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CONTROL OF A NONLINEAR SYSTEM BY LINEARIZATION

BY

DREW DOUGLAS NELSON
B.S., Pacific Lutheran University, 1981

RESEARCH REPORT

Submitted in partial fulfillment of the requirements
for the degree of Master of Science
in the Graduate Studies Program of the College of Engineering
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Orlando, Florida

Fall Term
1986

ABSTRACT

In today's linear control systems, exact solutions can be obtained by the use of Laplace Transforms in the frequency domain. In dealing with nonlinear systems, exact solutions are not always achievable. For this reason, it is necessary to linearize the system and then apply frequency response methods.

This paper shows the comparison of a nonlinear system with the linearized model of the same system. For both proportional and proportional-integral control, the response to a unit step change in the set point showed minimal difference between the linearized and nonlinear system.

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CHAPTER I

INTRODUCTION

Control Theory had its origins in the 1700s with James Watt's centrifugal governor for the speed control of a steam engine. Due to its proportional-control action, the Watt governor resulted in a static error of engine speed. To eliminate this error, an integral control action was implemented. In this instance, the integral action created an unstable condition. Without modern tools, stable solutions could only be obtained by experimentation and intuition.

In the 1900s, the advent of instruments and regulators for process and power industries created a need for theory to replace intuition in the design of control systems. The use of differential equations and the Routh-Hurwitz stability criteria became more widespread. These applications were still constrained to low-order and simple systems.

Due to World War II, a large interest in weapon position-control developed. This development spurred the subsequent development of the frequency-response and root-locus methods. These two methods form the core of classical control theory. Basic feedback control soon included such problems as sample-data control, random-signal systems and some phenomena caused by system nonlinearities and nonlinear control. Recent developments are geared towards finding optimal control for both deterministic and stochastic systems.

In today's linear control systems, exact solutions can be obtained by using Laplace Transforms. While dealing with non-linear systems, exact solutions are not always achievable. For this reason, it is necessary to linearize the system and then apply frequency response methods. In most practical cases, the main concern is with the stability of the non-linear system. Approximate solutions and stability checks are obtained by applying Laplace Transform techniques to the linearized system.

In this paper, a comparison of a nonlinear system with its linearized model will show how both differ in response to a unit step input. This comparison should show if the linear model fails and if so, by how much. A study of how any failure can be affected will be explored.

This paper will first describe the nonlinear system. The real-life tanks and pumps will be physically described and then modeled to arrive at a nonlinear system. A linearization will be conducted and linear control theory will be applied to arrive at an exact linear solution. A computer simulation will then numerically calculate the actual nonlinear numerical solution. A comparison of the results should show any failures of the linear model.

CHAPTER II

PHYSICAL SYSTEM

The nonlinear system to be controlled is a series of two tanks attached to a controller and pump. A diagram of the tank apparatus is depicted in Figure 2-1.

The two tanks are connected by four holes. Three of these holes are at a height of 3 cm above the base of the tank and can be closed off. Their diameters are 1.27 cm, .95 cm and .635 cm. The other hole, .317 cm in diameter, is at a height of 1.5 cm and remains open at all times. The total cross-sectional area of the four holes, a_1, is the cross sectional area of orifice one. This orifice also has a discharge coefficent of C_{D1}.

The second tank has two input flows, F_2 and F_L. F_2 is the flow of liquid between the two tanks and F_L is a load flow that is supplied from a source outside of the system. The output flow of the second tank, F_0, is through a valve 3 cm from the base of the tank. This valve is an adjustable tap that creates orifice two. Fully open, the diameter of the tap is .70 cm. Orifice two has a discharge coefficient C_{D2} and cross-sectional area a_2.

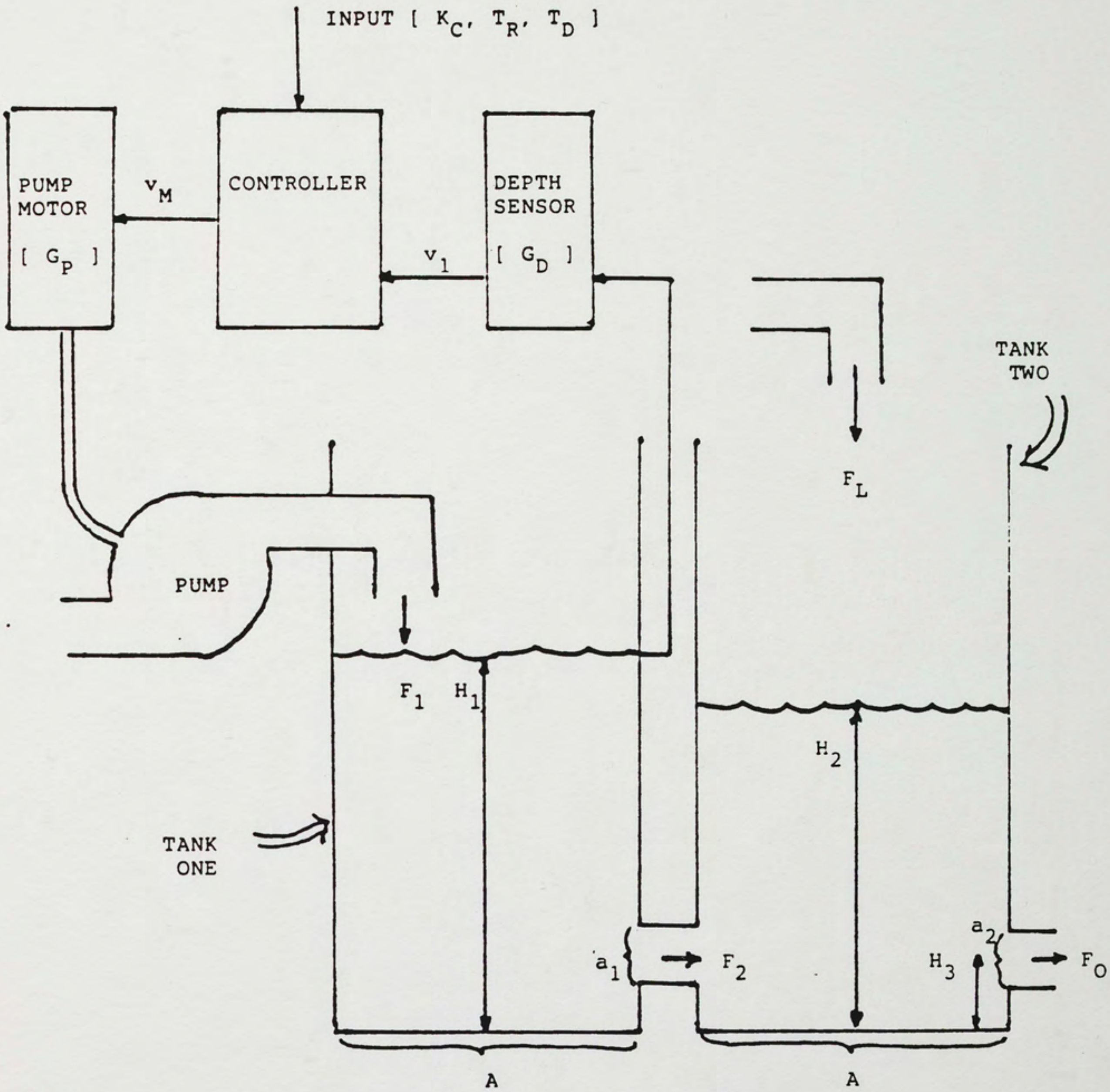


Figure 2-1.Tank Apparatus.

F_1, according to a voltage supplied to its motor, V_M. This voltage is supplied from a controller which receives a voltage input, V_1, from a depth sensor. The depth sensor transforms the variable height of tank one, H_1, into a voltage, V_1. The height of orifice two, H_3, remains fixed at 3 cm.

There are several different types of parameters in the system. Parameters such as a_1, a_2, C_{D1}, C_{D2}, H_3 and the cross-sectional area of each tank, A, (A = 200 cm^2 in this system) are fixed by the physical nature of the system.

Parameters such as the proportional gain, K_C, the reset time, T_R, and the derivative time, T_D, are input into the system. These three control parameters create which type of controller action the system will follow. This paper will only investigate two types of controller action. These two types are proportional control ($K_C = 10$, $T_R = \infty$, $T_D = 0$) and proportional-integral control ($K_C = 10$, $T_R = 10$, $T_D = 0$).

The two remaining parameters are used only in the linearized system. The pump motor constant, G_P, and the depth sensor constant, G_D, both depend on operating conditions. G_P is the small change in pump flow f_1 when a small change in voltage to the motor, v_M, is supplied at original voltage V_m. This is an approximation to the derivative of the relationship between F_1 and V_1 evaluated at some constant V_m. In much the same way, G_D is the small change in voltage to the controller, v_1, when a small change in the height of tank one, h_1, occurs at some original height H_1. The derivation of G_P begins by finding the relationship between V_M and F_1 by applying a least-squares fit polynomial approximation to actual data. From Figure 2-2 this relationship is given by:

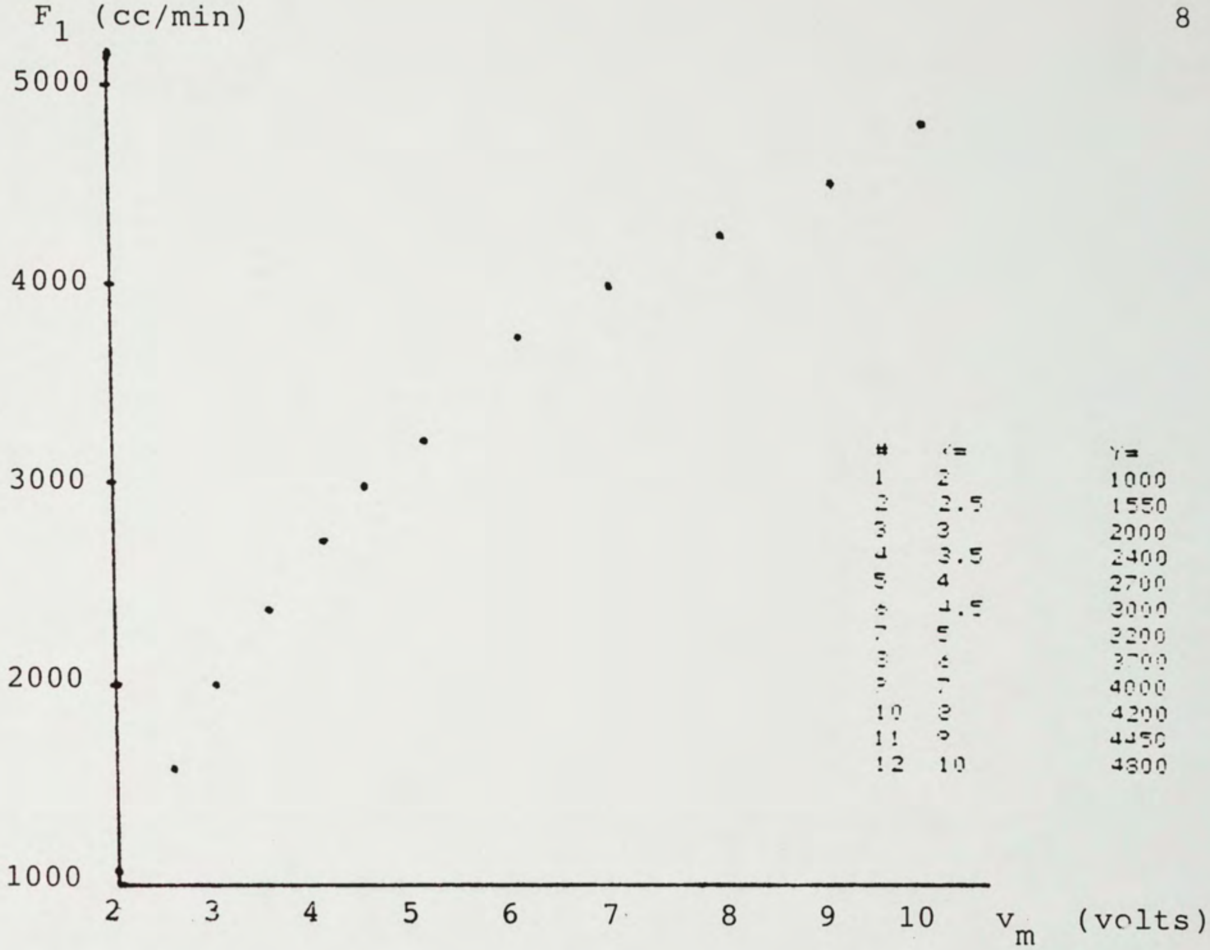
$$F_1 = f(V_M) = \frac{-49.176V_M^2 + 1023.8V_M - 687.28}{60} \qquad \text{(eq. 2-1)}$$

The next step is finding the rate of change by differentiating equation 2-1.

$$\frac{dF_1}{dV_M} = \frac{-98.352V_M + 1023.8}{60} \qquad \text{(eq. 2-2)}$$

G_P is then found by evaluating equation 2-2 at the original voltage $\overline{V}_M$.

$$G_P = \frac{f_1}{v_M} = \left.\frac{dF_1}{dV_M}\right|_{V_M = \overline{V}_M} = -1.6392\overline{V}_M + 17.063 \qquad \text{(eq. 2-3)}$$

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SATISFACTORY(Y:N)? <Y>
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Figure 2-2.Relationship of V_M and F_1.

