)) + y(t)^T g_u(t,\bar{x},\bar{u}) + z(t)^T h_u(t,\bar{x},\bar{u}) = 0, t \in I$$

$$y(t)^T g(t,\bar{x},\bar{u}) = 0\,, t \in I$$

$$w(t)^T B(t)w(t) \leqq 1, t \in I$$

$$y(t)) \geqq 0, t \in I$$

The following theorem gives the Fritz John type optimality conditions for (VCP) and will be required to establish the converse duality theorem.

Theorem 3.3 (Fritz John type optimality conditions): Let $(\bar{x},\bar{u})$ be an efficient solutions of (VCP) and the Frechet derivative Q' is surjective. Then there exist $\lambda^i \in R$, $i \in K$, piecewise smooth $y : I \to R^m$, $z : I \to R^n$ and $w^i : I \to R^n$, $i \in K$ such that

$$\sum \lambda^i \left(f^i_x(t,\bar{x},\bar{u}) - Df^i_{\dot{x}}(t,\bar{x},\bar{u})\right) + y(t)^T g_x(t,\bar{x},\bar{u}) + z(t)^T h_x(t,\bar{x},\bar{u}) + \dot{z}(t) = 0,\ t \in I$$

$$\sum \lambda^i \left(f^i_u(t,\bar{x},\bar{u}) + B^i(t)w^i(t)\right) + y(t)^T g_u(t,\bar{x},\bar{u}) + z(t)^T h_u(t,\bar{x},\bar{u}) = 0,\ t \in I$$

$$y(t)^T g(t,\bar{x},\bar{u}) \ = \ 0,\ t \in I$$

$$u(t)^T B^i(t)w^i(t) = \left(u(t)^T B^i(t)u(t)\right)^{1/2},\quad i \in K$$

$$w^i(t)^T B^i(t)w^i(t) \leqq 1,\quad t \in I,\ i \in K$$

$$(\lambda, y(t)) \geqq 0$$

$$(\lambda, y(t), z(t)) \neq 0,\quad t \in I$$

Proof. Since $(\bar{x},\bar{u})$ is an efficient solutions of (VCP), by Lemma3.1, $(\bar{x},\bar{u})$ is an optimal solutions $P_k(\bar{x},\bar{u})$ for each $p \in K$ and hence in particular of $P_1(\bar{x},\bar{u})$. Therefore, by Theorem 3.2, there exist $\lambda^i \in R$, $i \in K$, piecewise smooth functions $y : I \to R^m$, $z : I \to R^n$ and $w^i : I \to R^n$, $i \in K$ such that

$$\sum \lambda^i \left(f^i_x(t,\bar{x},\bar{u}) - Df^i_{\dot{x}}(t,\bar{x},\bar{u})\right) + y(t)^T g_x(t,\bar{x},\bar{u}) + z(t)^T h_x(t,\bar{x},\bar{u}) + \dot{z}(t) = 0,\ t \in I$$

$$\sum \lambda^i \left(f^i_u(t,\bar{x},\bar{u}) + B^i(t)w^i(t)\right) + y(t)^T g_u(t,\bar{x},\bar{u}) + z(t)^T h_u(t,\bar{x},\bar{u}) = 0,\ t \in I$$

$$y(t)^T g(t,\bar{x},\bar{u}) = 0,\ t \in I$$

$$u(t)^T B^i(t)w^i(t) = \left(u(t)^T B^i(t)u(t)\right)^{1/2},\ i \in K$$

$$w^i(t)^T B^i(t)w^i(t) \leqq 1,\ t \in I,\ i \in K$$

$$(\lambda, y(t)) \geqq 0$$

$$(\lambda, y(t), z(t)) \ \neq \ 0,\ t \in I$$

Thus the theorem follows. □

4. DUALITY

In this section, we propose the following Mond-Weir type dual to (VCP) and establish various duality results under suitable generalized invexity:

(VCD): Maximize

$$\left(\int_I \left(f^1(t,x,u) + \left(u(t)^T B^1(t)\, u(t)\right)\right)dt, ..., \int_I \left(f^p(t,x,u) + \left(u(t)^T B^p(t)\, u(t)\right)\right)dt\right)$$

subject to

$$x(a) = \alpha,\ x(b) = \beta \tag{4}$$

$$\sum_{i=1}^{p} \lambda^i \left(f_x^i(t,x,u)\right) + y(t)^T g_x(t,x,u) + z(t)^T h_x(t,x,u) + \dot{z}(t) = 0,\ t \in I \tag{5}$$

$$\sum_{i=1}^{p} \lambda^i \left(f_u^i(t,x,u) + B^i(t) w^i(t)\right) + y(t)^T g_u(t,x,u) + z(t)^T h_u(t,x,u) = 0,\ t \in I \tag{6}$$

