Faculty Bibliography 2010s

Faculty Bibliography

1-1-2012

Calculation of amplitudes in quantum electrodynamics

K. S. Karplyuk

O. O. Zhmudskyy
University of Central Florida

Find similar works at: https://stars.library.ucf.edu/facultybib2010

University of Central Florida Libraries http://library.ucf.edu

University of Central Florida

STARS
Showcase of Text, Archives, Research & Scholarship

Calculation of amplitudes in quantum electrodynamics

K. S. Karplyuk[1,*] and O. O. Zhmudskyy[2,†]

[1]*Department of Radiophysics, Taras Shevchenko University, Academic Glushkov prospect 2, building 5, Kyiv 03122, Ukraine*
[2]*Department of Physics, University of Central Florida, 4000 Central Florida Boulevard, Orlando, Florida, 32816, USA*

A new method of calculation of amplitudes of different processes in quantum electrodynamics is proposed. The method does not use the Feynman technique of trace of product of matrices calculation. The method strongly simplifies calculation of cross sections for different processes. The effectiveness of the method is shown on the cross-section calculation of Coulomb scattering, Compton scattering, and electron-positron annihilation.

I. INTRODUCTION

The most labor-intensive part of calculation of cross sections for different processes in quantum electrodynamics is the amplitude calculation for these processes. Such calculations for nonpolarized electrons, that is, electrons with no definite incoming or outgoing spin states, can be simplified by the Feynman trace technology of the traces of products of γ-matrices calculation [1]. In this paper we propose a method which strongly simplifies the amplitude calculation for polarized and unpolarized cross sections for any elementary processes.

Let us start from the identity which is satisfied for matrices with arbitrary complex elements:

$$\chi M \psi = \mathrm{Sp}\,\hat{\psi}\,\hat{\chi}\,M. \qquad (1)$$

Here M is an arbitrary square matrix, ψ is a matrix column, and χ is a matrix row. The square matrix $\hat{\psi}$ has only one nonzero column (let it be the nth) which is equal to ψ. The square matrix $\hat{\chi}$ has only one nth nonzero row which is equal to χ. In this paper we will use the $\hat{\psi}$ matrix with the first nonzero column and the $\hat{\chi}$ matrix with the first nonzero row. According to (1)

$$\bar{u}^{\,f} M u^i = \mathrm{Sp}\,\hat{u}^i\,\hat{\bar{u}}^f M = \mathrm{Sp}\,\hat{u}^i\,\hat{u}^{\dagger f}\gamma^0 M. \qquad (2)$$

Bispinors u^i and u^f represent the initial and final state of the fermions with momenta $\boldsymbol{p}^{i,f}$ and spins aligned along the unit vectors $\boldsymbol{s}^{i,f}$. They can be written as follows:

$$u^{i,f} = \sqrt{\frac{p_0^{i,f} + mc}{2p_0^{i,f} V}}\left(1 + \frac{\boldsymbol{p}^{i,f}\boldsymbol{s}_1}{p_0^{i,f} + mc}\right)\frac{1 + is^{i,f}\boldsymbol{s}_2}{\sqrt{2(1 + s_z^{i,f})}}\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}. \qquad (3)$$

These bispinors are normalized for one particle in a volume V, that is $\bar{u}^{i,f}\gamma^0 u^{i,f} = u^{i,f\dagger} u^{i,f} = \frac{1}{V}$.

Here and below for brevity the following designations are used:

$$
\begin{aligned}
\boldsymbol{c}\boldsymbol{\gamma} &= c_x\gamma^1 + c_y\gamma^2 + c_z\gamma^3, \\
\boldsymbol{c}\boldsymbol{s}_1 &= c_x\gamma^0\gamma^1 + c_y\gamma^0\gamma^2 + c_z\gamma^0\gamma^3, \\
\boldsymbol{c}\boldsymbol{s}_2 &= c_x\gamma^2\gamma^3 + c_y\gamma^3\gamma^1 + c_z\gamma^1\gamma^2, \\
\boldsymbol{c}\boldsymbol{\pi} &= c_x\gamma^0\gamma^2\gamma^3 + c_y\gamma^0\gamma^3\gamma^1 + c_z\gamma^0\gamma^1\gamma^2, \\
\pi^0 &= \gamma^1\gamma^2\gamma^3, \qquad \hat{\imath} = \gamma^0\gamma^1\gamma^2\gamma^3,
\end{aligned}
$$

