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MODAL ANALYSIS BY STATE SPACE APPROACH IN FREQUENCY DOMAIN

BY

FELICIANO FEIJO
B.S.M.E., Ricardo Palma University, 1973

RESEARCH REPORT

Submitted in partial fulfillment of the requirements
for the degree of Master of Science
in the Graduate Studies Program of the College of Engineering
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ABSTRACT

This report tests a new procedure to identify the eigenvalue problem matrix "A" by utilizing the frequency response of a system in state-space for a given sinusoidal forcing input.

Two sample identification tests are performed and a detailed explanation of both cases is presented. The method proved to be correct and the results obtained were accurate in both cases. A random error was then added to both sample tests in order to simulate real measuring conditions. Results indicate that it is more difficult to identify the "A" matrix of an undamped system. The error in the system must be very small for this type of system to be identified. However, when the identification is repeated several times and then averaged, the identification becomes more accurate.

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NOMENCLATURE

A	System's matrix in state space equation
B	Matrix with the added random error
b_1	Forcing input (sin wt)
b_2	Forcing input (cos wt)
C	Damping matrix
c_{ij}	Element of matrix C
D	Diagonal matrix
E	Error matrix
e_{ij}	Element of matrix E
F*	Dissipation function
G	Gyroscopic matrix
g_{ij}	Element of matrix G
H	Circulatory forces matrix
h_{ij}	Element of matrix H
I	Identity matrix
i	Integer number
K*	Stiffness matrix in 2n-dimensional state space representation
K	Stiffness matrix
k_{ij}	Element of matrix K
L	The Lagrangian

NOMENCLATURE (Continued)

M^*	Mass matrix in 2n-dimensional state space representation
M	Mass matrix
m_{ij}	Element of matrix M
n	Dimension of a vector matrix
Q	Generalized forces
q	Generalized coordinate
r	Number of constraints on the original system
T	Kinetic energy of the system
t	Time
U	Input vector
$\bar{u}$	r-dimensional input vector
V	Potential energy
X	Right eigenvectors matrix
x	Element of the right eigenvectors matrix, state vector
Y	Left eigenvectors matrix
y	Element of the left eigenvectors matrix
Z	m-dimensional output
α	Real part of complex eigenvalue
β	Imaginary part of complex eigenvalue
ε	A given number
λ	Eigenvalues of the system's matrix
ρ	Eigenvalues of the error matrix

NOMENCLATURE (Continued)

ϕ	Phase angle
ψ	Phase of the state variable
ω	Frequency

CHAPTER I

INTRODUCTION

Modal analysis can be defined as the theoretical or experimental analysis of the structure's dynamic characteristics of a mechanical system in terms of its modal parameters; that is, finding the eigenvalues and eigenvectors of the equations of motion which define the mechanical system. A review of the past, present and future of modal analysis technology will improve our knowledge on different methods and techniques that are being used for estimating, testing, and solving vibration problems.

Past Modal Analysis Technology

The development of modal testing techniques, which took place mainly after World War II, was made principally in the aircraft and space vehicle fields (Budd 1969). Before 1960, frequency response determination for modal analysis was limited to a discrete sine testing method, and mode shape was determined by a simple sine dwell method. Sand patterns and other similar testing techniques were also used to determine nodal (zero amplitude) patterns. During the early 1960's, the most important development was the tracking filter that allowed the use of the mechanical impedance measuring analyzer, mostly known as the transfer function analyzer. The transfer function analyzer measured frequency response by using a sinusoidal excitation signal to excite the mechanical system. This process made it possible to get frequency response plots, and from the poles, natural frequencies, and damping could be estimated. Another improvement was the development of a co-quad meter which allowed the real and imaginary components of the frequency response to be measured directly. Instead of using the total amplitude for the mode shape values, the quadrature response could be used to separate closely coupled modes.

Experimental modal analysis became divided into two fields: (1) the multi-input sine dwell (MISD), and (2) the single input frequency response (SIFRQ).

Multi-input Sine Dwell Method

The multi-input sine dwell test was the most popular modal testing method. Several classical papers were published which discuss the calculation of force ratios among shakers, like C.C. Kennedy (1947), R.C. Lewis (1950), G.W. Asher (1958) and R.E.D. Bishop (1963). In this method, a sine dwell is performed to determine the frequencies of the system poles. A number of exciters are connected to the structure and tuned to one of the pole frequencies. The force amplitude and phase are adjusted such that only one mode is excited by this combined forcing function. There are several checks which are performed to determine if the mode is properly tuned. Tuning each mode of vibration, which included positioning the shakers, adjusting the frequency and adjusting the forcing vector was the difficult procedure with this method. Only one frequency could be measured at a time, so it was required to have a large number of response transducers in parallel to minimize the test time.

Single-input Frequency Response Method

For the single-input frequency response method it was necessary to measure a large number of frequency responses. This was a very slow process using existing analog equipment at that time. But, in the late sixties, the minicomputer systems and the implementation of the fast Fourier transform algorithm in these computers made it possible to develop a Fourier analysis system which could be used to measure frequency response in a small fraction of the time required with the analog equipment. J.W. Cooley and J.W. Tukey (1965) developed the Fast Fourier Transform procedure which made all that possible and did some studies on complex Fourier series machine calculations.

Testing Methods

Many new excitation and testing methods were developed: impact testing, random, pseudo-random, etc., as a result of these new Fourier techniques. During the early seventies, a great deal of research was done to develop these testing methods to define their limitations and application areas. The main applications were made in trouble-shooting vibration problems and verifying finite element results. For the trouble-shooting applications it was not necessary to measure exact estimates of the modal vectors, since in most of the trouble-shooting applications these estimates were not used to build analytical models for finite element modeling.

However, it was difficult to compare experimental results with finite element results. Finite element analysis does not consider damping in most cases and damping is always present in experimental results.

Analytical Methods

Analytical methods were improved by the emergence of the finite element method, which was in turn made attractive by Fourier analysis and the digital computer. This method for dynamics got started in the early sixties with lumped mass finite element models of structural systems. The analysts were developing their modeling expertise: What kind of structures could be modeled and what elements? How should various boundary conditions be approximated, etc.? Methods for building analytical models by merging substructures based upon finite elements and experimental results were started. The results obtained during the sixties were limited by the ability to extract good modal data bases from measurements and by inadequate methods for including effects of damping and non-linearities. The seventies brought new reduction and solution algorithms, much better pre- and post-processing of the data, very good computer graphics, more powerful and cheaper computers, all of which greatly reduced the cost of constructing a finite element model.

Present Modal Analysis Technology

Experimentally, the present technology is a continuation of the technologies developed during the sixties but much more refined. With the introduction of low cost minicomputers, and the development of software programs for modal testing, for example General Radio Time/Data Division (1977) and Hewlett-Packard (1974), more tests are done using digital equipment. The key to conducting the modal test using a digital signal processing technique is the measurement of the frequency response function between the input forces and the responses at different locations of the structure. Modal parameters for defining the dynamic characteristics of the structure are then extracted from these measured functions. Substantial literature has dealt with this approach, for example, M. Richardson and R. Potter (1974); K.A. Ramsey (1975, 1976); A. Klosterman and R. Zimmerman (1975); D. Brown, R. Allemang, R. Zimmerman and M. Mergeay (1979). Single excitation frequency response methods have probably been developed nearly to their limit. The multi-shaker sine dwell method, as in R.R. Craig and Y.W.T. Su (1974), is also reaching its practical limits. Multi-input methods are currently being developed which are combining the best features of both methods, also new non-traditional methods have been developed (Brown 1982).

Testing Methods

Single-Input Frequency Response Function (SIFRF)

Multidegrees of freedom parameter estimation methods were developed which allowed a single frequency response measurement to be curve-fit reliably with minimum error. For trouble-shooting applications or frequency regions of low modal density, the single point excitation method has become very popular. In fact, more than ninety-five percent of the systems in service use this technology. We can find an example of this technique in Potter (1979).

Multi-Input Sine Dwell (MISD)

The MISD method has been limited to large aerospace companies. This method requires considerable investment in excitation and transducer systems. An approach to this method is studied by T. Comstock, J. Niebe, G. Wylie and F.H. Chu (1979). The method is only practical if a large number of transducers can be permanently mounted to the test structure, since only one mode can be determined at a time and it is necessary to tune each mode. Then, the testing time can become prohibitive.

Multi-Input Frequency Response (MIFRF)

The MIFRF method is an attempt at combining the advantages of the SIFRF method and the MISD method. The main advantage of the SIFRF method is that the complete dynamic response for a frequency band is determined between the input and output points while the major advantage of the MISD method is the excellent modal separation which is obtained. Also with the SIFRF method there is less chance of missing a mode, but a greater chance of measuring a contaminated mode. The data must be measured simultaneously with MIFRF method in order to separate the modes. The use of this procedure for accurate measurements of the frequency response function is explained by R.J. Allemang, R.W. Rost, and D.L. Brown (1982).

Ibrahim Time Domain (ITD)

The Ibrahim method is one of the new methods which uses free-decay responses in time domain to compute the modal parameters. It is not necessary to measure the system inputs, but impulse response functions can be used in the place of the free responses. The poles and modal vectors can be estimated from the free decays, but it is not possible to estimate the modal scale factors (modal mass or stiffness) without measuring the input force (unit impulse functions). The procedure is to measure the free decays at various points on the structure. A recurrence matrix is created from the free decays; the eigenvalues of this matrix are the poles of the system, the eigenvectors are the measured modal vectors. This method of identification of parameters is studied in S.R. Ibrahim and E.C. Mikulick (1973, 1976).

Autoregressive Moving Average (ARMA)

This method computes the modal characteristics by assuming that the response data is caused by a white random noise input to the system. The technique computes the best statistical model of the system in terms of its poles (from the autoregressive part), and zeros (from the moving average part), as well as statistical confidence factors on the parameters. Since in the general case the inputs are not measured, the modal vectors are determined by referencing each response function to a single response to provide relative magnitude and phase information. A method to estimate modal parameters using this technique has been developed by David H. Friedman (1982).

Direct Parameter Identification (DPI)

In this method, mass, stiffness and damping matrices are computed directly from the measured force input and response measurements taken from the test article. The system's modal parameters are computed directly from the mass, stiffness, and damping matrices by an eigenvalue decomposition method. This method does not require any restrictions on the relationships between input forces and can compute repeated roots, which none of the above methods do directly. An application of this procedure can be seen in Mauri Maattanen (1982).

Poly Reference (PR)

The poly reference method is a technique which uses measured unit impulse responses taken from a number of simultaneous exciter positions. It is a multi-dimensional complex exponential algorithm method. The complex exponential is a standard, multi-degree of freedom parameter estimation algorithm which is currently used to curve fit single input measurements. The method generates a companion matrix whose roots are the poles of the systems. Since it measures data from several exciter positions, it can be used to solve for repeated roots or in the ordinary measurement case pseudo-repeated roots. Pseudo-repeated roots and poles so closely coupled that they cannot, in a measurement sense, be distinguished from a true repeated root. An application of this method is shown by Havard Vold and G. Thomas Rocklin (1982).

Modal Assurance Criterion (MAC)

This technique measures the modal vector directly from the responses of a system to a series of randomly applied inputs. The procedure is to excite the system from a randomly distributed number of input points. The responses are measured at a minimum of two points on the structure per measurement cycle. For each input, the residues of both the reference and roving responses are estimated for each pole. Once the residues are estimated, a linear regression analysis is performed between the residues measured at the reference point and the roving point. The constant of proportionality between these two data sets is an estimate of the modal vector at the roving point. This method does not directly estimate poles (frequencies and damping) only modal vectors. An application of this technique has been done by G.D. Carbon, D.L. Brown and R.J. Allemang (1982).

Measurement Equipment

The measurement equipment that is currently used can be divided into three major categories: (a) Small dedicated two-channel Fourier analyzer systems that are dedicated systems which are not programmable and are very useful for the trouble-shooting type of problem. They, in general, use SIFRF method excitation and simple parameter estimation algorithms. (b) Larger multi-channel Fourier analyzer systems that are programmable general purpose computer systems with Fourier analysis software. These systems use the newer parameter estimation programs. The excitation signal used is of the random type, which tends to average out non-linear distortion errors so the measurements are more compatible with the parameter estimation algorithms. (c) Data acquisition systems are used to gather data for input into a more powerful computer system. This is done with some of the newer techniques which do not use frequency response information. Some studies on measurements have been done by K.A. Ramsey (1975, 1976) and R.W. Potter (1979).

Analytical Methods

The knowledge in analytical methods has been and is continuing to advance very rapidly. The finite element theory has been extensively developed with a large number of new programs for pre- and post-processing of data files, large new data base programs interfaced with drafting and solids generation programs used in the computer-aided manufacture areas. Finite element programs have been developed which run on typical desk top computers and small minicomputer systems for use in limited applications or by users with infrequent requirements. There are commercial programs available for taking modal measurements or data bases, and building an analytic model of the system which can be used to predict modifications. Some of these analytical methods are explained in the following papers: R.W. Mastain (1976): G.A. Hamma, S. Smith and R.C. Stroud (1976); R.G. Stroud, C.J. Bonner and G.J. Chambers (1978); and C.V. Stahle (1978).

Future Application of Modal Analysis

In the testing area, systems with a large number of input channels should be developed. The cost of transducers and data acquisition should be reduced so industry, in general, can afford the systems. The parameter estimation will use the complete data base in one pass to get the best modal model based upon this complete data set.

New transducers or procedures should be developed to measure moments and rotations which are not presently measured, but which are very important in building an analytical model based upon experimental data. Less expensive, mcre powerful and portable field test units should be developed for the trouble-shooting applications. Damping and non-linearities should be incorporated into the analytical models.

In the analytical area, the methods presented here and other advances in the area will, hopefully, help in making modal analysis more accurate, easier to perform and simpler to modify.

CHAPTER II

ANALYTICAL MODAL ANALYSIS

Analytical Determination of Modal Data

Methods of analytical determination of modal parameters are most readily examined in terms of the various steps of the analysis. The major steps in the analysis are the following:

1. Modeling of the system
2. Formulation of equations of motion
3. Solution for modal data

The derivation of the mathematical model of the system and the associated choice of coordinate system are linked with the formulation of the equations of motion. In all practical cases, at least parts of the system have distributed masses, and so the number of degrees of freedom required to represent exactly its dynamic motion is infinite. The reduction of a real structure to systems of finite degrees of freedom represents one of the basic simplifications in the modeling process. This can be done by such techniques as lumping of the masses (springs), by application of finite element techniques, or by a Ritz (1911) approach. The application of these techniques results finally in the equations of motion which reduce to a matrix eigenvalue problem by removal of the time factor. The methods by which these equations are solved are independent of the derivation of the mathematical model and the formulation of the equations of motion.

Modeling of the System

The mathematical model is the most important factor in obtaining satisfactory modal definition for the system. If the model is of poor quality, mathematical rigor in the solution of the equations of motions will not improve results. The following basic factors are given careful consideration in the synthesis of the mathematical model:

1. Stiffness distribution
2. Mass distribution
3. Boundary conditions

Neglect of any of these considerations may result in a model that is not dynamically similar to the actual system.

Stiffness Distribution

The definition of the stiffness distribution is the most difficult step in the synthesis of a mathematical system dynamic model. For systems having continuously distributed properties, a model exhibiting those same properties must be utilized (Hurty and Rubinstein 1964). Models of this type are used with displacement functions to obtain modal parameters.

Most structures are complex and contain discontinuities in stiffness. For these structures, mathematical models are used which are composed of independently modeled structural components (finite elements) joined at coordinate points through common displacement and force components. These are defined as finite element models.

In the analysis of systems with many components and large differences in stiffness parameters, the modal parameters of the components are used to synthesize the overall structural characteristics (Hurty 1965). Some advantages of using this technique are that it solves lower-order eigenvalue problems and generates the stiffness and mass parameters for smaller substructures.

Mass Distribution

Mass distribution depends on the physical system under consideration and the method of analysis. For a system with a not uniform mass distribution, commonly handled by an analysis based on continuous-system equations, the mass distribution is readily defined by the actual structural distribution.

Several methods are used to define the mass distribution. These include the lumped-mass method, the consistent-mass method, and a number of approaches that use various velocity-interpolation functions to define a mass matrix (Bisplinghoff, Ashley, and Halfman 1955). The lumped-mass method distributes the element masses in concentrations located at the coordinate points in a physically reasonable manner which maintains the center of mass of the structure (Fowler 1959). This method of distribution is well suited for analysis of structures with concentrated masses. The disadvantage of the method is the relatively large number of coordinate points required for accurate analysis of systems with preponderantly distributed masses. The consistent-mass represents the mass in agreement with the actual distribution of mass in the structure (Archer 1963; Melosh and Lang 1965). The consistent-mass distribution technique gives more accurate results than the concentrated-mass technique for systems where the mass is largely distributed in the structure.

Boundary Conditions

Because natural modes of a structure are sensitive to boundary conditions, the same boundary conditions are imposed on the model as on the actual structure, insofar as feasible. Frequently, static tests must be performed to determine the influence coefficients defining the boundary conditions at an interstage connection with a supporting stage structure.

Formulation of Equations of Motion

The methods of formulating the equations of motion can be classified under the following categories:

1. Integral equation methods
2. Differential equation methods
3. Energy methods

Each method may include either the theoretically exact or approximate approach and can be used to handle both the distributed and discrete system models. In all but a few special cases, however, the mathematically exact solution to the equations of motion cannot be found and the analyst must resort to numerical techniques. Examples of these methods are found in Melosh and Lang (1965), Melosh (1963), and Meirovitch (1967).

Integral Equation Methods

Integral equation methods of formulating the equations of motion make use of an influence-coefficient function that defines the displacement of any point of a structure in terms of the applied load. The displacement under the inertial loads is obtained through integral equations which become the equations of motion. The advantage of this method is that it includes the boundary conditions in the basic equation of motion.

Differential Equation Methods

The natural modes and frequencies of a system can be determined through differential equations which relate the structure's distortions to the inertial forces on the structure. This is a classical approach, and modal data obtained by this approach are available for a large variety of simple configurations. However, it is difficult to apply this method to a complex structure.

Energy Methods

Energy methods for formulation of the equations of motion are based on energy principles of mechanics, such as conservation of energy, virtual work, Lagrange's equation and Hamilton's principle. In this approach, displacement functions that approximate the mode shape are used to represent the system behavior. While the chosen functions are not theoretically restricted to those satisfying all the boundary conditions for the mode shapes, the accuracy of the solution depends strongly on how well the boundary conditions are satisfied. The advantage of energy methods lies in their versatility; they can be applied to any structural configuration and can approximate the system behavior to any desired degree of precision.

Solution for Modal Data

The equations of motion can be solved to obtain the modal data by several methods. Hand solution is usually limited to small order systems because of the overwhelming amount of numerical labor involved. The solution of large-order systems is generally achieved on electronic computers.

Exact solutions are available for linear differential equations with constant coefficients, such as equations used to characterize systems with both uniform mass and stiffness properties. Where these properties are not uniformly distributed, exact solutions are not always possible and approximate solutions are obtained. Studies on solution for modal data have been done by Hintz (1975), Caughey (1960), Hou (1969) and Meirovitch (1975).

System Equations of Motion

Mathematical models of dynamic systems can be divided into two large classes: discrete and distributed. The reference is to the system parameters such as mass, damping and stiffness. Discrete systems are described by functions depending on time alone, and distributed systems are described by functions depending on time and space. The minimum number of dependent variables required to fully describe the motion of a discrete system is known as the number of degrees of freedom of the system. These variables are called coordinates. Defining the system Lagrangian as:

$$L = T - V \qquad (1)$$

where:

T = kinetic energy of the system

V = potential energy of the system

The equations of motions take the form of Lagrange's equations (Meirovitch 1980):

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} + \frac{\partial F}{\partial q_i} = Q_i \qquad (2)$$

where:

F = dissipation function and Q_i = generalized forces

The solution of equation 2 consists of the n generalized coordinates $q_i(t)$ $(i = 1, 2, \ldots, n)$. This solution can be interpreted geometrically by conceiving of an n-dimensional Euclidean space with q_i as axes, where the space is known as the configuration space. The solution represents a point in the configuration space defined by the tip of a vector $\overline{q}(t)$ whose components are the generalized coordinates $q_i(t)$.

The configuration space is not very convenient for a geometric representation of the motion. The main reason is that two dynamical paths can intersect, so that a given point in the configuration space does not define the state of the system uniquely. If the generalized velocities $\dot{q}_i$ are used as a set of auxiliary variables, then the motion can be described in a 2n-dimensional Euclidean space defined by q_i and $\dot{q}_i$ and known as the state space. The set of 2n variables q_i and $\dot{q}_i$ define a state vector and the tip of this vector traces a trajectory in the state space. The advantage of the representation in the state space is that two trajectories never intersect, so that a given point in the state space corresponds to a unique trajectory.

The interest of this study lies in small motions about equilibrium points, and motions confined to small neighborhoods of the equilibrium points. The assumption of small motions about an equilibrium point implies linearization of the equations of motion, so we can ignore terms of degree higher than two. From this assumption, Meirovitch (1980) defines the mass, stiffness, damping, gyroscopic and circulatory forces coefficients such that:

$$M = [m_{ij}],\ K = [k_{ij}],\ C = [c_{ij}],\ G = [g_{ij}],\ H = [h_{ij}] \tag{3}$$

where:

m_{ij} = mass coefficient

k_{ij} = stiffness coefficient

c_{ij} = damping coefficient

g_{ij} = gyroscopic coefficient

h_{ij} = circulatory forces coefficient

M = mass matrix

K = stiffness matrix

C = damping matrix

G = gyroscopic matrix

H = circulatory forces matrix

M, K and C are symmetric matrices, then we have:

$$M = M^T,\ K = K^T,\ C = C^T \tag{4}$$

The equation of motion becomes:

$$M\ddot{\bar{q}} + (G + C)\dot{\bar{q}} + (K + H)\bar{q} = \bar{Q} \tag{5}$$

where Q is the generalized forcing vector.

<u>Forced Vibration and the Eigenvalue Problem</u>

A convenient way of deriving the solution of equation 5 is modal analysis, which requires the solution of the so-called eigenvalue problem for the system. Meirovitch (1980) derives the eigenvalue problem for various cases. Here we will be concerned with the two most frequent cases.

Conservative Non-gyroscopic System

In the absence of gyroscopic, damping, circulatory forces, we have G = C = H = 0. Then, equation 5 reduces to:

$$M\ddot{\bar{q}}(t) + K\bar{q}(t) = \bar{Q} \tag{6}$$

We can rewrite this equation as:

$$M^*\dot{\bar{x}}(t) + K^*\bar{x}(t) = \bar{U} \tag{7}$$

Where x(t) is the 2n-dimensional state vector:

$$\overline{x}(t) = [\dot{\overline{q}}(t)^T \mid \overline{q}(t)^T]^T$$

and (8)

$$M^* = \begin{bmatrix} M & 0 \\ 0 & I \end{bmatrix}, \qquad K^* = \begin{bmatrix} 0 & K \\ -I & 0 \end{bmatrix}$$

2n x 2n are real matrices and where $\overline{U}$ is a forcing vector.

$$\overline{U} = \left[\frac{\overline{Q}}{0}\right]$$

The final formulation is then given by:

$$\dot{\overline{x}} = A\overline{x} - M^{*-1}\overline{U}$$

where

$$A = -M^{*-1}K^* = \begin{bmatrix} 0 & -M^{-1}K \\ I & 0 \end{bmatrix} \tag{9}$$

Assuming that M* is not singular, A is an arbitrary real matrix. Because A is real, if the eigenvalues λ are complex, then they must occur in pairs of complex conjugates and so must the eigenvectors x.

Damped Non-gyroscopic Systems

If we let G and H be zero in equation 5, then the equation becomes:

$$M\ddot{\overline{q}}(t) + C\dot{\overline{q}}(t) + K\overline{q}(t) = \overline{Q} \tag{10}$$

It describes the forced vibration of a damped non-gyroscopic system, where all three matrices are real and symmetric. As it was done before, equation 10 can be rewritten in the form of equation 7:

$$M^*\dot{\bar{x}}(t) + K^*\bar{x}(t) = \bar{U}$$

where

$\bar{x}(t)$ is a 2n-dimensional state vector (equation 8) and

$$M^* = \left[\begin{array}{c|c} M & 0 \\ \hline 0 & I \end{array}\right], \quad K^* = \left[\begin{array}{c|c} C & K \\ \hline -I & 0 \end{array}\right] \tag{11}$$

2n x 2n are real matrices.

Assuming that M* is non-singular, the equation of motion becomes

$$\dot{\bar{x}} = A\bar{x} - M^{*-1}\bar{U}$$

where

$$A = -M^{*-1}K^* = \left[\begin{array}{c|c} -M^{-1}C & -M^{-1}K \\ \hline I & 0 \end{array}\right] \tag{12}$$

Again, A is real with complex eigenvalues which occur in pairs of complex conjugates, and so also the eigenvectors $\bar{x}$.

The Eigenvalue Problem

It can be easily seen (Meirovitch 1980) that the equation of the eigenvalue problem for damped non-gyroscopic systems can be written in the form:

$$\lambda M^*\bar{x} + K^*\bar{x} = 0 \tag{13}$$

in which $\bar{x}$ is a 2n-vector, where n represents the number of degrees of freedom of the system and:

$$M^* = \begin{bmatrix} M & | & 0 \\ -- & + & -- \\ 0 & | & -I \end{bmatrix} \qquad K^* = \begin{bmatrix} C & | & K \\ -- & + & -- \\ -I & | & 0 \end{bmatrix}$$

2n x 2n are real symmetric matrices which are clearly not positive definite.

For distinct eigenvalues, the orthogonality relations are:

$$x_j^T M^* x_i = 0, \quad \lambda_i \neq \lambda_j \quad i,j = 1, 2, \ldots, 2n \tag{14}$$

$$x_j^T K^* x_i = 0, \quad \lambda_i \neq \lambda_j \quad i,j = 1, 2, \ldots, 2n \tag{15}$$

The eigenvectors can also be "M" normalized by setting

$$\bar{x}_i^T M^* \bar{x}_i = 1 \quad i = 1, 2, \ldots, 2n \tag{16}$$

So that if $X = [\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_{2n}]$ represents the square matrix of the "M" normalized eigenvectors, then we can get:

$$X^T M^* X = I \tag{17}$$

and also

$$-X^T K^* X = D \tag{18}$$

where D is the diagonal matrix of the eigenvalues.

For the case where A (equation 12) is a real non-symmetric matrix, we define:

$$A^T \bar{y}_i = \lambda \bar{y}_i \tag{19}$$

$$A \bar{x}_i = \lambda \bar{x}_i \tag{20}$$

where:

X = right eigenvector matrix; $\bar{x}_i$ = element of matrix X

Y = left eigenvector matrix ; $\overline{y}_i$ = element of matrix Y

After minor manipulations, we find that:

$$X^T Y = I \tag{21}$$

and

$$Y^T AX = X^{-1} AX = \tag{22}$$

The two sets of eigenvectors $\overline{x}_i$ and $\overline{y}_i$ are then biorthogonal.

$$\overline{y}_i^T \overline{x}_i = 1 \quad i = 1, 2, \ldots, 2n \tag{23}$$

Eigensolution Error

If the matrix A is found with an error E such that

$$B = A + E$$

where:

B eigenvalues are $B_1, B_2, B_3, \ldots, B_n$

A eigenvalues are $\lambda_1, \lambda_2, \lambda_3, \ldots, \lambda_n$

E eigenvalues are $\rho_1, \rho_2, \rho_3, \ldots, \rho_n$

We find that according to Weyl's Theorem (Meirovitch 1980):

$$B_r - \lambda_r \leq I E \tag{24}$$

where

r = number of constraints on the original system providing a characterization of the eigenvalues

where

$$||E|| = ||B - A|| = \max. (-\rho_1, \rho_n) \tag{25}$$

where

$$||E|| \leq \left(\sum_{i=1}^{n} \sum_{j=1}^{n} |e_{ij}|^2 \right)^{\frac{1}{2}} \tag{26}$$

or

$$||E|| \leq \max_i \sum_{j=1}^{n} |e_{ij}| \qquad (27)$$

Where the maximum is with respect to any row i of E.

As an illustration, let us assume that all we know about E is that it is a small matrix and that the magnitudes of its elements do not exceed a given number ε. Then:

$$|e_{ij}| \leq \varepsilon, \quad ||E|| \leq n\varepsilon$$

Using Weyl's Theorem, we get:

$$|B_r - \lambda_r| \leq n\varepsilon \qquad (28)$$

Frequency Response in State Space

Conceptually, this method is more involved than the time domain approach. It requires the transformation of measured data from the time domain to the frequency domain and hence, involves the use of digital processing techniques. The key in this method is the measurement of a response function between the Fourier transform of the response to the Fourier transform of the input force. If a Laplace transform is used to transform the test data instead of a Fourier transform, this function is called a transfer function. Modal parameters of the system are then determined from these measured frequency response functions using curve fitting techniques. The major steps involved in this method can be summarized in general as follows:

1. Input a measurable force. This force can be sinusoidal, random, pseudo-random, transient or impulse in nature.

2. Measure the response at a selected point (or points) on the system.

3. Fourier transform the input force and output response measurements.

4. Compute the frequency response functions.

5. Estimate the modal parameters from these functions.

Due to the fact that the internal state of a system is represented by a vector, it is to be expected that the state-space approach will be more mathematically complex than the classical scalar input-output frequency-response technique. Although no design techniques have yet emerged from this new point of view that were not available in the classical frequency-response approach, the insight we gain into the internal state of linear systems in sinusoidal input-output steady-state may eventually lead to new design and analysis approaches.

The following procedure tries to narrow the gap between the classical control viewpoint based on the frequency-response technique and the contemporary approach using the state-space vector methods. Y. Takahashi, M.J. Rabins and D.M. Auslander (1972) develop this procedure where they consider a linear lumped system described by the following state and output equations:

$$\frac{d\overline{x}}{dt} = A\overline{x} + B\overline{u} \text{ or } \frac{d\overline{x}}{dt} = A\overline{x} + b\overline{u} \tag{29}$$

and

$$\overline{Z} = C\overline{x}$$

where:

$x(t)$ = an n-dimensional state

u = r-dimensional input

Z = m-dimensional output

The system is characterized by an 2n x 2n matrix A, input matrix B which is 2n x r and output matrix C which is m x 2n.

The derivative operator d/dt is replaced by $j\omega$ to deal with sinusoidal steady-state (or frequency response), where ω (rad/sec) is the angular frequency of the sinusoidal input and $j = \sqrt{-1}$. Finally, the following form for the ith state variable in sinusoidal steady-state is obtained:

$$x_i(\omega) = \alpha_i \sin \omega t + \beta_i \cos \omega t \tag{30}$$

with

$$\alpha_i = A_i \cos \psi_i, \ \beta_i = A_i \sin \psi_i \tag{31}$$

where:

A_i = amplitude

ψ_i = phase of the ith state space variable

From equation 30, we get:

$$\overline{x}(\omega) = [\overline{\alpha},\overline{\beta}]\begin{bmatrix}\sin \omega t\\ \cos \omega t\end{bmatrix} \tag{32}$$

where:

$$\overline{\alpha} = \begin{bmatrix}\alpha_1\\ \vdots\\ \alpha_{2n}\end{bmatrix}, \ \overline{\beta} = \begin{bmatrix}\beta_1\\ \vdots\\ \beta_{2n}\end{bmatrix} \tag{33}$$

The time derivative for $x(\omega)$ is given by:

$$\frac{d}{dt}\,\overline{x}(\omega) = [\overline{\alpha},\overline{\beta}]\begin{bmatrix}\omega\cos\omega t\\ -\omega\sin\omega t\end{bmatrix} = [-\omega\overline{\beta},\omega\overline{\alpha}]\begin{bmatrix}\sin\omega t\\ \cos\omega t\end{bmatrix} \tag{34}$$

Finally, the values for $\overline{\alpha}$ and $\overline{\beta}$ are found to be (Takahashi, Rabins and Auslander 1972):

$$\overline{\alpha} = (\omega^2 I + A^2)^{-1}\,(-A\overline{b}_1 + \omega\overline{b}_2) \tag{35}$$

$$\overline{\beta} = (\omega^2 I + A^2)^{-1}\,(-A\overline{b}_2 - \omega\overline{b}_1) \tag{36}$$

where:

$\overline{\alpha}$ = real part of the solution $\overline{x}$

$\overline{\beta}$ = imaginary part of the solution $\overline{x}$

ω = circular frequency (rad/sec)

A = eigenvalue problem matrix

$\overline{b}_1$ = forcing input ($\sin\omega t$) vector

$\overline{b}_2$ = forcing input ($\cos\omega t$) vector

The amplitude is given by:

$$|x_i| \text{ Amplitude} = \sqrt{\alpha_i^2 + \beta_i^2}\;,\; i = 1,2,\ldots 2n \tag{37}$$

The phase angle is obtained from:

$$\phi = \tan^{-1}\frac{\alpha_i}{\beta_i} \tag{38}$$

Eigenvalue Problem Matrix Identification

In frequency domain and for non-homogeneous equations, we start the procedure of identifying the eigenvalue problem matrix with equations 29 to 36. That process would describe relationships for sinusoidal steady-state.

Equations 35 and 36 were derived by generalizing a vector input, assuming:

$$\bar{u}(\omega) = [\bar{b}_1, \bar{b}_2]\begin{bmatrix}\sin \omega t \\ \cos \omega t\end{bmatrix} \tag{39}$$

where $\bar{b}_1$ and $\bar{b}_2$ are prescribed magnitudes with each as a 2n dimensional vector.

Substituting equations 32, 34 and 39 into equation 29, we find:

$$[-\omega\bar{\beta}, \omega\bar{\alpha}]\begin{bmatrix}\sin \omega t \\ \cos \omega t\end{bmatrix} = A[\bar{\alpha}, \bar{\beta}]\begin{bmatrix}\sin \omega t \\ \cos \omega t\end{bmatrix} + [\bar{b}_1, \bar{b}_2]\begin{bmatrix}\sin \omega t \\ \cos \omega t\end{bmatrix} \tag{40}$$

And from this relationship, we obtain:

$$[-\omega\bar{\beta}, \omega\bar{\alpha}] = A[\bar{\alpha}, \bar{\beta}] + [\bar{b}_1, \bar{b}_2] \tag{41}$$

or

$$-\omega\bar{\beta} = A\bar{\alpha} + \bar{b}_1 \tag{42}$$

$$\omega\bar{\alpha} = A\bar{\beta} + \bar{b}_2 \tag{43}$$

Metwalli (1982) shows that by measuring at 2n the ω_i, we can get $\bar{\alpha}$, and $\bar{\beta}$ for any ω_i, then arranging the different vectors in equation 43 such that:

$$A = [\omega_1\bar{\alpha}_1 - \bar{b}_{2,1} \mid \omega_2\bar{\alpha}_2 - \bar{b}_{2,2} \mid --- \mid \omega_{2n}\bar{\alpha}_{2n} - \bar{b}_{2,2n}][\bar{\beta}_1 \mid \bar{\beta}_2 \mid --- \mid \beta_{2n}]^{-1} \tag{44}$$

Also from equation 42,

$$A = [-\omega_1\overline{\beta}_1 - \overline{b}_{1,1} \mid -\omega_2\overline{\beta}_2 - \overline{b}_{1,2} \mid ---- \mid -\omega_{2n}\overline{\beta}_{2n} - \overline{b}_{2,2n}][\overline{\alpha}_1 \mid \overline{\alpha}_2 \mid ---- \mid \overline{\alpha}_{2n}]^{-1} \tag{45}$$

The following step is to add a random error to the response so that to simulate real measuring conditions.

Each of the ith columns of the first RH matrix in equation 43 has been changed to:

$$\text{ith column} = (\omega_i\overline{\alpha}_i - \overline{b}_{2,i})[1 + \%R.E.] \tag{46}$$

where:

R.E. = the percentage of random error added

The computer which was used to run the identification program was a Tektronix 4050 Series. The random function used causes an internal pseudorandom number generator to output a random number from 0 to 1. The random number generator contains approximately 140 trillion numbers within the range 0 to 1. These numbers are linked together in a chain in such a way as to appear random when they are output from the generator. Each time the random function is executed, one number in the chain is retrieved. The next time the random function is executed, another number in the chain is obtained and so on. The probability distribution of these random numbers is uniform, which has been used as such for simplicity.

CHAPTER III

SAMPLE IDENTIFICATION TESTS

This chapter gives the sample calculation of two different cases. The first one is a damped non-gyroscopic system, and the other is a conservative non-gyroscopic system. In this section we will derive the eigenvalue problem for both cases, and then we will check the properties of their solutions. Next, we will get the graphical representation of frequency versus amplitude, and phase angle versus amplitude in the frequency domain. Finally, a state space identification in the frequency domain for non-homogeneous equations will be done with a unit step sinusoidal input, and the addition of a certain percentage of random error.

Damped Non-gyroscopic System

From Figure 1, we can get the kinetic and potential energy for the system:

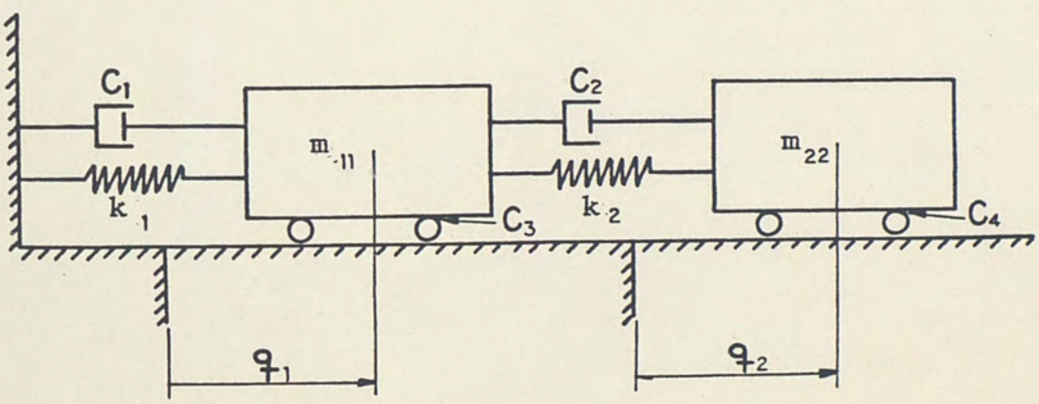

Fig. 1. Damped non-gyroscopic system.

$$T = \frac{1}{2} m_{11} \ddot{q}_1^2 + \frac{1}{2} m_{22} \ddot{q}_2^2$$

$$V = \frac{1}{2} k_1 q_1^2 + \frac{1}{2} k_2 (q_2 - q_1)^2 \tag{47}$$

And the dissipation function:

$$F^* = \frac{1}{2} c_1 \dot{q}_1^2 + \frac{1}{2} c_2 (\dot{q}_2 - \dot{q}_1)^2 + \frac{1}{2} c_3 \dot{q}_1^2 + \frac{1}{2} c_4 \dot{q}_2^2 \tag{48}$$

The system Lagrangian is:

$$L = T - V \tag{49}$$

$$L = \frac{1}{2} m_{11} \ddot{q}_1^2 + \frac{1}{2} m_{22} \ddot{q}_2^2 - \frac{1}{2} k_1 q_1^2 - \frac{1}{2} k_2 (q_2 - q_1)^2 \tag{50}$$

Solving by Lagrange's equations we have:

$$m_{11} \ddot{q}_1 + (c_1 + c_2 + c_3) \dot{q}_1 + (k_1 + k_2) q_1 - c_2 \dot{q}_2 - k_2 q_2 = 0$$

$$m_{22} \ddot{q}_2 + (c_2 + c_4) \dot{q}_2 + k_2 q_2 - k_2 q_1 - c_2 \dot{q}_1 = 0 \tag{51}$$

In a matrix form:

$$\begin{bmatrix} m_{11} & 0 \\ 0 & m_{22} \end{bmatrix} \begin{bmatrix} \ddot{q}_1 \\ \ddot{q}_2 \end{bmatrix} + \begin{bmatrix} c_1+c_2+c_3 & -c_2 \\ -c_2 & c_2+c_4 \end{bmatrix} \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \end{bmatrix} + \begin{bmatrix} k_1+k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{52}$$

Assuming:

$$m_{11} = m_{22} = 1$$

$$c_1 = 2,\ c_2 = 2.8,\ c_3 = 0,\ c_4 = 0 \tag{53}$$

$$k_1 = 4,\ k_2 = 4$$

The equations of motion are then:

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} \ddot{q}_1 \\ \ddot{q}_2 \end{bmatrix} + \begin{bmatrix} 4.8 & -2.8 \\ -2.8 & 2.8 \end{bmatrix}\begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \end{bmatrix} + \begin{bmatrix} 8 & -4 \\ -4 & 4 \end{bmatrix}\begin{bmatrix} q_1 \\ q_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

This is a damped non-gyroscopic system with a characteristic equation of motion (equation 10) as follows:

$$M\ddot{\bar{q}} + C\dot{\bar{q}} + K\bar{q} = 0$$

where:

$$M = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \qquad C = \begin{bmatrix} 4.8 & -2.8 \\ -2.8 & 2.8 \end{bmatrix}, \qquad K = \begin{bmatrix} 8 & -4 \\ -4 & 4 \end{bmatrix}$$

In state space, representation is given by equation 11:

$$M^* = \left[\begin{array}{c|c} M & 0 \\ \hline 0 & I \end{array}\right] = \left[\begin{array}{cc|cc} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \hline 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right]$$

$$K^* = \left[\begin{array}{c|c} C & K \\ \hline -I & 0 \end{array}\right] = \left[\begin{array}{cc|cc} 4.8 & -2.8 & 8 & -4 \\ -2.8 & 2.8 & -4 & 4 \\ \hline -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{array}\right]$$

where the matrix A of the eigenvalue problem is:

$$A = -M^{*-1} K^{*} = \left[\begin{array}{cc|cc} -4.8 & 2.8 & -8 & 4 \\ 2.8 & -2.8 & 4 & -4 \\ \hline 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{array}\right]$$

Using the computer program EIGRF from International Mathematical and Statistical Libraries, Inc. (1975)(Appendix A) we get the eigenvalues and normalized eigenvectors that are listed in Appendix B, Table 1. These eigenvectors are called the right eigenvectors.

To check the accuracy of the eigenvalues, we calculate the determinant of A:

$$\det A = \begin{vmatrix} -4.8 & 2.8 & -8 & 4 \\ 2.8 & -2.8 & 4 & -4 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{vmatrix}$$

$$\det A = 16$$

This must be equal to the product of the eigenvalues:

$$\prod_{i=1}^{4} \lambda_i = (-4.4094)(-0.42305 + 1.1699i)(-0.42305 - 1.1699i)(-2.3455)$$

$$\prod_{i=1}^{4} \lambda_i \cong 16 \cong \det A$$

The trace of matrix A must also be equal to the addition of all the eigenvalues.

$$\text{Trace } A = -4.8 - 2.8 = -7.6$$

$$\sum_{i=1}^{4} \lambda_i = -4.4094 + (-0.42305 + 1.1699i) + (-0.42305 - 1.1699i) - 2.3455$$

$$\sum_{i=1}^{4} \lambda_i = -7.6 = \text{Trace } A$$

In order to check the orthogonality, we use equations 14 and 15. The results we already have indicate that the solution is correct.

System Response

Next, we obtain a graphical representation of frequency versus amplitude and frequency versus phase angle for a unit step sinusoidal input, using equations 37 and 38, for a frequency between 0 and 10 cycles/sec. These graphs can be seen in Figures 2 through 9.

We have used a computer program for frequency response that is shown in Appendix A. The input for the frequency response was equations 35 and 36:

$$\bar{b}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad \bar{b}_2 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad \omega_{increment} = 0.1 \text{ cycles/sec.}$$

$$\omega_{maximum} = 10 \text{ cycles/sec.}$$

System Identification

The following step is to identify the matrix A (eigenvalue problem) for a given frequency range that usually contains all the set of eigenvalues of A. A unit step sinusoidal input is utilized

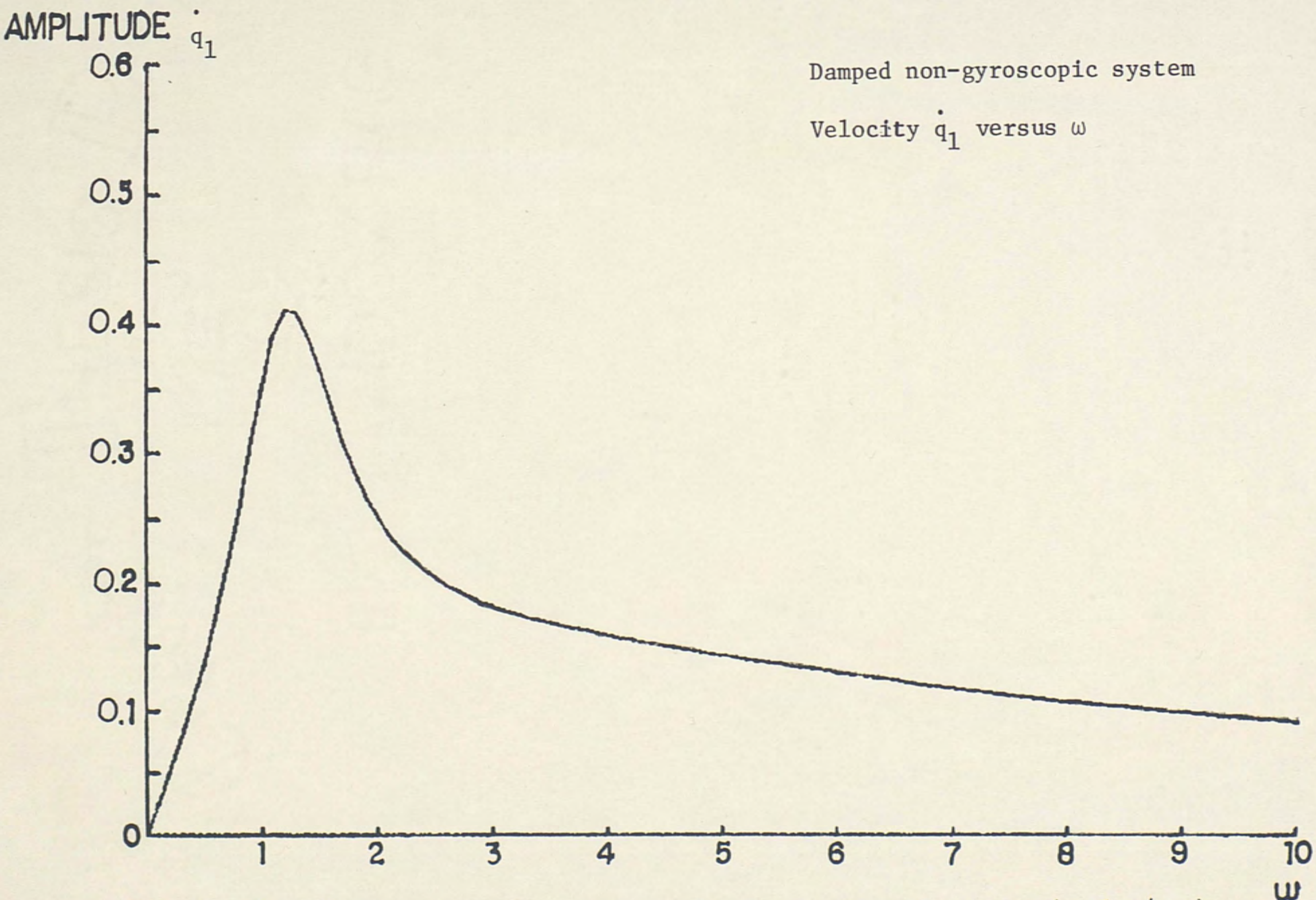

Fig. 2. Damped non-gyroscopic system amplitude I versus frequency (cycles/sec).

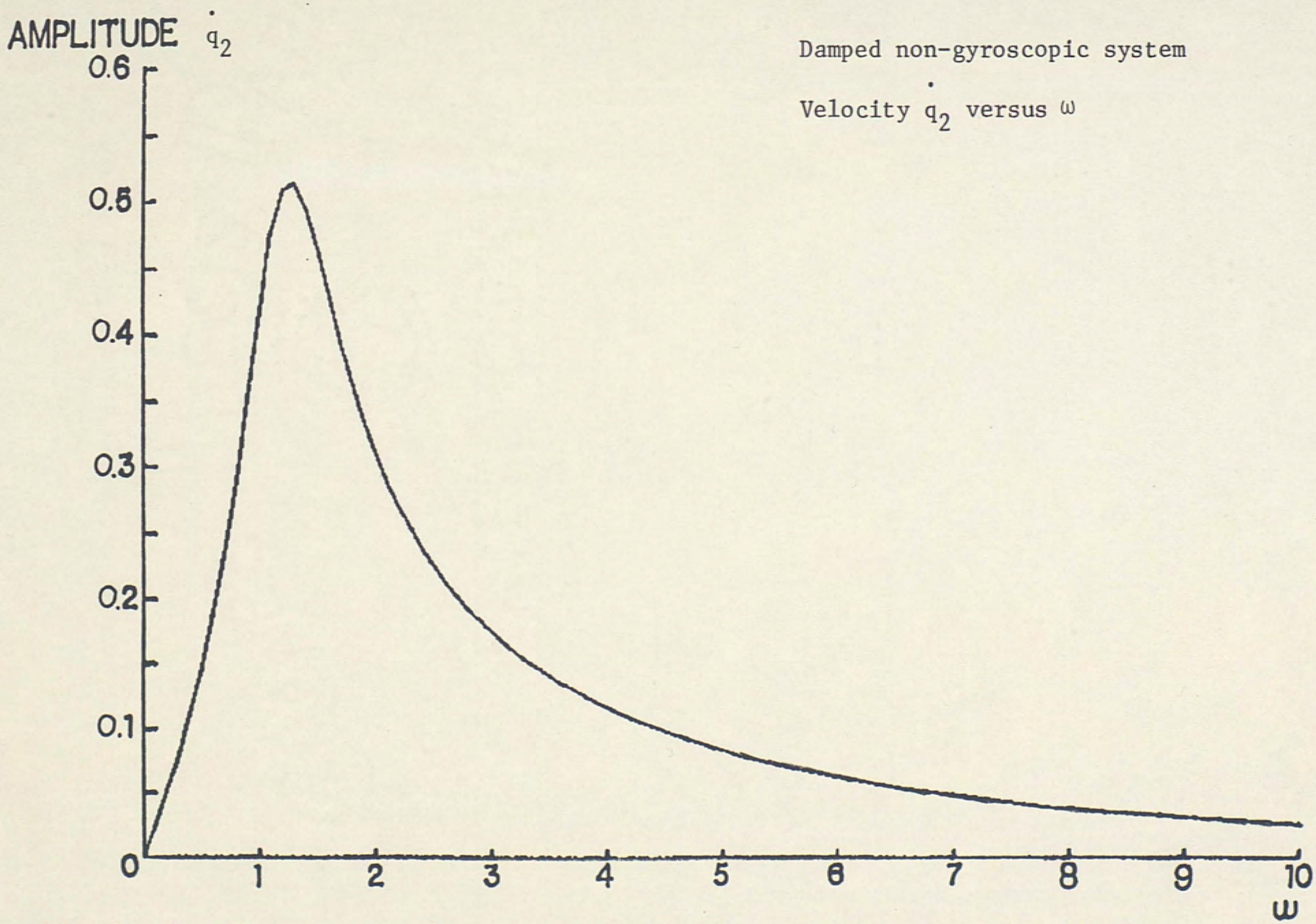

Fig. 3. Damped non-gyroscopic system amplitude II versus frequency (cycles/sec).

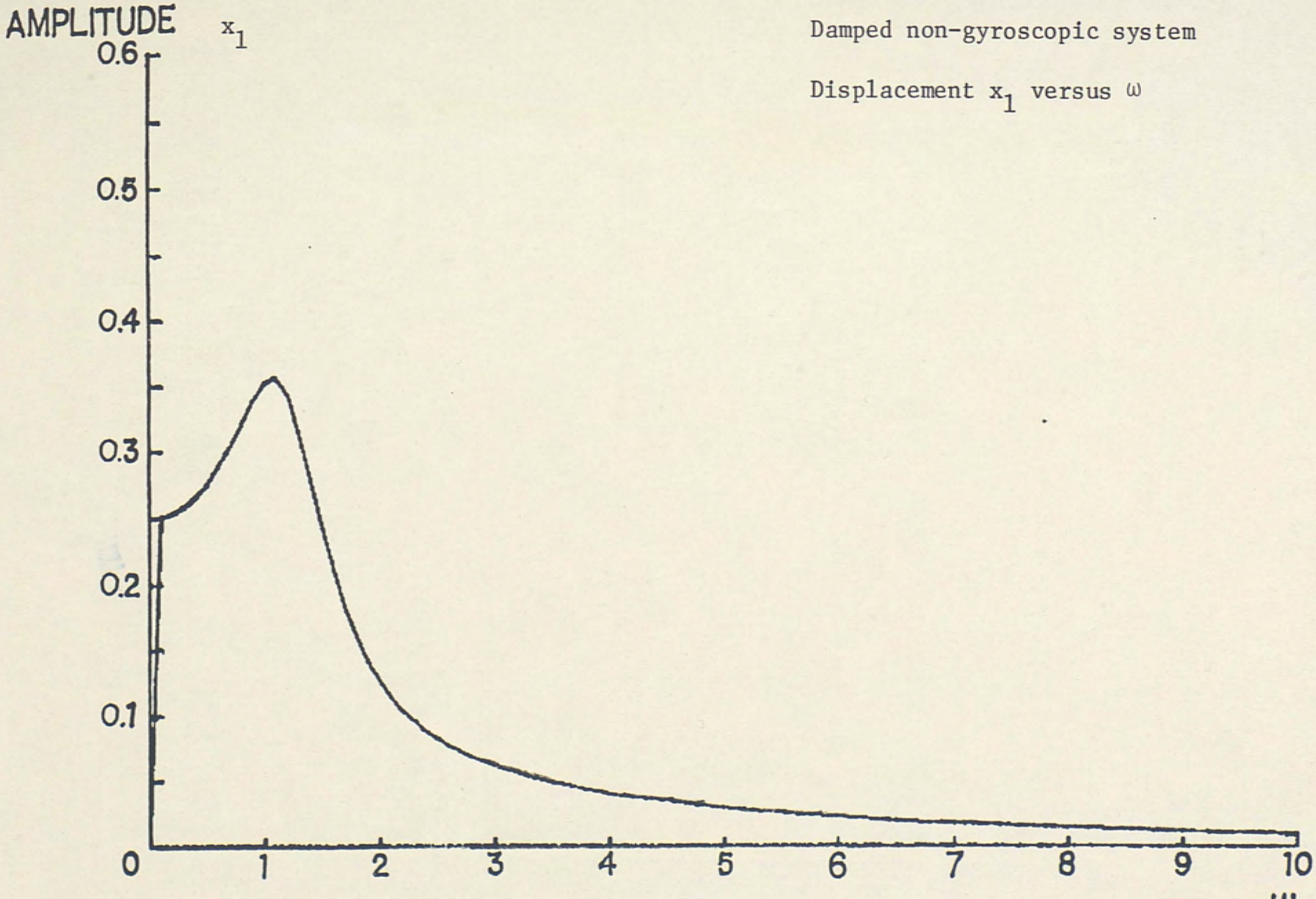

Fig. 4. Damped non-gyroscopic system amplitude III versus frequency (cycles/sec).

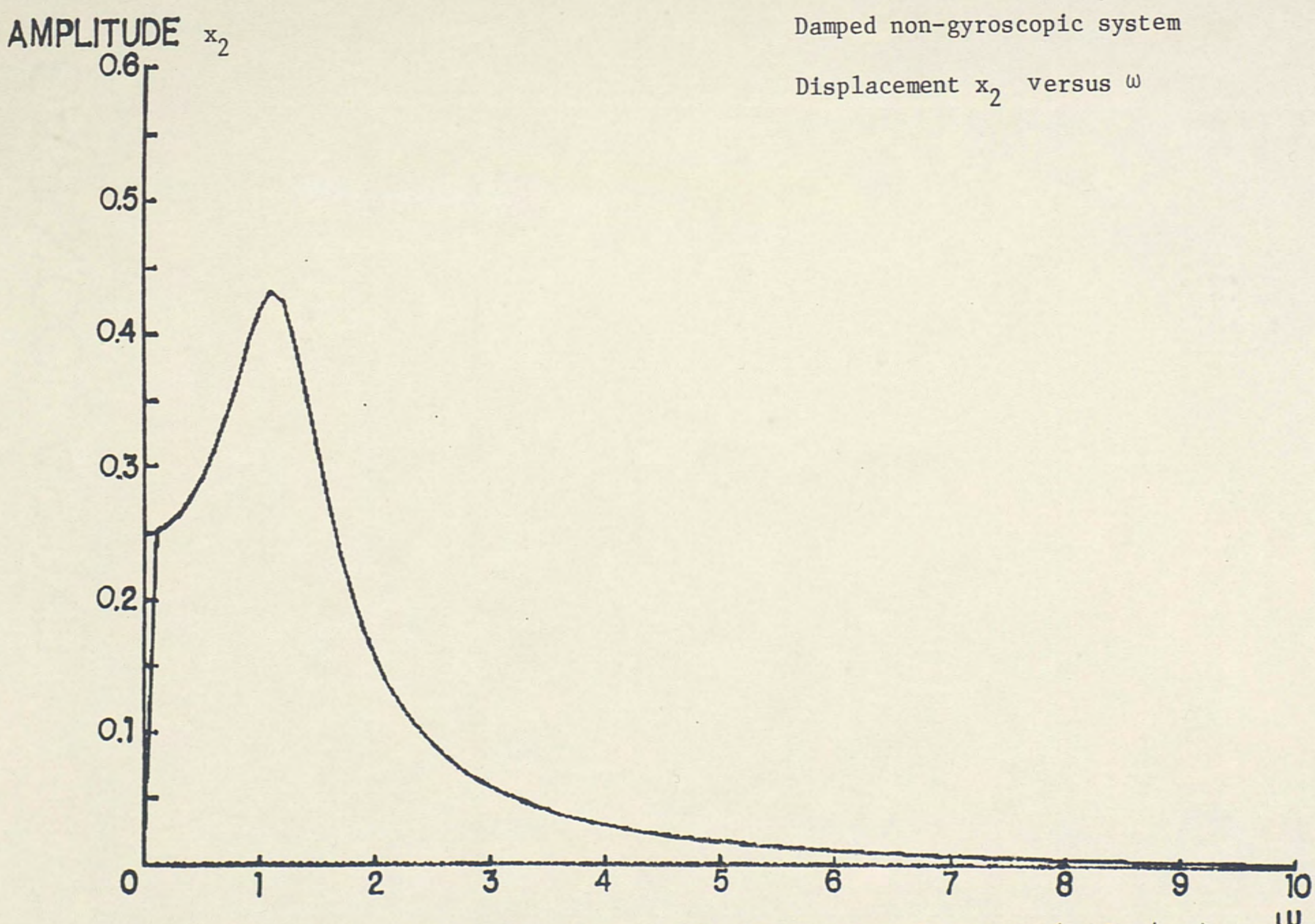

Fig. 5. Damped non-gyroscopic system amplitude IV versus frequency (cycles/sec).

