1-1-2014

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Robert A. Van Gorder
University of Central Florida

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Citation: Physics of Fluids **26**, 119102 (2014); doi: 10.1063/1.4900585
View online: https://doi.org/10.1063/1.4900585
View Table of Contents: http://aip.scitation.org/toc/phf/26/11
Published by the American Institute of Physics

PHYSICS OF FLUIDS **26**, 119102 (2014)

Response to "Comment on 'General rotating quantum vortex filaments in the low-temperature Svistunov model of the local induction approximation'" [Phys. Fluids 26, 119101 (2014)]

Robert A. Van Gorder[a)]
Department of Mathematics, University of Central Florida, Orlando, Florida 32816-1364, USA

(Received 24 September 2014; accepted 17 October 2014; published online 4 November 2014)

[http://dx.doi.org/10.1063/1.4900585]

I. INTRODUCTION

In Sec. II of Ref. 1, I discuss the Svistunov[2] model of the local induction approximation (LIA) in some generality, taking into account all possible motions, and discussing features of the model which permit translations (motion along the x-axis) and rotations (around the x-axis). However, the topic of the paper was on the structure of rotating filaments, so the translational motion was dropped in the subsequent sections. That said, the title and abstract of the paper were very clear on this point, where it was emphasized that rotating filaments were sought. Indeed, such solutions are of a more narrow class that one could possibly obtain, however it is worth studying all such rotating filaments in a unified manner.

The authors of Ref. 3 are correct in that the solutions discussed in Sec. III (and beyond) exhibit no translational motion. Essentially, a vortex filament of the type which may be described in complex potential form has both translational motion and rotational motion, with the rotational motion strongly influencing structure of the filament. The resulting structure when translated along the x-axis with whatever translational velocity one calculates. It is possible to study the rotational motion of standing filaments which do not translate. This makes sense, as it allows one to study the geometry or structure of the filaments without dealing with effects due to motion of the overall filament structure. This is exactly what I do in Ref. 1, which was essentially a study of the various structures possible in standing vortex filaments. One can certainly include translational effects if one wishes to see these filaments move along the x-axis, however this greatly complicated the mathematics (as shall be shown in Sec. II), rendering any qualitative analytical results ineffective. On the other hand, the numerical results are similar for both the non-translating and translating filaments (with only adjustments to parameter values, as mentioned below). Therefore, I would argue that my paper[1] adequately studies a variety of filament structures possible in the case of purely rotating filaments.[5]

All of that said, the case of filaments exhibiting both translational and rotational motions is also interesting. Hence, I will devote most of this reply to giving an overview of how one can mathematically study such solutions.

Before proceeding, we should note that there are circumstances under which translational motion is essential to the vortex filament structure. In the case where mutual friction effects are considered (for temperatures above the absolute zero limit), translational and rotational motions are both tied to normal fluid velocity in a rather fundamental way. See, for instance, Ref. 4. In such cases, growth or decay of the filament should be considered. In Ref. 6, the authors of the present Comment attempted

a)Author to whom correspondence should be addressed. Electronic mail: rav@knights.ucf.edu

a potential formulation for the mutual friction case, however as I pointed out in Ref. 7 they did not appropriately apply the complex modulus to the real exponential (so they took $|e^{rt+ist}| = 1$ whereas they needed $|e^{rt+ist}| = e^{rt}$). In my paper,[4] a non-potential formulation is used so that the filament is allowed to grow or decay. It was shown that the decay or growth rate is not generally linear.

In Sec. II, I provide a brief errata to Ref. 1. In Sec. III, I discuss how one can account for rotational and translational motions of the filaments in question, and how the purely rotating filaments are still solutions to the Svistunov model. In Sec. IV, I discuss why one might want to study the purely rotating filaments of Ref. 1 or the rotating and translating filaments, and how one might expect to find them. Finally, in Sec. V, I discuss the case where the motion cannot be partitioned into pure rotational and translational motions. In this case, a non-stationary solution must be sought.

II. ERRATA TO REF. 1

(i) In Eq. (14) of Ref. 1, there should be a factor of γ multiplying the second term (so as to keep the notation consistent with the rest of the paper).

(ii) In Eq. (17) of Ref. 1, the imaginary unit "i" should be omitted from the second line, hence this term should read $\mathbf{r}(x,t) = (x, R(x)\cos(\Theta(x) - \gamma t), R(x)\sin(\Theta(x) - \gamma t))$.