where matrices γ are used in standard Dirac-Pauli representation. Let us use (3) and calculate the product $\hat{u}^i\,\hat{\bar{u}}^f$:

$$
\begin{aligned}
\hat{u}^i\hat{\bar{u}}^f = {} & \frac{1}{4V}\sqrt{\frac{(p_0^i + mc)(p_0^f + mc)}{p_0^i p_0^f(1 + s_z^i)(1 + s_z^f)}}\left(1 + \frac{\boldsymbol{p}^i\boldsymbol{s}_1}{p_0^i + mc}\right)[a_0(1 + \gamma^0) + \boldsymbol{a}(\boldsymbol{\pi} + \boldsymbol{s}_2)]\left(1 + \frac{\boldsymbol{p}^f\boldsymbol{s}_1}{p_0^f + mc}\right)\gamma^0 \\
= {} & \frac{1}{4V}\sqrt{\frac{(p_0^i + mc)(p_0^f + mc)}{p_i^0 p_0^f(1 + s_z^i)(1 + s_z^f)}}\Bigg\{a_0(1 + \gamma^0) + \boldsymbol{a}(\boldsymbol{\pi} + \boldsymbol{s}_2) + \frac{(\boldsymbol{a}\cdot\boldsymbol{p}^i)(\hat{\imath} - \pi^0) + (a^0\boldsymbol{p}^i - \boldsymbol{a}\times\boldsymbol{p}_i)(\boldsymbol{s}_1 - \boldsymbol{\gamma})}{p_0^i + mc} \\
& - \frac{(\boldsymbol{a}\cdot\boldsymbol{p}^f)(\hat{\imath} + \pi^0) + (a^0\boldsymbol{p}^f + \boldsymbol{a}\times\boldsymbol{p}^f)(\boldsymbol{s}^1 + \boldsymbol{\gamma})}{p_0^f + mc} - \frac{a_0(\boldsymbol{p}^i\cdot\boldsymbol{p}^f) - \boldsymbol{a}\cdot(\boldsymbol{p}^i\times\boldsymbol{p}^f)}{(p_0^i + mc)(p_0^f + mc)}(1 - \gamma^0) \\
& - \frac{a^0(\boldsymbol{p}^i\times\boldsymbol{p}^f) + (\boldsymbol{p}^i\cdot\boldsymbol{p}^f)\boldsymbol{a} - (\boldsymbol{a}\cdot\boldsymbol{p}^f)\boldsymbol{p}^i - (\boldsymbol{a}\cdot\boldsymbol{p}^i)\boldsymbol{p}^f}{(p_0^i + mc)(p_0^f + mc)}(\boldsymbol{\pi} - \boldsymbol{s}_2)\Bigg\}. \qquad (4)
\end{aligned}
$$

*karpks@hotmail.com
†ozhmudsky@physics.ucf.edu

Here

$$a^0 = \frac{i}{4}\boldsymbol{e}_z \cdot (\boldsymbol{s}^f \times \boldsymbol{s}^i) + \frac{1}{4}(1 + \boldsymbol{s}^i \cdot \boldsymbol{s}^f + \boldsymbol{e}_z \cdot \boldsymbol{s}^i + \boldsymbol{e}_z \cdot \boldsymbol{s}^f),$$

(5)

$$\boldsymbol{a} = \frac{i}{4}[\boldsymbol{e}_z + \boldsymbol{s}^i + \boldsymbol{s}^f + (\boldsymbol{e}_z \cdot \boldsymbol{s}^f)\boldsymbol{s}^i + (\boldsymbol{e}_z \cdot \boldsymbol{s}^i)\boldsymbol{s}^f - \boldsymbol{e}_z(\boldsymbol{s}^i \cdot \boldsymbol{s}^f)]$$

$$+ \frac{1}{4}(\boldsymbol{s}^f \times \boldsymbol{s}^i - \boldsymbol{e}_z \times \boldsymbol{s}^f + \boldsymbol{e}_z \times \boldsymbol{s}^i),$$

(6)

where $\boldsymbol{e}_z$ is the unit vector along the z axis. Note that a_0 and $\boldsymbol{a}$ depend on the direction of the spins $\boldsymbol{s}^i$ and $\boldsymbol{s}^f$ only, and do not depend on energies and momenta of initial and final fermions.

