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The Applications of Utility Theory in Insurance Industry

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Abstract: In this paper, The Applications of Utility Theory in insurance industry are discussed from two ways.First of all we consider the insurance pricing from both insurers and insured , and makes the strict explanation from the value example to the St. Petersburg paradox. .Then we discuss insurance pricing between the risk swap agreement insurers and give the value example.
Key words: Utility Theory, Utility function, Insurance premium,expected Utility, Risk Theory（LIU, WANG & GUO. 2007）

1. INTRODUCTION

The insurance pricing is always the core of insurance business. Although the price pattern is commonly fixed by “pure insurance premium and attachment insurance premium” in insurance practice and books , theoretically speaking, the insurance product is the same as other commodity. Its price is essentially decidied by the market supply-demand relation. What is particularly is that it is not to fix price for the visible product merely, but to invisible “risk”. Here the risk can be understanded as the adjustment or the loss random variable(S.M.Ross. 2005). As the matter stands, the insurance pricing in formally is to establish one kind of price measures, which is possible to use one kind of precise quantity (insurance premium) to weigh an indefinite loss. So we discuss the insurance pricing question from the economic utility theory in this paper.

2. DISCUSSING INSURANCE PRICING SEPARATELY FROM INSURER AND ISURED'S ANGLE(QIN GUI-XIA.2008)

First, we analyse the insurance pricing from the insurer and insured's value structure separately. Suppose somebody has the property valuing w, but this property faces some kind of latent loss, which is

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expressed as a random variable X , $0 \le X \le w$ .The probability distribution records is $F(x)$ .Our question is how many insurance premiums he have to take out for this insurance? According to Utility theory (WANG, JIANG & LIU. 2003), the fewer the insurance premium H is, the better for the insured The highest insurance premium is the solution when "insurance effectiveness" was equal to "insurance effectiveness not to take out".

If the insured is willing to take out insurance, he only loses the insurance premium whether the loses occur or not. And the insured still has $w-H$, supposes its effectiveness for the insured is $u(w-H)$;If the insured does not take out insurance, in fact its property is the random variable $w-X$, we record the effectiveness of this random variable as $U([w-X])$. Therefore, to the property owner, the insurance premium should satisfy:

$$u(w-H) \ge U([w - X])$$

H bigger, $w-H$ is smaller, and insurance effectiveness $u(w-H)$ is also smaller. When the equal sign is established, it does not matter whether to participate or not . The highest insurance premium H^* which can be accepted by the insured is the solution when the equation equal sign is established.

In another inspect, considering from insurer's angle, if insuring, the insurer may increase an insurance premium income G in original wealth foundation v , but undertake the risk for the insured. Its wealth becomes the random variable $v+G-X$. How many insurance premiums should the insurer charge to insure the property owner's risk? Similarly, the higher G is, the better is to the insurer. Suppose the insurer records the determination quantity and random variable effectiveness for u_1 and U_1 separately .Then the reasonable premiums should satisfy the following effectiveness inequality:

$$U_1([\mathrm{v} + \mathrm{G} - \mathrm{X}]) \ge u_1(v)$$

The smaller G is, the smaller $U_1([\mathrm{v} + \mathrm{G} - \mathrm{X}])$ is, When the equal sign establishes, the insurance has not any attraction. Therefore the insurer is willing to accept the lowest insurance premium G_* G_* which can be accepted by the insurer. And G is the solution when the equation equal sign establishes.

Therefore, only the highest insurance premium H^* which the insurer is willing to pay is more than the lowest insurance premium G_* which the insurer is willing to accept , could a reasonable insurance contract be situated between H^* and G_* . Figure 1 shows the relations among critical insurance premium H^* , G_* and pure insurance premium $E[X]$ as well as actual price P .

By Utility Theory, most people hate the risk. By the Jensen inequality (XIE, HAN. 2000), the loathing risk's policy holder is willing to pay higher insurance premium to take out insurance, namely $H^* > E[X]$. If $H^* < G_*$, it is unable to finalize a deal.

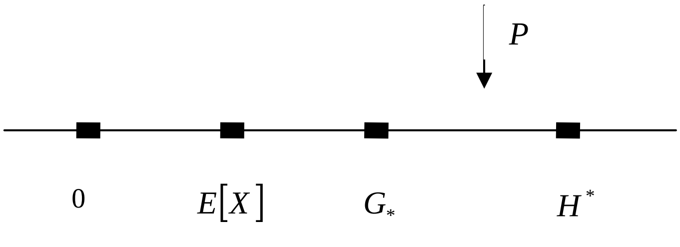

Figure 1 $E[X]$, H^*, G_* and P

The following is a famous gambling example using the utility function to fix the safe product price. Although it is not a direct safe policy-making question, it contains the same essence.

