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Exact solution for the self-induced motion of a vortex filament in the arc-length representation of the local induction approximation

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We review two formulations of the fully nonlinear local induction equation approximating the self-induced motion of the vortex filament (in the local induction approximation), corresponding to the Cartesian and arc-length coordinate systems. The arc-length representation put forth by Umeki [Theor. Comput. Fluid Dyn. **24**, 383 (2010)] results in a type of $1+1$ derivative nonlinear Schrödinger (NLS) equation describing the motion of such a vortex filament. We obtain exact stationary solutions to this derivative NLS equation; such exact solutions are a rarity. These solutions are periodic in space and we determine the nonlinear dependence of the period on the amplitude.

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The self-induced velocity of a vortex filament has been described by the approximation $\mathbf{v} = \gamma\kappa\mathbf{t}\times\mathbf{n}$ (in the work of Da Rios [1] and Arms and Hama [2]), where $\mathbf{t}$ and $\mathbf{n}$ are unit tangent and unit normal vectors to the vortex filament, respectively, κ is the curvature, and γ is the strength of the vortex filament. The Da Rios equations have an interesting history stretching back over the last century; for an interesting account of the history of the Da Rios equations, see Ref. [3]. A discussion of the mathematical formulation of the problem governing the self-induced motion of a vortex filament can be found in Ref. [4], where applications to vortices trailing aircraft are also discussed.

A number of methods have been employed to study the Da Rios equations, particularly the local induction approximation (LIA). Shivamoggi and van Heijst [5] recently reformulated the Da Rios equations in the extrinsic vortex filament coordinate space and were able to find exact solutions to an approximate equation governing a localized stationary solution in the LIA. Exact stationary solutions to the LIA in extrinsic coordinate space have been found by Kida [6] in the case of torus knots, and these solutions were given in terms of elliptic integrals. By rewriting the LIA in cylindrical-polar coordinates, Ricca also obtained torus knot solutions—which were asymptotically equivalent to Kida's solutions—in explicit analytic form and derived a stability criterion (see, e.g., Refs. [7–10]). Static solutions to the LIA have also been found by Lipniacki [11]. See also Ricca's discussion [12] of the physical invariants obtained under LIA.

While solutions under various approximations to the LIA are indeed useful for certain applications, the study of the fully nonlinear equations governing the self-induced motion of a vortex filament in the LIA is itself with merit. We previously derived such an equation in the Cartesian coordinate space [13,14]. To this end, we consider the vortex filament essentially aligned along the x axis: $\mathbf{r} = x\mathbf{i}_x + y(x,t)\mathbf{i}_y + z(x,t)\mathbf{i}_z$. We then have that

$$\mathbf{t} = \frac{d\mathbf{r}}{ds} = \frac{d\mathbf{r}}{dx}\frac{dx}{ds} = (\mathbf{i}_x + y_x\mathbf{i}_y + z_x\mathbf{i}_z)\frac{dx}{ds},$$

and $\mathbf{v} = y_t\mathbf{i}_y + z_t\mathbf{i}_z$, where $\frac{dx}{ds} = 1/\sqrt{1+y_x^2+z_x^2}$. From the governing equation $\mathbf{v} = \gamma\kappa\mathbf{t}\times\mathbf{n}$, we compute the quantities

$$\begin{aligned} y_t &= -\gamma z_{xx}\left(\frac{dx}{ds}\right)^3 = -\gamma z_{xx}\left(1+y_x^2+z_x^2\right)^{-3/2}, \\ z_t &= \gamma y_{xx}\left(\frac{dx}{ds}\right)^3 = \gamma y_{xx}\left(1+y_x^2+z_x^2\right)^{-3/2}, \end{aligned} \tag{1}$$

and, upon defining $\Phi(x,t) = y(x,t) + iz(x,t)$, it was shown in Ref. [14] that the coupled system of real partial differential equations (1) reduces to the single complex partial differential equation

$$i\Phi_t + \gamma(1+|\Phi_x|^2)^{-3/2}\Phi_{xx} = 0. \tag{2}$$

Dmitriyev [15] considered the approximation $i\Phi + \gamma\Phi_{xx} = 0$, while Shivamoggi and van Heijst [5] considered a quadratic approximation to the nonlinearity in Eq. (2). The full nonlinear equation was obtained in Ref. [13]. To recover y and z once a solution Φ to Eq. (2) is known, note that $y = \mathrm{Re}\Phi$ and $z = \mathrm{Im}\Phi$. Some mathematical properties of Eq. (2) were discussed in Ref. [14] in the case where periodic stationary solutions are possible, though a systematic study of all such stationary solutions was not considered. In Ref. [16] a more systematic approach was taken to classify all such stationary solutions $\Phi(x,t) = e^{-i\gamma t}\psi(x)$ to Eq. (2). Spatially periodic solutions (2) were shown to be governed by an implicit relation involving the sum of elliptic integrals of differing kinds. The amplitude of such periodic solutions was shown to obey $|\psi| < \sqrt{2}$.