$$\int_I y(t)^T g(t,x,u)dt \geqq 0 \tag{7}$$

$$\int_I z(t)^T \left(h(t,x,u) - \dot{x}(t)\right)dt \geqq 0 \tag{8}$$

$$y(t) \geqq 0,\ t \in I \tag{9}$$

$$w^i(t)^T B^i(t) w^i(t) \leqq 1,\ t \in I,\ i \in K \tag{10}$$

$$\lambda > 0 \tag{11}$$

Definition 4.1 A feasible solution $(\bar{x}, \bar{u})$ for (VCP) is efficient if there is no other feasible (x, u) for (VCP) such that

$$\int_I f^i(t,x,u)dt < \int_I f^i(t,\bar{x},\bar{u})dt \quad \text{for some} \quad i \in \{1,2,...,r\}$$

$$\int_I f^j(t,x,u)dt \leqq \int_I f^j(t,\bar{x},\bar{u})dt \quad \text{for some} \quad j \in \{1,2,...,r\}$$

In case of maximization, the signs of the above inequalities are reversed. We require the following lemma in the subsequent analysis.

Theorem 4.1 (Weak Duality): Assume that all feasible $(\bar{x}, \bar{u})$ for (VCP) and all (x, u, λ, y, z, w) for (VCD) that with respect to the same functions η and ζ.

(A_1): $\sum \lambda^i \int_I \left(f^i(t,x,u) + u(t)^T B^i(t)\, w^i(t)\right)dt$ is pseudoinvex with respect to the functions η and ζ.

(A_2): $\int\limits_I y(t)^T g(t,x,u)dt$ is quasi-invex η and ζ.

(A_3): $\int\limits_I z(t)^T \left(h(t,x,u) - \dot{x}(t)\right)dt$ is quasi-invex.

Then the following cannot hold

$$\int\limits_I \left(f^i\left(t,\bar{x},\bar{u}\right) + \left(\bar{u}(t)^T B^i\left(t\right)\bar{u}\left(t\right)\right)^{1/2}\right)dt < \int\limits_I \left(f^i\left(t,x,u\right) + u(t)^T B^i\left(t\right)w^i\left(t\right)\right)dt \quad \text{for some } i \in \{1,2,...,p\} \tag{12}$$

$$\int\limits_I \left(f^j\left(t,\bar{x},\bar{u}\right) + \left(\bar{u}(t)^T B^j\left(t\right)\bar{u}\left(t\right)\right)^{1/2}\right)dt \leqq \int\limits_I \left(f^j\left(t,x,u\right) + u(t)^T B^j\left(t\right)w^j\left(t\right)\right)dt \quad \text{for some } j \in \{1,2,...,p\} \tag{13}$$

Proof. Suppose, contrary to the result, that (12) and (13) hold. Then (A_1) yields

$$\int\limits_I \left(\eta^T(t,\bar{x},x)f_x^i\left(t,\bar{x},u\right) + \zeta^T(t,\bar{u},u)\left(f_u^i\left(t,\bar{x},u\right) + B^i(t)w^i(t)\right)\right)dt \leqq 0 \quad \text{for all } i \in \{1,2,...,p\} \tag{14}$$

Multiplying each inequality of (14) by $\lambda^i > 0$ and summing up for all $i = 1,2,...,p$, we get

$$\int\limits_I \left\{\eta^T(t,\bar{x},x)\left(\sum_{i=1}^p \lambda^i f_x^i\left(t,\bar{x},u\right)\right) + \zeta^T(t,\bar{u},u)\left(\sum_{i=1}^p \lambda^i\left(f_u^i\left(t,\bar{x},u\right) + B^i(t)w^i(t)\right)\right)\right\}dt \leqq 0 \tag{15}$$

Using the feasibility of (VCP) and (VCD), we have

$$\int\limits_I y(t)^T g(t,\bar{x},\bar{u})dt \leqq \int\limits_I y(t)^T g(t,x,\bar{u})dt$$

This, because of (A_2) implies

$$\int\limits_I \left\{\eta^T(t,\bar{x},x)\left(y(t)^T g_x(t,\bar{x},u)\right) + \zeta^T(t,\bar{u},u)\left(y(t)^T g_u(t,x,u)\right)\right\}dt \leqq 0 \tag{16}$$

Also

$$\int\limits_I z(t)^T \left(h(t,\bar{x},\bar{u}) - \dot{\bar{x}}(t)\right)dt \leqq \int\limits_I z(t)^T \left(h(t,x,u) - \dot{x}(t)\right)dt$$

From (A_3), it implies that

$$0 > \int_I \left\{ \eta^T(t,\bar{x},u) \left(z(t)^T h_x(t,x,u) \right) - \frac{d}{dt}\eta^T(t,\bar{x},u)z(t) + \zeta^T(t,\bar{u},u) \left(z(t)^T h_u(t,x,u) \right) \right\} dt$$

$$0 > \int_I \left\{ \eta^T(t,\bar{x},u) \left(z(t)^T h_x(t,x,u) \right) + \zeta^T(t,\bar{u},u) \left(z(t)^T h_u(t,x,u) \right) \right\} dt - \eta^T(t,\bar{x},u)z(t)\Big|_{t=a}^{t=b} + \int_I \eta^T(t,\bar{x},u)\dot{z}(t)dt$$