To derive G_D, the same procedure is followed. First, the relationship between H_1 and V_1 must be found by applying a least-squares fit polynomial approximation to actual data. From Figure 2-3, this equation is given by equation 2-4.

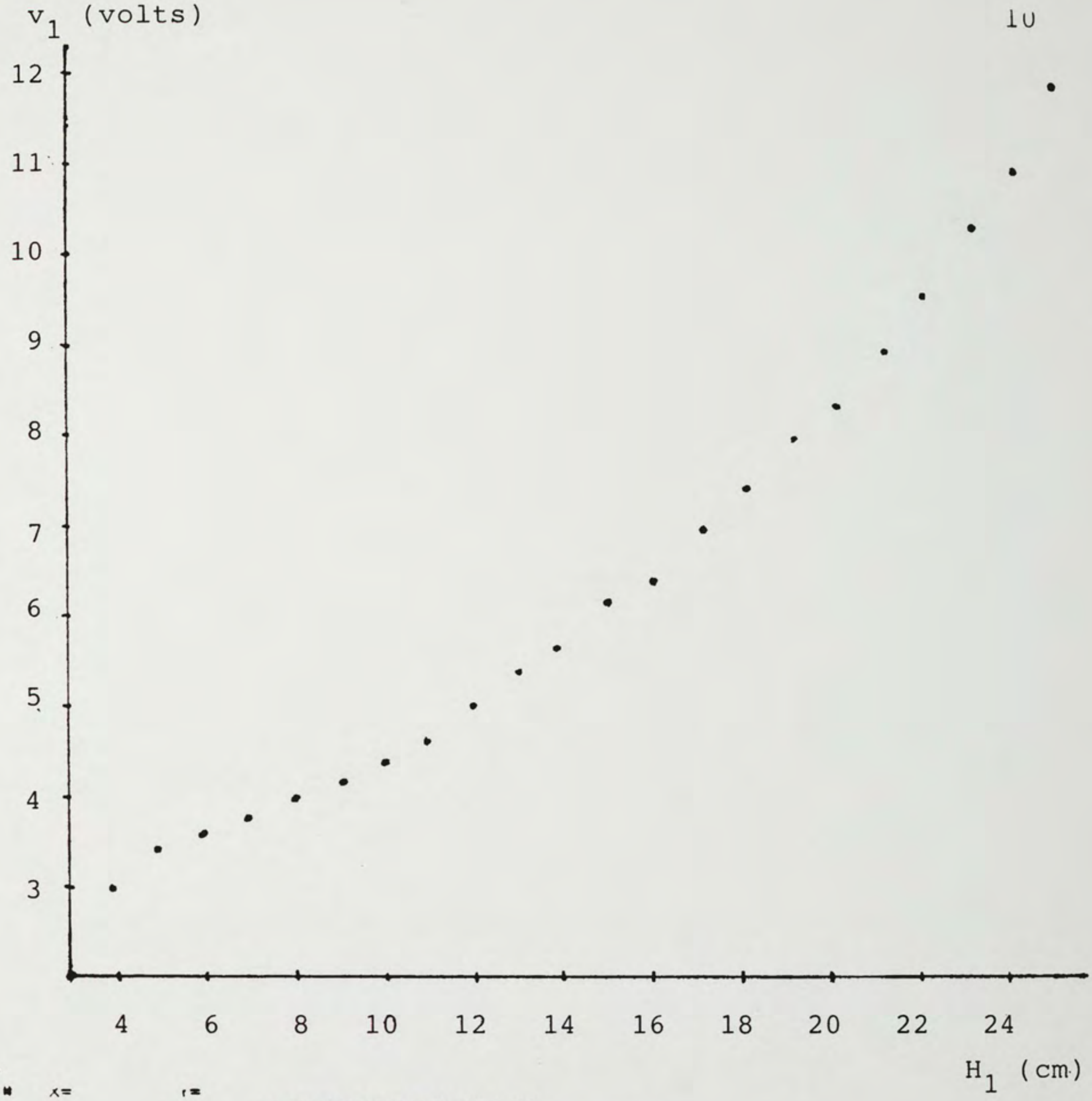
$$V_1 = f(H_1) = .00081H_1^3 - .02214H_1^2 + .47795H_1 + 1.1766 \qquad \text{(eq. 2-4)}$$

Differentiating equation 2-4 yields equation 2-5.

$$\frac{dV_1}{dH_1} = .00243H_1^2 - .04428H_1 + .47795 \qquad \text{(eq. 2-5)}$$

And then evaluating equation 2-5 at $\bar{H}_1$ gives equation 2-6.

$$G_D = \frac{v_1}{h_1} = \left.\frac{dV_1}{dH_1}\right|_{H_1=\bar{H}_1} = .00243\bar{H}_1^2 - .04428\bar{H}_1 + .47795 \qquad \text{(eq. 2-6)}$$



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2	4	3
3	5	[illegible]
4	6	3.6
5	7	3.8
6	8	4
7	9	4.2
8	10	4.4
9	11	4.8
10	12	5
11	13	5.4
12	14	5.6
13	15	6.2
14	16	6.4
15	17	7
16	18	7.4
17	19	8
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Figure 2-3. Relationship of H_1 and V_1.

CHAPTER III

SYSTEM MODELING

Since the system has already been described physically, it will now be described with dynamic equations. The first equations will be those that describe the time rate of change of the heights, H_1 and H_2, in terms of the flow variables, F_1, F_2, F_L and F_O, and physical parameters. Equations for the flows, F_O and F_2, in terms of physical parameters and height variables, H_1 and H_2, will be next derived. Finally, the steady state relationship for H_2 in terms of H_1 and F_L will be found. Assuming the system is initially in steady state, this will enable the initial height of tank two to be found from the height in tank one and the load variable. For this model, orifice one shall have all but the largest hole open ($a_1 = 1.109$) and the tap on orifice two will be wide open ($a_2 = .384$). No load variable will be applied ($F_L = 0$) and tank one is initially at a height of 10 cm ($H_1 = 10$). With these two sets of equations, the goal of comparing linear and non-linear solutions can be met. The time rate of change of H_1 and H_2 is found by deriving $\frac{dH_1}{dt}$ and $\frac{dH_2}{dt}$.

The rate of change of volume of a tank is equal to the rate of volume into the tank minus the rate of volume out of the tank. This implies the rate of change in the volume of tank one (V_1) is described by equation 3-1.

$$\frac{dV_1}{dt} = F_1 - F_2 \qquad \text{(eq. 3-1)}$$

In the same manner the rate of change in the volume of tank two (V_2) is given by equation 3-2.

$$\frac{dV_2}{dt} = (F_2 + F_L) - F_O \qquad \text{(eq. 3-2)}$$

Since the volume of a rectangular tank is the area times the height of the tank, equations 3-3 and 3-4 hold for this example.

$$V_1 = A*H_1 \qquad \text{(eq. 3-3)}$$

$$V_2 = A*H_2 \qquad \text{(eq. 3-4)}$$

Since the area A is constant, equations 3-5 and 3-6 apply.

$$\frac{dV_1}{dt} = A*\frac{dH_1}{dt} \qquad \text{(eq. 3-5)}$$

$$\frac{dV_2}{dt} = A*\frac{dH_2}{dt} \qquad \text{(eq. 3-6)}$$

By substituting equation 3-5 into equation 3-1 and solving for the rate of change of height one, equation 3-7 holds.

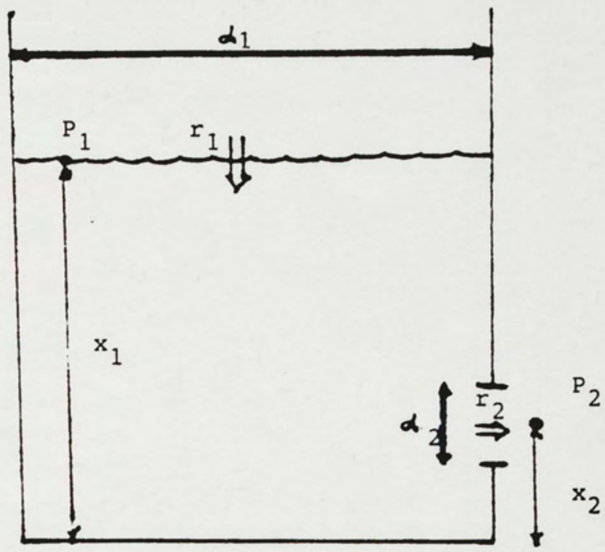
$$\frac{dH_1}{dt} = \frac{F_1 - F_2}{A} \qquad \text{(eq. 3-7)}$$

Following the same procedure, the rate of change in height two is given by substituting equation 3-6 into equation 3-2 and is shown by equation 3-8.

$$\frac{dH_2}{dt} = \frac{F_2 + F_L - F_O}{A} \qquad \text{(eq. 3-8)}$$

To find F_1 and F_2 in terms of height variables and physical parameters, several steps are completed.

First, Bernoulli's Theorem states that for Figure 3-1, equations 3-9 and 3-10 hold.



WHERE P = PRESSURE

x = HEIGHT

r = RATE OF FLUID FLOW

ρ = DENSITY OF FLUID

α = CROSS-SECTIONAL AREA

Figure 3-1. Single Tank With Orifice.

$$P_1 + \rho g X_1 + \tfrac{1}{2} \rho r_1{}^2 = P_2 + \rho g X_2 + \tfrac{1}{2} \rho r_2{}^2 \quad \text{(eq. 3-9)}$$

$$P_2 = P_1 + \rho g(X_1 - X_2) \quad \text{(eq. 3-10)}$$

By the equation of continuity, equation 3-11 yields the relationship between rates of flow and cross-sectional areas.

$$\alpha_1 * r_1 = \alpha_2 * r_2 \quad \text{(eq. 3-11)}$$

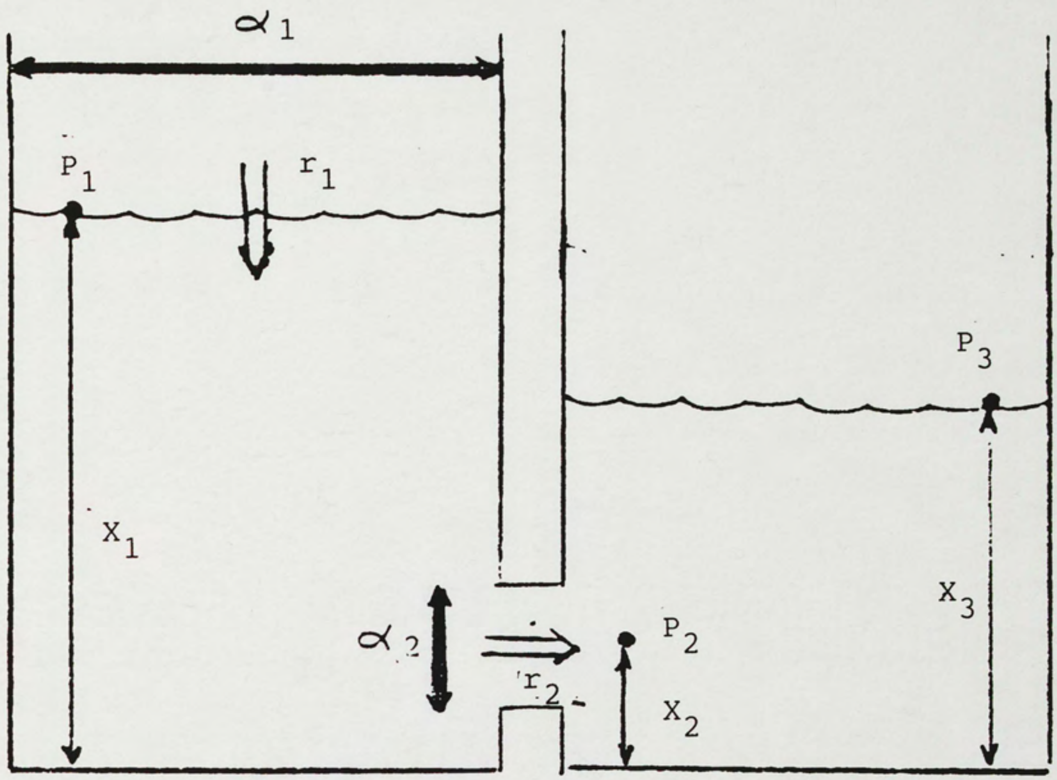
For the situation depicted in Figure 3-2, applying the relationship in equation 3-10 yields equation 3-12.

$$P_2 = P_1 + \rho g(X_3 - X_2) \quad \text{(eq. 3-12)}$$

Substituting equation 3-11 into equation 3-9 yields equation 3-13.

$$P_1 + \rho g X_1 + \tfrac{1}{2}\rho r_1{}^2 = P_1 + \rho g(X_3 - X_2) + \rho g X_2 + \tfrac{1}{2}\rho r_2{}^2 \quad \text{(eq. 3-13)}$$

Subtracting P_1 from both sides, regrouping and dividing by the density yields equation 3-14.



