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Path-Independence of Work Done Theorem Is Invalid in Center-Bound Force Fields

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Abstract: The notion of work done, and the corresponding to it concept of potential energy, was incompletely defined making the path independence theorem of work done by center-bound force fields invalid for other than radial/conservative forces. Hence nonradial effects along equipotential surfaces, whose presence was suggested by experiments, can exist. New, mathematically complete representation of work done by center-bound force fields (generated by a single source) is offered.

Key words: Path independence theorem of work done; Potentials; Nonradial effects

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1. INTRODUCTION: FORMER DEFINITION OF WORK DONE VIOLATES PRODUCT DIFFERENTIATION RULE

In a first, introductory course on differential calculus students must learn the absolutely mandatory product differentiation rule (PDR), which is binding for all differentials involving any products:

$$d(F * r) = F * dr + r * dF \tag{1}$$

where F and r stand for arbitrary functions and the asterisk denotes a certain multiplication. Proof of the obligatory PDR is quite convincing and its universal validity indisputable (Bers, 1967, p.216).

Then in a subsequent course on advanced calculus, they are tacitly forced to forget about the PDR when they learn the traditional formula for calculating rates dW of the work done W by a force **F**:

$$dW = \mathbf{F} \bullet d\mathbf{r} \tag{2}$$

defined as scalar/dot product of a force vector **F** and the rate of a displacement vector d**r**. The rate of work done (2) is clearly incomplete, for if the force vector can also vary along with the distance **r**, as is does in center-bound force fields, then this formula is not accounting for an effect of that. Scalar/dot product of vectors is denoted by bold dot, and vector/cross product will be denoted by cross, whereas the regular multiplication by numbers or functions/functionals/operators is implied.

Since it is impossible a feat to defend violation of proven theorem representing an operational law, the PDR was tacitly ignored in virtually all traditional presentations of work done and energy. It would take a miracle to disprove proven mathematical theorem by using yet another theorem that asserts path independence (and is also proven but in different setting) which gives rationalization for the violation that could not be justified. To waive theorem by another theorem without turning mathematics into farce or a kind of "mathemagic" required sophistry disguised as sophistication.

Yet mathematics is not just an abstract language but is inherent in the Nature (Capra, 1975, p.32), and if the laws of nature are supposed to be implementations of laws of mathematics (Wenzl, 1954, p.127), then the definition of work done may have been wrongly implemented. It can take centuries to uncover buried mathematical inconsistencies (Vyal'tsev, 1965, p.131), because theorems depend on chosen interpretations, which in turn rely on our paradigms, i.e. on our views on how the world works and how it was constructed—compare (Heitsch, 1978, p.247).

2. PATH INDEPENDENCE OF WORK DONE THEOREM IS INVALID IN CENTER-BOUND FORCE FIELDS

In 1773 AD Lagrange observed that gravitational force **F** satisfies Laplace equation and so it could be expressed as gradient $\mathbf{F} = -\nabla V$ of a scalar function $V(r) = 1/r$ of the distance r between the gravity centers of interacting bodies in a radial/center-bound, attractive gravitational force field generated by one of the bodies, presumed as locally dominant for the sake of simplicity of reasonings (Birkhoff, 1973, p.335). From definition of gradient we obtain $dV=\nabla V(r)\bullet d\mathbf{r}$ in the usual vectorial notation (Seaborn, 2001, p.35).

His observation led to development of mathematical potential theory, whose two central notions are potential V and "work done" W by the force field **F** over distance r = **|r|** (Hart, 1955, p.279). The amount of work done along a smooth curve C(A,B) between its end points A and B was usually defined as

$$\mathrm{W}(A \rightarrow B) = \int_A^B \mathbf{F} \bullet \mathrm{d}\mathbf{r} = -\int_A^B \nabla \mathrm{V}(\mathrm{r}) \bullet \mathrm{d}\mathbf{r} = \int_B^A \nabla \mathrm{V}(\mathrm{r}) \bullet \mathrm{d}\mathbf{r} \tag{3}$$

where **r** is a vector function pointing to path along the curve C (Beatrous & Curjel, 2002, p.290). The gradient of attractive force field is negative, because the force vector **F** acts in direction opposite to that of the gradient (Toretti, 1996, p.7).

The center-bound distance r is the value of the vector **r** pointing from the gravity center of M to that of m, and d**r** is an infinitesimal displacement that represents also generic vector function d**r**().