(By integrating by parts)

Using $\eta^T = 0$, at $t = a$ and $t = b$ we have,

$$0 > \int_I \left\{ \eta^T \left(z(t)^T h_x(t,x,u) + \dot{z}(t) \right) + \zeta^T(t,\bar{u},u)z(t)^T h_u(t,x,u) \right\} dt \qquad (17)$$

Combining (15), (16) and (17), we have

$$\int_I \left\{ \eta^T(t,\bar{x},x) \left(\sum_{i=1}^{p} \lambda^i \left(f_x^i(t,\bar{x},u) \right) + y(t)^T g_x(t,\bar{x},u) + z(t)^T h_x(t,x,u) + \dot{z}(t) \right) + \zeta^T(t,\bar{u},u) \left(\sum_{i=1}^{p} \lambda^i \left(f_u^i(t,\bar{x},u) + B^i(t)w^i(t) \right) \right) + y(t)^T g_u(t,x,u) + z(t)^T h_u(t,x,u) \right\} dt < 0$$

This contradicts (5) and (6). The result follows. □

Corollary 4.1 Assume that weak duality (Theorem 4.1) holds between (VCP) and (VCD). If (x,u) is feasible for (VCP) and $\left(x,u,\lambda,y,z,w^1,...,w^p\right)$ is feasible for (VCD) with $y(t)^T g(t,\bar{x},\bar{u}) = 0$, $t \in I$, then (x,u) is efficient for (VCP) and $\left(x,u,\lambda,y,z,w^1,...,w^p\right)$ is efficient for (VCD).

Proof. Suppose (x,u) is not efficient for (VCP). Then there exists some $(\bar{x},\bar{u})$ for (VCP) such that

$$\int_I \left(f^i(t,\bar{x},\bar{u}) + \left(\bar{u}(t)^T B^i(t)\,\bar{u}(t) \right)^{1/2} \right) dt < \int_I \left(f^i(t,x,u) + \left(u(t)^T B^i(t)\,u(t) \right)^{1/2} \right) dt$$

for some $i \in \{1,2,...,p\}$

$$\int_I \left(f^j(t,\bar{x},\bar{u}) + \left(\bar{u}(t)^T B^j(t)\,\bar{u}(t) \right)^{1/2} \right) dt$$

$$\leqq \int_I \left(f^j(t,x,u) + \left(\bar{u}(t)^T B^j(t)\,\bar{u}(t) \right)^{1/2} \right) dt$$

for all $j \in \{1,2,...,p\}$

$$\bar{u}(t)^T B^j(t)\,w^j(t) \leqq \left(\bar{u}(t)^T B^j(t)\,\bar{u}(t) \right)^{1/2} \left(w^j(t)^T B^j(t)\,w^j(t) \right)^{1/2}, \quad t \in I$$

$$\leqq \left(\bar{u}(t)^T B^j(t)\,\bar{u}(t) \right)^{1/2}, \quad j \in \{1,2,...,p\}$$

(using $\left(w^j(t)^T B^j(t)\,w^j(t) \right)^{1/2} \leqq 1$)

Using $\bar{u}(t)^T B^j(t)\,w^j(t) \leqq \left(\bar{u}(t)^T B^j(t)\,\bar{u}(t) \right)^{1/2}$, $t \in I$, for all $j \in \{1,2,...,p\}$ these give

$$\int_I \left(f^i(t,\bar{x},\bar{u}) + \left(\bar{u}(t)^T B^i(t)\,w^i(t) \right) \right) dt < \int_I \left(f^i(t,x,u) + \left(\bar{u}(t)^T B^i(t)\,\bar{u}(t) \right)^{1/2} \right) dt$$

for some $i \in \{1,2,...,p\}$

$$\int_I \left(f^j(t,\bar{x},\bar{u}) + \left(\bar{u}(t)^T B^j(t)\,w^j(t) \right) \right) dt \leqq \int_I \left(f^j(t,x,u) + \left(\bar{u}(t)^T B^j(t)\,\bar{u}(t) \right)^{1/2} \right) dt$$

for all $j \in \{1,2,...,p\}$

This contradicts weak duality. Hence $(\bar{x},\bar{u})$ is efficient for (VCP).