$P_1 = P_3$

Figure 3-2.Dual Connected Tanks.

$$gX_1 + \tfrac{1}{2}r_1^2 = gX_3 - gX_2 + gX_2 + \tfrac{1}{2}r_2^2 \qquad \text{(eq. 3-14)}$$

Equation 3-11 can be rewritten as shown in equation 3-15.

$$r_1 = \frac{\alpha_2 * r_2}{\alpha_1} \qquad \text{(eq. 3-15)}$$

Substituting the simplified equation 3-14 into equation 3-15 and solving for r_2^2 yields equation 3-16.

$$r_2^2 = \frac{2g(X_1 - X_3)}{1 - (\alpha_2/\alpha_1)^2} \qquad \text{(eq. 3-16)}$$

By finding the square root of both sides of equation 3-16, r_2 is shown by equation 3-17 after some regrouping.

$$r_2 = \alpha_1 \frac{\sqrt{2g(X_1 - X_3)}}{\alpha_1^2 - \alpha_2^2} \qquad \text{(eq. 3-17)}$$

Since Figure 3-2 is a sharp-edged orifice and the volume flowrate through such an orifice equals the cross-sectional area of the orifice times the rate of flow through the orifice, the volume flowrate for Figure 3-2 is given by equation 3-18.

$$\text{Volume Flowrate} = (\alpha_2) \frac{1}{\alpha_1^2 - \alpha_2^2} \sqrt{2g(X_1 - X_3)} \qquad \text{(eq. 3-18)}$$

Therefore the tank in Figure 2-1 has a flowrate out of the first tank given by equation 3-19.

$$F_2 = a_1 \frac{A}{A^2 - a_1^2} \sqrt{2g}\sqrt{H_1 - H_2} \qquad \text{(eq. 3-19)}$$

Substituting the variables that describe the flowrate out of the second tank into equation 3-18 yields equation 3-20.

$$F_o = a_2 \frac{A}{A^2 - a_2^2} \sqrt{2g}\sqrt{H_2 - H_3} \qquad \text{(eq. 3-20)}$$

Upon regrouping equation 3-19 and calling the second factor the discharge coefficient C_{D1}, equation 3-21 describes F_2.

$$F_2 = C_{D1} a_1 \sqrt{2g}\sqrt{H_1 - H_2} \qquad \text{(eq. 3-21)}$$

F_O is found in the same manner with the second factor as the discharge coefficient C_{D2} and is calculated with equation 3-22.

$$F_O = C_{D2} a_2 \sqrt{2g} \sqrt{H_2 - H_3} \qquad \text{(eq. 3-22)}$$

Since $A >> a_1$ and $A >> a_2$ for Figure 2-1, both C_{D1} and C_{D2} approximately equal 1.

Finally, using equation 3-22, the steady state relationship for H_2 in terms of H_1 and F_L can be found. Since at steady state $\frac{dH_2}{dt} = 0$, equation 3-8 yields equation 3-23.

$$0 = \frac{dH_2}{dt} = \frac{F_2 + F_L - F_O}{A} \qquad \text{(eq. 3-23)}$$

Solving for F_O in terms of F_2 and F_L gives the relationship in equation 3-24.

$$F_O = F_2 + F_L \qquad \text{(eq. 3-24)}$$

Substituting equations 3-21 and 3-22 into equation 3-24 gives the following equation.

$$C_{D2}a_2\sqrt{2g}\ \sqrt{\bar{H}_2 - H_3} = C_{D1}a_1\sqrt{2g}\ \sqrt{\bar{H}_1 - \bar{H}_2} + F_L$$

(eq. 3-25)

To solve this let $y = \sqrt{\bar{H}_1 - \bar{H}_2}$, then $y^2 = \bar{H}_1 - \bar{H}_2$ and $\bar{H}_2 = \bar{H}_1 - y^2$. Substituting these last two equations into equation 3-25 gives equation 3-26.

$$C_{D2}a_2\sqrt{2g}\sqrt{(\bar{H}_1 - H_3) - y^2} = C_{D1}a_1\sqrt{2g}y + F_L \quad \text{(eq. 3-26)}$$

Squaring both sides of equation 3-25 yields equation 3-27.

$$C_{D2}{}^2a_2{}^22g\left[(\bar{H}_1 - H_3) - y^2\right] = C_{D1}{}^2a_1{}^22gy^2 + 2F_LC_{D1}a_1\sqrt{2g}y + F_L{}^2 \quad \text{(eq. 3-27)}$$

Rewriting equation 3-27 as a quadratic gives equation 3-28.

$$0=(C_{D1}{}^2a_1{}^2+C_{D2}{}^2a_2{}^2)y^2+\frac{F_LC_{D1}a_1\sqrt{2}y}{\sqrt{g}}+\frac{F_L{}^2}{2g}-C_{D2}{}^2a_2{}^2(\bar{H}_1-H_3)$$

(eq. 3-28)

Solving equation 3-28 for the unknown y and simplifying yields equation 3-29.

$$y=\frac{-F_L C_{D1} a_1 \pm C_{D2} a_2 \sqrt{(\bar{H}_1 - H_3)(C_{D1}{}^2 a_1{}^2 + C_{D2}{}^2 a_2{}^2)2g - F_L{}^2}}{\sqrt{2g}\,(C_{D1}{}^2 a_1{}^2 + C_{D2}{}^2 a_2{}^2)} \qquad \text{(eq. 3-29)}$$

Equation 3-29 gives two solutions. But since by definition y is the square root of the differences in height, it must be greater than zero. The negative solution can then be disregarded and equation 3-30 gives the proper result.

$$y=\frac{C_{D2} a_2 \sqrt{(\bar{H}_1 - H_3)(C_{D1}{}^2 a_1{}^2 + C_{D2}{}^2 a_2{}^2)2g - F_L{}^2} - F_L C_{D1} a_1}{\sqrt{2g}\,(C_{D1}{}^2 a_1{}^2 + C_{D2}{}^2 a_2{}^2)} \qquad \text{(eq. 3-30)}$$

Applying the definition of y and solving for $\bar{H}_2$, the desired solution is achieved and given by equation 3-31.

$$\bar{H}_2=\bar{H}_1-\frac{C_{D2} a_2 \sqrt{(\bar{H}_1 - H_3)(C_{D1}{}^2 a_1{}^2 + C_{D2}{}^2 a_2{}^2)2g - F_L{}^2} - F_L C_{D1} a_1}{\sqrt{2g}\,(C_{D1}{}^2 a_1{}^2 + C_{D2}{}^2 a_2{}^2)} \qquad \text{(eq. 3-31)}$$

CHAPTER IV

LINEAR SYSTEM

The non-linear system has been found in terms of physical parameters, operating parameters, control parameters and variables. To control a linearization of the system in Figure 2-1 several steps must be completed. First, a feedback loop is described as in Figure 4-1. Using this loop, the unknown characteristics, G_D and G_P, must be found as shown back in Chapter Two. A Taylor's series will be used to find the unknown system transfer function, G(s). The feedback loop will then be applied and the control height $h_1(s)$ will be found in response to the change in reference height, $h_{1R}(s)$. Laplace Transform techniques will be used in the two different types of control. For proportional only ($K_C = 10$, $T_R = \infty$, $T_D = 0$) and proportional-integral ($K_C = 10$, $T_R = 10$, $T_D = 0$) control, $h_1(t)$ and $h_2(t)$ will be found. After finding these small changes, $H_1(t)$ and $H_2(t)$ will be found.

Finding the depth constant, G_D, and pump constant, G_P for this system begins by applying the initial conditions $H_1 = 10$ and $F_L = 0$. By substituting these values into equation 3-34, H_2 is 9.24 cm. Inserting this value into equation 3-21, yields F_1 as 42.6 cc/sec. After finding this

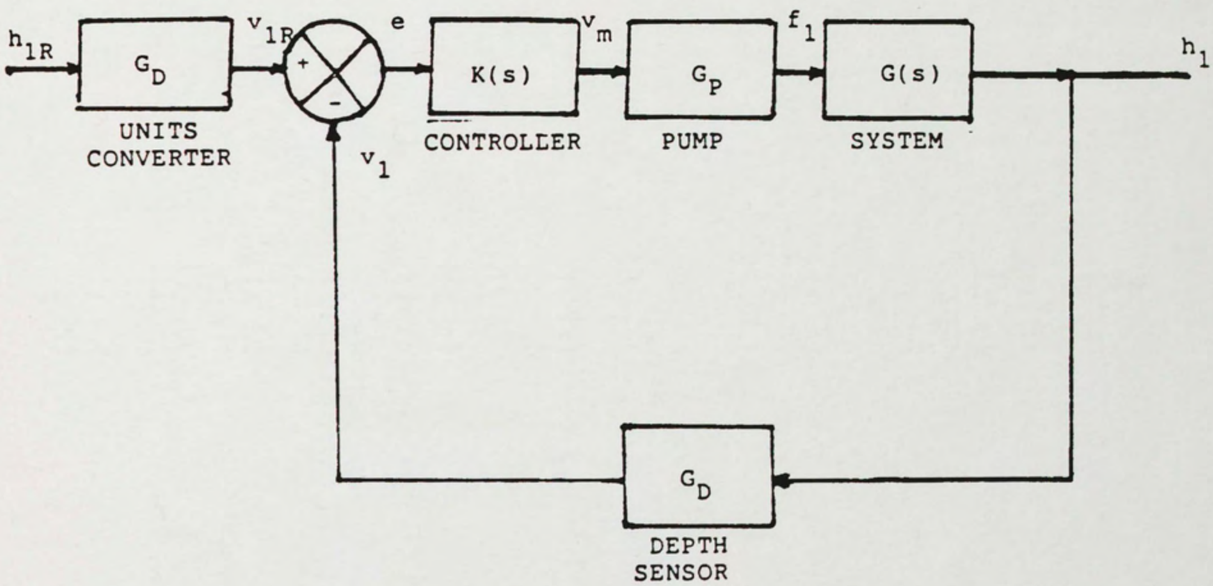


Figure 4-1. Feedback Loop For Linear System.

flow and placing this into equation 2-4, V_M is found to be 3.9 volts. Finally from equations 2-3 and 2-6, G_D is .278 volts/cm and G_P is 10.67 cc/sec per volt.

