CSCanada
**Studies in Mathematical Sciences**
Vol. 3, No. 2, 2011, pp. 16-20
**DOI:** 10.3968/j.sms.1923845220110302.205

ISSN 1923-8444 [Print]
ISSN 1923-8452 [Online]
www.cscanada.net
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# Exact Travelling Wave Solutions of Zakharov-Kuznetsov(ZK) Equation by the First Integral Method

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Received 18 September, 2011; accepted 26 November, 2011

## Abstract
The first integral method is an efficient method for obtaining exact solutions of nonlinear partial differential equations. The aim of this letter is to find exact solutions of the Zakharov-Kuznetsov(ZK) equation by the first integral method.

## Key words
First integral method; Exact solution; ZK equation

A. Asaraai, M. B. Mehrlatifan, S. Khaleghizadeh (2011). Exact Travelling wave solutions of Zakharov-Kuznetsov(ZK) equation by the First Integral Method. *Studies in Mathematical Sciences*, *3*(2), 16-20. Available from: URL: http://www.cscanada.net/index.php/sms/article/view/j.sms.1923845220110302.205
DOI: http://dx.doi.org/10.3968/j.sms.1923845220110302.205

## 1. INTRODUCTION

Nonlinear evolution equations have a major role in various scientific and engineering fields, such as fluid mechanics, plasma physics, optical fibers, solid state physics, chemical kinematics, chemical physics and geochemistry. Nonlinear wave phenomena of dispersion, dissipation, diffusion, reaction and convection are very important in nonlinear wave equations. In recent years, quite a few methods for obtaining explicit traveling and solitary wave solutions of nonlinear evolution equations have been proposed. A variety of powerful methods, such as, tanh-sech method [1, 2, 3],extended tanh method [4, 5, 6], hyperbolic function method [7], sine-cosine method [8, 9, 10], Jacobi elliptic function expansion method [11], F-expansion method [12] ,and the first integral method [13, 14]. The first integral method was first proposed by Feng [13] in solving Burgers-KdV equation which is based on the ring theory of commutative algebra.The Zakharov-Kuznetsov(ZK) equation is in the following form:

$$u_t + auu_x + b(u_{xx} + u_{yy})_x = 0,$$

where a, b and c are real constants. Wazwaz in [15] applied the extended tanh method to obtain exact solutions of the generalized Zakharov-Kuznetsov (gZK) equation in the form

$$u_t + au^n u_{\{x\}} + b(u_{xx} + u_{yy})_x = 0, \quad n > 1. \tag{1}$$

If $n = 2$ the Eq.(1) becomes

$$u_t + au^2 u_x + b(u_{xx} + u_{yy})_x = 0. \tag{2}$$

The ZK equation, presented in [15], governs the behavior of weakly nonlinear ion-acoustic waves in a plasma comprising cold ions and hot isothermal electrons in the presence of a uniform magnetic field [16, 17]. The ZK equation, which is a more isotropic two-dimensional, was first derived for describing weakly nonlinear ion-acoustic waves in a strongly magnetized lossless plasma in two dimensions [15]. The aim of this paper is to find exact solutions of Eq.(2).

## 2. FIRST INTEGRAL METHOD

Consider the nonlinear partial differential equation in the form

$$F(u, u_x, u_y, u_t, u_{xx}, u_{xy}, \cdots) = 0, \tag{3}$$

where $u = u(x, y, t)$ is the solution of nonlinear partial differential equation Eq.(3). We use the transformations,

$$u(x, y, t) = f(\xi), \tag{4}$$

where $\xi = x + y - st$. This enables us to use the following changes:

$$\frac{\partial}{\partial t}(.) = -s\frac{\partial}{\partial \xi}(.), \quad \frac{\partial}{\partial x}(.) = \frac{\partial}{\partial \xi}(.), \quad \frac{\partial^2}{\partial x^2}(.) = \frac{\partial^2}{\partial \xi^2}(.), \cdots \tag{5}$$

Using (5) to transfer the nonlinear partial differential equation Eq.(3) to nonlinear ordinary differential equation

$$G(f(\xi), f_\xi(\xi), f_{\xi\xi}(\xi), \cdots) = 0 \tag{6}$$

Next, we introduce a new independent variable

$$X(\xi) = f(\xi), \quad Y = \frac{\partial f(\xi)}{\partial \xi}. \tag{7}$$

which leads a system of nonlinear ordinary differential equations

$$\begin{aligned} X_\xi(\xi) &= Y(\xi), \\ Y_\xi(\xi) &= F_1(X(\xi), Y(\xi)). \end{aligned} \tag{8}$$