Now, Suppose $\left(x,u,\lambda,y,z,w^1,...,w^p \right)$ is not efficient for (VCD). Then there exists some feasible $\left(\hat{x},\hat{u},\hat{\lambda},\hat{y},\hat{z},\hat{w}^1,...,\hat{w}^p \right)$ for (VCD) such that

$$\int_I \left(f^i(t,\hat{x},\hat{u}) + \hat{u}(t)^T B^i(t)\,\hat{w}^i(t) \right) dt > \int_I \left(f^i(t,x,u) + u(t)^T B^i(t)\,w^i(t) \right) dt$$

for some $i \in \{1,2,...,p\}$

$$\int_I \left(f^j(t,\hat{x},\hat{u}) + \hat{u}(t)^T B^j(t)\,\hat{w}^j(t) \right) dt \geqq \int_I \left(f^j(t,x,u) + \left(u(t)^T B^j(t)\,w^j(t) \right)^{1/2} \right) dt$$

for all $j \in \{1,2,...,p\}$

Using $\left(\hat{u}(t)^T B^j(t)\,\hat{u}(t) \right)^{1/2} \geqq \left(\hat{u}(t)^T B^j(t)\,w^j(t) \right)$, for all $j \in \{1,2,...,p\}$

We have

$$\int_I \left(f^i(t,\hat{x},\hat{u}) + \left(\hat{u}(t)^T B^i(t)\,\hat{u}(t) \right)^{1/2} \right) dt > \int_I \left(f^i(t,x,u) + u(t)^T B^i(t)\,w^i(t) \right) dt$$

for some $i \in \{1, 2, ..., p\}$

$$\int_I \left(f^j\left(t, \hat{x}, \hat{u}\right) + \left(\hat{u}(t)^T B^j\left(t\right) \hat{u}\left(t\right)\right)^{1/2} \right) dt \geqq \int_I \left(f^j\left(t, x, u\right) + u(t)^T B^j\left(t\right) w^j\left(t\right) \right) dt$$

for all $j \in \{1, 2, ..., p\}$

This contradicts weak duality. Hence $\left(x, u, \lambda, y, z, w^1, ..., w^p\right)$ is efficient for (VCD). □

Theorem 4.2 (Strong Duality): Let $(\bar{x}, \bar{u})$ is efficient for (VCP) and assume that $(\bar{x}, \bar{u})$ is normal and $Q' = \left[D - H_x\left(\bar{x}, \bar{u}\right), -H_u\left(\bar{x}, \bar{u}\right)\right]$ is surjective for at least one $k \in \{1, 2, ..., p\}$. Then there exists $\lambda' \in R^k$ and piecewise smooth $y : I \to R^m$, $z : I \to R^n$, $w^i : I \to R^n$, $i = 1, 2, ..., p$ such that $\left(\bar{x}, \bar{u}, \bar{\lambda}, \bar{y}, \bar{z}, \bar{w}^1, ..., \bar{w}^p\right)$ is feasible for (VCD). If also weak duality holds between (VCP) and (VCD), then $\left(\bar{x}, \bar{u}, \bar{\lambda}, \bar{y}, \bar{z}, \bar{w}^1, ..., \bar{w}^p\right)$ is efficient for (VCD).

Proof. As $(\bar{x}, \bar{u})$ satisfy the constraint qualifications of Theorem 3.2 for at least one $k \in \{1, 2, ..., p\}$, it follows from Theorem 3.2 that there exist $\lambda' \in R^k$, and piecewise smooth $y' : I \to R^m$, $z' : I \to R^n$, $w^i \in R^n$, $i = 1, 2, ..., p$ satisfying

$$f_x^k(t, \bar{x}, \bar{u}) + \sum_{\substack{i=1 \\ i \neq k}}^{p} {\lambda'}^i f_x^i(t, \bar{x}, \bar{u}) + y'(t)^T g_x(t, \bar{x}, \bar{u}) + z'(t)^T h_x(t, \bar{x}, \bar{u}) + \dot{z}(t) = 0, t \in I \tag{18}$$

$$\begin{aligned} &\left(f_u^k(t, \bar{x}, \bar{u}) + B^r(t) w^r(t)\right) \\ &+ \sum_{\substack{i=1 \\ i \neq k}}^{p} {\lambda'}^i \left(f_u^i(t, \bar{x}, \bar{u}) + B^i(t) w^i(t)\right) + y'(t)^T g_u(t, \bar{x}, \bar{u}) + z'(t)^T h_u(t, \bar{x}, \bar{u}) = 0,\ t \in I \end{aligned} \tag{19}$$

$$\begin{gathered} \bar{u}(t)^T B^i(t) w^i(t) = \left(\bar{u}(t)^T B^i(t) \bar{u}(t)\right)^{1/2}, \\ w^i(t)^T B^i(t) w^i(t) \leqq 1, i = 1, 2, ..., p \\ y'(t)^T g(t, \bar{x}, \bar{u}) = 0\,, t \in I \\ {\lambda'}^i > 0, \\ y'(t)) \geqq 0, t \in I \end{gathered}$$