To find G(s), the pump flow as a function of time first must be found. For a small change in pump flow, $f_1(t)$, the total flow of fluid into tank one, $F_1(t)$, is described by equation 4-1 where $\bar{F}_1$ is some original flow.

$$F_1(t) = \bar{F}_1 + f_1(t) \qquad \text{(eq. 4-1)}$$

In the same manner equations 4-2 and 4-3 describe the heights of tank one and two for small changes in height, $h_1(t)$ and $h_2(t)$, and original heights $\bar{H}_1$ and $\bar{H}_2$.

$$H_1(t) = \bar{H}_1 + h_1(t) \qquad \text{(eq. 4-2)}$$

$$H_2(t) = \bar{H}_2 + h_2(t) \qquad \text{(eq. 4-3)}$$

By differentiating equations 4-2 and 4-3, equations 4-4 and 4-5 are found.

$$\frac{dH_1}{dt} = \frac{dh_1}{dt} \qquad \text{(eq. 4-4)}$$

$$\frac{dH_2}{dt} = \frac{dh_2}{dt} \qquad \text{(eq. 4-5)}$$

By using a Taylor's series with small changes h_1 and h_2, the flow F_2 is given by equation 4-6.

$$F_2 = \bar{F}_2 + \left.\frac{\delta F_2}{\delta H_1}\right|_{\bar{H}_1,\bar{H}_2} h_1 + \left.\frac{\delta F_2}{\delta H_2}\right|_{\bar{H}_1,\bar{H}_2} h_2 \qquad \text{(eq. 4-6)}$$

As equation 4-7 shows, F_0 can also be written in a Taylor's series.

$$F_0 = \bar{F}_0 + \left.\frac{\delta F_0}{\delta H_1}\right|_{\bar{H}_1,\bar{H}_2} h_1 + \left.\frac{\delta F_0}{\delta H_2}\right|_{\bar{H}_1,\bar{H}_2} h_2 \qquad \text{(eq. 4-7)}$$

By differentiating equation 3-21 and substituting into equation 4-6, equation 4-8 is found.

$$F_2 = \bar{F}_2 + C_{D1}a_1\sqrt{\frac{g}{2(\bar{H}_1 - \bar{H}_2)}}\,h_1 - C_{D1}a_1\sqrt{\frac{g}{2(\bar{H}_1 - \bar{H}_2)}}\,h_2 \qquad \text{(eq. 4-8)}$$

Regrouping like terms and rewriting yields equation 4-9

$$F_2 = \bar{F}_2 + k_1(h_1 - h_2)$$
$$\text{where } k_1 = C_{D1}a_1\sqrt{\frac{g}{2(\bar{H}_1 - \bar{H}_2)}} \qquad \text{(eq. 4-9)}$$

For the tank described in Chapter Two, k_1 is 28.03. In the same manner, the flow out is given by equation 4-10.

$$F_O = \bar{F}_O + k_2 h_2$$
$$\text{where } k_2 = C_{D2} a_2 \sqrt{\frac{g}{2(\bar{\bar{H}}_2 - \bar{\bar{H}}_3)}} \qquad \text{(eq. 4-10)}$$

For the system in Chapter Two, k_2 is 3.41.

Taking equation 3-7 and substituting into equations 4-4, 4-1 and 4-9 yields equation 4-11.

$$\frac{dh_1}{dt} = \frac{\bar{F}_1 + f_1}{A} - \frac{\bar{F}_2 + k_1(h_1 - h_2)}{A} \qquad \text{(eq. 4-11)}$$

Since the system is initially in steady state, $\bar{F}_1$ is equal to $\bar{F}_2$ and equation 4-11 can be rewritten as equation 4-12.

$$\frac{dh_1}{dt} = -\frac{k_1}{A} h_1 + \frac{k_1}{A} h_2 + \frac{f_1}{A} \qquad \text{(eq. 4-12)}$$

In the same manner, equation 3-8 can be rewritten as shown in equation 4-13 by inserting equations 4-5, 4-9 and 4-10.

$$\frac{dh_2}{dt} = \frac{F_L}{A} + \frac{\bar{F}_2 + k_1h_1 - k_1h_2}{A} - \frac{\bar{F}_O + k_2h_2}{A} \qquad \text{(eq. 4-13)}$$

Since this system is also initially in steady state and F_L remains constant throughout the problem F_L plus F_2 equals F_O and equation 4-14 holds.

$$\frac{dh_2}{dt} = \frac{k_1h_1}{A} - \frac{k_1 + k_2}{A} h_2 \qquad \text{(eq. 4-14)}$$

Rewriting equations 4-12 and 4-14 into matrix form yields equation 4-15.

$$\frac{d}{dt}\begin{bmatrix} h_1 \\ h_2 \end{bmatrix} = \begin{bmatrix} \frac{-k_1}{A} & \frac{k_1}{A} \\ \frac{k_1}{A} & \frac{-k_1 + k_2}{A} \end{bmatrix} \begin{bmatrix} h_1 \\ h_2 \end{bmatrix} + \begin{bmatrix} \frac{1}{A} \\ 0 \end{bmatrix} f_1 \qquad \text{(eq. 4-15)}$$

Since the system transfer function G(s) is $h_1(s)$ over $f_1(s)$, Laplace Transforms will be needed. By transforming equation 4-15 and since $h_1(t=0)$, $h_2(t=0)$ and $f_1(t=0)$ all are zero, equation 4-16 holds.

$$s\begin{bmatrix} h_1(s) \\ h_2(s) \end{bmatrix} = \begin{bmatrix} \frac{-k_1}{A} & \frac{k_1}{A} \\ \frac{k_1}{A} & -\frac{k_1+k_2}{A} \end{bmatrix} \begin{bmatrix} h_1(s) \\ h_2(s) \end{bmatrix} + \begin{bmatrix} \frac{1}{A} \\ 0 \end{bmatrix} f_1(s) \qquad \text{(eq. 4-16)}$$

Regrouping, inverting and solving yields equation 4-17.

$$\begin{bmatrix} \frac{h_1(s)}{f_1(s)} \\ \frac{h_2(s)}{f_1(s)} \end{bmatrix} = \frac{1}{s^2+\frac{2k_1+k_2}{A}s-\frac{k_1k_2}{A^2}} \begin{bmatrix} \frac{s}{A} + \frac{k_1+k_2}{A^2} \\ \frac{k_1}{A^2} \end{bmatrix} \qquad \text{(eq. 4-17)}$$

Therefore the system transfer function is given by equation 4-18.

$$G(s) = \frac{\frac{s}{A} + \frac{k_1+k_2}{A^2}}{s^2+\frac{2k_1+k_2}{A}s-\frac{k_1k_2}{A^2}} \qquad \text{(eq. 4-18)}$$

The next step is finding the change in height $h_1(s)$ in response to a small change in the reference height $h_{1R}(s)$. From Figure 4-1, it can be seen that the reference voltage to reference height relationship is given by equation 4-19.

$$v_{1R} = G_D h_{1R}(s) \qquad \text{(eq. 4-19)}$$

Also for Figure 4-1, the voltage/height relationship is given by equation 4-20.

$$v_1 = G_D h_1(s) \qquad \text{(eq. 4-20)}$$

By completing the loop around Figure 4-1, equation 4-21 is found.

$$(v_{1R} - v_1)K(s)G_P G(s) = h_1(s) \qquad \text{(eq. 4-21)}$$

Substituting 4-19 and 4-20 into 4-21 yields equation 4-22.

$$\left[G_D h_{1R}(s) - G_D h_1(s)\right] K(s)G_P G(s) = h_1(s) \qquad \text{(eq. 4-22)}$$

Solving for the first height in the frequency domain gives equation 4-23.

$$h_1(s) = \frac{G_D G_P K(s)G(s)}{1 + G_D G_P K(s)G(s)} h_{1R}(s) \qquad \text{(eq. 4-23)}$$

Therefore, for the system in Chapter Two, this equation can be rewritten as seen in equation 4-24.

$$h_1(s) = \frac{.0148K(s)(s+.157)}{s^2+.297s+.00239+.0148K(s)(s+.157)} h_{1R}(s)$$

(eq. 4-24)

Now the change in height one in response to a unit step change in the reference height with proportional control ($K_C = 10$, $T_R = \infty$, $T_D = 0$) can be found. Since it is a unit step change $h_{1R}(s)$ is 1/s. And since this is proportional only control, K(s) is 10. Inserting these values into equation 4-24 yields equation 4-25.

$$h_1(s) = \frac{.148(s+.157)}{s(s^2+.445s+.0257)} \qquad \text{(eq. 4-25)}$$

$h_1(t)$ will be the inverse Laplace Transform of equation 4-25. To find this inverse, a Heaviside expansion will be used. The roots of the denominator are -.0682 and -.3768, therefore:

$$h_1(t) = .148\mathcal{L}^{-1}[Z(s)]$$

$$\text{where } Z(s) = \frac{s + .157}{s(s+.0682)(s+.3768)} \qquad \text{(eq. 4-26)}$$

Rewriting this into a manageable form gives equation 4-27.

$$h_1(t) = .148\,\mathcal{L}^{-1}\left[\frac{A}{s} + \frac{B}{s+.0682} + \frac{C}{s+.3768}\right]$$

where A is the limit of n as s goes to zero, 6.109, B is the limit of $(s+.0682)Z(s)$ as s goes to $-.0682$, -4.219 and C is the limit of $(s+.3768)Z(s)$ as s goes to $-.3768$, -1.89. (eq. 4-27)

Finally, inverting equation 4-27 gives equation 4-28.

$$h_1(t) = (.904 - .624e^{-.0682t} - .28e^{-.3768t})\,\delta_1(t)$$

(eq. 4-28)

By inserting equation 4-28 into equation 4-2, the time response for the height of water in tank one is:

$$H_1(t) = 10 + (.904 - .624e^{-.0682t} - .28e^{-.377t})\,\delta_1(t)$$

(eq. 4-29)

To find $h_2(s)$ and subsequently its response to the step change as a function of time, $H_2(t)$, equation 4-14 will be transformed by a Laplace Transform.

$$sh_2(s) = \frac{k_1}{A}h_1(s) - \frac{k_1+k_2}{A}h_2(s)$$

(eq. 4-30)

Solving for $h_2(s)$ and inserting the values for the physical system gives equation 4-31.

$$h_2(s) = \frac{.14}{s + .157} h_1(s) \qquad \text{(eq. 4-31)}$$

By inserting $h_1(s)$ as found in equation 4-25 and inverting yields equation 4-32.

$$h_2(t) = (.148) \quad \mathcal{L}^{-1}\left[Y(s) \right]$$

$$\text{where } Y(s) = \frac{.14(s+.157)}{(s+.157)s(s^2+.445s+.0257)} \qquad \text{(eq. 4-32)}$$

Going ahead and expanding equation 4-32, yields equation 4-33.

$$h_2(t) = (.148) \quad \mathcal{L}^{-1}\left[\frac{A}{s} + \frac{B}{s + .0682} + \frac{C}{s + .3768} \right]$$

where A is the limit of sY(s) as s goes to zero, 5.45, B is the limit of (s+.0682)Y(s) as s goes to -.0682, -6.65, and C is the limit of (s+.3768)Y(s) as s goes to -.3768, 1.20. (eq. 4-33)

Inverting equation 4-33 yields the final result for $h_2(t)$.

$$h_2(t) = (.806 - .984e^{-.0682t} + .178e^{-.3768t}) \quad \delta_1(t)$$

(eq. 4-34)

Putting equation 4-34 into equation 4-3 yields equation 4-35.

$$H_2(t)=9.24+(.806 -.984e^{-.0682t}+.178e^{-.3768t})\ \delta_1(t) \quad \text{(eq. 4-35)}$$

Figure 4-2 lists the results for $H_1(t)$ and $H_2(t)$ in response to a step change with proportional control.

To find the change in height one in response to a unit step change in the reference height with integral control, the same steps are followed. For the problem, let $K_C = 10$, $T_R = 10$ and $T_D = 0$. Then, as before, $h_{1R}(s)$ is 1/s and since this is an integral control example: $K(s)= 10\ (1 + 1/10s)$. Substituting these values into equation 4-24, gives equation 4-36.

$$h_1(s)= \frac{.148(s^2+.257s+.0157)}{s(s^3+.445s^2+.0404s+.00232)} \quad \text{(eq. 4-36)}$$

Separating equation 4-36 into a workable form yields equation 4-37.

$$h_1(s) = \frac{1}{s} - \frac{s^2 + .297s + .00236}{s^3+.445s^2+.0404s+.00232} \quad \text{(eq. 4-37)}$$

Figure 4-2. $H_1(t)$, $H_2(t)$ With Proportional Control.