The path independence theorem (PIT) of the integrals in the equation (3) is defined as their having the same value along any path linking its endpoints A-B within the given region, which is assumed to be the force field (Edwards & Penney, 1999, p.977). The most commonly offered proof of the PIT of the work done function W() in the radial/center-bound attractive force field **F** was usually presented as follows (Edwards & Penney, 2002, p.1032):

$$\text{PIT: } \mathrm{W}() = \int_{\mathrm{B}}^{\mathrm{A}} \nabla \mathrm{V}(\mathrm{r}) \bullet \mathrm{d}\mathbf{r} \Leftrightarrow \quad \text{FTC: } \int_{\mathrm{B}}^{\mathrm{A}} \mathrm{F}(\mathrm{r})\mathrm{dr} = \mathrm{V}(\mathrm{A}) - \mathrm{V}(\mathrm{B}) \tag{4}$$

where the vector **r** points to the trajectory path (or contour/curve C leading from a point A to point B on C) of a point particle influenced by the local force field (Barr, 1997, p.405), although not necessarily driven by the field. As counterpart of potential energy, the amount of generic work done function W() can be viewed as either function of the force (F = |**F**|) and displacement d**r**, or as function of gradient of the (scalar) potential function V(r) times displacement d**r**. The integration turns the function W() into its functional W with respect to both F and r, which becomes also a function of the endpoints A, B. For value of line integral depends upon both: the path of integration and the integrand (Chirgwin & Plumpton, 1964, p.24). The PIT was invented for generic force fields whose spin and rotation are either negligibly small or nonexistent. Center-bound fields with single uniform source mass exhibit spheroidal shape.

The equation (4) imply that the amount of work done W in the PIT shown on the left-hand side (LHS) is equivalent to the fundamental theorem of calculus (FTC) standing on the right-hand side (RHS), which can be written in vectorial notation (Eisenman, 2005, p.122). Although validity of the FTC remains uncontested, its alleged equivalence to the PIT is not only questionable logically, but formally inadmissible.

Rather than proving the PIT, in some textbooks the FTC is proved instead for functions expressed by gradients of scalar-valued functions (Hughes-Hallett, et al., 2005, p.890), which is fine, but when interpreted in physical terms the PDR is usually disregarded (Finney, Thomas, Demana, & Waits, 1995, p.APP-26; Swokowski, 1992, p.982; Stewart, 1999, p.1060), even when the force is varying (Salas & Hille, 1990, p.1016). Then the FTC is transferred onto work done by (assumed as conservative) radial force fields and then morphed into the PIT (McCallum, et al., 2005, p.890; Feynman, Leighton, & Sands, 1977, pp.13-14;

Halliday & Resnick, 1974, p.266) as if these two theorems were equivalent, which they are not.

The problem is that while the FTC specifically asserts that the integrated function F(r) is the total (complete) derivative (Cronin-Scanlon, 1967, p.140; Fitzpatrick, 1996, pp.127-144; Stewart, 1999, p.395/851), this very requirement is not really respected in the PIT since the work done has already been incompletely defined in the equation (2), which defined away the term **r**•d**F** defying thus the precondition of the FTC. Hence the PIT is invalid in general when force and distance vectors can vary independently of each other, as it is within any center-bound force field.

Some authors specifically made the inference that if an integral between any two points in a strip is independent of the path of integration, then its integrand is an exact [or total] differential (Postnikov, 1983, p.129).

The PIT would be true if energy expense incurred along equipotential surfaces could be recovered when the path is reversed, as it is with the conservative forces that can act only in radial directions (Thomas & Finney, 1996, p.1076). But energy exchange on equipotential surfaces is directed toward the dominant source of the force field (Czajko, 2000). The energy could be regained only if the lesser mass m would somehow become the source of the locally dominant field, i.e. the mass m would have to become greater than M.

Since work done by force fields corresponds to potential energy, which is indestructible and thus must be conserved (either directly as in the case of conservative forces or within isolated systems), the PIT virtually prevents any nonradial exchange of energy (without proving its impossibility) by assuming that radial force cannot cause nonradial effects, in defiance of Frenet-Serret results (Czajko, 2011).