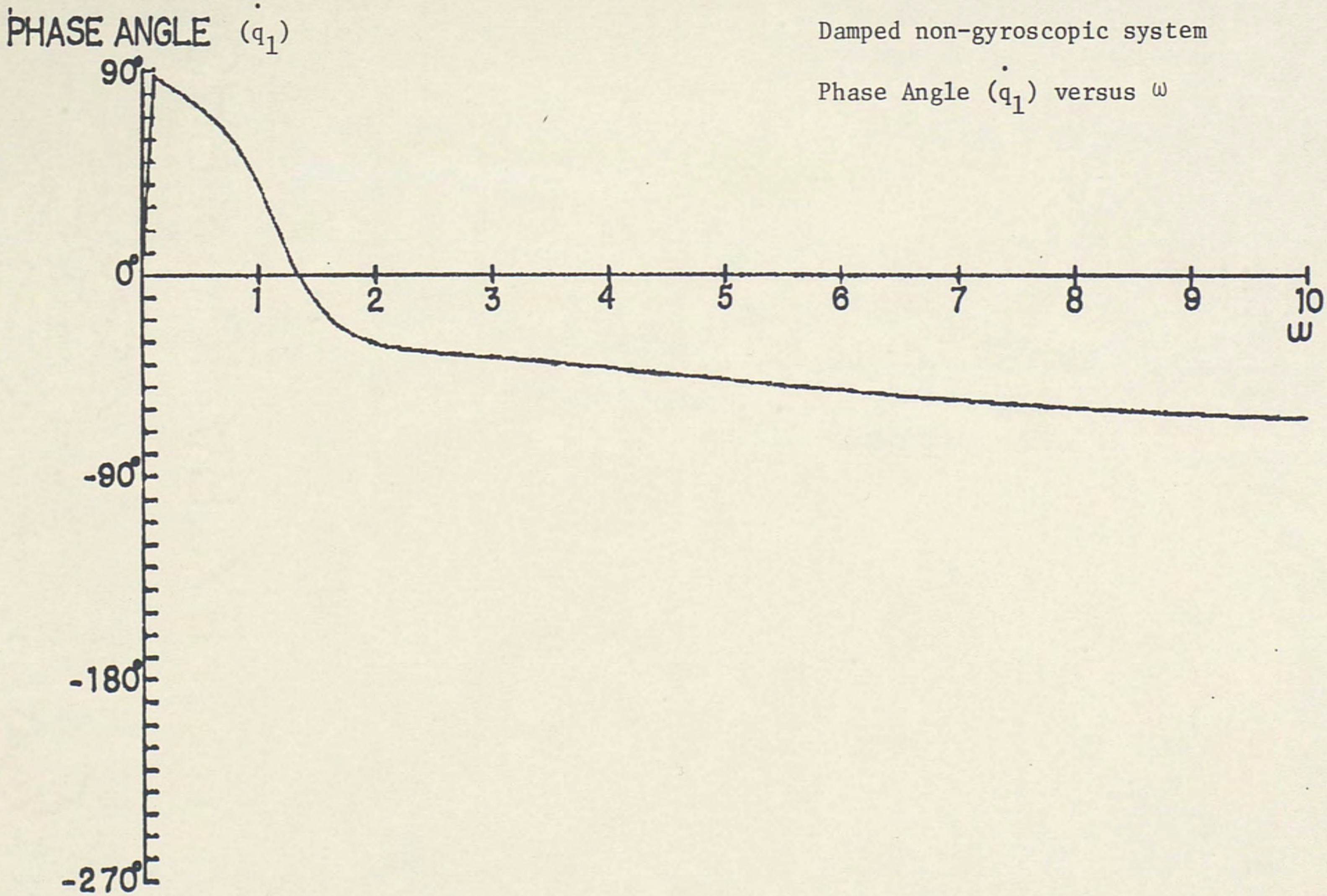

Fig. 6. Damped non-gyroscopic system phase angle I versus frequency (cycles/sec).

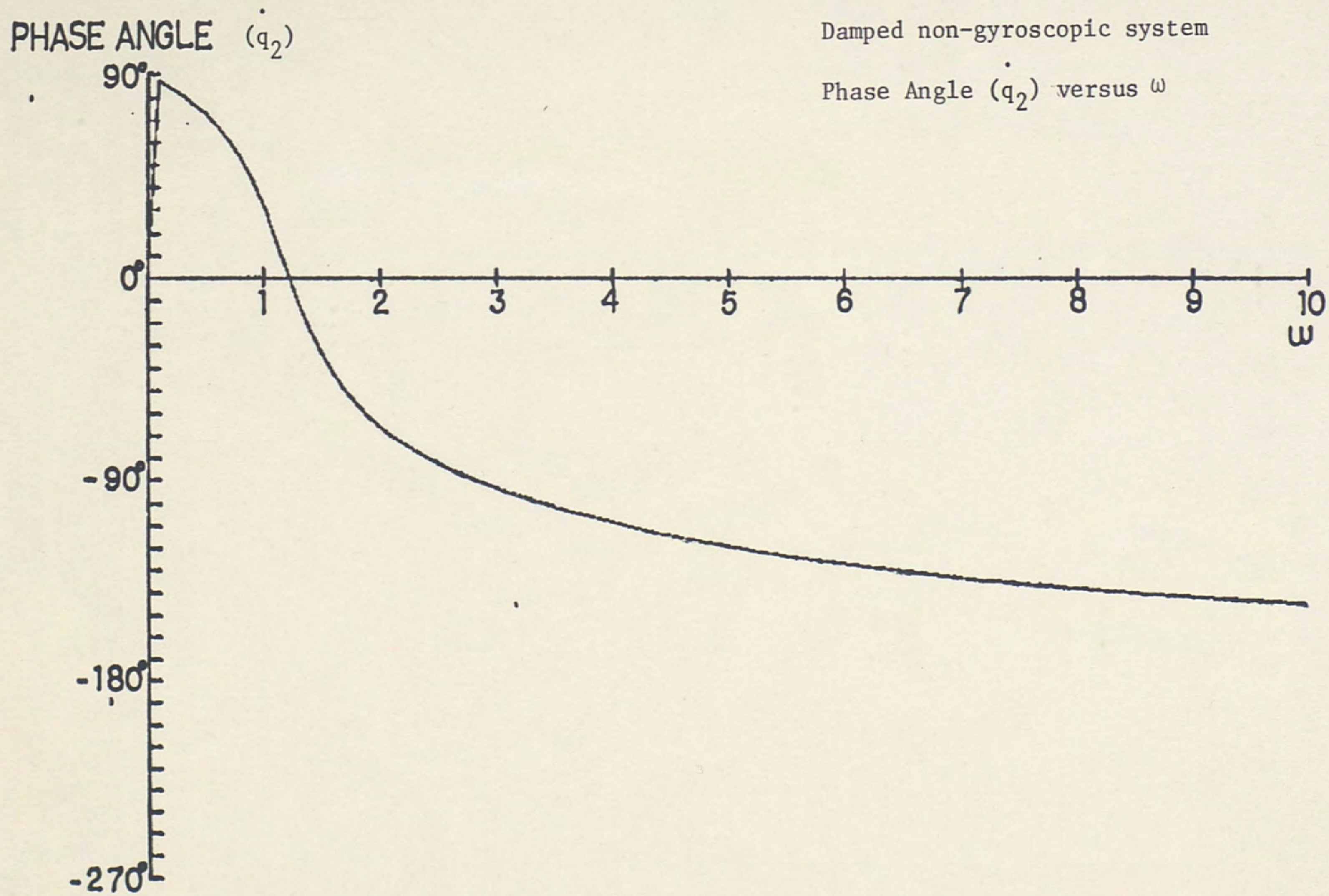

Fig. 7. Damped non-gyroscopic system phase angle II versus frequency (cycles/sec).

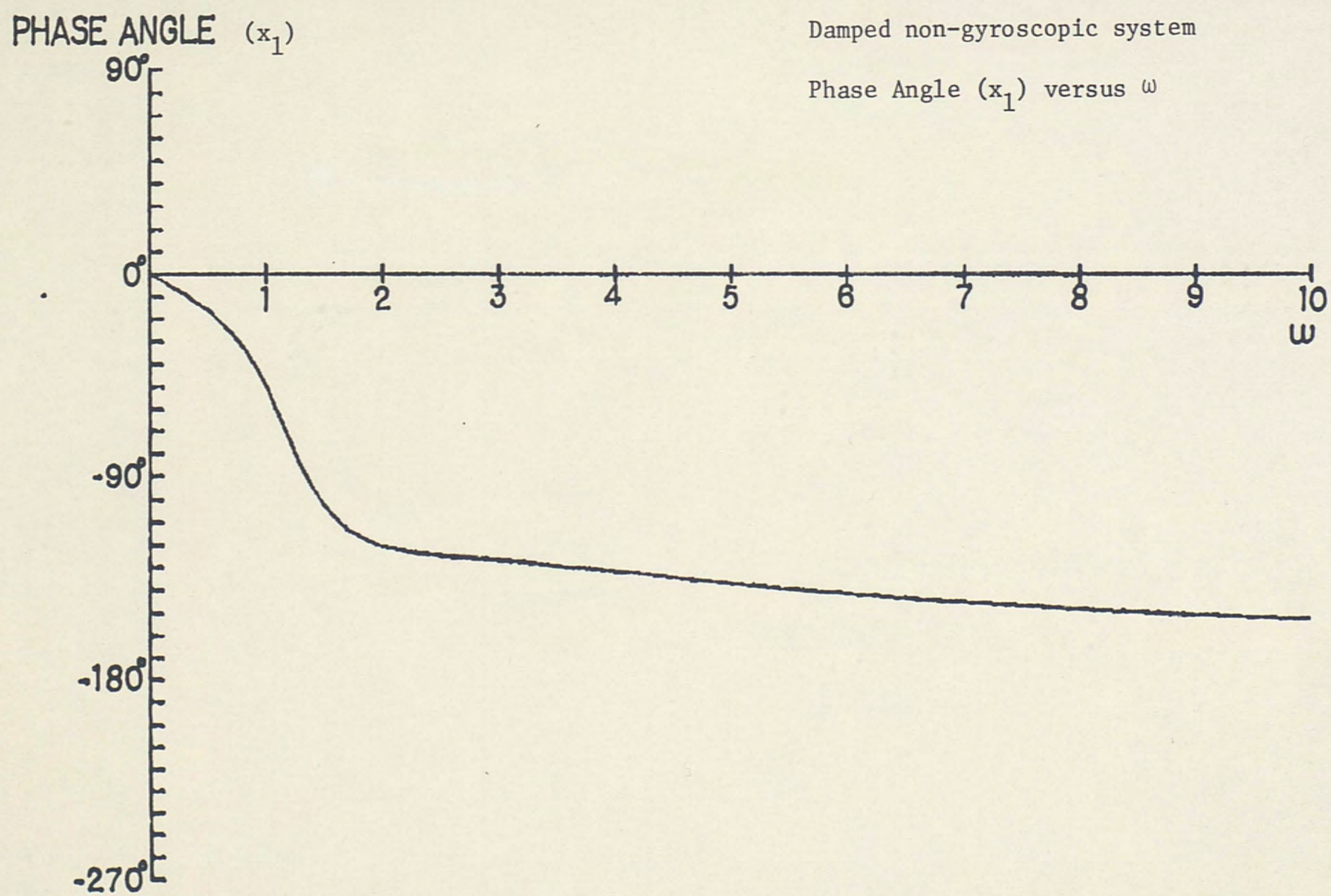

Fig. 8. Damped non-gyroscopic system phase angle III versus frequency (cycles/sec).

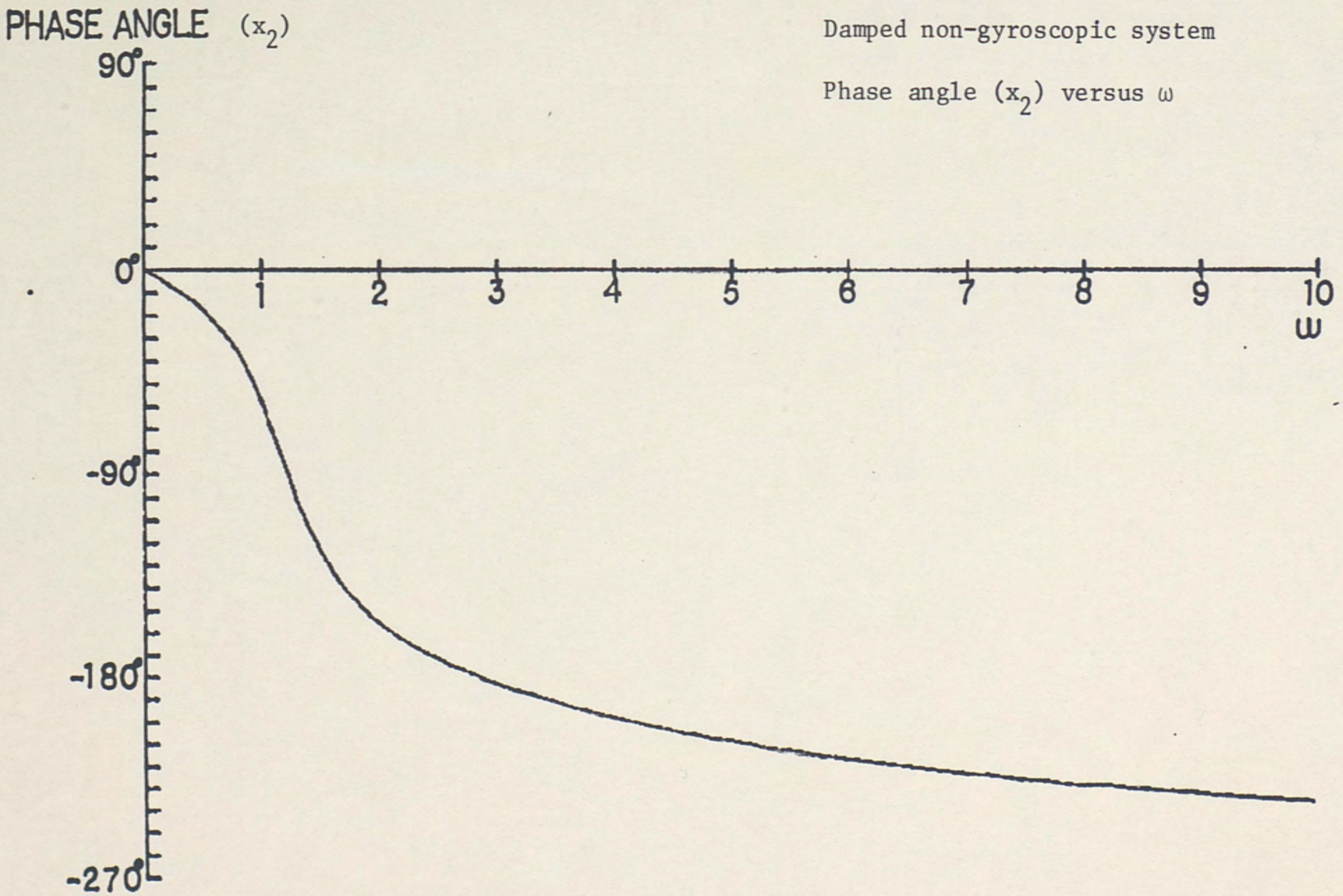

Fig. 9. Damped non-gyroscopic system phase angle IV versus frequency (cycles/sec).

to excite the system, as indicated before. The input data for the computer program for identification (Appendix A) is:

$$\bar{b}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad \bar{b}_2 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad \begin{matrix} \omega_1 = 1.3 \text{ cycles/sec.} \\ \omega_2 = 5.2 \text{ cycles/sec.} \end{matrix}$$

where:

ω_1 = initial frequency

ω_2 = maximum frequency

$\bar{b}_1$ = unit input (sine)

$\bar{b}_2$ = unit input (cosine)

In addition, we have added a random error of 0.2%, 0.4%, 0.6%, 0.8% and 1.0% in every case, and we identified the matrix A for every percentage of random error. Then, we applied equation 25 that gives the error matrix:

$$E = B - A$$

where:

B = matrix with an assumed random error

A = real system matrix without random error

E = error matrix

The eigenvalues and eigenvectors of the error matrix are shown in Tables 2, 3, 4, 5 and 6 of Appendix B. We notice that the eigenvalues of the error matrix do not exceed 5% of the eigenvalues of the original matrix A.

Finally, we averaged 100 times the identification of the matrix A for a fixed random error (0.5%). We obtained a very good approximation to the original matrix A, i.e., B after averaging 100 times is:

$$B = \begin{bmatrix} -4.797540 & 2.805658 & -8.023725 & 4.017355 \\ 2.800209 & -2.799554 & 3.998017 & -3.999138 \\ 0.998862 & -0.002508 & 0.010779 & -0.008241 \\ -0.002132 & 0.995271 & 0.020753 & -0.015628 \end{bmatrix} \approx A$$

The computer program for the frequency response was used again for the case when the system is critically damped. This case occurs when we have 95.266% of the original damping. We then have two equal real roots and one pair of complex conjugate roots that are shown in Table 7 of Appendix B. The frequency response versus amplitude and phase angle graphs are very similar to those of the original system. The plots can be seen in Figures 10 through 17.

The identification computer program was used again with a 0.5% of random error. It was performed 100 times and then averaged. The matrix that was obtained is practically the same as the original (with 95.266% of the initial damping), i.e. , for

$$A = \begin{bmatrix} -4.572753 & 2.667439 & -8 & 4 \\ 2.667439 & -2.667439 & 4 & -4 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

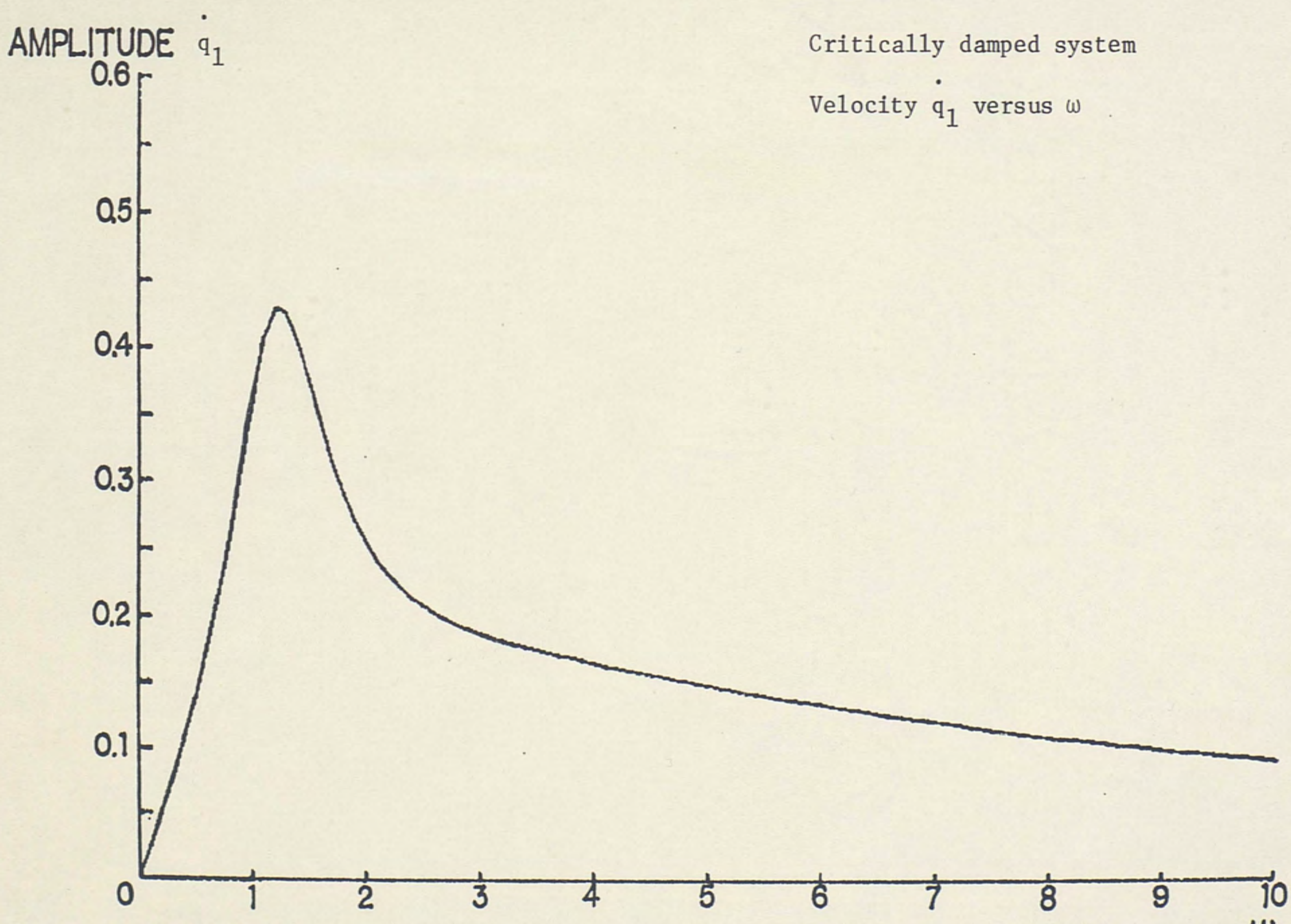

Fig. 10. Critically damped system amplitude I versus frequency (cycles/sec).

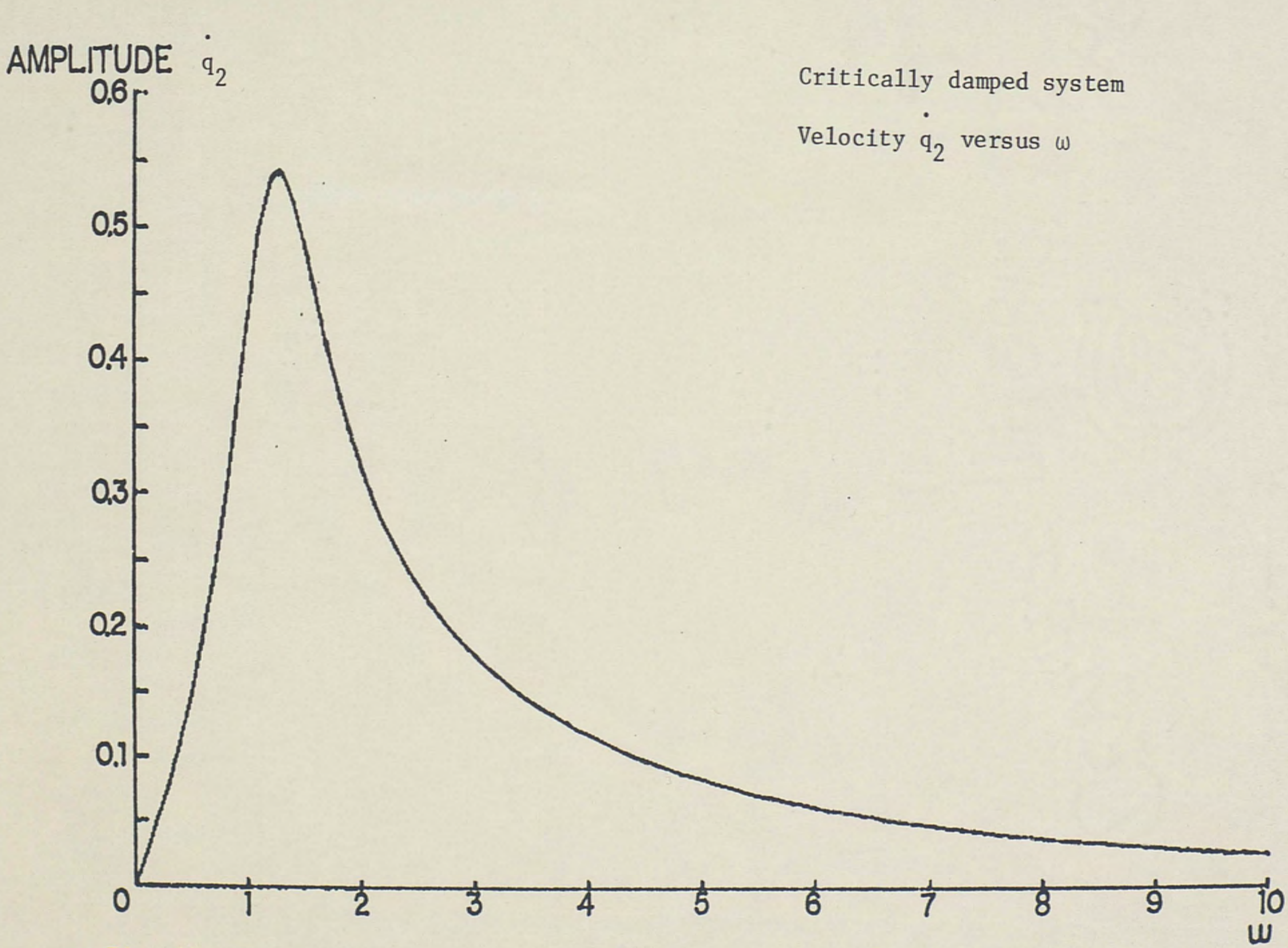

Fig. 11. Critically damped system amplitude II versus frequency (cycles/sec).

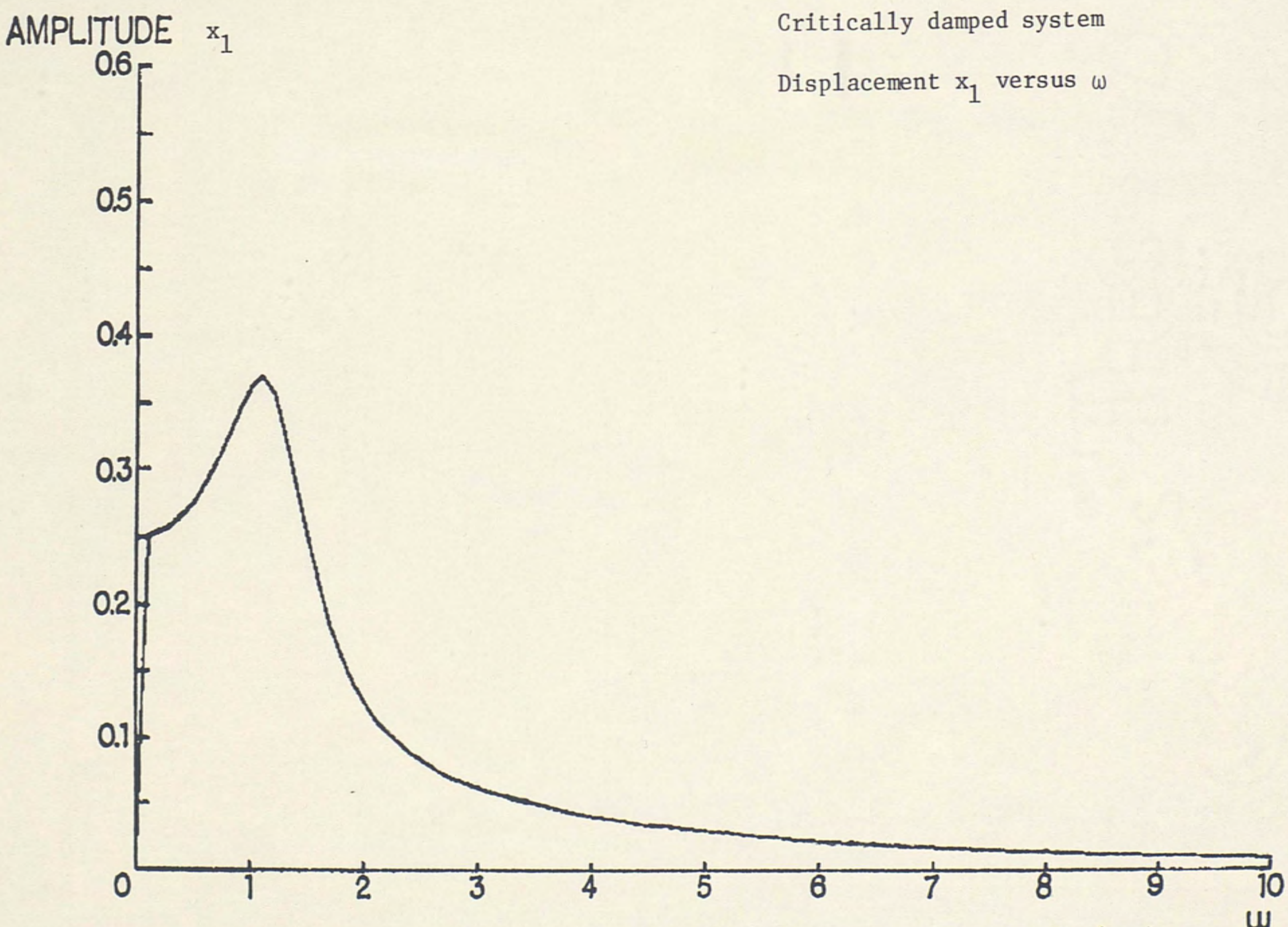

Fig. 12. Critically damped system amplitude III versus frequency (cycles/sec).

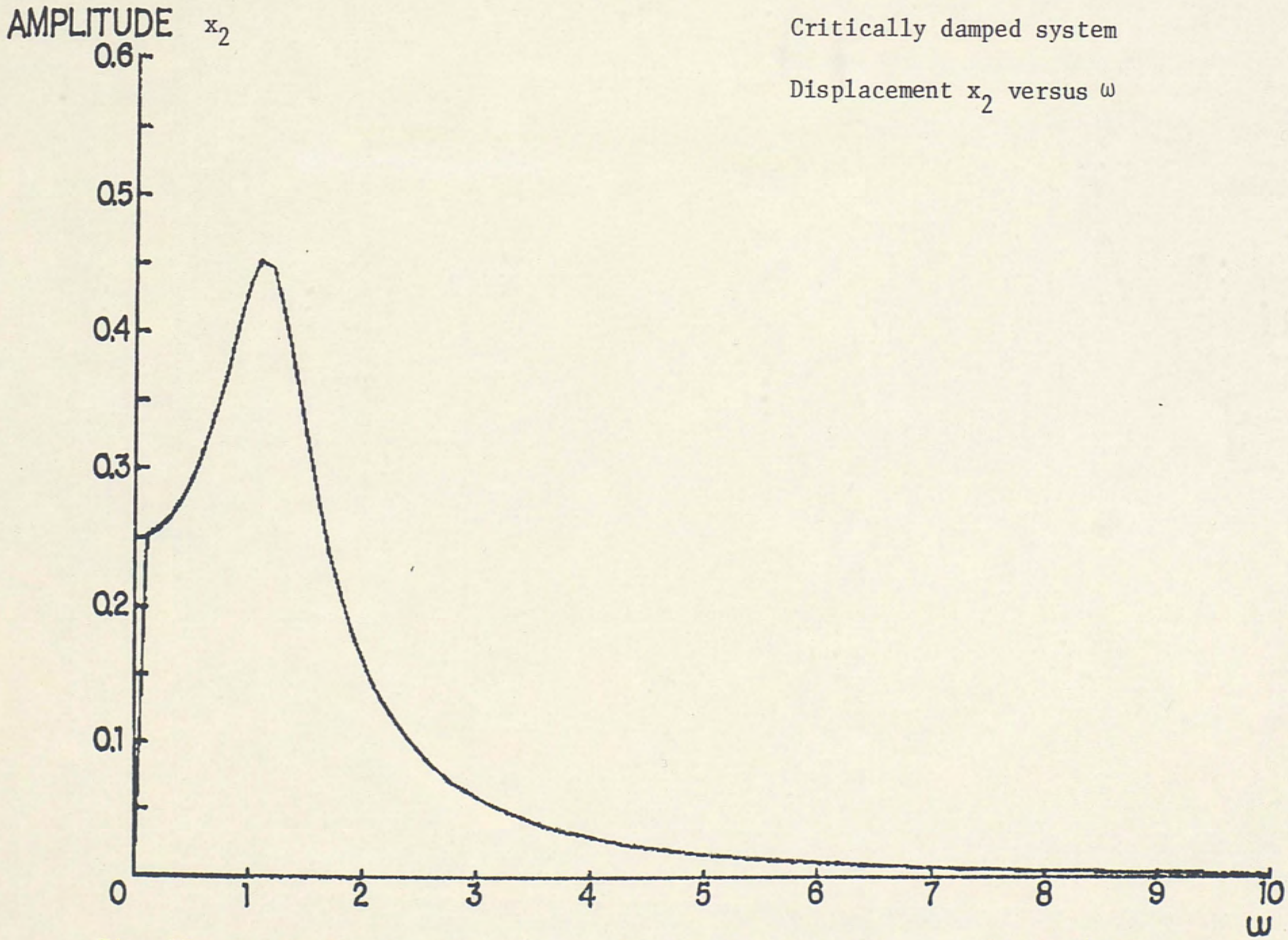

Fig. 13. Critically damped system amplitude IV versus frequency (cycles/sec).

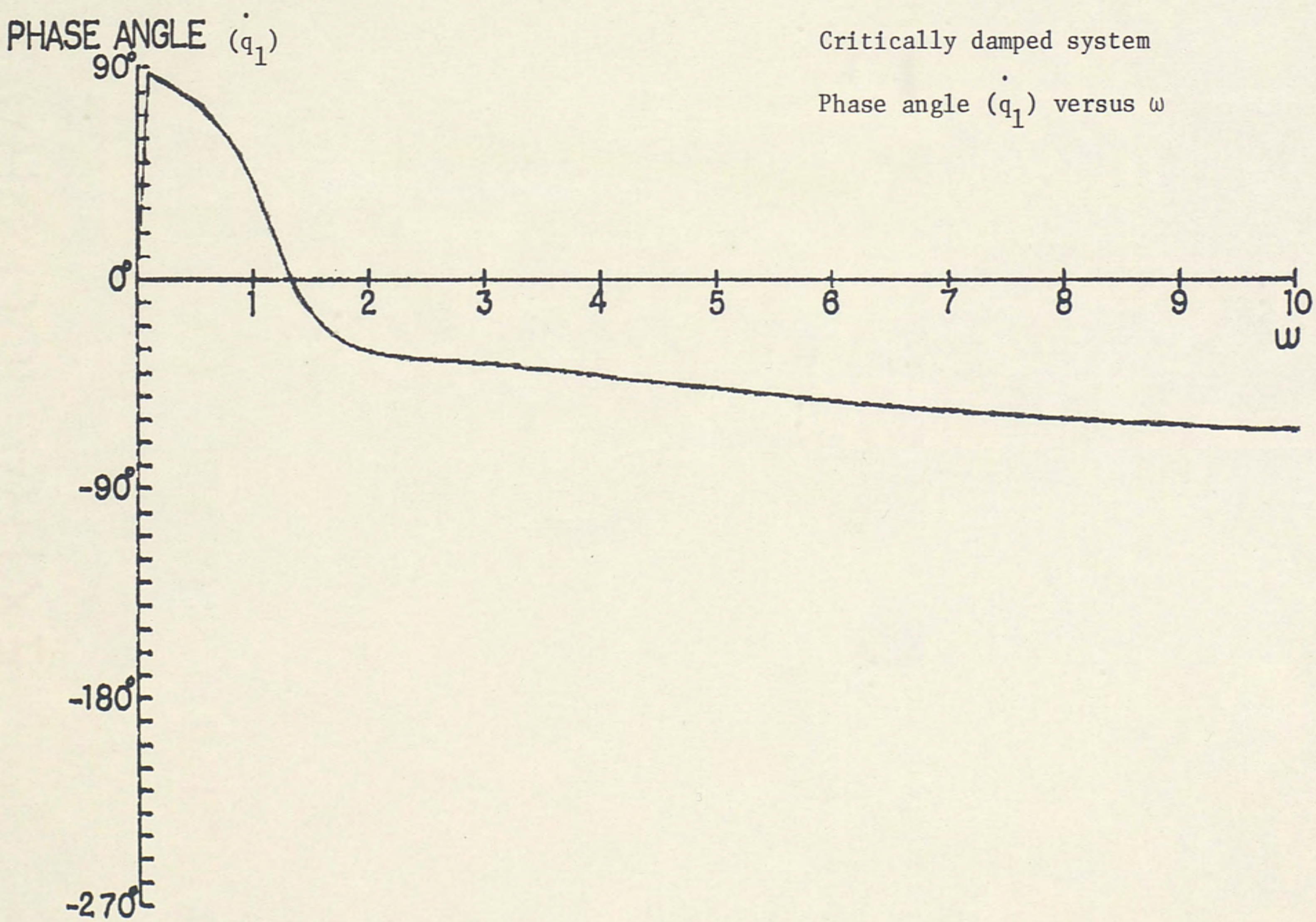

Fig. 14. Critically damped system phase angle I versus frequency (cycles/sec).

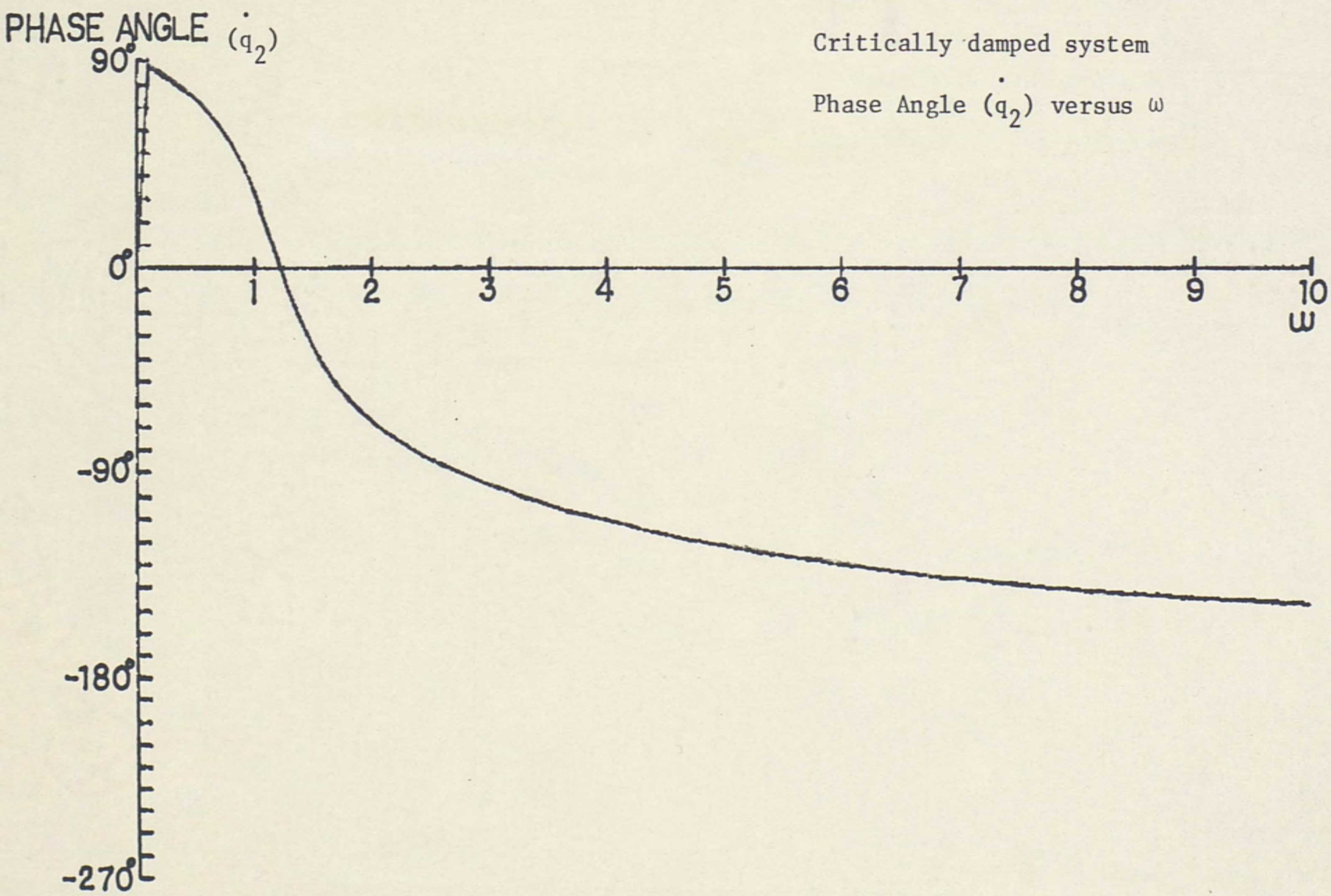

Fig. 15. Critically damped system phase angle II versus frequency (cycles/sec).

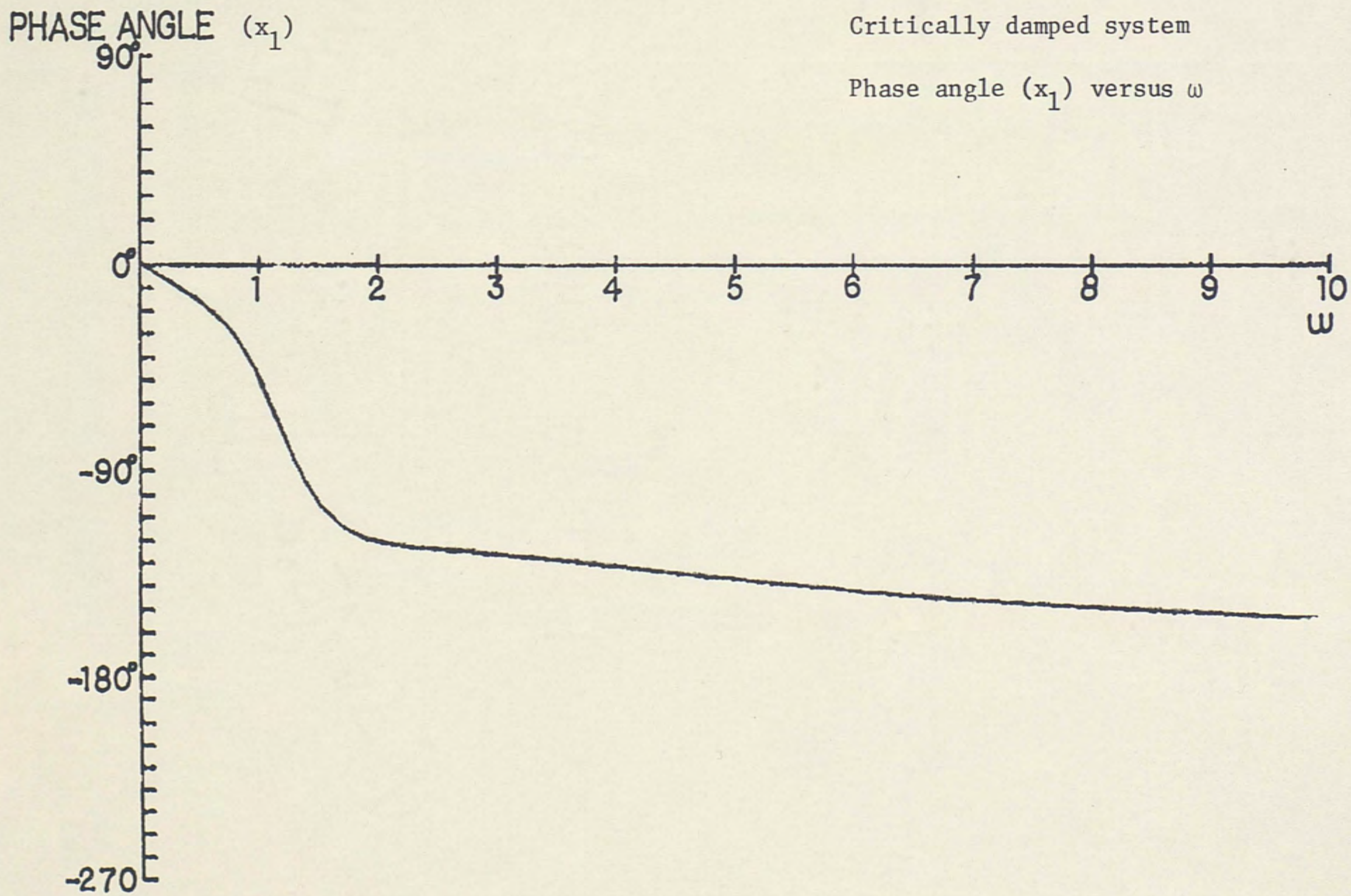

Fig. 16. Critically damped system phase angle III versus frequency (cycles/sec).

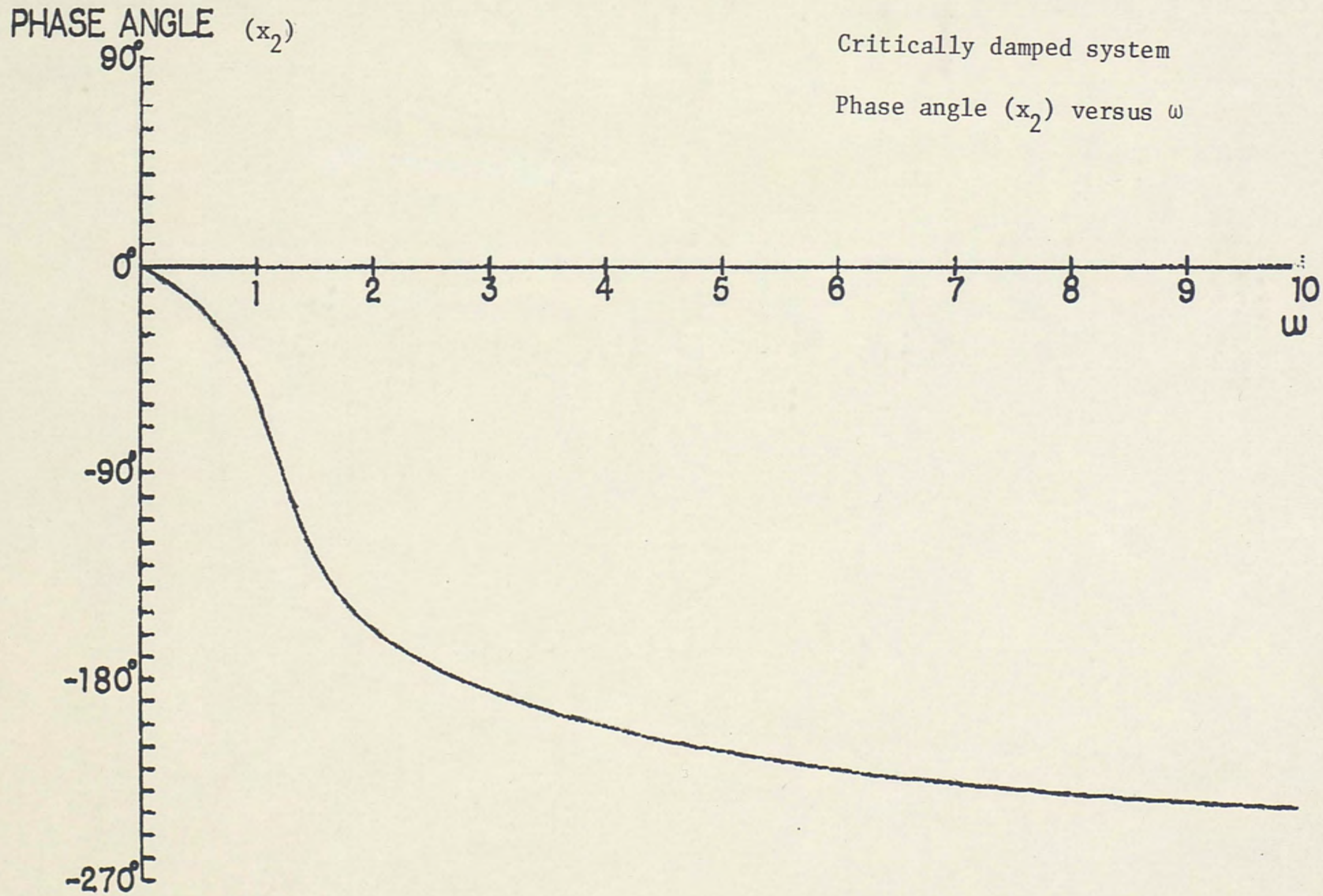

Fig. 17. Critically damped system phase angle IV versus frequency (cycles/sec).

$$B = \begin{bmatrix} -4.578310 & 2.660487 & -7.964574 & 3.973146 \\ 2.664597 & -2.670542 & 4.016645 & -4.012823 \\ 1.000007 & -0.000675 & 0.001206 & -0.000794 \\ -0.000666 & 0.999987 & 0.002201 & -0.001730 \end{bmatrix} \underset{\sim}{\tau} A$$

Another interesting case which has been examined is the underdamped system with 10% of the original damping in the matrix A. The eigenvalues and eigenvectors were obtained, and we can see that the eigenvalues are two pairs of complex conjugate roots, as shown in Table 8 of Appendix B.

The graphs obtained for amplitude and phase angle versus frequency show an increase in magnitude for amplitude and a decrease in the values of the phase angle with respect to the original matrix A. It can be seen in Figures 18 through 25.

The identification process was performed 100 times with 0.5% of random error and the matrix obtained was similar to the original (with 10% of the initial damping), i.e., for

$$A = \begin{bmatrix} -0.48 & 0.28 & -8 & 4 \\ 0.28 & -0.28 & 4 & -4 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix},$$

$$B = \begin{bmatrix} -0.480265 & 0.278332 & -7.993251 & 3.999450 \\ 0.280299 & -0.279460 & 3.999220 & -4.000215 \\ 1.000632 & 0.000869 & -0.002930 & -0.000438 \\ -0.000012 & 0.999676 & 0.000034 & 0.000291 \end{bmatrix}$$

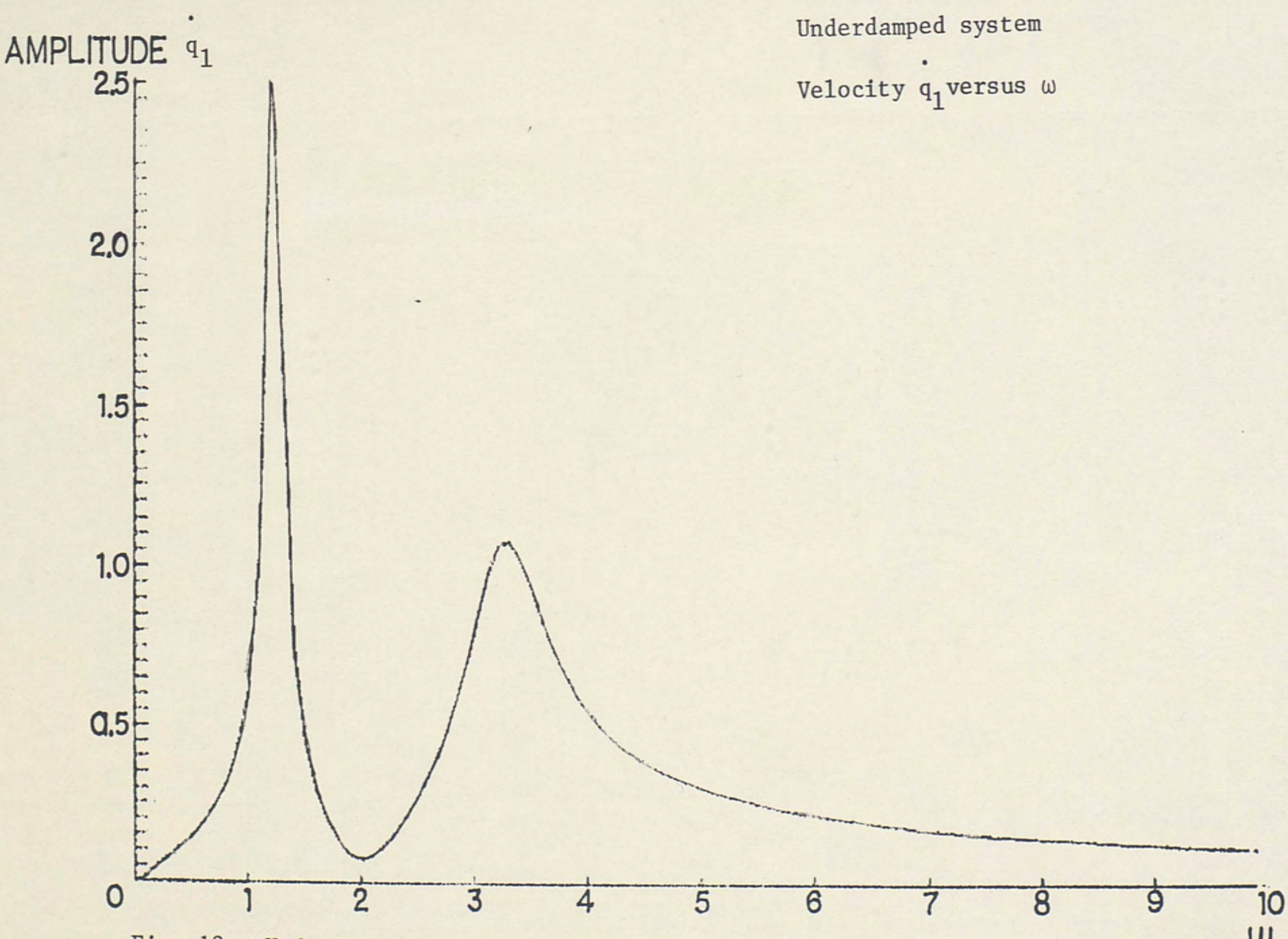

Fig. 18. Underdamped system amplitude I versus frequency (cycles/sec).

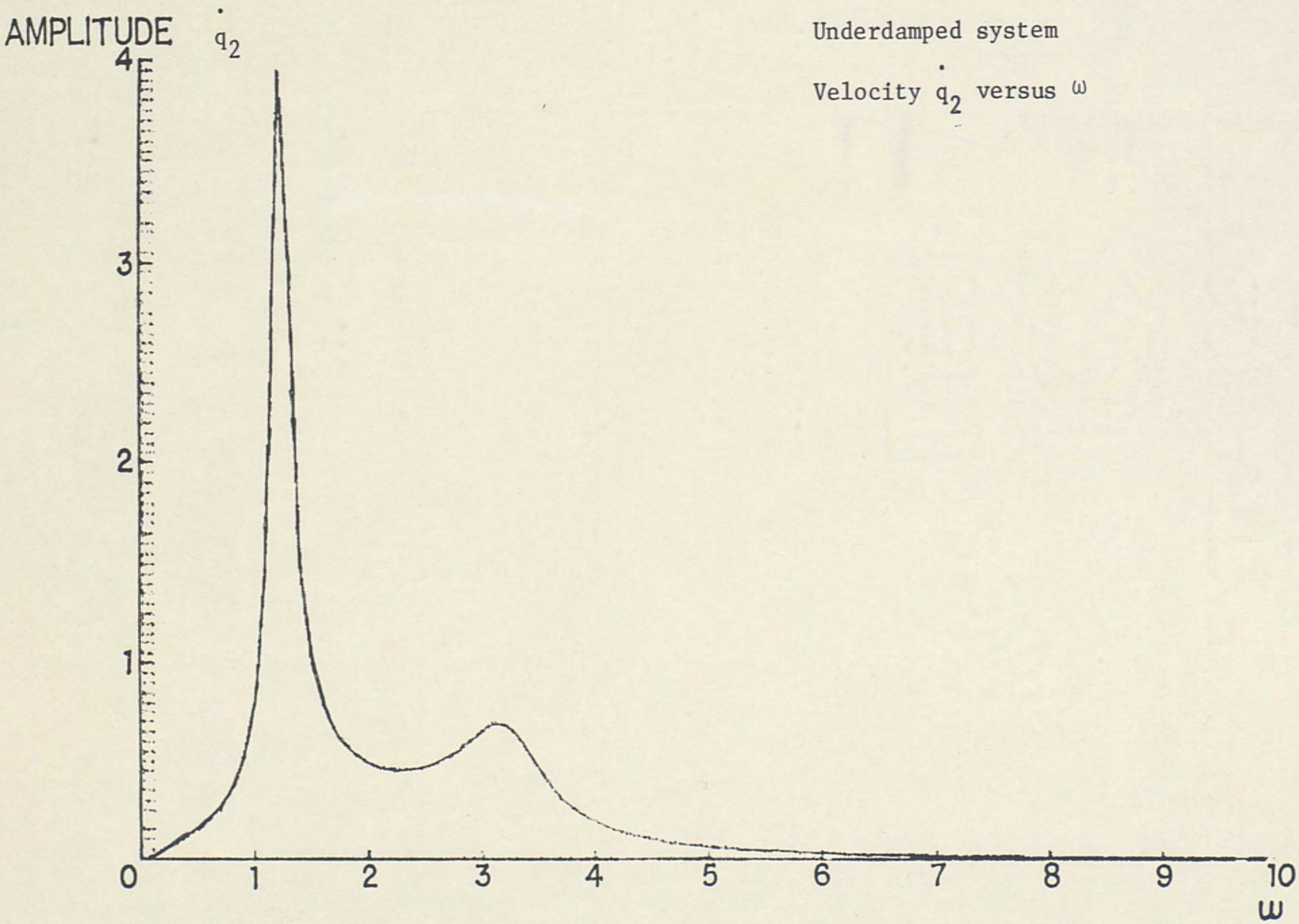

Fig. 19. Underdamped system amplitude II versus frequency (cycles/sec).