LINEAR SYSTEM

TIME	TANK1	TANK2
0	10	9.24
1	10.12	9.24
2	10.22	9.27
3	10.3	9.3
4	10.36	9.33
5	10.41	9.37
6	10.46	9.41
7	10.49	9.44
8	10.52	9.48
9	10.55	9.51
10	10.58	9.55
11	10.6	9.58
12	10.62	9.61
13	10.64	9.64
14	10.66	9.66
15	10.67	9.69
16	10.69	9.71
17	10.7	9.73
18	10.72	9.75
19	10.73	9.77
20	10.74	9.79
21	10.75	9.81
22	10.76	9.82
23	10.77	9.84
24	10.78	9.85
25	10.79	9.86
26	10.79	9.87
27	10.8	9.88
28	10.81	9.9
29	10.81	9.9
30	10.82	9.91
31	10.82	9.92
32	10.83	9.93
33	10.83	9.94
34	10.84	9.94
35	10.84	9.95
36	10.85	9.96
37	10.85	9.96
38	10.85	9.97
39	10.86	9.97
40	10.86	9.98
41	10.86	9.98
42	10.86	9.98
43	10.87	9.99
44	10.87	9.99
45	10.87	10
46	10.87	10
47	10.87	10
48	10.88	10
49	10.88	10.01
50	10.88	10.01
51	10.88	10.01
52	10.88	10.01
53	10.88	10.01
54	10.88	10.02
55	10.88	10.02
56	10.89	10.02
57	10.89	10.02
58	10.89	10.02
59	10.89	10.02
60	10.89	10.02
61	10.89	10.03
62	10.89	10.03
63	10.89	10.03
64	10.89	10.03
65	10.89	10.03
66	10.89	10.03
67	10.89	10.03
68	10.89	10.03
69	10.89	10.03
70	10.89	10.03
71	10.89	10.03
72	10.89	10.03
73	10.89	10.03
74	10.89	10.03
75	10.9	10.04
76	10.9	10.04
77	10.9	10.04
78	10.9	10.04
79	10.9	10.04
80	10.9	10.04

By numerical methods $s = -.348$ is a root of the denominator of the second term. Therefore rewriting using Heaviside expansion yields equation 4-38.

$$h_1(s) = \frac{1}{s} - \frac{.214}{s+.348} - \frac{.786s+.00271}{s^2+.097s+.00664} \qquad \text{(eq. 4-38)}$$

Rewriting equation 4-38 as equation 4-39, allows easy conversion through inverse Laplace Transforms.

$$h_1(s)= \frac{1}{s} - \frac{.214}{s+.348} - \frac{.786(s+.0485)}{(s+.0485)^2+.0655^2} + \frac{.0354}{(s+.0485)^2+.0655^2}$$

(eq. 4-39)

The final solution for the small change in height one as function of time is given by equation 4-40.

$$h_1(t) =(1-.214e^{-.348t}-.786e^{-.0485t}\cos(.0655t) +.541e^{-.0485t}\sin(.0655t))\delta_1(t)$$

(eq. 4-40)

Substituting equation 4-40 into equation 4-2, gives the final solution.

$$H_1(t)=10+ (1- .214e^{-.348t} -.786e^{-.0485t}\cos(.0655t) +.541e^{-.0485t}\sin(.0655t))\delta_1(t)$$

(eq. 4-41)

Substituting equation 4-36 into equation 4-31 and then factoring yields:

$$h_2(s) = \frac{.0207(s + .157)(s + .1)}{(s + .157)(s + .348)s(s^2 + .097s + .00664)}$$

(eq. 4-42)

Rewriting equation 4-42 by factoring and using Heaviside expansions again gives the following:

$$h_2(s) = \frac{.896}{s} + \frac{.157}{s + .348} - 1.05\left[\frac{s + .0453}{s^2+.097s+.00664}\right]$$

(eq. 4-43)

Writing equation 4-43 into the proper form for inverse Laplace Transforms gives:

$$h_2(s)=\frac{.896}{s}+\frac{.157}{s+.348}-1.05\left[\frac{s + .011}{(s+.0485)^2+.0655^2} - \frac{.0032}{(s+.0485)^2+.0655^2}\right]$$

(eq. 4-44)

Inverting equation 4-44 yields the following solution for a small change in the second tank height as a function of time.

$$h_2(t) = \left[.896 + .157e^{-.348t} - 1.05e^{-.0485t}\cos(.0655t) + .0513e^{-.0485t}\sin(.0655t)\right]\delta_1(t)$$

(eq. 4-45)

Substituting equation 4-45 into equation 4-3 gives the final solution for the second tank height withproportional-integral control.

$$H_2(t) = 9.24 + \left[.896 + .157e^{-.348t} - 1.05e^{-.0485t}\cos(.0655t) + .0513e^{-.0485t}\sin(.0655t)\right]\delta_1(t)$$

(eq. 4-46)

Results for $H_1(t)$ and $H_2(t)$ in response to a unit step change in the set point with integral control are supplied in Figure 4-3.

Figure 4-3. $H_1(t)$, $H_2(t)$ With Proportional-Integral Control.

LINEAR SYSTEM

TIME	TANK1	TANK2
0	10	9.24
1	10.13	9.25
2	10.25	9.27
3	10.34	9.3
4	10.43	9.35
5	10.51	9.39
6	10.58	9.44
7	10.64	9.49
8	10.7	9.54
9	10.76	9.59
10	10.81	9.64
11	10.85	9.69
12	10.9	9.74
13	10.93	9.78
14	10.97	9.83
15	11	9.87
16	11.03	9.91
17	11.06	9.95
18	11.08	9.98
19	11.1	10.02
20	11.12	10.05
21	11.13	10.08
22	11.14	10.1
23	11.16	10.13
24	11.16	10.15
25	11.17	10.17
26	11.18	10.18
27	11.18	10.2
28	11.18	10.21
29	11.18	10.23
30	11.18	10.24
31	11.18	10.24
32	11.18	10.25
33	11.17	10.26
34	11.17	10.26
35	11.16	10.27
36	11.16	10.27
37	11.15	10.27
38	11.15	10.27
39	11.14	10.27
40	11.13	10.27
41	11.12	10.26
42	11.12	10.26
43	11.11	10.26
44	11.1	10.25
45	11.09	10.25
46	11.09	10.24
47	11.08	10.24
48	11.07	10.23
49	11.06	10.23
50	11.06	10.22
51	11.05	10.22
52	11.04	10.21
53	11.04	10.21
54	11.03	10.2
55	11.03	10.19
56	11.02	10.19
57	11.02	10.18
58	11.01	10.18
59	11.01	10.17
60	11	10.17
61	11	10.16
62	11	10.16
63	10.99	10.16
64	10.99	10.15
65	10.99	10.15
66	10.99	10.15
67	10.98	10.14
68	10.98	10.14
69	10.98	10.14
70	10.98	10.13
71	10.98	10.13
72	10.98	10.13
73	10.98	10.13
74	10.98	10.13
75	10.98	10.12
76	10.98	10.12
77	10.98	10.12
78	10.98	10.12
79	10.98	10.12
80	10.98	10.12

CHAPTER V

NONLINEAR SYSTEM

The system of equations found in Chapter Three will be used to evaluate the nonlinear system. These differential equations will be rewritten as difference equations with step size Δt. During each Δt, the equations will be used to find the flow rates (equations 3-21 and 3-22), use these rates to find the rate of change in height (equations 3-7 and 3-8) and then calculate the new height. This cycle will be repeated PTSP times and the heights will be output to the user each time segment of length (PTSP)*(Δt). After 80 of these print steps, the program ends. This cycle is illustrated in Figure 5-1.

The use of this program involves several steps by the user. First, the user must describe the tank settings for orifice one and two. For the purpose of this paper, orifice one is set so the 1.27 cm hole is plugged (Ans. "Y") and holes .95 cm and .625 cm are open (Ans. "N"). The tap for orifice two will be wide open (Ans. "1"). The physical parameters for this configuration are now displayed. The next entries by the user are the control parameters. The user is asked for the proportional gain, K_C, and the reset

FIND VOLTAGE OF TANK 1	$v_1(0)=f(H_1(0))$	eqn 2-4
FIND ERROR	$e(0)=v_{1R}-v_1(0)$	
FIND NEW MOTOR VOLTAGE	$v_m(0)=v_m+K_C(e(0)+T_D e(0)/\Delta t)$	
FIND FLOW INTO TANK 1	$F_1(0)=f(v_m(0))$	eqn 2-1
FIND $\frac{dH_1(0)}{dt}$	$\frac{dH_1(0)}{dt}=\frac{F_1(0)-F_1}{A}$	
FIND $\frac{dH_2(0)}{dt}$	$\frac{dH_2(0)}{dt}=0$	
AS $n=1,2,3\ldots$ $[t_j=j\,\Delta t]$		
FIND NEW HEIGHT 1	$H_1(t_n)=H_1(t_{n-1})+\frac{dH_1(t_{n-1})}{dt}\Delta t$	
FIND NEW HEIGHT 2	$H_2(t_n)=H_1(t_{n-1})+\frac{dH_1(t_{n-1})}{dt}\Delta t$	
FIND FLOW BETWEEN TANKS	$F_2(t_n)=f(H_1(t_n),H_2(t_n))$	eqn 3-21
FIND FLOW OUT OF TANK 2	$F_0(t_n)=f(H_2(t_n))$	eqn 3-22
FIND VOLTAGE OF TANK 1	$v_1(t_n)=f(H_1(t_n))$	eqn 2-4
FIND ERROR	$e(t_n)=v_{1R}-v_1(t_n)$	
FIND NEW MOTOR VOLTAGE	$v_m(t_n)=v_m+K_C\left(e(t_n)+\sum_{i=0}^{n}e(t_i)/T_R+T_D\frac{e(t_n)-e(t_{n-1})}{\Delta t}\right)$	
FIND FLOW INTO TANK 1	$F_1(t_n)=f(v_m(t_n))$	
FIND $\frac{dH_1(t_n)}{dt}$	$\frac{dH_1(t_n)}{dt}=\frac{F_1(t_n)-F_2(t_n)}{A}$	eqn 3-7
FIND $\frac{dH_2(t_n)}{dt}$	$\frac{dH_2(t_n)}{dt}=\frac{F_2(t_n)+F_L-F_0(t_n)}{A}$	eqn 3-8
NEXT n		

Figure 5-1. Cycle of steps for Solving the Nonlinear System.

time, T_R. Throughout this paper K_C is ten. For proportional only control, as in the first example, $T_R = 0$. For this paper's proportional-integral control problem, example two, $T_R = 10$. The final control input derivative time, T_D, is input. $T_D = 0$ in both of these examples.

The initial values are then input to begin the simulation. These consist of the initial level in tank one (10 cm in both examples), the step change in the control variable (1 cm in both examples) and the load variable that describes the input into tank two (0 cc/min for these cases). Several informational values are output and the simulation begins.

A reference listing is supplied in the Appendix. Sample outputs are supplied in Figure 5-2 (Proportional Only Control) and Figure 5-3 (Proportional-Integral Control).

As a check for accuracy of this numerical solution by Euler's method, another method was attempted. The second method to be chosen was a fourth order Runge-Kutta solution. The equations that describe this solution are given as depicted in equation 5-1 for H_1.

$$H_1(t_0 + \Delta t) = H_1(t_0) + \frac{k_1 + 2k_2 + 2k_3 + k_4}{6} \qquad \text{(eq. 5-1)}$$

```
 RUN
CHOOSE A TANK CONFIGURATION
WHICH HOLES DO YOU WISH PLUGGED?
(Y=YES,N=NO)
1.27 CM ?Y
.95 CM ?N
.635 CM?N

HOW DO YOU WANT THE TAP OF THE SECOND TANK SET ?
1. FULL OPEN
2. 3/4 OPEN
3. 1/2 OPEN
4. 1/4 OPEN
?1

PHYSICAL PARAMETERS

H3 = 3           G  = 980
CD1= 1           CD2= 1
A1 = 1.1         A2 = .38
A  = 200

CONTROL PARAMETERS

ENTER KC?10
ENTER TR FOR INTEGRAL CONTROL (0 IF NONE DESIRED)?10
ENTER TD FOR DIFFERENTIAL CONTROL(0 FOR NONE)?0

INITIAL VALUES

WHAT IS YOUR INITIAL LEVEL IN TANK 1 (3 TO 25 CM)?10
INPUT THE STEP CHANGE IN THIS CONTROL VARIABLE?1
INPUT THE LOAD VARIABLE (CC/MIN)?0

VARIABLE VALUES

H1(0) = 10        H2(0) =9.24
F1D = .2552       FL = 0
SET POINT 11
SET POINT VOLTAGE 4.83
DESIGN VOLTAGE 3.88
```

Figure 5-2. Results For Proportional Control Of
The Nonlinear System.