St. Petersburg paradox (GUO.2004) There is a fable that one kind of gambling is popular in the St. Petersburg in the past street corner. The rule is all participant prepaid certain number money, for instance 100 rubles, then threw the cent, the gambling was terminated when the person surface dynasty presented first time; If the person surface dynasty did not present until the n talent, the participant took back 2^n rubles. The question is that whether the policymaker take part in the gambling

Suppose the cent is even. The probability that the person surface dynasty does not present until the n talent is $p(n) = 2^{-n}$.The corresponding repayment value is $2^n - 100$, $n = 1,2,3\ldots$.Therefore, the average repayment of "participating the gambling" is $E = \sum_{n=1}^{\infty}(2^n - 100)2^{-n} = +\infty$, but the average repayment of " not participating the gambling" is obviously 0.It looks like that the policymaker can win (on average) "the infinite many rubles" by spending 100 rubles. It seems that participating the gambling is absolutely worthwhile. But the actual situation was contrary; extremely few can take back 100 rubles above situations.

In fact, according to utility theory, what we should consider is the Utility function $u(x_n)$ of policymaker , not the amount value x_n itself, and policymaker's wealth level (recorded as w) will also affect his effectiveness. Generally, suppose the policy-maker is willing to pay the price p to attend this game, recorded as $x_n = w + 2^n - p$, by now, the probability of "participating in the gambling" is still $p(x_n) = 2^{-n}$, $x_n = w + 2^n - 100$, $n = 1,2,\ldots$ the expected utility value of "participating in the gambling" is $EU = \sum_{n=1}^{\infty} u(x_n)p(x_n) = \sum_{n=1}^{\infty} u(w + 2^n - p) \times 2^{-n}$.

Generally speaking, the most policy-makers are loathe the risk, only when it could bring bigger utility than expected, the policymaker is willing to take part in the gambling. Namely:

$$u(w) \le \sum_{n=1}^{\infty} u(w + 2^n - p) \times 2^{-n}$$

We might select a model risk loathing function $y = \ln x$ to take policy-maker's utility function. As simplified computation, here suppose policy-maker's wealth level is for $w = 10000$ rubles, therefore the expected utility of participating in the gambling is:

$$EU = \sum_{n=1}^{\infty} u(x_n)p(x_n) = \sum_{n=1}^{\infty} \ln(10000 + 2^n - p) \times 2^{-n}$$

When $u(10000) \le \sum_{n=1}^{\infty} u(10000 + 2^n - p) \times 2^{-n}$, namely $p \le 14.25$

policy-maker will choose "participating in the gambling".

That is, although this game's expectation repayment is infinite , the policy-maker is only willing to pay the minimum price to attend this game. If "participating in the gambling" is regarded as insurance product, policymakers with 10000 rubles is willing to pay 14.25 rubles to take out insurance at most.

3. INSURANCE PRICING BETWEEN INSURERS

In the reinsurance arrangement, stopping the loss reinsurance (LIU.2007). is the most superior. But in reinsurance practice, what needs to consider is not only the benefit original insurance company but the reinsurance company. In safe practice, to ensure the security, often two or more insurance companies sign one risk agreement which is advantageous for both through the negotiations , namely the two companies takes the original insurer and the reinsurance person's dual statuses appears at the same time.

Supposes Insurance company A and Insurance company B has a chit respectively, random variable X_1 and X_2 stand for their loss separately. And

$F_1(x)$ and $F_2(x)$ stand for the distribution function separately .Moreover, supposes initial reserve fund of company A and the company B respectively for w_1 and w_2 .For simplifing model, supposes the insurers only charge the insurance premium from the insured, namely $p_1 = \int_0^{\infty} x dF_1(x)$, $p_2 = \int_0^{\infty} x dF_2(x)$ .Insurance company A and the B utility function was standed for $u_1(x)$ and $u_2(x)$ separately. If both the two companys' services have the indemnity with their amount respectively for x_1 and x_2 .According to the contract provision, the amount which insurance company A will pay is $y(x_1, x_2)$, Insurance company B pays the surplus indemnity $x_1 + x_2 - y(x_1, x_2)$. Because these two company's benefit is opposite, therefore they have to carry on the negotiations in the function $y(x_1, x_2)$, making the bilateral expected utility value as big as possible.In which,

$$U_1(y) = \int_0^{\infty}\int_0^{\infty} u_1(w_1 + p_1 - y(x_1, x_2)) dF(x_1)F(x_2)$$

$$U_2(y) = \int_0^{\infty}\int_0^{\infty} u_2(w_2 + p_2 - x_1 - x_2 + y(x_1, x_2)) dF(x_1)F(x_2)$$

Obviously, both two companies are seeking to achieving the biggest effectiveness. According to the Pareto thought , the necessary and sufficient condition of optimal solution $y(x_1, x_2)$ is: $u_1'(w_1 + p_1 - y(x_1, x_2)) = k u_2'(w_2 + p_2 - x_1 - x_2 + y(x_1, x_2))$ in which $k \geq 0$. (1)

the proof for details sees (WANG Gang.2003).