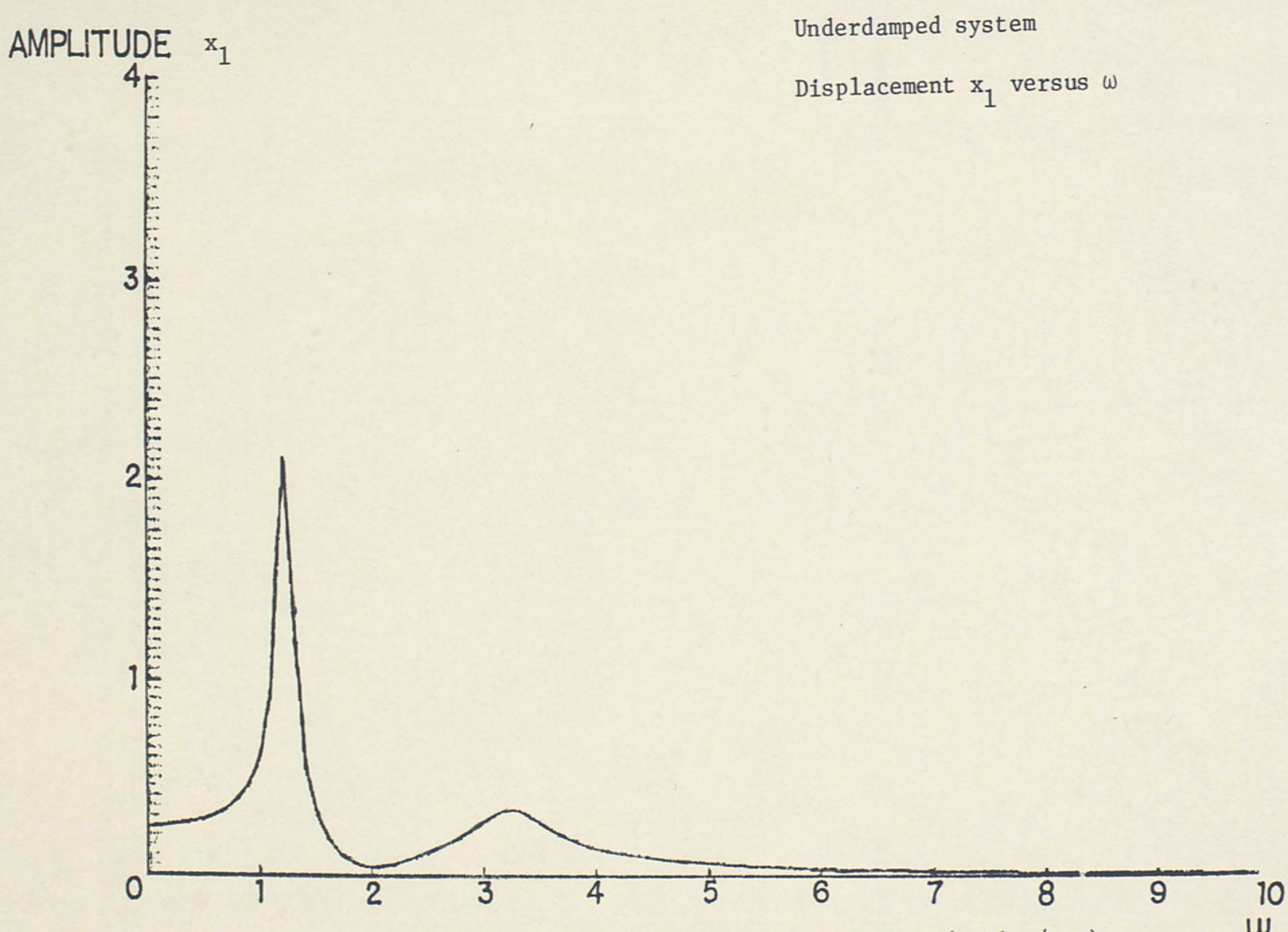

Fig. 20. Underdamped system amplitude III versus frequency (cycles/sec).

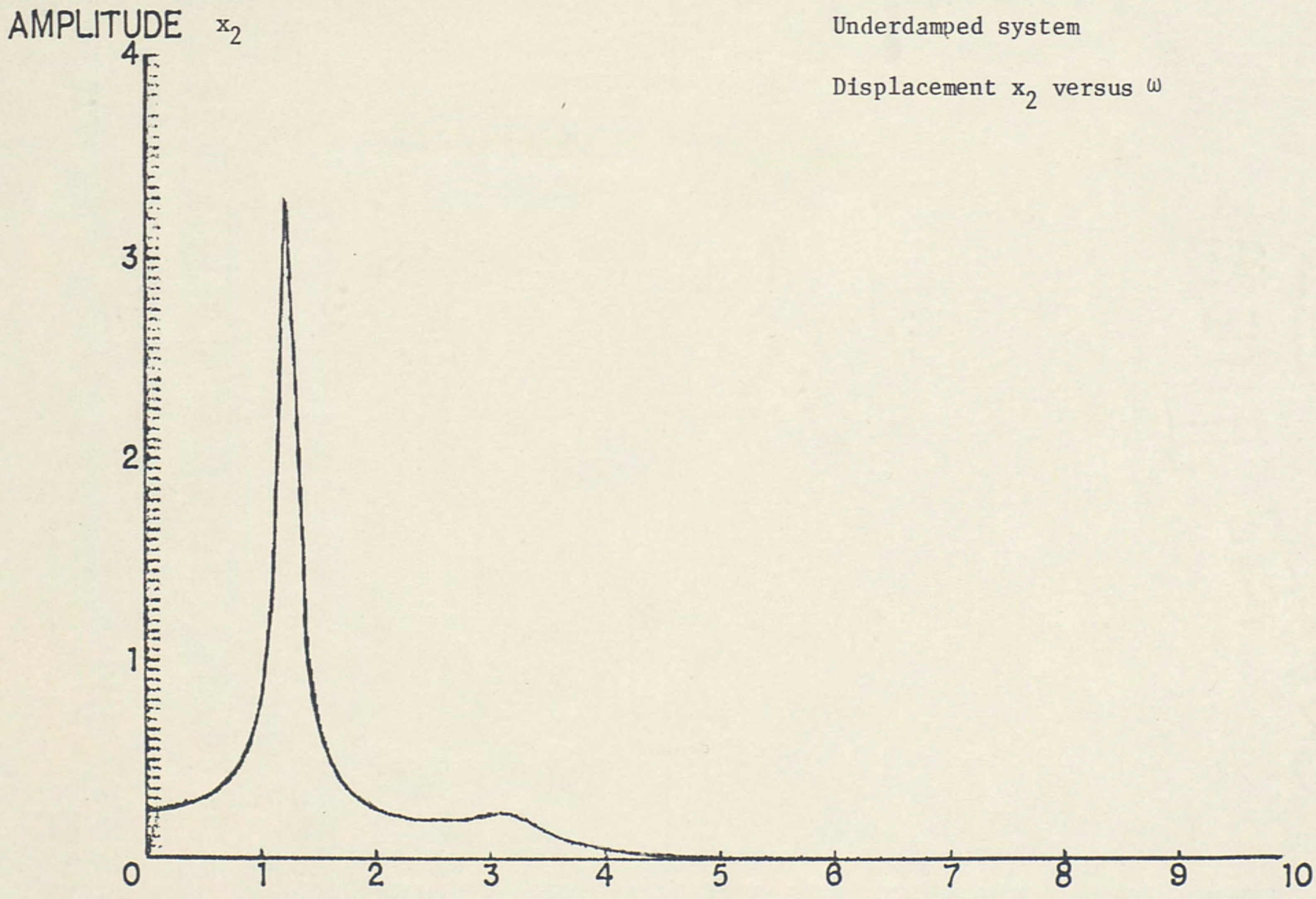

Fig. 21. Underdamped system amplitude IV versus frequency (cycles/sec).

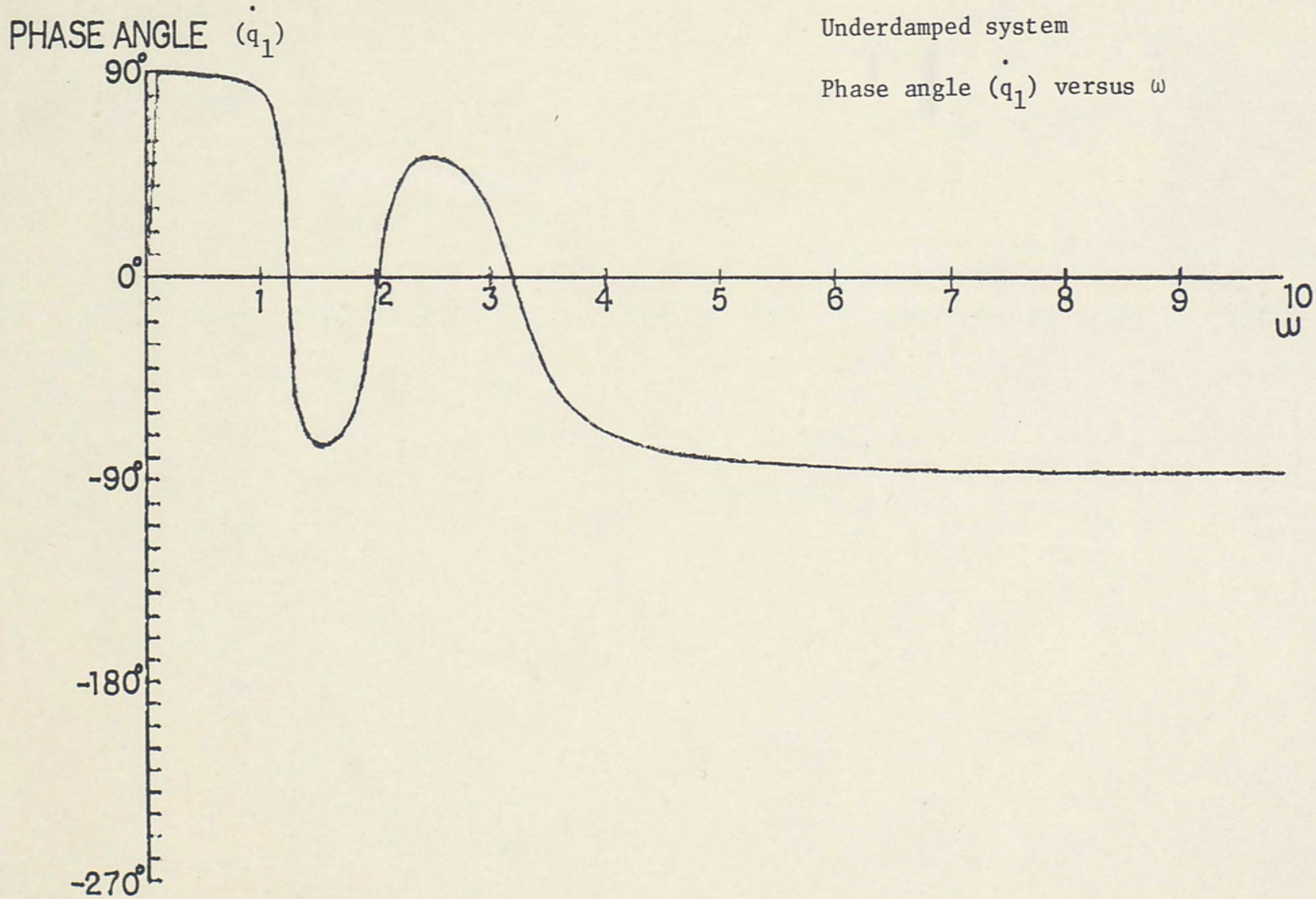

Fig. 22. Underdamped system phase angle I versus frequency (cycles/sec).

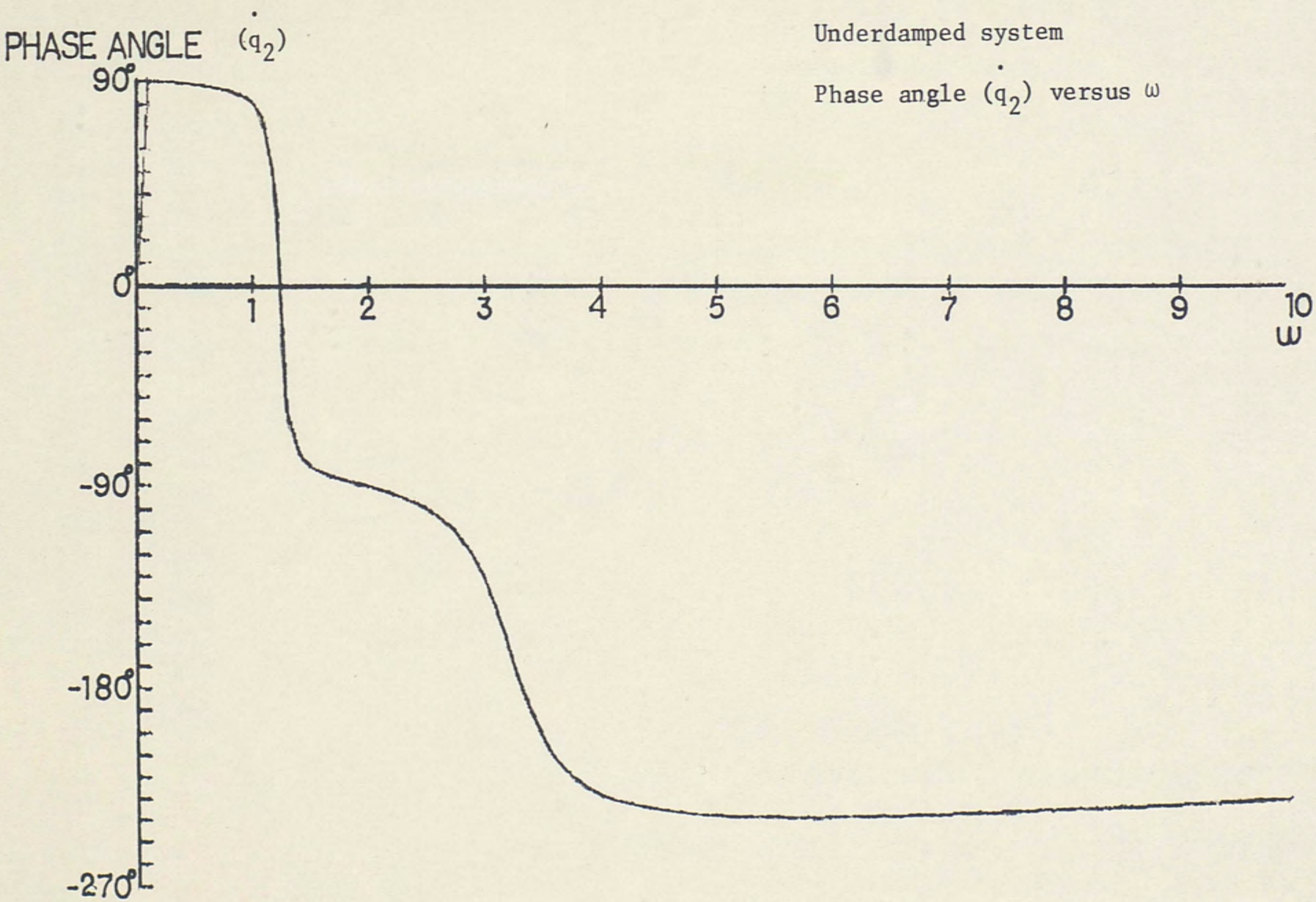

Fig. 23. Underdamped system phase angle II versus frequency (cycles/sec).

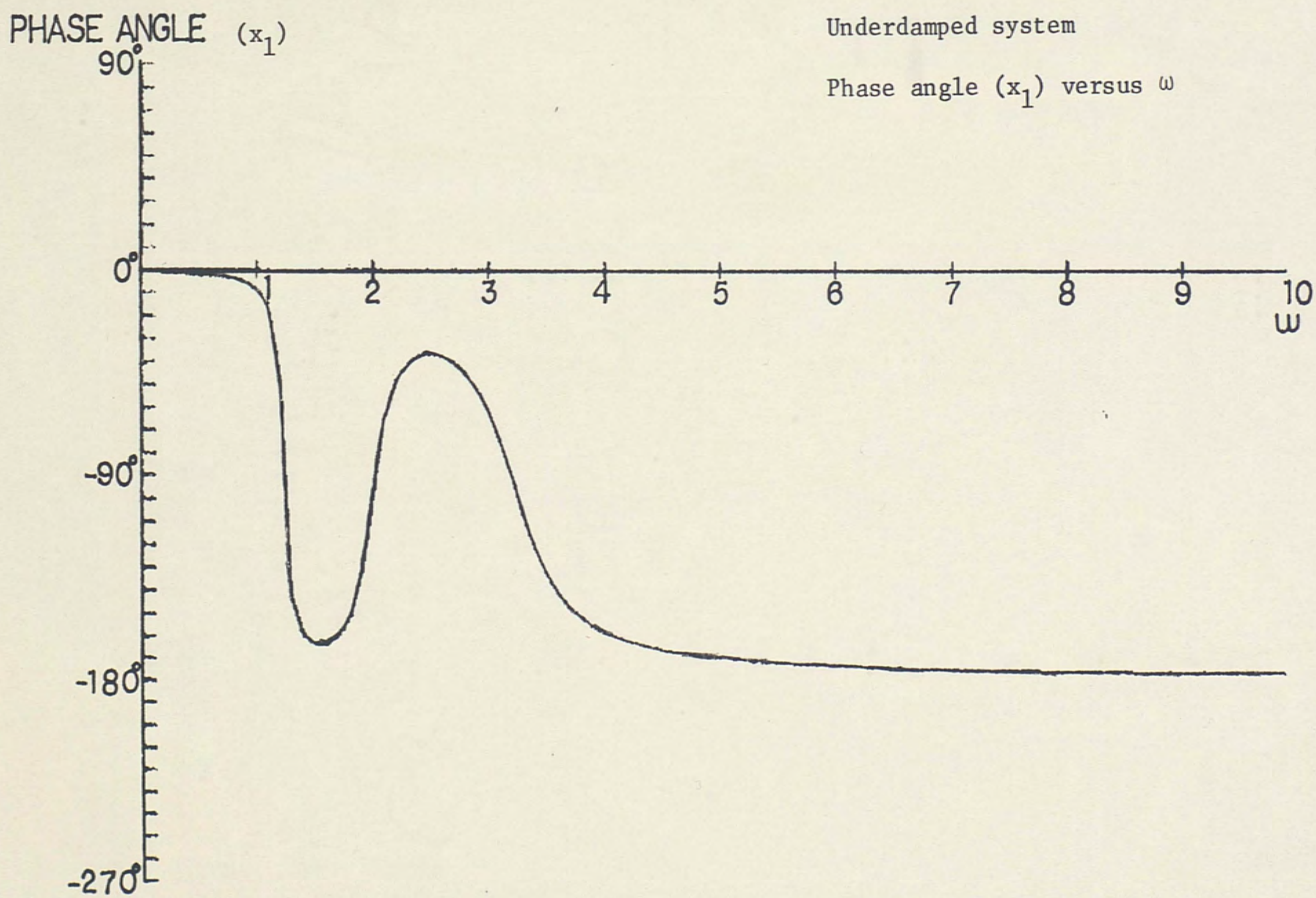

Fig. 24. Underdamped system phase angle III versus frequency (cycles/sec).

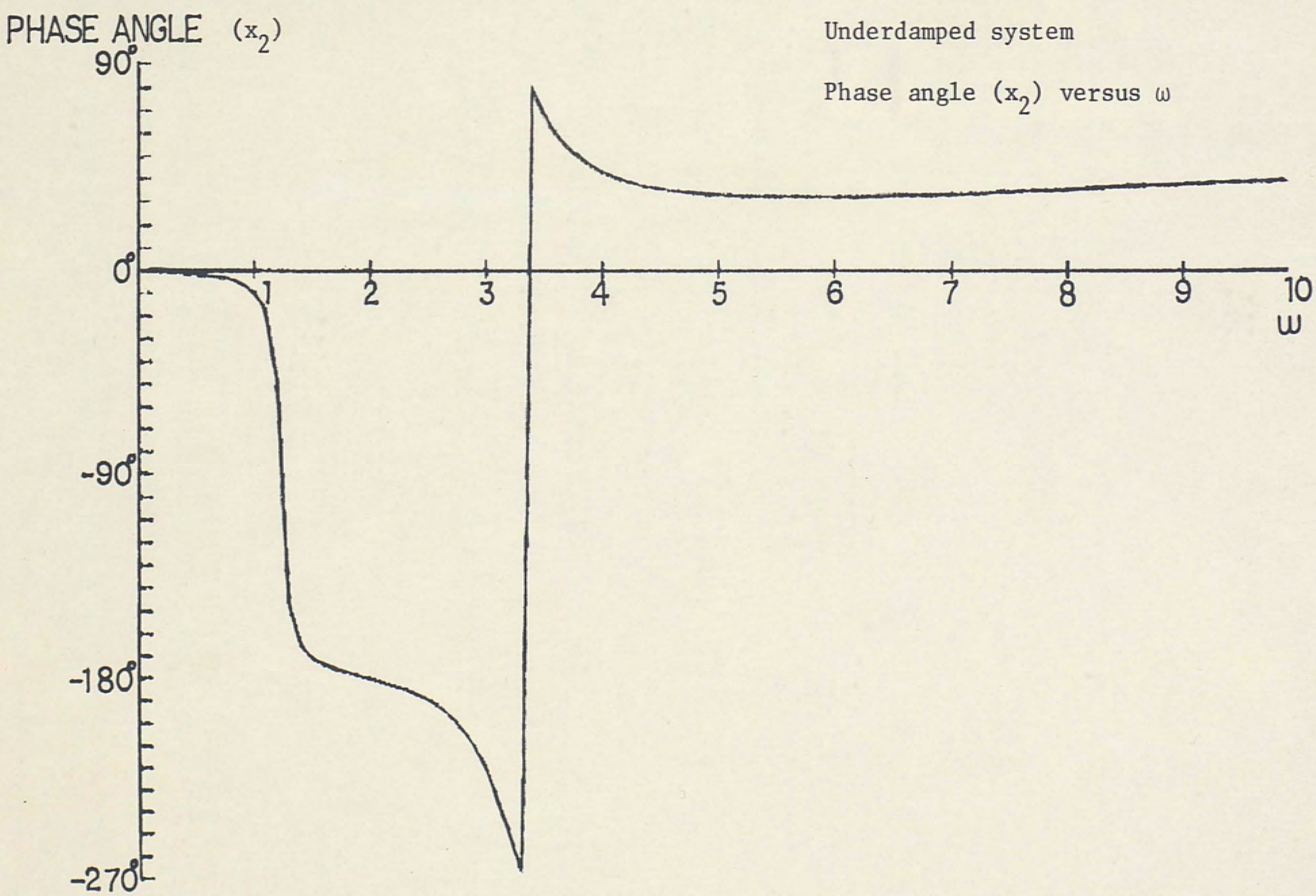

Fig. 25. Underdamped system phase angle IV versus frequency (cycles/sec).

Conservative Non-gyroscopic System

A number of Cam follower systems that have been used for automotive and aircraft piston engines have a complex follower system involving two levers for motion amplification. In this example, we will consider a double lever Cam mechanism with a four degree of freedom dynamic system model. A study about modeling this system has been done by Delbert Tesar and Gary K. Matthew (1976), and the equations of motion and detailed modeling procedures are also explained by Barkan (1953).

The details of the double lever Cam follower system and system model are shown in Figure 26. The differential equations of motion of this system are written in terms of displacement variables q_i, lumped masses m_i, and linear spring stiffness K_i.

Damping coefficients are neglected by Tesar and Matthew (1976) for simplicity. The potential and kinetic energy for the system are:

$$T = \tfrac{1}{2} m_1 \ddot{q}_1^2 + \tfrac{1}{2} m_2 \ddot{q}_2^2 + \tfrac{1}{2} m_3 \ddot{q}_3^2 + \tfrac{1}{2} m_4 \ddot{q}_4^2 \tag{54}$$

$$V = \tfrac{1}{2}k_1(q_1-q_o)^2 + \tfrac{1}{2}k_2(q_2-q_1)^2 + \tfrac{1}{2}k_3(q_3-q_2)^2 + \tfrac{1}{2}k_4(q_4-q_3)^2 + \tfrac{1}{2}k_5(q_5-q_4)^2 = 0$$

Solving by Lagrange's equation and setting $q_5 = 0$, and $q_o = y$, we get:

$$m_1 \ddot{q}_1 + (k_1 + k_2)q_1 - k_2 q_2 = k_1 y$$

$$m_2 \ddot{q}_2 - k_2 q_1 + (k_2 + k_3)q_2 - k_3 q_3 = 0 \tag{55}$$

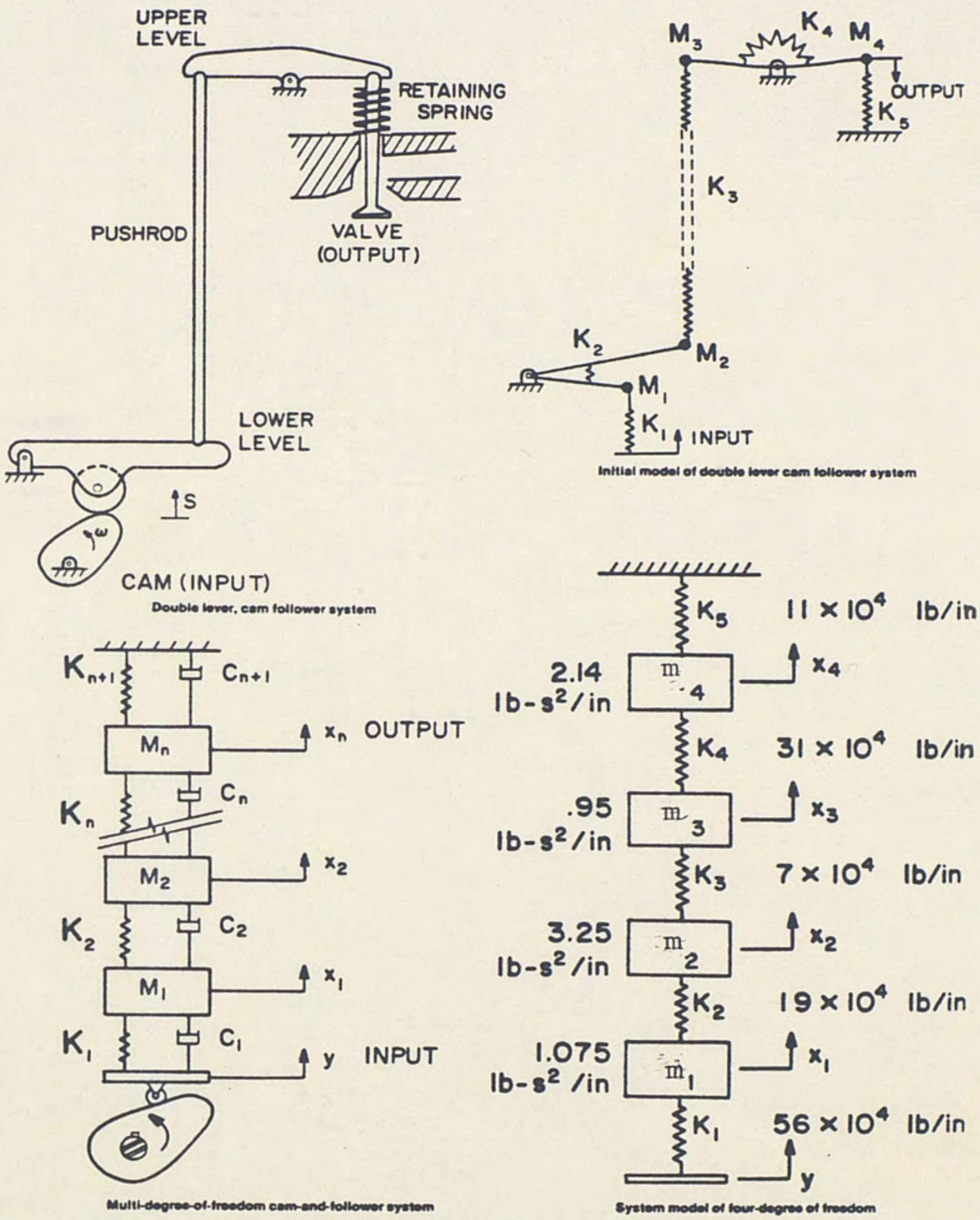

Fig. 26. Conservative non-gyroscopic system double lever cam follower system and model (Young and Shoup 1982).

$$m_3 \ddot{q}_3 - k_3 q_2 + (k_3 + k_4) q_3 - k_4 q_4 = 0$$

(55)
(continued)

$$m_4 \ddot{q}_4 - k_4 q_3 + (k_4 + k_5) q_4 = 0$$

In a matrix form:

(56)

$$\begin{bmatrix} m_1 & 0 & 0 & 0 \\ 0 & m_2 & 0 & 0 \\ 0 & 0 & m_3 & 0 \\ 0 & 0 & 0 & m_4 \end{bmatrix} \begin{bmatrix} \ddot{q}_1 \\ \ddot{q}_2 \\ \ddot{q}_3 \\ \ddot{q}_4 \end{bmatrix} + \begin{bmatrix} k_1+k_2 & -k_2 & 0 & 0 \\ -k_2 & k_2+k_3 & 0 & 0 \\ 0 & -k_3 & k_3+k_4 & -k_4 \\ 0 & 0 & -k_4 & k_4+k_5 \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{bmatrix} = \begin{bmatrix} k_1 y \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Next, we consider the data of a sample calculation given by S.S.D. Young and T.E. Shoup (1982) where they use the following values:

$$m_1 = 1.075 \text{ lb-s}^2/\text{in}$$

$$m_2 = 3.25 \text{ lb-s}^2/\text{in}$$

$$m_3 = 0.95 \text{ lb-s}^2/\text{in}$$

$$m_4 = 2.14 \text{ lb-s}^2/\text{in}$$

$$k_1 = 56 \times 10^4 \text{ lb/in}$$

$$k_2 = 19 \times 10^4 \text{ lb/in}$$

$$k_3 = 7 \times 10^4 \text{ lb/in}$$

$$k_4 = 31 \times 10^4 \text{ lb/in}$$

$$k_5 = 11 \times 10^4 \text{ lb/in}$$

$$y = 1 \text{ in}$$

Replacing these values in equation 56, the equations of motion are:

$$\begin{bmatrix} 1.075 & 0 & 0 & 0 \\ 0 & 3.25 & 0 & 0 \\ 0 & 0 & 0.95 & 0 \\ 0 & 0 & 0 & 2.14 \end{bmatrix} \begin{bmatrix} \ddot{q}_1 \\ \ddot{q}_2 \\ \ddot{q}_3 \\ \ddot{q}_4 \end{bmatrix} + \begin{bmatrix} 75 & -19 & 0 & 0 \\ -19 & 26 & -7 & 0 \\ 0 & -7 & 38 & -31 \\ 0 & 0 & -31 & 42 \end{bmatrix} \times 10^4 \begin{bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{bmatrix} = \begin{bmatrix} 56 \times 10^4 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad (57)$$

This is a conservative non-gyroscopic system with a characteristic equation of motion as follows:

$$M\ddot{q} + Kq = 0$$

where:

$$M = \begin{bmatrix} 1.075 & 0 & 0 & 0 \\ 0 & 3.25 & 0 & 0 \\ 0 & 0 & 0.95 & 0 \\ 0 & 0 & 0 & 2.14 \end{bmatrix} \quad (58)$$

$$K = \begin{bmatrix} 75 & -19 & 0 & 0 \\ -19 & 26 & -7 & 0 \\ 0 & -7 & 38 & -31 \\ 0 & 0 & -31 & 42 \end{bmatrix} \times 10^4$$

In state space representation, the system matrix (equation 8) is given by:

$$
M^* = \left[\begin{array}{cccc|cccc}
1.0715 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 3.25 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0.95 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 2.14 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{array}\right] \qquad (59)
$$

$$
K^* = \left[\begin{array}{cccc|cccc}
0 & 0 & 0 & 0 & 750,000 & -190,000 & 0 & 0 \\
0 & 0 & 0 & 0 & -190,000 & 260,000 & -70,000 & 0 \\
0 & 0 & 0 & 0 & 0 & -70,000 & 380,000 & -310,000 \\
0 & 0 & 0 & 0 & 0 & 0 & -310,000 & 420,000 \\
\hline
-1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 & 0 & 0 & 0
\end{array}\right] \qquad (60)
$$

The matrix A of the eigenvalue problem is then (see equation 9):

$$
A = \begin{bmatrix}
0 & 0 & 0 & 0 & -697,674 & 176,744 & 0 & 0 \\
0 & 0 & 0 & 0 & 58,461 & -80,000 & 21,538 & 0 \\
0 & 0 & 0 & 0 & 0 & 73,684 & -400,000 & 326,315 \\
0 & 0 & 0 & 0 & 0 & 0 & 144,859 & -192,261 \\
1 & -1.776\times10^{-15} & 2.664\times10^{-15} & 1.776\times10^{-15} & 0 & 0 & 0 & 0 \\
0 & 1 & -1.065\times10^{-14} & -7.105\times10^{-15} & 0 & 0 & 0 & 0 \\
-3.552\times10^{-15} & 7.105\times10^{-15} & 1 & 0 & 0 & 0 & 0 & 0 \\
-3.552\times10^{-15} & 5.329\times10^{-15} & -2.842\times10^{-14} & 1 & 0 & 0 & 0 & 0
\end{bmatrix} \qquad (61)
$$

With the computer program EIGRF from International Mathematical and Statistical Libraries, Inc. (1975)(Appendix A), we obtained the eigenvalues and normalized eigenvectors that are listed in Table 9 of Appendix B.

The eigenvectors were taken to be M orthogonal so that:

$$\phi_i^T M\phi_i = I \qquad i = 1, \ldots, n$$

and

$$\phi_i^T K\phi_i = \omega_i^2 \qquad i = 1, \ldots, n$$

In state space,

$$\phi_i^T M^*\phi_i = I \qquad i = 1, \ldots, 2n$$

$$\phi_i^T K^*\phi_i = \omega_i \qquad i = 1, \ldots, 2n$$

and these relations are checked and found to be correct.

System Response

The computer program for frequency response was used to get the graphs of frequency versus amplitude and phase angle that can be seen in Figures 27 through 42.

The input data was the matrix A (equation 61) and the input is defined by:

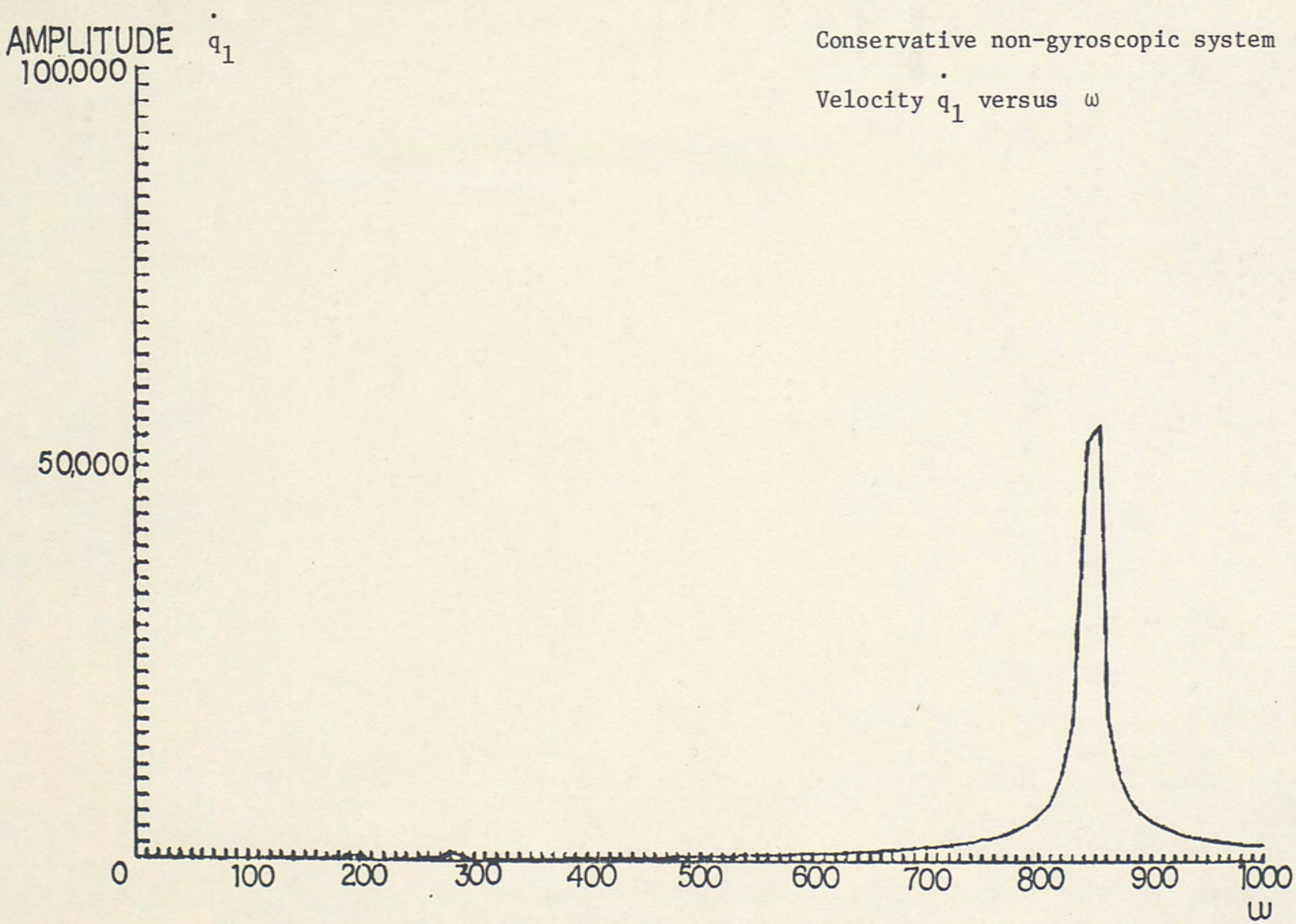

Fig. 27. Conservative non-gyroscopic system amplitude I versus frequency (cycles/sec).

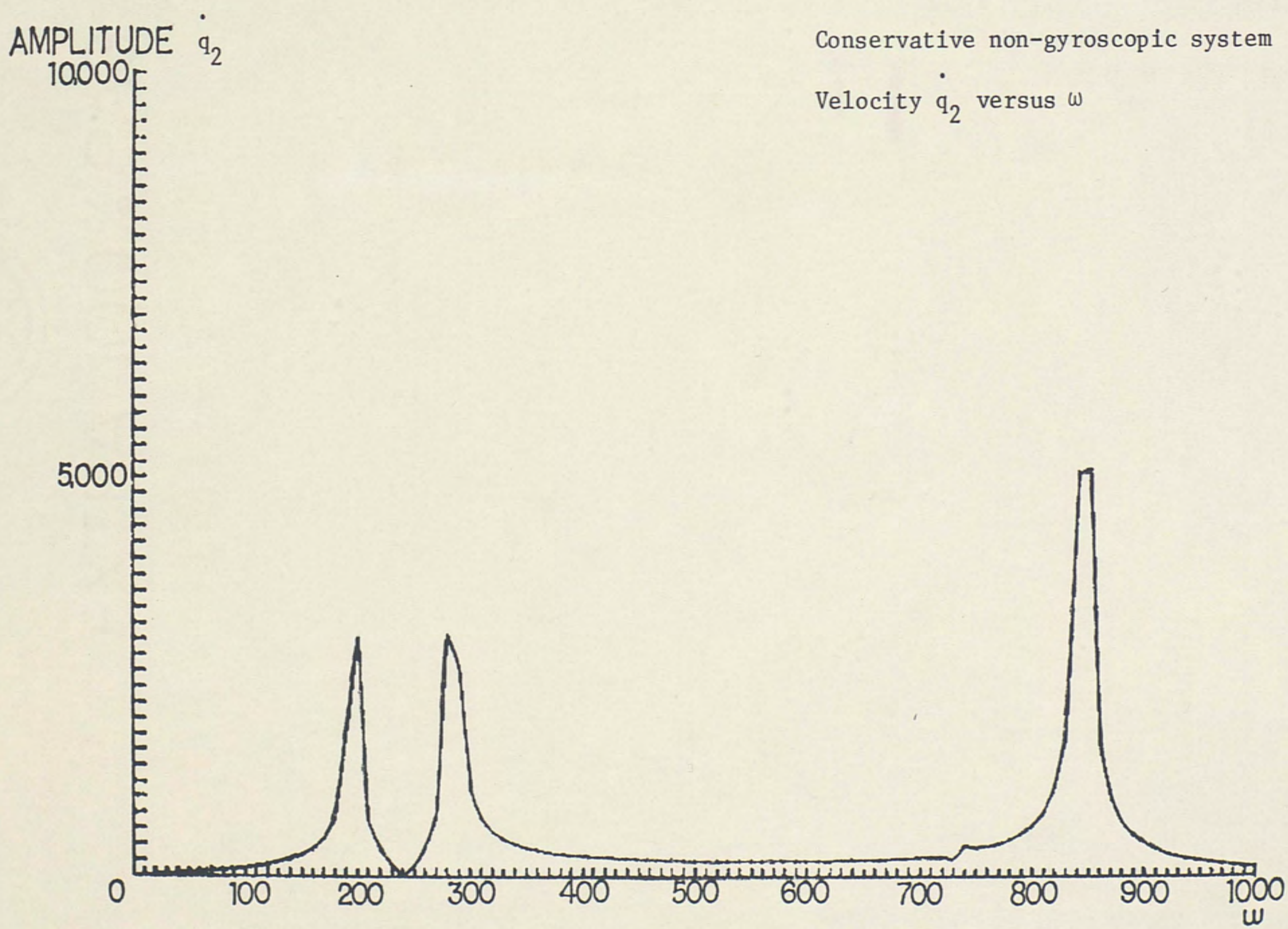

Fig. 28. Conservative non-gyroscopic system amplitude II versus frequency (cycles/sec).

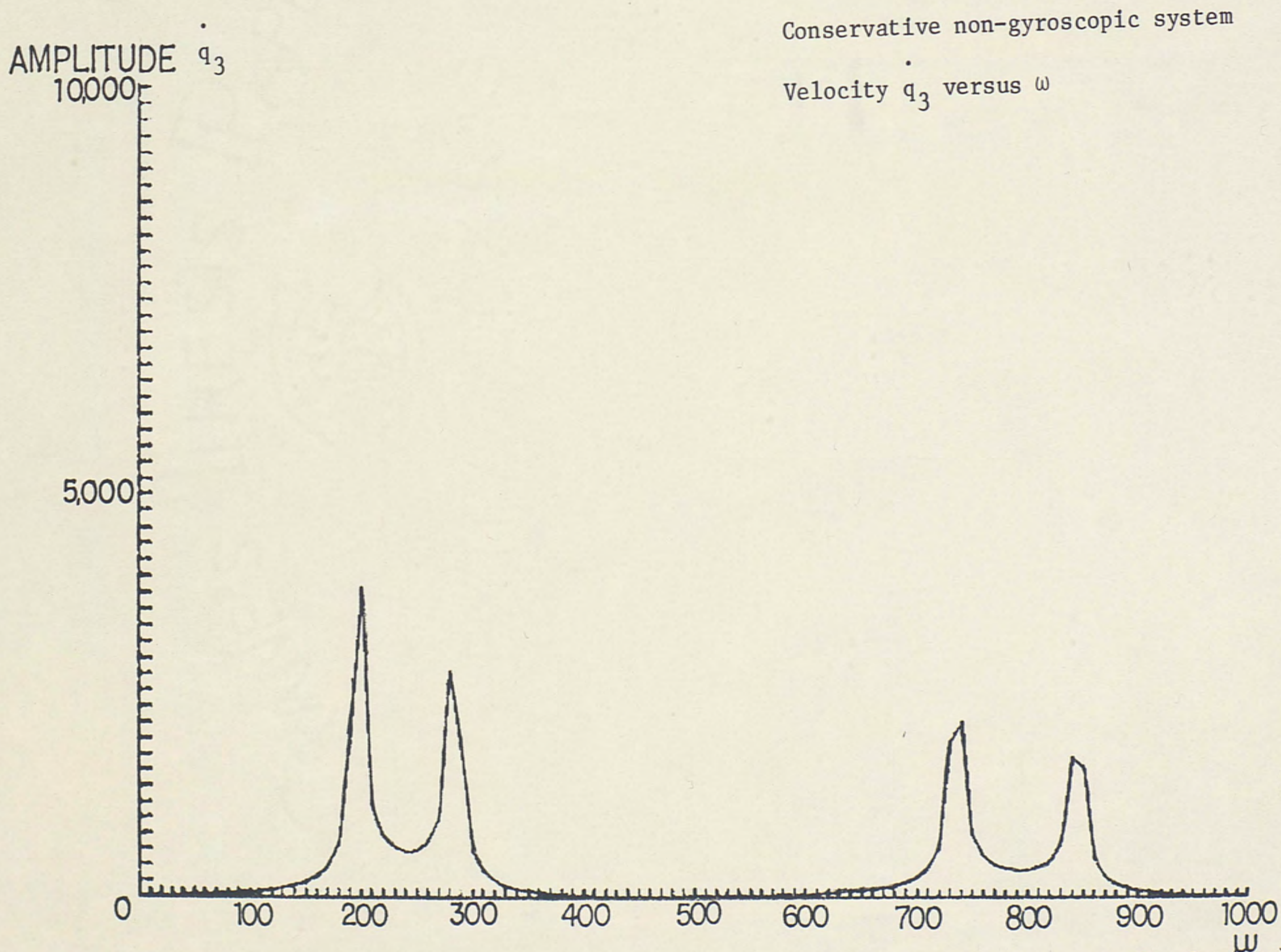

Fig. 29. Conservative non-gyroscopic system amplitude III versus frequency (cycles/sec).

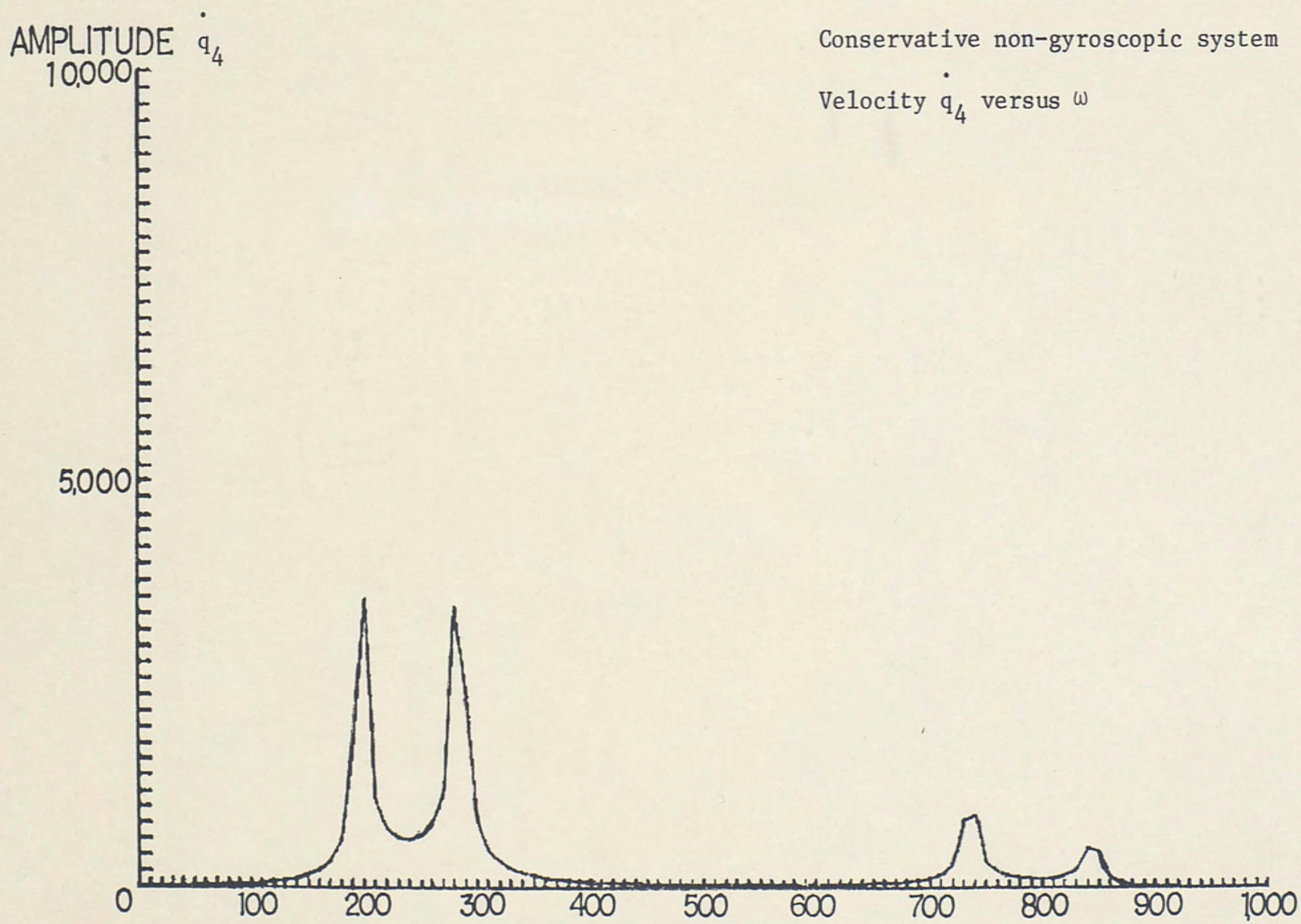

Fig. 30. Conservative non-gyroscopic system amplitude IV versus frequency (cycles/sec).

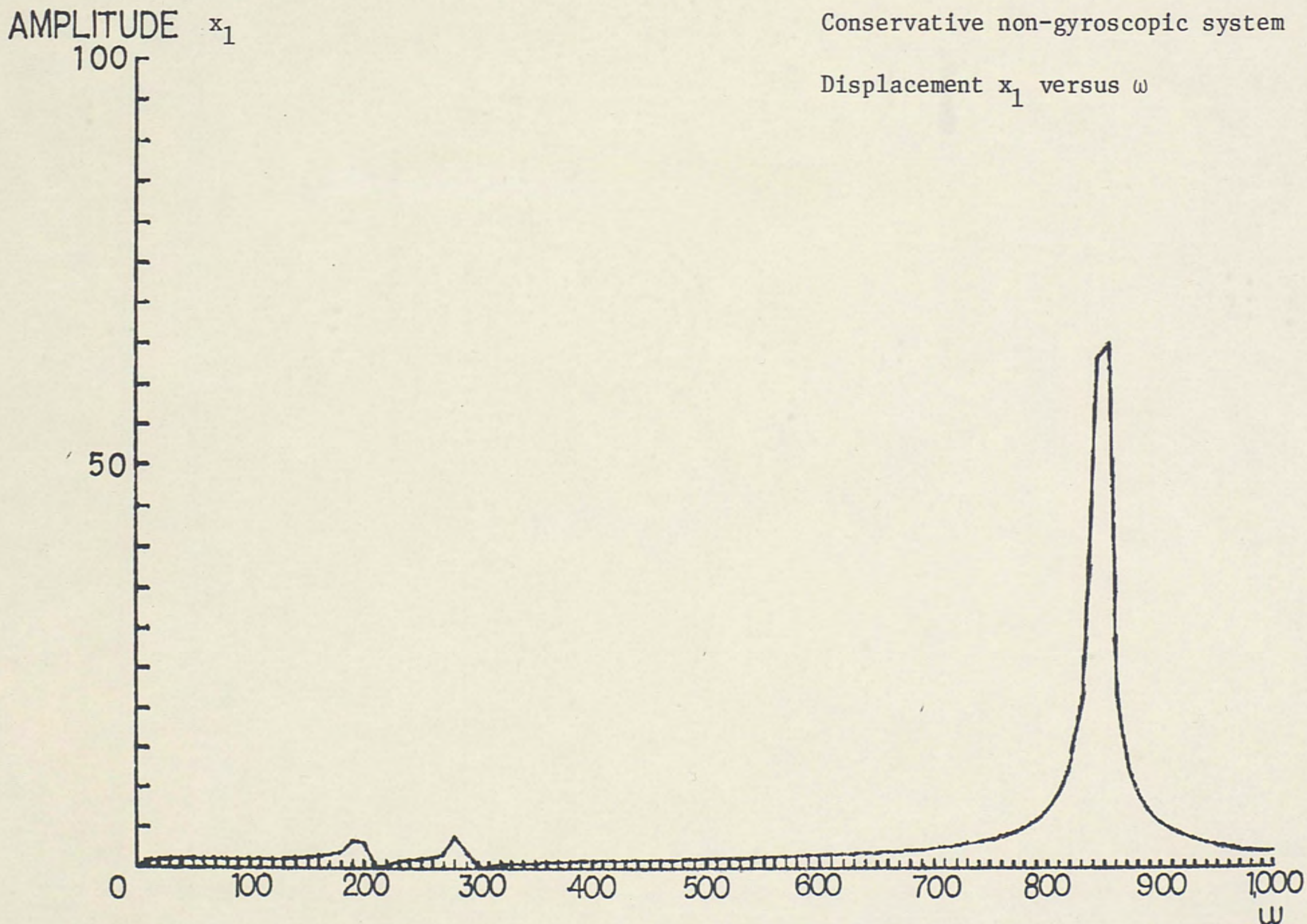

Fig. 31. Conservative non-gyroscopic system amplitude V versus frequency (cycles/sec).

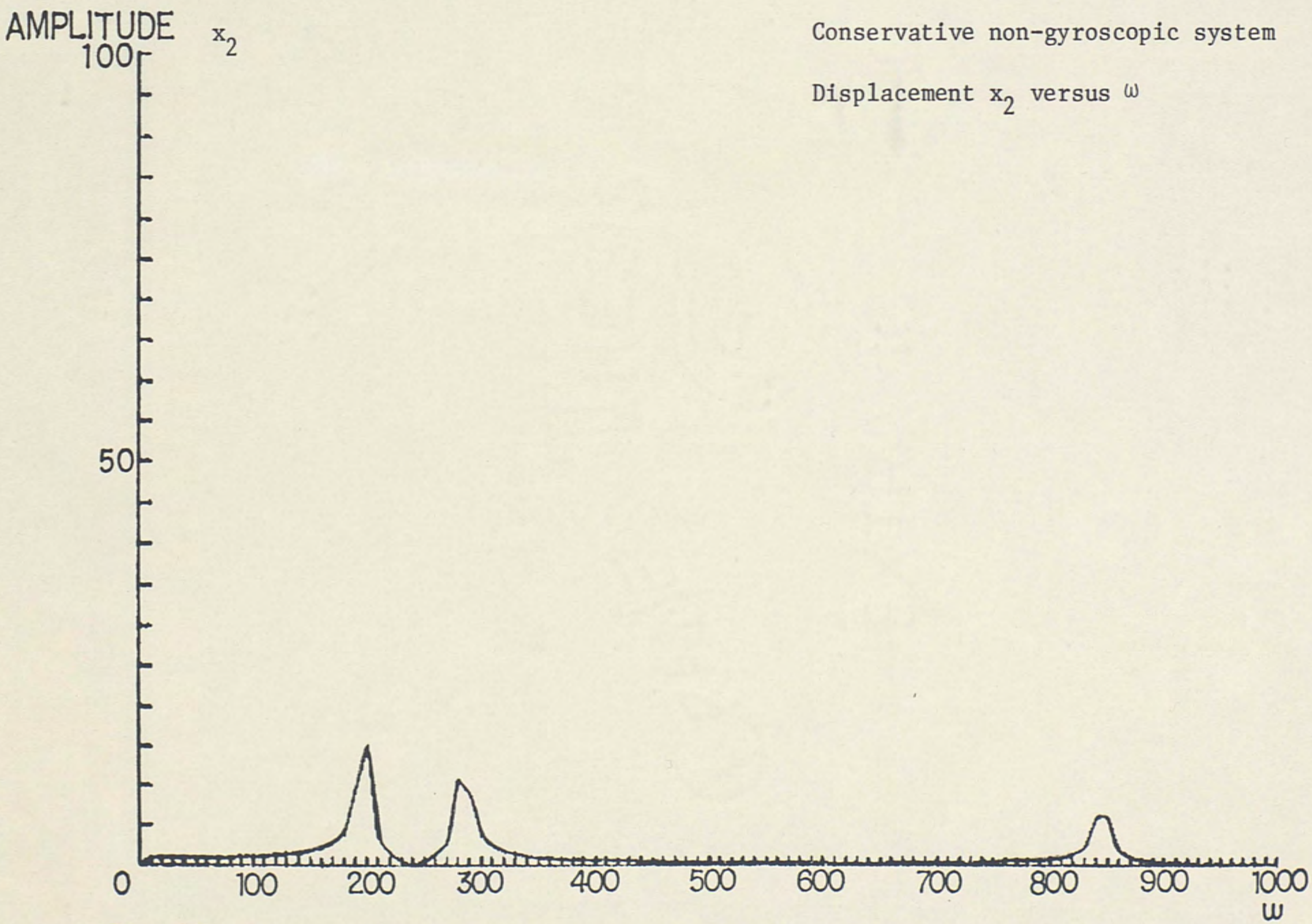

Fig. 32. Conservative non-gyroscopic system amplitude VI versus frequency (cycles/sec).