The main underlying (though usually unspoken) premise of the PIT is that no force field vectors perpendicular to the radial Newtonian ones exist, whereas in the radial/center-bound attractive force fields of gravity (or electrostatics, for instance) the effective neighboring force field vectors can always be decomposed into radial and certain nonradial subcomponents as well. Hamilton's theorem ensures that (Doran, Lasenby, & Gul, 1993, p.1186). If so, however, then a tangential and also a binormal component of the force vectors could arise (Czajko, 2011). In general, force vectors must not be compared directly, but their comparison should proceed on coordinate by coordinate basis. It is its unspoken premise what the PIT effectively tries to establish/prove, however. No theorem should "prove" what had already been implicitly assumed as one of its premises, even if inadvertently. It is unacceptable.

The equation (4) suggest that any work done in excess will be compensated by some work undone by the field. This could certainly be true if all forces of the field would be parallel, not center-bound.

The PIT is invalid within any center-bound force fields, whether gravitational or electromagnetic.

The unfair generalization of the Newton's inspired idea (which identified standalone generic force with the resultant force of gravity) prompted some authors to express doubts as to what is potential energy, while recognizing logical necessity of presence of a mixed energy term in the Hamiltonian function that usually contains explicitly only the purely radial parts of potential energy (Mercier, 1959, p.122; Mercier, 1977).

3. ALTERNATIVE PROOFS OF PATH INDEPENDENCE OF WORK DONE THEOREM ARE INADMISSIBLE

Since functional is a single mapping of its generating function to a number, whereas function is a mapping of a whole space into another space (hence a sequence of single mappings of numbers) (Ryder, 1996, p.172), the former approach to work done and potential was logically (and conceptually) deficient.

Functional is just a value, even if expressed by an equation comprising several variables (hence value to be calculated yet); it is like single picture. Function, on the other hand, is a sequence of values – like whole film/movie, which contains sequences of single pictures. We can differentiate functions (or animate film), but we cannot meaningfully differentiate a single value, because the differential of a single value (which resolves to constant) equals to zero. Differentiation involves tendency towards a limit, which is an attribute of sequences of values/numbers, but is irrelevant to any single value that could tend only to itself, if generalization of the term limit would be made for the sake of the argument. Differential means rate of change. In animated films the pictures/frames do change. With single picture there is no change; hence any attempt to differentiate functional disguised as its function is pointless. Distinction between function and its functional is essential.

Mixing functionals with their functions is rampant in differentials to be integrated, such as these:

$$\mathbf{F} \bullet d\mathbf{r} = \mathbf{F} \bullet \mathbf{T}ds = -\nabla V(r) \bullet d\mathbf{r} = -\left(\frac{\partial V}{\partial x}dx + \frac{\partial V}{\partial y}dy + \frac{\partial V}{\partial z}dz\right) \tag{5}$$

which are commonly used in some more direct, alternative proofs of the PIT (Stein & Barcellos, 1992, p.988; Anton, 1999, p.738; Varberg, Purcell, & Rigdon, 2000; Larson, R. E., & Hostetler, 1986, p.929).

In order to salvage whatever could make some sense in the equation (5) they could be rewritten as

$$\mathbf{F} \bullet \partial\mathbf{r} = -\nabla V(r) \bullet \partial\mathbf{r} = -\frac{\partial V}{\partial r}dr \tag{6}$$

but in this case an alert student (remembering that partial differentials are not total ones) could ask the (embarrassing) question: how could partial = incomplete differentials represent total energy?

The reader has already seen above that these differentials are actually incomplete. But the alleged equivalence $d\mathbf{r} = \mathbf{T}ds$ has not been debunked yet. These two vectors do point in the same direction tangent to the path ds, which runs along the trajectory curve. Hence it seems like we got a certain answer, but to what question? What was the function whose (proper) differentiation yielded this particular equivalence relation? And why the y and z, if the radial potential was actually defined along x = r? One could see it clearly if instead of using y and z one would write angles in polar or cylindrical or spherical coordinates. If the radial force is supposed to have only radial effects, let us say in the direction of x, and no tangential (y) or binormal (z) effects could exist, then their appearance there is unnecessary and confusing distraction. When both y and z are present, they suggest that the radius r is being decomposed into x,y,z, which would be fine, were it not for the fact that all the expressions to the left mean composition, not decomposition. This is inconsistent.

Trying to prove the PIT in order to disprove rigorously proven theorem like the PDR, one has to break many other laws, rules and conventions. The equation (5) is an example of what might be called "collage mathematics". The purported differential **T**ds has not really been calculated from any function of the pointing vector **r**(). Hence it is just functional of some unspecified function and as such is ineligible for being differentiated. It is usually copied from a book on differential geometry and pasted into the expression d**r** = **T**ds pretending to be genuinely obtained differential.

