

STEADY-STATE ANALYSIS OF THE GI/M/1 QUEUE WITH MULTIPLE VACATIONS AND SET-UP TIME

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Abstract: In this paper, we consider a GI/M/1 queueing model with multiple vacations and set-up time. We derive the distribution and the generating function and the stochastic decomposition of the steady-state queue length, meanwhile, we get the waiting time distributions.

Key words: multiple vacations, set-up time, stochastic decomposition

INTRODUCTION

Vacation queues servers to stop the customers' service at some periods, and the time during which the service is interrupted is called the vacation time. Vacation queue research originated from Levy and Yechial, then many researchers on queueing theory deal with this fields. So far, the theory frame whose core is the stochastic decomposition is developed and vacation queues have been applied successfully to many fields, such as computer systems, communication networking, flexible manufacture, electronic and call centers. Details can be seen in the surveys of Doshi and the monographs of Tian. For GI/M/1 type queues with server vacations, Tian used the matrix geometric solution method to analyze and obtained the expressions of the rate matrix and proved the stochastic decomposition properties for queue length and waiting time in a GI/M/1 vacation model with multiple exponential vacations. Using matrix-geometric approach, we give expressions of distributions for queue length at an arrival epoch and the steady-state distribution for the waiting time.

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1. DESCRIPTION OF THE MODEL

Consider a classical GI/M/1 queue, inter-arrival times are i.i.d.r.v.s. Let $A(x)$ and $a^*(s)$ be the distribution function and L.S transform of the inter-arrival time A of customers. The mean inter-arrival time is $E(A) = -a^{*'}(0) = \lambda^{-1}$, Service times during service period, vacation times and set-up times are assumed to be exponentially distributed with rate μ, θ, β , respectively. We assume that the service discipline is FCFS.

Suppose τ_n be the arrival epoch of nth customers with $\tau_0=0$. Let $L_n = L_v(\tau_n^-)$ be the number of the customers before the nth arrival. Define

$$J_n = J(\tau_n) = \begin{cases} 0, & \text{the nth arrival occurs during a service period,} \\ 1, & \text{the nth arrival occurs during a set - up period,} \\ 2, & \text{the nth arrival occurs during a vacation period.} \end{cases}$$

The process $\{(L_n, J_n), n \geq 1\}$ is a Markov chain with the state space

$$\Omega = \{(0, 2) \cup (k, j), k \geq 1, j = 0, 1, 2\}.$$

Meanwhile, we introduce the expressions below

$$a_k = \int_0^\infty \frac{(\mu t)^k}{k!} e^{-\mu t} dA(t), k \geq 0, \quad b_k = \int_0^t \int_0^t \beta e^{-\beta x} \frac{[\mu(t-x)]^k}{k!} e^{-\mu(t-x)} dx dA(t), k \geq 0,$$

$$c_k = \int_0^t \int_0^{t-y} \int_0^{t-x-y} \theta e^{-\theta x} \beta e^{-\beta y} \frac{[\mu(t-x-y)]^k}{k!} e^{-\mu(t-x-y)} dx dy dA(t), k \geq 0.$$

First, the transition from $(i, 0)$ to $(j, 0)$ occur if $i + 1 - j$ services complete during an inter-arrival time. Therefore, we have

$$p_{(i,0)(j,0)} = a_{i+1-j}, i \geq 1, j = 1, \dots, i + 1.$$

Similarly,

$$p_{(i,1)(j,0)} = b_{i+1-j}, i \geq 1, j = 1, \dots, i + 1. \quad p_{(i,1)(i+1,1)} = \int_0^\infty e^{-\beta t} dA(t) = a^*(\beta) = \gamma_2.$$

$$p_{(i,2)(i+1,2)} = \int_0^\infty e^{-\theta t} dA(t) = a^*(\theta) = \gamma_3. \quad p_{(i,2)(j,0)} = c_{i+1-j}, i \geq 0, j = 1, \dots, i + 1.$$

$$p_{(i,2)(i+1,1)} = \int_0^t \int_0^t \theta e^{-\theta x} e^{-\beta(t-x)} dA(t) = \theta(a^*(\beta) - a^*(\theta)) / (\theta - \beta) = \alpha(\gamma_2 - \gamma_3).$$