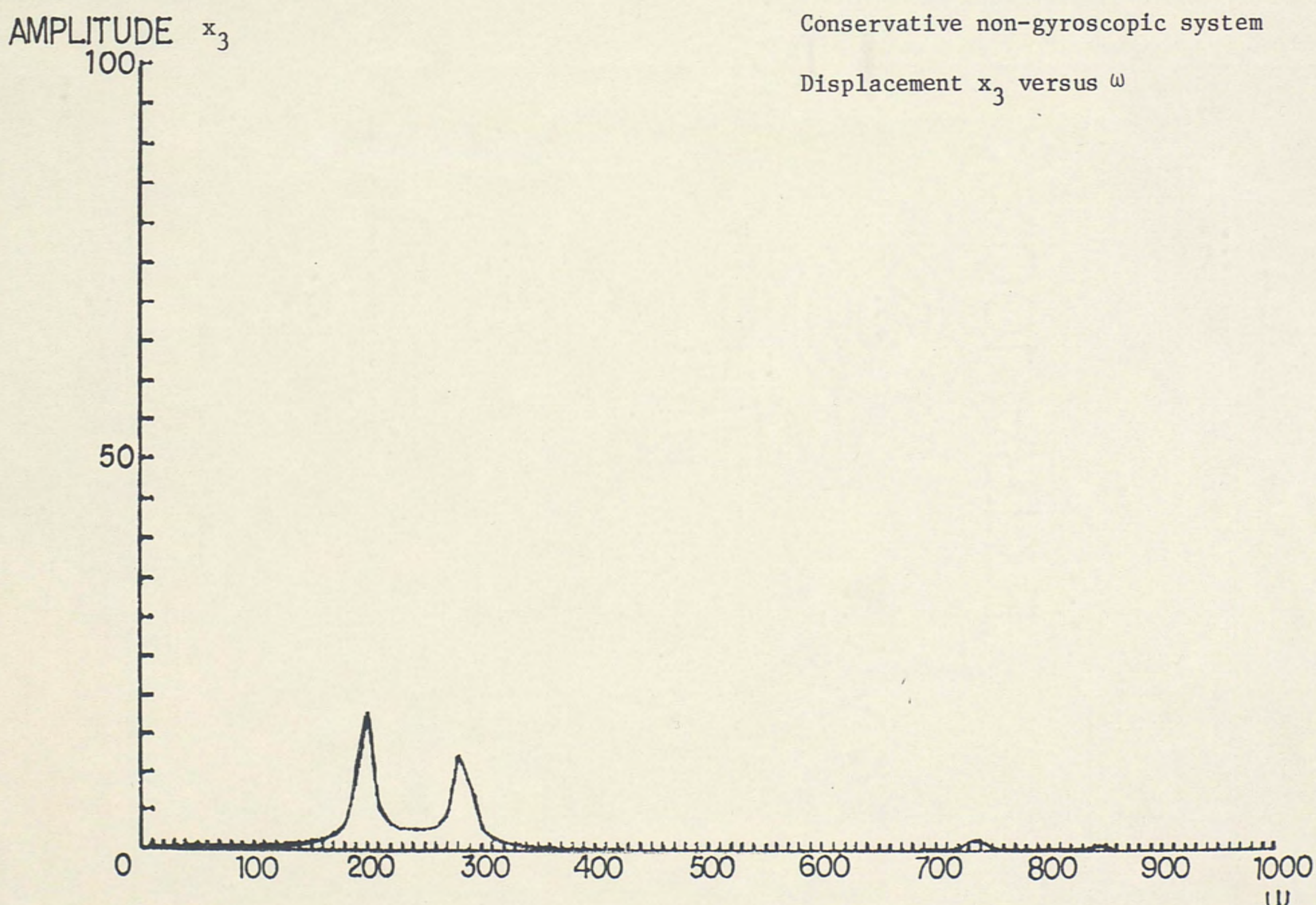

Fig. 33. Conservative non-gyroscopic system amplitude VII versus frequency (cycles/sec).

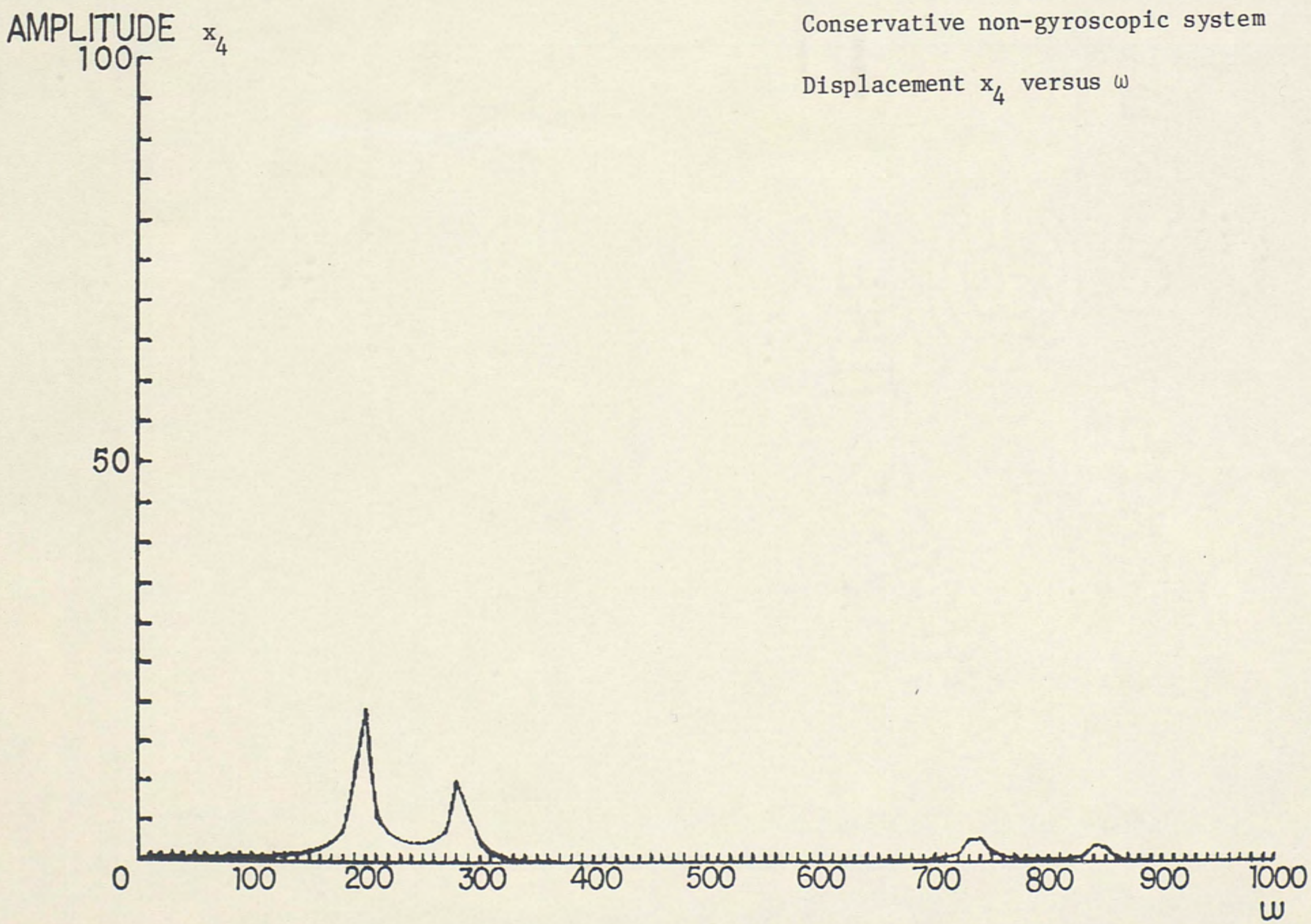

Fig. 34. Conservative non-gyroscopic system amplitude VIII versus frequency (cycles/sec).

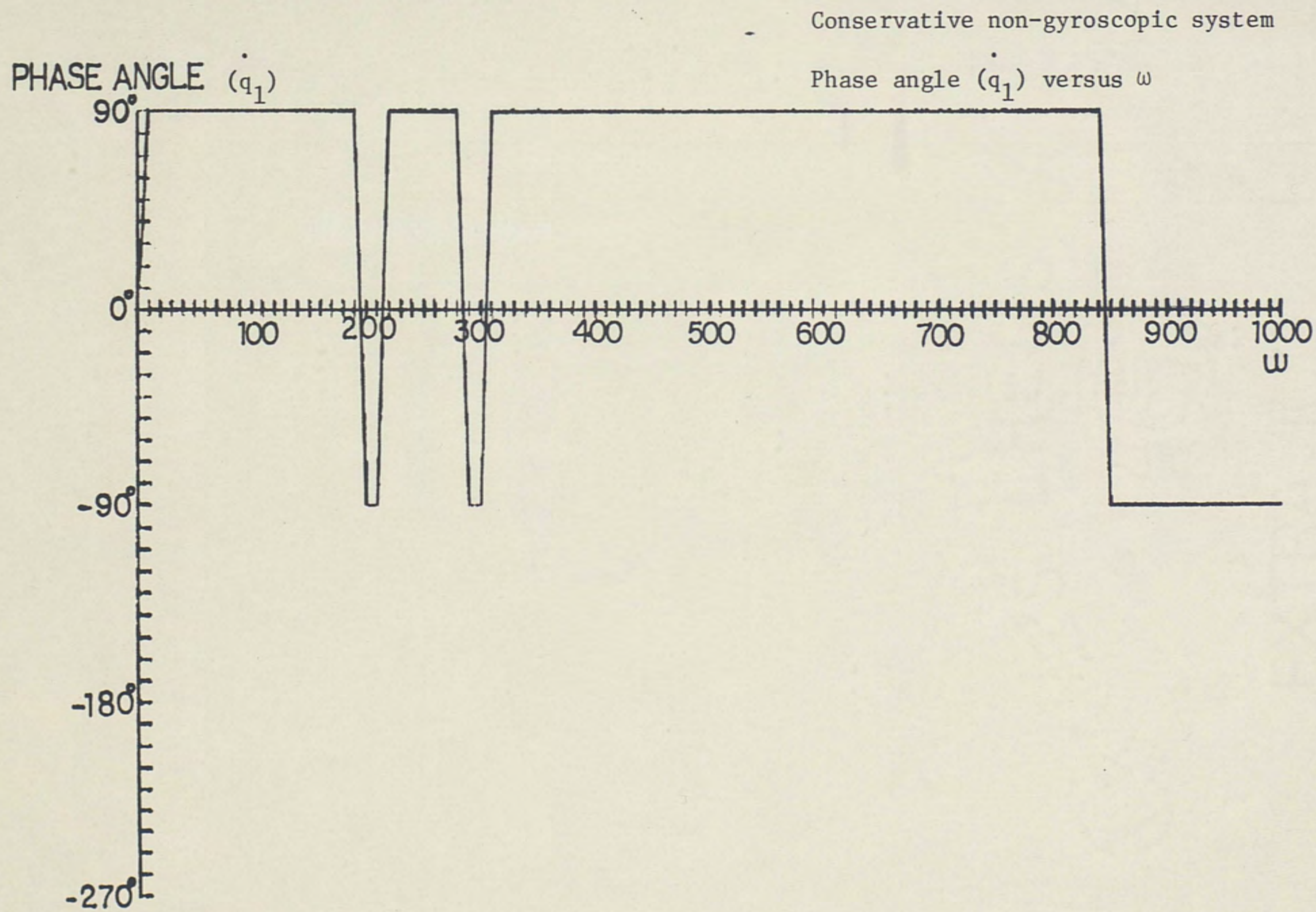

Fig. 35. Conservative non-gyroscopic system phase angle I versus frequency (cycles/sec).

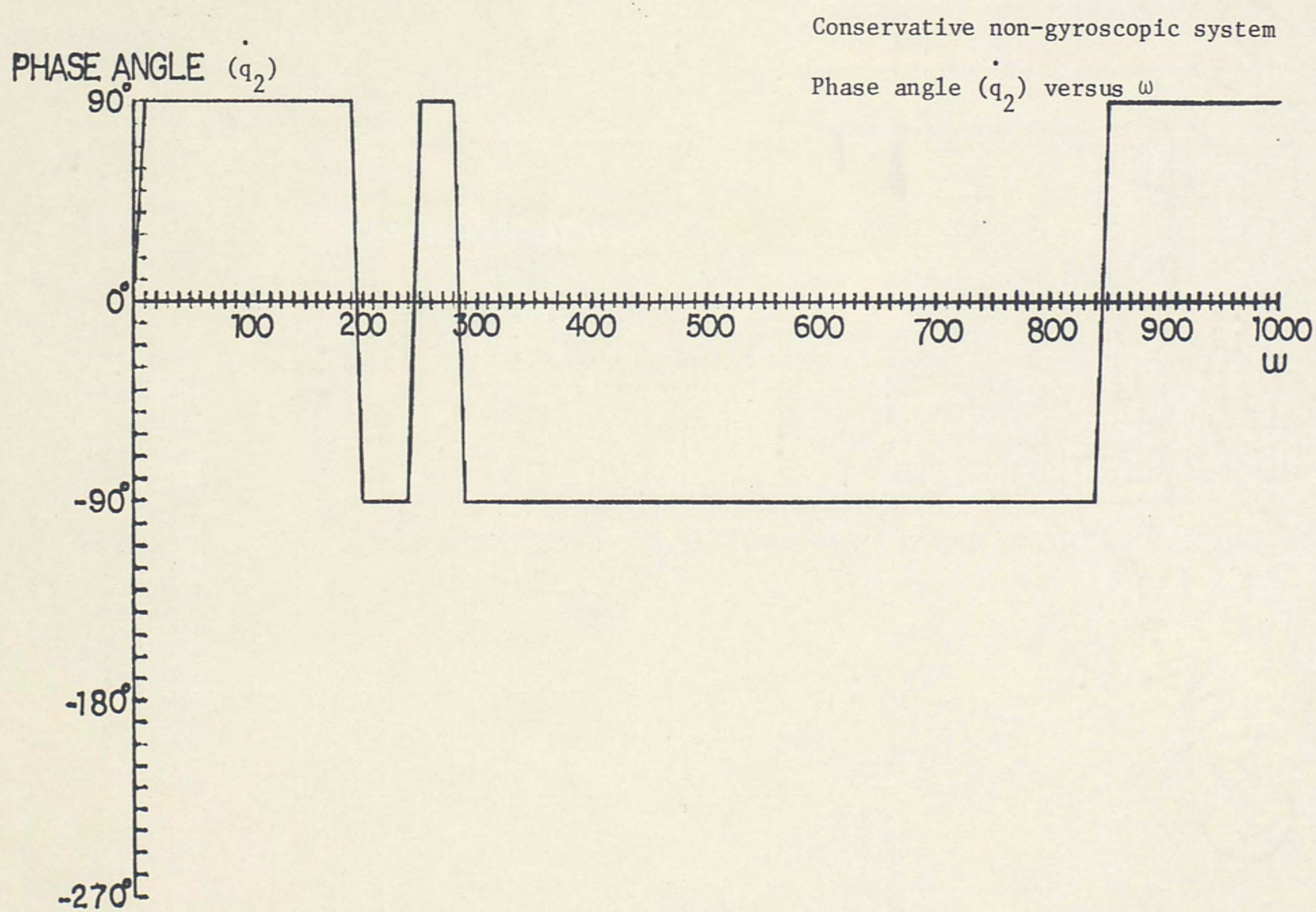

Fig. 36. Conservative non-gyroscopic system phase angle II versus frequency (cycles/sec).

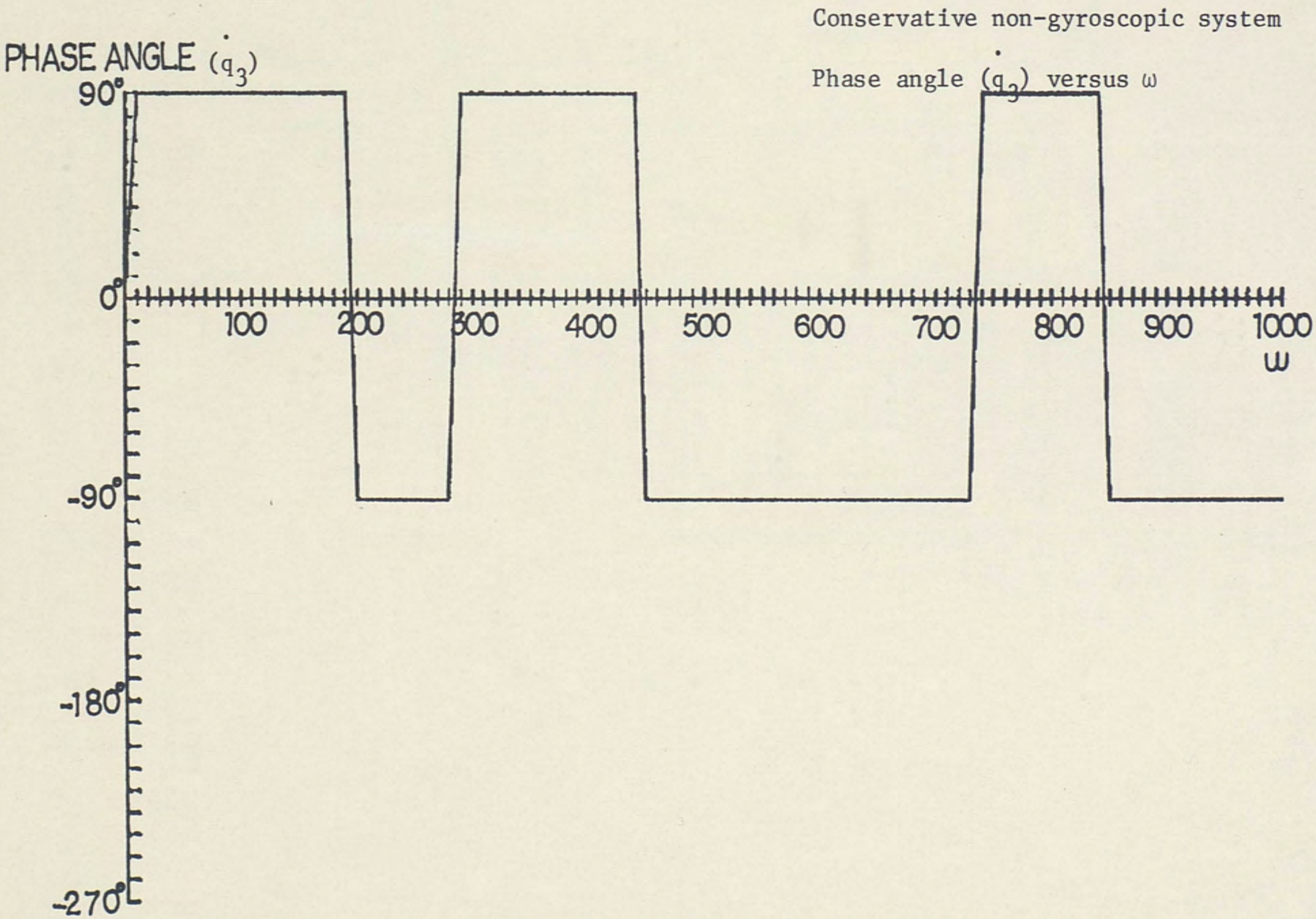

Fig. 37. Conservative non-gyroscopic system phase angle III versus frequency (cycles/sec).

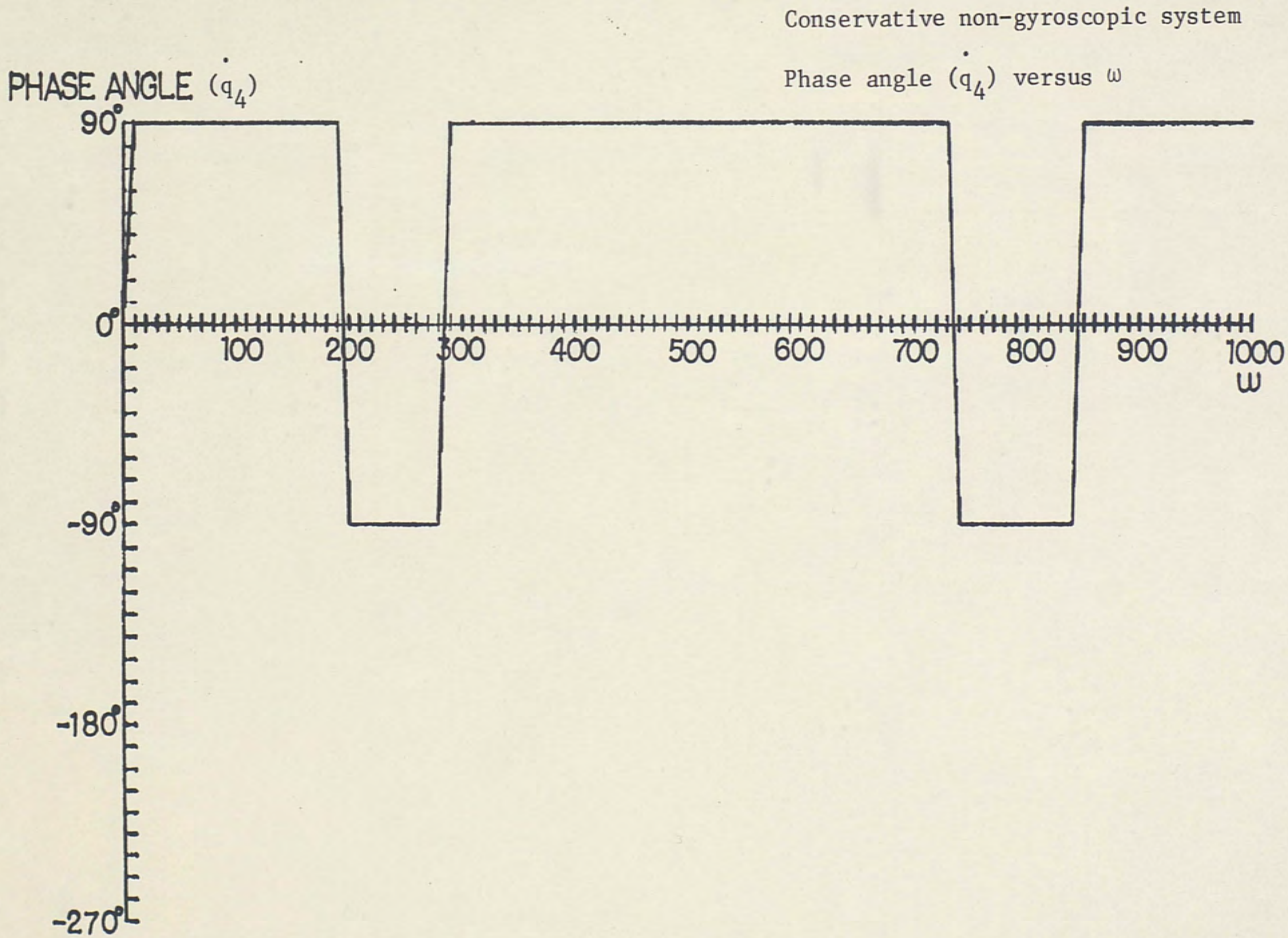

Fig. 38. Conservative non-gyroscopic system phase angle IV versus frequency (cycles/sec).

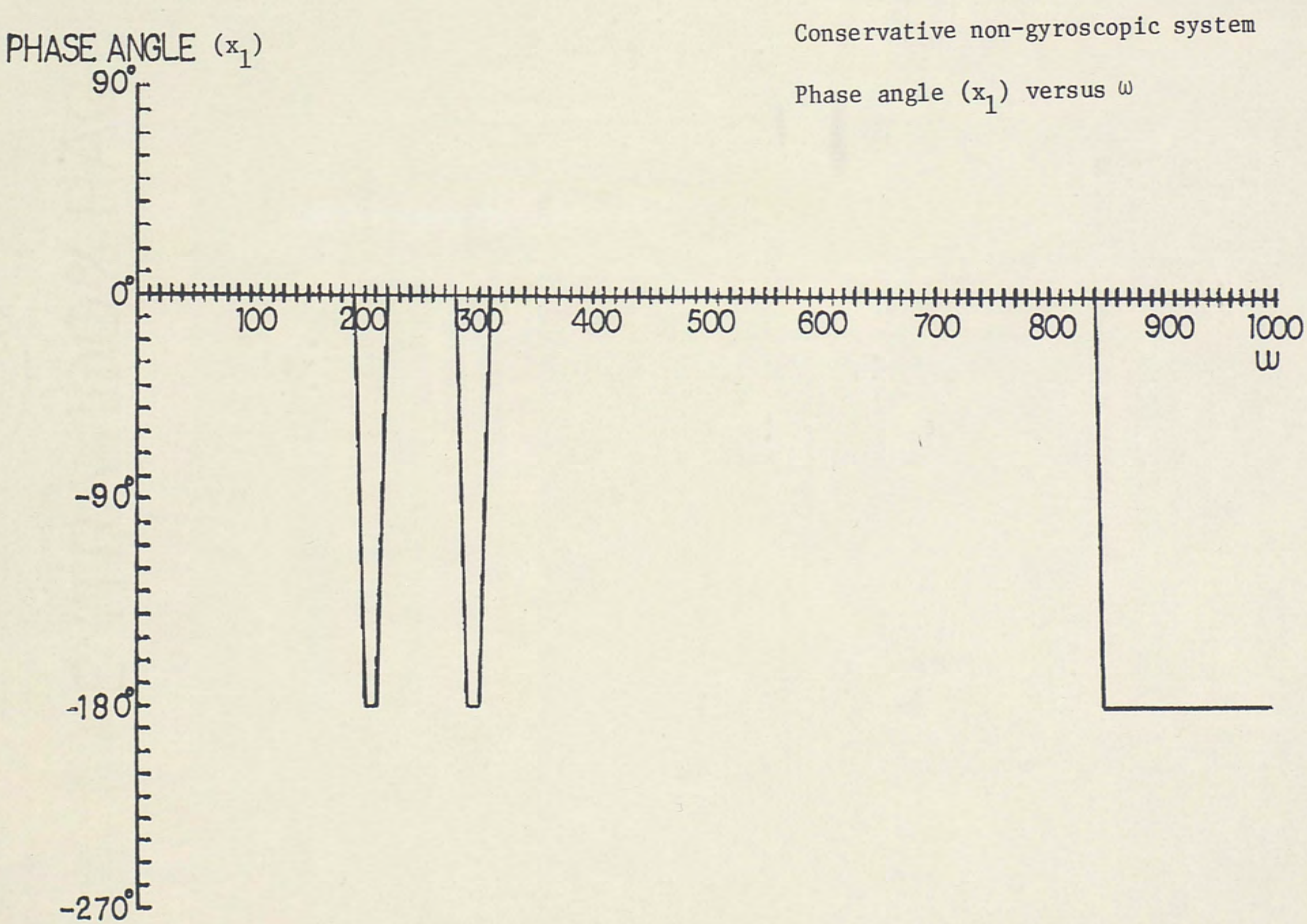

Fig. 39. Conservative non-gyroscopic system phase angle V versus frequency (cycles/sec).

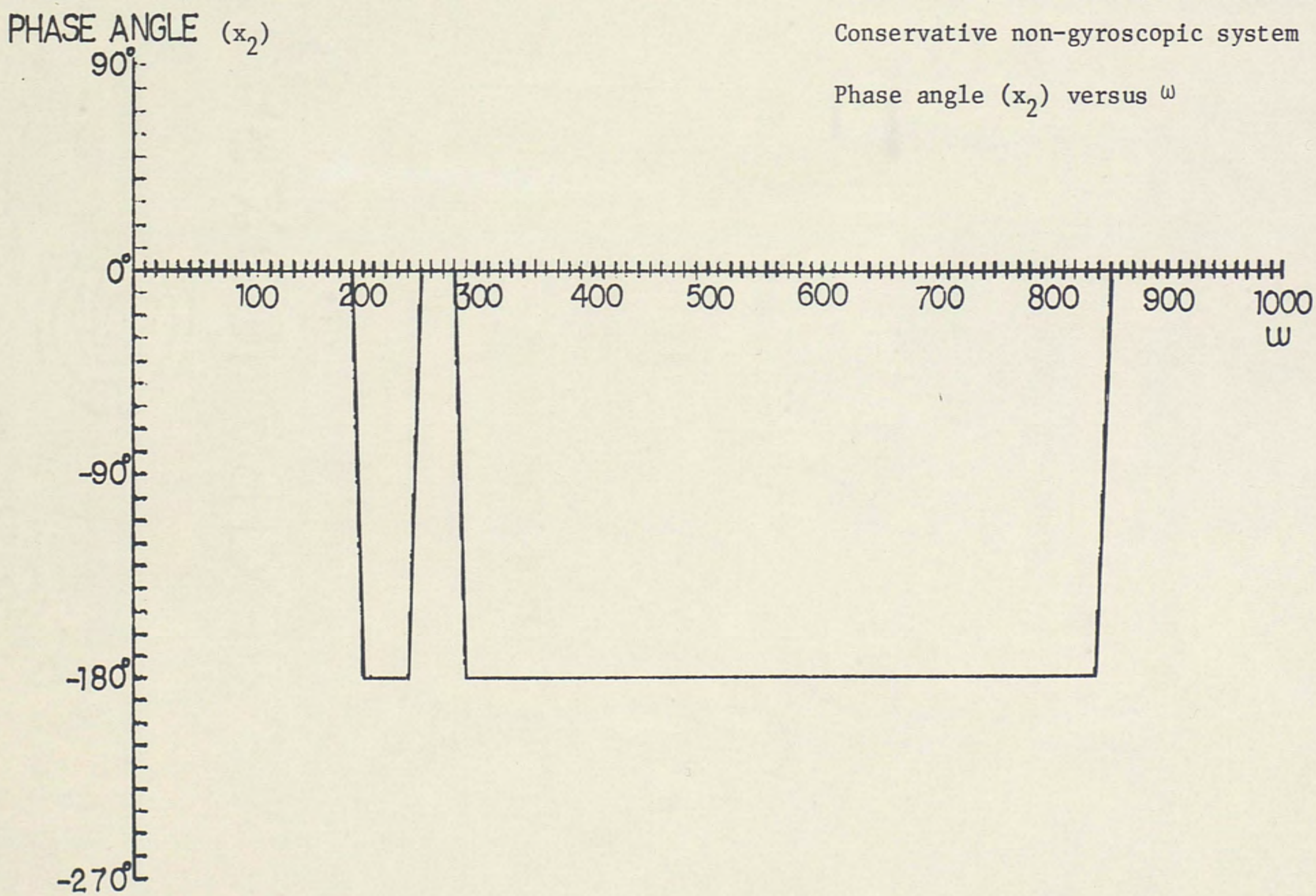

Fig. 40. Conservative non-gyroscopic system phase angle VI versus frequency (cycles/sec).

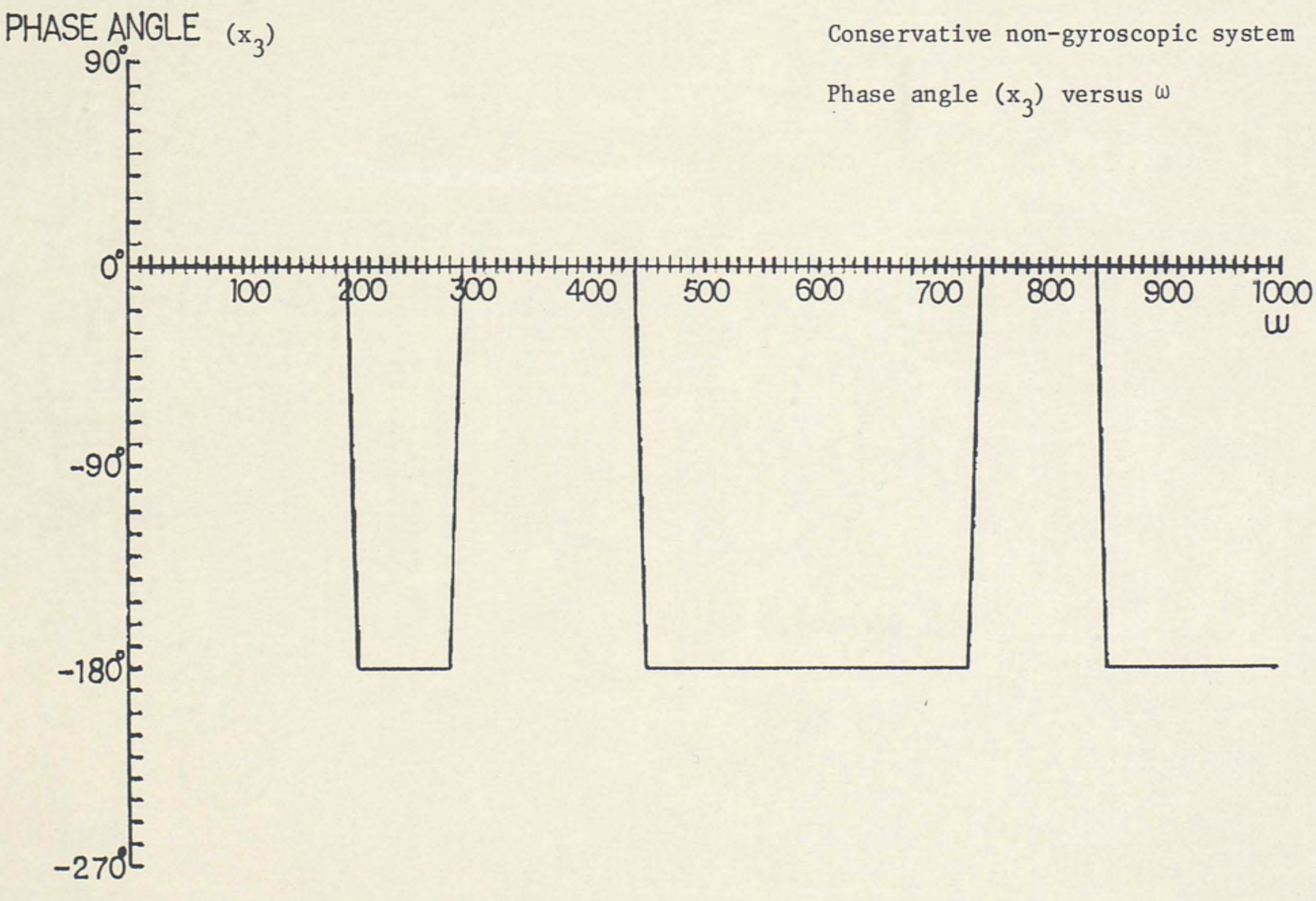

Fig. 41. Conservative non-gyroscopic system phase angle VII versus frequency (cycles/sec).

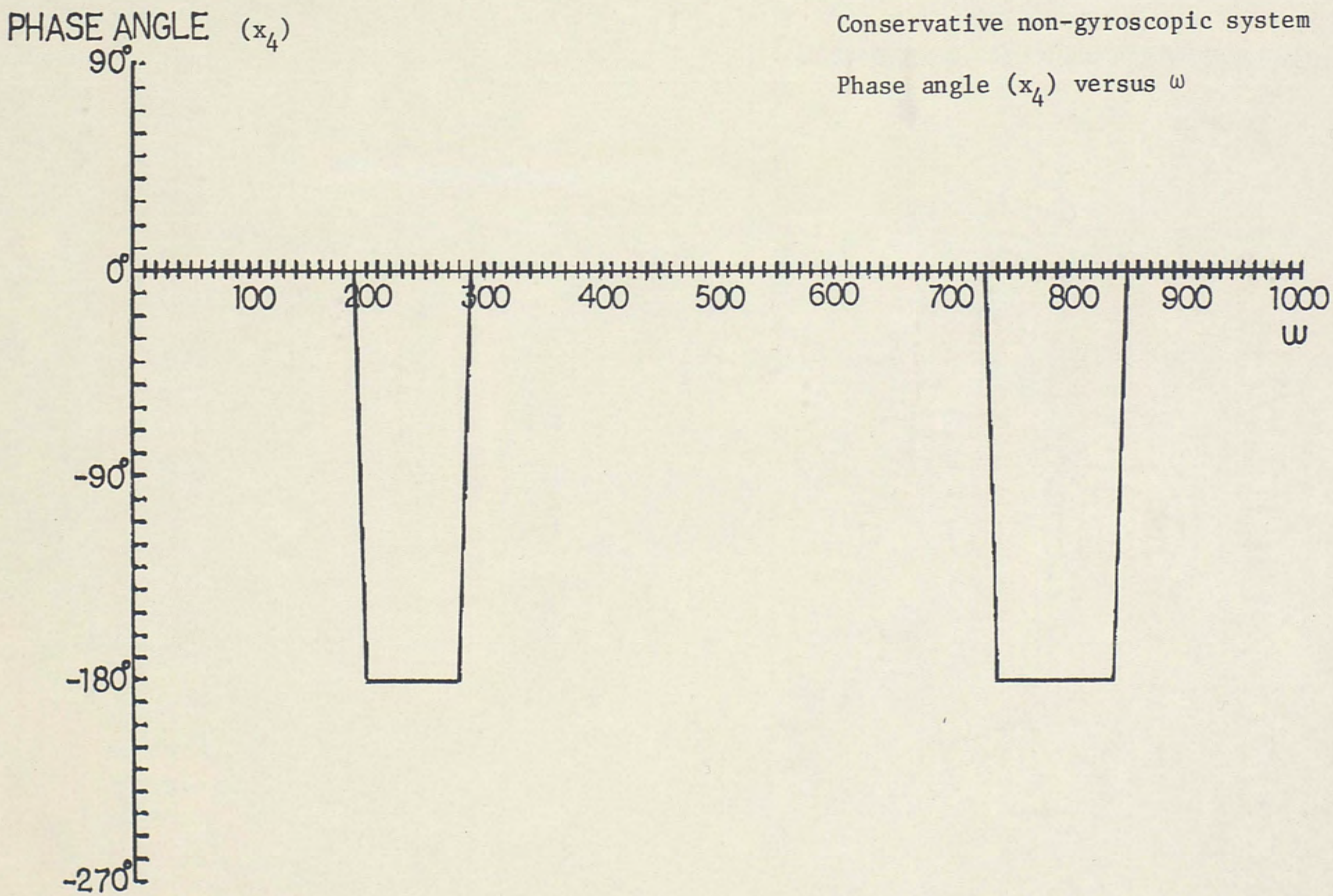

Fig. 42. Conservative non-gyroscopic system phase angle VIII versus frequency (cycles/sec).

$$\overline{b}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad \overline{b}_2 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad \begin{matrix} \omega_{increment} = 10 \text{ cycles/sec.} \\ \omega_{maximum} = 1{,}000 \text{ cycles/sec.} \end{matrix}$$

As can be noted, we used a unit step sinusoidal input and a range of frequencies that includes the values of the eigenvalues. We can see in the graphs of frequency versus amplitude that the four peaks of the response curves correspond exactly to the values of the eigenvalues. In the graphs of frequency versus phase angle, it is easily seen that the sudden changes in slope take place for the frequencies of the eigenvalues.

System Identification

The computer program for matrix identification was used in order to get the original matrix A of this example. A $0.5/1 \times 10^4$ or 0.005% random error was added to the frequency response solution, but the original matrix was not obtained.

Finally, different percentages of random error were added to the solution. The following is a list of the percentages considered:

0	or	0%	R.E.
$0.5/1 \times 10^4$	or	0.005%	R.E.
$0.5/1 \times 10^5$	or	0.0005%	R.E.
$0.5/1 \times 10^6$	or	0.00005%	R.E.
$0.5/1 \times 10^7$	or	0.000005%	R.E.
$0.5/1 \times 10^8$	or	0.0000005%	R.E.
$0.5/1 \times 10^{10}$	or	0.000000005%	R.E.

Only in the last case, $0.5/1 \times 10^{10}$, the identified matrix is exactly the same as the original. For the other cases between $0.5/1 \times 10^{4}$ and $0.5/1 \times 10^{8}$, a graph was plotted to observe the behavior of both the real and complex part of the eigenvalues. This is shown in Figure 43.

The input used in these previous cases was:

$$\bar{b}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad \bar{b}_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \qquad \begin{matrix} \omega_1 = 120 \text{ cycles/sec.} \\ \omega_2 = 960 \text{ cycles/sec.} \end{matrix}$$

It can be noticed that the input is a unit sine and unit cosine, because if we input only sine or cosine, we will have a singular matrix during the manipulation process.

As in the case of the damped non-gyroscopic system, the identification computer program was used 100 times and averaged for the added $0.5/1 \times 10^{10}$ random error. The matrix obtained was very similar to the original matrix A, i.e. (see equation 61):

$$B = \begin{bmatrix}
-2.214\times10^{-7} & 2.609\times10^{-5} & -3.256\times10^{-4} & 3.927\times10^{-4} & -697{,}673.999 & 176{,}743.970 & 0.251523 & -0.259750 \\
-2.211\times10^{-7} & 2.318\times10^{-5} & -2.813\times10^{-4} & 3.345\times10^{-4} & 58{,}461.000 & -80{,}000.025 & 21{,}538.216 & -0.222124 \\
3.776\times10^{-8} & -3.882\times10^{-6} & 4.630\times10^{-5} & -5.454\times10^{-5} & -1.368\times10^{-4} & 73{,}684.004 & -400{,}000.035 & 326{,}315.036 \\
7.813\times10^{-9} & -7.755\times10^{-7} & 9.114\times10^{-6} & -1.076\times10^{-5} & -2.769\times10^{-5} & 8.416\times10^{-4} & 144{,}858.992 & -196{,}260.992 \\
1.000 & -9.571\times10^{-9} & 7.214\times10^{-8} & -6.556\times10^{-8} & -4.623\times10^{-7} & 8.107\times10^{-6} & -5.310\times10^{-5} & 5.009\times10^{-5} \\
-5.171\times10^{-11} & 1.000 & -4.176\times10^{-8} & 4.700\times10^{-8} & 1.596\times10^{-7} & -4.009\times10^{-6} & 3.178\times10^{-5} & -3.206\times10^{-5} \\
-4.361\times10^{-12} & 2.317\times10^{-10} & 0.9999 & 1.119\times10^{-9} & 1.160\times10^{-8} & -1.812\times10^{-7} & 1.131\times10^{-6} & -9.743\times10^{-7} \\
-4.172\times10^{-11} & 4.577\times10^{-9} & -5.613\times10^{-8} & 1.000 & 1.566\times10^{-7} & -5.091\times10^{-6} & 4.329\times10^{-5} & -4.453\times10^{-5}
\end{bmatrix}$$

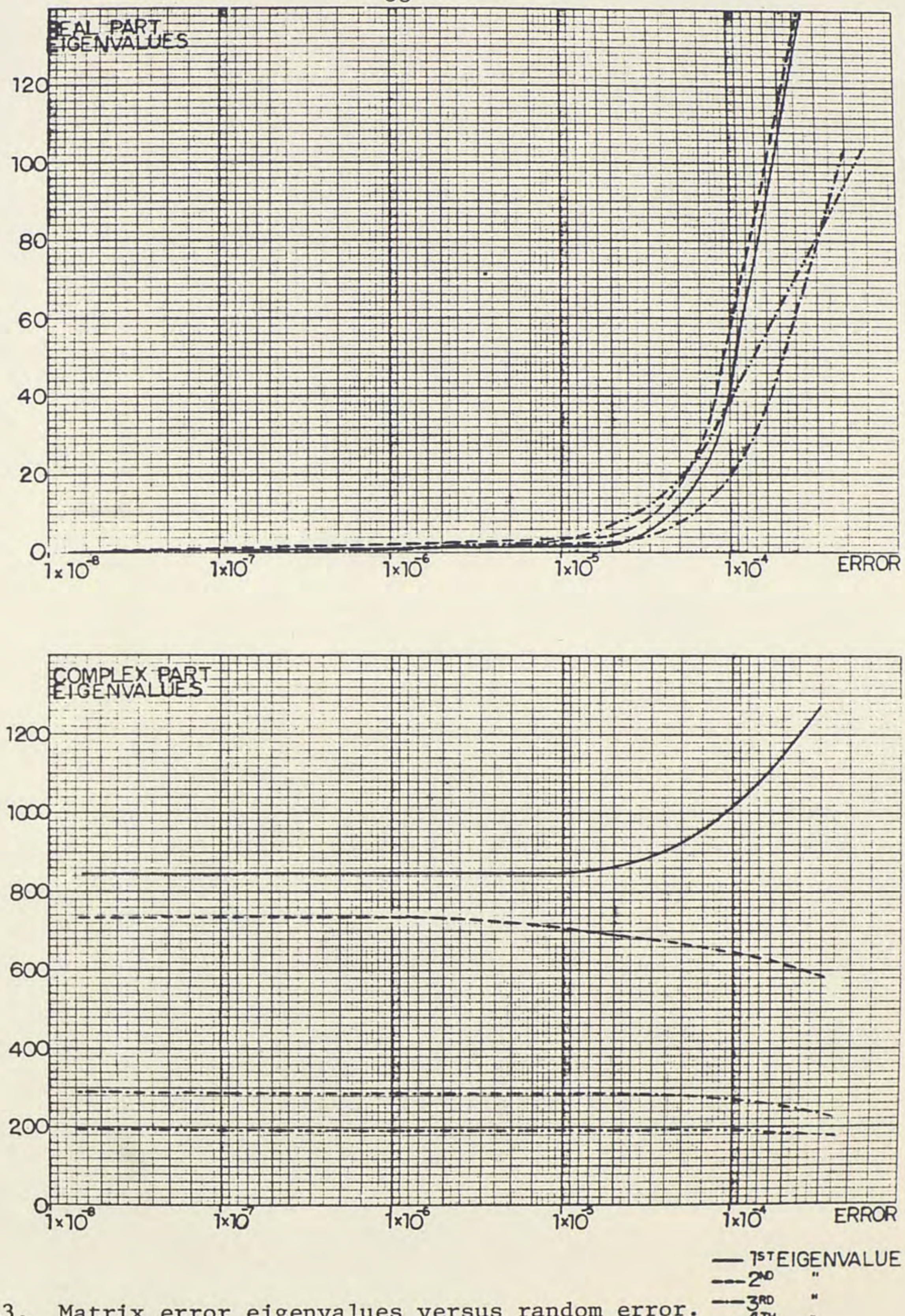

Fig. 43. Matrix error eigenvalues versus random error.

In addition to that, the error matrix E was calculated for every case, and the eigenvalues and eigenvectors obtained are depicted in Tables 10 through 14 of Appendix B. It can be noticed that the complex part of the eigenvalues of the error matrix is very small compared with the eigenvalues of the original matrix A. The eigenvalues of the real part are larger compared with the original eigenvalues, and for a $0.5/1 \times 10^4$ random error there is one eigenvalue which is very large (493.6) compared with the values of the original matrix A.

Discussion of Results

The simulated system response for the damped non-gyroscopic system shows that the system is stable. It can be seen that the peak occurs for the frequency corresponding to the value of the imaginary part of the two complex conjugate roots, in the frequency versus amplitude graph of the original system. For the critically damped case, we obtain practically the same results for amplitude and phase angle because the original system is only approximately 5% overdamped compared with the critically damped.

Reducing the damping to a 10% of the initial value caused an increase in amplitude up to 400% of the unit sinusoidal input to the system initially.

Also for the underdamped case the peaks occur at the frequency that corresponds to the imaginary part of the four complex conjugate roots of the system.

The error eigenvalues obtained for the added random errors (0.2% to 1.0%) are not bigger than 5% of the original eigenvalues. The identification of the system matrix was performed one hundred times and averaged with a 0.5% added random error for all the previous cases. The matrices obtained through the new identification process are very good approximations to the original system matrix. For the case of the conservative non-gyroscopic system, the response shows the peaks occurring, as expected, at a frequency corresponding to the complex part of the eight complex conjugate eigenvalues. But for the phase angle, the graphs show sudden changes in slope because there is no damping in the system.

In order to identify the system matrix, several percentages of random error were assumed. Only good results are obtained for very low percentages of error. The identified system matrix was obtained, but for a $0.5/10^{10}$ random error. The identification results are improved by performing the identification process 100 times and then averaging it.

CHAPTER IV

CONCLUSIONS AND RECOMMENDATIONS

The purpose of this work was to present a method to determine the modal parameters of a system. This method was verified by finding the response of two different sample cases and the identification tests were then performed for both of them.

The first case was a damped non-gyroscopic system. The modeling of this system, setting of the equations of motion, obtaining the modal data, and identifying the matrix of the system were easily performed. The results obtained for identification were accurate. The input used for the identification process was a unit sinusoidal force. The process could also be done with a unit cosinusoidal force. Both inputs can be used at the same time, with same results obtained. For the case of measurement error, results indicate that averaging can improve the accuracy of the identification process.

The second case of a conservative non-gyroscopic system was also modeled and solved as in the first case with the same method. The identification of the system matrix was then obtained in this case with both unit sinusoidal and cosinusoidal forces as inputs. If we use only a unit sinusoidal or cosinusoidal force as an input, we will find singularities during the manipulation process that will not permit us to get a solution. This is caused because of the lack of damping in this particular case. The error added in order to simulate real measuring conditions had to be much smaller than in the first case in order to get the original matrix A. Therefore, measurements for such systems should be very accurate. Based on the methods studied in this report, it is recommended that future research is needed in the following directions.

1. Use different sets of frequencies to perform the identification of the system matrix, and perform an averaging process over these sets.

2. Experimentally confirm the method to identify the system matrix from the real and complex part of the frequency response.

APPENDICES

APPENDIX A

COMPUTER PROGRAMS FOR DATA ANALYSIS

PROGRAM 1

COMPUTER PROGRAM FOR EIGENVALUES AND NORMALIZED EIGENVECTORS

```
      INTEGER N,IA,IJOB,IZ,IER
      REAL*8 A(4,4),WK(24),RW(8),RZ(32),H(4,4),F(4,4)
      COMPLEX*16 W(4),Z(4,4),D(4,4),S(4,4),E(4,4),ZN,CDSQRT
     *,G(4,4),Z1(4,4),Y(4,4)
      EQUIVALENCE (W(1),RW(1)),(Z(1,1),RZ(1))
      DATA  A/

      N=4
      IA=4
      IZ=4
      IJOB=2
      CALL EIGRF (A,N,IA,IJOB,RW,RZ,IZ,WK,IER)
      WRITE(6,20)IER,WK(1)
   20 FORMAT(1X,'IER=',I2,10X,'WK(1)=',E12.5//)
      WRITE(6,10)
   10 FORMAT(1X,'EIGENVALUES',/)
      DO 50 I=1,N
      WRITE (6,30)I,W(I)
   30 FORMAT(1X,'W( ',I2,' )= ',E12.5,8X,E12.5,1X,'I')
   50 CONTINUE
      WRITE(6,51)
   51 FORMAT(//,1X,'EIGENVECTORS',/ )
      DO 9 K=1,N
      WRITE(6,40)(Z(K,J),J=1,N)
   40 FORMAT(1X,'|',4(2X,E12.5,2X,E12.5,1X,'I'),2X,'|',
     */1X,'|',4(2X,E12.5,2X,E12.5,1X,'I'),2X,'|',/1X,'|',122X,'|')
    9 CONTINUE
C           NORMALIZE EIGENVECTORS
      DO 60 I=1,N
      ZN=(0.,0.)
      DO 70 J=1,N
   70 ZN=ZN+Z(J,I)*Z(J,I)
      DO 60 K=1,N
   60 Z(K,I)=Z(K,I)/CDSQRT(ZN)
      WRITE(6,80)
   80 FORMAT(//,1X,'NORMALIZED EIGENVECTORS',/)
      DO 90 K=1,N
      WRITE(6,40)(Z(K,J),J=1,N)
   90 CONTINUE
      STOP
      END
```

PROGRAM 2

COMPUTER PROGRAM FOR FREQUENCY RESPONSE

```
100 INIT
110 PAGE
120 REM:          ***** -.---.---.-      *****
130 REM:
140 PRINT "       L_ENTER THE TYPE OF DATA  <G_"
150 REM:
160 REM:          ****    -.---.---.-    *****
170 PRINT "J_  K1=1  FOR  DATA FILE  G_"
180 PRINT "J_  K1=2  FOR  FUNCTIONS  G_"
190 PRINT "J_  K1=3  FOR  DIRECT INPUT G_"
200 PRINT "J_  ENTER THE TYPE OF DATA ,K1>G_";
210 INPUT K1
220 REM:
230 REM:            ***** -.---.---.-    *****
240 REM:
250 PRINT "L_        WHAT TYPE OF OUTPUT DO YOU WANT? "
260 REM
270 REM:            ***** -.---.---.-    *****
280 PRINT "J_  K2=1   FOR TABLES  G_"
290 PRINT "J_  K2=2   FOR GRAPH   G_"
300 PRINT "J_  K2=3   FOR OUTPUT FILE G_"
310 PRINT "J_ ENTER THE TYPE OF OUTPUT ,K2>G_";
320 INPUT K2
330 IF K1=1 THEN 680
340 IF K1=2 THEN 870
350 IF K1=3 THEN 360
360 PRINT "J_  DATA FROM DIRECT INPUT "
370 PRINT "L_ENTER THE DIMENSION SIZE N  >G_";
```

PROGRAM 2 (Continued)

```
380 INPUT N
390 DIM A(N,N),A2(N,N),C(N,N),D1(N,1),D2(N,1),U1(N,1),U2(N,1)
400 DIM F(N,1),B(N,1)
410 PRINT "J_ENTER THE VALUES OF MATRIX A G_"
420 FOR I=1 TO N
430 FOR J=1 TO N
440 PRINT USING 460:"ENTER A(",I,",",J,")  XG_ ";
450 INPUT A(I,J)
460 IMAGE 8A,FD,1A,FD,6A,S
470 NEXT J
480 NEXT I
490 PRINT "J_ENTER THE VALUES OF U1 J_G_"
500 FOR I=1 TO N
510 PRINT USING 530:"U1(",I,")  XG_";
520 INPUT U1(I,1)
530 IMAGE 3A,FD,6A,S
540 NEXT I
550 PRINT "J_ENTER THE VALUES OF U2 J_G_"
560 FOR I=1 TO N
570 PRINT USING 590:"U2(",I,")  XG_";
580 INPUT U2(I,1)
590 IMAGE 3A,FD,5A,S
600 NEXT I
610 PRINT "J_ENTER THE OMEGA INCREMENT"
620 PRINT "J_G_ OMEGA=";
630 INPUT W1
640 PRINT "J_ ENTER THE FINAL OMEGA, OMEGA MAX"
650 PRINT "J_G_ OMAX=";
```

PROGRAM 2 (Continued)

```
660 INPUT W2
670 GO TO 910
680 PRINT "J_      DATA FROM DATA FILE G_"
690 CALL "OPEN","FEIJ014 LFN:1 TACCESS:FULL"
700 READ #1,1:N
710 DIM A(N,N),A2(N,N),C(N,N),D1(N,1),D2(N,1),U1(N,1),U2(N,1)
720 DIM F(N,1),B(N,1)
730 FOR I=1 TO N
740 FOR J=1 TO N
750 READ #1:A(I,J)
760 NEXT J
770 NEXT I
780 FOR I=1 TO N
790 READ #1:U1(I,1)
800 NEXT I
810 FOR I=1 TO N
820 READ #1:U2(I,1)
830 NEXT I
840 READ #1:W1
850 READ #1:W2
860 GO TO 910
870 REM  FUNCTIONS FOR INPUTS
880 REM.
890 PRINT "J_ CLOSED FORM FUNCTIONS  G_"
900 REM: GET YOUR DATA FROM FUNCTIONS
910 PRINT "J_ THE MAIN PROGRAM  J_G_"
920 M=W2/W1
930 DIM E(M,N),H(M,N),W(M)
940 FOR L=1 TO M
950 W(L)=W1*L
960 A2=A MPY A
970 FOR I=1 TO N
980 A2(I,I)=A2(I,I)+W(L)*W(L)
990 NEXT I
```

PROGRAM 2 (Continued)

```
1000 C=INV(A2)
1010 D1=0
1020 D2=0
1030 D1=A MPY U1
1040 D2=A MPY U2
1050 FOR I=1 TO N
1060 D1(I,1)=-D1(I,1)+W(L)*U2(I,1)
1070 D2(I,1)=-D2(I,1)-W(L)*U1(I,1)
1080 NEXT I
1090 F=0
1100 B=0
1110 F=C MPY D1
1120 B=C MPY D2
1130 FOR I=1 TO N
1140 E(L,I)=SQR(F(I,1)*F(I,1)+B(I,1)*B(I,1))
1150 SET DEGREES
1152 IF F(I,1)=0 THEN 1162
1155 IF B(I,1)=0 THEN 1166
1160 H(L,I)=ATN(B(I,1)/F(I,1))
1161 GO TO 1170
1162 H(L,I)=90
1163 IF B(I,1)>0 THEN 1200
1164 H(L,I)=-90
1165 GO TO 1200
1166 H(L,I)=0
1167 IF F(I,1)>0 THEN 1200
1168 H(L,I)=-180
1169 GO TO 1200
1170 IF B(I,1)<0 AND F(I,1)<0 THEN 1190
1171 IF B(I,1)>0 AND F(I,1)<0 THEN 1190
1180 GO TO 1200
1190 H(L,I)=H(L,I)-180
1200 NEXT I
1210 NEXT L
1220 PAGE
1230 PRINT "OMEGA   ";
1240 FOR I=1 TO N
1250 PRINT "    ";"A";I;"     ";
1260 NEXT I
1270 PRINT "J_J_"
```

PROGRAM 2 (Continued)

```
1280 FOR I=1 TO M
1290 PRINT USING 1295:W(I)
1295 IMAGE 5D,X,S
1300 FOR J=1 TO N
1310 PRINT USING 1315:E(I,J)
1315 IMAGE 2E, X,S
1320 NEXT J
1330 PRINT
1340 NEXT I
1350 PRINT "J_J_"
1360 PRINT "PRESS[RETURN] TO CONTINUE";
1370 INPUT A$
1380 PAGE
1390 PRINT "OMEGA  ";
1400 FOR I=1 TO N
1410 PRINT " ";"FAY";I;"    ";
1420 NEXT I
1430 PRINT "J_J_"
1440 FOR I=1 TO M
1450 PRINT USING 1295:W(I)
1460 FOR J=1 TO N
1470 PRINT USING 1475:H(I,J)
1475 IMAGE 4D.2D,X,S
1480 NEXT J
1490 PRINT
1500 NEXT I
1510 FOR J=1 TO N
1520 PRINT "J_J_"
1530 PRINT " PRESS [RETURN] TO CONTINUE";
1540 INPUT A$
1550 PAGE
1560 VIEWPORT 10,120,20,90
1570 WINDOW 0,1000,0,100
1580 AXIS 50,5
1590 MOVE 0,0
```

PROGRAM 2 (Continued)

```
1600 FOR I=1 TO M
1610 DRAW W(I),E(I,J)
1620 NEXT I
1630 HOME
1640 PRINT "I_      AMPLITUTE      A";J;"      VERSUS OMEGA ";
1650 PRINT @32,21:0,4
1660 PRINT "                    PRESS [RETURN] TO CONTINUE ";
1670 INPUT A$
1680 NEXT J
1690 FOR J=1 TO N
1700 PRINT "J_J_"
1710 INPUT A$
1720 PAGE
1730 VIEWPORT 10,120,20,90
1740 WINDOW 50,1000,-270,90
1750 AXIS 50,10
1760 MOVE 0,0
1770 FOR I=1 TO M
1780 DRAW W(I),H(I,J)
1790 NEXT I
1800 HOME
1810 PRINT "I_            PHASE ANGLE  FAY(";J;")VERSUS OMEGA";
1820 PRINT @32,21:0,4
1830 PRINT "               PRESS [RETURN] TO CONTINUE";
1840 INPUT A$
1850 NEXT J
1860 IF K2<3 THEN 1960
1870 REM:FILE SUBROUTINE FOR OUTPUT
1880 CALL "OPEN","FEIJ015 LFN:2 TACCESS:FULL"
1890 WRITE #2,1:N,M
1900 FOR I=1 TO M
1910 WRITE #2:W(I)
1920 FOR J=1 TO N
1930 WRITE #2:E(I,J),H(I,J)
1940 NEXT J
END
```