Figure 5-2. - Continued.

TIME (SEC)	TANK1 (CM)	TANK2 (CM)
0	10	9.24
1	10.09	9.24
2	10.18	9.26
3	10.25	9.28
4	10.31	9.31
5	10.36	9.34
6	10.41	9.38
7	10.45	9.41
8	10.48	9.44
9	10.51	9.48
10	10.54	9.51
11	10.56	9.54
12	10.59	9.57
13	10.61	9.59
14	10.63	9.62
15	10.64	9.65
16	10.66	9.67
17	10.68	9.69
18	10.69	9.71
19	10.7	9.73
20	10.72	9.75
21	10.73	9.77
22	10.74	9.79
23	10.75	9.8
24	10.76	9.82
25	10.77	9.83
26	10.78	9.85
27	10.78	9.86
28	10.79	9.87
29	10.8	9.88
30	10.81	9.89
31	10.81	9.9
32	10.82	9.91
33	10.82	9.92
34	10.83	9.93
35	10.83	9.93
36	10.84	9.94
37	10.84	9.95
38	10.84	9.95
39	10.85	9.96
40	10.85	9.97
41	10.85	9.97
42	10.86	9.98
43	10.86	9.98
44	10.86	9.98
45	10.87	9.99
46	10.87	9.99
47	10.87	9.99
48	10.87	10
49	10.87	10
50	10.88	10
51	10.88	10.01
52	10.88	10.01
53	10.88	10.01
54	10.88	10.01
55	10.88	10.02
56	10.88	10.02
57	10.88	10.02
58	10.89	10.02
59	10.89	10.02
60	10.89	10.02
61	10.89	10.03
62	10.89	10.03
63	10.89	10.03
64	10.89	10.03
65	10.89	10.03
66	10.89	10.03
67	10.89	10.03
68	10.89	10.03
69	10.89	10.03
70	10.89	10.03
71	10.89	10.03
72	10.9	10.04
73	10.9	10.04
74	10.9	10.04
75	10.9	10.04
76	10.9	10.04
77	10.9	10.04
78	10.9	10.04
79	10.9	10.04
80	10.9	10.04

```
CHOOSE A TANK CONFIGURATION
WHICH HOLES DO YOU WISH PLUGGED?
(Y=YES,N=NO)
1.27 CM ?Y
.95 CM ?N
.635 CM?N

HOW DO YOU WANT THE TAP OF THE SECOND TANK SET ?
1. FULL OPEN
2. 3/4 OPEN
3. 1/2 OPEN
4. 1/4 OPEN
?1

PHYSICAL PARAMETERS

H3 = 3              G   = 980
CD1= 1              CD2= 1
A1 = 1.1            A2 = .38
A  = 200

CONTROL PARAMETERS

ENTER KC?10
ENTER TR FOR INTEGRAL CONTROL (O IF NONE DESIRED)?0
ENTER TD FOR DIFFERENTIAL CONTROL(O FOR NONE)?0

INITIAL VALUES

WHAT IS YOUR INITIAL LEVEL IN TANK 1 (3 TO 25 CM)?10
INPUT THE STEP CHANGE IN THIS CONTROL VARIABLE?1
INPUT THE LOAD VARIABLE (CC/MIN)?0

VARIABLE VALUES

H1(O) = 10          H2(O) =9.24
F1D =  2552         FL = 0
SET POINT 11
SET POINT VOLTAGE 4.83
DESIGN VOLTAGE 3.88
```

Figure 5-3. Results For Proportional-Integral Control Of The Nonlinear System.

Figure 5-3. - Continued.

TIME (SEC)	TANK1 (CM)	TANK2 (CM)
0	10	9.24
1	10.1	9.24
2	10.19	9.26
3	10.28	9.29
4	10.36	9.32
5	10.44	9.36
6	10.5	9.4
7	10.57	9.44
8	10.63	9.48
9	10.69	9.53
10	10.74	9.57
11	10.79	9.62
12	10.84	9.66
13	10.88	9.71
14	10.92	9.75
15	10.96	9.79
16	10.99	9.84
17	11.03	9.88
18	11.06	9.91
19	11.08	9.95
20	11.11	9.98
21	11.13	10.02
22	11.15	10.05
23	11.16	10.08
24	11.18	10.1
25	11.19	10.13
26	11.2	10.15
27	11.2	10.17
28	11.21	10.19
29	11.21	10.21
30	11.21	10.22
31	11.21	10.24
32	11.21	10.25
33	11.21	10.26
34	11.21	10.27
35	11.2	10.27
36	11.2	10.28
37	11.19	10.28
38	11.18	10.28
39	11.17	10.29
40	11.17	10.29
41	11.16	10.28
42	11.15	10.28
43	11.14	10.28
44	11.13	10.28
45	11.12	10.27
46	11.11	10.27
47	11.11	10.26
48	11.1	10.26
49	11.09	10.25
50	11.08	10.25
51	11.07	10.24
52	11.06	10.23
53	11.06	10.23
54	11.05	10.22
55	11.04	10.21
56	11.04	10.21
57	11.03	10.2
58	11.02	10.2
59	11.02	10.19
60	11.01	10.18
61	11.01	10.18
62	11	10.17
63	11	10.17
64	11	10.16
65	10.99	10.16
66	10.99	10.15
67	10.99	10.15
68	10.99	10.15
69	10.98	10.14
70	10.98	10.14
71	10.98	10.14
72	10.98	10.13
73	10.98	10.13
74	10.98	10.13
75	10.98	10.13
76	10.97	10.12
77	10.97	10.12
78	10.97	10.12
79	10.97	10.12
80	10.97	10.12

Equation 5-2 gives the value for H_2.

$$H_2(t_0+\Delta t) = H_2(t_0) + \frac{n_1+2n_2+2n_3+n_4}{6} \qquad \text{(eq. 5-2)}$$

The value of k_1 is found from equation 5-3.

$$k_1 = \Delta t*f(H_1(t_0),H_2(t_0)) \qquad \text{(eq. 5-3)}$$

Where $f(H_1,H_2)$ is the time derivative of H_1 calculated by equation 3-7. In the same manner n_1 is found by equation 5-4.

$$n_1 = \Delta t*g(H_1(t_0),H_2(t_0)) \qquad \text{(eq. 5-4)}$$

Where $g(H_1,H_2)$ is the time rate of change of H_2 as found in equation 3-8. The definition of k_2 is shown in equation 5-5.

$$k_2 = \Delta t*f(H_1(t_0)+k_1/2,H_2(t_0)+n_1/2) \qquad \text{(eq. 5-5)}$$

In the same manner, n_2 is given by equation 5-6.

$$n_2 = \Delta t*g(H_1(t_0)+k_1/2,H_2(t_0)+n_1/2) \qquad \text{(eq. 5-6)}$$

k_3 is given by equation 5-7.

$$k_3 = \Delta t*f(H_1(t_0)+k_2/2,H_2(t_0)+n_2/2) \qquad \text{(eq. 5-7)}$$

Equation 5-8 yields n_3.

$$n_3 = \Delta t*g(H_1(t_0)+k_2/2,H_2(t_0)+n_2/2) \qquad \text{(eq. 5-8)}$$

The value of k_4 is found by equation 5-9.

$$k_4 = \Delta t*f(H_1(t_0)+k_3/2,H_2(t_0)+n_3/2) \qquad \text{(eq. 5-9)}$$

To find n_4, equation 5-10 is used.

$$n_4 = \Delta t*g(H_1(t_0)+k_3/2,H_2(t_0)+n_3/2) \qquad \text{(eq. 5-10)}$$

Results of the Runge-Kutta solution and a comparison to the previously calculated Euler's method are shown in Figure 5-4 for Proportional Only Control and Figure 5-5 for Proportional-Integral Control. As can be seen by these results there are no major differences in the two solutions. Since an analytic solution does not exist, this lack of a difference is an opportunity for future study of the two nonlinear solutions. But for this paper's purpose it is not required and all future references to the nonlinear system will imply the Euler's solution.

	H_1		H_2	
Time	Euler's	Runge-Kutta	Euler's	Runge-Kutta
0	10	10	9.24	9.24
5	10.36	10.36	9.34	9.35
10	10.54	10.53	9.51	9.52
15	10.64	10.64	9.65	9.65
20	10.72	10.72	9.75	9.76
25	10.77	10.77	9.83	9.84
30	10.81	10.81	9.89	9.9
35	10.83	10.83	9.93	9.95
40	10.85	10.85	9.97	9.98
45	10.87	10.87	9.99	10
50	10.88	10.88	10	10.02
55	10.88	10.89	10.02	10.03
60	10.89	10.89	10.02	10.04
65	10.89	10.9	10.03	10.05
70	10.89	10.9	10.03	10.05
75	10.9	10.9	10.04	10.05
80	10.9	10.9	10.04	10.06

Figure 5-4. Proportional Only Control Comparison For The Nonlinear System.

Time	H_1 Euler's	H_1 Runge-Kutta	H_2 Euler's	H_2 Runge-Kutta
0	10	10	9.24	9.24
5	10.44	10.41	9.36	9.37
10	10.74	10.71	9.57	9.58
15	10.96	10.93	9.79	9.79
20	11.11	11.08	9.98	9.98
25	11.19	11.17	10.13	10.12
30	11.21	11.2	10.22	10.22
35	11.2	11.19	10.27	10.27
40	11.17	11.16	10.29	10.28
45	11.12	11.12	10.27	10.27
50	11.08	11.07	10.25	10.25
55	11.04	11.03	10.21	10.22
60	11.01	11	10.18	10.19
65	10.99	10.98	10.16	10.16
70	10.98	10.97	10.14	10.14
75	10.98	10.96	10.13	10.13
80	10.97	10.96	10.12	10.12

Figure 5-5. Proportional-Integral Control Comparison For The Nonlinear System.

CHAPTER VI

COMPARISON OF RESULTS

Plotting the results of the proportional only control solutions in figures 6-1 and 6-2 show a comparison of the linear and nonlinear solutions. In this example, no noticeable difference occurred between the two systems. The nonlinear system reacts slightly slower, but at no point does it differ by more than 1/2%. In both instances the final steady state values are the same. By these results the linearization appears to hold throughout. In proportional only control, the linear model displayed no major faults.

For Proportional-Integral control, as displayed in figures 6-3 and 6-4, no major difference between the linear and nonlinear solutions appear. Once again the nonlinear system lags behind the linear model. No difference appeared in the steady state values either. From these results, the linear model appears to also hold for proportional-integral control.

Figure 6-1. Height Of Tank 1 vs Time Proportional Control Only.