Now setting, for $i = 1, 2, ..., p,\ i \neq k$

$${\lambda'}^i = {\lambda'}^i \Bigg/ \left(1 + \sum_{\substack{i=1 \\ i \neq k}}^{p} {\lambda'}^i \right),\ \ \bar{\lambda}^k = 1 \Bigg/ \left(1 - \sum_{\substack{i=1 \\ i \neq k}}^{p} {\lambda'}^i \right)$$

$$\bar{y}^i = y'(t) \Bigg/ \left(1 + \sum_{\substack{i=1 \\ i \neq k}}^{p} {\lambda'}^i \right),\ \ \bar{z}(t) = z'(t) \Bigg/ \left(1 + \sum_{\substack{i=1 \\ i \neq k}}^{p} {\lambda'}^i \right)$$

Dividing (18) and (19) by $\left(1+\sum\limits_{\substack{i=1\\i\neq k}}^{p}\lambda'^{i}\right)$, we get

$$\sum_{i=1}^{p}\bar{\lambda}^{i}f_{x}^{i}(t,\bar{x},\bar{u})+y(t)^{T}g(t,\bar{x},\bar{u})+z(t)^{T}h_{x}(t,\bar{x},\bar{u})+\dot{z}(t)=0,t\in I$$

$$\sum_{i=1}^{p}\lambda^{i}(f_{u}^{i}(t,\bar{x},\bar{u})+B^{i}(t)w^{i}(t))+y(t)^{T}g(t,\bar{x},\bar{u})+z(t)^{T}h_{u}(t,\bar{x},\bar{u})=0,t\in I$$

$$y(t)^{T}g(t,\bar{x},\bar{u})=0\,,t\in I$$

$$\bar{u}(t)^{T}B^{i}(t)w^{i}(t)=\left(\bar{u}(t)^{T}B^{i}(t)\bar{u}(t)\right)^{1/2},i=1,2,...,p$$

$$w^{i}(t)^{T}B^{i}(t)w^{i}(t)\leqq 1,i=1,2,...,p$$

The relations $\int\limits_{I}\bar{y}(t)^{T}g(t,x,u)dt\geqq 0$ and $\int\limits_{I}z(t)^{T}\left(h(t,x,u)-\dot{x}(t)\right)dt\geqq 0$ are obvious.

The above relations imply that $\left(\bar{x},\bar{u},\bar{\lambda},\bar{y},\bar{z},\bar{w}^{1},...,\bar{w}^{p}\right)$ is feasible for (VCD). The result now follows from Corollary 4.1. □

Theorem 4.3 (Converse Duality): Let $\left(\bar{x},\bar{u},\bar{\lambda},\bar{y},\bar{z},\bar{w}^{1},...,\bar{w}^{p}\right)$ be an efficient solution at which

(A_1) $\int\limits_{I}\sigma(t)^{T}M(t)\sigma(t)dt=0\Rightarrow\sigma(t)=0$, where some vector $\sigma(t)$ of appropriate dimension

(A_2) (a) The vectors $y(t)^{T}g_{x}$, $z(t)^{T}h_{x}+\dot{z}(t)$ are linearly independent. Or

(b) The vectors $y(t)^{T}g_{x}$, $z(t)^{T}h_{u}$ are linearly independent.

(A_3) $z(a)=0=z(b)$.

Then $(\bar{x},\bar{u})$ is feasible for (VCP) and value of the objective functional are the same. If also weak duality (Theorem 4.1) holds between (VCP) and (VCD) holds, then $(\bar{x},\bar{u})$ is an efficient solution for (VCP).

Proof. By Theorem 3.3, there exist $\alpha\in R^{p},\ \mu_{1}\in R,\ \mu_{2}\in R,\ \zeta\in R$

$$\begin{aligned}&\sum_{i=1}^{p}\alpha^{i}\left(f_{x}^{i}\right)+\theta(t)^{T}\left(\lambda^{T}f_{xx}+y(t)^{T}g_{xx}+z(t)^{T}h_{xx}\right)\\&\quad+\phi(t)^{T}\left(\lambda^{T}f_{ux}+y(t)^{T}g_{ux}+z(t)^{T}h_{ux}\right)+\mu_{1}y(t)^{T}g_{x}+\mu_{2}z(t)^{T}h_{x}=0\end{aligned}\tag{20}$$

$$\begin{aligned}&\sum_{i=1}^{p}\alpha^{i}\left(f_{u}^{i}+B^{i}(t)w^{i}(t)\right)+\theta(t)^{T}\left(\lambda^{T}f_{xu}+y(t)^{T}g_{xu}+z(t)^{T}h_{xu}\right)\\&\quad+\phi(t)^{T}\left(\lambda^{T}f_{uu}+y(t)^{T}g_{uu}+z(t)^{T}h_{uu}\right)+\mu_{1}y(t)^{T}g_{u}+\mu_{2}z(t)^{T}h_{u}=0\end{aligned}\tag{21}$$

$$\theta(t)^{T}f_{x}^{i}+\phi(t)^{T}\left(f_{u}^{i}+B^{i}(t)w^{i}(t)\right)+\zeta^{i}=0\tag{22}$$