By the qualitative theory of ordinary differential equations [13] ,if we can find the integrals to Eq.(8) under the same conditions, then the general solutions to Eq.(8) can be solved directly. However, in general, it is really difficult for us to realize this even for one first integral, because for a given plane autonomous system, there is no systematic theory that can tell us how to find its first integrals, nor is there a logical way for telling us what these first integrals are. We will apply the Division Theorem to obtain one first integral to Eq.(8) which reduces Eq.(6) to a first order integrable ordinary differential equation. An exact solution to Eq.(3) is then obtained by solving this equation. Now, let us recall the Division Theorem:

Division Theorem:

Suppose that $P(w, z)$ and $Q(w, z)$ are polynomials in $C[w, z]$ and $P(w, z)$ is irreducible to $C[w, z]$. If $Q(w, z)$ vanishes through all zero points of $P(w, z)$, then there exists a polynomial $F_2(w, z)$ in $C(w, z)$ such that

$$q(w, z) = P(w, z)F_2(w, z).$$

## 3. EXACT SOLUTIONS OF ZK EQUATION

In this section we study the ZK equation in the form

$$u_t + au^2u_x + b(u_{xx} + u_{yy}) = 0. \tag{9}$$

By make the transformation $u(x, y, t) = f(\xi), \xi = x + y - st$, the Eq.(9) becomes

$$-s\frac{\partial f(\xi)}{\partial \xi} + a(f(\xi))^2\frac{\partial f(\xi)}{\partial \xi} + b\frac{\partial}{\partial \xi}(\frac{\partial^2 f(\xi)}{\partial \xi^2} + \frac{\partial^2 f(\xi)}{\partial \xi^2}) = 0, \tag{10}$$

by integrating Eq.(10) and neglecting the constant of integration we obtain

$$-sf(\xi) + \frac{a}{3}(f(\xi))^3 + 2b\frac{\partial^2 f(\xi)}{\partial \xi^2} = 0. \tag{11}$$

Using (7) we get

$$\dot{X}(\xi) = Y(\xi), \tag{12}$$

$$\dot{Y}(\xi) = \frac{s}{2b}X(\xi) - \frac{a}{6b}(X(\xi))^3. \tag{13}$$

According to the first integral method, we suppose the $X(\xi)$ and $Y(\xi)$, are the nontrivial solutions of (12) and (13) also

$$q(X, Y) = \sum_{i=0}^{N} a_i(X)Y^i = 0,$$

is an irreducible polynomial in the complex domain C(X, Y), such that

$$q(X(\xi), Y(\xi)) = \sum_{i=0}^{N} a_i(X(\xi))Y(\xi)^i = 0, \tag{14}$$

where $a_i(X)(i = 0, 1, \ldots, N)$, are polynomials of X and $a_N(X) \neq 0$. Equation (14) is called the first integral to (12), (13). Due to the Division Theorem, there exists a polynomial $g(X) + h(X)Y$, in the complex domain $C(X, Y)$, such that

$$\frac{dq}{d\xi} = \frac{dq}{dX}\frac{dX}{d\xi} + \frac{dq}{dY}\frac{dY}{d\xi} = (g(X) + h(X)Y)\sum_{i=0}^{m} a_i(X)Y^i \tag{15}$$

In this example, we take two different cases, assuming that N = 1, and N = 2, in (14).
**Case A:** Suppose that $N = 1$, by comparing with the coefficients of $Y^i(i = 2, 1, 0)$, of both sides of (15), we have

$$\dot{a}_1(X) = h(X)a_1(X), \tag{16}$$

$$\dot{a}_0(X) = g(X)a_1(X) + h(X)a_0(X), \tag{17}$$

$$a_1(X)[\frac{s}{2b}X(\xi) - \frac{a}{6b}(X(\xi))^3] = g(X)a_0(X). \tag{18}$$

We obtain that $a_1(X)$, is constant and $h(X) = 0$, take $a_1(X) = 1$, and balancing the degrees of $g(X), a_1(X)$ and $a_0(X)$, we conclude that $deg\, g(X) = 1$, only. Suppose that $g(X) = A_1X + B_0$, then we find $a_0(X)$.

$$a_0(X) = A_0 + B_0X + \frac{1}{2}A_1X^2. \tag{19}$$

Substituting $a_0(X)$, $a_1(X)$ and $g(X)$, in the last equation in (18) and setting all the coefficients of $X$ to be zero, then we obtain a system of nonlinear equations and by solving it, we obtain

$$B_0 = 0, \quad A_1 = \pm\sqrt{-\frac{a}{3b}}, \quad S = \pm 2bA_0\sqrt{-\frac{a}{3b}}, \tag{20}$$

where $A_0$ is arbitrary constant.