(iii) There was a typo in the helical reduction (in the curvature-torsion frame) which carried through, and the representation given in the Comment[3] is correct. This was a last minute addition at the behest of a reviewer, and I did not check this as carefully as I should have.

III. MATHEMATICS OF THE TRANSLATING FILAMENT

Consider the equation

$$i\Psi_t + \gamma \frac{\Psi_{xx}}{(1+|\Psi_x|^2)^{3/2}} + \frac{\gamma}{2}\frac{\Psi_x\left(\Psi_x^*\Psi_{xx} - \Psi_x\Psi_{xx}^*\right)}{(1+|\Psi_x|^2)^{3/2}} = 0\,, \tag{1}$$

which was shown to be equivalent to the equation of motion for a vortex filament under the Svistunov model of the LIA. Here, the last factor is due to translational motion of the filament. As discussed in Ref. 1, translational velocity scales as $\beta \sim \Psi_x^*\Psi_{xx} - \Psi_x\Psi_{xx}^*$, so when the motion is purely rotational (i.e., $\beta \to 0$) the latter terms necessarily vanish. As such, the solutions of Ref. 1 are indeed the purely rotational solutions.

Keeping the translational motion (so that we no longer have purely rotating filaments), and defining

$$\Psi(x,t) = e^{-\gamma i t}\psi(x)\,, \quad \text{where} \quad \psi(x) = R(x)\exp(i\Theta(x))\,, \tag{2}$$

we find that (1) gives

$$\begin{aligned} R &+ \frac{R'' - R\Theta'^2 + (2R'\Theta' + R\Theta'')i}{[1 + R'^2 + R^2\Theta'^2]^{3/2}} + \frac{R'^2(2R'\Theta' + R\Theta'') - RR'\Theta'(R'' - R\Theta'^2)}{[1 + R'^2 + R^2\Theta'^2]^{3/2}} \\ &- \frac{RR'\Theta'(2R'\Theta' + R\Theta'') - (R\Theta')^2(R'' - R\Theta'^2)}{[1 + R'^2 + R^2\Theta'^2]^{3/2}} i = 0\,. \end{aligned} \tag{3}$$

The final two terms are rather complicated, but these are simply the terms used to account for translational effects. Separating this equation into real and imaginary parts, we obtain a system of two real ordinary differential equations (ODEs) for the unknown functions R and Θ, to wit,

$$R + \frac{R'' - R\Theta'^2 + R'^2(2R'\Theta' + R\Theta'') - RR'\Theta'(R'' - R\Theta'^2)}{[1 + R'^2 + R^2\Theta'^2]^{3/2}} = 0\,, \tag{4}$$

$$2R'\Theta' + R\Theta'' - RR'\Theta'(2R'\Theta' + R\Theta'') + (R\Theta')^2(R'' - R\Theta'^2) = 0\,. \tag{5}$$

Clearly, (4) and (5) govern the motion of a vortex filament with both translational and rotational motions under the Svistunov model. In the absence of translational motion, the system reduces to that studied in Ref. 1, which was valid for purely rotational motion.

While a variety of analytical properties of the solutions discussed in Ref. 1 were given, the far more complicated structure of the system (4) and (5) precludes and such elegant results, even in the qualitative sense. Still, we can numerically simulate the solutions R, Θ to the system (4) and (5). While the formulation which takes into account the translational motion of the filament is far more complicated, in a structural sense we can obtain the same types of filaments, although some parameter values will need to be selected differently. In general, the initial condition $R(0)$ can be taken smaller to get the same type of results seen in Ref. 1. The reason for this is that in the case where translation is neglected, we have pure rotation alone, while in the case where translation is included, we have similar structures, with the primary difference being that such structures are moving along the x-axis instead of remaining stationary with respect to this axis. With translation, the larger amplitude solutions tend to be even less stable than those in Ref. 1, so very specialized structures such as helical or planar filaments occur for smaller amplitude deflections from the x-axis. However, in reference to the variety of structures found, the vortex filament shapes found in Ref. 1 can be found in the translational model (with appropriate modifications to the model parameters. As such, the results of Ref. 1 are indeed sufficient to give us a qualitative understanding of the structure of general rotating vortex filaments under the Svistunov model (which was indeed the point of that paper).