The formulation (2), corresponding to the Cartesian coordinate system, is one possible way to describe the fully nonlinear self-induced motion of a vortex filament in the LIA. Umeki [17] obtained an alternative formulation, applying an arc-length-based coordinate system as opposed to a Cartesian coordinate system. Umeki defines $\mathbf{r} = \mathbf{t}\times\mathbf{t}_s$, where s is the arc-length element. Now, $\mathbf{t}_t = \mathbf{t}\times\mathbf{t}_{ss}$. Let us write $\mathbf{t} = (\tau_x,\tau_y,\tau_z)$. Then Umeki defines the complex field v by

$$\tau_x + i\tau_y = \frac{2v}{1+|v|^2}, \quad \tau_z = \frac{1-|v|^2}{1+|v|^2}. \tag{3}$$

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The relation $\mathbf{t}_t = \mathbf{t} \times \mathbf{t}_{ss}$ then implies

$$(\tau_x + i\tau_y)_t = i((\tau_x + i\tau_y)_{ss}\tau_z - (\tau_x + i\tau_y)\tau_{zss}),$$
$$2\tau_{z_t} = i((\tau_x^* + i\tau_y^*)_{ss}(\tau_x + i\tau_y) - (\tau_x^* + i\tau_y^*)(\tau_x + i\tau_y)_{ss}).$$

From here, Umeki [17] then found

$$iv_t + v_{ss} - 2v^* v_s^2/(1 + |v|^2) = 0, \quad (4)$$

where v denotes directly the tangential vector of the filament. While the Cartesian and arc-length formulations are obtained through different derivations, both formulations are equivalent to the localized induction equation (LIE). Umeki [18] showed that there exists a transformation between solutions to Eq. (2) and solutions to Eq. (4). A plane wave solution to Eq. (4) exists [18], and Umeki [18] was also able to show that the famous one-soliton solution of Hasimoto [19] is given by

$$v(s,t) = \frac{\nu\,\mathrm{sech}\,[k(s-ct)]}{\nu\,\mathrm{sech}^2\,[k(s-ct)] - 2}\left(\tanh\,[k(s-ct)] - \frac{ic}{2k}\right),$$

$\nu = 2k^2/(4k^2 + c^2)$, $0 < \nu < 1/2$ in the arc-length representation.

We now turn our attention to obtaining stationary solutions, which has not been done for the local induction equation in the arc-length representation. Let us consider the ansatz

$$v(s,t) = e^{-i\alpha^2 t} q(\alpha s), \quad (5)$$

where q is assumed to be a real-valued function, which puts Eq. (4) into the form

$$q + q_{ss} - \frac{2qq_s^2}{1+q^2} = 0. \quad (6)$$

Hence, the solution (5) is invariant under $\alpha \in \mathbb{R}$, so without loss of generality we shall consider $\alpha = 1$ henceforth. We should remark that a factor of $e^{+i\alpha^2 t}$ in Eq. (5) results in unstable solutions, so the '−' case in the exponent is what we limit our attention to. Also note that Eq. (6) is essentially a nonlinear oscillator provided $2q_s^2 < 1 + q^2$.

Our goal is to obtain an exact solution for Eq. (6), and defining a conserved quantity will greatly help in constructing a second integral. To this end, let us define the quantity

$$E = -\frac{q_s^2 - q^2 - 1}{(1+q^2)^2}, \quad (7)$$

$E \in (0,1)$. Observe that the quantity is conserved:

$$\frac{dE}{ds} = -\frac{2q_s}{(1+q^2)}\left(q + q_{ss} - \frac{2qq_s^2}{1+q^2}\right) = 0. \quad (8)$$