PROGRAM 3

COMPUTER PROGRAM FOR SYSTEM MATRIX IDENTIFICATION

```
100 INIT
110 PAGE
120 PRINT "THE TYPE OF DATA"
130 PRINT "K1=1 FOR DIRECT INPUT"
140 PRINT "K1=2 FOR FUNCTIONS"
150 PRINT "K1=3 FOR DATA FILE"
160 PRINT "ENTER THE TYPE OF DATA K1";
170 INPUT K1
180 PRINT "TYPE OF OUTPUT REQUIRED"
190 PRINT "K2=1 FOR TABLES"
200 PRINT "K2=2 FOR GRAPHS"
210 PRINT "K2=3 FOR OUTPUT FILE"
220 PRINT "ENTER THE TYPE OF OUTPUT REQUIRED K2";
230 INPUT K2
240 IF K1=1 THEN 270
250 IF K1=2 THEN 780
260 IF K1=3 THEN 590
270 PRINT "DATA FROM DIRECT INPUT"
280 PRINT "ENTER THE DIMENSION SIZE N";
290 INPUT N
300 DIM A(N,N),A2(N,N),C(N,N),D1(N,1),D2(N,1),U1(N,1),U2(N,1)
310 DIM F(N,1),B(N,1)
320 PRINT "ENTER THE MATRIX A"
330 FOR I=1 TO N
340 FOR J=1 TO N
350 PRINT USING 370:"ENTER A(",I,",",J,")";
360 INPUT A(I,J)
370 IMAGE 8A,FD,1A,FD,6A,S
380 NEXT J
390 NEXT I
400 PRINT "ENTER THE MATRIX U1"
410 FOR I=1 TO N
420 PRINT USING 440:"U1(",I,")";
430 INPUT U1(I,1)
440 IMAGE 3A,FD,6A,S
450 NEXT I
460 PRINT "ENTER THE MATRIX U2"
470 FOR I=1 TO N
480 PRINT USING 500:"U2(",I,")";
490 INPUT U2(I,1)
```

PROGRAM 3 (Continued)

```
500 IMAGE 3A,FD,5A,S
510 NEXT I
520 PRINT "ENTER THE OMEGA INCREMENT"
530 PRINT "OMEGA";
540 INPUT W1
550 PRINT "ENTER THE FINAL OMEGA, OMEGA MAX"
560 PRINT "OMEGAMAX=";
570 INPUT W2
580 GO TO 820
590 PRINT "DATA FROM DATA FILE"
600 CALL "OPEN","FEIJ014 LFN:1 TACCESS:FULL"
610 READ #1,1:N
620 DIM A(N,N),A2(N,N),C(N,N),D1(N,1),D2(N,1),U1(N,1),U2(N,1)
630 DIM F(N,1),B(N,1)
640 FOR I=1 TO N
650 FOR J=1 TO N
660 READ #1:A(I,J)
670 NEXT J
680 NEXT I
690 FOR I=1 TO N
700 READ #1:U1(I,1)
710 NEXT I
720 FOR I=1 TO N
730 READ #1:U2(I,1)
740 NEXT I
750 READ #1:W1
760 READ #1:W2
770 GO TO 820
780 REM: FUNCTIONS FOR INPUTS
790 REM:
800 PRINT "CLOSED FORM FUNCTIONS"
810 REM: GET YOUR DATA FROM FUNCTIONS
820 PRINT "THE PROGRAM"
830 M=W2/W1
840 DIM E(N,M),H(N,M),W(M),A3(N,M),A4(N,M),T1(N,M)
844 T1=0
848 FOR T=1 TO 100
850 FOR L=1 TO M
860 W(L)=W1*L
870 A2=A MPY A
```

PROGRAM 3 (Continued)

```
880 FOR I=1 TO N
890 A2(I,I)=A2(I,I)+W(L)*W(L)
900 NEXT I
910 C=INV(A2)
920 D1=0
930 D2=0
940 D1=A MPY U1
950 D2=A MPY U2
960 FOR I=1 TO N
970 D1(I,1)=-D1(I,1)+W(L)*U2(I,1)
980 D2(I,1)=-D2(I,1)-W(L)*U1(I,1)
990 NEXT I

1000 F=0
1010 B=0
1020 F=C MPY D1
1030 B=C MPY D2
1040 Z5=RND(-1)
1050 FOR I=1 TO N
1060 E(I,L)=(F(I,1)*W(L)-U2(I,1))*(1+(RND(1)-0.5)/100)
1070 H(I,L)=B(I,1)
1090 NEXT I
1100 NEXT L
1110 A3=INV(H)
1120 A4=E MPY A3
1130 T1=T1+A4
1135 PRINT T;" ... ";
1140 NEXT T
1150 T1=T1/100
1160 PRINT "L_THE AVERAGED MATRIX=J_";T1
1170 STOP
1180 END
```

APPENDIX B

TABLES

TABLE 1

DAMPED NON-GYROSCOPIC SYSTEM EIGENVALUES AND EIGENVECTORS

EIGENVALUES

W(1)=	-0.44094D 01	0.0	I
W(2)=	-0.42305D 00	0.11699D 01	I
W(3)=	-0.42305D 00	-0.11699D 01	I
W(4)=	-0.23445D 01	0.0	I

EIGENVECTORS

-0.77938D 00	0.0	I	0.12238D 00	0.49501D 00	I	0.12238D 00	-0.49501D 00	I	0.39496D 01	0.0	I
0.58622D 00	0.0	I	0.25752D 00	0.74178D 00	I	0.25752D 00	-0.74178D 00	I	-0.34546D 01	0.0	I
0.17675D 00	0.0	I	0.34073D 00	-0.22781D 00	I	0.34073D 00	0.22781D 00	I	-0.16846D 01	0.0	I
-0.13295D 00	0.0	I	0.49033D 00	-0.39742D 00	I	0.49033D 00	0.39742D 00	I	0.14735D 01	0.0	I

NORMALIZED EIGENVECTORS

-0.77938D 00	0.0	I	-0.64992D 00	0.18627D 00	I	-0.64992D 00	-0.18627D 00	I	0.69235D 00	0.0	I
0.58622D 00	0.0	I	-0.97031D 00	0.37735D 00	I	-0.97031D 00	-0.37735D 00	I	-0.60557D 00	0.0	I
0.17675D 00	0.0	I	0.31845D 00	0.44037D 00	I	0.31845D 00	-0.44037D 00	I	-0.29531D 00	0.0	I
-0.13295D 00	0.0	I	0.55046D 00	0.63032D 00	I	0.55046D 00	-0.63032D 00	I	0.25830D 00	0.0	I

TABLE 2

DAMPED NON-GYROSCOPIC SYSTEM - ERROR MATRIX EIGENVALUES AND EIGENVECTORS -
0.2% RANDOM ERROR

EIGENVALUES

W(1)=	0.42146D-01	0.47386D-01 I
W(2)=	0.42146D-01	-0.47386D-01 I
W(3)=	0.18084D-02	0.50168D-02 I
W(4)=	0.18084D-02	-0.50168D-02 I

EIGENVECTORS

0.50295D 00	0.68732D 0 I	0.50295D 00	-0.68732D 00 I	-0.34833D 00	0.15454D 01 I	-0.34833D 00	-0.15454D 01 I
0.16082D 00	-0.58037D 0 I	0.16082D 00	0.58037D 00 I	-0.14124D 00	0.17172D 01 I	-0.14124D 00	-0.17172D 01 I
0.72722D-01	0.61495D 0 I	0.72722D-01	-0.61495D 00 I	0.11687D 01	0.87467D 00 I	0.11687D 01	-0.87467D 00 I
0.20937D 00	0.59606D 0 I	0.20937D 00	-0.59606D 00 I	0.16235D 01	0.41860D 00 I	0.16235D 01	-0.41860D 00 I

NORMALIZED EIGENVECTORS

0.66293D 00	-0.22572D 0 I	0.66293D 00	0.22572D 00 I	0.79036D 00	0.51526D 00 I	0.79036D 00	-0.51526D 00 I
-0.41587D 00	-0.26883D 0 I	-0.41587D 00	0.26883D 00 I	0.92927D 00	0.43533D 00 I	0.92927D 00	-0.43533D 00 I
0.50041D 00	0.94076D- 1 I	0.50041D 00	-0.94076D-01 I	0.73089D 00	-0.47084D 00 I	0.73089D 00	0.47084D 00 I
0.51918D 00	-0.17792D- 1 I	0.51918D 00	0.17792D-01 I	0.57071D 00	-0.81940D 00 I	0.57071D 00	0.81940D 00 I

TABLE 3

DAMPED NON-GYROSCOPIC SYSTEM - ERROR MATRIX EIGENVALUES AND EIGENVECTORS - 0.4% RANDOM ERROR

EIGENVALUES

W(1)=	0.26127D-01	0.0	I
W(2)=	0.419060D-02	0.781250D-02	I
W(3)=	0.419060D-02	-0.781250D-02	I
W(4)=	-0.25587D-02	0.0	I

EIGENVECTORS

0.57952D 00	0.0	I	0.59642D 00	0.80440D 00	I	0.59642D 00	-0.80440D 00	I	-0.72136D 01	0.0	I
-0.59949D 00	0.0	I	-0.25139D 01	-0.86796D 00	I	-0.25139D 01	0.86796D 00	I	-0.47229D 01	0.0	I
-0.42314D 00	0.0	I	-0.24675D 01	0.57451D 00	I	-0.24675D 01	-0.57451D 00	I	-0.20611D 01	0.0	I
-0.35457D 00	0.0	I	-0.25421D 01	0.10272D 01	I	-0.25421D 01	-0.10272D 01	I	-0.46873D 00	0.0	I

NORMALIZED EIGENVECTORS

0.57952D 00	0.0	I	0.12938D 00	0.20839D 00	I	0.12938D 00	-0.20839D 00	I	-0.81257D 00	0.0	I
-0.59949D 00	0.0	I	-0.59619D 00	-0.26257D 00	I	-0.59619D 00	0.26257D 00	I	-0.53201D 00	0.0	I
-0.42314D 00	0.0	I	-0.61395D 00	0.90503D-01	I	-0.61395D 00	-0.90503D-01	I	-0.23217D 00	0.0	I
-0.35457D 00	0.0	I	-0.64129D 00	0.19950D 00	I	-0.64129D 00	-0.19950D 00	I	-0.52800D-01	0.0	I

TABLE 4

DAMPED NON-GYROSCOPIC SYSTEM - ERROR MATRIX EIGENVALUES AND EIGENVECTORS - 0.6% RANDOM ERROR

EIGENVALUES

W(1)=	-0.12408D 00	0.0	I
W(2)=	0.36492D-01	0.0	I
W(3)=	0.25189D-02	0.0	I
W(4)=	-0.11136D-01	0.0	I

EIGENVECTORS

-0.16667D 01	0.0	I	-0.22818D 01	0.0	I	0.19512D 01	0.0	I	0.52673D 00	0.0	I
-0.29007D 00	0.0	I	0.11594D 01	0.0	I	0.35126D 01	0.0	I	-0.42632D 00	0.0	I
-0.29360D 00	0.0	I	0.89983D 00	0.0	I	0.17975D 01	0.0	I	0.19384D 01	0.0	I
-0.90488D 00	0.0	I	0.93307D 00	0.0	I	0.10631D 01	0.0	I	0.28246D 01	0.0	I

NORMALIZED EIGENVECTORS

-0.85873D 00	0.0	I	-0.79534D 00	0.0	I	0.43088D 00	0.0	I	0.15083D 00	0.0	I
-0.14945D 00	0.0	I	0.40410D 00	0.0	I	0.77567D 00	0.0	I	-0.12208D 00	0.0	I
-0.15127D 00	0.0	I	0.31364D 00	0.0	I	0.39694D 00	0.0	I	0.55507D 00	0.0	I
-0.46622D 00	0.0	I	0.32523D 00	0.0	I	0.23475D 00	0.0	I	0.80885D 00	0.0	I

TABLE 5

DAMPED NON-GYROSCOPIC SYSTEM - ERROR MATRIX EIGENVALUES AND EIGENVECTORS -
0.8% RANDOM ERROR

EIGENVALUES

W(1)=	-0.22089D-01	0.46262D-01	I
W(2)=	-0.22089D-01	-0.46262D-01	I
W(3)=	-0.10207D-01	0.0	I
W(4)=	0.92538D-03	0.0	I

EIGENVECTORS

0.14276D 02	0.38062D 1	I	0.14276D 02	-0.38062D 01	I	0.25222D 01	0.0	I	0.13706D 01	0.0	I
0.42315D 01	0.45207D 1	I	0.42315D 01	-0.45207D 01	I	0.32246D 01	0.0	I	0.11185D 01	0.0	I
0.41935D 01	0.42970D 1	I	0.41935D 01	-0.42970D 01	I	0.38545D 01	0.0	I	-0.20663D 01	0.0	I
0.38299D 01	0.30977D 1	I	0.38299D 01	-0.30977D 01	I	0.39634D 01	0.0	I	-0.32273D 01	0.0	I

NORMALIZED EIGENVECTORS

0.87059D 00	-0.13327D 0	I	0.87059D 00	0.13327D 00	I	0.36664D 00	0.0	I	0.32474D 00	0.0	I
0.33911D 00	0.14577D 0	I	0.33911D 00	-0.14577D 00	I	0.46874D 00	0.0	I	0.26500D 00	0.0	I
0.33169D 00	0.13446D 0	I	0.33169D 00	-0.13446D 00	I	0.56030D 00	0.0	I	-0.48956D 00	0.0	I
0.28318D 00	0.77654D- 1	I	0.28318D 00	-0.77654D-01	I	0.57613D 00	0.0	I	-0.76463D 00	0.0	I

TABLE 6

DAMPED NON-GYROSCOPIC SYSTEM - ERROR MATRIX EIGENVALUES AND EIGENVECTORS - 1.0% RANDOM ERROR

EIGENVALUES

W(1)=	0.18939D 00	0.0	I
W(2)=	0.68012D-02	0.17049D-01	I
W(3)=	0.68012D-02	-0.17049D-01	I
W(4)=	0.19382D-02	0.0	I

EIGENVECTORS

0.11542D 01	0.0	I	0.21497D 01	-0.32496D 01	I	0.21497D 01	0.32496D 01	I	0.64900D 00	0.0	I
-0.79758D 00	0.0	I	0.23093D 01	0.26072D 01	I	0.23093D 01	-0.26072D 01	I	-0.28050D 01	0.0	I
0.40437D-01	0.0	I	0.27354D 00	0.10279D 01	I	0.27354D 00	-0.10279D 01	I	0.25848D 01	0.0	I
-0.59752D 00	0.0	I	-0.67621D 00	0.10151D 01	I	-0.67621D 00	-0.10151D 01	I	0.41021D 01	0.0	I

NORMALIZED EIGENVECTORS

0.75663D 00	0.0	I	0.11539D 01	0.53762D 00	I	0.11539D 01	-0.53762D 00	I	0.11509D 00	0.0	I
-0.52286D 00	0.0	I	-0.73088D 00	0.87219D 00	I	-0.73088D 00	-0.87219D 00	I	-0.49743D 00	0.0	I
0.26508D-01	0.0	I	-0.31892D 00	0.13806D 00	I	-0.31892D 00	-0.13806D 00	I	0.45838D 00	0.0	I
-0.39171D 00	0.0	I	-0.36070D 00	-0.16945D 00	I	-0.36070D 00	0.16945D 00	I	0.72746D 00	0.0	I

TABLE 7

CRITICALLY DAMPED NON-GYROSCOPIC SYSTEM EIGENVALUES AND EIGENVECTORS

EIGENVALUES

```
W(  1 )= -0.40312D 00    0.11762D 01 I
W(  2 )= -0.40312D 00   -0.11762D 01 I
W(  3 )= -0.32199D 01    0.0         I
W(  4 )= -0.32141D 01    0.0         I
```

EIGENVECTORS

0.70228D 00	-0.28605D 00 9	0.70228D 00	0.28605D 00 9	-0.75327D 00	0.0	9	-0.14901D 04	0.0	9
0.10423D 01	-0.53707D 00 9	0.10423D 01	0.53707D 00 9	0.59816D 00	0.0	9	0.11837D 04	0.0	9
-0.40074D 00	-0.45971D 00 9	-0.40074D 00	0.45971D 00 9	0.23395D 00	0.0	9	0.46360D 03	0.0	9
-0.68037D 00	-0.65294D 00 9	-0.68037D 00	0.65294D 00 9	-0.18577D 00	0.0	9	-0.36829D 03	0.0	9

NORMALIZED EIGENVECTORS

0.65910D 00	-0.18806D 00 9	0.65910D 00	0.18806D 00 9	-0.74789D 00	0.0	9	-0.74765D 00	0.0	9
0.98925D 00	-0.38021D 00 9	0.98925D 00	0.38021D 00 9	0.59388D 00	0.0	9	0.59394D 00	0.0	9
-0.31493D 00	-0.45241D 00 9	-0.31493D 00	0.45241D 00 9	0.23227D 00	0.0	9	0.23262D 00	0.0	9
-0.54720D 00	-0.65349D 00 9	-0.54720D 00	0.65349D 00 9	-0.18444D 00	0.0	9	-0.18479D 00	0.0	9

TABLE 8

UNDERDAMPED NON-GYROSCOPIC SYSTEM EIGENVALUES AND EIGENVECTORS

EIGENVALUES

```
W(  1 )= -0.33758D 00        0.32182D 01 I
W(  2 )= -0.33758D 00       -0.32182D 01 I
W(  3 )= -0.42418D-01        0.12354D 01 I
W(  4 )= -0.42418D-01       -0.12354D 01 I
```

EIGENVECTORS

-0.15369D 01	-0.14098D 01 9	-0.15369D 01	0.14098D 01 9	-0.49141D 00	-0.35279D 00 9	-0.49141D 00	0.35279D 00 9
0.92751D 00	0.90121D 00 9	0.92751D 00	-0.90121D 00 9	-0.80093D 00	-0.56171D 00 9	-0.80093D 00	0.56171D 00 9
-0.38375D 00	0.51781D 00 9	-0.38375D 00	-0.51781D 00 9	-0.27158D 00	0.40709D 00 9	-0.27158D 00	-0.40709D 00 9
0.24709D 00	-0.31413D 00 9	0.24709D 00	0.31413D 00 9	-0.43190D 00	0.66313D 00 9	-0.43190D 00	-0.66313D 00 9

NORMALIZED EIGENVECTORS

-0.89246D 00	0.16872D-01 9	-0.89246D 00	-0.16872D-01 9	-0.88759D 00	0.50036D-01 9	-0.88759D 00	-0.50036D-01 9
0.55349D 00	0.54314D-02 9	0.55349D 00	-0.54314D-02 9	-0.14344D 01	0.96751D-01 9	-0.14344D 01	-0.96751D-01 9
0.33959D-01	0.27376D 00 9	0.33959D-01	-0.27376D 00 9	0.65092D-01	0.71621D 00 9	0.65092D-01	-0.71621D 00 9
-0.16176D-01	-0.17029D 00 9	-0.16176D-01	0.17029D 00 9	0.11804D 00	0.11570D 01 9	0.11804D 00	-0.11570D 01 9

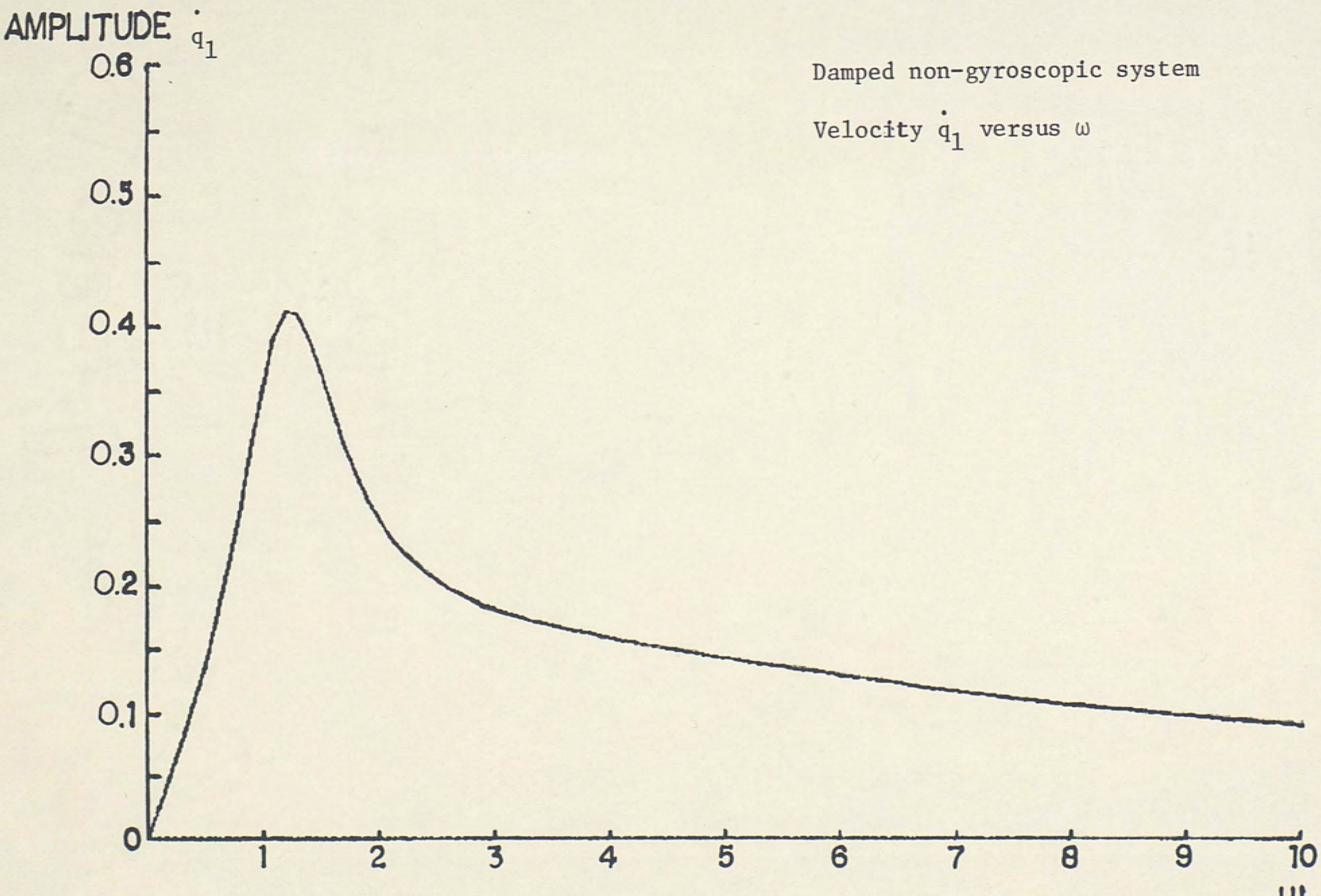

Fig. 2. Damped non-gyroscopic system amplitude I versus frequency (cycles/sec).

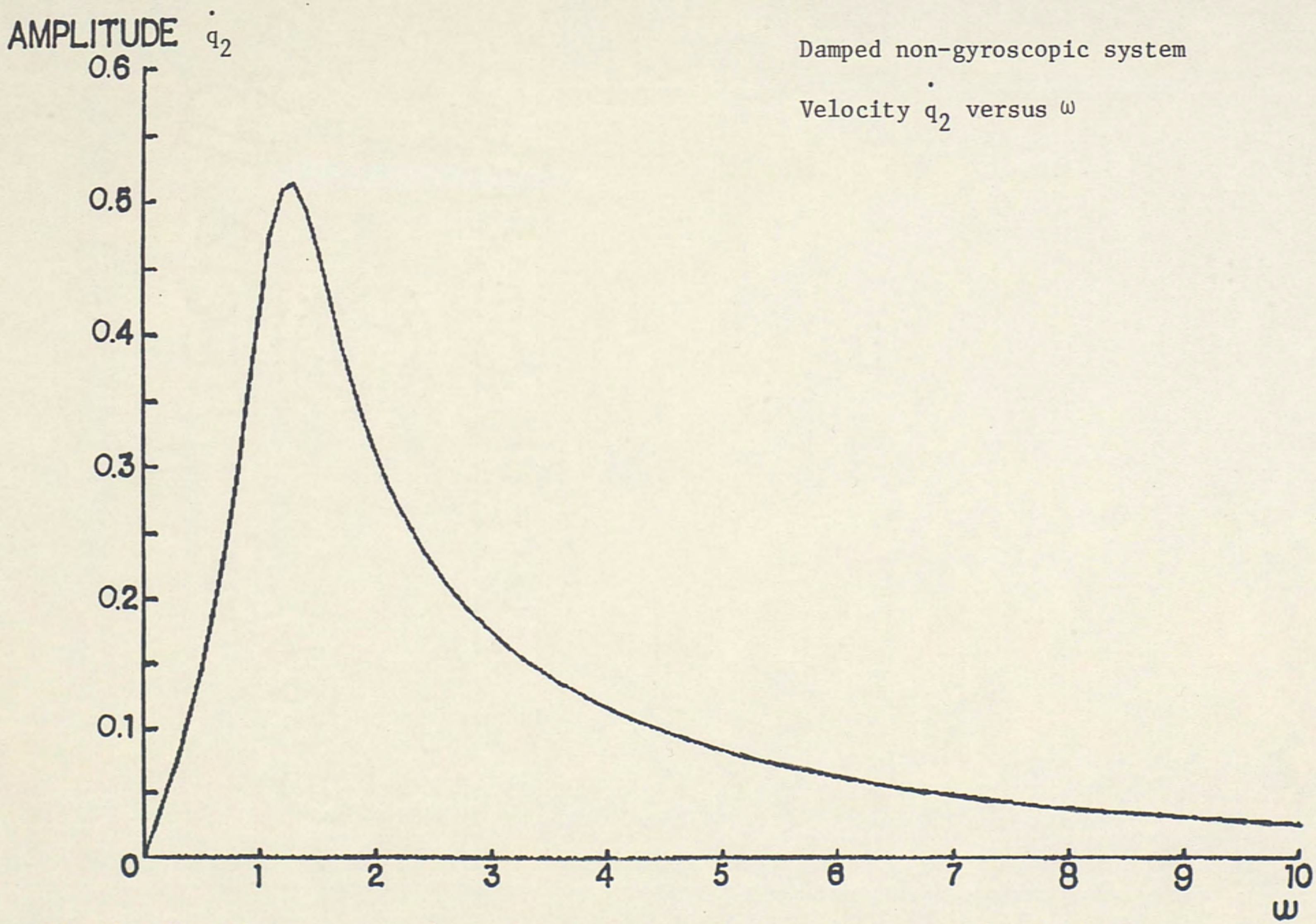

Fig. 3. Damped non-gyroscopic system amplitude II versus frequency (cycles/sec).

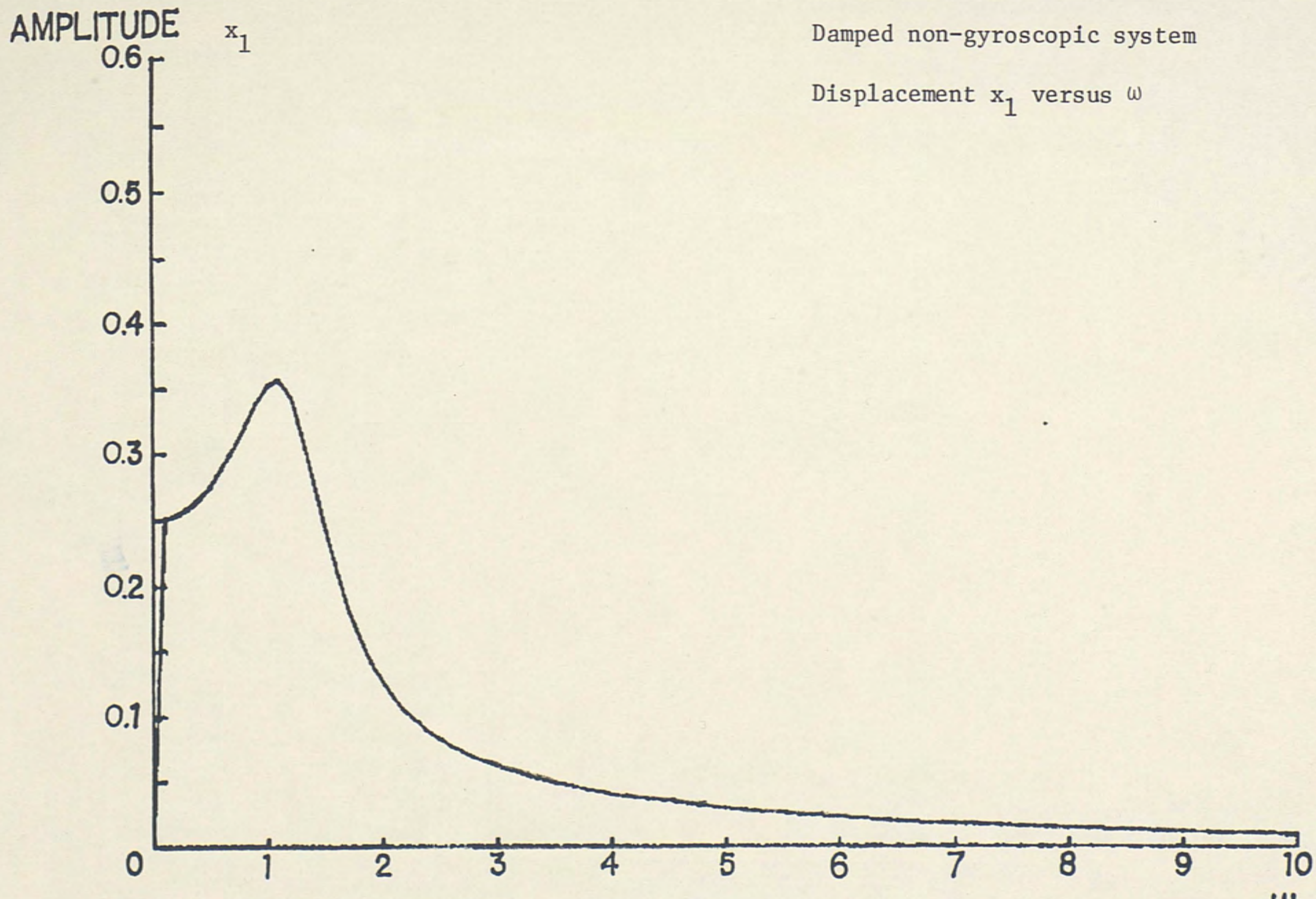

Fig. 4. Damped non-gyroscopic system amplitude III versus frequency (cycles/sec).

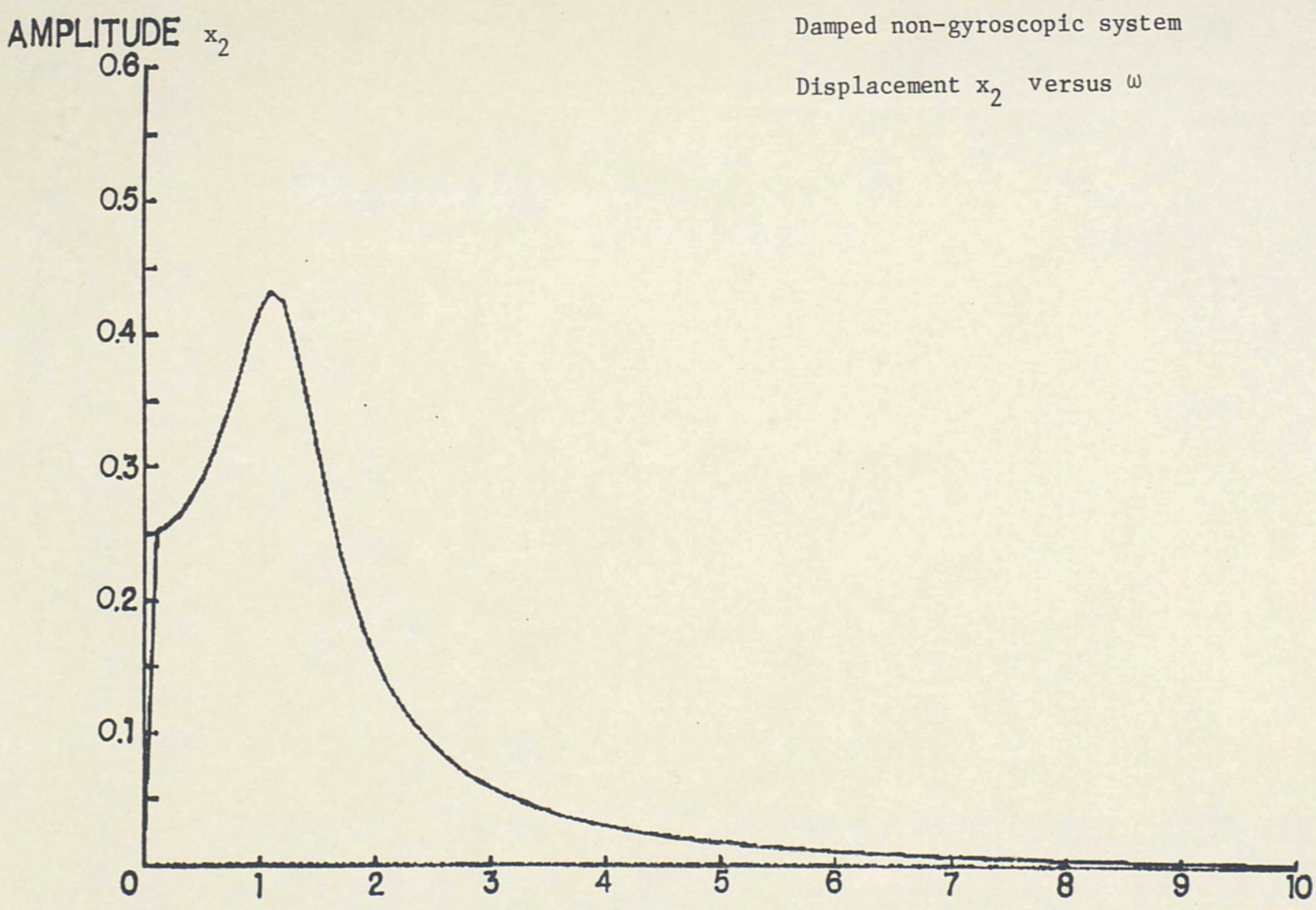

Fig. 5. Damped non-gyroscopic system amplitude IV versus frequency (cycles/sec).

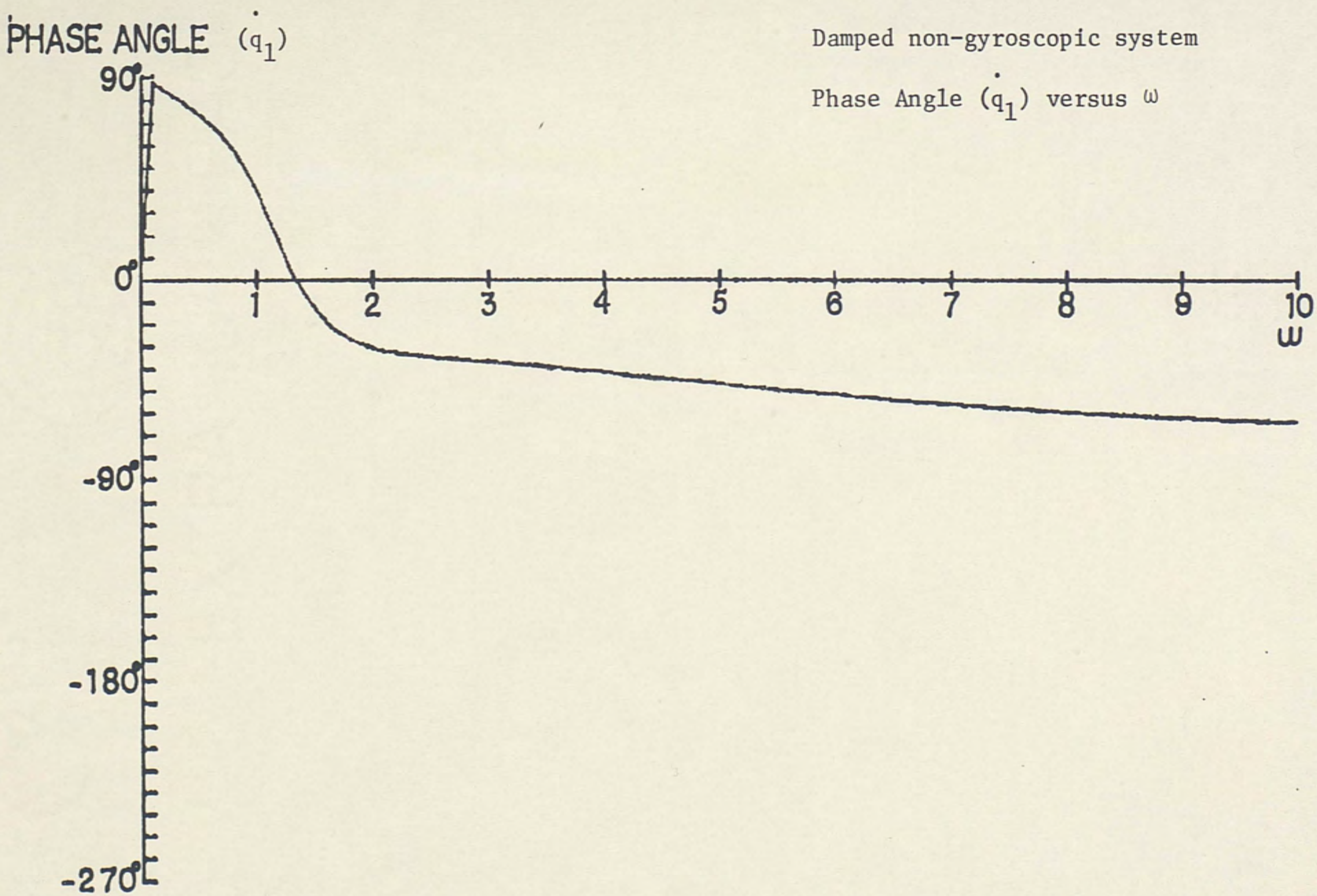

Fig. 6. Damped non-gyroscopic system phase angle I versus frequency (cycles/sec).

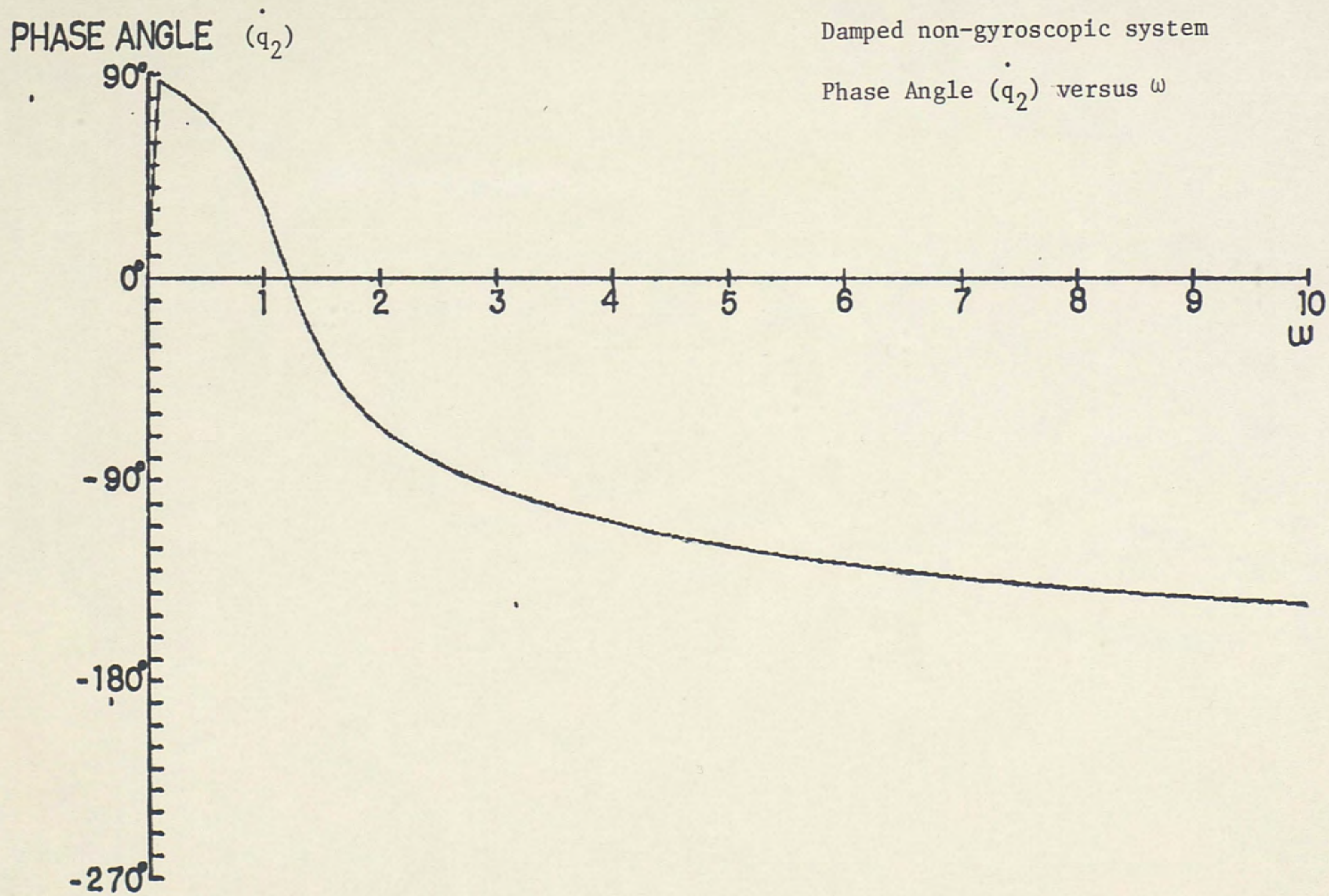

Fig. 7. Damped non-gyroscopic system phase angle II versus frequency (cycles/sec).

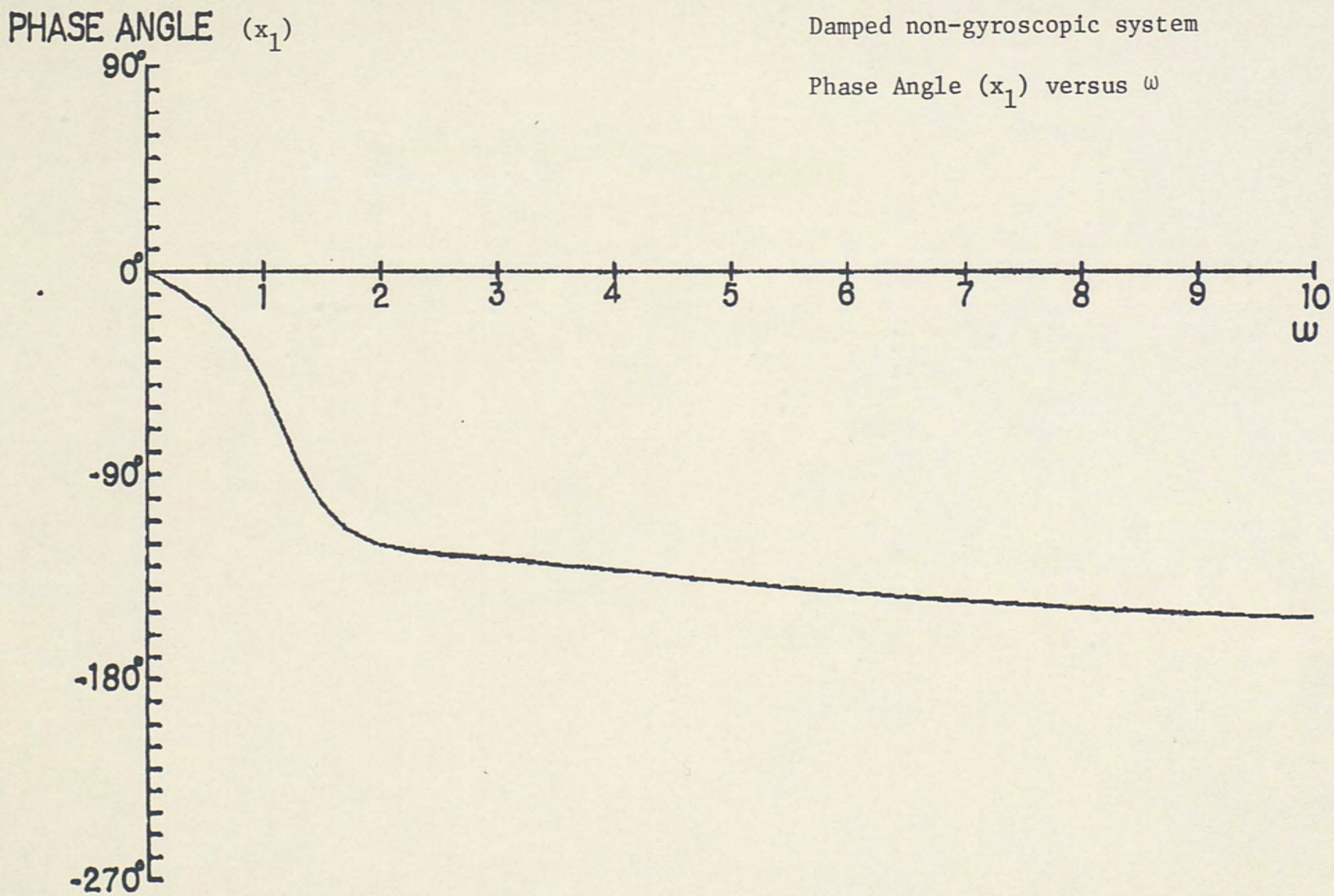

Fig. 8. Damped non-gyroscopic system phase angle III versus frequency (cycles/sec).

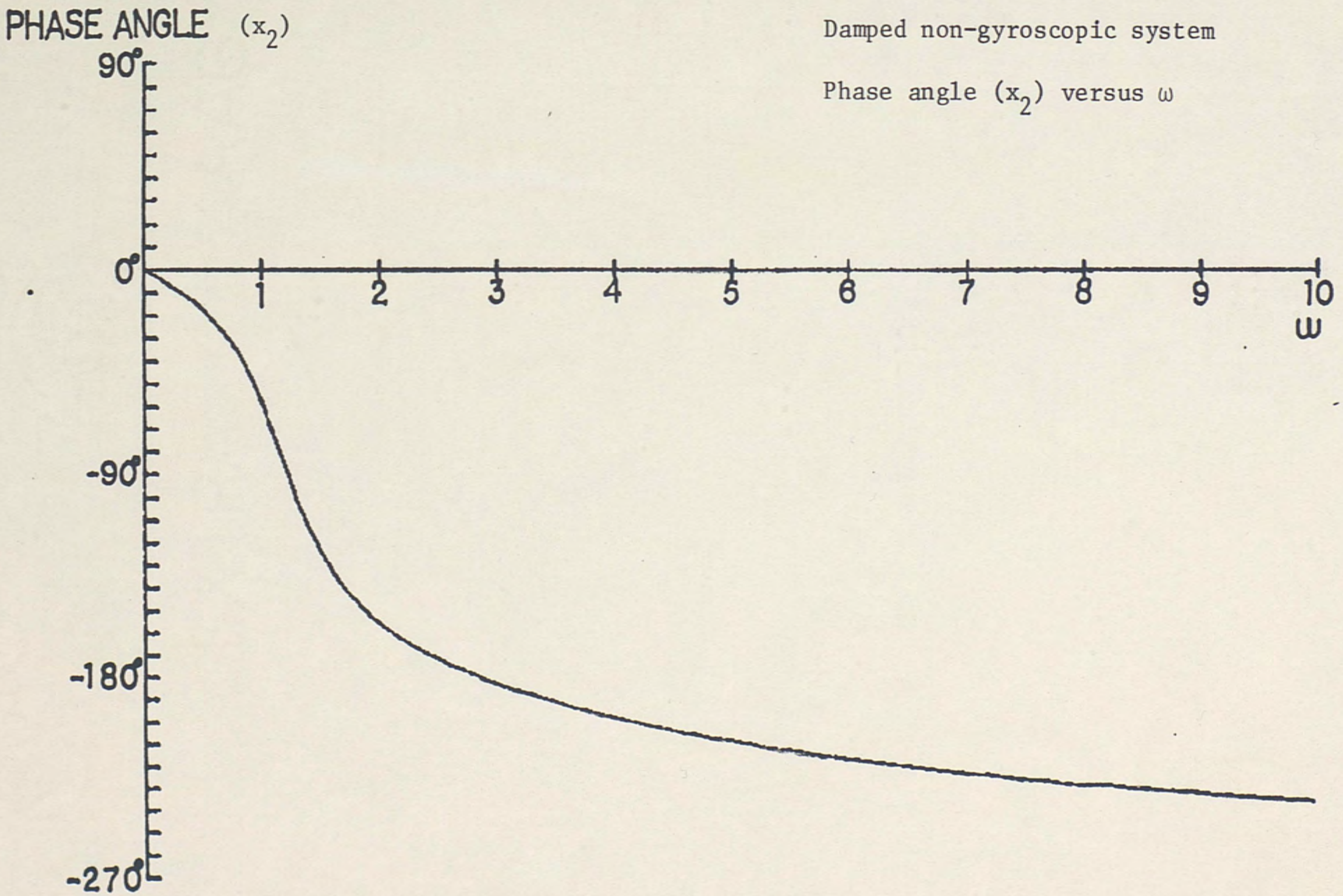

Fig. 9. Damped non-gyroscopic system phase angle IV versus frequency (cycles/sec).

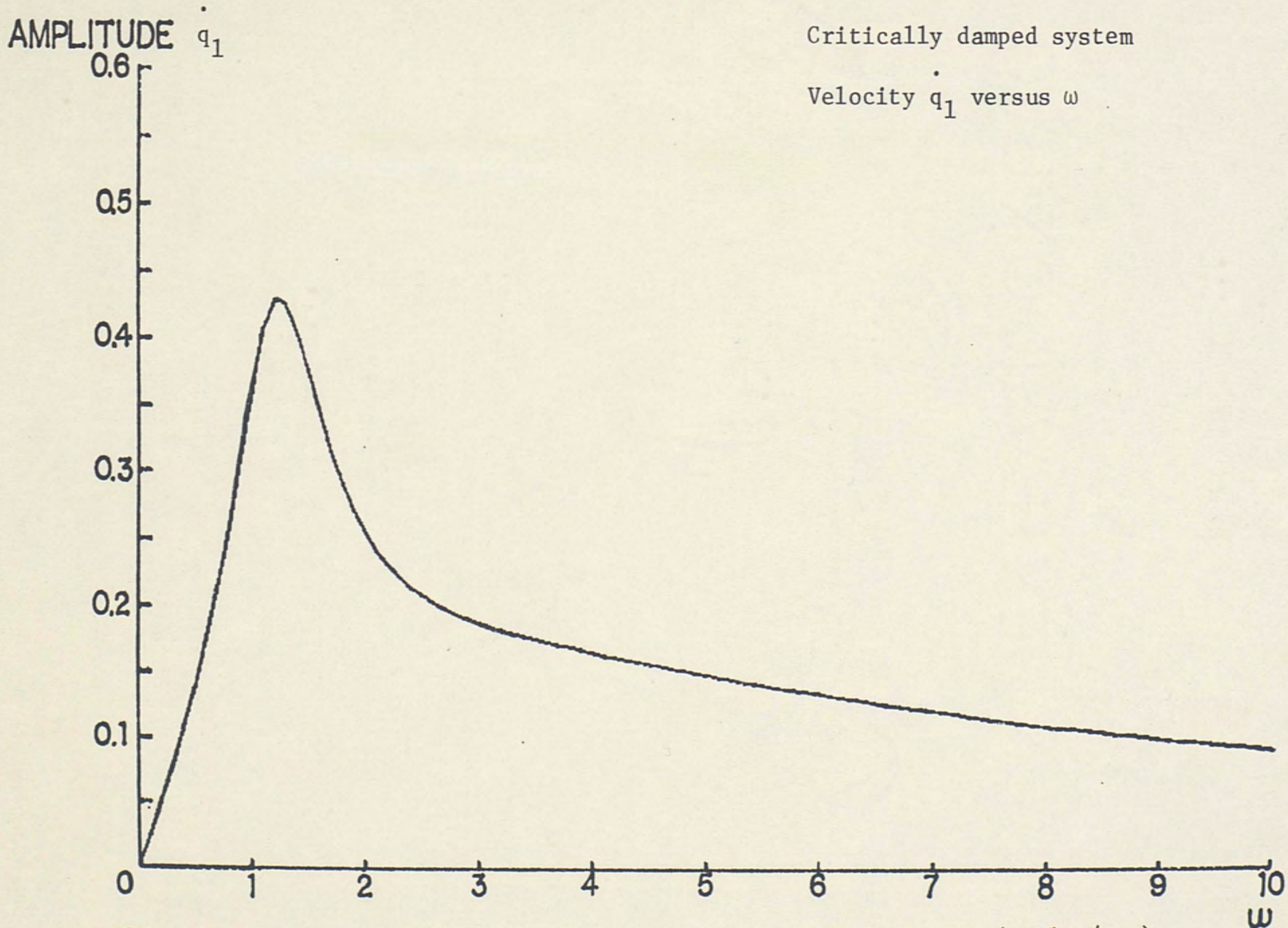

Fig. 10. Critically damped system amplitude I versus frequency (cycles/sec).

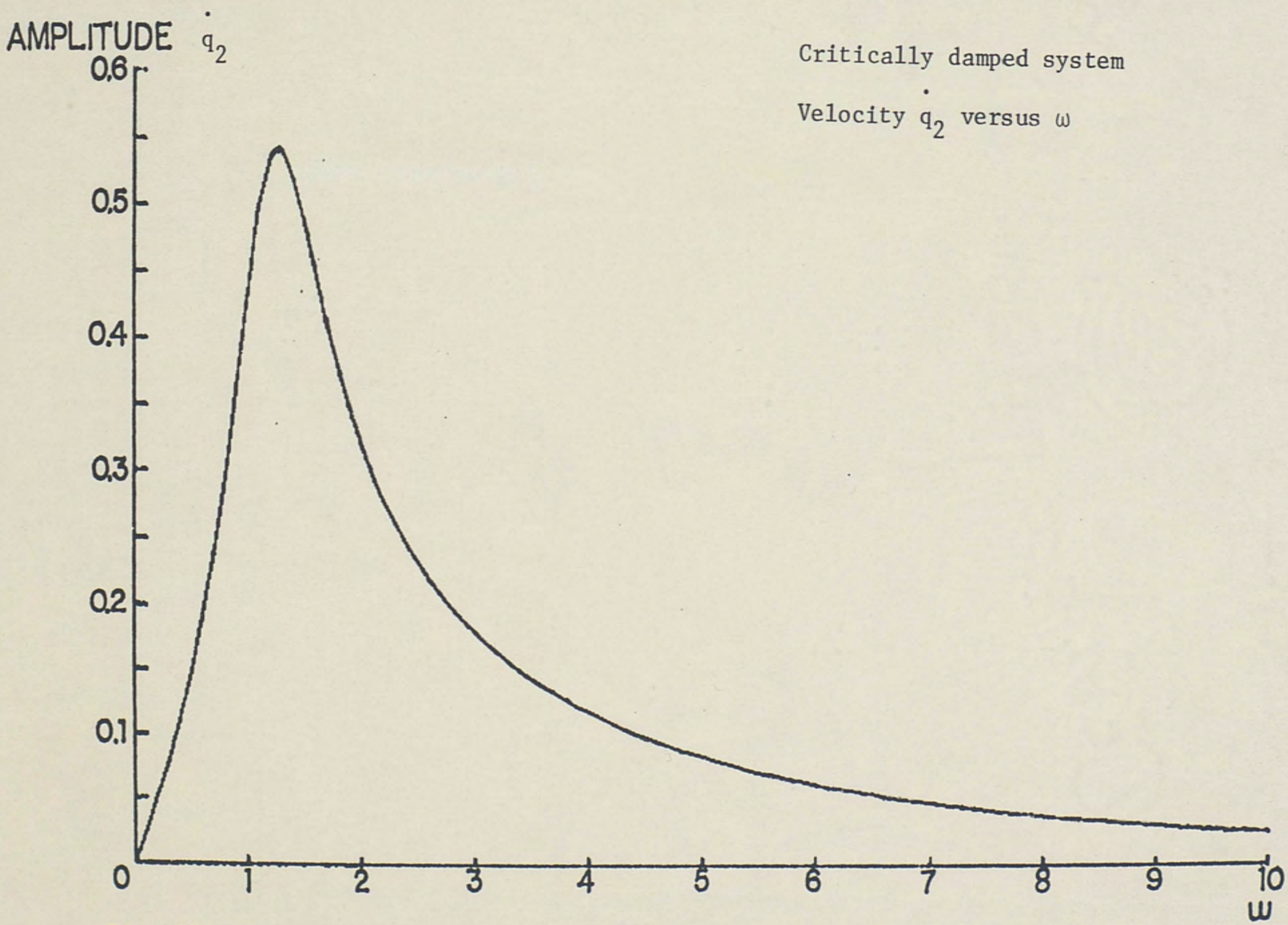

Fig. 11. Critically damped system amplitude II versus frequency (cycles/sec).