As far as $\hat{u}^i \hat{\bar{u}}^f$ is known, the trace of the matrix $\hat{u}^i \hat{\bar{u}}^f M$ can be calculated. In the general case matrix M has the following form:

$$M = I + V_0\gamma^0 + \boldsymbol{V}\boldsymbol{\gamma} + W_0\pi^0 + \boldsymbol{W}\boldsymbol{\pi} + \boldsymbol{E}\boldsymbol{s}_1 + \boldsymbol{B}\boldsymbol{s}_2 + J\hat{\imath}.$$

(7)

The unit matrix is the only one of the 16 Dirac matrices which has nonzero trace. That is why in the matrix product of $\hat{u}^i \hat{\bar{u}}^f$ and M it is enough to take into account only those terms which are proportional to the unit matrix. Such terms appear only for a multiplication of the same matrices. This remark simplifies the multiplication and trace calculation:

$$\bar{u}^f M u^i = \mathrm{Sp}\,\hat{u}^i \hat{\bar{u}}^f M$$

$$= \sqrt{\frac{(p_0^i + mc)(p_0^f + mc)}{p_0^i p_0^f (1 + s_z^i)(1 + s_z^f)}} \frac{1}{V}(a_0 K_0 - \boldsymbol{a} \cdot \boldsymbol{K}).$$

(8)

In three dimensions, the scalar K_0 and the pseudovector $\boldsymbol{K}$ can be written as

$$K_0 = (I + V_0) + \frac{\boldsymbol{p}^i \cdot (\boldsymbol{E} + \boldsymbol{V})}{p_0^i + mc} - \frac{\boldsymbol{p}^f \cdot (\boldsymbol{E} - \boldsymbol{V})}{p_0^f + mc}$$

$$+ \frac{(\boldsymbol{p}^i \times \boldsymbol{p}^f) \cdot (\boldsymbol{W} - \boldsymbol{B}) - (\boldsymbol{p}^i \cdot \boldsymbol{p}^f)(I - V_0)}{(p_0^i + mc)(p_0^f + mc)},$$

(9)

$$\boldsymbol{K} = (\boldsymbol{W} + \boldsymbol{B}) + \frac{\boldsymbol{p}^i(W_0 + J) + \boldsymbol{p}^i \times (\boldsymbol{E} + \boldsymbol{V})}{p_0^i + mc} + \frac{\boldsymbol{p}^f(W_0 - J) + \boldsymbol{p}^f \times (\boldsymbol{E} - \boldsymbol{V})}{p_0^f + mc}$$

$$+ \frac{\boldsymbol{p}^i[\boldsymbol{p}^f \cdot (\boldsymbol{W} - \boldsymbol{B})] + \boldsymbol{p}^f[\boldsymbol{p}^i \cdot (\boldsymbol{W} - \boldsymbol{B})] - (\boldsymbol{p}^i \cdot \boldsymbol{p}^f)(\boldsymbol{W} - \boldsymbol{B})}{(p_0^i + mc)(p_0^f + mc)} - \frac{(\boldsymbol{p}^i \times \boldsymbol{p}^f)(I - V_0)}{(p_0^i + mc)(p_0^f + mc)}.$$

(10)

Note that K_0 and $\boldsymbol{K}$ depend on energies and momenta of initial and final fermion states only and do not depend on their polarization states $\boldsymbol{s}^i$ and $\boldsymbol{s}^f$.