If the function is defined as **r**() = r**R** where r = |**r**| is the length/magnitude/value of the vector **r** and **R** is unit vector pointing in the direction of the vector **r**, then differentiation respecting the PDR gives d**r**() = rd**R**()+**R**dr() where the parentheses following name of variable indicate that the very variable maintains the status of a varying function, whereas absence of parentheses indicates that the variable has status of functional = value of its (certain but implied) function. Since d**R**() = **T**() we get d**r**() = r**T**()+**R**dr(). One can see that, in general, so-defined pointing vector function **r**() does not really yield **T**ds as its legitimate differential regardless of whether we consider the unit versor **T** as just functional or as function **T**(). The distinction is important for differentiations and integrations.

Moreover, because the vector functions d**R**() and **R** are orthogonal/perpendicular, hence also the vector function d**r**() must have certain nonradial components in addition to the purely radial ones.

Because the expression d**r** = **T**ds does not emerge naturally from legitimate operation, for it to exist (in physical or just abstract mathematical reality) its existence had to be postulated. But postulated objects exist only on paper or in mind. To exist in reality, an object has to be either defined (which only applies to simple objects represented by variables determined by single numbers), or it should arise from proper operations. The operational definition applies to compound objects. To presume that postulated objects could somehow attain an actual/constructible existence (even in the abstract mathematical reality) is a conceptual mistake. Usually the expression d**r** = **T**ds implicitly assigns conventional meanings to its variables. In differential geometry by traditionally accepted common convention the variable **r** denotes the radius pointing to the path s and **T** is perceived as unit vector tangent to the path. The equation d**r** = **T**ds is wrong even though it is used in differential geometry too. With such illegitimately obtained equations one could "prove" anything, not only the PIT.

To see it more clearly let us post the inverse question: Under what conditions could the equation d**r** = **T**ds be valid? From second Frenet-Serret formula (Struik, 1988, p.18) one might get the following condition:

$$d\mathbf{N}(\quad) = -\kappa\mathbf{T}ds + \tau\mathbf{B}ds \Rightarrow rd\mathbf{N}(\quad) = -\mathbf{T}ds \Rightarrow \int rd\mathbf{N}(\quad) = -\int \mathbf{T}ds \Rightarrow r\mathbf{N}$$

$$= -s\mathbf{T} \Rightarrow ||r|| = ||s||$$

where torsion τ = 0 and curvature κ = 1/r. And from first Frenet-Serret formula one might obtain

$$d\mathbf{T}(\quad) = \kappa\mathbf{N}ds \Rightarrow rd\mathbf{T}(\quad) = \mathbf{N}ds \Rightarrow \int rd\mathbf{T}(\quad) = \int \mathbf{N}ds \Rightarrow r\mathbf{T}$$

$$= s\mathbf{N} \Rightarrow |r| = |s|$$

where **T**, **N, B** are the tangent, principal normal, and binormal unit vectors, respectively, of a 3D moving Frenet frame (trihedron), κ is primary curvature and τ is torsion, i.e. the second curvature that always stands perpendicular to the primary one (**N**) and to the tangent unit vector as well (Struik, 1988, p.18).

Yet such thinking rigs the proof of the PIT by making tacit assumption to the effect that no other than radial (hence conservative) forces are admitted whereas nonradial ones are just disregarded. The differential equation d**r** = **T**ds is just manipulation disguised as lawful operation. It is evident that to disprove/ waive the proven PDR theorem by proving the PIT instead, one has to cheat a lot.

Posting differential equation without differentiating their functions is just fraud or a guess at best. Manipulations virtually redefine the actual reality into one that does not exist outside one's mind.

Wherever there is an interaction involving two or more objects, no postulative definitions should be employed, and the interacting objects should be defined upon the same vector/coordinate basis.

4. RESOLUTION/DECOMPOSITION OF THE POINTING VECTOR WITHIN A MOVING 3D TRIHEDRON

If there were no local source, such as our Sun, of the locally dominant gravitational field to disturb trajectories passing near it, then photon emitted from a distant star would follow in Earth's vicinity nearly straightlinear path towards it. Hence the force field vector of the Sun's gravitational field and the radial pointing vector may be assumed as being originally two-dimensional (2D), as they operate in the plane formed by the Sun's gravity center and the path. Thence in order to make any two interacting vector functions entirely comparable we should cast them both in the 3D trihedron which moves with the photon and is fixed to it so that it does not move relative to the photon ever.