For a fixed value of E, we find that

$$q_s^2 = (1+q^2)[1-(1+q^2)E], \quad (9)$$

and, upon separating variables,

$$\int_{q_0}^{q} \frac{d\xi}{\sqrt{(1+\xi^2)[1-(1+\xi^2)E]}} = \pm(s - s_0), \quad (10)$$

where $q_0 = q(s_0)$ is a second arbitrary constant. Performing the required integration, we obtain the expression

$$\frac{1}{\sqrt{E}} F\left(\frac{\sqrt{E}}{\sqrt{1-E}}q, \frac{\sqrt{1-E}}{\sqrt{E}}i\right) = \pm(s - \hat{s}), \quad (11)$$

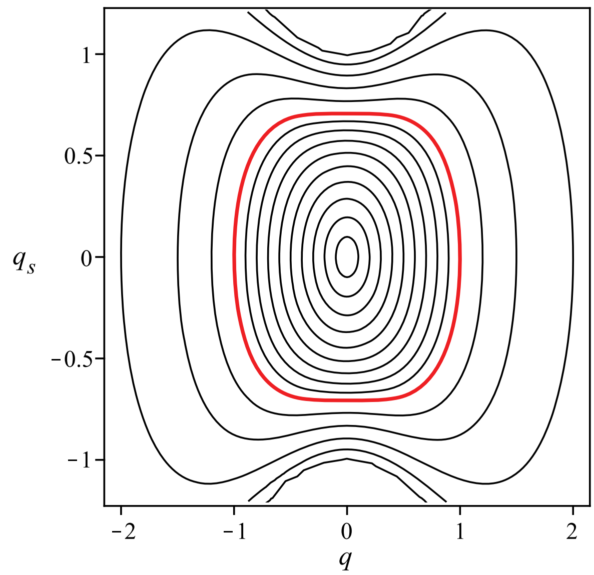

FIG. 1. (Color online) Phase portrait in (q,q_s) for the solution to the fully nonlinear oscillator equation modeling the LIE under the arc-length representation.

where $\hat{s}$ is a constant involving s_0 and q_0. Here, F is the elliptic integral of the first kind.

Inverting Eq. (11) to obtain $q(s)$, we find that

$$q(s) = \frac{\sqrt{1-E}}{\sqrt{E}}\,\mathrm{sn}\left(\pm\sqrt{E}(s-\hat{s}), \frac{\sqrt{1-E}}{\sqrt{E}}i\right), \quad (12)$$

where $\mathrm{sn}(a,b)$ denotes the Jacobi elliptic function. While Eq. (12) is a closed-form expression, it involves the conserved quantity E, which is perhaps not so satisfying. Note that the amplitude of q may be found from Eq. (9); setting $q_s = 0$, we find that the amplitude $A = A(E)$ is given by

$$A = \max_s |q(s)| = \frac{\sqrt{1-E}}{\sqrt{E}}. \quad (13)$$

It follows that $E = 1/(1 + A^2)$, hence Eq. (12) becomes

$$q(s) = A\,\mathrm{sn}\left(\pm\frac{1}{\sqrt{1+A^2}}(s-\hat{s}), Ai\right). \quad (14)$$

With this we have obtained an exact stationary solution $q(s)$ in terms of amplitude A. In Fig. 1 we plot the phase portrait for q versus q_s, which demonstrates the exact periodic solutions. In Fig. 2, we display solution profiles for various values of the amplitude A. We should remark that in the Cartesian case, solutions to models which are low-order approximations to the fully nonlinear model agree well for small amplitudes [16], and we expect the same will hold here (though we omit the details of any approximating models here).

A similar exact solution was obtained by Hasimoto [20], through a different derivation, for a two-dimensional model (recall that our model is three dimensions). Hasimoto's derivation started with $\mathbf{v} = Y\mathbf{i}_y$, as opposed to $\mathbf{v} = y_t\mathbf{i}_y + z_t\mathbf{i}_z$. Assuming a stationary solution, Hasimoto's assumption leads to the equation $Y_{xx} + \frac{\Omega}{\gamma}(1 + Y_x^2)^{3/2}Y = 0$. Hasimoto finds a solution $Y = A\,\mathrm{cn}(\xi,k)$ [where $x = x(\xi)$, ξ is a parametrization linking Y and x implicitly, cn denotes the elliptic cosine function (sn being the elliptic sine function)], which has initial