In order to evaluate the probability and cross section of the process, the square of the amplitude of transaction must be calculated. Let us calculate $|(a_0 K_0 - \boldsymbol{a}\boldsymbol{K})|^2$ supposing that all coefficients in (7) are real. This is usually the case in quantum electrodynamics. Thus the modulus square $|\bar{u}^f M u^i|^2$ is

$$\bar{u}^f M u^i|^2 = \frac{(p_0^i + mc)(p_0^f + mc)}{p_0^i p_0^f (1 + s_z^i)(1 + s_z^f)V^2}(a_0 K_0 - \boldsymbol{a} \cdot \boldsymbol{K})$$

$$\times (a_0^* K_0 - \boldsymbol{a}^* \cdot \boldsymbol{K})$$

$$= \frac{(p_0^i + mc)(p_0^f + mc)}{8V^2 p_0^i p_0^f}[(1 + \boldsymbol{s}^i\boldsymbol{s}^f)K_0^2$$

$$+ (1 - \boldsymbol{s}^i\boldsymbol{s}^f)\boldsymbol{K} \cdot \boldsymbol{K} + 2(\boldsymbol{s}^i\boldsymbol{K})(\boldsymbol{s}^f\boldsymbol{K})$$

$$+ 2(\boldsymbol{s}^i \times \boldsymbol{s}^f)\boldsymbol{K}K_0].$$

(11)

Expressions (8)–(11) are universal. They determine $\bar{u}^f M u^i$ and $|\bar{u}^f M u^i|^2$ for any processes in quantum electrodynamics. Different processes differ by matrix M only. The only thing we need to do in order to calculate $\bar{u}^f M u^i$ and $|\bar{u}^f M u^i|^2$ is to represent the interaction matrix M in a form (7). Then substitute coefficients from matrix (7) into expressions (9) and (10). These expressions entirely determine

K_0 and $\boldsymbol{K}$. Equations (8) and (11) give the algebraic expressions for $\bar{u}^f M u^i$ and $|\bar{u}^f M u^i|^2$. All that remains is the simplification of $\bar{u}^f M u^i$ and $|\bar{u}^f M u^i|^2$ as much as possible.

Expression (11) determines the square of the amplitude of the process which corresponds to any desirable spin states of the incoming and outgoing fermions. This expression explicitly represents dependence on the fermion polarization, which is why this dependence can be easily analyzed. If a detector is blind to polarization, i.e., for the final state both polarizations of fermions in $\boldsymbol{s}^f$ direction and in $-\boldsymbol{s}^f$ direction are registered, expression (11) must include the sum for both directions $\boldsymbol{s}^f$ and $-\boldsymbol{s}^f$:

$$|\bar{u}^f M u^i|^2 = \frac{(p_0^i + mc)(p_0^f + mc)}{4V^2 p_0^i p_0^f}[K_0^2 + \boldsymbol{K} \cdot \boldsymbol{K}].$$

(12)

Let us demonstrate the effectiveness of the above method on three examples: Coulomb scattering, Compton scattering and electron-positron annihilation.

II. COULOMB SCATTERING

We will define the cross section of an electron of charge e scattering on the Coulomb center of charge Ze versus the square of the amplitude $|\bar{u}^f \gamma^0 u^i|^2$ in a usual way:

$$\frac{d\sigma}{d\Omega} = \frac{(2Zr_0mc^2)^2}{(2p\sin\frac{\theta}{2})^4}\left(\frac{Vp_o}{c}\right)^2|\bar{u}^f\gamma^0 u^i|^2$$

$$= \left(\frac{Zr_0}{2}\right)^2\left(\frac{c}{v}\frac{1}{\sin\frac{\theta}{2}}\right)^4\left(1-\frac{v^2}{c^2}\right)V^2|\bar{u}^f\gamma^0 u^i|^2. \quad (13)$$

Here r_0 is the classical electron radius and θ is the scattering angle. The amplitude square $|\bar{u}^f\gamma^0 u^i|^2$ is defined by the universal expression (11), in which K_0 and $\mathbf{K}$ must be calculated for the matrix $M = \gamma^0$. Hence, in expressions (9) and (10) we must set $V_0 = 1$. All other coefficients must be set to 0. We must also take into account that $p_0^i = p_0^f = p_0$ because, for the Coulomb scattering, energy is conserved:

$$K_0 = 1 + \frac{\mathbf{p}^i\cdot\mathbf{p}^f}{(p_0+mc)^2}, \qquad \mathbf{K} = \frac{\mathbf{p}^i\times\mathbf{p}^f}{(p_0+mc)^2}. \quad (14)$$