The moving trihedron comprises three fixed unit vectors/versors $(\mathbf{e}_{\parallel}, \mathbf{e}_{\perp}, \mathbf{e}_{\vdash})$ which form the base of our reference frame. If the trihedron is lefthanded, and the versor is originally parallel (and after that always tangent) to the (eventually curvilinear) trajectory path of the photon, then we get:

$$d\mathbf{e}_{\parallel} = \mathbf{e}_{\perp} \text{ and } d\mathbf{e}_{\perp} = \mathbf{e}_{\vdash} \text{where } \mathbf{e}_{\vdash} = \mathbf{e}_{\parallel} \times \mathbf{e}_{\perp} \tag{7}$$

which correspond to the (geometrically) natural order of twisting of (arms of) the trihedron.

The local gravitational field affects forces and energy, but its impact on a moving particle is being determined at the point of impact (i.e. where the particle is), which is pointed to by the vector **r**:

$$\mathbf{r}():=\ \mathbf{r}\cos\left(\mathbf{r_p},\mathbf{r}\right)+\mathbf{r}\sin\left(\mathbf{r_p},\mathbf{r}\right)\ =\ \mathrm{r}\cos\alpha\mathbf{e}_{\perp}+\mathrm{r}\sin\alpha\mathbf{e}_{\parallel} \tag{8}$$

where r = |**r**| is length of the pointing/radius vector. The angle $\alpha\ =\ \measuredangle(\mathbf{r_p},\mathbf{r})\ =\ 0$ at the perihelion and tends to ±90° as the particle's trajectory extends away from the perihelion (Czajko, 2011). The constant perihelion vector $\mathbf{r_p}$ is fixed by the "time of day", because it shadows the elapsing time every day.

The constant radius $\mathbf{r}_p$ points to perihelion of the moving particle and **r** is a variable radius vector pointing to generic point along the path. The substitution : = is also an implicit definition. Active (operationally) functions usually may be identified by following them parentheses, to distinguish them from operationally inactive (but only during given differentiation or integration) functionals.

Differentiation of the equation (8) while applying the PDR to compound functions (in brackets) yields

$$\begin{aligned}\mathrm{d}\mathbf{r}\ &=\ \mathrm{d}[\mathrm{r}\cos\alpha\mathbf{e}_{\perp}]+\mathrm{d}[\mathrm{r}\sin\alpha\mathbf{e}_{\parallel}]=\ \mathrm{d}[\mathrm{r}\cos\alpha]\mathbf{e}_{\perp}\\&+\mathrm{r}\cos\alpha\mathrm{d}\mathbf{e}_{\perp}+\mathrm{d}[\mathrm{r}\sin\alpha]\mathbf{e}_{\parallel}+\mathrm{r}\sin\alpha\mathrm{d}\mathbf{e}_{\parallel}\end{aligned} \tag{9}$$

and then applying the PDR again to still compound functions (in brackets on the RHS) gives

$$\begin{aligned}\mathrm{d}\mathbf{r}\ &=\ \{\cos\alpha\mathbf{e}_{\perp}\mathrm{dr}-\mathrm{r}\sin\alpha\mathbf{e}_{\perp}\mathrm{d}\alpha\}+\mathrm{r}\cos\alpha\mathrm{d}\mathbf{e}_{\perp}\\&+\{\sin\alpha\mathbf{e}_{\parallel}\mathrm{dr}+\mathrm{r}\cos\alpha\mathbf{e}_{\parallel}\mathrm{d}\alpha\}+\mathrm{r}\sin\alpha\mathrm{d}\mathbf{e}_{\parallel}\end{aligned} \tag{10}$$

and after substituting differentials of the versors according to the equations (7) we finally obtain:

$$\begin{aligned}\mathrm{d}\mathbf{r}\ &=\ \cos\alpha\mathbf{e}_{\perp}\mathrm{dr}-\mathrm{r}\sin\alpha\mathbf{e}_{\perp}\mathrm{d}\alpha+\mathrm{r}\cos\alpha\mathbf{e}_{\vdash}\\&+\sin\alpha\mathbf{e}_{\parallel}\mathrm{dr}+\mathrm{r}\cos\alpha\mathbf{e}_{\parallel}\mathrm{d}\alpha+\mathrm{r}\sin\alpha\mathbf{e}_{\perp}\end{aligned} \tag{11}$$

where besides the vector components parallel/tangential and normal/perpendicular to the path, the displacement vector gained vector in direction $\vdash$ perpendicular to both the other components (Czajko, 2011).