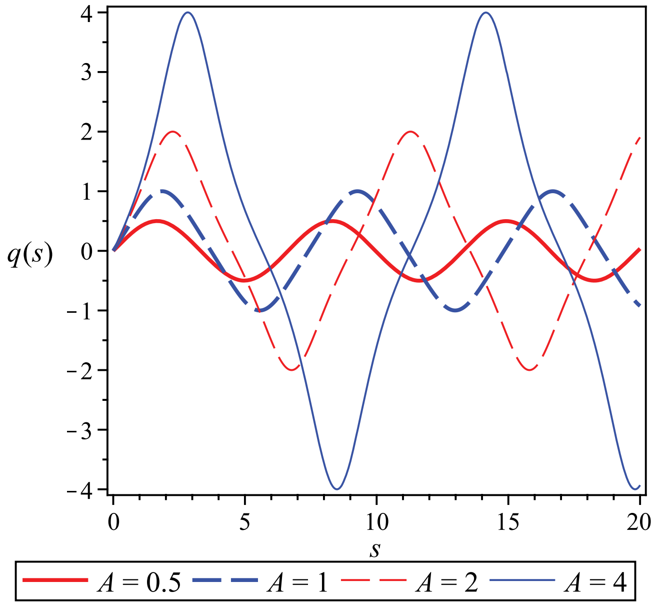

FIG. 2. (Color online) Plots of the solution $q(s)$ given in Eq. (14) for various values of the amplitude A. Note that the period of the solutions is strongly influenced by the amplitude. The nonlinear dependence of the period T with the amplitude A is shown graphically in Fig. 3.

conditions $Y(0,k) = A$ and $Y'(0) = 0$. Hence, Hasimoto's solution for the two-dimensional problem is a direct analogy to the solution for the three-dimensional problem we have found here under the arc-length representation.

Observe the nonlinear dependence of the period on the amplitude. From this exact relation, we see that the period $T = T(A)$ obeys the relation

$$T(A) = 4\sqrt{1+A^2}\int_0^{\frac{\pi}{2}} \frac{d\theta}{\sqrt{1+A^2\sin^2\theta}} = 4K\left(\frac{A}{\sqrt{1+A^2}}\right), \tag{15}$$

where K is the elliptic quarter period. Recalling the asymptotic expansion

$$K(m) \approx \frac{\pi}{2} + \frac{\pi}{8}\frac{m^2}{1-m^2} - \frac{\pi}{16}\frac{m^4}{1-m^2}, \tag{16}$$

which is a good approximation for $m < 1/2$, we have

$$T(A) \approx 2\pi + \frac{\pi}{2}A^2 - \frac{\pi}{4}\frac{A^4}{1+A^2}, \tag{17}$$

which in turn is a good approximation for the small-amplitude regime $A < 1/\sqrt{3}$. The large-amplitude asymptotics are slightly less standard. For $m > 2$, there exists an accurate asymptotic expansion

$$4K\left(1-\frac{1}{m}\right) \approx J(m), \tag{18}$$

where

$$J(m) = 4\left(1 + \frac{1}{m} + \frac{5}{16m^2} + \frac{7}{32m^3}\right)\ln(2\sqrt{2m}) - \left(\frac{1}{m} + \frac{7}{8m^2} + \frac{17}{24m^3}\right). \tag{19}$$

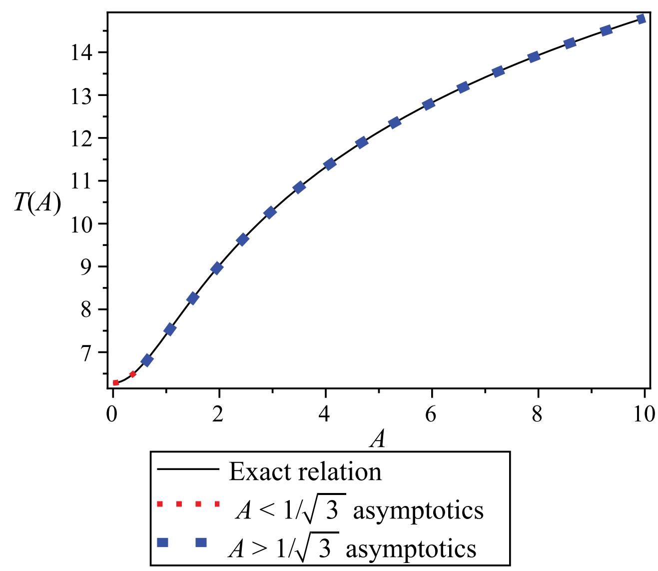

FIG. 3. (Color online) Plot of the period $T(A)$ of the solution (14) vs the amplitude A. The exact relation is found by numerically plotting Eq. (15). Note that both the $A < 1/\sqrt{3}$ and $A > 1/\sqrt{3}$ asymptotic expansions are excellent fits to the exact relation.