So, the cross section becomes

$$\frac{d\sigma}{d\Omega} = \frac{1}{2}\left(\frac{Zr_0}{2}\right)^2\left(\frac{c}{v}\frac{1}{\sin\frac{\theta}{2}}\right)^4\left(1-\frac{v^2}{c^2}\right)\left(\frac{p_0+mc}{2p_0}\right)^2$$

$$\times [(1+\mathbf{s}^i\cdot\mathbf{s}^f)K_0^2 + (1-\mathbf{s}^i\cdot\mathbf{s}^f)K^2$$

$$+ 2(\mathbf{s}^i\cdot\mathbf{K})(\mathbf{s}^f\cdot\mathbf{K}) + 2(\mathbf{s}^i\times\mathbf{s}^f)\cdot\mathbf{K}K_0]. \quad (15)$$

In expression (15) K_0 and $\mathbf{K}$ are determined according to Eq. (14).

Expression (15) determines the differential cross section $d\sigma/d\Omega$ in the case with definite incoming and outgoing electron spin states. If the polarization of the final electron is not registered, expression (12) must be used:

$$\frac{d\sigma}{d\Omega} = \left(\frac{Zr_0}{2}\right)^2\left(\frac{c}{v}\frac{1}{\sin\frac{\theta}{2}}\right)^4\left(1-\frac{v^2}{c^2}\right)\left(\frac{p_0+mc}{2p_0}\right)^2(K_0^2+K^2).$$
$$(16)$$

Expression $(\frac{p_0+mc}{2p_0})^2(K_0^2 + \mathbf{K}\cdot\mathbf{K})$ can be simplified thus:

$$\left(\frac{p_0+mc}{2p_0}\right)^2(K_0^2 + \mathbf{K}\cdot\mathbf{K})$$

$$= \left(\frac{p_0+mc}{2p_0}\right)^2\left\{\left[1+\frac{p^2\cos\theta}{(p_0+mc)^2}\right]^2 + \frac{p^4\sin^2\theta}{(p_0+mc)^4}\right\}$$

$$= \frac{p_0^2 + p_0^2 - p^2(1-\cos\theta)}{2p_0^4} = \left(1-\frac{v^2}{c^2}\sin^2\frac{\theta}{2}\right). \quad (17)$$

Recall that θ is an angle between $\mathbf{p}^i$ and $\mathbf{p}^f$. After substituting (17) into (16) we come up with an unpolarized cross section for Coulomb scattering:

$$\frac{d\sigma}{d\Omega} = \left(\frac{Zr_0}{2}\right)^2\left(\frac{c}{v}\frac{1}{\sin\frac{\theta}{2}}\right)^4\left(1-\frac{v^2}{c^2}\right)\left(1-\frac{v^2}{c^2}\sin^2\frac{\theta}{2}\right). \quad (18)$$

This is the well-known Mott scattering cross section [2]. Note that for the polarized cross-section calculation (15) and the unpolarized cross-section calculation (18) instead of the Feynman technique (of trace of product of matrices calculation) we use expressions (9)–(12) which strongly simplify calculations.

III. COMPTON SCATTERING

It is well known that Compton scattering in the first order of probability theory is represented by two Feynman diagrams. Call them a and b. According to the a diagram, an electron absorbs a photon of frequency ω_1 first, and then emits a photon of frequency ω_2. According to the b diagram, an electron emits a photon of frequency ω_2 first, and then absorbs a photon of frequency ω_1. The amplitudes for the two diagrams must be added and their sum squared.

Assume that the incoming electron is at rest; hence, $\mathbf{p}^i = 0$, $p_0^i = mc$. Let us express the scattering cross section versus the square of the sum of the amplitudes $|\bar{u}^f M u^i|^2$ in a usual way:

$$\frac{d\sigma}{d\Omega} = r_0^2 mc p_0^f V^2\left(\frac{\omega^2}{\omega^1}\right)^2|\bar{u}^f M u^i|^2. \quad (19)$$

Here r_0 is the classical electron radius. Matrix M for the two diagrams is

$$M = \mathfrak{e}_2\boldsymbol{\gamma}\frac{p_{0a}\gamma^0 - \mathbf{p}_a\boldsymbol{\gamma} + mc}{2mc\hbar k_1}\mathfrak{e}_1\boldsymbol{\gamma}$$

$$+ \mathfrak{e}_1\boldsymbol{\gamma}\frac{p_{0b}\gamma^0 - \mathbf{p}_b\boldsymbol{\gamma} + mc}{(-2mc\hbar k_2)}\mathfrak{e}_2\boldsymbol{\gamma}.$$