5. NONROTATIONAL CENTER-BOUND FORCE FIELDS CAN INDUCE ROTATIONAL EFFECTS

Although vectors in force fields defined as gradient of generic scalar potential function (**F = -**∇V) are termed irrotational (Corben & Stehle, 1950, p.37), such force fields can induce rotations in particles moving within them.

When an airplane needs to make a turn in flight, it has to lean on the wing pointing in the direction of the attempted turn. This tilts its tail, however. What was supposed to be just a planar turn (in the horizontal plane fixed to the flying airplane) has an unintended auxiliary angular effect in the third (vertical) dimension signified by the tail. The airplane effectively behaves just like a screw which may be surprising for planar turn. In this abstract mathematical analysis of the turning airplane we can ignore the impact of gravity on its altered vertical position when its tail is changing its angle in the third/vertical direction. Since vectors are entities comprising line segment and angle, changing one alters the whole entity. In fact, similar behavior is exhibited also by fast moving motorcycles.

When road turns, for example, the motorcycle rider must lean towards the incoming curve in order to stay on the road. Most highways designed for high speeds are tilted toward the incoming curve so that drivers need not react to the road's curving because the tilt automatically enforces the twist they need to accommodate the incoming curve without reducing their speed, within the speed limit allowed on the road. Every curvilinear motion proceeds as if the moving object were screw (Czajko, 2011).

Based upon these analogies, one can imagine that when ray coming from a distant star is deflected by our Sun (while passing near Sun on its way to Earth), it has to exhibit the nonradial twist imposed by the—radial by its definition—deflection. Since nothing in the physical world comes for free, the twist drains the ray's energy, which is the currency exchanged in all physical interactions, at the expense of work done by the gravitational field of our Sun on the ray (Czajko, 2000).

6. NEW, MATHEMATICALLY COMPLETE DEFINITION OF WORK DONE

For an originally straightlinear or planar trajectory, normal and tangential subcomponents of the force field vector determined at each point (or pointed to) by the radius vector $\mathbf{r}$ can be written as

$$\begin{aligned}\mathbf{F}(\quad) &:= \mathbf{F_N} + \mathbf{F_T} = \mathbf{F}_\perp + \mathbf{F}_\parallel = \mathbf{F}\cos(\mathbf{r}, \mathbf{r_p}) \\ &+\mathbf{F}\sin(\mathbf{r}, \mathbf{r_p}) = -\mathrm{F}\cos\alpha\mathbf{e}_\perp + \mathrm{F}\sin\alpha\mathbf{e}_\parallel\end{aligned} \tag{12}$$

respectively. Since magnitudes of inwards acting forces decrease outwards, the force must have minus sign. Also unlike the radius, which covaries with the angle, the force field vector diminishes when the angle widens farther away from the perihelion. Hence negative angle must be taken for the force vector to keep it on par value with the radius. Recall that cos(-α) = cosα but sin(-α) = -sinα.

Since representation of the force field vector function given by equation (12) is identical in form to that of the pointing vector in equation (8), the rate of change of the force field vector function is analogous:

$$\begin{aligned}\mathrm{d}\mathbf{F} &= -\cos\alpha\mathbf{e}_\perp\mathrm{dF} + \mathrm{F}\sin\alpha\mathbf{e}_\perp\mathrm{d}\alpha - \mathrm{F}\cos\alpha\mathbf{e}_\vdash \\ &+\sin\alpha\mathbf{e}_\parallel\mathrm{dF} + \mathrm{F}\cos\alpha\mathbf{e}_\parallel\mathrm{d}\alpha + \mathrm{F}\sin\alpha\mathbf{e}_\perp\end{aligned} \tag{13}$$

and from the equations (11) and (12) follows the usual, radially acquired part, of the rate of work done

$$\mathbf{F} \bullet \mathrm{d}\mathbf{r} = -\mathrm{Fcos}2\alpha\mathrm{dr} + \mathrm{Frsin}2\alpha\mathrm{d}\alpha - \frac{1}{2}\mathrm{Frsin}2\alpha \qquad (14)$$

and the equations (8) and (13) yield the formerly missing, nonradial part of the rate of work done