$$\theta(t)^{T}g_{x}+\phi(t)^{T}g_{u}+\mu_{1}g+\psi(t)=0\tag{23}$$

$$\theta(t)^T h_x - \dot{\theta}(t) + \theta(t)^T h_u + \mu_2 (h - \dot{x}) = 0 \tag{24}$$

$$\alpha^i u(t)^T B^i(t) + \phi(t)^T B^i(t) - 2\gamma(t) B^i(t) w^i(t) = 0 \tag{25}$$

$$\mu_1 y g = 0 \tag{26}$$

$$\mu_2 z (h - \dot{x}) = 0 \tag{27}$$

$$\gamma^i \left(1 - w^i(t)^T B^i(t) w^i(t)\right) = 0\,, i = 1, 2, ..., p \tag{28}$$

$$\lambda^T \zeta = 0 \tag{29}$$

$$\psi(t)^T y(t) = 0 \tag{30}$$

$$(\alpha, \theta(t), \phi(t), \gamma(t), \mu_1, \mu_2, \zeta, \psi) \neq 0 \tag{31}$$

$$(\alpha, \gamma(t), \mu_1, \mu_{2,} \zeta, \psi) \geqq 0 \tag{32}$$

Multiplying (23) and (24) respectively by $y(t)$ and $z(t)$, we have

$$\theta(t)^T \left(y(t)^T g_x\right) + \phi(t)^T \left(y(t)^T g_u\right) + \mu_1 y(t)^T g + \psi(t)^T y(t) = 0,\ t \in I \tag{33}$$

$$\theta(t)^T \left(z(t)^T h_x\right) - z(t)^T \dot{\theta}(t) + \phi(t)^T \left(z(t)^T h_u\right) + \mu_2 z(t)^T (h - \dot{x}(t)) = 0,\ t \in I \tag{34}$$

Thus by using (26) and (30), from (33), we have

$$\theta(t)^T \left(y(t)^T g_x\right) + \phi(t)^T \left(y(t)^T g_u\right) = 0,\ t \in I$$

which can be written as

$$\int_I \left(\theta(t)^T, \phi(t)^T\right) \begin{pmatrix} y(t)^T g_x \\ y(t)^T g_u \end{pmatrix} dt = 0 \tag{35}$$

From (34), we have

$$\int_I \left(\theta(t)^T \left(z(t)^T h_x\right) + \phi(t)^T \left(z(t)^T h_u\right)\right) dt - \int_I z(t)^T \dot{\theta}(t) dt = 0$$

Integrating by parts, we have

$$\int_I \left(\theta(t)^T \left(z(t)^T h_x\right) + \phi(t)^T \left(z(t)^T h_u\right)\right) dt - z(t)^T \dot{\theta}(t)\Big|_{t=a}^{t=b} + \int_I \dot{z}(t)^T \theta(t) dt = 0$$

Using the hypothesis (A_3), we have

$$\int_I \left[\theta(t)^T \left(z(t)^T h_x + \dot{z}(t)\right) + \phi(t)^T \left(z(t)^T h_u\right)\right] dt = 0$$

or

$$\int_I \left(\theta(t)^T, \phi(t)^T\right)^T \begin{pmatrix} z(t)^T h_x + \dot{z}(t) \\ z(t)^T h_u \end{pmatrix} dt = 0 \tag{36}$$

Using equality constraints (5) and (6) in (20) and (21), we have

$$\begin{aligned} &\sum_{i=1}^{p} \left(\alpha^i - \lambda^i \mu_1\right) y(t)^T g_x + \sum_{i=1}^{p} \left(\alpha^i - \lambda^i \mu_2\right) \left(z(t)^T h_x + \dot{z}(t)\right) \\ &+ \left(\sum_{i=1}^{p} \lambda^i\right) \theta(t)^T \left(\lambda^T f_{xx} + y(t)^T g_{xx} + z(t)^T h_{xx}\right) \\ &+ \left(\sum_{i=1}^{p} \lambda^i\right) \phi(t)^T \left(\lambda^T f_{ux} + y(t)^T g_{ux} + z(t)^T h_{ux}\right) = 0 \end{aligned} \tag{37}$$

$$\begin{aligned} &\sum_{i=1}^{p} \left(\alpha^i - \lambda^i \mu_1\right) y(t)^T g_u + \sum_{i=1}^{p} \left(\alpha^i - \lambda^i \mu_2\right) \left(z(t)^T h_u\right) \\ &+ \left(\sum_{i=1}^{p} \lambda^i\right) \theta(t)^T \left(\lambda^T f_{xu} + y(t)^T g_{xu} + z(t)^T h_{xu}\right) \\ &+ \left(\sum_{i=1}^{p} \lambda^i\right) \phi(t)^T \left(\lambda^T f_{uu} + y(t)^T g_{uu} + z(t)^T h_{uu}\right) = 0 \end{aligned} \tag{38}$$