The transition matrix of (L_n, J_n) can be written as the Block-Jacobi matrix

$$\tilde{P} = \begin{bmatrix} B_{00} & A_{01} & & & & \\ B_1 & A_1 & A_0 & & & \\ B_2 & A_2 & A_1 & A_0 & & \\ B_3 & A_3 & A_2 & A_1 & A_0 & \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

where $B_{00} = 1 - c_0 - \alpha(\gamma_2 - \gamma_3) - \gamma_3$, $A_{01} = (c_0, \alpha(\gamma_2 - \gamma_3), \gamma_3)$,

$$A_0 = \begin{pmatrix} a_0 & 0 & 0 \\ b_0 & \gamma_2 & 0 \\ c_0 & \alpha(\gamma_2 - \gamma_3) & \gamma_3 \end{pmatrix}, \quad A_k = \begin{pmatrix} a_k & 0 & 0 \\ b_k & 0 & 0 \\ c_k & 0 & 0 \end{pmatrix}, k \geq 1$$

$$B_k = \begin{pmatrix} 1 - \sum_{i=0}^k a_i \\ 1 - \sum_{i=0}^k b_i - \gamma_2 \\ 1 - \sum_{i=0}^k c_i - \alpha(\gamma_2 - \gamma_3) - \gamma_3 \end{pmatrix}, k \geq 1,$$

The matrix \tilde{P} is a GI/M/1 type matrix.

2. STEADY-STATE DISTRIBUTION

Lemma1. If $\rho = \lambda\mu^{-1} < 1$, $\theta, \beta > 0$, then $\delta > 0$, $\frac{\theta\beta}{\theta - \beta}(\delta - \Delta) > 0$.

where $\delta = \frac{\gamma_1 - a^*(\beta)}{\beta - \mu(1 - a^*(\beta))} = \frac{\gamma_1 - \gamma_2}{\beta - \mu(1 - \gamma_2)}$, $\Delta = \frac{\gamma_1 - a^*(\theta)}{\theta - \mu[1 - a^*(\theta)]} = \frac{\gamma_1 - \gamma_3}{\theta - \mu(1 - \gamma_3)}$. (1)

Theorem 1. If $\rho < 1$, $\theta, \beta > 0$, then the matrix equation $R = \sum_{k=0}^{\infty} R^k A_k$ has the minimal non-negative solution

$$R = \begin{pmatrix} \gamma_1 & 0 & 0 \\ \beta\delta & \gamma_2 & 0 \\ \alpha\beta(\delta - \Delta) & \alpha(\gamma_2 - \gamma_3) & \gamma_3 \end{pmatrix}$$

where γ_1 is the unique roots in the range $0 < z < 1$ of the equation $z = a^*(\mu(1 - z))$.

$\alpha = \theta/(\theta - \beta)$, δ and Δ are defined as in (1).

Proof. Because all $A_k, k \geq 0$ are upper triangular, we assume that R has the same structure as

$$R = \begin{pmatrix} r_{11} & 0 & 0 \\ r_{21} & r_{22} & 0 \\ r_{31} & r_{32} & r_{33} \end{pmatrix}$$

we easily obtain

$$\begin{cases} r_{11} = \sum_{k=0}^{\infty} r_{11}^k a_k = a^*(\mu(1-r_{11})), & r_{22} = \gamma_2, & r_{33} = \gamma_3 \\ r_{21} = \sum_{k=1}^{\infty} r_{21} \left(\sum_{i=0}^{k-1} r_{11}^i r_{22}^{k-1-i} \right) a_k + \sum_{k=0}^{\infty} r_{22}^k b_k, & r_{32} = \alpha(\gamma_2 - \gamma_3) \\ r_{31} = \sum_{k=1}^{\infty} r_{31} \left(\sum_{i=0}^{k-1} r_{11}^i r_{33}^{k-1-i} \right) a_k + \sum_{k=2}^{\infty} r_{32} r_{21} \left(\sum_{i=0}^{k-2} r_{11}^i \sum_{j=0}^{k-2-i} r_{22}^j r_{33}^{k-2-i-j} \right) a_k \\ \quad + \sum_{k=1}^{\infty} r_{32} \left(\sum_{i=0}^{k-1} r_{22}^i r_{33}^{k-1-i} \right) b_k + \sum_{k=0}^{\infty} r_{33}^k c_k \end{cases} \quad (2)$$

As we known, if $\rho < 1, \theta > 0$, the first equation has the unique roots $r_{11} = \gamma_1$ in the range $0 < r_{11} < 1$.

We can compute

$$\sum_{k=0}^{\infty} r_{22}^k b_k = \frac{\beta[a^*(\mu(1-r_{22})) - r_{22}]}{\beta - \mu(1-r_{22})} = \frac{\beta[a^*(\mu(1-\gamma_2)) - \gamma_2]}{\beta - \mu(1-\gamma_2)},$$

$$1 - \sum_{k=1}^{\infty} \left(\sum_{i=0}^{k-1} r_{11}^i r_{22}^{k-1-i} \right) a_k = 1 - \sum_{k=1}^{\infty} \frac{\gamma_1^k - \gamma_2^k}{\gamma_1 - \gamma_2} a_k = 1 - \frac{\gamma_1 - a^*(\mu(1-\gamma_2))}{\gamma_1 - \gamma_2} = \frac{a^*(\mu(1-\gamma_2)) - \gamma_2}{\gamma_1 - \gamma_2},$$

Finally, we obtain $r_{21} = \beta\delta$, $r_{31} = \alpha\beta(\delta - \Delta)$ and the expression for R .