$$\mathbf{r} \bullet \mathrm{d}\mathbf{F} = -\mathrm{rcos}2\alpha\mathrm{dF} + \mathrm{Frsin}2\alpha\mathrm{d}\alpha + \frac{1}{2}\mathrm{Frsin}2\alpha \qquad (15)$$

and from the equations (14) and (15) we obtain the new, mathematically complete rate of work done:

$$\mathrm{dW}() = \mathbf{F} \bullet \mathrm{d}\mathbf{r} + \mathbf{r} \bullet \mathrm{d}\mathbf{F} = -\mathrm{Fcos}2\alpha\mathrm{dr} + 2\mathrm{Frsin}2\alpha\mathrm{d}\alpha - \mathrm{rcos}2\alpha\mathrm{dF} \qquad (16)$$

which indicates that work done involves more energy than previously admitted. The extra angular part dα (in the total rate of work done) emerged autonomously/unpostulated from operations (Czajko, 2011).

For rays coming from a distant star (and is then intercepted on the Earth) the doubled angle means one-sided equipotential trip of up to 2×90° = 180° around our Sun's surface. For conservative/radial (hence center-bound paths with outgoing α = 0° and incoming α = 180°) the angular part vanishes.

The equation (16) is only mathematically complete, however, for its physical completeness should also include characteristics of the field (Czajko, 2000). Notice that the positive term, which is opposite to the two regular/negative work done components, contributes to the decrease of energy in the rays that was observed in the Sadeh experiments (Czajko, 2000; Sadeh, Knowles, & Yaplee, 1968; Sadeh, Knowles, & Au, 1968). This happens on equipotential surfaces where the magnitudes of force and radius vectors are constant and only angle changes along the surface.

If net force would act outwards (making thus the term—Fcos2αdr positive) then the rays' radial energy would decrease too, as it happens when rays go away from the local center of gravity (Beiser, 1973, p.67).

7. IMPACT OF THE NEW, MATHEMATICALLY COMPLETE DEFINITION OF WORK DONE ON PHYSICS

The extra two contributions to the rate of work done by center-bound force fields are considerably smaller than the radial contribution to energy. Yet with the exception of an initial formation of a star or planet, for all the already formed celestial bodies their mass increase (or decrease, for that matter) does not really start from zero-mass. The mass can change also with its varying relative speed and accelerations. Moreover, the diameter of our Sun increases from time to time too, which coincides with higher levels of solar activity (Basu, 1998). This could affect the nonradial effects within its field (Gigolashvili, Gogoladze, & Khutsishvili, 1995), because they depend on angular distances measured along equipotential surfaces (Czajko, 2000).

Yet even such a small discrepancy is conceptually important, because it codetermines also other factors. For happenings in vicinity of large massive bodies like our Sun it is both operationally and numerically significant, because when the total exchange of energy within its force field can be determined very precisely (as is the case with radio waves and rays whose frequency/ energy can be fairly accurately measured), one gets over 25% discrepancy in solar spectra (Czajko, 2000). The nonradial effects retrodicted the observed excess over Einstein's "flagship" prediction of deflection of light (Merat, 1974) and retrodicted frequency decreases found in the aforesaid Sadeh experiments (with rays from Taurus A and in radio waves triggered by atomic clocks (Czajko, 2000)), which effectively defeated the PIT.

Since $\cos(\mathbf{r},\mathbf{r_p}) = \mathbf{r}_p/\mathbf{r}$ in the equation (12) then the force vector that is tangent to the path is expressed as

$$\begin{aligned} \mathbf{F_T} &= \mathbf{F}_{\|} = -\mathbf{F}\sin(\mathbf{r},\mathbf{r_p}) = -\mathbf{F}\sqrt{1-\left(\cos(\mathbf{r},\mathbf{r_p})\right)^2} \\ &= -\mathbf{F}\sqrt{1-\left(\frac{\mathbf{r}_p}{\mathbf{r}}\right)^2} = -\mathbf{F}\sqrt{1-\left(\frac{\boldsymbol{\lambda}}{\boldsymbol{\lambda}_p}\right)^2} \end{aligned} \tag{17}$$

with inverse radius of perihelion $\lambda_p = 1/\mathbf{r}_p$ limiting distances like relativistic contraction factor does.