Here $p_{0a} = mc + \hbar k_1$, $\mathbf{p}_a = \hbar\mathbf{k}_1$, $p_{0b} = mc - \hbar k_2$, $\mathbf{p}_b = -\hbar\mathbf{k}_2$, $\mathbf{k}_1$ and $\mathbf{k}_2$ are the wave vectors of photons 1 and 2, $k_1 = \omega_1/c$, $k_2 = \omega_2/c$, $\mathfrak{e}_1$ and $\mathfrak{e}_2$ are the unit vectors of polarization of photons 1 and 2, $\mathfrak{e}_1\cdot\mathbf{k}_1 = 0$, and $\mathfrak{e}_2\cdot\mathbf{k}_2 = 0$. The polar angle in $d\Omega$ is measured from the $\mathbf{k}_1$ direction. Matrix multiplication in the expression for M leads to the coefficients in Eq. (7):

$$I = \frac{(\mathfrak{e}_1\cdot\mathfrak{e}_2)}{2\hbar}\left(\frac{1}{k_2}-\frac{1}{k_1}\right),$$

$$V_0 = \frac{(\mathfrak{e}_1\cdot\mathfrak{e}_2)}{2mc\hbar}\left(\frac{p_{0a}}{k_1}-\frac{p_{0b}}{k_2}\right),$$

$$\mathbf{V} = \frac{(\mathbf{p}_a\cdot\mathfrak{e}_1)\mathfrak{e}_2 - \mathfrak{e}_2\times(\mathbf{p}_a\times\mathfrak{e}_1)}{2mc\hbar k_1}$$

$$+ \frac{(\mathbf{p}_b\cdot\mathfrak{e}_2)\mathfrak{e}_1 - \mathfrak{e}_1\times(\mathbf{p}_b\times\mathfrak{e}_2)}{(-2mc\hbar k_2)},$$

$$\mathbf{W} = \frac{(\mathfrak{e}_1\times\mathfrak{e}_2)}{2mc\hbar}\left(\frac{p_{0a}}{k_1}+\frac{p_{0b}}{k_2}\right),$$

$$\mathbf{B} = -\frac{(\mathfrak{e}_1\times\mathfrak{e}_2)}{2\hbar}\left(\frac{1}{k_1}+\frac{1}{k_2}\right),$$

$$W_0 = \frac{-\mathfrak{e}_2\cdot(\mathbf{p}_a\times\mathfrak{e}_1)}{2mc\hbar k_1} + \frac{\mathfrak{e}_1\cdot(\mathbf{p}_b\times\mathfrak{e}_2)}{2mc\hbar k_2},$$

$$\mathbf{E} = 0, \qquad J = 0.$$

Substitution of these coefficients into (9) and (10) gives us expressions for K_0 and $\mathbf{K}$:

$$K_0 = (I + V_0) + \frac{\boldsymbol{p}^f \cdot \boldsymbol{V}}{p_0^f + mc}$$

$$= 2\frac{e_1 \cdot e_2}{2mc} - \frac{1}{2mc}\frac{\boldsymbol{p}^f \cdot [e_2 \times (\mathfrak{f}_1 \times e_1)]}{p_0^f + mc}$$

$$- \frac{1}{2mc}\frac{\boldsymbol{p}^f \cdot [e_1 \times (\mathfrak{f}_2 \times e_2)]}{p_0^f + mc}, \tag{20}$$

$$\boldsymbol{K} = (\boldsymbol{W} + \boldsymbol{B}) + \frac{\boldsymbol{p}^f W_0 - \boldsymbol{p}^f \times \boldsymbol{V}}{p_0^f + m}$$

$$= \frac{1}{2mc}\frac{\boldsymbol{p}^f \times [e_2 \times (\mathfrak{f}_1 \times e_1)] - \boldsymbol{p}^f[\mathfrak{f}_1 \times e_1)]}{p_0^f + m}$$

$$+ \frac{1}{2mc}\frac{\boldsymbol{p}^f \times [e_1 \times (\mathfrak{f}_2 \times e_2)] - \boldsymbol{p}^f[e_1 \cdot (\mathfrak{f}_2 \times e_2)]}{p_0^f + m}. \tag{21}$$