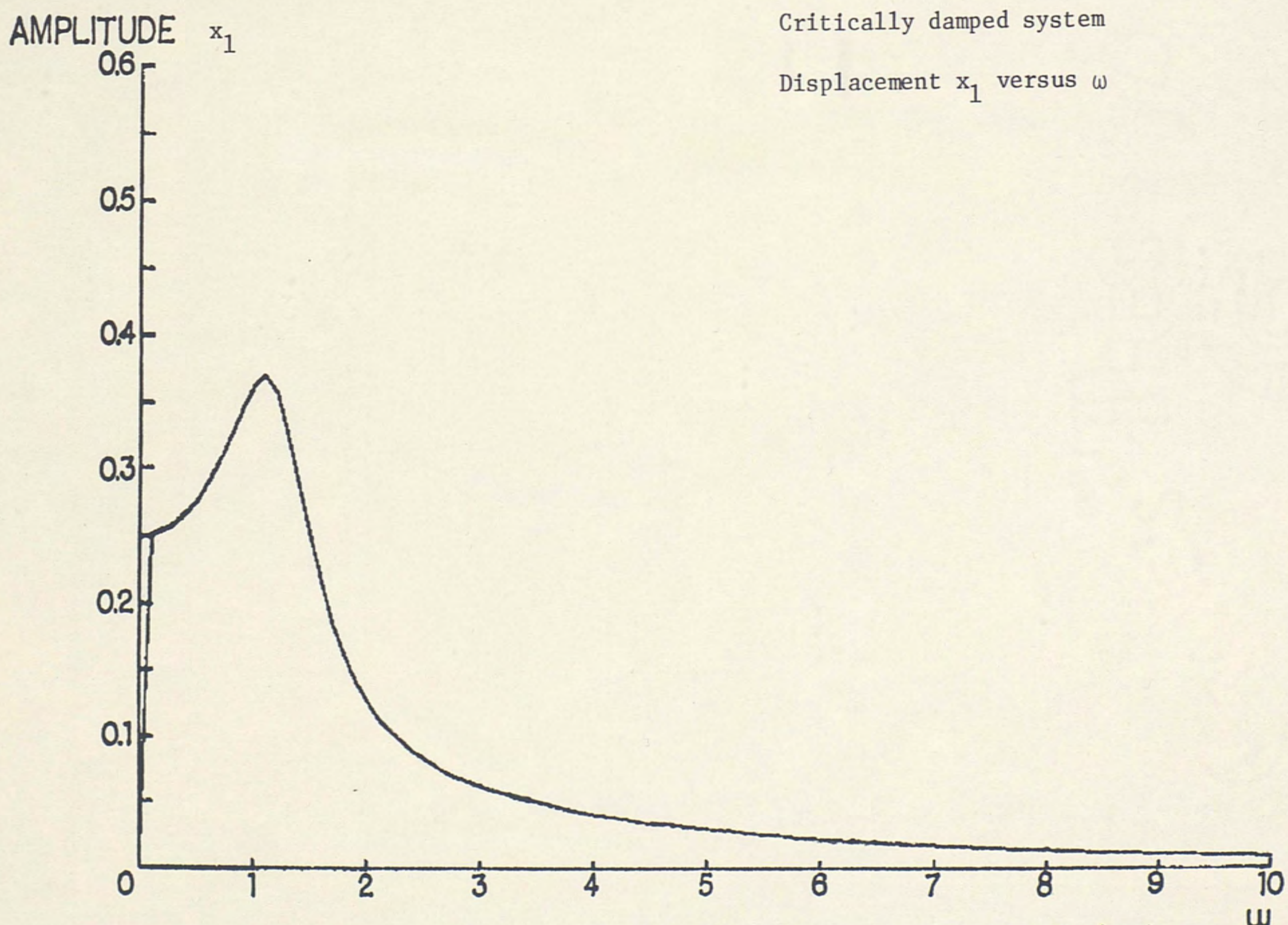

Fig. 12. Critically damped system amplitude III versus frequency (cycles/sec).

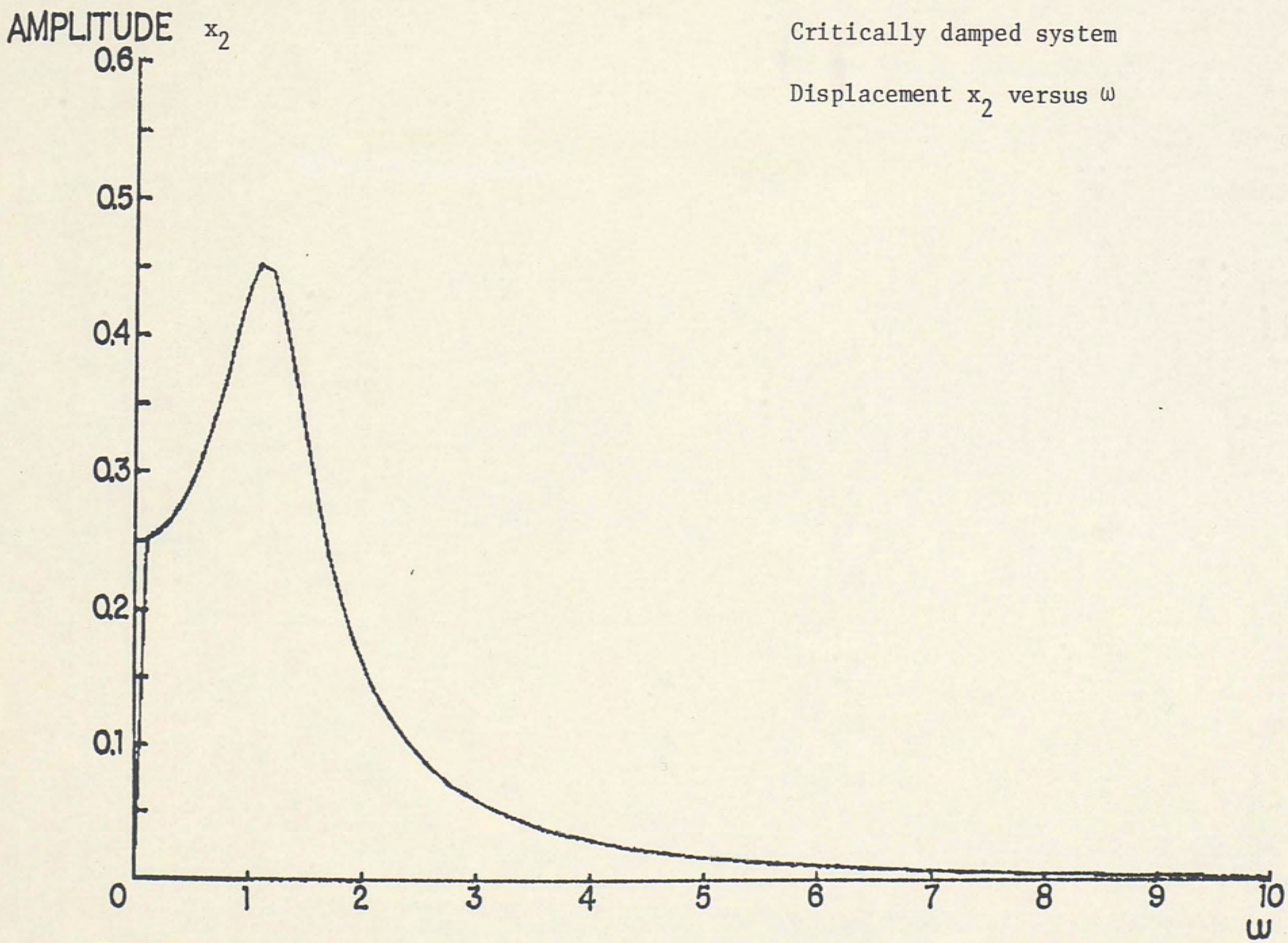

Fig. 13. Critically damped system amplitude IV versus frequency (cycles/sec).

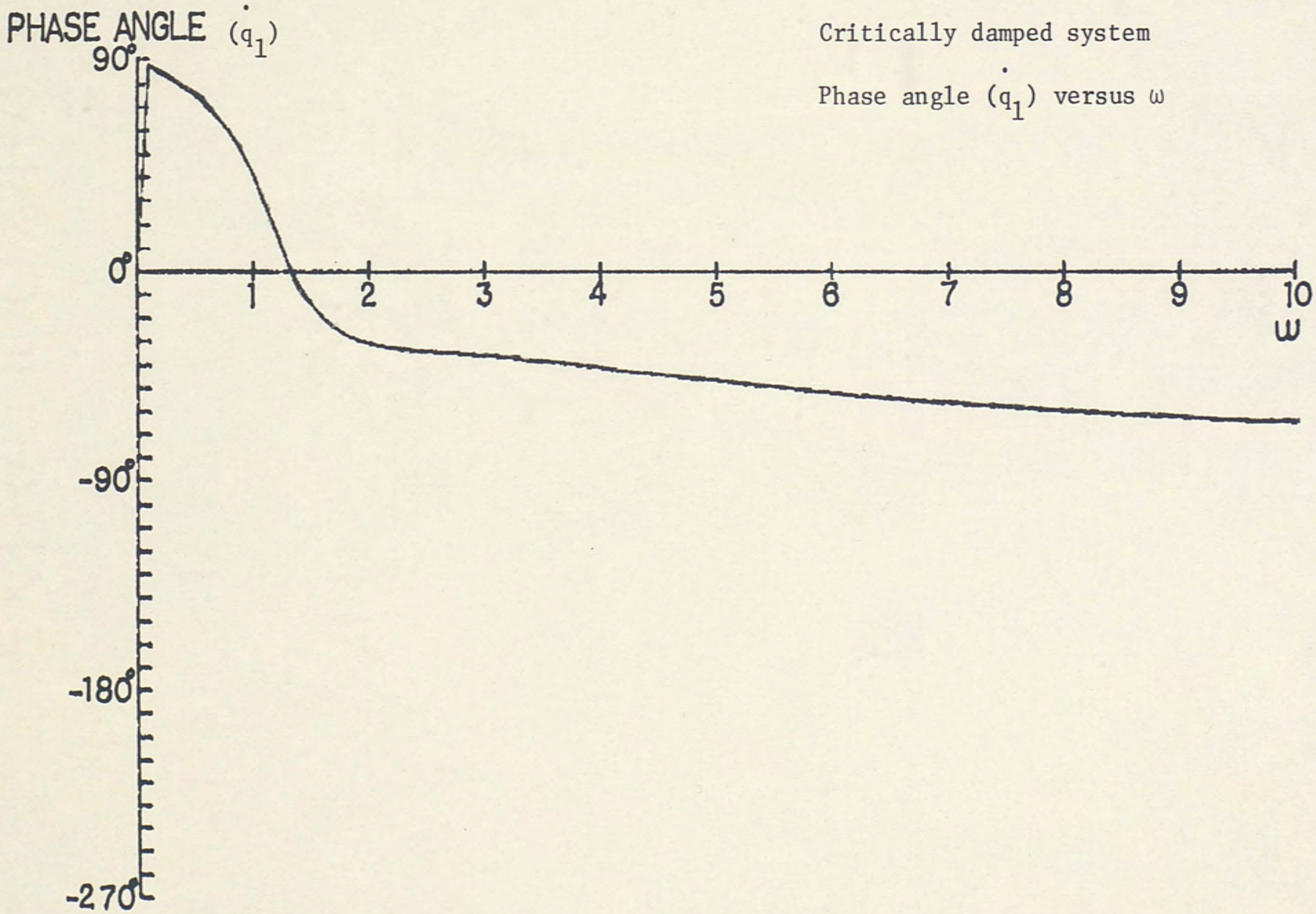

Fig. 14. Critically damped system phase angle I versus frequency (cycles/sec).

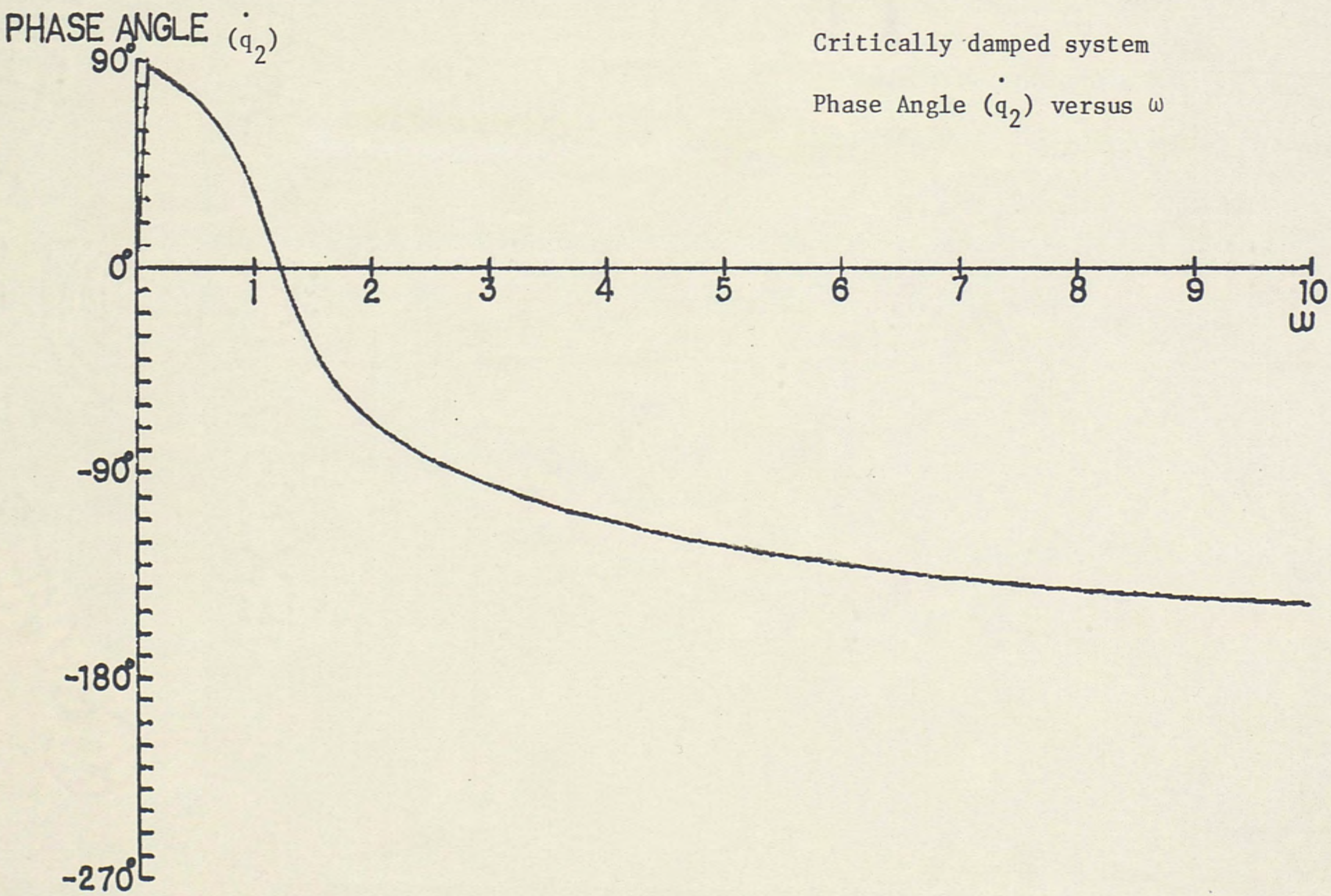

Fig. 15. Critically damped system phase angle II versus frequency (cycles/sec).

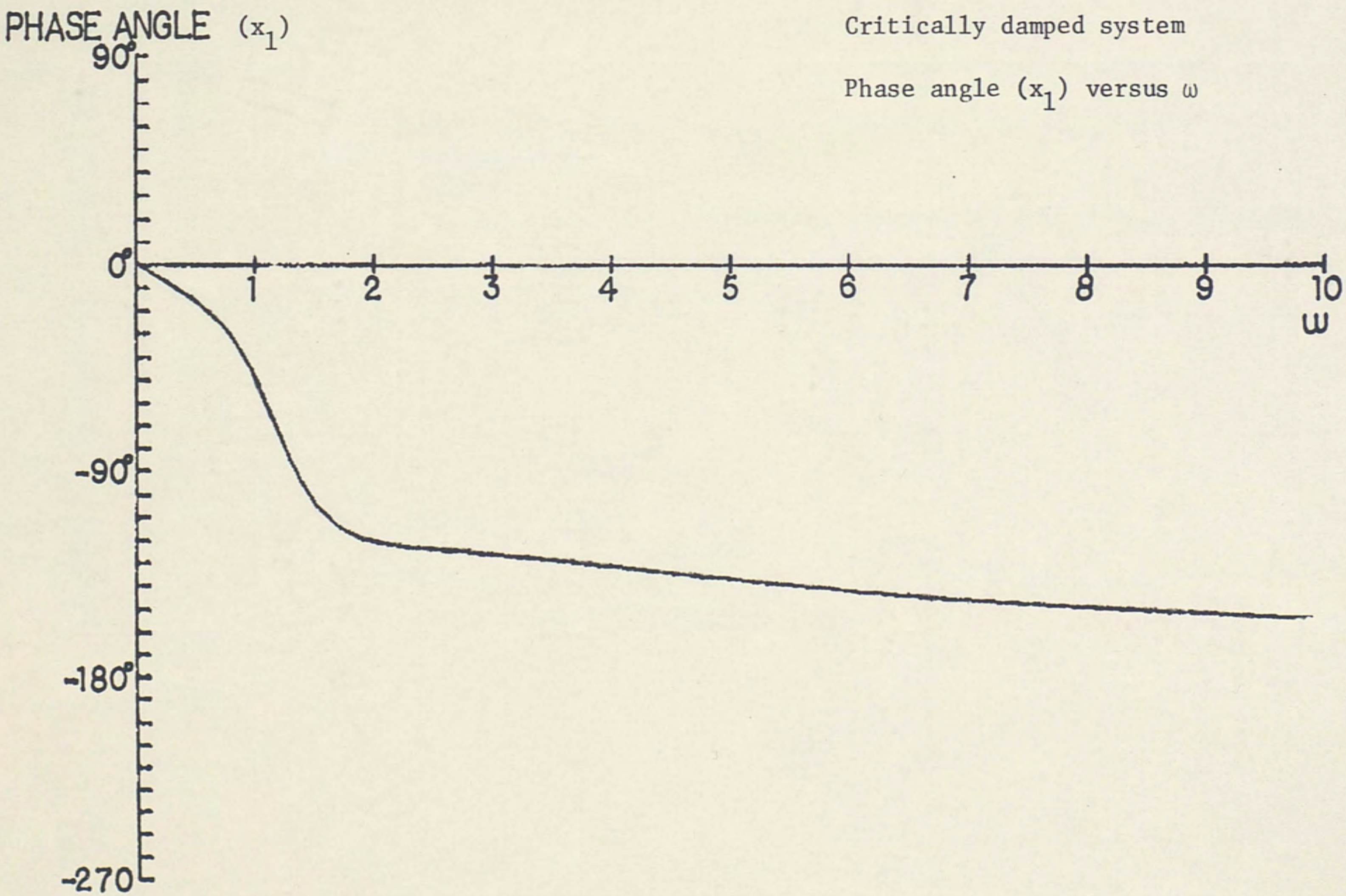

Fig. 16. Critically damped system phase angle III versus frequency (cycles/sec).

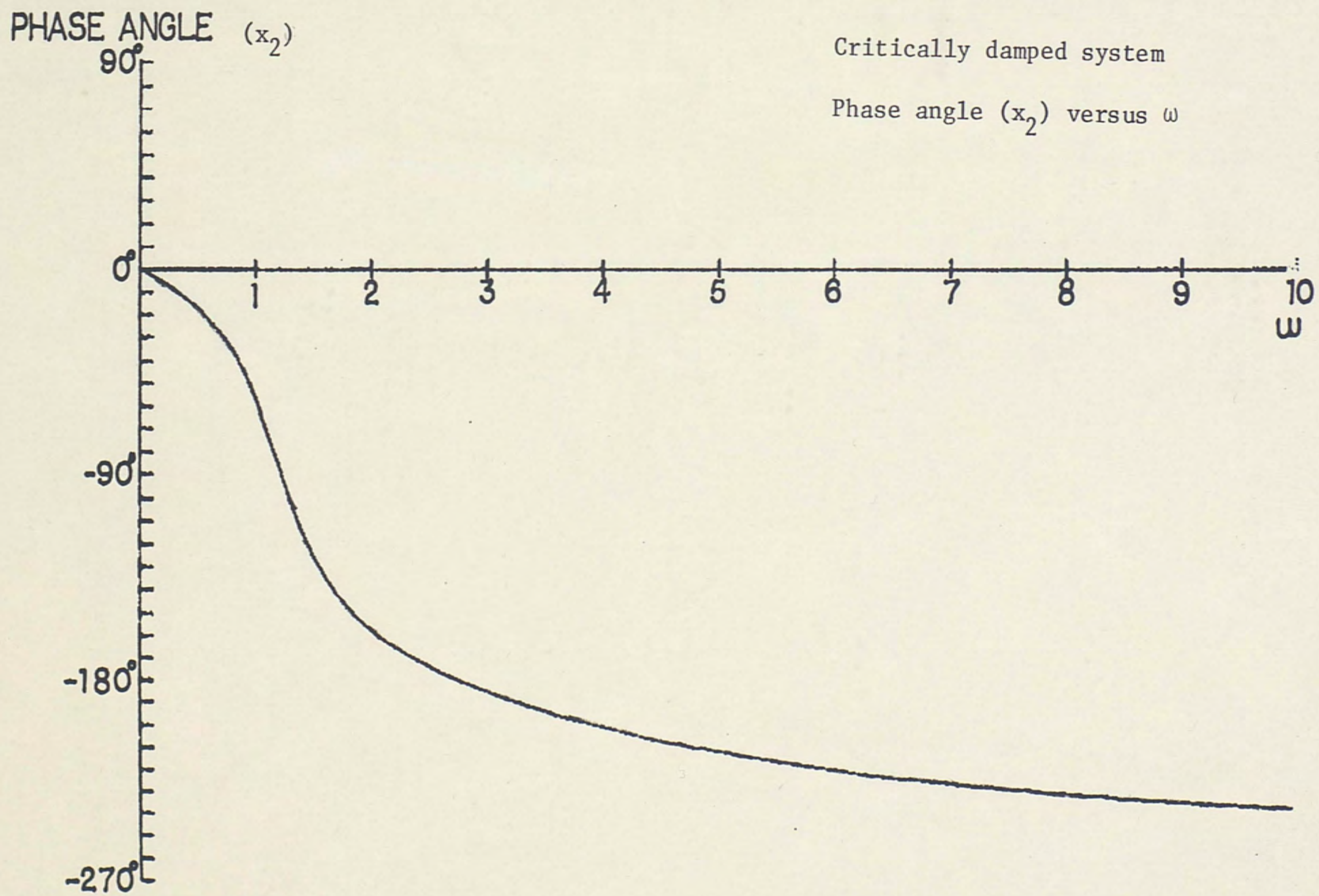

Fig. 17. Critically damped system phase angle IV versus frequency (cycles/sec).

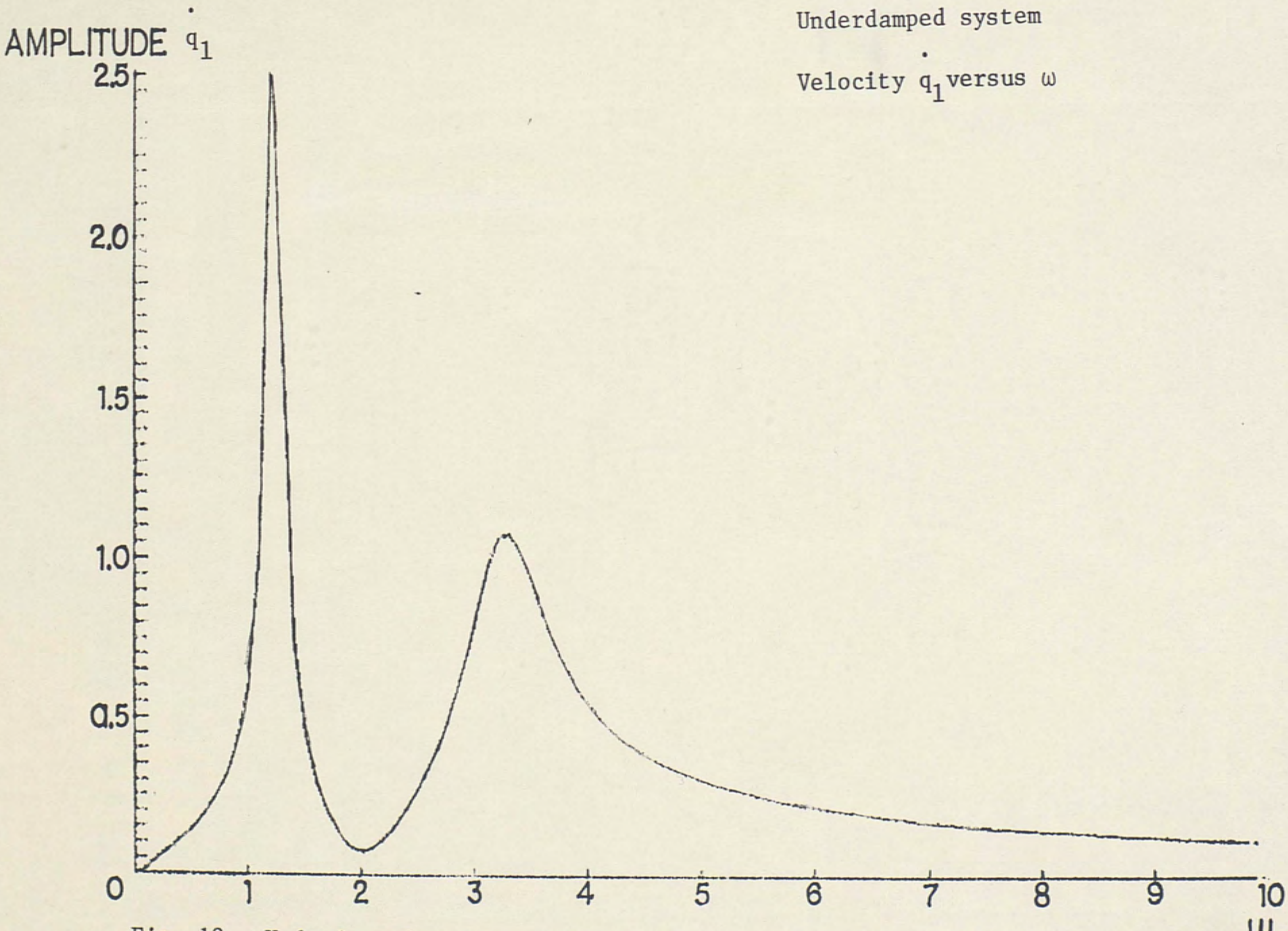

Fig. 18. Underdamped system amplitude I versus frequency (cycles/sec).

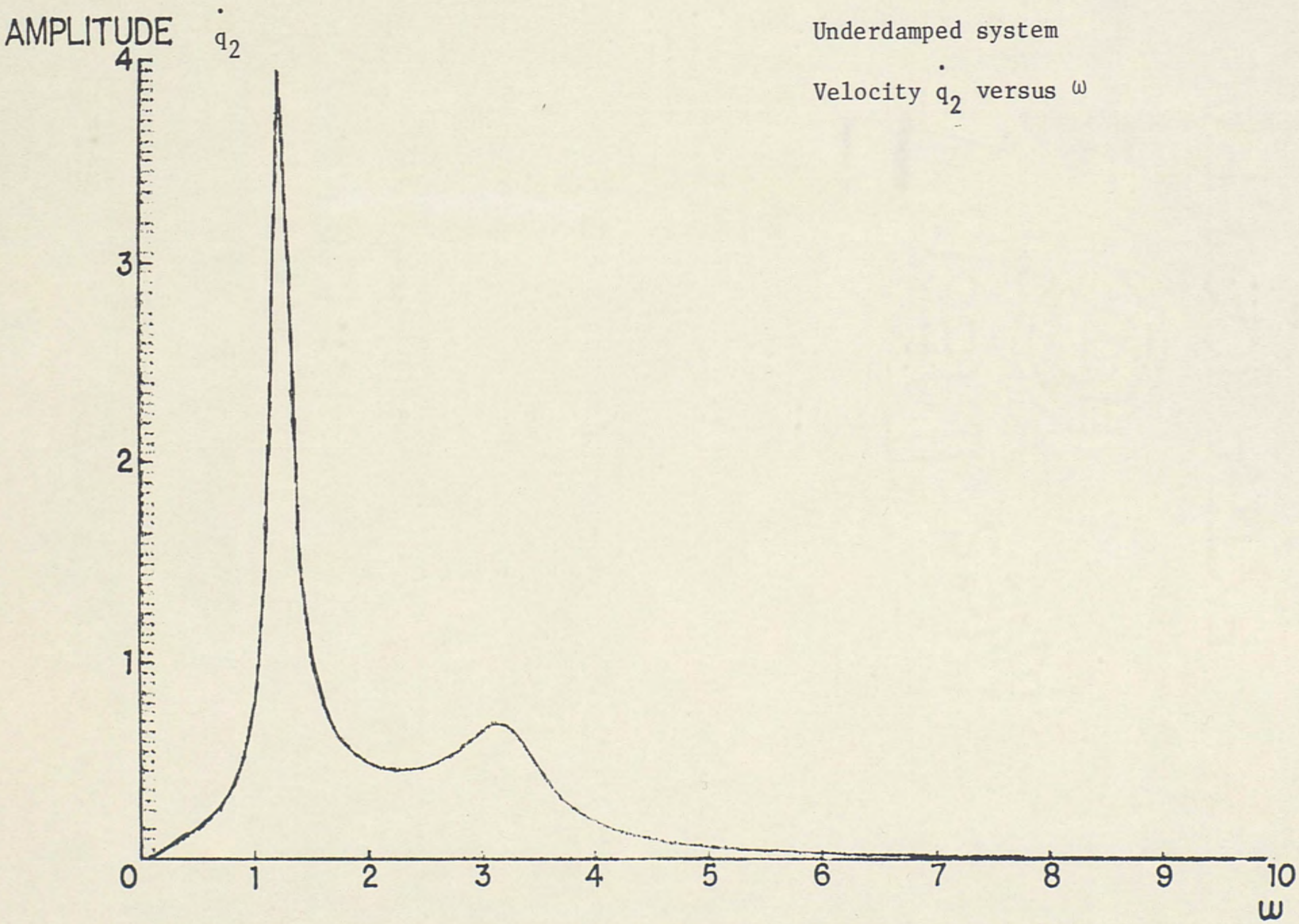

Fig. 19. Underdamped system amplitude II versus frequency (cycles/sec).

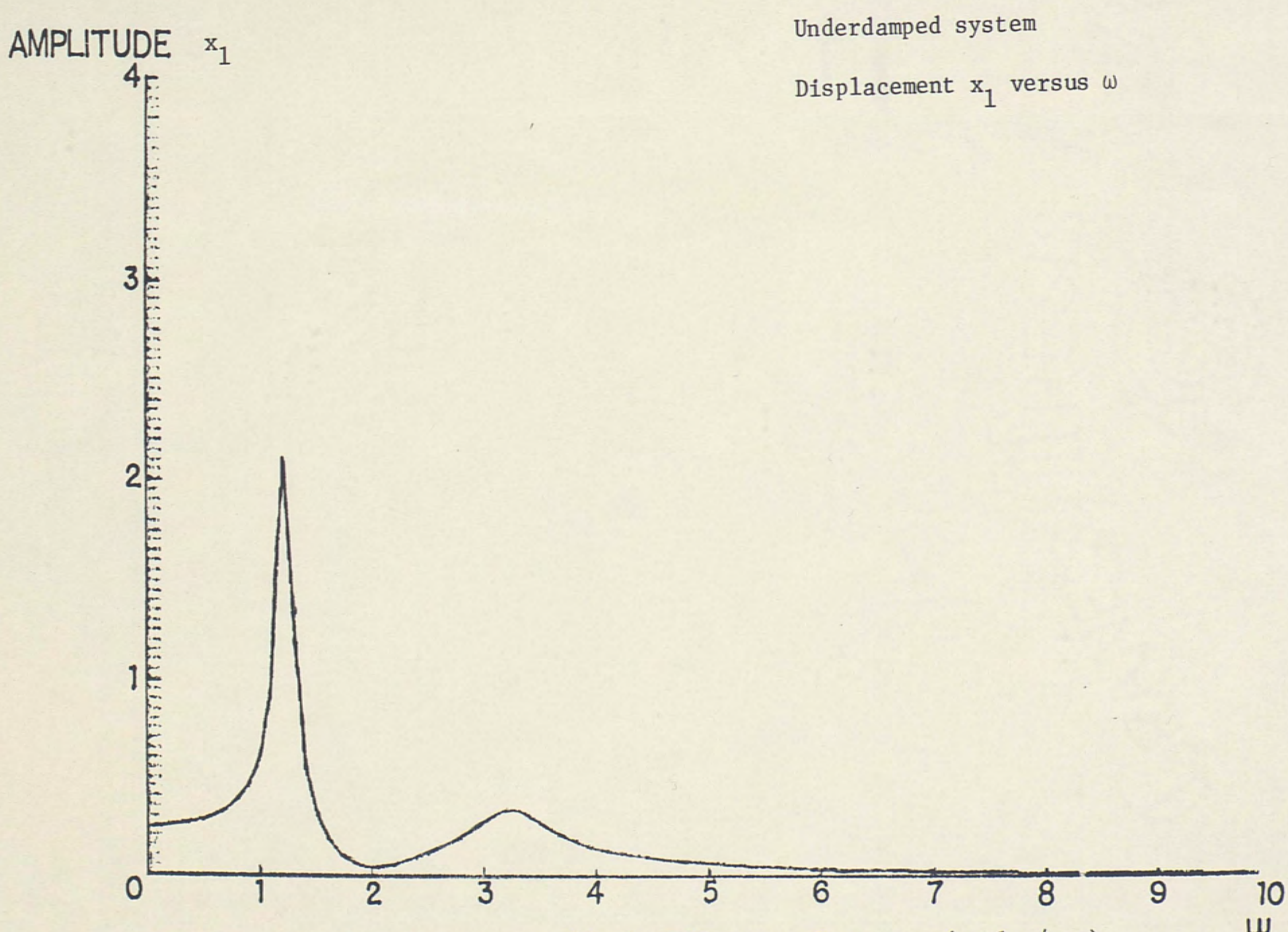

Fig. 20. Underdamped system amplitude III versus frequency (cycles/sec).

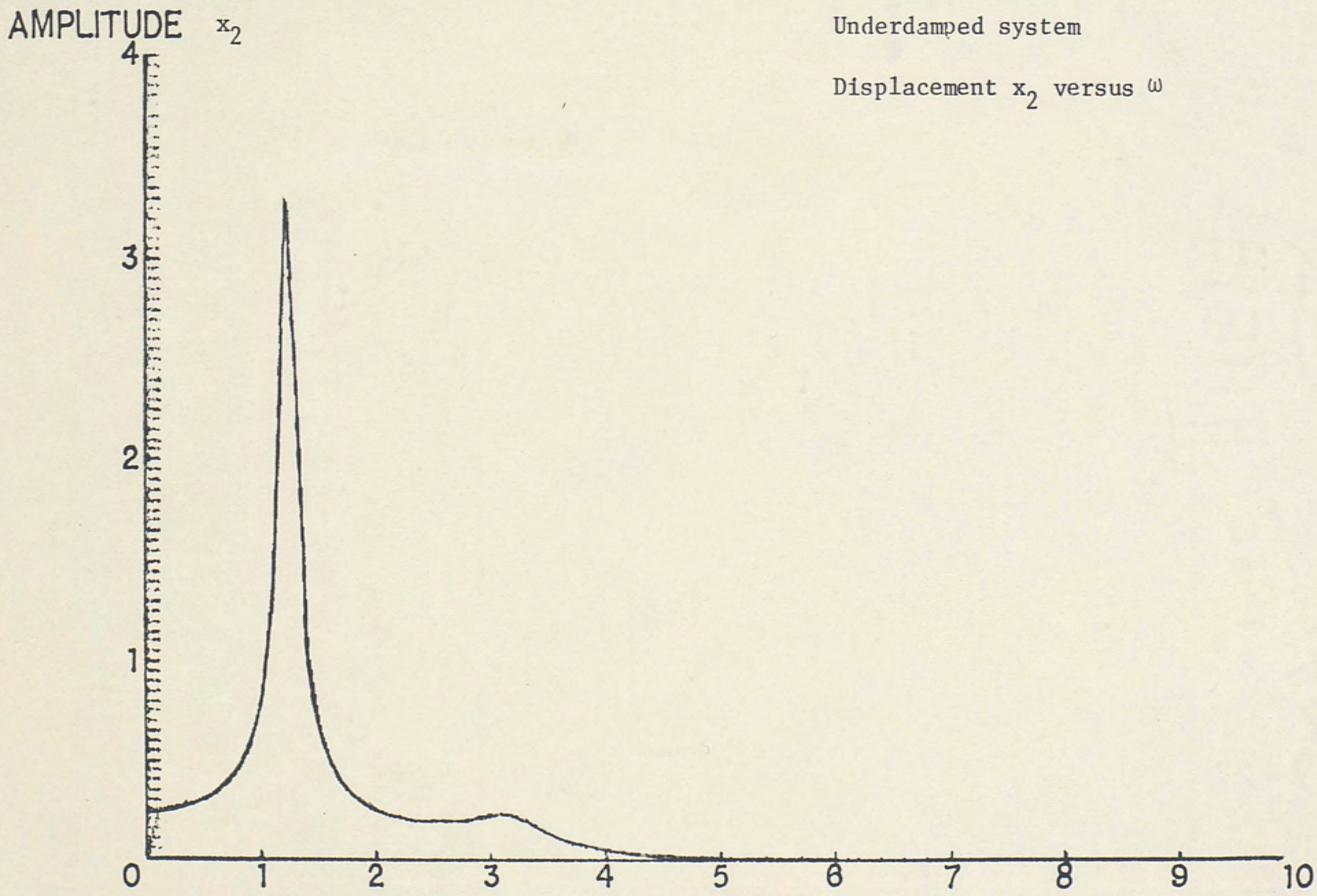

Fig. 21. Underdamped system amplitude IV versus frequency (cycles/sec).

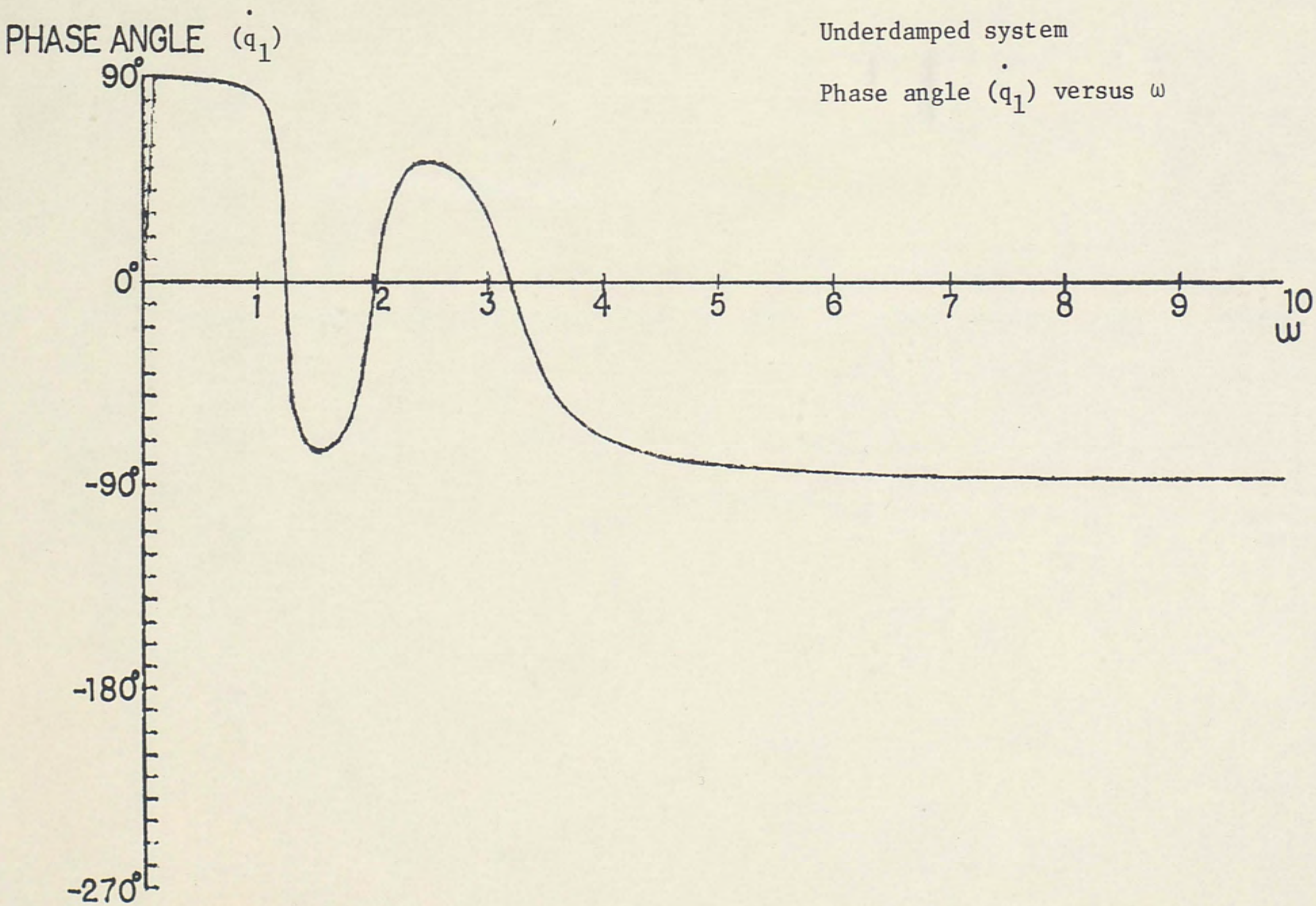

Fig. 22. Underdamped system phase angle I versus frequency (cycles/sec).

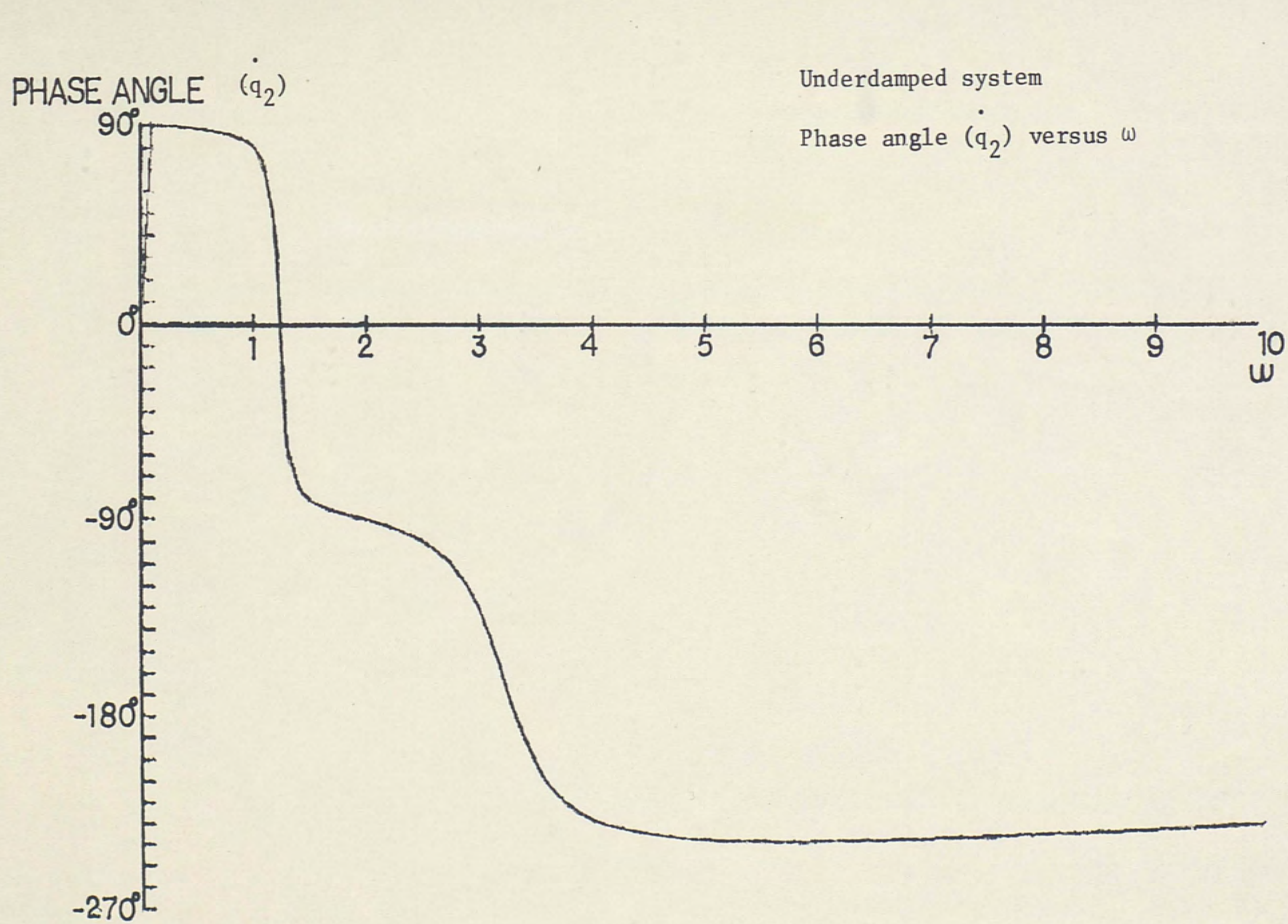

Fig. 23. Underdamped system phase angle II versus frequency (cycles/sec).

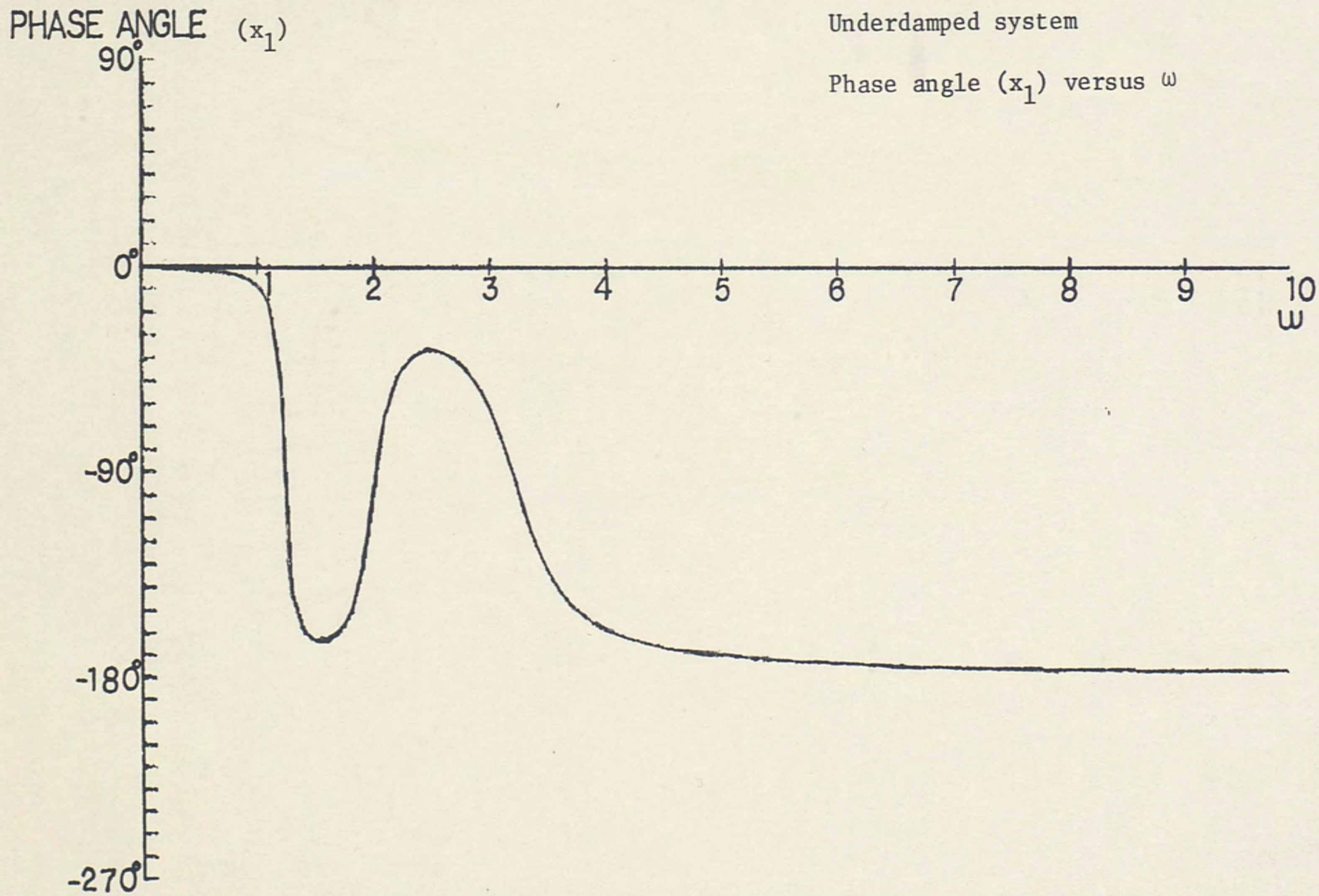

Fig. 24. Underdamped system phase angle III versus frequency (cycles/sec).

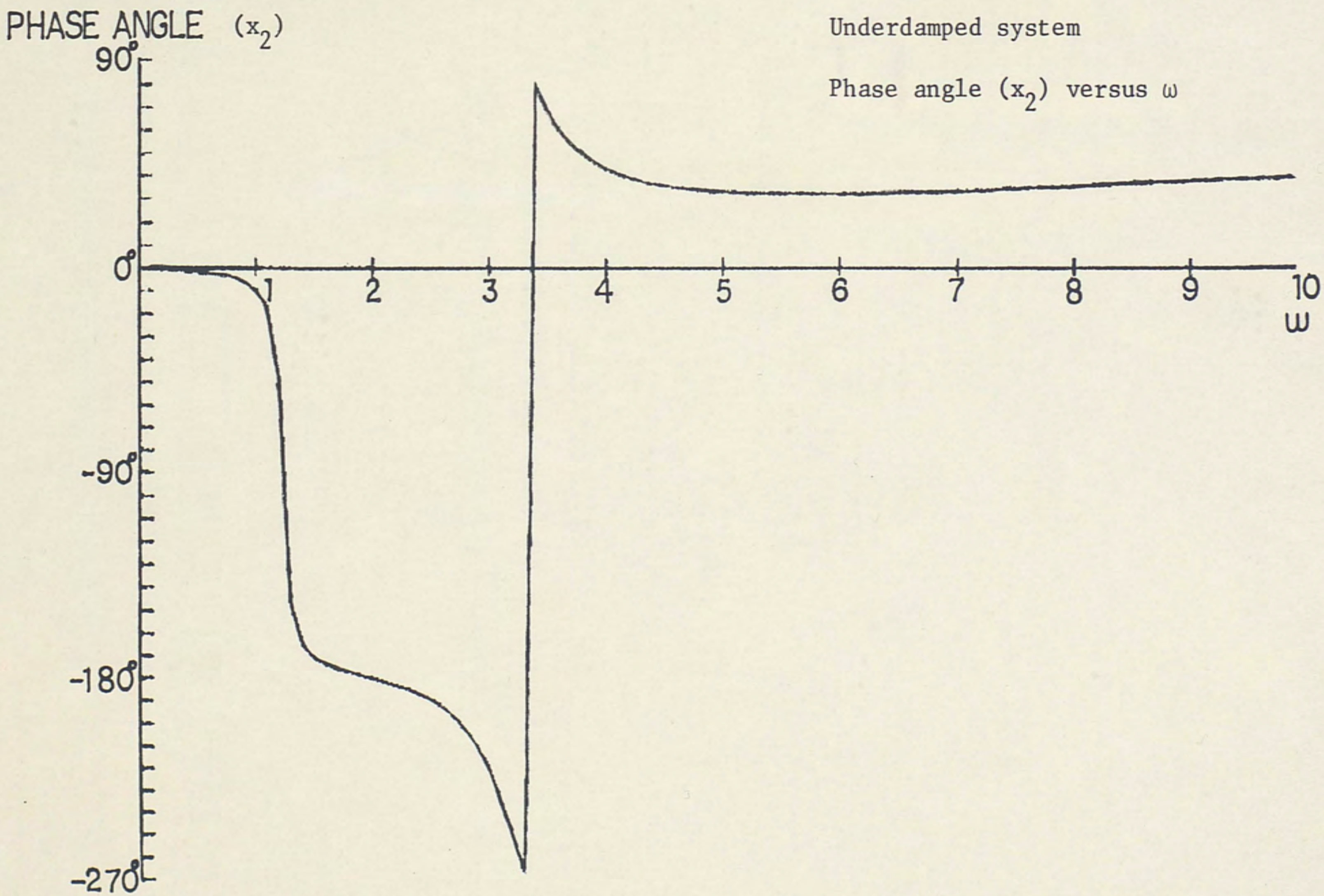

Fig. 25. Underdamped system phase angle IV versus frequency (cycles/sec).

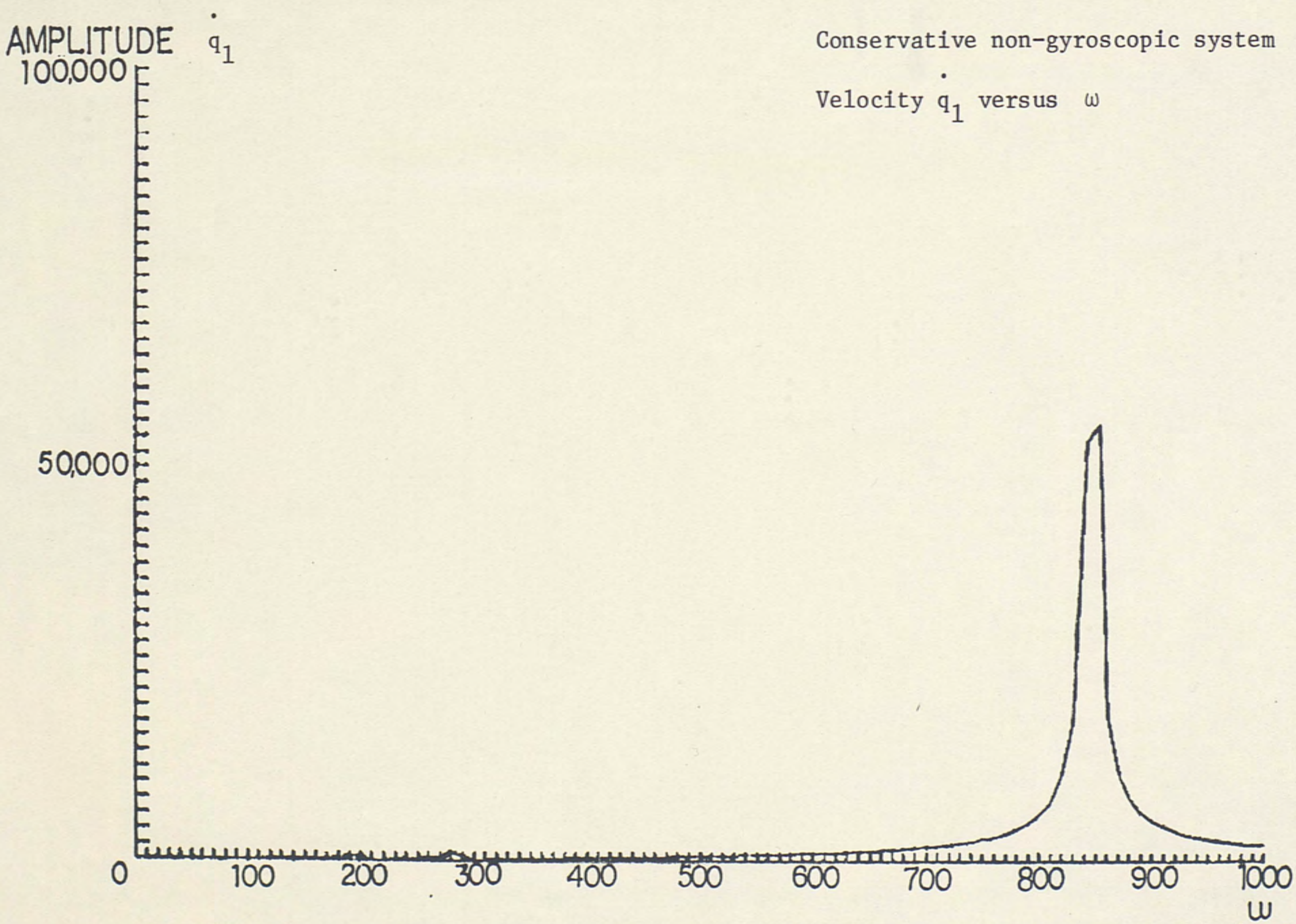

Fig. 27. Conservative non-gyroscopic system amplitude I versus frequency (cycles/sec).

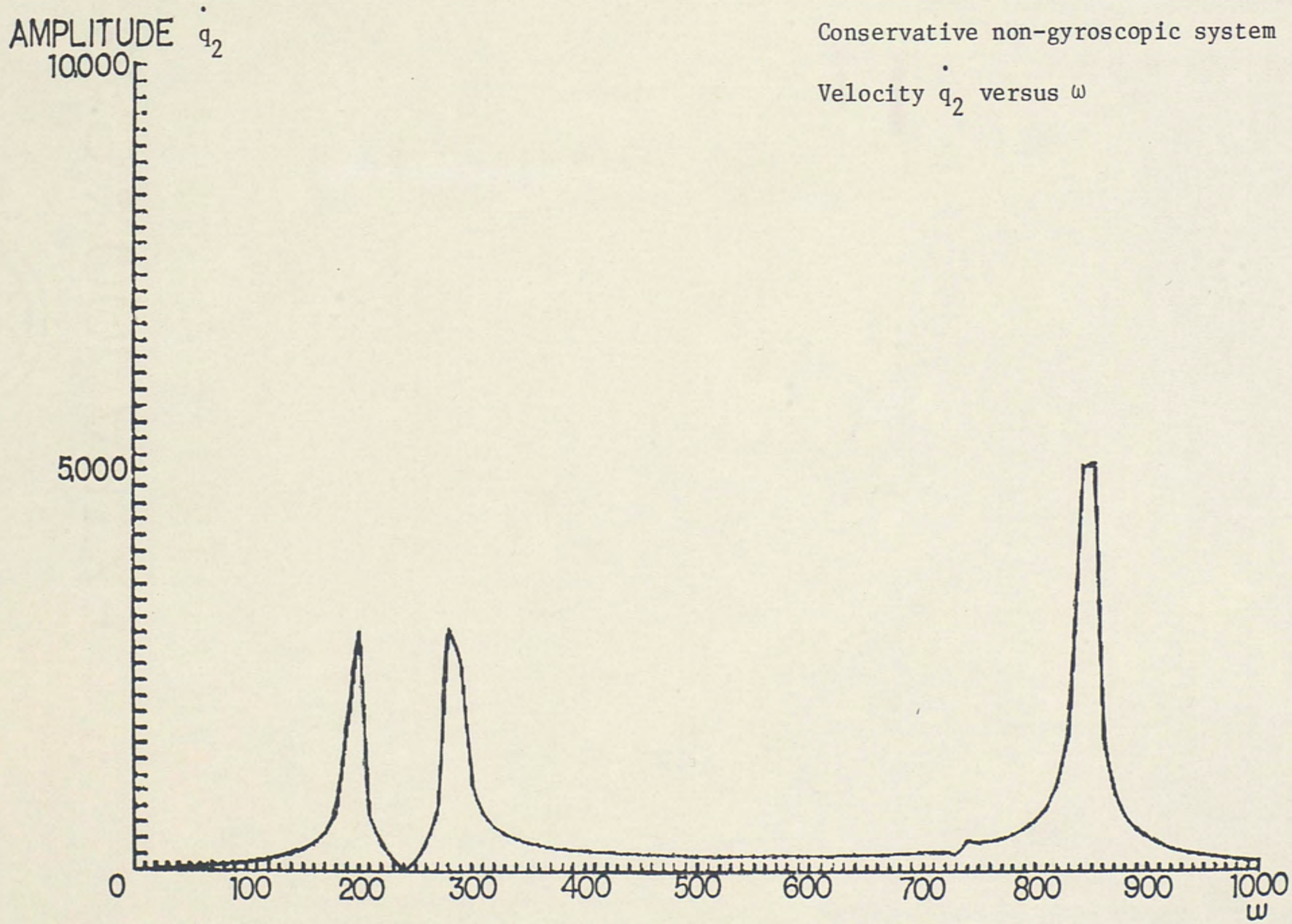

Fig. 28. Conservative non-gyroscopic system amplitude II versus frequency (cycles/sec).