8. CONSERVATION VERSUS RETRIEVAL OF POTENTIAL ENERGY IN CENTER-BOUND FORCE FIELDS

Angle-dependent contributions to potential energy in rotational motion are known (Jaeger, 1951, p.251). And work done in rotary motion was defined as $W = Fr\Delta\theta$ (Williams, Franklin, & Metcalfe, 1984, p.118). Also for energy transport in plasma contained in very strong magnetic field slowly varying in both space and time one finds components parallel and perpendicular to the field (Johnson, 1999). But the PIT precluded nonradial effects in nonrotating fields.

Although total energy within center-bound fields is conserved (Lerner, & Trigg, 2005, p.720), only radial components of force vectors are conservative in the sense that energy given by photon running away from the field's center can be recovered if the photon reverses its path and starts approaching the center.

But energy acquired by the field from photon traveling along equipotential surfaces cannot be recovered by reversing the photon's path, because path is not active/differentiated variable in the $r\bullet d\mathbf{F}$ term. Energy lost by photon suffering nonradial frequency decrease is assigned to the locally dominant field whose source curves the space most at the given point on the equipotential surface.

Hypothetically however, if a photon would somehow keep on acquiring mass and eventually grew bigger than the Sun, it would then curve the local space towards itself and consequently could start sucking energy from the Sun, because the overgrown mass of what was initially the photon would become dominant source of the local field and thereby the effective force (codetermined by mass) would become the active variable $d\mathbf{F}$ in the process of energy acquisition by the oversized photon.

Energy is always interchangeable, but in radial part of work done (**F**•d**r**) the active displacement d**r** of radius drives its exchange, whereas in the formerly ignored nonradial part (**r**•d**F**) its energy exchange is driven by change d**F** to the active force for each value of the pointing vector, which is inactive/functional there. Only active/differentiated functions do act; inactive/functionals do not.

Since former rate of work done was incompletely defined, previous interaction energy could never be conserved. Schrödinger equation, whose inefficiency for atoms with compound nuclei is due to the incompleteness of the potential energy that feeds it, attests to that. For energy to be conserved, one must define it (and work done) completely, at least mathematically. And to compare energies exchanged in different fields, the definition of energy must be physically complete too (Czajko, 2000).

9. PATH INDEPENDENCE OF WORK DONE THEOREM IMPAIRED GENERAL THEORY OF RELATIVITY

Though Einstein made significant changes to Newtonian mechanics, he did not alter the unspoken radial-only paradigm that permeated theories of former physics. He has specifically omitted any deviations [from radial gravity] due to some [possible, at least theoretically] tangential effects, because they would be too slight if measured on Earth's surface (Einstein, 1916, p.161). Apparently the PIT impaired Einstein who saw nonradial = tangential/binormal effects as just insignificant local aberration from the purely radial ones, even though neglecting nonradial/equipotential phenomena spoiled predictions of many not exclusively radial relativistic experiments (Czajko, 2000; Merat, 1974; Dyson, 1921; Czajko, 1990; Czajko, 1991).

Since notion of absolute teleparallelism in a Riemannian manifold is independent of the idea of metric (Cartan, 1984a), the teleparallelism of Levi-Civita permits one to assume that in infinitesimally small surrounding no significant discrepancy between two tangent planes will arise (Cartan, 1984b), because angles of two parallel tangents are preserved and the metric ds^2 remains invariant there (Cartan, 1984c).

A curvature tensor emerges autonomously from the vectorial change that shifted vectors undergo upon infinitesimal parallel translation (Einstein, 1948) as a result of the operation. And the 4D curvature arising from Einstein's gravitational law in vacuum is of the same geometric nature as rotational curvature of a normal, generalized conformal space (Cartan, 1986, p.199).

Despite such astounding mathematical advantages of general theory of relativity, however, due to the Einstein's deliberate omission/neglect of tangential/binormal/equipotential/nonradial effects, it must be complemented by the experimentally confirmed theory of nonradial/equipotential effects of gravity (Czajko, 2000) whose presence is mathematically unavoidable (Czajko, 2011).

10. CONCLUSION

Traditional definition of rates of work done has violated the mandatory product differentiation rule causing its incompleteness, which invalidates path independence theorem of work done. It ignored angular potential energy exchanged along equipotential surfaces within center-bound force fields.

All theories that rely on the invalid theorem are deficient for nonradial/ tangential/binormal effects, even though they can be used for purely radial phenomena. New mathematically complete formula for calculating rates of work done and potential energy in center-bound force fields was derived.

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