We also find that, for the very large amplitude regime, the translational effects can have a disruptive effect on the filaments. For such large amplitude deflections occurring in actual experiments, we in general expect a smoothing over time. This is, in particular, true of the superfluid filaments that contend with mutual friction effects leading to dissipation. Dissipating helical filaments were studied in Ref. 4.

IV. USEFULNESS OF THE ROTATING FILAMENTS

The authors of the Comment[3] pose a question on the practical utility of studying these generalized filaments. While the planar and helical filaments have previously occupied a prominent place in the literature, there are several reasons why one should consider more generalized rotating filaments.

The helical filament is rather idealized. In practice, there may be perturbations or deformities along the helical filament, meaning that the filament is not purely helical. As we demonstrated in Ref. 1, these "almost helical" (or "almost planar," for that matter) filaments can still be described as generalized rotating filament solutions, and hence can be studied under this framework.

A planar filament is highly idealized, yet still interesting from the aspect that it is a stationary solution (it maintains its shape as it moves in space). However, the planar filament is an example of a completely torsion-less solution. Realistically, in experiments there will be at least some friction and loss of energy, and the planar solution would be expected to bend and deform. While such solutions are no longer planar, they can still be described under the framework given here (since they are still within the class of generalized rotating filaments). Therefore, realistic experimental "approximations" to the planar filament can be described in the framework.

For the large-amplitude regime, we were able to find that far more exotic dynamics are possible. This is useful for two reasons. First, it suggests that any initial large-amplitude helical or planar filament can give way to a more complicated structure, as we suspect that the large amplitude solutions would realign due to instability. Second, in the case of strong torsion effects (corresponding to experiments with more viscous fluids), we also find some more of these more exotic solutions. So, there are two obvious possible routes to obtaining such solutions experimentally. It is likely that such solutions will be relatively short lived, and will eventually give way to turbulence. The same could be said of large amplitude solutions even in the presence of translational motion.

In summary, studying more general mathematical vortex filaments than just purely planar or helical structures can prove useful, since these more general structures can account for greater variability in experimental vortex filament structure. Meanwhile, the larger amplitude solutions might give way to more complicated (non-stationary) dynamics, and in that sense might be useful for investigating the small time dynamics of such problems. We say more on this latter point in Sec. V.

V. EXTENSIONS FOR THE NON-STATIONARY VORTEX FILAMENT CASE

The translating and rotating vortex filaments exhibit translational motion along a single axis (which we refer to as the x-axis) and rotate about this axis in the other two axes (which we refer to as the y- and z-axes). When the motion is completely rotational, as was the case considered in Ref. 1, note that the assumption (2) makes complete sense, as it gives completely rotational motion (due to the $U(1)$ symmetry of (1)). If one then includes translation (as suggested in Ref. 3 and discussed above), we still obtain a valid system of Eqs. (4) and (5). However, there is a subtle issue. In deriving this system, one must assume that the motion can perfectly be decomposed into transverse and rotational parts, that is, one assumes that the vortex filament takes the form

$$\mathbf{r}(x,t) = (x + \beta(t),\, R(x)\cos(\Theta(x) - \gamma t),\, R(x)\sin(\Theta(x) - \gamma t))\,. \tag{6}$$

However, it may be the case that the translational and rotational motions cannot be decomposed so easily.

In summary, for the purely rotating filaments (as discussed in Ref. 1), the assumption of a "stationary solution" (stationary in the sense that only $U(1)$ rotations are permitted) as given in (2) makes complete sense. However, if the translational motion is particularly quick, or if there are other motions (perhaps there are time-dependent effects in the y- and z-coordinates, as well), this assumption of a stationary solution for the translational case may break down. This makes sense, as the purely rotating solutions are but one special case, and as varied as the solutions presented in Ref. 1 were, there are still many other possible solutions that do not exhibit pure rotation (which were not considered in Ref. 1, as they did not fit the theme of that paper). Some of these solutions may not even take the form (2), since this assumption presupposes that rotational and translational effects can be partitioned in some nice way.