These can be written as

$$\begin{aligned} &\sum_{i=1}^{p} \left(\alpha^i - \lambda^i \mu_1\right) \begin{pmatrix} y(t)^T g_x \\ y(t)^T g_u \end{pmatrix} + \sum_{i=1}^{p} \left(\alpha^i - \lambda^i \mu_2\right) \begin{pmatrix} z(t)^T h_x + \dot{z}(t) \\ z(t)^T h_u \end{pmatrix} \\ &+ \left(\sum_{i=1}^{p} \lambda^i\right) \begin{pmatrix} \left(\lambda^T f_{xx} + y(t)^T g_{xx} + z(t)^T h_{xx}\right) & \left(\lambda^T f_{ux} + y(t)^T g_{ux} + z(t)^T h_{ux}\right) \\ \left(\lambda^T f_{xu} + y(t)^T g_{xu} + z(t)^T h_{xu}\right) & \left(\lambda^T f_{uu} + y(t)^T g_{uu} + z(t)^T h_{uu}\right) \end{pmatrix} \\ &\cdot \begin{pmatrix} \theta(t) \\ \phi(t) \end{pmatrix} = 0 \end{aligned}$$

or

$$\begin{aligned} &\sum_{i=1}^{p} \left(\alpha^i - \lambda^i \mu_1\right) \left(\theta(t)^T, \phi(t)^T\right) \begin{pmatrix} y(t)^T g_x \\ y(t)^T g_u \end{pmatrix} \\ &+ \sum_{i=1}^{p} \left(\alpha^i - \lambda^i \mu_2\right) \left(\theta(t)^T, \phi(t)^T\right) \begin{pmatrix} z(t)^T h_x + \dot{z}(t) \\ z(t)^T h_u \end{pmatrix} \\ &+ \left(\sum_{i=1}^{p} \lambda^i\right) \left(\theta(t)^T, \phi(t)^T\right) \\ &\begin{pmatrix} \left(\lambda^T f_{xx} + y(t)^T g_{xx} + z(t)^T h_{xx}\right) & \left(\lambda^T f_{ux} + y(t)^T g_{ux} + z(t)^T h_{ux}\right) \\ \left(\lambda^T f_{xu} + y(t)^T g_{xu} + z(t)^T h_{xu}\right) & \left(\lambda^T f_{uu} + y(t)^T g_{uu} + z(t)^T h_{uu}\right) \end{pmatrix} \begin{pmatrix} \theta(t) \\ \phi(t) \end{pmatrix} = 0 \end{aligned}$$

or

$$\sum_{i=1}^{p}\left(\alpha^i-\lambda^i\mu_1\right)\int_I\left(\theta(t)^T,\phi(t)^T\right)\begin{pmatrix} y(t)^T g_x \\ y(t)^T g_u \end{pmatrix}dt$$

$$+\sum_{i=1}^{p}\left(\alpha^i-\lambda^i\mu_2\right)\int_I\left(\theta(t)^T,\phi(t)^T\right)\begin{pmatrix} z(t)^T h_x+\dot{z}(t) \\ z(t)^T h_u \end{pmatrix}dt$$

$$+\left(\sum_{i=1}^{p}\lambda^i\right)\int_I\left(\theta(t)^T,\phi(t)^T\right)$$

$$\begin{pmatrix} \left(\lambda^T f_{xx}+y(t)^T g_{xx}+z(t)^T h_{xx}\right) & \left(\lambda^T f_{ux}+y(t)^T g_{ux}+z(t)^T h_{ux}\right) \\ \left(\lambda^T f_{xu}+y(t)^T g_{xu}+z(t)^T h_{xu}\right) & \left(\lambda^T f_{uu}+y(t)^T g_{uu}+z(t)^T h_{uu}\right) \end{pmatrix}\begin{pmatrix}\theta(t)\\ \phi(t)\end{pmatrix}dt=0 \tag{39}$$

Using (34) and (35), we have

$$\int_I\left(\theta(t)^T,\phi(t)^T\right)$$

$$\begin{pmatrix} \left(\lambda^T f_{xx}+y(t)^T g_{xx}+z(t)^T h_{xx}\right) & \left(\lambda^T f_{ux}+y(t)^T g_{ux}+z(t)^T h_{ux}\right) \\ \left(\lambda^T f_{xu}+y(t)^T g_{xu}+z(t)^T h_{xu}\right) & \left(\lambda^T f_{uu}+y(t)^T g_{uu}+z(t)^T h_{uu}\right) \end{pmatrix}\begin{pmatrix}\theta(t)\\ \phi(t)\end{pmatrix}dt=0$$

By the hypothesis (A_2), we have

$$\left(\theta(t)^T,\phi(t)^T\right)=0 \text{ i.e. } \theta(t)=0=\phi(t),\quad t\in I$$