Theorem 2. The Markov chain (L_n, J_n) is positive recurrent if and only if $\rho < 1, \theta, \beta > 0$.

Proof. Based on Neuts, the Markov chain (L_n, J_n) is positive recurrent if and only if the spectral radius $SP(R) = \max\{\gamma_1, \gamma_2, \gamma_3\}$ of R is less than 1, and the matrix

$$B[R] = \begin{pmatrix} B_{00} & A_{01} \\ \sum_{k=1}^{+\infty} R^{k-1} B_k & \sum_{k=1}^{+\infty} R^{k-1} A_k \end{pmatrix}$$

has a positive left invariant vector. Evidently, $SP(R) = \max\{\gamma_1, \gamma_2, \gamma_3\} < 1$. Substituting the expressions for R, A_k and B_k in $B[R]$, we obtain

$$B[R] = \begin{pmatrix} 1 - c_0 & c_0 & 0 & 0 \\ \frac{a_0}{\gamma_1} & 1 - \frac{a_0}{\gamma_1} & 0 & 0 \\ 1 - \frac{\beta\delta a_0}{\gamma_1\gamma_2} + \frac{b_0}{\gamma_2} & \frac{\beta\delta a_0}{\gamma_1\gamma_2} - \frac{b_0}{\gamma_2} & 0 & 0 \\ A & B & \alpha \frac{\gamma_2 - \gamma_3}{\gamma_3} & 1 \end{pmatrix}$$

Where

$$A = -\alpha\beta \left(\frac{\delta - \Delta}{\gamma_1\gamma_3} + \frac{\delta(\gamma_2 - \gamma_3)}{\gamma_1\gamma_2\gamma_3} \right) a_0 - \alpha \frac{\gamma_2 - \gamma_3}{\gamma_2\gamma_3} + \frac{c_0}{\gamma_3} - \alpha \frac{\gamma_2 - \gamma_3}{\gamma_3}$$

$$B = \alpha\beta \left(\frac{\delta - \Delta}{\gamma_1\gamma_3} + \frac{\delta(\gamma_2 - \gamma_3)}{\gamma_1\gamma_2\gamma_3} \right) a_0 + \alpha \frac{\gamma_2 - \gamma_3}{\gamma_2\gamma_3} - \frac{c_0}{\gamma_3}.$$

It can be verify that $B[R]$ has the left invariant vector

$$\pi_0 = K(1, \alpha\beta(\delta - \Delta), \alpha(\gamma_2 - \gamma_3), \gamma_3). \quad (3)$$

Thus, if $\rho < 1$, $\theta, \beta > 0$, the Markov chain (L_n, J_n) is positive recurrent.

If $\rho < 1, \theta, \beta > 0$, let (L_v, J) be the stationary limit of the process (L_n, J_n) . Let

$$\pi_0 = \pi_{02}; \quad \pi_k = (\pi_{k0}, \pi_{k1}, \pi_{k2}), k \geq 1,$$

$$\pi_{kj} = P\{L = k, J = j\} = \lim_{n \rightarrow \infty} P\{L_n = k, J_n = j\}, (k, j) \in \Omega.$$

Theorem 3. If $\rho < 1$, the stationary probability distribution of (L_v, J) is

$$\begin{cases} \pi_{k0} = K\alpha\beta \left(\frac{\gamma_1^k - \gamma_2^k}{\gamma_1 - \gamma_2} \delta - \frac{\gamma_1^k - \gamma_3^k}{\gamma_1 - \gamma_3} \Delta \right), k \geq 1, \\ \pi_{k1} = K\alpha(\gamma_2^k - \gamma_3^k), k \geq 1, \\ \pi_{k2} = K\gamma_3^k, k \geq 0, \end{cases}$$

where $K = \frac{(1 - \gamma_1)(1 - \gamma_2)(1 - \gamma_3)}{\alpha\beta [(1 - \gamma_3)\delta - (1 - \gamma_2)\Delta] + \alpha(\gamma_2 - \gamma_3)(1 - \gamma_1) + (1 - \gamma_1)(1 - \gamma_2)}$