In expressions (20) and (21), $\mathfrak{f}_1 = \frac{\boldsymbol{k}_1}{k_1}$, $\mathfrak{f}_2 = \frac{\boldsymbol{k}_2}{k_2}$. Expressions for K_0 and $\boldsymbol{K}$ together with (11) and (19) determine the polarized cross section. In order to get the unpolarized cross section, expressions (12) and (19) must be used. The sum $K_0^2 + \boldsymbol{K}^2$ can be simplified and expressed as

$$K_0^2 + \boldsymbol{K}^2 = \frac{1}{2mc}\frac{1}{p_0^f + mc}\left[4(e_1 \cdot e_2)^2 + \frac{(\omega_1 - \omega_2)^2}{\omega_1\omega_2}\right]. \tag{22}$$

Thus, the unpolarized cross section is

$$\frac{d\sigma}{d\Omega} = \frac{r_0^2}{4}\left(\frac{\omega_2}{\omega_1}\right)^2\left[4(e_1 \cdot e_2)^2 + \frac{(\omega_1 - \omega_2)^2}{\omega_1\omega_2}\right]. \tag{23}$$

This is the well-known Klein-Nishina scattering cross section [3]. As in the previous case we don't use the Feynman technique of trace of product of matrices calculation. Using the universal expressions (11) and (12) instead strongly simplified calculations.

IV. ANNIHILATION

Two Feynman diagrams represent the annihilation process in the first order of perturbation theory. The first diagram (call it *a*) corresponds to the process in which an incoming electron emits a photon γ_1 of frequency ω_1, then a photon γ_2 of frequency ω_2, and transfers to the state with negative energy. The second diagram (call it *b*) corresponds to the process in which γ_1 and γ_2 interchange. In order to calculate the annihilation probability we must add the amplitudes of these processes and then square it.

For simplicity, assume that the electron is at rest, so $\boldsymbol{p}^i = 0$, $p_0^i = mc$. Following Feynman, we treat the positron as an electron with negative energy moving backward in time. This electron has linear momentum and spin opposite in direction to the positron's momentum and spin. It allows us to describe the positron by the same bispinor (3)

if we set up $p_0^f = -p^+$, $\boldsymbol{p}^f = -\boldsymbol{p}^+$, $\boldsymbol{s}^f = -\boldsymbol{s}^+$, where the index "+" designates positron quantities. The same substitutions have to be done in expressions (9) and (10). Let us express the annihilation cross section versus the square of the sum of the amplitudes $|\bar{u}^f M u^i|^2$ in a usual way:

$$\frac{d\sigma}{d\Omega} = r_0^2 \frac{m\hbar^2\omega_1^2 p_0^+ V^2}{|p^+|(p_0^+ + mc)c}|\bar{u}^f M u^i|^2. \tag{24}$$

Matrix M, which corresponds to the sum of two diagrams, is

$$M = e_2\boldsymbol{\gamma}\frac{p_{0a}\gamma^0 - \boldsymbol{p}_a\boldsymbol{\gamma} + mc}{(-2mc\hbar k_1)}e_1\boldsymbol{\gamma}$$

$$+ e_1\boldsymbol{\gamma}\frac{p_{0a}\gamma^0 - \boldsymbol{p}_b\boldsymbol{\gamma} + mc}{(-2mc\hbar k_2)}e_2\boldsymbol{\gamma}.$$

Here $p_{0a} = mc - \hbar k_1$, $\boldsymbol{p}_a = -\hbar\boldsymbol{k}_1$, $p_{0b} = mc - \hbar k_2$, $\boldsymbol{p}_b = -\hbar\boldsymbol{k}_2$, $\boldsymbol{k}_1$ and $\boldsymbol{k}_2$ are the wave vectors of the photons 1 and 2, $k_1 = \omega_1/c$, $k_2 = \omega_2/c$, e_1 and e_2 are the unit vectors of the polarization of the photons 1 and 2, $e_1 \cdot \boldsymbol{k}_1 = 0$, and $e_2 \cdot \boldsymbol{k}_2 = 0$. The polar angle in the $d\Omega$ is measured from the $\boldsymbol{k}_1$ direction. After transformation of the matrix M to the form (7) we can find coefficients in (7) in the reference frame in which the incoming electron is at rest $\boldsymbol{p}^i = 0$:

$$I + V_0 = 2\frac{e_1 \cdot e_2}{2mc}, \qquad \boldsymbol{W} + \boldsymbol{B} = 0,$$

$$\boldsymbol{E} = 0, \qquad \boldsymbol{J} = 0,$$

$$\boldsymbol{V} = -\frac{e_1 \times (\mathfrak{f}_2 \times e_2) + e_2 \times (\mathfrak{f}_1 \times e_1)}{2mc}$$

$$= -\frac{(\mathfrak{f}_1 + \mathfrak{f}_2)(e_1 \cdot e_2) - e_1(e_2 \cdot \mathfrak{f}_1) - e_2(e_1 \cdot \mathfrak{f}_2)}{2mc},$$

$$W_0 = -\frac{e_1 \cdot (\mathfrak{f}_2 \times e_2) + e_2 \cdot (\mathfrak{f}_1 \times e_1)}{2mc}.$$

Here $\mathfrak{f}_{1,2} = \frac{\boldsymbol{k}_{1,2}}{k_{1,2}}$, and $k_{1,2} = \frac{\omega_{1,2}}{c}$. Using (9) and (10), scalar K_0 and pseudovector $\boldsymbol{K}$ can be calculated:

$$K_0 = \frac{[2(p_0^f + mc) + \hbar(k_1 + k_2)(1 + \cos\theta)](e_1 \cdot e_2)}{2mc(p_0^f + mc)}$$

$$- \frac{\hbar(k_1 + k_2)(\mathfrak{f}_1 \cdot e_2)(\mathfrak{f}_2 \cdot e_1)}{2mc(p_0^f + mc)}, \tag{25}$$

$$\boldsymbol{K} = \frac{\boldsymbol{p}^f \times [e_1 \times (\mathfrak{f}_2 \times e_2)] + \boldsymbol{p}^f \times [e_2 \times (\mathfrak{f}_1 \times e_1)]}{2mc(p_0^f + mc)}$$

$$+ \frac{\boldsymbol{p}^f[e_1 \cdot (e_2 \times \mathfrak{f}_2)] + \boldsymbol{p}^f[e_2 \cdot (e_1 \times \mathfrak{f}_1)]}{2mc(p_0^f + mc)}. \tag{26}$$

These expressions for K_0 and $\boldsymbol{K}$ together with (11) and (24) determine the polarized cross section (both the electron and the positron have a given direction of spin). In order to calculate the unpolarized cross section expressions, (12) and (24) must be used. The sum $K_0^2 + \boldsymbol{K}^2$ can be reduced to

$$K_0^2 + K^2 = \frac{1}{2mc}\frac{1}{p_0^f + mc}\left[4(e_1 \cdot e_2)^2 - \frac{(\omega_1 + \omega_2)^2}{\omega_1\omega_2}\right].$$

(27)

Thus, the unpolarized cross section is

$$\frac{d\sigma}{d\Omega} = \frac{r_0^2}{4}\frac{\hbar^2 k_1^2}{|p^+|(p_0^+ + mc)}\left[\frac{(\omega_1 + \omega_2)^2}{\omega_1\omega_2} - 4(e_1 \cdot e_2)^2\right].$$

(28)

This result entirely coincides with the one calculated by the technique of trace of product of matrices calculation [1] and with the one calculated by Dirac [4].

As shown in the above examples, the method proposed in this paper allows us to strongly simplify calculation of the polarized and unpolarized fermion cross sections in quantum electrodynamics. It is free from the necessity of calculation of trace of product of a great amount of matrices.

[1] R. P. Feynman, *Quantum Electrodynamics* (Benjamin, New York, 1961).

[2] N. F. Mott, Proc. R. Soc. London, Ser. A **124**, 425 (1929).

[3] O. Klein and Y. Nishina, Z. Phys. **52**, 853 (1929).

[4] P. A. M. Dirac, Math. Proc. Cambridge Philos. Soc. **26**, 361 (2008).