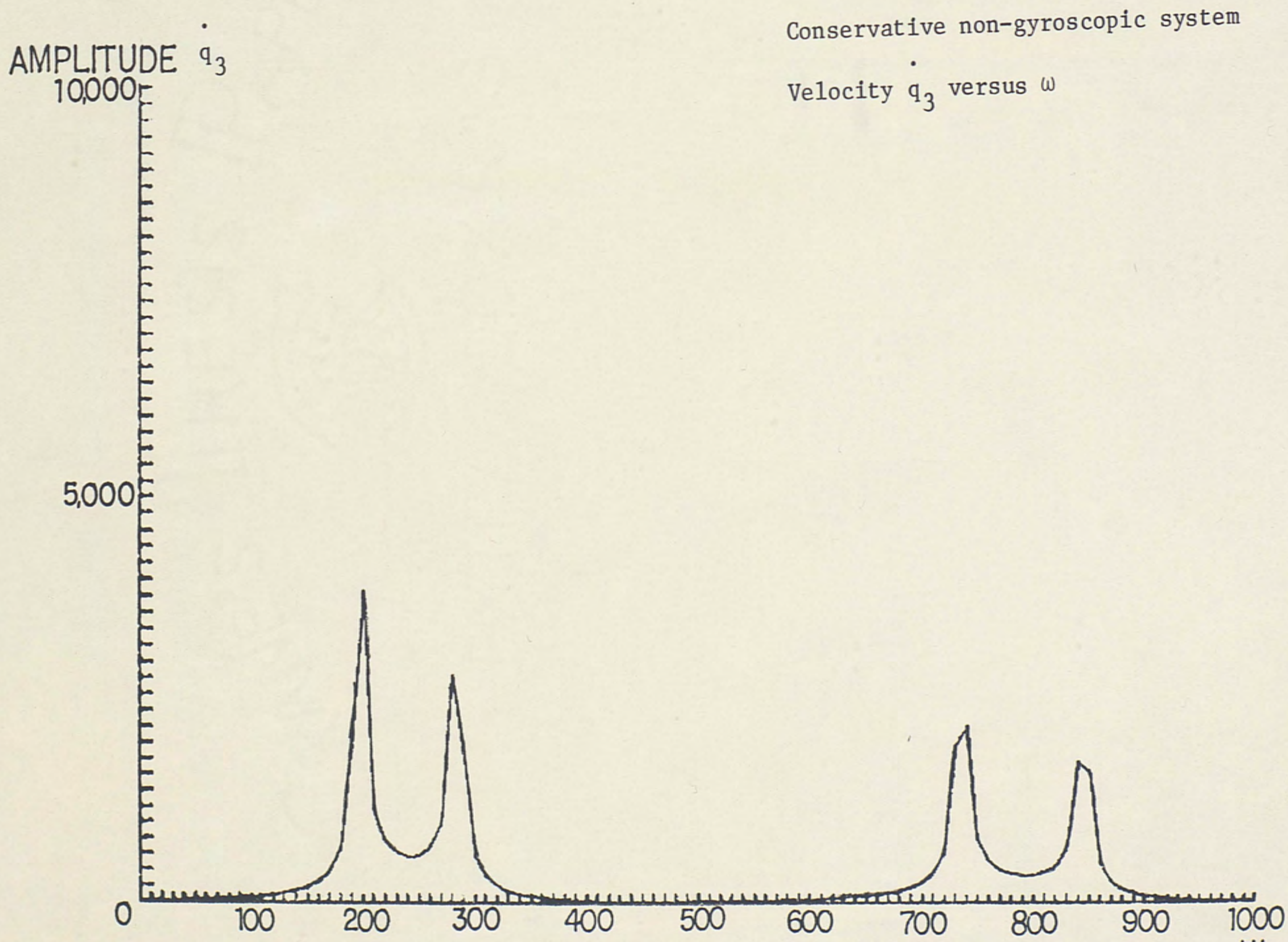

Fig. 29. Conservative non-gyroscopic system amplitude III versus frequency (cycles/sec).

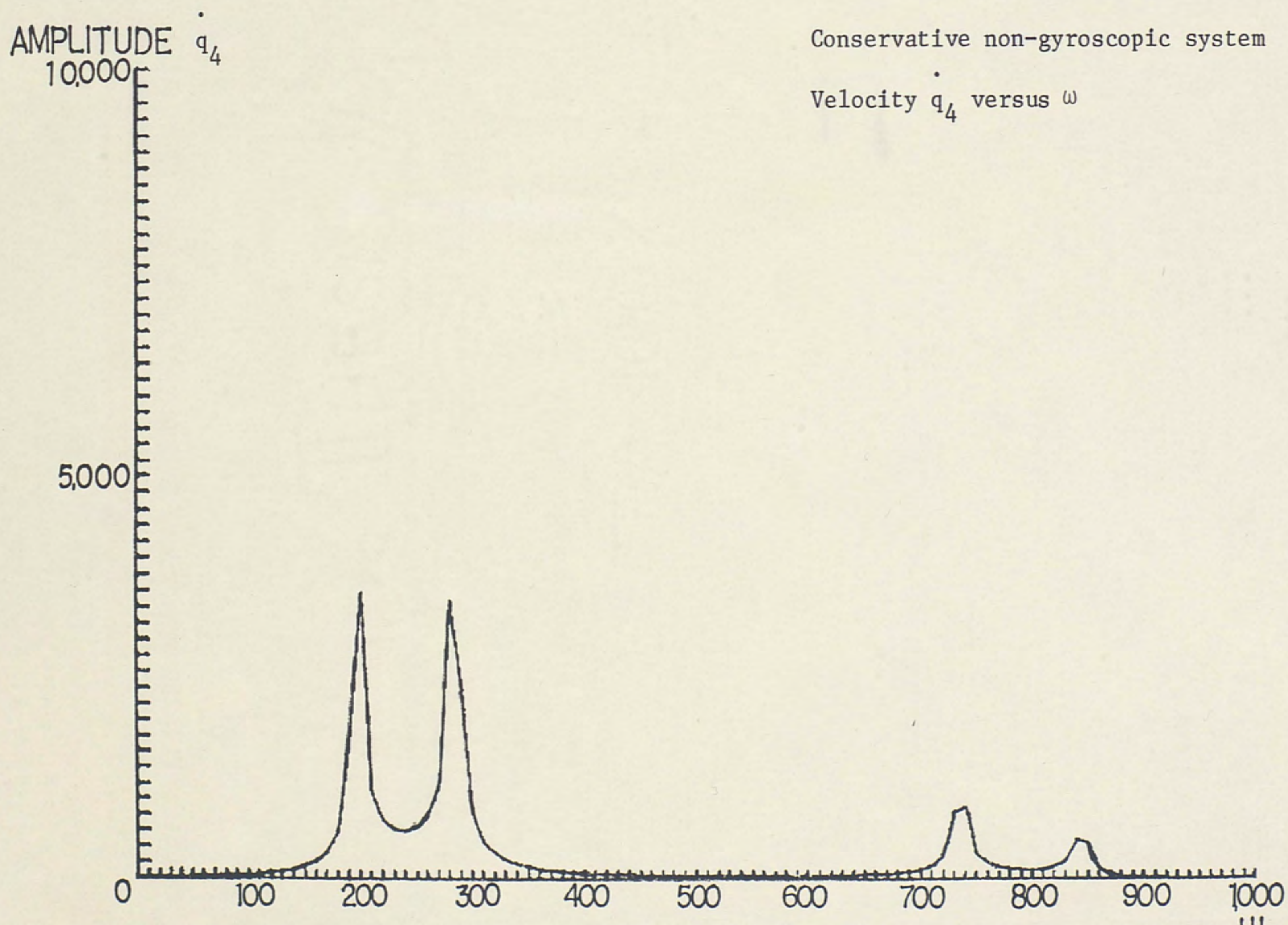

Fig. 30. Conservative non-gyroscopic system amplitude IV versus frequency (cycles/sec).

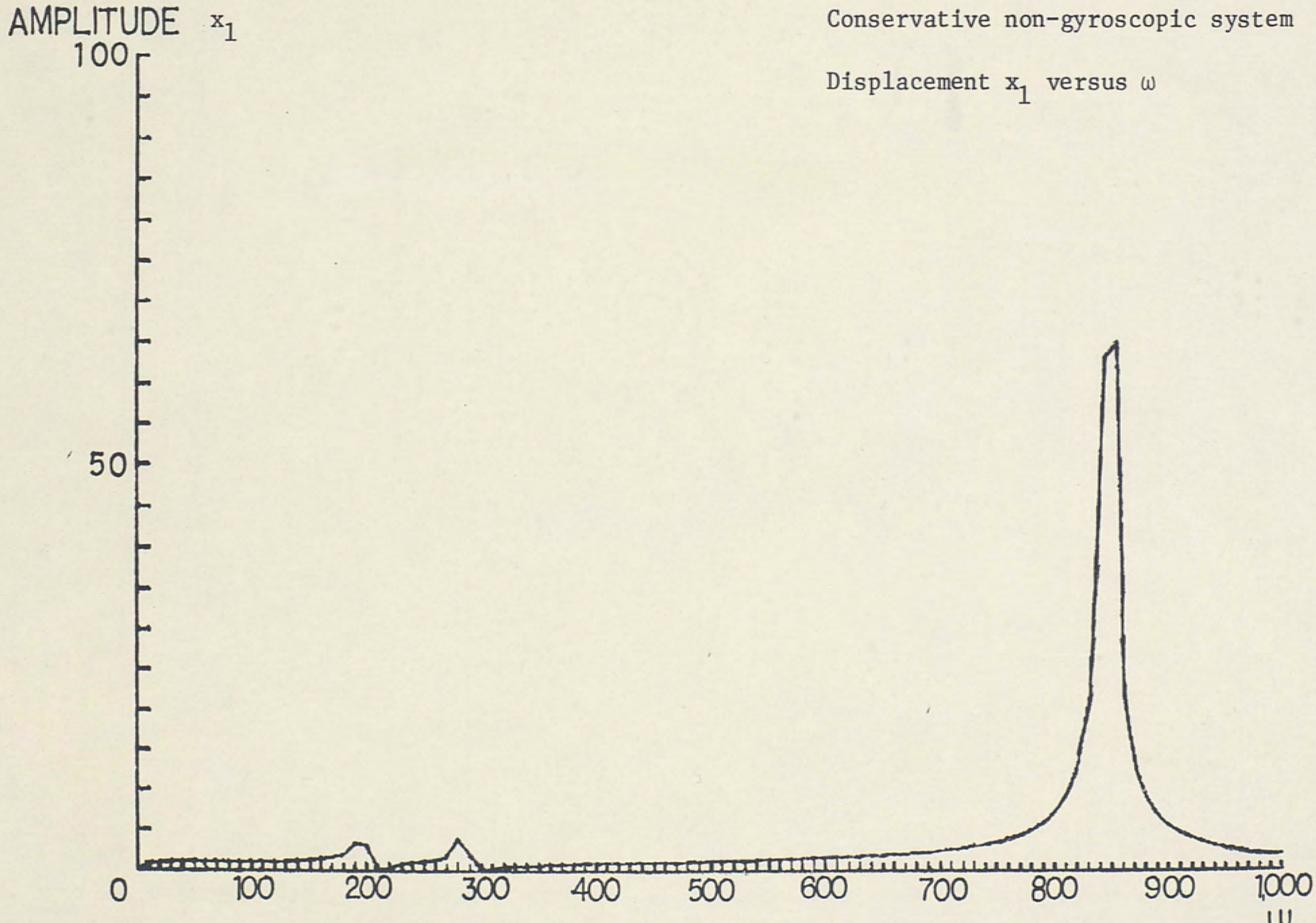

Fig. 31. Conservative non-gyroscopic system amplitude V versus frequency (cycles/sec).

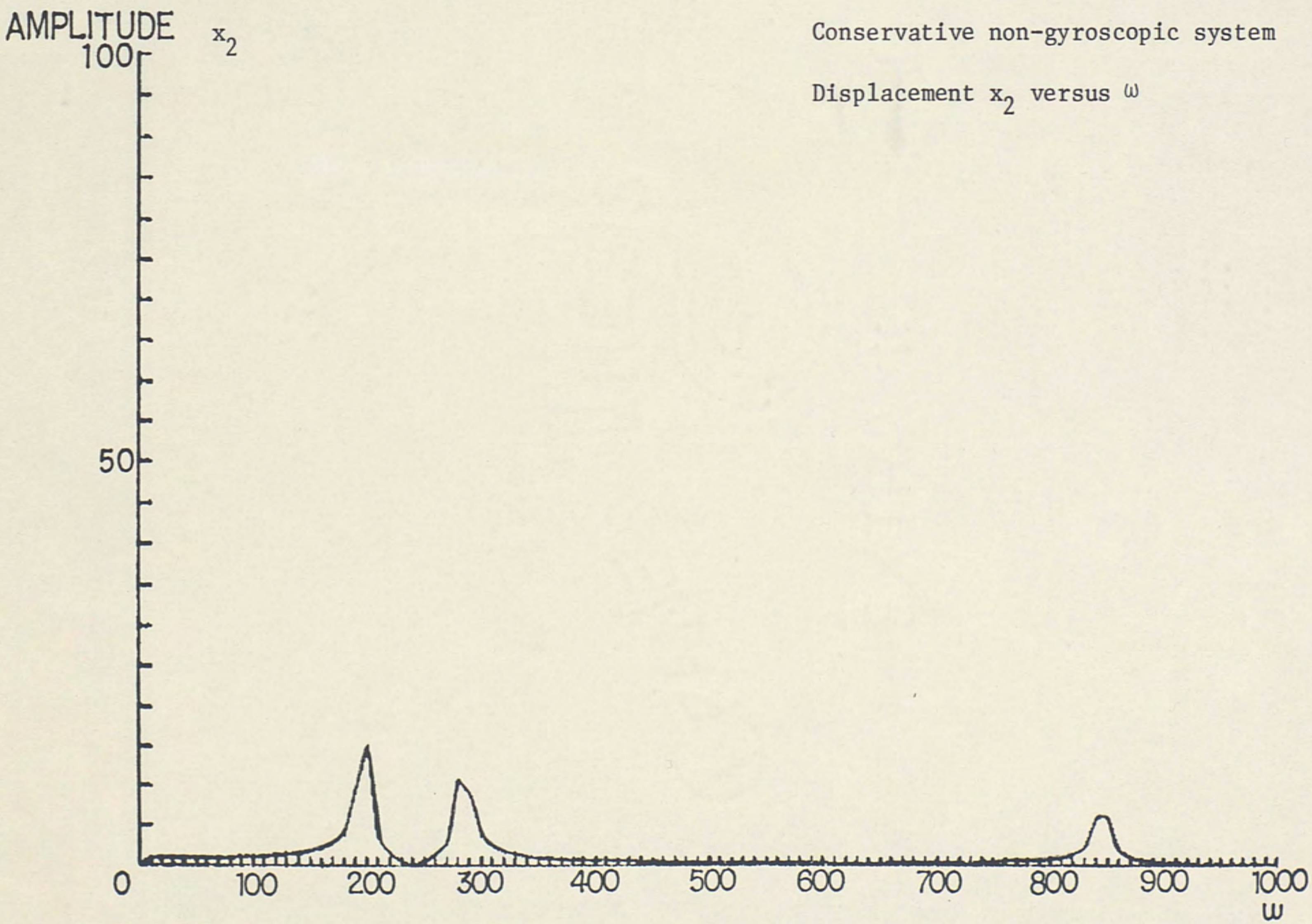

Fig. 32. Conservative non-gyroscopic system amplitude VI versus frequency (cycles/sec).

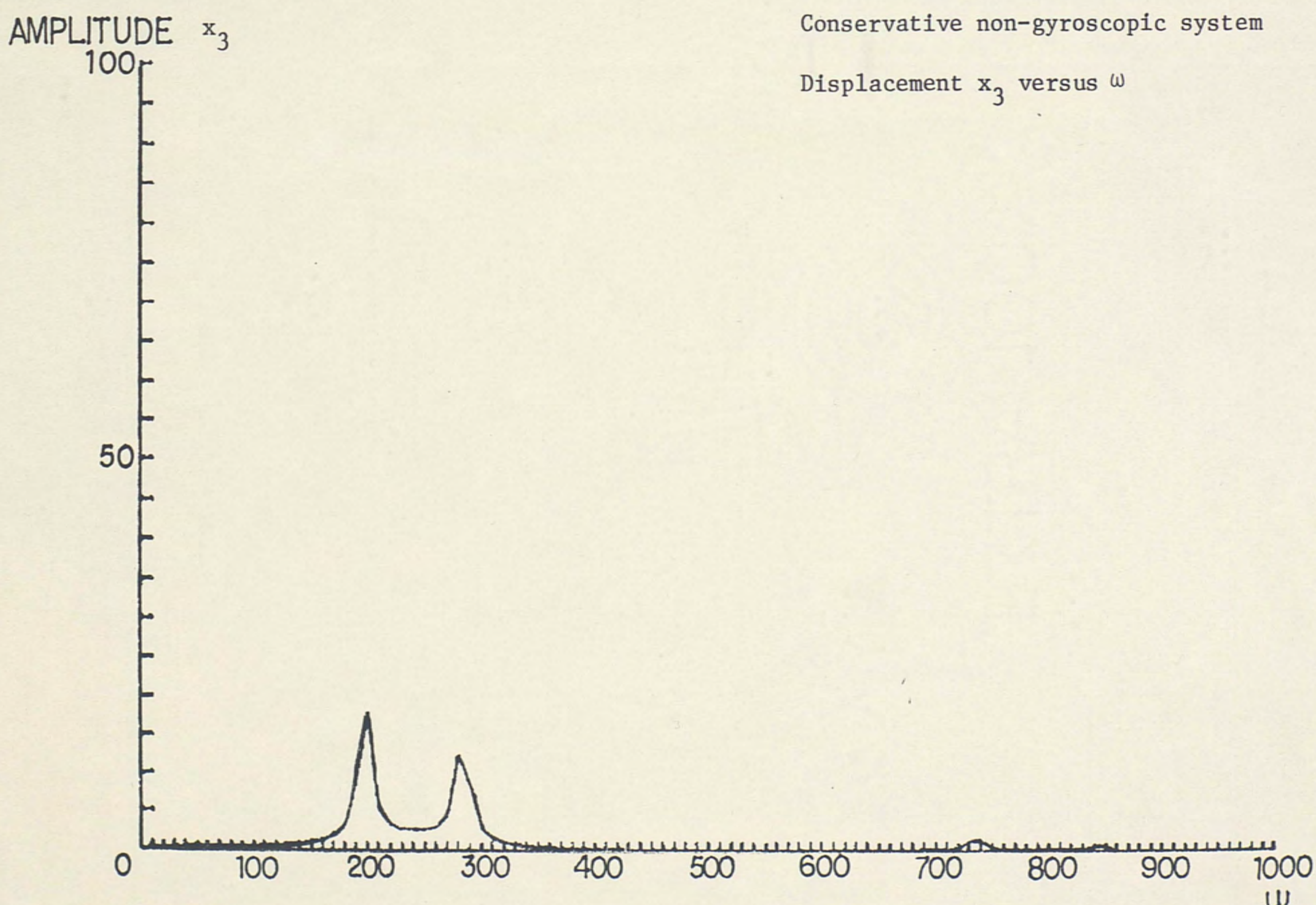

Fig. 33. Conservative non-gyroscopic system amplitude VII versus frequency (cycles/sec).

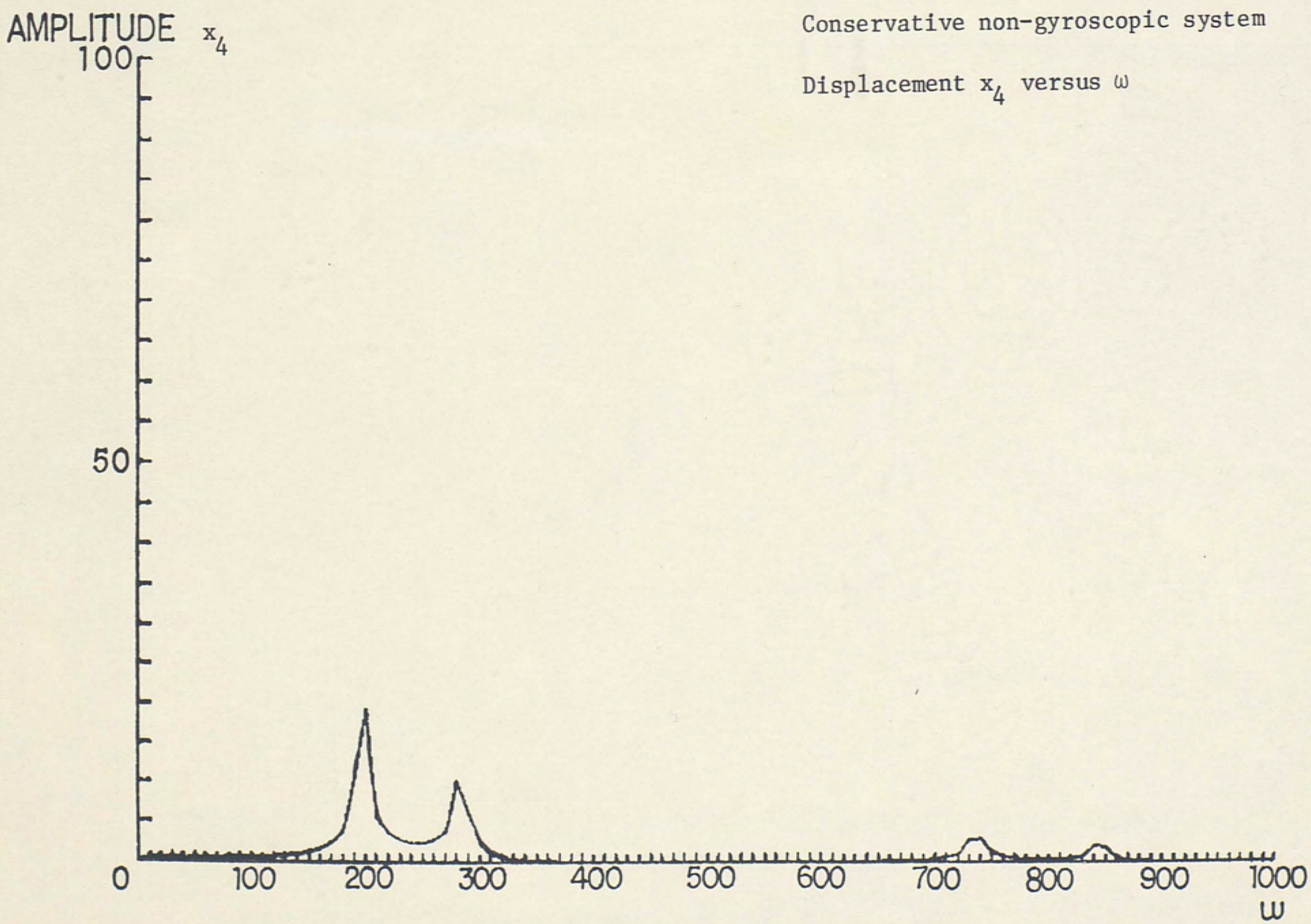

Fig. 34. Conservative non-gyroscopic system amplitude VIII versus frequency (cycles/sec).

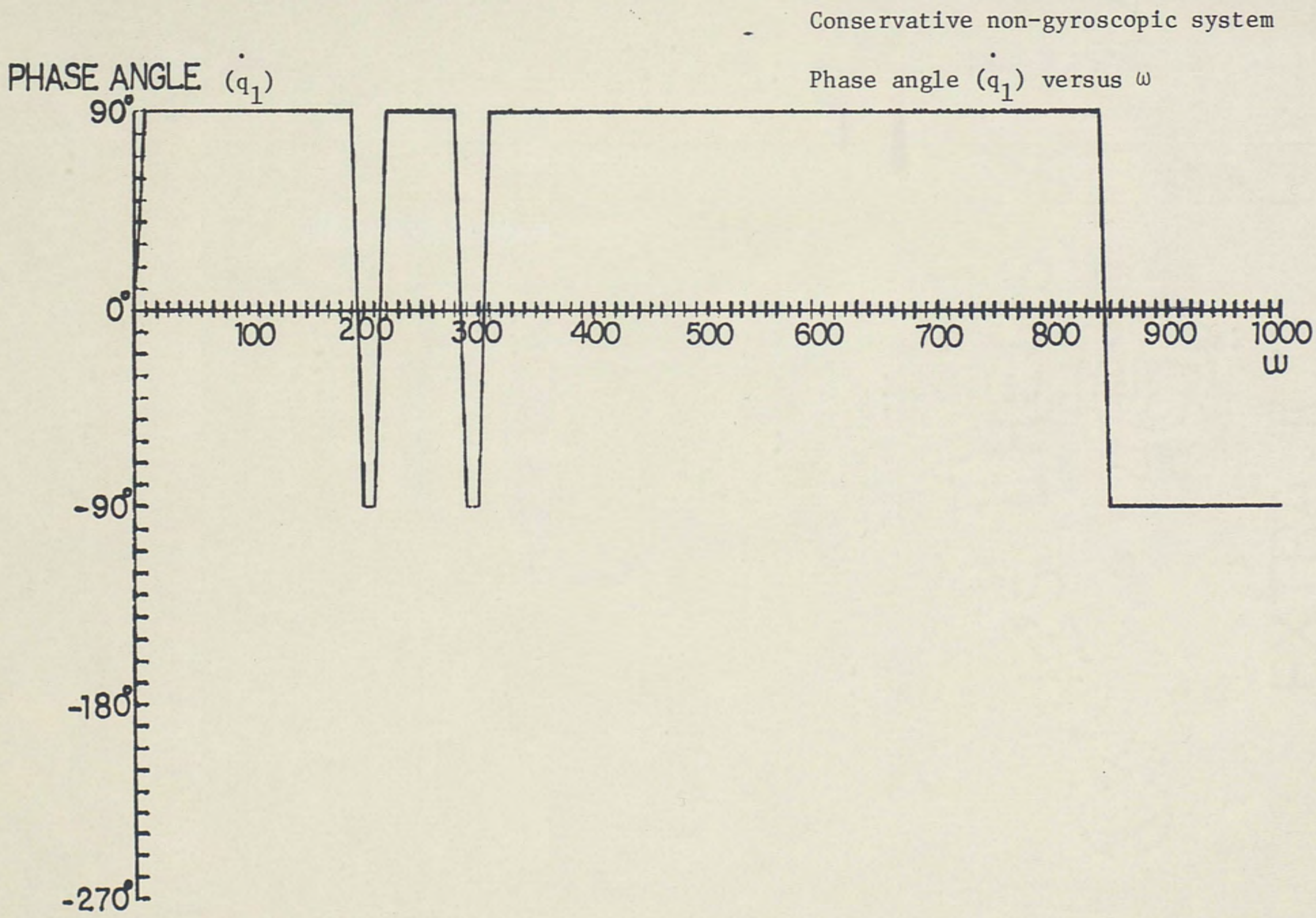

Fig. 35. Conservative non-gyroscopic system phase angle I versus frequency (cycles/sec).

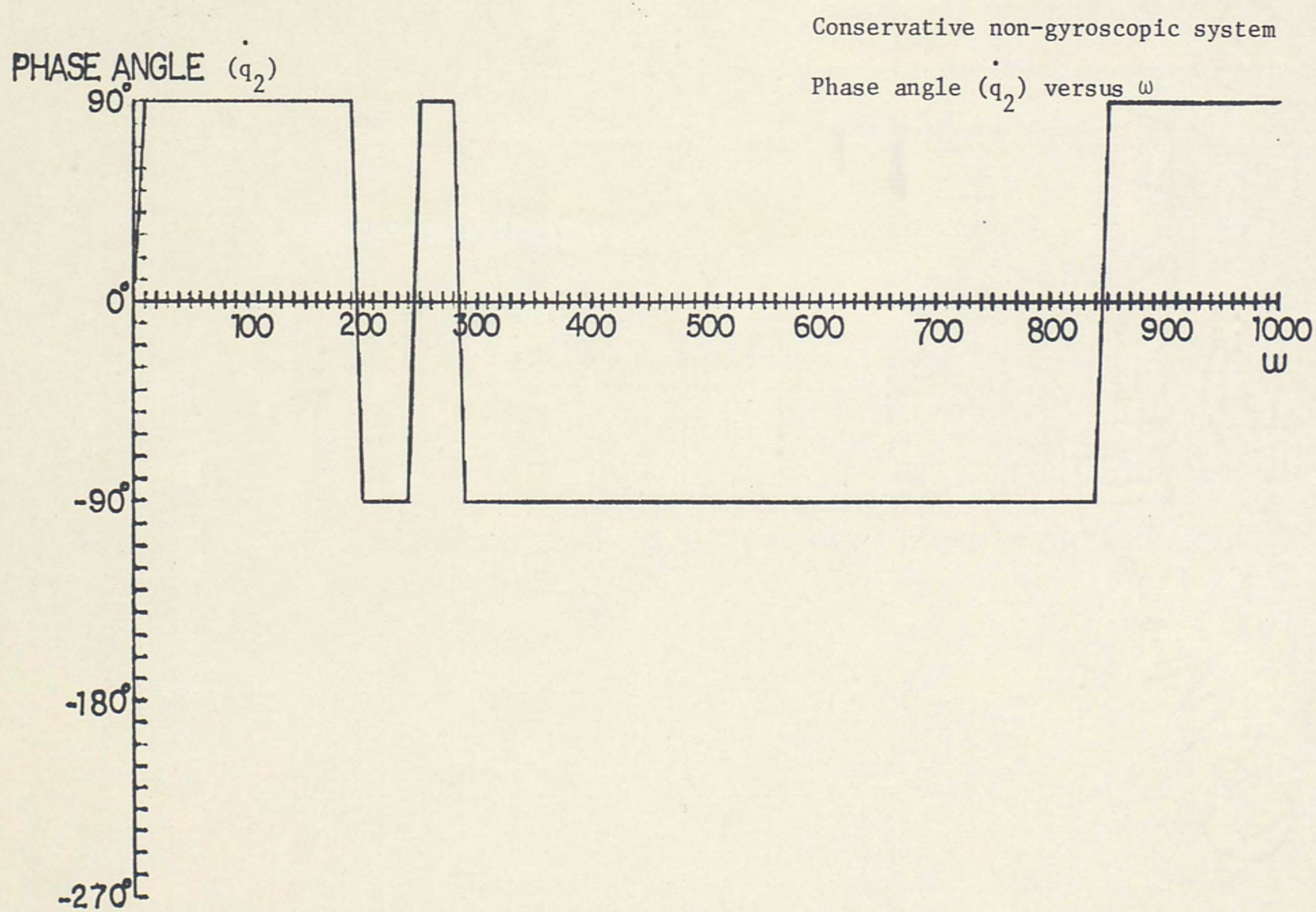

Fig. 36. Conservative non-gyroscopic system phase angle II versus frequency (cycles/sec).

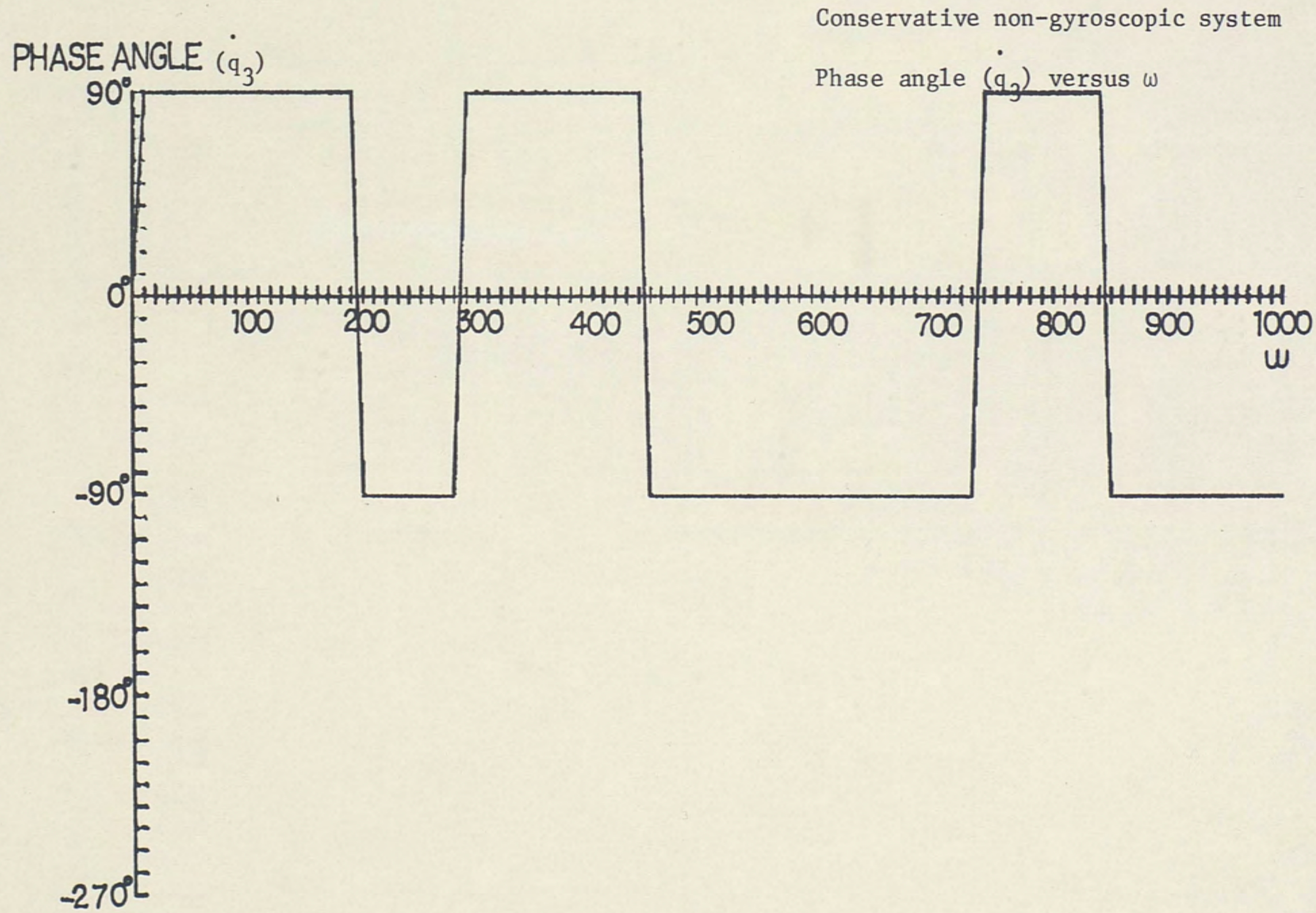

Fig. 37. Conservative non-gyroscopic system phase angle III versus frequency (cycles/sec).

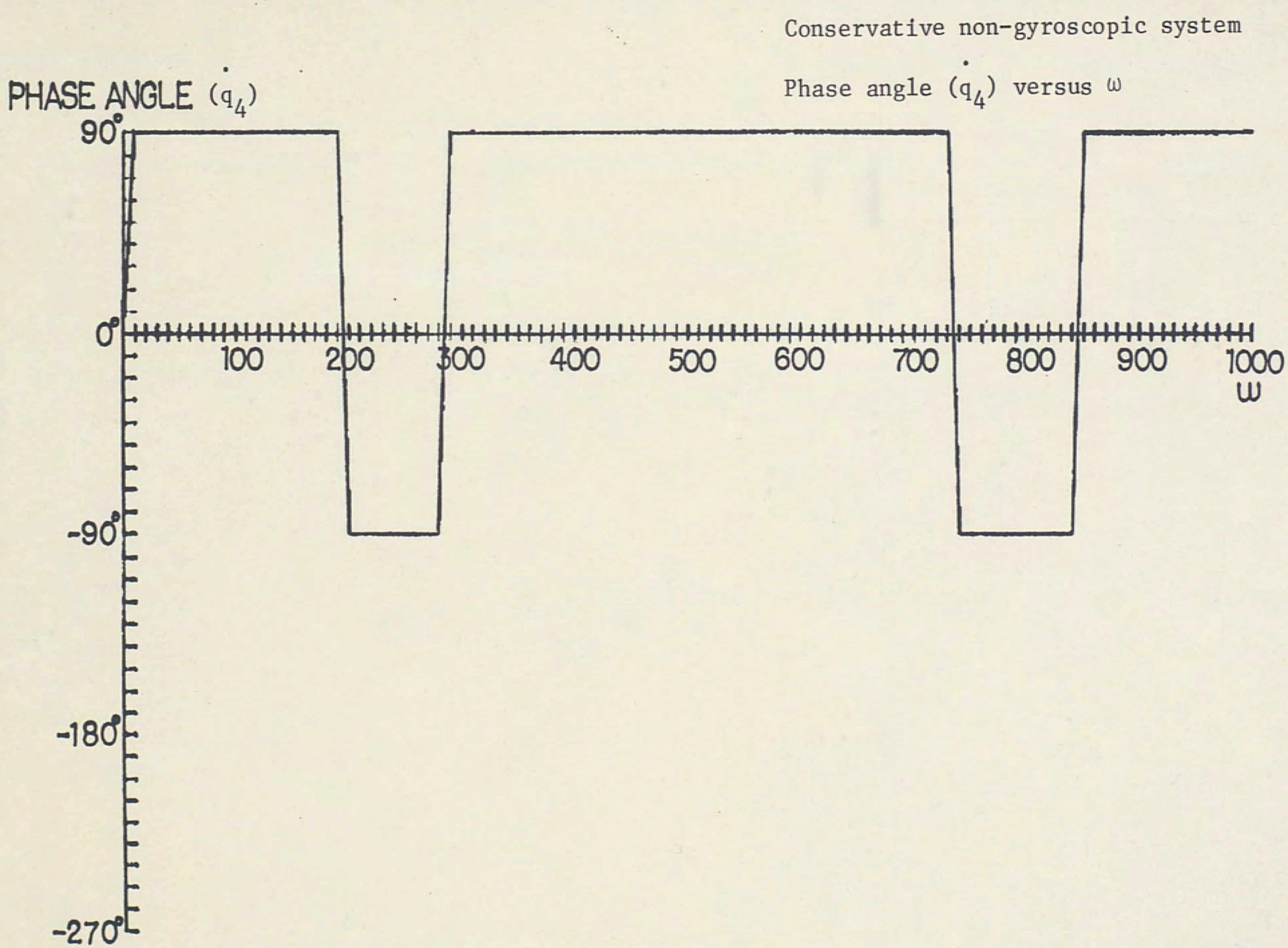

Fig. 38. Conservative non-gyroscopic system phase angle IV versus frequency (cycles/sec).

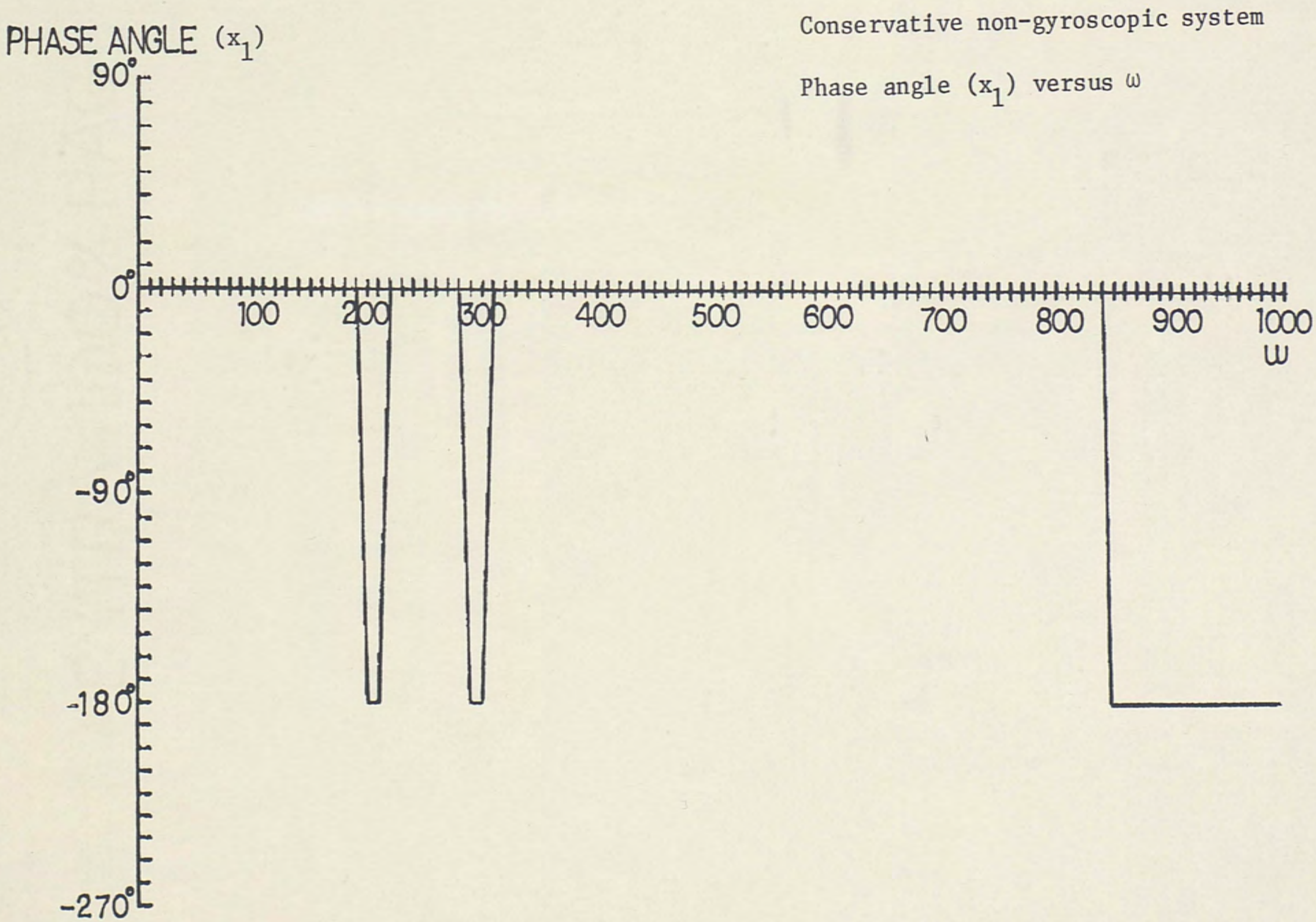

Fig. 39. Conservative non-gyroscopic system phase angle V versus frequency (cycles/sec).

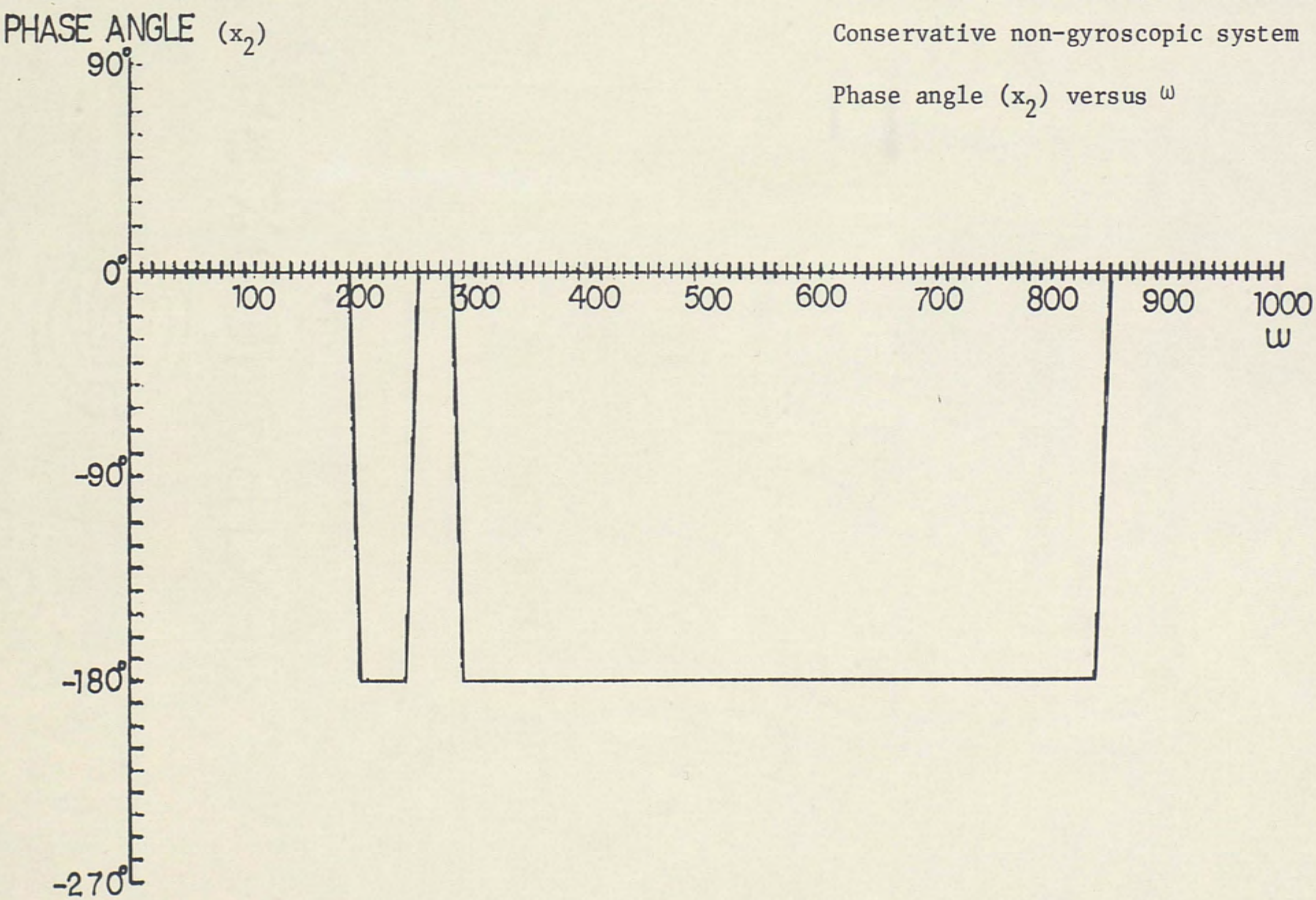

Fig. 40. Conservative non-gyroscopic system phase angle VI versus frequency (cycles/sec).

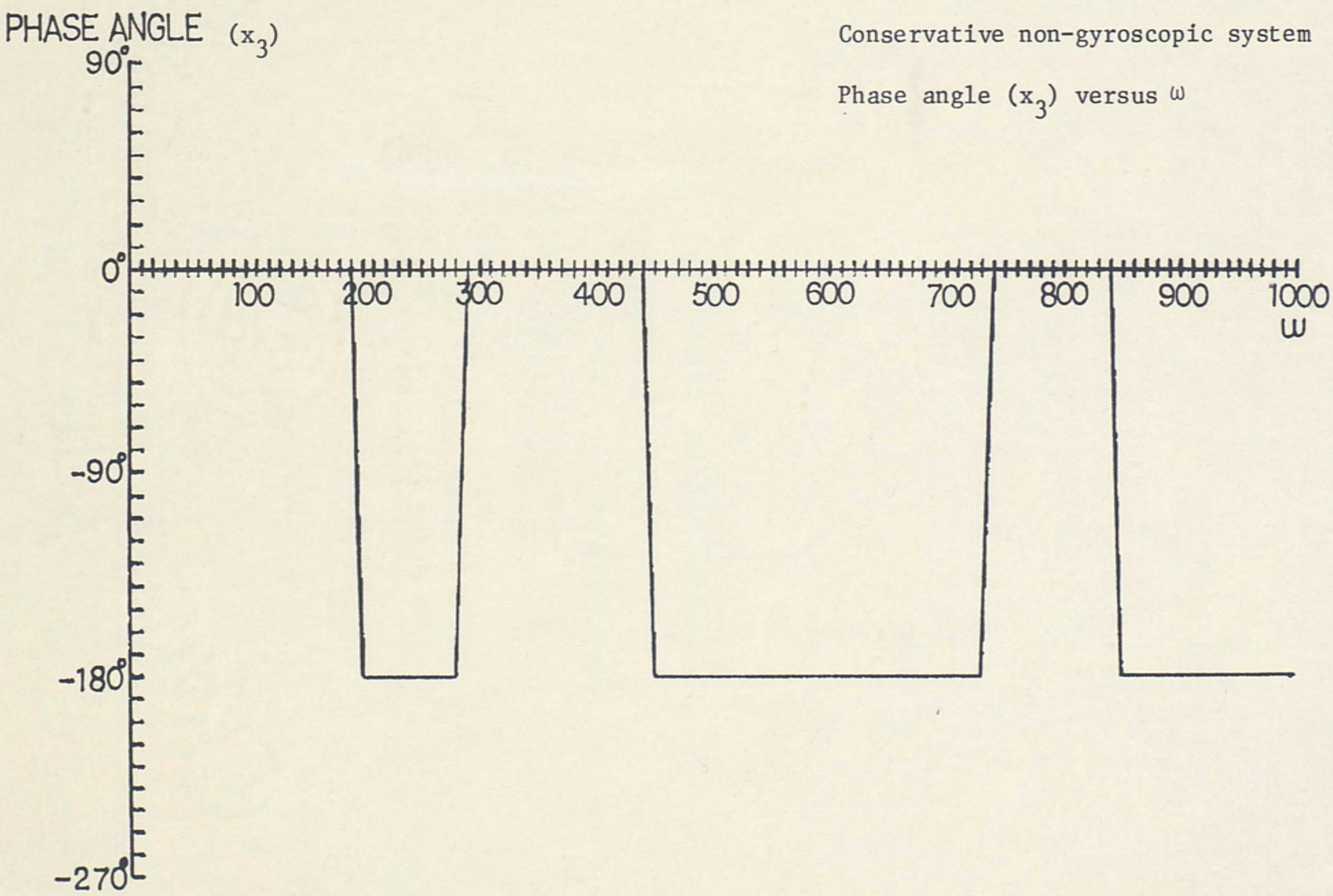

Fig. 41. Conservative non-gyroscopic system phase angle VII versus frequency (cycles/sec).

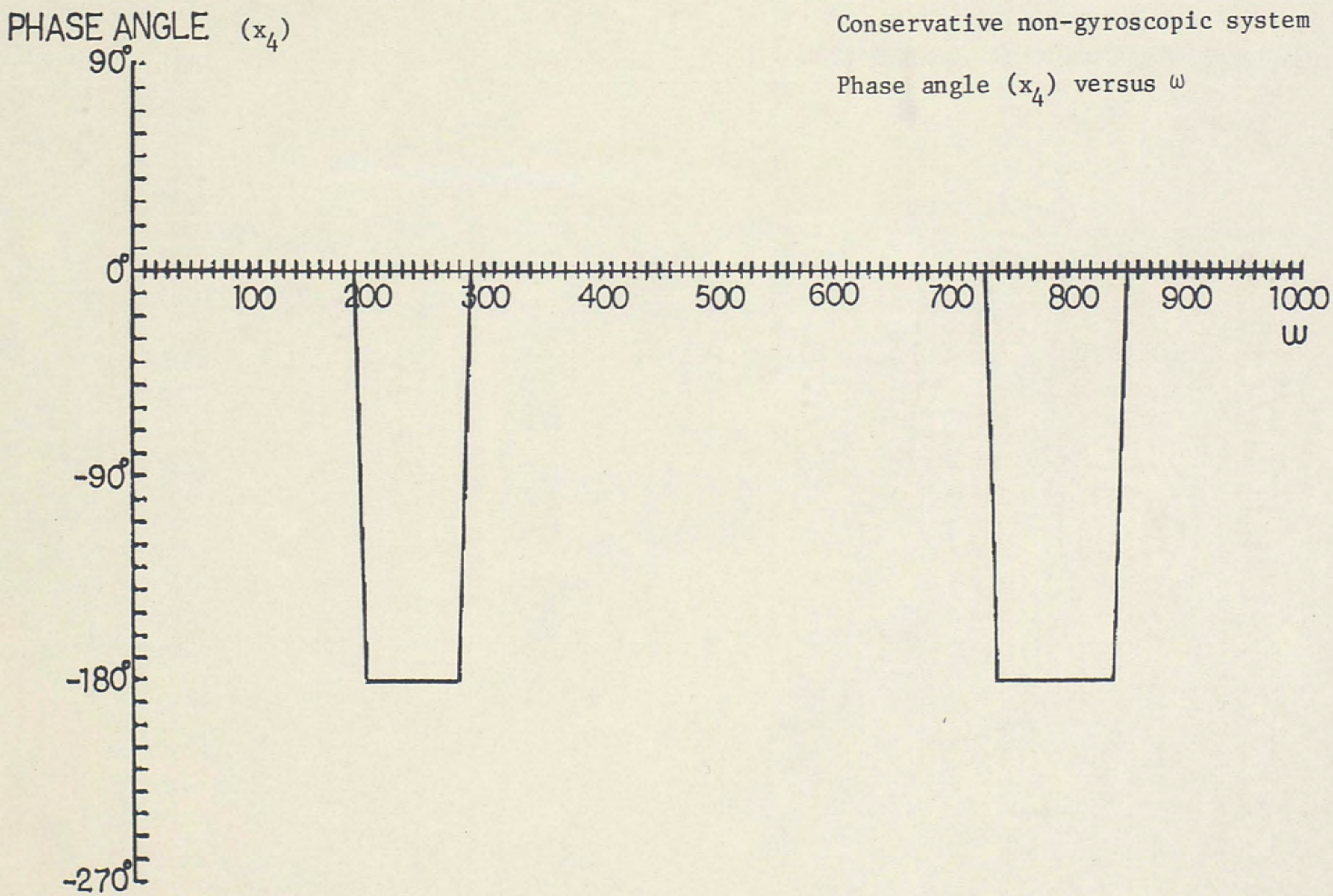

Fig. 42. Conservative non-gyroscopic system phase angle VIII versus frequency (cycles/sec).