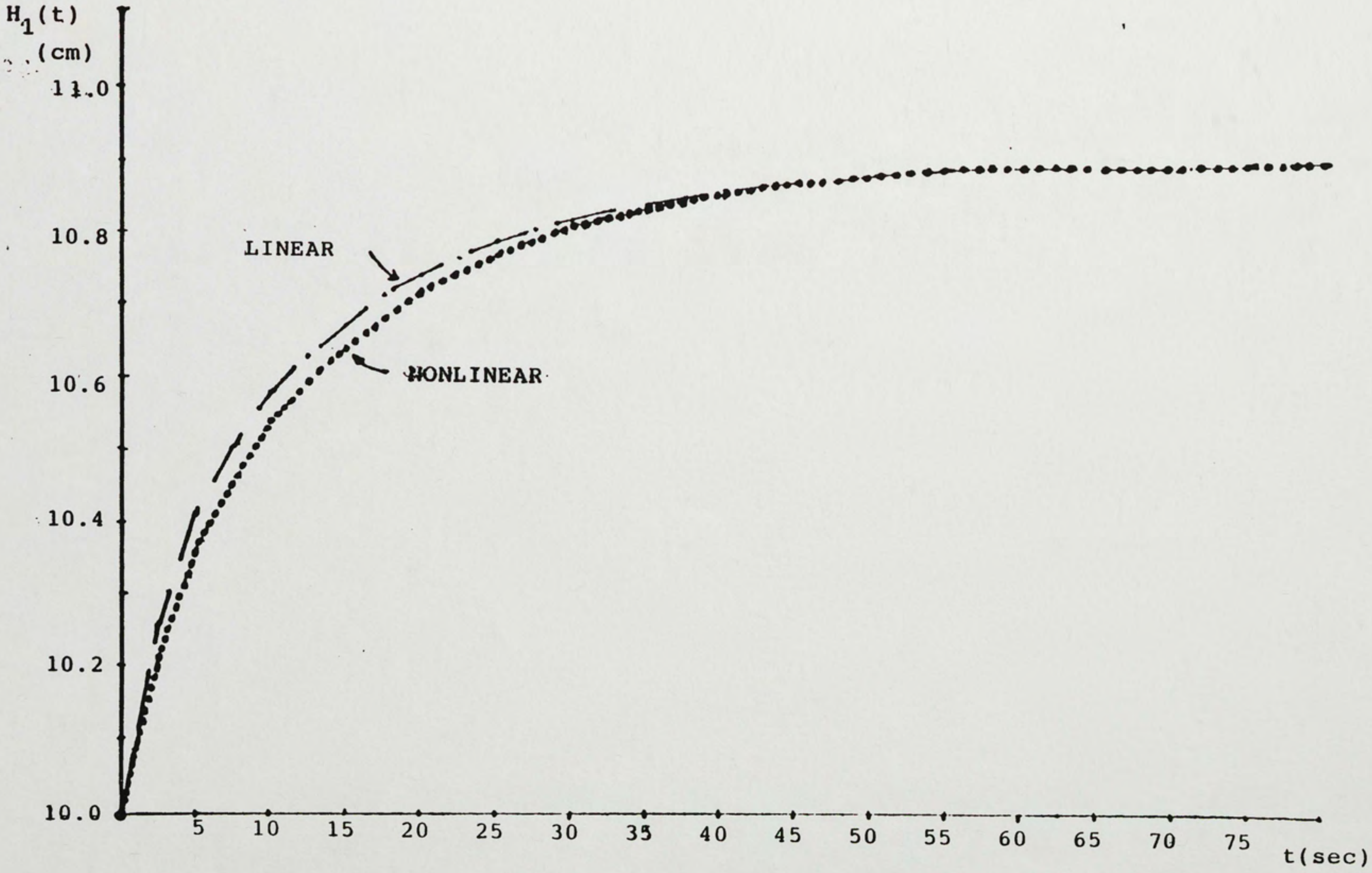


Figure 6-2. Height Of Tank 2 Vs Time Proportional Control Only.

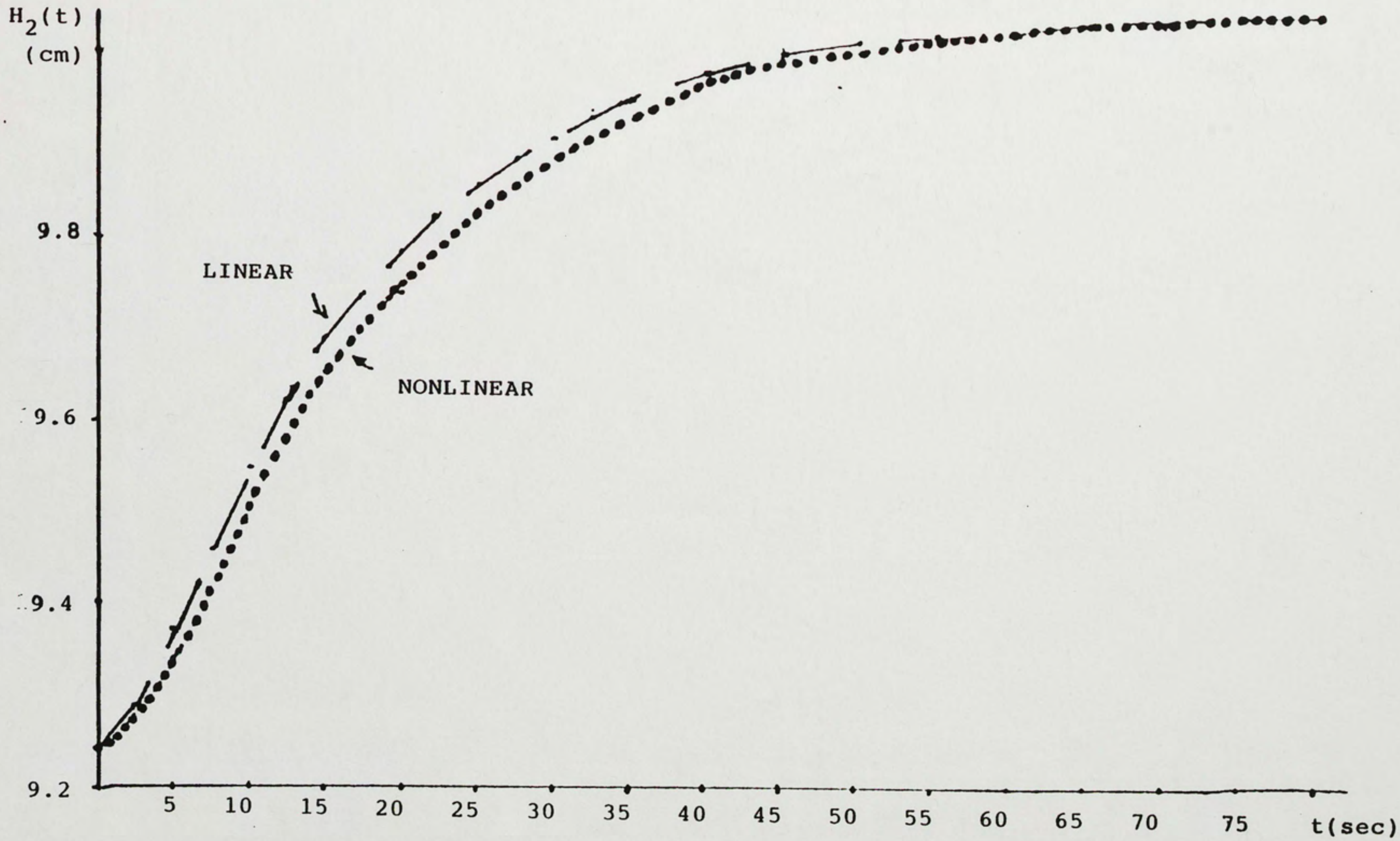


Figure 6-3. Height Of Tank 1 Vs Time Proportional-Integral Control.

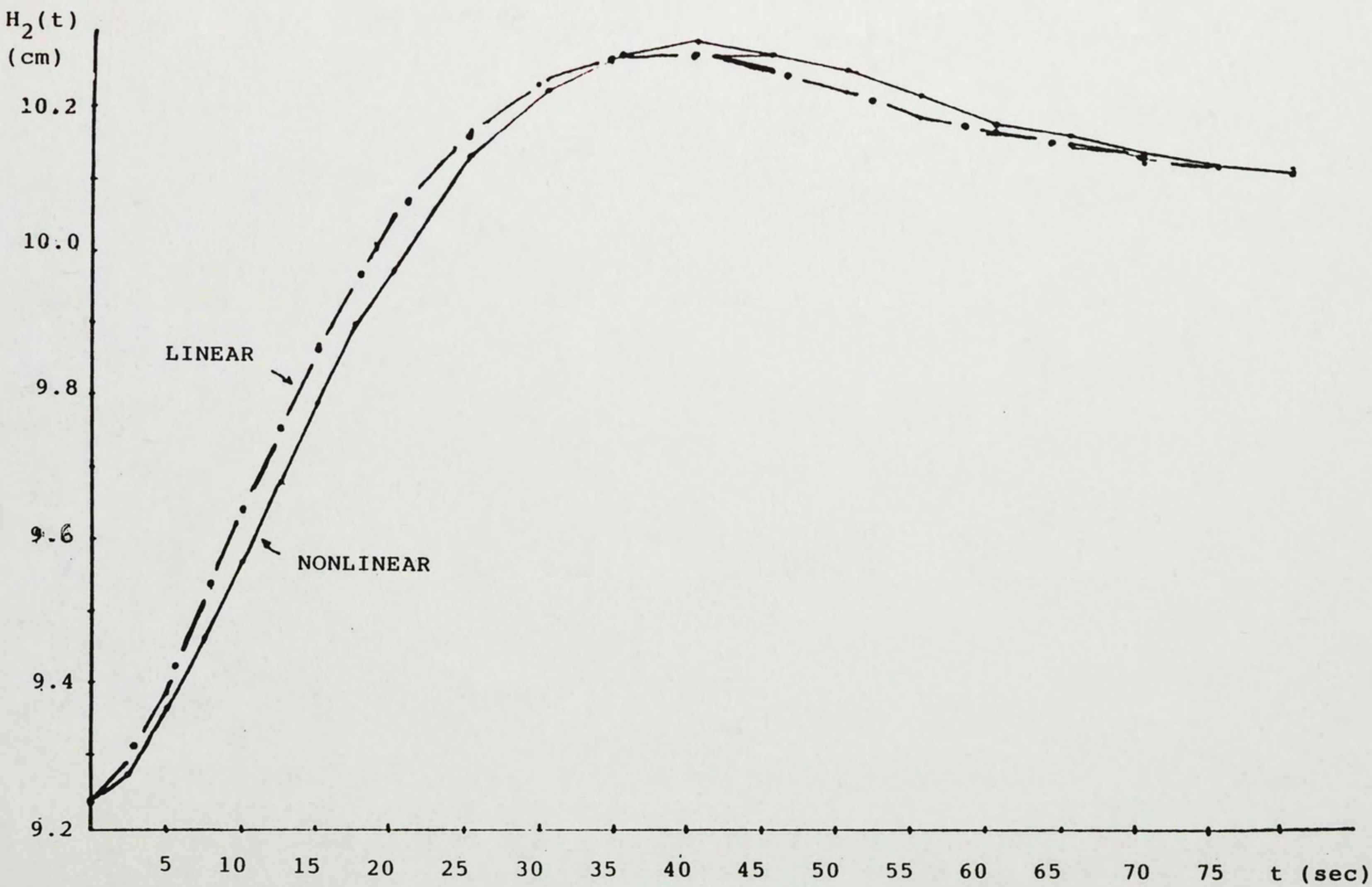
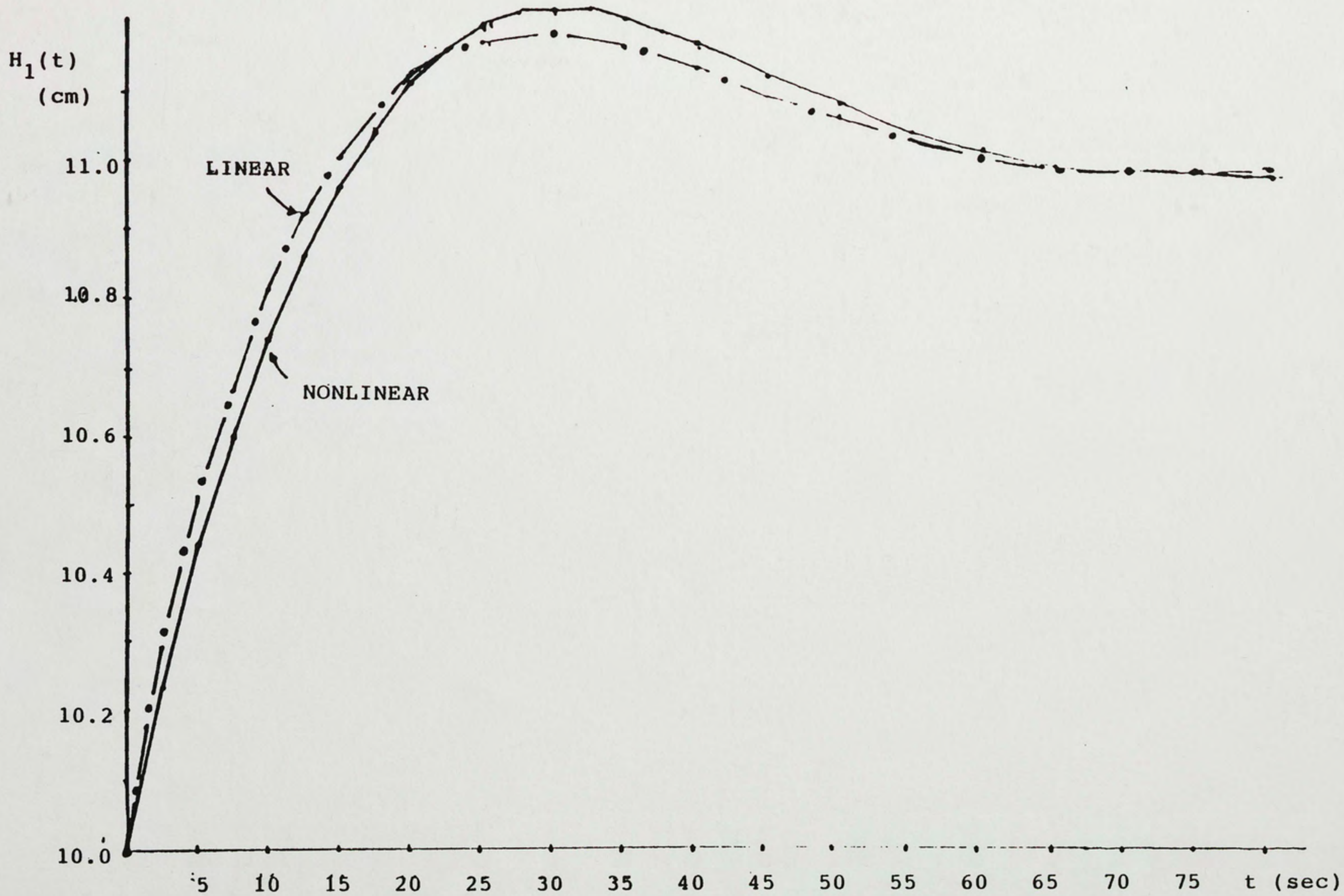


Figure 6-4. Height Of Tank 2 Vs Time Proportional-Integral Control.