Only when the expected utility is bigger than do not cooperate ,can companies choose the cooperation. Namely:

$$U_1(0) \leq U_1(y) \quad U_2(0) \leq U_2(y)$$

From this we may obtain the value scope of $y(x_1, x_2)$ which satisfies the condition

Suppose the two insurance companies are known for the effectiveness of monetary :

$$u_1(x) = -a_1 x^2 + x \quad u_2(x) = -a_2 x^2 + x$$

by the type (1), the necessary and sufficient condition of optimal solution become:

$$2a_1(w_1 + p_1 - y(x_1, x_2)) - 1 = 2a_2 k(w_2 + p_2 - x_1 - x_2 + y(x_1, x_2)) - k$$

by the above equation,

$$y(x_1,x_2)=\frac{2a_1(w_1+p_1)-2a_2k(w_2+p_2)+2a_2k(x_1+x_2)+k-1}{2(a_1+a_2k)}$$

$$=\frac{a_2k}{a_1+a_2k}(x_1+x_2)+\frac{a_1}{a_1+a_2k}p_1-\frac{a_2k}{a_1+a_2k}p_2+\frac{2a_1w_1-2a_2kw_2+k-1}{a_1+a_2k}$$

Making $h=\frac{a_1}{a_1+a_2k}$, $1-h=\frac{a_2k}{a_1+a_2k}$,

$$Q=\frac{2a_1w_1-2a_2kw_2+k-1}{2(a_1+a_2k)}=(1-h)(\frac{1}{2a_2}-w_2)-h(\frac{1}{2a_1}-w_1)$$

Therefore, $y(x_1,x_2)=(1-h)(x_1+x_2)+hp_1-(1-h)p_2+Q$

By the above equation, we can see that if in the company A has the amount for claim, then it only pays a corresponding round number, other parts are paid by company B.

When the Insurance company A utility function is $u_1(x)=-a_1x^2+x$, company's initial utility is

$$U_1(0)=\int_0^\infty u_1(w_1+p_1-x_1)dF_1(x_1)=-a_1w_1^2-a_1\int_0^\infty (p_1-x_1)^2dF_1(x_1)+w_1$$

Reorganized this type may write

$$U_1(0)=\frac{1}{4a_1}-a_1(\frac{1}{2a_1}-w_1)^2-a_1V_1 \text{ in which } V_1=\int_0^\infty (x-p_1)^2dF_1(x)$$

Similarly initial utility of company B is:

$$U_2(0)=\int_0^\infty u_2(w_2+p_2-x_2)dF_2(x_2)=\frac{1}{4a_2}-a_2(\frac{1}{2a_2}-w_2)^2-a_2V_2$$

in which $V_2=\int_0^\infty (x-p_2)^2dF_2(x)$

making $w_1=3$, $w_2=4$, $V_1=2$, $V_2=3$, $a_1=1/6$, $a_2=1/6$

then $U_1(0)=3/2-6(3-3)^2-1/6\times 2=7/6$

$$U_2(0)=3/2-6(3-4)^2-1/6\times 3=5/6$$

$$U_1(y)=3/2-1/6(1-h)^2\left[(6-7)^2+5\right]=3/2-(1-h)^2$$

$$U_2(y)=3/2-1/6h^2\left[(6-7)^2+5\right]=3/2-h^2$$

By $U_1(0)\le U_1(y)$ and $U_2(0)\le U_2(y)$, we can see :

$$1-\sqrt{3}/3\le h\le \sqrt{6}/3$$

The Nash solution has gave the maximization of h:

$$Max = [U_1(y) - U_1(0)][U_2(y) - U_2(0)] = \left[\tfrac{1}{3} - (1-h)^2\right]\left[\tfrac{2}{3} - h^2\right]$$

Solution: $h \approx 0.613$ $y(x_1, x_2)$

The optimal solution namely: $y(x_1, x_2) = 0.387(x_1 + x_2) + 0.613 p_1 - 0.387 p_2$ - 0.387

So the most superior effectiveness of two company is:

$U_1(y) = 1.350231$ $U_2(y) = 1.124231$

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