PROGRAM 1

COMPUTER PROGRAM FOR EIGENVALUES AND NORMALIZED EIGENVECTORS

```
      INTEGER N,IA,IJOB,IZ,IER
      REAL*8 A(4,4),WK(24),RW(8),RZ(32),H(4,4),E(4,4)
      COMPLEX*16 W(4),Z(4,4),D(4,4),S(4,4),F(4,4),ZN,CDSQRT
     *,G(4,4),Z1(4,4),Y(4,4)
      EQUIVALENCE (W(1),RW(1)),(Z(1,1),RZ(1))
      DATA   A/

      N=4
      IA=4
      IZ=4
      IJOB=2
      CALL EIGRF (A,N,IA,IJOB,RW,RZ,IZ,WK,IER)
      WRITE(6,20)IER,WK(1)
   20 FORMAT(1X,'IER=',I2,10X,'WK(1)=',E12.5//)
      WRITE(6,10)
   10 FORMAT(1X,'EIGENVALUES',/)
      DO 50 I=1,N
      WRITE (6,30)I,W(I)
   30 FORMAT(1X,'W( ',I2,' )= ',E12.5,8X,E12.5,1X,'I')
   50 CONTINUE
      WRITE(6,51)
   51 FORMAT(//,1X,'EIGENVECTORS',/ )
      DO 9 K=1,N
      WRITE(6,40)(Z(K,J),J=1,N)
   40 FORMAT(1X,'|',4(2X,E12.5,2X,E12.5,1X,'I'),2X,'|',
     */1X,'|',4(2X,E12.5,2X,E12.5,1X,'I'),2X,'|',/1X,'|',122X,'|')
    9 CONTINUE
C            NORMALIZE EIGENVECTORS
      DO 60 I=1,N
      ZN=(0.,0.)
      DO 70 J=1,N
   70 ZN=ZN+Z(J,I)*Z(J,I)
      DO 60 K=1,N
   60 Z(K,I)=Z(K,I)/CDSQRT(ZN)
      WRITE(6,80)
   80 FORMAT(//,1X,'NORMALIZED EIGENVECTORS',/)
      DO 90 K=1,N
      WRITE(6,40)(Z(K,J),J=1,N)
   90 CONTINUE
      STOP
      END
```

PROGRAM 2

COMPUTER PROGRAM FOR FREQUENCY RESPONSE

```
100 INIT
110 PAGE
120 REM:          ***** -.---.---.-      *****
130 REM:
140 PRINT "        L_ ENTER THE TYPE OF DATA  <G_"
150 REM:
160 REM:          ****     -.---.---.-    *****
170 PRINT "J_  K1=1  FOR  DATA FILE  G_"
180 PRINT "J_  K1=2  FOR  FUNCTIONS  G_"
190 PRINT "J_  K1=3  FOR  DIRECT INPUT G_"
200 PRINT "J_  ENTER THE TYPE OF DATA ,K1>G_";
210 INPUT K1
220 REM:
230 REM:            *****  -.---.---.-    *****
240 REM:
250 PRINT "L_       WHAT TYPE OF OUTPUT DO YOU WANT?  "
260 REM
270 REM:            *****  -.---.---.-    *****
280 PRINT "J_  K2=1   FOR TABLES  G_"
290 PRINT "J_  K2=2   FOR GRAPH   G_"
300 PRINT "J_  K2=3   FOR OUTPUT FILE G_"
310 PRINT "J_ ENTER THE TYPE OF OUTPUT ,K2>G_";
320 INPUT K2
330 IF K1=1 THEN 680
340 IF K1=2 THEN 870
350 IF K1=3 THEN 360
360 PRINT "J_  DATA FROM DIRECT INPUT "
370 PRINT "L_ENTER THE DIMENSION SIZE N  >G_";
```

PROGRAM 2 (Continued)

```
380 INPUT N
390 DIM A(N,N),A2(N,N),C(N,N),D1(N,1),D2(N,1),U1(N,1),U2(N,1)
400 DIM F(N,1),B(N,1)
410 PRINT "J_ENTER THE VALUES OF MATRIX A G_"
420 FOR I=1 TO N
430 FOR J=1 TO N
440 PRINT USING 460:"ENTER A(",I,",",J,")  )G_ ";
450 INPUT A(I,J)
460 IMAGE 8A,FD,1A,FD,6A,S
470 NEXT J
480 NEXT I
490 PRINT "J_ENTER THE VALUES OF U1 J_G_"
500 FOR I=1 TO N
510 PRINT USING 530:"U1(",I,")  )G_";
520 INPUT U1(I,1)
530 IMAGE 3A,FD,6A,S
540 NEXT I
550 PRINT "J_ENTER THE VALUES OF U2 J_G_"
560 FOR I=1 TO N
570 PRINT USING 590:"U2(",I,")  )G_";
580 INPUT U2(I,1)
590 IMAGE 3A,FD,5A,S
600 NEXT I
610 PRINT "J_ENTER THE OMEGA INCREMENT"
620 PRINT "J_G_ OMEGA=";
630 INPUT W1
640 PRINT "J_ ENTER THE FINAL OMEGA, OMEGA MAX"
650 PRINT "J_G_ OMAX=";
```

PROGRAM 2 (Continued)

```
660 INPUT W2
670 GO TO 910
680 PRINT "J_      DATA FROM DATA FILE G_"
690 CALL "OPEN","FEIJ014 LFN:1 TACCESS:FULL"
700 READ #1,1:N
710 DIM A(N,N),A2(N,N),C(N,N),D1(N,1),D2(N,1),U1(N,1),U2(N,1)
720 DIM F(N,1),B(N,1)
730 FOR I=1 TO N
740 FOR J=1 TO N
750 READ #1:A(I,J)
760 NEXT J
770 NEXT I
780 FOR I=1 TO N
790 READ #1:U1(I,1)
800 NEXT I
810 FOR I=1 TO N
820 READ #1:U2(I,1)
830 NEXT I
840 READ #1:W1
850 READ #1:W2
860 GO TO 910
870 REM: FUNCTIONS FOR INPUTS
880 REM.
890 PRINT "J_ CLOSED FORM FUNCTIONS  G_"
900 REM: GET YOUR DATA FROM FUNCTIONS
910 PRINT "J_ THE MAIN PROGRAM  J_G_"
920 M=W2/W1
930 DIM E(M,N),H(M,N),W(M)
940 FOR L=1 TO M
950 W(L)=W1*L
960 A2=A MPY A
970 FOR I=1 TO N
980 A2(I,I)=A2(I,I)+W(L)*W(L)
990 NEXT I
```

PROGRAM 2 (Continued)

```
1000 C=INV(A2)
1010 D1=0
1020 D2=0
1030 D1=A MPY U1
1040 D2=A MPY U2
1050 FOR I=1 TO N
1060 D1(I,1)=-D1(I,1)+W(L)*U2(I,1)
1070 D2(I,1)=-D2(I,1)-W(L)*U1(I,1)
1080 NEXT I
1090 F=0
1100 B=0
1110 F=C MPY D1
1120 B=C MPY D2
1130 FOR I=1 TO N
1140 E(L,I)=SQR(F(I,1)*F(I,1)+B(I,1)*B(I,1))
1150 SET DEGREES
1152 IF F(I,1)=0 THEN 1162
1155 IF B(I,1)=0 THEN 1166
1160 H(L,I)=ATN(B(I,1)/F(I,1))
1161 GO TO 1170
1162 H(L,I)=90
1163 IF B(I,1)>0 THEN 1200
1164 H(L,I)=-90
1165 GO TO 1200
1166 H(L,I)=0
1167 IF F(I,1)>0 THEN 1200
1168 H(L,I)=-180
1169 GO TO 1200
1170 IF B(I,1)<0 AND F(I,1)<0 THEN 1190
1171 IF B(I,1)>0 AND F(I,1)<0 THEN 1190
1180 GO TO 1200
1190 H(L,I)=H(L,I)-180
1200 NEXT I
1210 NEXT L
1220 PAGE
1230 PRINT "OMEGA   ";
1240 FOR I=1 TO N
1250 PRINT "    ";"A";I;"     ";
1260 NEXT I
1270 PRINT "J_J_"
```

PROGRAM 2 (Continued)

```
1280 FOR I=1 TO M
1290 PRINT USING 1295:W(I)
1295 IMAGE 5D,X,S
1300 FOR J=1 TO N
1310 PRINT USING 1315:E(I,J)
1315 IMAGE 2E, X,S
1320 NEXT J
1330 PRINT
1340 NEXT I
1350 PRINT "J_J_"
1360 PRINT "PRESS[RETURN] TO CONTINUE";
1370 INPUT A$
1380 PAGE
1390 PRINT "OMEGA  ";
1400 FOR I=1 TO N
1410 PRINT " ";"FAY";I;"    ";
1420 NEXT I
1430 PRINT "J_J_"
1440 FOR I=1 TO M
1450 PRINT USING 1295:W(I)
1460 FOR J=1 TO N
1470 PRINT USING 1475:H(I,J)
1475 IMAGE 4D.2D,X,S
1480 NEXT J
1490 PRINT
1500 NEXT I
1510 FOR J=1 TO N
1520 PRINT "J_J_"
1530 PRINT " PRESS [RETURN] TO CONTINUE";
1540 INPUT A$
1550 PAGE
1560 VIEWPORT 10,120,20,90
1570 WINDOW 0,1000,0,100
1580 AXIS 50,5
1590 MOVE 0,0
```

PROGRAM 2 (Continued)

```
1600 FOR I=1 TO M
1610 DRAW W(I),E(I,J)
1620 NEXT I
1630 HOME
1640 PRINT "I_      AMPLITUTE      A";J;"      VERSUS OMEGA ";
1650 PRINT @32,21:0,4
1660 PRINT "                      PRESS [RETURN] TO CONTINUE ";
1670 INPUT A$
1680 NEXT J
1690 FOR J=1 TO N
1700 PRINT "J_J_"
1710 INPUT A$
1720 PAGE
1730 VIEWPORT 10,120,20,90
1740 WINDOW 50,1000,-270,90
1750 AXIS 50,10
1760 MOVE 0,0
1770 FOR I=1 TO M
1780 DRAW W(I),H(I,J)
1790 NEXT I
1800 HOME
1810 PRINT "I_          PHASE ANGLE  FAY(";J;")VERSUS OMEGA";
1820 PRINT @32,21:0,4
1830 PRINT "                PRESS [RETURN] TO CONTINUE";
1840 INPUT A$
1850 NEXT J
1860 IF K2<3 THEN 1960
1870 REM:FILE SUBROUTINE FOR OUTPUT
1880 CALL "OPEN","FEIJO15 LFN:2 TACCESS:FULL"
1890 WRITE #2,1:N,M
1900 FOR I=1 TO M
1910 WRITE #2:W(I)
1920 FOR J=1 TO N
1930 WRITE #2:E(I,J),H(I,J)
1940 NEXT J
END
```

PROGRAM 3

COMPUTER PROGRAM FOR SYSTEM MATRIX IDENTIFICATION

```
100 INIT
110 PAGE
120 PRINT "THE TYPE OF DATA"
130 PRINT "K1=1 FOR DIRECT INPUT"
140 PRINT "K1=2 FOR FUNCTIONS"
150 PRINT "K1=3 FOR DATA FILE"
160 PRINT "ENTER THE TYPE OF DATA K1";
170 INPUT K1
180 PRINT "TYPE OF OUTPUT REQUIRED"
190 PRINT "K2=1 FOR TABLES"
200 PRINT "K2=2 FOR GRAPHS"
210 PRINT "K2=3 FOR OUTPUT FILE"
220 PRINT "ENTER THE TYPE OF OUTPUT REQUIRED K2";
230 INPUT K2
240 IF K1=1 THEN 270
250 IF K1=2 THEN 780
260 IF K1=3 THEN 590
270 PRINT "DATA FROM DIRECT INPUT"
280 PRINT "ENTER THE DIMENSION SIZE N";
290 INPUT N
300 DIM A(N,N),A2(N,N),C(N,N),D1(N,1),D2(N,1),U1(N,1),U2(N,1)
310 DIM F(N,1),B(N,1)
320 PRINT "ENTER THE MATRIX A"
330 FOR I=1 TO N
340 FOR J=1 TO N
350 PRINT USING 370:"ENTER A(",I,",",J,")";
360 INPUT A(I,J)
370 IMAGE 8A,FD,1A,FD,6A,S
380 NEXT J
390 NEXT I
400 PRINT "ENTER THE MATRIX U1"
410 FOR I=1 TO N
420 PRINT USING 440:"U1(",I,")";
430 INPUT U1(I,1)
440 IMAGE 3A,FD,6A,S
450 NEXT I
460 PRINT "ENTER THE MATRIX U2"
470 FOR I=1 TO N
480 PRINT USING 500:"U2(",I,")";
490 INPUT U2(I,1)
```

PROGRAM 3 (Continued)

```
500 IMAGE 3A,FD,5A,S
510 NEXT I
520 PRINT "ENTER THE OMEGA INCREMENT"
530 PRINT "OMEGA";
540 INPUT W1
550 PRINT "ENTER THE FINAL OMEGA, OMEGA MAX"
560 PRINT "OMEGAMAX=";
570 INPUT W2
580 GO TO 820
590 PRINT "DATA FROM DATA FILE"
600 CALL "OPEN","FEIJO14 LFN:1 TACCESS:FULL"
610 READ #1,1:N
620 DIM A(N,N),A2(N,N),C(N,N),D1(N,1),D2(N,1),U1(N,1),U2(N,1)
630 DIM F(N,1),B(N,1)
640 FOR I=1 TO N
650 FOR J=1 TO N
660 READ #1:A(I,J)
670 NEXT J
680 NEXT I
690 FOR I=1 TO N
700 READ #1:U1(I,1)
710 NEXT I
720 FOR I=1 TO N
730 READ #1:U2(I,1)
740 NEXT I
750 READ #1:W1
760 READ #1:W2
770 GO TO 820
780 REM: FUNCTIONS FOR INPUTS
790 REM:
800 PRINT "CLOSED FORM FUNCTIONS"
810 REM: GET YOUR DATA FROM FUNCTIONS
820 PRINT "THE PROGRAM"
830 M=W2/W1
840 DIM E(N,M),H(N,M),W(M),A3(N,M),A4(N,M),T1(N,M)
844 T1=0
848 FOR T=1 TO 100
850 FOR L=1 TO M
860 W(L)=W1*L
870 A2=A MPY A
```

PROGRAM 3 (Continued)

```
880 FOR I=1 TO N
890 A2(I,I)=A2(I,I)+W(L)*W(L)
900 NEXT I
910 C=INV(A2)
920 D1=0
930 D2=0
940 D1=A MPY U1
950 D2=A MPY U2
960 FOR I=1 TO N
970 D1(I,1)=-D1(I,1)+W(L)*U2(I,1)
980 D2(I,1)=-D2(I,1)-W(L)*U1(I,1)
990 NEXT I

1000 F=0
1010 B=0
1020 F=C MPY D1
1030 B=C MPY D2
1040 Z5=RND(-1)
1050 FOR I=1 TO N
1060 E(I,L)=(F(I,1)*W(L)-U2(I,1))*(1+(RND(1)-0.5)/100)
1070 H(I,L)=B(I,1)
1090 NEXT I
1100 NEXT L
1110 A3=INV(H)
1120 A4=E MPY A3
1130 T1=T1+A4
1135 PRINT T;" ... ";
1140 NEXT T
1150 T1=T1/100
1160 PRINT "L_THE AVERAGED MATRIX=J_";T1
1170 STOP
1180 END
```

TABLE 1

DAMPED NON-GYROSCOPIC SYSTEM EIGENVALUES AND EIGENVECTORS

EIGENVALUES

W(1)=	-0.44094D 01	0.0	I
W(2)=	-0.42305D 00	0.11699D 01	I
W(3)=	-0.42305D 00	-0.11699D 01	I
W(4)=	-0.23445D 01	0.0	I

EIGENVECTORS

-0.77938D 00	0.0	I	0.12238D 00	0.49501D 00	I	0.12238D 00	-0.49501D 00	I	0.39496D 01	0.0	I
0.58622D 00	0.0	I	0.25752D 00	0.74178D 00	I	0.25752D 00	-0.74178D 00	I	-0.34546D 01	0.0	I
0.17675D 00	0.0	I	0.34073D 00	-0.22781D 00	I	0.34073D 00	0.22781D 00	I	-0.16846D 01	0.0	I
-0.13295D 00	0.0	I	0.49033D 00	-0.39742D 00	I	0.49033D 00	0.39742D 00	I	0.14735D 01	0.0	I

NORMALIZED EIGENVECTORS

-0.77938D 00	0.0	I	-0.64992D 00	0.18627D 00	I	-0.64992D 00	-0.18627D 00	I	0.69235D 00	0.0	I
0.58622D 00	0.0	I	-0.97031D 00	0.37735D 00	I	-0.97031D 00	-0.37735D 00	I	-0.60557D 00	0.0	I
0.17675D 00	0.0	I	0.31845D 00	0.44037D 00	I	0.31845D 00	-0.44037D 00	I	-0.29531D 00	0.0	I
-0.13295D 00	0.0	I	0.55046D 00	0.63032D 00	I	0.55046D 00	-0.63032D 00	I	0.25830D 00	0.0	I

TABLE 2

DAMPED NON-GYROSCOPIC SYSTEM - ERROR MATRIX EIGENVALUES AND EIGENVECTORS - 0.2% RANDOM ERROR

EIGENVALUES

W(1)=	0.42146D-01	0.47386D-01 I
W(2)=	0.42146D-01	-0.47386D-01 I
W(3)=	0.18084D-02	0.50168D-02 I
W(4)=	0.18084D-02	-0.50168D-02 I

EIGENVECTORS

0.50295D 00	0.68732D 0 I	0.50295D 00	-0.68732D 00 I	-0.34833D 00	0.15454D 01 I	-0.34833D 00	-0.15454D 01 I
0.16082D 00	-0.58037D 0 I	0.16082D 00	0.58037D 00 I	-0.14124D 00	0.17172D 01 I	-0.14124D 00	-0.17172D 01 I
0.72722D-01	0.61495D 0 I	0.72722D-01	-0.61495D 00 I	0.11687D 01	0.87467D 00 I	0.11687D 01	-0.87467D 00 I
0.20937D 00	0.59606D 0 I	0.20937D 00	-0.59606D 00 I	0.16235D 01	0.41860D 00 I	0.16235D 01	-0.41860D 00 I

NORMALIZED EIGENVECTORS

0.66293D 00	-0.22572D 0 I	0.66293D 00	0.22572D 00 I	0.79036D 00	0.51526D 00 I	0.79036D 00	-0.51526D 00 I
-0.41587D 00	-0.26883D 0 I	-0.41587D 00	0.26883D 00 I	0.92927D 00	0.43533D 00 I	0.92927D 00	-0.43533D 00 I
0.50041D 00	0.94076D- 1 I	0.50041D 00	-0.94076D-01 I	0.73089D 00	-0.47084D 00 I	0.73089D 00	0.47084D 00 I
0.51918D 00	-0.17792D- 1 I	0.51918D 00	0.17792D-01 I	0.57071D 00	-0.81940D 00 I	0.57071D 00	0.81940D 00 I

TABLE 3

DAMPED NON-GYROSCOPIC SYSTEM - ERROR MATRIX EIGENVALUES AND EIGENVECTORS - 0.4% RANDOM ERROR

EIGENVALUES

W(1)=	0.26127D-01	0.0	I
W(2)=	0.41906D-02	0.78125D-02	I
W(3)=	0.41906D-02	-0.78125D-02	I
W(4)=	-0.25587D-02	0.0	I

EIGENVECTORS

0.57952D 00	0.0	I	0.59642D 00	0.80440D 00	I	0.59642D 00	-0.80440D 00	I	-0.72136D 01	0.0	I
-0.59949D 00	0.0	I	-0.25139D 01	-0.86796D 00	I	-0.25139D 01	0.86796D 00	I	-0.47229D 01	0.0	I
-0.42314D 00	0.0	I	-0.24675D 01	0.57451D 00	I	-0.24675D 01	-0.57451D 00	I	-0.20611D 01	0.0	I
-0.35457D 00	0.0	I	-0.25421D 01	0.10272D 01	I	-0.25421D 01	-0.10272D 01	I	-0.46873D 00	0.0	I

NORMALIZED EIGENVECTORS

0.57952D 00	0.0	I	0.12938D 00	0.20839D 00	I	0.12938D 00	-0.20839D 00	I	-0.81257D 00	0.0	I
-0.59949D 00	0.0	I	-0.59619D 00	-0.26257D 00	I	-0.59619D 00	0.26257D 00	I	-0.53201D 00	0.0	I
-0.42314D 00	0.0	I	-0.61395D 00	0.90503D-01	I	-0.61395D 00	-0.90503D-01	I	-0.23217D 00	0.0	I
-0.35457D 00	0.0	I	-0.64129D 00	0.19950D 00	I	-0.64129D 00	-0.19950D 00	I	-0.52800D-01	0.0	I

TABLE 4

DAMPED NON-GYROSCOPIC SYSTEM - ERROR MATRIX EIGENVALUES AND EIGENVECTORS - 0.6% RANDOM ERROR

EIGENVALUES

W(1)= -0.124080D 00	0.0	I
W(2)= 0.364920D-01	0.0	I
W(3)= 0.251890D-02	0.0	I
W(4)= -0.111360D-01	0.0	I

EIGENVECTORS

-0.166670D 01	0.0	I	-0.228180D 01	0.0	I	0.195120D 01	0.0	I	0.526730D 00	0.0	I
-0.290070D 00	0.0	I	0.115940D 01	0.0	I	0.351260D 01	0.0	I	-0.426320D 00	0.0	I
-0.293600D 00	0.0	I	0.899930D 00	0.0	I	0.179750D 01	0.0	I	0.193840D 01	0.0	I
-0.904880D 00	0.0	I	0.933070D 00	0.0	I	0.106310D 01	0.0	I	0.282460D 01	0.0	I

NORMALIZED EIGENVECTORS

-0.858730D 00	0.0	I	-0.795340D 00	0.0	I	0.430880D 00	0.0	I	0.150830D 00	0.0	I
-0.149450D 00	0.0	I	0.404100D 00	0.0	I	0.775670D 00	0.0	I	-0.122080D 00	0.0	I
-0.151270D 00	0.0	I	0.313640D 00	0.0	I	0.396940D 00	0.0	I	0.555070D 00	0.0	I
-0.466220D 00	0.0	I	0.325230D 00	0.0	I	0.234750D 00	0.0	I	0.808850D 00	0.0	I

TABLE 5

DAMPED NON-GYROSCOPIC SYSTEM - ERROR MATRIX EIGENVALUES AND EIGENVECTORS -
0.8% RANDOM ERROR

EIGENVALUES

W(1)=	-0.22089D-01	0.46262D-01	I
W(2)=	-0.22089D-01	-0.46262D-01	I
W(3)=	-0.10207D-01	0.0	I
W(4)=	0.92538D-03	0.0	I

EIGENVECTORS

0.14276D 02	0.38062D 1	I	0.14276D 02	-0.38062D 01	I	0.25222D 01	0.0	I	0.13706D 01	0.0	I
0.42315D 01	0.45207D 1	I	0.42315D 01	-0.45207D 01	I	0.32246D 01	0.0	I	0.11185D 01	0.0	I
0.41935D 01	0.42970D 1	I	0.41935D 01	-0.42970D 01	I	0.38545D 01	0.0	I	-0.20663D 01	0.0	I
0.38299D 01	0.30977D 1	I	0.38299D 01	-0.30977D 01	I	0.39634D 01	0.0	I	-0.32273D 01	0.0	I

NORMALIZED EIGENVECTORS

0.87059D 00	-0.13327D 0	I	0.87059D 00	0.13327D 00	I	0.36664D 00	0.0	I	0.32474D 00	0.0	I
0.33911D 00	0.14577D 0	I	0.33911D 00	-0.14577D 00	I	0.46874D 00	0.0	I	0.26500D 00	0.0	I
0.33169D 00	0.13446D 0	I	0.33169D 00	-0.13446D 00	I	0.56030D 00	0.0	I	-0.48956D 00	0.0	I
0.28318D 00	0.77654D- 1	I	0.28318D 00	-0.77654D-01	I	0.57613D 00	0.0	I	-0.76463D 00	0.0	I

TABLE 6

DAMPED NON-GYROSCOPIC SYSTEM - ERROR MATRIX EIGENVALUES AND EIGENVECTORS - 1.0% RANDOM ERROR

EIGENVALUES

W(1)=	0.18939D 00	0.0	I
W(2)=	0.68012D-02	0.17049D-01	I
W(3)=	0.68012D-02	-0.17049D-01	I
W(4)=	0.19382D-02	0.0	I

EIGENVECTORS

0.11542D 01	0.0	I	0.21497D 01	-0.32496D 01	I	0.21497D 01	0.32496D 01	I	0.64900D 00	0.0	I
-0.79758D 00	0.0	I	0.23093D 01	0.26072D 01	I	0.23093D 01	-0.26072D 01	I	-0.28050D 01	0.0	I
0.40437D-01	0.0	I	0.27354D 00	0.10279D 01	I	0.27354D 00	-0.10279D 01	I	0.25848D 01	0.0	I
-0.59752D 00	0.0	I	-0.67621D 00	0.10151D 01	I	-0.67621D 00	-0.10151D 01	I	0.41021D 01	0.0	I

NORMALIZED EIGENVECTORS

0.75663D 00	0.0	I	0.11539D 01	0.53762D 00	I	0.11539D 01	-0.53762D 00	I	0.11509D 00	0.0	I
-0.52286D 00	0.0	I	-0.73088D 00	0.87219D 00	I	-0.73088D 00	-0.87219D 00	I	-0.49743D 00	0.0	I
0.26508D-01	0.0	I	-0.31892D 00	0.13806D 00	I	-0.31892D 00	-0.13806D 00	I	0.45838D 00	0.0	I
-0.39171D 00	0.0	I	-0.36070D 00	-0.16945D 00	I	-0.36070D 00	0.16945D 00	I	0.72746D 00	0.0	I

TABLE 7

CRITICALLY DAMPED NON-GYROSCOPIC SYSTEM EIGENVALUES AND EIGENVECTORS

EIGENVALUES

```
W( 1 )= -0.40312D 00   0.11762D 01 I
W( 2 )= -0.40312D 00  -0.11762D 01 I
W( 3 )= -0.32199D 01   0.0         I
W( 4 )= -0.32141D 01   0.0         I
```

EIGENVECTORS

0.70228D 00	-0.28605D 00 9	0.70228D 00	0.28605D 00 9	-0.75327D 00	0.0	9	-0.14901D 04	0.0	9
0.10423D 01	-0.53707D 00 9	0.10423D 01	0.53707D 00 9	0.59816D 00	0.0	9	0.11837D 04	0.0	9
-0.40074D 00	-0.45971D 00 9	-0.40074D 00	0.45971D 00 9	0.23395D 00	0.0	9	0.46360D 03	0.0	9
-0.68037D 00	-0.65294D 00 9	-0.68037D 00	0.65294D 00 9	-0.18577D 00	0.0	9	-0.36829D 03	0.0	9

NORMALIZED EIGENVECTORS

0.65910D 00	-0.18806D 00 9	0.65910D 00	0.18806D 00 9	-0.74789D 00	0.0	9	-0.74765D 00	0.0	9
0.98925D 00	-0.38021D 00 9	0.98925D 00	0.38021D 00 9	0.59388D 00	0.0	9	0.59394D 00	0.0	9
-0.31493D 00	-0.45241D 00 9	-0.31493D 00	0.45241D 00 9	0.23227D 00	0.0	9	0.23262D 00	0.0	9
-0.54720D 00	-0.65349D 00 9	-0.54720D 00	0.65349D 00 9	-0.18444D 00	0.0	9	-0.18479D 00	0.0	9

TABLE 8

UNDERDAMPED NON-GYROSCOPIC SYSTEM EIGENVALUES AND EIGENVECTORS

EIGENVALUES

```
W(  1 )= -0.33758D 00      0.32182D 01 I
W(  2 )= -0.33758D 00     -0.32182D 01 I
W(  3 )= -0.42418D-01      0.12354D 01 I
W(  4 )= -0.42418D-01     -0.12354D 01 I
```

EIGENVECTORS

-0.15369D 01	-0.14098D 01 9	-0.15369D 01	0.14098D 01 9	-0.49141D 00	-0.35279D 00 9	-0.49141D 00	0.35279D 00 9
0.92751D 00	0.90121D 00 9	0.92751D 00	-0.90121D 00 9	-0.80093D 00	-0.56171D 00 9	-0.80093D 00	0.56171D 00 9
-0.38375D 00	0.51781D 00 9	-0.38375D 00	-0.51781D 00 9	-0.27158D 00	0.40709D 00 9	-0.27158D 00	-0.40709D 00 9
0.24709D 00	-0.31413D 00 9	0.24709D 00	0.31413D 00 9	-0.43190D 00	0.66313D 00 9	-0.43190D 00	-0.66313D 00 9

NORMALIZED EIGENVECTORS

-0.89246D 00	0.16872D-01 9	-0.89246D 00	-0.16872D-01 9	-0.88759D 00	0.50036D-01 9	-0.88759D 00	-0.50036D-01 9
0.55349D 00	0.54314D-02 9	0.55349D 00	-0.54314D-02 9	-0.14344D 01	0.96751D-01 9	-0.14344D 01	-0.96751D-01 9
0.33959D-01	0.27376D 00 9	0.33959D-01	-0.27376D 00 9	0.65092D-01	0.71621D 00 9	0.65092D-01	-0.71621D 00 9
-0.16176D-01	-0.17029D 00 9	-0.16176D-01	0.17029D 00 9	0.11804D 00	0.11570D 01 9	0.11804D 00	-0.11570D 01 9

TABLE 9

CONSERVATIVE NON-GYROSCOPIC SYSTEM EIGENVALUES AND EIGENVECTORS

EIGENVALUES

W(1)=	-0.10119D-13	0.84508D 03 I
W(2)=	-0.10119D-13	-0.84508D 03 I
W(3)=	-0.47545D-13	0.73511D 03 I
W(4)=	-0.47545D-13	-0.73511D 03 I
W(5)=	-0.19904D-13	0.19641D 03 I
W(6)=	-0.19904D-13	-0.19641D 03 I
W(7)=	-0.68141D-13	0.28429D 03 I
W(8)=	-0.68141D-13	-0.28429D 03 I

EIGENVECTORS

0.84100D 03	-0.20804D- 1 I	0.84100D 03	0.20804D-11 I	0.31120D 02	-0.17725D-11 I	0.31120D 02	0.17725D-11 I
-0.29114D 02	-0.24041D- 2 I	-0.29114D 02	0.24041D-12 I	0.52530D 02	-0.68553D-12 I	0.52530D 02	0.68553D-12 I
-0.78411D 02	0.20214D- 2 I	-0.78411D 02	-0.20214D-12 I	0.27695D 02	-0.14242D-11 I	0.27695D 02	0.14242D-11 I
-0.10857D 03	-0.80503D- 2 I	-0.10857D 03	0.80503D-12 I	0.18334D 03	-0.26458D-11 I	0.18334D 03	0.26458D-11 I
0.25923D 02	-0.59002D- 3 I	0.25923D 02	0.59002D-13 I	-0.67645D 03	0.34254D-10 I	-0.67645D 03	-0.34254D-10 I
-0.12977D 03	-0.81242D- 2 I	-0.12977D 03	0.81242D-12 I	-0.14959D 03	0.20246D-11 I	-0.14959D 03	-0.20246D-11 I
-0.72509D 01	0.12091D- 3 I	-0.72509D 01	-0.12091D-13 I	0.28476D 03	-0.14372D-10 I	0.28476D 03	0.14372D-10 I
-0.11922D 03	-0.65740D- 2 I	-0.11922D 03	0.65740D-12 I	-0.18772D 03	0.26842D-11 I	-0.18772D 03	-0.26842D-11 I
-0.24590D-14	-0.99518D 0 I	-0.24590D-14	0.99518D 00 I	-0.24210D-14	-0.42334D-01 I	-0.24210D-14	0.42334D-01 I
-0.10663D-14	0.14823D 0 I	-0.10663D-14	-0.14823D 00 I	-0.25865D-14	-0.18477D 00 I	-0.25865D-14	0.18477D 00 I
0.24297D-15	0.92786D- 1 I	0.24297D-15	-0.92786D-01 I	-0.22114D-14	-0.37675D-01 I	-0.22114D-14	0.37675D-01 I
-0.38685D-14	0.55276D 0 I	-0.38685D-14	-0.55276D 00 I	-0.92448D-14	-0.64489D 00 I	-0.92448D-14	0.64489D 00 I
-0.73544D-16	-0.30675D- 1 I	-0.73544D-16	0.30675D-01 I	0.46784D-13	0.92021D 00 I	0.46784D-13	-0.92021D 00 I
-0.40334D-14	0.66072D 0 I	-0.40334D-14	-0.66072D 00 I	0.72250D-14	0.52618D 00 I	0.72250D-14	-0.52618D 00 I
0.17551D-16	0.85801D- 2 I	0.17551D-16	-0.85801D-02 I	-0.19633D-13	-0.38737D 00 I	-0.19633D-13	0.38737D 00 I
-0.33999D-14	0.60698D 0 I	-0.33999D-14	-0.60698D 00 I	0.94850D-14	0.66029D 00 I	0.94850D-14	-0.66029D 00 I

NORMALIZED EIGENVECTORS

0.99518D 00	0.64938D- 8 I	0.99518D 00	-0.64938D-18 I	0.42333D-01	-0.26810D-15 I	0.42333D-01	0.26810D-15 I
-0.13929D 00	-0.26322D- 5 I	-0.13929D 00	0.26322D-15 I	0.17135D 00	0.18464D-15 I	0.17135D 00	-0.18464D-15 I
-0.92786D-01	0.96160D- 7 I	-0.92786D-01	-0.96160D-17 I	0.37674D-01	-0.30150D-16 I	0.37674D-01	0.30150D-16 I
-0.51943D 00	-0.54386D- 5 I	-0.51943D 00	0.54386D-15 I	0.59802D 00	-0.19168D-15 I	0.59802D 00	0.18168D-15 I
0.30675D-01	0.60827D- 7 I	0.30675D-01	-0.60827D-17 I	-0.92019D 00	0.11463D-16 I	-0.92019D 00	-0.11463D-16 I
-0.62087D 00	0.66757D- 6 I	-0.62087D 00	-0.66757D-16 I	-0.48795D 00	-0.28951D-15 I	-0.48795D 00	0.28951D-15 I
-0.85801D-02	-0.69227D- 7 I	-0.85801D-02	0.69227D-17 I	0.38736D 00	0.59462D-16 I	0.38736D 00	-0.59462D-16 I
-0.57038D 00	0.48690D- 5 I	-0.57038D 00	-0.48690D-15 I	-0.61231D 00	0.10494D-15 I	-0.61231D 00	-0.10494D-15 I
0.41063D-20	-0.11776D- 2 I	0.41063D-20	0.11776D-02 I	-0.37790D-18	-0.57588D-04 I	-0.37790D-18	0.57588D-04 I
-0.58577D-18	0.70917D- 3 I	-0.58577D-18	-0.70917D-03 I	0.78020D-19	-0.60272D-03 I	0.78020D-19	0.60272D-03 I
0.15839D-19	0.10980D- 3 I	0.15839D-19	-0.10980D-03 I	-0.41364D-18	-0.51250D-04 I	-0.41364D-18	0.51250D-04 I
-0.16678D-17	0.26446D- 2 I	-0.16678D-17	-0.26446D-02 I	-0.43751D-18	-0.21035D-02 I	-0.43751D-18	0.21035D-02 I
0.27891D-20	-0.36299D- 4 I	0.27891D-20	0.36299D-04 I	0.27052D-18	0.12518D-02 I	0.27052D-18	-0.12518D-02 I
0.83242D-18	0.31611D- 2 I	0.83242D-18	-0.31611D-02 I	-0.68095D-18	0.17164D-02 I	-0.68095D-18	-0.17164D-02 I
-0.43540D-20	0.10153D- 4 I	-0.43540D-20	-0.10153D-04 I	-0.31403D-19	-0.52694D-03 I	-0.31403D-19	0.52694D-03 I
0.22261D-17	0.29040D- 2 I	0.22261D-17	-0.29040D-02 I	0.51078D-18	0.21538D-02 I	0.51078D-18	-0.21538D-02 I

TABLE 10

CONSERVATIVE NON-GYROSCOPIC SYSTEM - ERROR MATRIX EIGENVALUES AND EIGENVECTORS
WITH 0.005% RANDOM ERROR

EIGENVALUES

W(1)=	-0.49316D 03	0.0
W(2)=	0.76432D 01	0.0
W(3)=	-0.15948D 01	0.34858D 00
W(4)=	-0.15948D 01	-0.34858D 00
W(5)=	-0.38645D 00	0.0
W(6)=	0.28631D 00	0.0
W(7)=	0.12157D-02	0.33498D-01
W(8)=	0.12157D-02	-0.33498D-01

EIGENVECTORS

-0.76780D 03	0.0	0.11675D 04	0.0	-0.18058D 04	-0.57872D 04	-0.18058D 04	0.57872D 04
0.12023D 04	0.0	-0.11928D 04	0.0	-0.12231D 03	0.19205D 03	-0.12231D 03	-0.19205D 03
0.10608D 03	0.0	-0.12886D 03	0.0	0.26618D 03	0.60143D 03	0.26618D 03	-0.60143D 03
0.17921D 03	0.0	-0.29001D 03	0.0	-0.12203D 02	0.47445D 02	-0.12203D 02	-0.47445D 02
0.85561D 01	0.0	0.81953D 01	0.0	0.10923D 03	-0.25231D 03	0.10923D 03	0.25231D 03
-0.58814D 02	0.0	-0.54657D 02	0.0	-0.25466D 02	0.18688D 02	-0.25466D 02	-0.18688D 02
-0.27739D 01	0.0	-0.25724D 01	0.0	-0.28349D 02	0.76375D 02	-0.28349D 02	-0.76375D 02
0.30928D 02	0.0	0.23165D 02	0.0	-0.28023D 02	-0.25702D 02	-0.28023D 02	0.25702D 02
-0.52563D 00	0.0	0.98993D 00	0.0	0.15625D 01	-0.14972D 02	0.15625D 01	0.14972D 02
-0.19821D 00	0.0	0.26733D 01	0.0	0.51632D 00	-0.13334D 01	0.51632D 00	0.13334D 01
0.63663D-01	0.0	-0.32699D-02	0.0	0.50394D 00	-0.34329D-01	0.50394D 00	0.34329D-01
0.99383D-01	0.0	0.70943D-01	0.0	0.10754D 00	-0.41996D 00	0.10754D 00	0.41996D 00
-0.36025D-01	0.0	0.45183D-01	0.0	0.72203D-01	-0.43479D 00	0.72203D-01	0.43479D 00
-0.16896D 00	0.0	0.41092D-01	0.0	-0.75339D-01	-0.17552D 00	-0.75330D-01	0.17552D 00
-0.23364D-01	0.0	0.19300D-01	0.0	-0.13302D 00	0.28387D-01	-0.13302D 00	-0.28387D-01
-0.25570D-01	0.0	0.11654D 00	0.0	-0.94984D-01	-0.18506D 00	-0.94984D-01	0.18506D 00

NORMALIZED EIGENVECTORS

-0.99052D 00	0.0	0.99394D 00	0.0	-0.99414D 00	-0.22189D-03	-0.99414D 00	0.22189D-03
0.98717D 00	0.0	-0.97055D 00	0.0	-0.98732D 00	-0.11501D-01	-0.98732D 00	0.11501D-01
0.13686D 00	0.0	-0.10971D 00	0.0	0.10715D 00	-0.12266D-01	0.10715D 00	0.12266D-01
0.14715D 00	0.0	-0.23599D 00	0.0	-0.20272D 00	0.63533D-01	-0.20272D 00	-0.63533D-01
0.11038D-01	0.0	0.69773D-02	0.0	-0.34155D-01	-0.29431D-01	-0.34155D-01	0.29431D-01
-0.56503D-01	0.0	-0.44475D-01	0.0	-0.12709D 00	-0.51100D-01	-0.12709D 00	0.51100D-01
-0.35785D-02	0.0	-0.21903D-02	0.0	0.10569D-01	0.81708D-02	0.10569D-01	-0.81708D-02
0.25395D-01	0.0	0.18843D-01	0.0	0.30623D-01	-0.16203D 00	0.30623D-01	0.16203D 00
-0.67811D-03	0.0	0.84277D-03	0.0	-0.22672D-02	-0.97646D-03	-0.22672D-02	0.97646D-03
-0.16275D-03	0.0	0.21752D-02	0.0	0.60937D-02	-0.11467D-02	0.60937D-02	0.11467D-02
0.82131D-04	0.0	-0.27839D-05	0.0	0.19260D-04	-0.80560D-04	0.19260D-04	0.80560D-04
0.81602D-04	0.0	0.57726D-04	0.0	0.17933D-02	-0.56411D-03	0.17933D-02	0.56411D-03
-0.46475D-04	0.0	0.38457D-04	0.0	-0.64529D-04	-0.32556D-04	-0.64529D-04	0.32556D-04
-0.13873D-03	0.0	0.33423D-04	0.0	0.47448D-03	-0.67894D-03	0.47448D-03	0.67894D-03
-0.30141D-04	0.0	0.16431D-04	0.0	-0.20592D-05	0.22210D-04	-0.20592D-05	-0.22210D-04
-0.20996D-04	0.0	0.99453D-04	0.0	0.46468D-03	-0.77317D-03	0.46468D-03	0.77317D-03

TABLE 11

CONSERVATIVE NON-GYROSCOPIC SYSTEM - ERROR MATRIX EIGENVALUES AND EIGENVECTORS WITH 0.0005% RANDOM ERROR

EIGENVALUES

W(1)=	0.15542D 01	0.28017D 01
W(2)=	0.15542D 01	-0.28017D 01
W(3)=	0.31358D-01	0.0
W(4)=	0.60609D-02	0.0
W(5)=	-0.21635D-02	0.0
W(6)=	0.15501D-02	0.16339D-02
W(7)=	0.15501D-02	-0.16339D-02
W(8)=	-0.20521D-03	0.0

EIGENVECTORS

0.32012D 03	-0.46783D 4	0.32012D 03	0.46783D 04	0.19013D 03	0.0	-0.55259D 04	0.0
0.23647D 04	0.0	0.12579D 04	-0.15318D 05	0.12579D 04	0.15318D 05	-0.10112D 04	0.0
-0.52652D 02	0.10129D 3	-0.52652D 02	-0.10129D 03	0.64825D 02	0.0	-0.21355D 03	0.0
-0.21853D 03	0.0	-0.25442D 04	0.38845D 04	-0.25442D 04	-0.38845D 04	0.34388D 03	0.0
-0.28042D 02	0.12284D 3	-0.28042D 02	-0.12294D 03	-0.81841D 02	0.0	-0.22742D 03	0.0
-0.47446D 03	0.0	-0.16101D 04	0.77471D 01	-0.16101D 04	-0.77471D 01	-0.28955D 03	0.0
0.74607D 01	-0.51657D 2	0.74607D 01	0.51657D 02	0.26543D 02	0.0	0.10212D 02	0.0
0.79780D 02	0.0	0.10409D 04	0.16312D 03	0.10409D 04	-0.16312D 03	-0.16865D 03	0.0
0.19931D 00	-0.41206D 1	0.19931D 00	0.41206D 01	-0.33657D 01	0.0	-0.46495D 01	0.0
0.17217D 01	0.0	-0.80769D 01	-0.36159D 02	-0.80769D 01	0.36159D 02	-0.51896D 01	0.0
-0.14686D 00	0.13325D 1	-0.14686D 00	-0.13325D 01	-0.59214D 00	0.0	-0.94277D 00	0.0
-0.10739D 01	0.0	-0.15627D 02	0.26621D 01	-0.15627D 02	-0.26621D 01	-0.13095D 01	0.0
-0.88489D-01	0.43629D 0	-0.88489D-01	-0.43629D 00	-0.29346D 00	0.0	-0.61764D 00	0.0
-0.12298D 01	0.0	-0.63903D 01	-0.18975D 00	-0.63903D 01	0.18975D 00	-0.16059D 01	0.0
0.35305D-03	0.27661D- 1	0.35305D-03	-0.27661D-01	-0.80865D-01	0.0	-0.22354D 00	0.0
-0.37252D 00	0.0	-0.11163D 01	0.34109D-01	-0.11163D 01	-0.34109D-01	-0.12828D 01	0.0

NORMALIZED EIGENVECTORS

0.99939D 00	-0.33686D- 3	0.99939D 00	0.33696D-03	0.85994D 00	0.0	-0.99841D 00	0.0
0.97543D 00	0.0	0.98125D 00	-0.36862D-01	0.98125D 00	0.36862D-01	-0.90333D 00	0.0
-0.22307D-01	-0.97140D- 2	-0.22307D-01	0.97140D-02	0.29661D 00	0.0	-0.38583D-01	0.0
-0.90190D-01	0.0	-0.26579D 00	-0.13181D 00	-0.26579D 00	0.13181D 00	0.30720D 00	0.0
-0.26530D-01	-0.41748D- 2	-0.26530D-01	0.41748D-02	-0.37447D 00	0.0	-0.41090D-01	0.0
-0.19582D 00	0.0	-0.12754D-01	-0.10208D 00	-0.12754D-01	0.10208D 00	-0.25866D 00	0.0
0.11093D-01	0.83102D- 3	0.11093D-01	-0.83102D-03	0.12145D 00	0.0	0.18450D-02	0.0
0.32926D-01	0.0	-0.24203D-02	0.67271D-01	-0.24203D-02	-0.67271D-01	-0.15066D 00	0.0
0.87906D-03	-0.17870D- 4	0.87906D-03	0.17870D-04	-0.15400D-01	0.0	-0.84005D-03	0.0
0.71659D-03	0.0	0.22322D-02	-0.78774D-03	0.22322D-02	0.78774D-03	-0.46361D-02	0.0
-0.28546D-03	-0.11743D- 4	-0.28546D-03	0.11743D-04	-0.26637D-02	0.0	-0.17034D-03	0.0
-0.44323D-03	0.0	-0.28789D-03	-0.97103D-03	-0.28789D-03	0.97103D-03	-0.11698D-02	0.0
-0.94059D-04	-0.12436D- 4	-0.94059D-04	0.12436D-04	-0.13427D-02	0.0	-0.11159D-03	0.0
-0.51757D-03	0.0	-0.36631D-04	-0.40681D-03	-0.36631D-04	0.40681D-03	-0.14346D-02	0.0
-0.58762D-05	0.47950D- 6	-0.58762D-05	-0.47950D-06	-0.37000D-03	0.0	-0.40389D-04	0.0
-0.15374D-03	0.0	-0.19665D-04	-0.70550D-04	-0.19665D-04	0.70550D-04	-0.11460D-02	0.0

TABLE 12

CONSERVATIVE NON-GYROSCOPIC SYSTEM - ERROR MATRIX EIGENVALUES AND EIGENVECTORS
WITH 0.00005% RANDOM ERROR

EIGENVALUES

W(1)=	0.103740D 02	0.0
W(2)=	-0.16448D 00	0.0
W(3)=	-0.19366D-01	0.0
W(4)=	0.10632D-01	0.0
W(5)=	0.55431D-02	0.0
W(6)=	-0.12404D-02	0.65366D-03
W(7)=	-0.12404D-02	-0.65366D-03
W(8)=	0.16583D-02	0.0

EIGENVECTORS

0.64897D 03	0.0	-0.12538D 03	0.0	0.66043D 03	0.0	0.61600D 03	0.0
0.25023D 04	0.0	-0.66169D 04	0.33490D 04	-0.66169D 04	-0.33490D 04	0.25705D 04	0.0
0.42529D 02	0.0	-0.10154D 03	0.0	-0.17944D 03	0.0	0.15136D 02	0.0
-0.25527D 02	0.0	-0.83913D 02	-0.48051D 02	-0.83913D 02	0.48051D 02	0.11971D 03	0.0
0.60811D 02	0.0	-0.15211D 02	0.0	0.95086D 01	0.0	0.56191D 01	0.0
-0.93180D 01	0.0	-0.16177D 02	-0.34073D 02	-0.16177D 02	0.34073D 02	0.28472D 02	0.0
0.53761D 00	0.0	-0.11074D 02	0.0	-0.12216D 02	0.0	0.10793D 02	0.0
0.44591D 02	0.0	-0.16568D 01	0.11740D 03	-0.16568D 01	-0.11740D 03	-0.33958D 02	0.0
0.22424D 01	0.0	-0.25285D 01	0.0	-0.21002D 01	0.0	0.11519D 01	0.0
0.43260D 01	0.0	-0.94664D 01	-0.46990D 01	-0.94664D 01	0.46990D 01	-0.42730D 01	0.0
-0.15111D 00	0.0	0.24474D 00	0.0	-0.27124D 00	0.0	-0.22108D 00	0.0
-0.48128D 00	0.0	-0.32159D 01	-0.23075D 01	-0.32159D 01	0.23075D 01	-0.26466D 01	0.0
-0.53856D-01	0.0	0.65590D-01	0.0	0.10858D 00	0.0	-0.14283D 00	0.0
-0.33612D 00	0.0	-0.11473D 01	-0.55689D 00	-0.11473D 01	0.55689D 00	-0.10852D 01	0.0
0.20229D-01	0.0	0.20372D-01	0.0	0.79754D-01	0.0	-0.99316D-01	0.0
-0.18240D 00	0.0	-0.76598D 00	-0.82965D-01	-0.76598D 00	0.82965D-01	-0.84502D 00	0.0

NORMALIZED EIGENVECTORS

0.99352D 00	0.0	-0.77180D 00	0.0	0.96476D 00	0.0	0.99950D 00	0.0
0.99978D 00	0.0	-0.10001D 01	-0.37672D-04	-0.10001D 01	0.37672D-04	0.99877D 00	0.0
0.65105D-01	0.0	-0.62534D 00	0.0	-0.26213D 00	0.0	0.24559D-01	0.0
-0.10199D-01	0.0	-0.61556D-02	-0.10937D-01	-0.61556D-02	0.10937D-01	0.46514D-01	0.0
0.93098D-01	0.0	-0.93633D-01	0.0	0.13890D-01	0.0	0.91174D-02	0.0
-0.37229D-02	0.0	0.46359D-03	-0.48733D-02	0.46359D-03	0.48733D-02	0.11063D-01	0.0
0.82304D-03	0.0	-0.68146D-01	0.0	-0.17846D-01	0.0	0.17512D-01	0.0
0.17816D-01	0.0	-0.79938D-02	0.12974D-01	-0.79938D-02	-0.12974D-01	-0.13194D-01	0.0
0.34329D-02	0.0	-0.15565D-01	0.0	-0.39679D-02	0.0	0.18690D-02	0.0
0.17284D-02	0.0	-0.74244D-03	-0.11534D-02	-0.74244D-03	0.11534D-02	-0.16602D-02	0.0
-0.23133D-03	0.0	0.15964D-02	0.0	-0.39624D-03	0.0	-0.35873D-03	0.0
-0.19229D-03	0.0	-0.20491D-03	-0.47108D-03	-0.20491D-03	0.47108D-03	-0.10283D-02	0.0
-0.82449D-04	0.0	0.40375D-03	0.0	0.15861D-03	0.0	-0.23175D-03	0.0
-0.13429D-03	0.0	-0.90824D-04	-0.13838D-03	-0.90824D-04	0.13838D-03	-0.42163D-03	0.0
0.30969D-04	0.0	0.12540D-03	0.0	0.11650D-03	0.0	-0.16115D-03	0.0
-0.72875D-04	0.0	-0.79847D-04	-0.60197D-04	-0.79847D-04	0.60197D-04	-0.32833D-03	0.0

TABLE 13

CONSERVATIVE NON-GYROSCOPIC SYSTEM - ERROR MATRIX EIGENVALUES AND EIGENVECTORS WITH 0.000005% RANDOM ERROR

EIGENVALUES

W(1)=	-0.44132D 00	0.0
W(2)=	0.81050D-02	0.0
W(3)=	-0.84913D-03	0.0
W(4)=	-0.26155D-04	0.41608D-03
W(5)=	-0.26155D-04	-0.41608D-03
W(6)=	-0.45279D-03	0.0
W(7)=	-0.11112D-03	0.16600D-03
W(8)=	-0.11112D-03	-0.16600D-03

EIGENVECTORS

-0.16100D 04	0.0	0.17451D 04	0.0	0.40707D 04	0.0	-0.57050D 04	-0.14772D 05
-0.57050D 04	0.14772D 5	-0.21415D 03	0.0	-0.26147D 04	-0.61707D 04	-0.26147D 04	0.61707D 04
0.44392D 03	0.0	-0.26335D 03	0.0	-0.48589D 03	0.0	-0.66009D 03	-0.28054D 03
-0.66009D 03	0.28054D 3	0.41572D 03	0.0	-0.85679D 03	-0.16986D 03	-0.85679D 03	0.16986D 03
0.48913D 02	0.0	0.64593D 01	0.0	0.42744D 02	0.0	-0.19285D 03	0.21277D 03
-0.19285D 03	-0.21277D 3	-0.48336D 01	0.0	-0.41977D 03	0.24535D 03	-0.41977D 03	-0.24535D 03
-0.12504D 03	0.0	0.26757D 02	0.0	0.54977D 02	0.0	0.26939D 03	-0.34982D 03
0.26939D 03	0.34982D 3	-0.21579D 03	0.0	-0.28972D 03	-0.32396D 02	-0.28972D 03	0.32396D 02
0.26783D 01	0.0	-0.70207D 00	0.0	0.31007D 01	0.0	0.18049D 02	-0.12895D 02
0.18049D 02	0.12895D 2	0.29427D 01	0.0	0.10292D 02	-0.24046D 02	0.10292D 02	0.24046D 02
0.57517D 00	0.0	-0.56556D-01	0.0	-0.10746D 01	0.0	0.29315D 00	-0.81680D 00
0.29315D 00	0.81680D 0	0.22063D 01	0.0	0.34766D 00	-0.66126D 01	0.34766D 00	0.66126D 01
0.10967D-01	0.0	0.44091D-01	0.0	0.10447D 00	0.0	-0.49719D 00	0.12899D 00
-0.49719D 00	-0.12899D 0	0.14892D 00	0.0	-0.11912D 01	-0.14118D 01	-0.11912D 01	0.14118D 01
-0.35516D-01	0.0	0.51163D-01	0.0	0.21037D 00	0.0	0.13442D 00	-0.63751D 00
0.13442D 00	0.63751D 0	-0.37461D 00	0.0	-0.11547D 01	-0.10724D 01	-0.11547D 01	0.10724D 01

NORMALIZED EIGENVECTORS

-0.96092D 00	0.0	0.99968D 00	0.0	0.99281D 00	0.0	-0.10003D 01	-0.54835D-03
-0.10003D 01	0.54835D- 3	-0.41544D 00	0.0	-0.10061D 01	-0.81773D-02	-0.10061D 01	0.81773D-02
0.26496D 00	0.0	-0.14920D 00	0.0	-0.11850D 00	0.0	-0.31573D-01	0.32496D-01
-0.31573D-01	-0.32496D- 1	0.80647D 00	0.0	-0.74546D-01	0.10789D 00	-0.74546D-01	-0.10789D 00
0.29154D-01	0.0	0.36598D-02	0.0	0.10425D-01	0.0	0.81401D-02	0.16211D-01
0.81401D-02	-0.16211D- 1	-0.93769D-02	0.0	0.87393D-02	0.72472D-01	0.87393D-02	-0.72472D-01
-0.74634D-01	0.0	0.15159D-01	0.0	0.13498D-01	0.0	-0.14470D-01	-0.23844D-01
-0.14470D-01	0.23844D- 1	-0.42056D 00	0.0	-0.21759D-01	0.37976D-01	-0.21759D-01	-0.37976D-01
0.15986D-02	0.0	-0.39775D-03	0.0	0.75624D-03	0.0	-0.34836D-03	-0.13572D-02
-0.34836D-03	0.13572D- 2	0.57086D-02	0.0	-0.26980D-02	-0.28532D-02	-0.26980D-02	0.28532D-02
0.34330D-03	0.0	-0.32942D-04	0.0	-0.26209D-03	0.0	-0.41441D-04	-0.35897D-04
-0.41441D-04	0.35887D- 4	0.42796D-02	0.0	-0.89016D-03	-0.44264D-03	-0.89016D-03	0.44264D-03
0.65460D-05	0.0	0.24980D-04	0.0	0.25480D-04	0.0	-0.37317D-05	0.32232D-04
-0.37317D-05	-0.32232D- 4	0.28883D-03	0.0	-0.26559D-03	0.79819D-04	-0.26559D-03	-0.79819D-04
-0.21198D-04	0.0	0.28997D-04	0.0	0.51307D-04	0.0	-0.34496D-04	-0.22449D-04
-0.34496D-04	0.22449D- 4	-0.72671D-03	0.0	-0.21666D-03	0.95040D-04	-0.21666D-03	-0.95040D-04

TABLE 14

CONSERVATIVE NON-GYROSCOPIC SYSTEM - ERROR MATRIX EIGENVALUES AND EIGENVECTORS
WITH 0.0000005% RANDOM ERROR

EIGENVALUES

W(1)=	0.97138D-01	0.0
W(2)=	-0.10271D-01	0.0
W(3)=	0.21748D-02	0.52336D-02
W(4)=	0.21748D-02	-0.52336D-02
W(5)=	0.66794D-03	0.0
W(6)=	0.14369D-03	0.74611D-04
W(7)=	0.14369D-03	-0.74611D-04
W(8)=	-0.71727D-03	0.0

EIGENVECTORS

-0.19283D 05	0.0	-0.15748D 05	0.0	-0.88547D 04	-0.20798D 05	-0.88547D 04	0.20798D 05
0.25872D 03	0.0	-0.24871D 03	-0.81317D 03	-0.24871D 03	0.81317D 03	-0.10755D 04	0.0
-0.10316D 04	0.0	-0.10397D 04	0.0	-0.11977D 04	-0.39679D 03	-0.11977D 04	0.39679D 03
0.64367D 02	0.0	-0.12176D 02	0.22625D 02	-0.12176D 02	-0.22625D 02	-0.13417D 03	0.0
-0.18715D 01	0.0	0.34985D 02	0.0	0.19925D 03	-0.12526D 03	0.19925D 03	0.12526D 03
0.31564D 03	0.0	0.37931D 03	0.28363D 04	0.37931D 03	-0.28363D 04	-0.40117D 03	0.0
-0.10429D 03	0.0	-0.74856D 02	0.0	-0.41641D 02	-0.16287D 03	-0.41641D 02	0.16287D 03
-0.91631D 02	0.0	0.25589D 03	0.19060D 03	0.25589D 03	-0.19060D 03	0.36098D 03	0.0
0.16860D 02	0.0	-0.15982D 03	0.0	0.37736D 03	0.93330D 01	0.37736D 03	-0.93330D 01
-0.61794D 02	0.0	0.41855D 02	-0.20869D 02	0.41855D 02	0.20869D 02	-0.13690D 03	0.0
0.10888D 01	0.0	-0.10445D 02	0.0	0.12133D 02	-0.14798D 02	0.12133D 02	0.14798D 02
-0.12192D 02	0.0	-0.20367D 02	-0.52835D 02	-0.20367D 02	0.52835D 02	-0.16389D 02	0.0
-0.81703D 00	0.0	-0.53723D 00	0.0	-0.56611D-01	-0.12931D 01	-0.56611D-01	0.12931D 01
-0.65804D 00	0.0	-0.10485D 02	-0.11931D 02	-0.10485D 02	0.11931D 02	0.38421D 00	0.0
-0.86437D-01	0.0	-0.25192D 00	0.0	-0.52543D 00	0.43689D 00	-0.52543D 00	-0.43689D 00
0.96097D-02	0.0	-0.78109D 01	-0.89009D 01	-0.78109D 01	0.89009D 01	0.28022D 01	0.0

NORMALIZED EIGENVECTORS

-0.99856D 00	0.0	-0.99776D 00	0.0	-0.10003D 01	-0.16265D-02	-0.10003D 01	0.16265D-02
0.60467D 00	0.0	-0.28239D 00	0.41245D-01	-0.28239D 00	-0.41245D-01	-0.88257D 00	0.0
-0.53421D-01	0.0	-0.65874D-01	0.0	-0.36985D-01	0.41826D-01	-0.36985D-01	-0.41826D-01
0.15044D 00	0.0	0.68880D-02	0.51876D-02	0.68880D-02	-0.51876D-02	-0.11011D 00	0.0
-0.96916D-04	0.0	0.22166D-02	0.0	-0.16293D-02	-0.10287D-01	-0.16293D-02	0.10287D-01
0.73770D 00	0.0	0.96019D 00	0.18089D-01	0.96019D 00	-0.18089D-01	-0.32921D 00	0.0
-0.54009D-02	0.0	-0.47429D-02	0.0	-0.73516D-02	-0.11398D-02	-0.73516D-02	0.11398D-02
-0.21416D 00	0.0	0.76218D-01	-0.75220D-01	0.76218D-01	0.75220D-01	0.29623D 00	0.0
0.87308D-03	0.0	-0.10126D-01	0.0	0.69461D-02	-0.15191D-01	0.69461D-02	0.15191D-01
-0.14442D 00	0.0	-0.47995D-02	-0.14945D-01	-0.47995D-02	0.14945D-01	-0.11235D 00	0.0
0.56362D-04	0.0	-0.66176D-03	0.0	-0.39097D-03	-0.75116D-03	-0.39097D-03	0.75116D-03
-0.28455D-01	0.0	-0.18562D-01	0.40758D-02	-0.18562D-01	-0.40758D-02	-0.13450D-01	0.0
-0.42310D-04	0.0	-0.34038D-04	0.0	-0.53598D-04	-0.20198D-04	-0.53598D-04	0.20198D-04
-0.15379D-02	0.0	-0.44902D-02	0.28730D-02	-0.44902D-02	-0.28730D-02	0.31530D-03	0.0
-0.44762D-05	0.0	-0.15961D-04	0.0	0.86332D-05	0.28981D-04	0.86332D-05	-0.28981D-04
0.22459D-04	0.0	-0.33493D-02	0.21396D-02	-0.33493D-02	-0.21396D-02	0.22996D-02	0.0

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