CHAPTER VII

SUMMARY

This report investigated the comparison of linearized and nonlinearized solutions of a nonlinear physical situation. The system was modeled through the relationships of height to voltage, voltage to pump flow, tank flow rates and orifice flows. The system was then linearized and an analytic solution was calculated using linear control theory and Laplace Transform techniques. A nonlinear system simulation found the exact solution using the model's flow rate equations as difference equations.

Results for these two different solution techniques were obtained and compared. No major differences were apparent between the linearized and nonlinearized solutions. The only perceptable difference between these solutions was a slight lag between the linear and actual nonlinear result. This lag arose in both proportional only and proportional-integral control situations. In both of these situations, the linearized and nonlinearized solutions attained the same steady state value.

In conclusion, this paper has shown that for a nonlinear system under proportional only or proportional-integral control, a linearization can be used to find an analytical solution that closely resembles the exact numerical solution. This analytical solution sufficently replicates the exact numerical system to use modern control theory techniques for subsequent studies in the variation of proportional gain, reset time and the effects of a change in step size of the reference variable on the system.

APPENDIX

REFERENCE LISTING FOR COMPUTER MODEL OF THE NONLINEAR SYSTEM AND ASSOCIATED VARIABLE DESCRIPTIONS

H3 = HEIGHT OF ORIFICE 3 (H_3)

CD1 = DISCHARGE COEFFICIENT ORIFICE 1 (C_{D1})

CD2 = DISCHARGE COEFFICIENT ORIFICE 2 (C_{D2})

G = GRAVITATIONAL CONSTANT (g)

A = AREA OF TANK (A)

A1 = CROSS SECTIONAL AREA OF ORIFICE 1 (a_1)

A2 = CROSS SECTIONAL AREA OF ORIFICE 2 (a_2)

KC = PROPORTIONAL GAIN (K_C)

TR = RESET TIME (T_R)

TD = DERIVATIVE TIME (T_D)

H1 = HEIGHT OF TANK 1 (H_1)

H2 = HEIGHT OF TANK 2 (H_2)

HSP = SET POINT HEIGHT (H_{1R})

FL = LOAD VARIABLE INTO TANK 2 (F_L)

F1D = ORIGINAL PUMP FLOW (F_1)

VD = ORIGINAL MOTOR VOLTAGE (v_m)

VS = REFERENCE VOLTAGE (v_{1R})

V = VOLTAGE OF TANK 1 (v_1)

EV = DIFFERENCE FROM REFERENCE (e)

VM = VOLTAGE TO THE MOTOR (v_m)

D1 = RATE OF CHANGE OF HEIGHT IN TANK 1 ($\frac{dH_1}{dt}$)

D2 = RATE OF CHANGE OF HEIGHT IN TANK 2 ($\frac{dH_2}{dt}$)

Definitions of Nonlinear System Variables With Their Linear System Equivalent in Parentheses

DT = TIME STEP

PTSP = NUMBER OF TIME STEPS PER PRINTING

O(A) = VOLTAGE-HEIGHT RELATIONSHIP eqn. 2-3

X(A) = FLOW-VOLTAGE RELATIONSHIP eqn. 2-1

A = DUMMY VARIABLE

CNF = TOTAL OF OPEN RADIUS^2 FOR ORIFICE 1

A$ = INPUT VARIABLE TO SIGNIFY OPEN HOLE "Y" PLUGGED

"N" OPEN

B = SIGNIFIES OPENING OF TAP

C1 = CONSTANT DESCRIBING ORIFICE 1

C2 = CONSTANT DESCRIBING ORIFICE 2

HSTP = STEP CHANGE IN THE REFERENCE VARIABLE

PEV = PREVIOUS TIME STEP'S ERROR

TTE = TOTAL ERROR

T = RUNNING TIME

L = COUNTS PRINT STEPS

W = COUNTS TIME STEPS

Description of Other Variables

```
10  REM
20  REM  STEP SIZE AND PRINT STEP

30  REM
40 DT = .1
50 PTSP = 10
60  REM
70  REM  VOLTAGE-HEIGHT RELATIONS
     HIP FOR TANK 1
80  REM
90  DEF  FN O(A) = .00081 * A ^ 3
      - .02214 * A ^ 2 + .47795 *
     A + 1.1766
100  REM
110  REM  PUMP-VOLTAGE RELATIONSH
     IP
120  REM
130  DEF  FN X(A) = ( - 49.176 *
     A ^ 2 + 1023.3 * A - 687.28)
      / 60
140  REM
150  REM  PHYSICAL PARAMETERS
160  REM
170 H3 = 3
180 CD1 = 1
190 CD2 = 1
200 G = 980
210 A = 200
220  PRINT "CHOOSE A TANK CONFIGU
     RATION"
230 CNF = (.317) ^ 2
240  PRINT "WHICH HOLES DO YOU WI
     SH PLUGGED?"
250  PRINT "(Y=YES,N=NO)"
260  PRINT "1.27 CM ";
270  INPUT A$
280  IF A$ = "N" THEN CNF = CNF +
     (1.27) ^ 2
290  PRINT ".95 CM ";
300  INPUT A$
310  IF A$ = "N" THEN CNF = CNF +
     (.95) ^ 2
320  PRINT ".635 CM";
330  INPUT A$
340  IF A$ = "N" THEN CNF = CNF +
     (.635) ^ 2
350 A1 = CNF * 3.14 / 4
360  PRINT
370  PRINT "HOW DO YOU WANT THE T
     AP OF THE SECOND TANK SET ?"
380  PRINT "1. FULL OPEN"
390  PRINT "2. 3/4 OPEN"
400  PRINT "3. 1/2 OPEN"
410  PRINT "4. 1/4 OPEN"
420  INPUT B
430  IF 0 < B AND B < 5 AND B =  INT
     (B) THEN 460
440  PRINT "THAT WAS NOT ONE OF Y
     OUR CHOICES,TRY AGAIN"
450  GOTO 370
460 A2 = (.875 - .175 * B) ^ 2 *
     3.14 / 4
470  PRINT
480  PRINT "PHYSICAL PARAMETERS"
490  PRINT
500  PRINT "H3 = "H3,"G  = "G
510  PRINT "CD1= "CD1,"CD2= "CD2
520  PRINT "A1 = " INT (A1 * 100)
      / 100,"A2 = " INT (A2 * 100
     ) / 100
530  PRINT "A  = "A
540 C1 = CD1 * A1 *  SQR (G * 2)
550 C2 = CD2 * A2 *  SQR (G * 2)
560  PRINT
570  PRINT "CONTROL PARAMETERS"
580  PRINT
590  PRINT "ENTER KC";
600  INPUT KC
610  PRINT "ENTER TR FOR INTEGRAL
      CONTROL (0 IF NONE DESIRED)
     ";
620  INPUT TR
630  IF TR = 0 THEN TR = 99999999
     9999
640  PRINT "ENTER TD FOR DIFFEREN
     TIAL CONTROL(0 FOR NONE)";
650  INPUT TD
660  PRINT
670  PRINT "INITIAL VALUES"
680  PRINT
690  PRINT "WHAT IS YOUR INITIAL
     LEVEL IN TANK 1 ("H3" TO 25
     CM)";
700  INPUT H1
710  IF H3 <  = H1 AND H1 <  = 25
      THEN 740
720  PRINT "THAT IS OUT OF RANGE,
     TRY AGAIN"
730  GOTO 700
```

```
740  PRINT "INPUT THE STEP CHANGE
       IN THIS CONTROL VARIABLE";
750  INPUT HSTP
760 HSP = H1 + HSTP
770  PRINT "INPUT THE LOAD VARIAB
     LE (CC/MIN)";
780  INPUT FL
790 FL = FL / 60
800 H2 = H1 - ((C2 *  SQR ((C1 ^
     2 + C2 ^ 2) * (H1 - H3) - FL
      ^ 2) - C1 * FL) / (C1 ^ 2 +
     C2 ^ 2)) ^ 2
810 F1D = C1 *  SQR (H1 - H2)
820 VD = 10.4 - 1.105 *  SQR (77.
     356 - F1D)
830 VS =  FN Q(HSP)
840  PRINT
850  PRINT "VARIABLE VALUES"
860  PRINT
870  PRINT "H1(0) = " INT (H1 * 1
     00) / 100,"H2(0) =" INT (H2 *
     100) / 100
880  PRINT "F1D = " INT (F1D *600
     0) / 100,"FL = " INT (FL * 6
     000) / 100
890  PRINT "SET POINT " INT (HSP *
     100) / 100
900  PRINT "SET POINT VOLTAGE " INT
     (VS * 100) / 100
910  PRINT "DESIGN VOLTAGE " INT
     (VD * 100) / 100
920  PRINT
930  REM  SET OTHER INITIAL VALUE
     S
940  REM
950 V =  FN Q(H1)
960 EV = VS - V
970 VM = VD + KC * (EV + TD * EV /
     DT)
980 PEV = 0
990 TTE = 0
1000  IF VM > 10 THEN VM = 10
1010  IF VM < 2 THEN 1030 AND F1 =
     0
1020 F1 =  FN X(VM)
1030 D1 = (F1 - F1D) / A
1040 D2 = 0
1050 T = 0
1060  PRINT "TIME","TANK1","TANK2
     "
1070  PRINT "(SEC)","(CM)","(CM)"
1080  PRINT  INT (T), INT (H1 * 1
     00) / 100, INT (H2 * 100) /
     100
1090  REM
1100  REM  STEP WITH TIMESTEP DT
1110  REM             PRINTSTEP PT
     SP
1120  FOR L = 1 TO 80
1130  FOR W = 1 TO PTSP
1140  REM
1150  REM  FIND NEW VALUES
1160  REM
1170 T = T + DT
1180 H1 = H1 + D1 * DT
1190 H2 = H2 + D2 * DT
1200 F0 = C2 *  SQR (H2 - H3)
1210 F2 = C1 *  SQR (H1 - H2)
1220 V =  FN Q(H1)
1230 EV = VS - V
1240 TTE = TTE + (PEV + EV) * DT /
     2
1250 VM = VD + KC * (EV + TTE / T
     R + TD * (EV - PEV) / DT)
1260 PEV = EV
1270  IF VM > 10 THEN VM = 10
1280  IF VM < 2 THEN 1300 AND F1 =
     0
1290 F1 =  FN X(VM)
1300 D1 = (F1 - F2) / A
1310 D2 = (F2 + FL - F0) / A
1320  NEXT W
1330  PRINT  INT (T + .5), INT (H
     1 * 100) / 100, INT (H2 * 10
     0) / 100
1340  NEXT L
```

BIBLIOGRAPHY

Conte, S. D., and deBoor, C., 1972. Elementary Numerical Analysis: An Algorithmic Approach, McGraw Hill.

D'Azzo, J. J., and Houpis, C. H., 1975. Linear Control System Analysis And Design, McGraw-Hill.

Kreyszig, E., 1979. Advanced Engineering Mathematics, John Wiley & Sons.

Weber, T. W., 1973. An Introduction To Process Dynamics And Control, John Wiley & Sons.

Weidner, R., and Sells, R., 1973. Elementary Classical Physics, Allyn and Bacon Inc.

Wellstead, P. E., 1981. Coupled Tank Apparatus. Tecquipment.