When $m > 2$, the argument of K is less than or equal to 1/2. Thus,

$$T(A) \approx J\left(\frac{\sqrt{1+A^2}}{\sqrt{1+A^2}-A}\right) \tag{20}$$

is a good approximation for $A > 1/\sqrt{3}$.

In Fig. 3, we plot the the period $T(A)$ of the solution (14) versus the amplitude A. The approximate asymptotic solutions

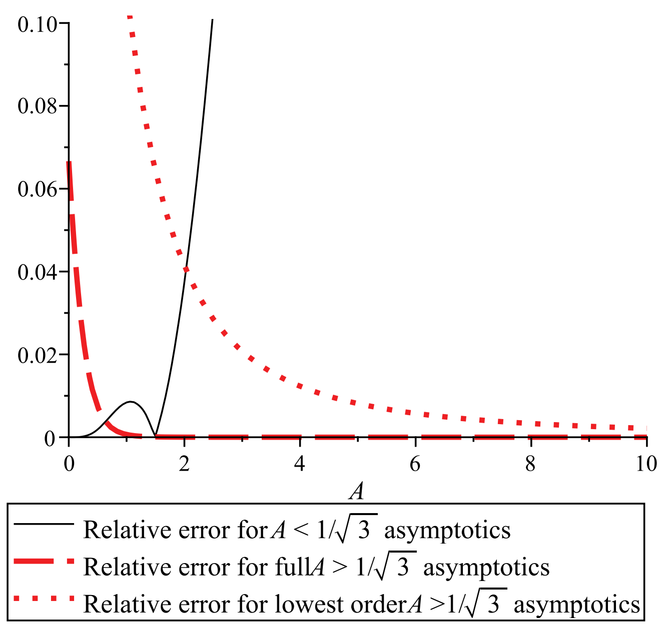

FIG. 4. (Color online) Relative error $|T(A) - T_{\rm approx}|/|T(A)|$ of the approximations to $T(A)$. We also include the lowest order approximation $T(A) \approx 4\ln(2\sqrt{2m})$ for the $A > 1/\sqrt{3}$ case. We see the good agreement with the $A < 1/\sqrt{3}$ asymptotics and $A > 1/\sqrt{3}$ asymptotics where needed.

are also included in their valid regions. Then in Fig. 4 we plot the relative error in these approximations, showing the agreement between the exact and asymptotic solutions. For the $A > 1/\sqrt{3}$ asymptotics, only retaining the logarithmic term (as a lowest order approximation) is not completely sufficient, as demonstrated in Fig. 4.

We have found an exact stationary solution for the self-induced motion of a vortex filament in the arc-length representation of the LIA. Such a formula is interesting in both its simplicity and its potential applications. Note that this representation is simpler than that found in the Cartesian representation; in particular, the integral representation permits a clean inversion so that we may obtain solutions in the form of Eq. (14). In the Cartesian case, however, the solutions were defined implicitly by a linear combination of elliptic integrals, which was then inverted numerically. Umeki [18] gives a relation between the arc-length and Cartesian representations which can be used to map the arc-length formula into a formula for the Cartesian representation. This involves complicated mathematical expressions and we omit the details of this inversion here.

We should remark that, while interesting, the physical scenario considered here is certainly not the only case of interest. The behavior of a vortex filament in a superfluid is another area of current research [21–29], since it grants us a model of superfluid turbulence. The nonlinear motion of a vortex filament in a superfluid has been previously studied in the case of a Cartesian coordinate system in the form of a partially linearized model [30] and, more recently, a fully nonlinear model [31]. In developing these models, one begins with the local induction equation and adds terms due to the ambient superfluid; see [30]. It is possible that, in an arc-length coordinate system, the solution representation for the fully nonlinear model can be simplified. The application of the present results to the study of the motion of a vortex filament in a superfluid, under the arc-length formulation, is certainly possible. In particular, it becomes clear that the solution presented here would serve as the order-zero perturbation theory for the superfluid case, with higher-order corrections resulting from the superfluid friction parameters [31,32]. Along these lines, see also Ref. [33]. Since we were able to obtain an explicit exact stationary solution in the present geometry, perhaps the arc-length formulation will prove most useful in the study of such superfluid models. This is one potential area of future work.

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