Let us consider what would be required to study non-stationary solutions to (1). Assuming that there is rotational motion (but that this is not the only motion in the y- or z-axes), we can take

$$\Psi(x,t) = e^{-i\omega T} u(x,T)\,, \quad \text{where} \quad T = \gamma t\,, \tag{7}$$

where ω is a spectral parameter. Then, (1) becomes

$$\omega u + i u_t + \frac{\partial}{\partial x}\left(\frac{u_x}{\sqrt{1+|u_x|^2}}\right) = 0\,. \tag{8}$$

Instead of solving a system of real ODEs, we must instead solve a system of real partial differential equations (PDEs). Letting $u(x, t) = A(x, t) + iB(x, t)$ for real-valued functions $A(x, t)$ and $B(x, t)$, we find

$$A_t + \omega B + \frac{\partial}{\partial x}\left(\frac{B_x}{\sqrt{1+A_x^2+B_x^2}}\right) = 0\,, \tag{9}$$

$$B_t - \omega A - \frac{\partial}{\partial x}\left(\frac{A_x}{\sqrt{1+A_x^2+B_x^2}}\right) = 0\,. \tag{10}$$

Solving this system, one obtains the vortex filament solution

$$\mathbf{r}(x,t) = (x+\beta(t), \cos(\omega T)A(x,T) + \sin(\omega T)B(x,T), \cos(\omega T)B(x,T) - \sin(\omega T)A(x,T))\,. \tag{11}$$

To see the connection with these solutions and those of Ref. 1, let us assume that a solution $\Psi(x, t) = e^{-\gamma it}R(x)e^{i\Theta(x)}$ found in Ref. 1 is taken at time $t = 0$, and then as time increases the filament is given by the evolution equations (9) and (10). Then, $\Psi(x, 0) = R(x)\exp(i\Theta(x))$. Yet, $\Psi(x, 0) = A(x, 0) + iB(x, 0)$ from the choice of $u(x, t)$. Therefore, we can pick initial conditions

$$A(x,0) = R(x)\cos(\Theta(x)) \quad \text{and} \quad B(x,0) = R(x)\sin(\Theta(x))\,. \tag{12}$$

Then, solving the evolution equations (9) and (10) subject to initial conditions (12) (where the functions $R(x)$ and $\Theta(x)$ are those found for the purely rotating vortex filaments), we allow for new solutions which initially match the purely rotating filaments but which exhibit translational and other motions as time increases. Doing so, we shall in general obtain unsteady solutions (as opposed to stationary solutions) which depend on time in some non-uniform way (as opposed to the uniform rotation or translation discussed above). Such solutions could include cases that cannot be considered through the previous framework, such as self-similar solutions (quasi-stationary states) or solutions giving more complicated temporal dynamics.

This technique could also be used to study the time-evolution of unstable large amplitude solutions previously studied in Ref. 1. Indeed, one can treat the more exotic solutions found in Ref. 1 as initial conditions, and then study the evolution of a solution pair $A(x, t)$ and $B(x, t)$. This would be useful in determining which (if any) of the solutions in Ref. 1 may lead to chaos or other complicated dynamics.

[1] R. A. Van Gorder, "General rotating quantum vortex filaments in the low-temperature Svistunov model of the local induction approximation," Phys. Fluids **26**, 065105 (2014).
[2] B. Svistunov, "Superfluid turbulence in the low-temperature limit," Phys. Rev. B **52**, 3647 (1995).
[3] N. Hietala and R. Hänninen "Comment on 'General rotating quantum vortex filaments in the low-temperature Svistunov model of the local induction approximation' [Phys. Fluids **26**, 065105 (2014)]," Phys. Fluids **26**, 119101 (2014).
[4] R. A. Van Gorder, "Decay of helical Kelvin waves on a quantum vortex filament," Phys. Fluids **26** , 075101 (2014).
[5] G. Boffetta, A. Celani, D. Dezzani, J. Laurie, and S. Nazarenko, "Modeling Kelvin wave cascades in superfluid helium," J. Low Temp. Phys. **156**, 193 (2009).
[6] N. Hietala and R. Hänninen, "Comment on 'Motion of a helical vortex filament in superfluid 4He under the extrinsic form of the local induction approximation' [Phys. Fluids **25**, 085101 (2013)]," Phys. Fluids **26**, 019101 (2014).
[7] R. A. Van Gorder, "Response to 'Comment on "Motion of a helical vortex filament in superfluid 4He under the extrinsic form of the local induction approximation' [Phys. Fluids **26**, 019101 (2014)]," Phys. Fluids **26**, 019102 (2014).