Using the conditions (20), into Eq.(14), we obtain

$$Y(\xi) = -A_0 \pm \frac{1}{2}\sqrt{-\frac{a}{3b}}(X(\xi))^2 \tag{21}$$

Combining (21) with (12),we obtain the exact solution to equation (12), (13) and exact solutions to Eq.(9) can be written as:

$$u_1(x,y,t) = -\sqrt{\frac{2A_0}{\sqrt{-\frac{a}{3b}}}}tanh[\sqrt{\frac{A_0\sqrt{-\frac{a}{3b}}}{2}}(x+y \pm 2bA_0\sqrt{-\frac{a}{3b}}t+\xi_0)].$$

$$u_2(x,y,t) = -\sqrt{\frac{2A_0}{\sqrt{-\frac{a}{3b}}}}tan[\sqrt{\frac{A_0\sqrt{-\frac{a}{3b}}}{2}}(x+y \pm 2bA_0\sqrt{-\frac{a}{3b}}t+\xi_0)].$$

**Case B:** Suppose that $N = 2$, by equating the coefficients of $Y^i(i = 3, 2, 1, 0)$ on both sides of (15), we have

$$\dot{a}_2(X) = h(X)a_2(X), \tag{22}$$

$$\dot{a}_1(X) = g(X)a_2(X) + h(X)a_1(X), \tag{23}$$

$$\dot{a}_0(X) = -2a_2(X)[\frac{s}{2b}X(\xi) - \frac{a}{6b}(X(\xi))^3)] + g(X)a_1(X) + h(X)a_0(X), \tag{24}$$

$$a_1(X)[\frac{s}{2b}X(\xi) - \frac{a}{6b}(X(\xi))^3] = g(X)a_0(X). \tag{25}$$

We obtain that $a_2(X)$, is constant and $h(X) = 0$, take $a_2(X) = 1$, and balancing the degrees of $g(X)$, $a_0(X)$, and $a_0(X)$, we conclude that $deg\, g(X) = 1$, only. Suppose that $g(X) = A_1X + B_0$, then we find $a_1(X)$ and $a_0(X)$, as

$$a_1(X) = A_0 + B_0X + \frac{1}{2}A_1X^2, \tag{26}$$

$$a_0(X) = d + B_0A_0X + \frac{1}{2}\left(-\frac{s}{b} + B_0^2 + A_0A_1\right)X^2 + \frac{1}{2}A_1B_0X^3 + \frac{1}{4}\left(\frac{a}{3b} + \frac{1}{2}A_1^2\right)X^4. \tag{27}$$

Substituting $a_0(X)$, $a_1(X)$, $a_1(X)$ and $g(X)$, in the last equation in (25) and setting all the coefficients of $X$ to be zero, then we obtain a system of nonlinear equations and by solving with aid Maple, we obtain

$$d = \frac{1}{4}A_0^2, \quad B_0 = 0, \quad A_1 = \pm\frac{2\sqrt{-3ba}}{3b}, \quad S = \pm bA_0\sqrt{-\frac{a}{3b}}, \tag{28}$$

where $A_0$ is arbitrary constant.

Using the conditions (28), into Eq.(14), we obtain

$$Y(\xi) = -\frac{\sqrt{-3ba}(x(\xi))^2 + 3A_0b}{6b}. \tag{29}$$

Combining (29) with (12),we obtain the exact solution to equation (12), (13) and the exact solution to Eq.(9) can be written as:

$$u_3(x,y,t) = -\frac{\sqrt{A_0b\sqrt{-3ba}}}{\sqrt{-ba}}tanh[\frac{\sqrt{-A_0b\sqrt{-3ba}}}{2b\sqrt{3}}(x+y \pm bA_0\sqrt{-\frac{a}{3b}}t+\xi_0)].$$

The exact solutions of the Zakharov-Kuznetsov (ZK) equation, $u_1(x,y,t)$ and $u_3(x,y,t)$ are soliton solutions.

# 4. CONCLUSION

In this paper, the first integral method has been successfully applied to find the solutions for Zakharov-Kuznetsov(ZK) equation.Thus, we can say that the proposed methods can be extended to solve the problems of nonlinear partial differential equations arising in the theory of solitons and other areas.

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