Using $\theta(t)=0=\phi(t),\quad t\in I$ in (37), we have

$$\sum_{i=1}^{p}\left(\alpha^i-\lambda^i\mu_1\right)y(t)^T g_x+\sum_{i=1}^{p}\left(\alpha^i-\lambda^i\mu_2\right)\left(z(t)^T h_x+\dot{z}(t)\right)=0$$

By the hypothesis (A_3) $y(t)^T g_x$, $z(t)^T h_x+\dot{z}(t)$, we have

$$\sum_{i=1}^{p}\left(\alpha^i-\lambda^i\mu_1\right)=0,\quad \sum_{i=1}^{p}\left(\alpha^i-\lambda^i\mu_2\right)=0$$

$$\sum_{i=1}^{p}\alpha^i=\mu_1\sum_{i=1}^{p}\lambda^i,\quad \sum_{i=1}^{p}\alpha^i=\mu_2\sum_{i=1}^{p}\lambda^i$$

Let $\alpha^i=0\;\; i\in K$. Then $\mu_1=0$ and $\mu_2=0$. The relation (22) and (23) implies

$$\zeta=0 \text{ and } \psi(t)=0,\; t\in I$$

From (25) and (28) implies $\gamma^i(t)=0, i=1,2,...,p$ and $t\in I$.
$(\alpha,\gamma(t),\theta(t),\phi(t),\mu_1,\mu_2,\zeta,\psi(t))=0$, implying a contradiction.
Hence $\alpha^i>0,\; i=1,2,...,p$ giving $\mu_1>0$ and $\mu_2>0$ consequently (23) and (24) imply

$$g(t,\bar{x},\bar{u})\leqq 0,\; t\in I,\; h(t,\bar{x},\bar{u})=0,\; t\in I$$

Thus $(\bar{x}, \bar{u})$ feasible for (VCP).

Now (25) gives

$$B^i(t)u(t) = \frac{2\gamma^i(t)}{\alpha_i} B^i(t)w^i(t)\ , t \in I \tag{40}$$

The Schwartz inequality

$$u(t)^T B^i(t)w^i(t) \leqq \left(u(t)^T B^i(t)u(t)\right)^{1/2} \left(w(t)^T B^i(t)w(t)\right)^{1/2}, i \in \{1, 2, ..., p\}\ , t \in I$$

In view of (40) yields

$$\bar{u}(t)^T B^i(t)w^i(t) = \left(\bar{u}(t)^T B^i(t)\bar{u}(t)\right)^{1/2} \left(w^i(t)^T B^i(t)w^i(t)\right)^{1/2}$$

If $\gamma^i(t) > 0,\ t \in I$, (28) implies

$$w^i(t)^T B^i(t)w^i(t) = 1, \quad i = 1, 2, ..., p$$

Consequently

$$\bar{u}(t)B^i(t)w^i(t) = \left(\bar{u}(t)^T B^i(t)\bar{u}(t)\right)^{1/2},\ i = 1, 2, ..., p$$

If $\gamma^i(t) = 0,\ t \in I$, then (40) yields $B^i(t)u(t) = 0,\ t \in I$. So we still have

$$\bar{u}(t)^T B^i(t)w^i(t) = \left(\bar{u}(t)^T B^i(t)\bar{u}(t)\right)^{1/2}$$

$$f^i(t, \bar{x}, \bar{u}) + \left(\bar{u}(t)^T B^i(t)\bar{u}(t)\right)^{1/2} = f^i(t, \bar{x}, \bar{u}) + \bar{u}(t)^T B^i(t)w^i(t),\ i = 1, 2, ..., k$$

This implies that objective functions have the same value. By Corollary 4.1, the efficiency of $\left(\bar{x}, \bar{u}, \bar{\lambda}, \bar{y}, \bar{z}, \bar{w}^1, ..., \bar{w}^p\right)$ for (VCD) follows. □

5. RELATED PROBLEM

If (VCP) and (VCD) are independent of t and x these essentially reduce to the static cases of non- differentially multiobjective programming recently studied by Husain and Jain [10]. Putting $b - a = 1$, (VCP) and (VCD) become the following problems.

(VCP$_0$): Minimize

$$\left(f^1(u) + \left(u^T B^1 u\right)^{1/2}, ..., f^p(u) + \left(u^T B^p u\right)^{1/2}\right)$$

subject to

$$g(u) \leqq 0$$

$$h(u) = 0$$

(VCD$_0$): Maximize $\left(f^1(u) + u^T B^1 w^1, ..., f^p(u) + u^T B^p w^p\right)$
subject to

$$\sum_{i=1}^{k} \lambda^i \left(f_u(u) + B^i w^i\right) + y^T g_u + z^T h_u = 0$$

$$y^T g(u) \geqq 0$$

$$z^T h(u) \geqq 0$$

$$w^{i^T} B^i w^i \leqq 1, \quad i = 1, 2, ..., p$$

$$\lambda > 0, \quad y \geqq 0.$$

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