Proof. $(\pi_{02}, \pi_{10}, \pi_{11}, \pi_{12})$ is given by the positive left invariant vector (3) and satisfies the normalizing condition

$$\pi_{02} + (\pi_{k0}, \pi_{k1}, \pi_{k2})(I - R)^{-1} e = 1$$

Then, we get

$$K = \frac{(1 - \gamma_1)(1 - \gamma_2)(1 - \gamma_3)}{\alpha\beta [(1 - \gamma_3)\delta - (1 - \gamma_2)\Delta] + \alpha(\gamma_2 - \gamma_3)(1 - \gamma_1) + (1 - \gamma_1)(1 - \gamma_2)}.$$

We obtain

$$\pi_{02} = K, \quad (\pi_{10}, \pi_{11}, \pi_{12}) = K(\alpha\beta(\delta - \Delta), \alpha(\gamma_2 - \gamma_3), \gamma_3).$$

We have

$$\pi_k = (\pi_{k0}, \pi_{k1}, \pi_{k2}) = (\pi_{10}, \pi_{11}, \pi_{12})R^{k-1}, \quad k \geq 1,$$

Finally, we easily obtain the theorem.

Theorem 4. If $\rho < 1$, the stationary queue length L_v can be decomposed into the sum of two independent random variables: $L_v = L + L_d$, where L is the stationary queue length of a classical GI/M/1 queue without vacation, follows a geometric distribution with parameter γ_1 ; L_d follows the discrete PH distributions (φ, T) of order 2, where

$$\varphi = \frac{K}{1-\gamma_1} \left(\alpha \frac{\gamma_2 - \gamma_1 + \beta\delta}{1-\gamma_2} - \frac{(\alpha-1)(\gamma_3 - \gamma_1) + \alpha\beta\Delta}{1-\gamma_3} \right), \varphi_3 = \frac{K}{1-\gamma_1}, T = \begin{pmatrix} \gamma_2 & \\ & \gamma_3 \end{pmatrix}, T^0 = \begin{pmatrix} 1-\gamma_2 & \\ & 1-\gamma_3 \end{pmatrix}$$

Proof. The PGF of L_v is as follows:

$$\begin{aligned} L_v(z) &= \sum_{k=0}^{\infty} z^k P(L_v = k) \\ &= K\alpha \left[\frac{z^k \gamma_3^k}{\alpha} + (\gamma_2^k - \gamma_3^k) z^k + \frac{\beta\delta(\gamma_1^k - \gamma_2^k)}{\gamma_1 - \gamma_2} z^k - \frac{\beta\Delta(\gamma_1^k - \gamma_3^k)}{\gamma_1 - \gamma_3} z^k \right] \\ &= \frac{1-\gamma_1}{1-z\gamma_1} \frac{K}{1-\gamma_1} \left(\alpha \frac{1-z\gamma_1+z\beta\delta}{1-z\gamma_2} - \frac{(\alpha-1)(1-z\gamma_1)+z\alpha\beta\Delta}{1-z\gamma_3} \right) = L(z)L_d(z) \end{aligned}$$

Where $L(z)$ is the PGF of L of a classical GI/M/1queue without vacation.

$$L_d(z) = \frac{K}{1-\gamma_1} \left(\alpha \frac{1-z\gamma_1+z\beta\delta}{1-z\gamma_2} - \frac{(\alpha-1)(1-z\gamma_1)+z\alpha\beta\Delta}{1-z\gamma_3} \right) \quad (4)$$

$$\frac{1-z\gamma_1+z\beta\delta}{1-z\gamma_2} = (1-z\gamma_1+z\beta\delta) \sum_{k=0}^{\infty} \gamma_2^k z^k = 1 + (\gamma_2 + \beta\delta - \gamma_1) \sum_{k=1}^{\infty} \gamma_2^{k-1} z^k.$$

Substituting the above equation into (4), we obtain the distribution of L_d .

We can easily get means

$$E(L_v) = \frac{\gamma_1}{1-\gamma_1} + \frac{K}{1-\gamma_1} \left[\alpha \frac{\gamma_2 - \gamma_1 + \beta\delta}{(1-\gamma_2)^2} - \frac{(\alpha-1)(\gamma_3 - \gamma_1) + \alpha\beta\Delta}{(1-\gamma_3)^2} \right].$$

3. WAITING TIME DISTRIBUTION

Let W and $\tilde{W}(s)$ be the steady-state waiting time and its LST, respectively. Firstly, let H_0, H_1, H_2 be the probability that the server is in the service(set-up, vacation) period when a new customer arrives. We can compute

$$\begin{aligned} H_0 &= \frac{\alpha\beta[(1-\gamma_3)\delta - (1-\gamma_2)\Delta]}{\alpha\beta[(1-\gamma_3)\delta - (1-\gamma_2)\Delta] + \alpha(\gamma_2 - \gamma_3)(1-\gamma_1) + (1-\gamma_1)(1-\gamma_2)}, \\ H_1 &= \frac{\alpha(\gamma_2 - \gamma_3)(1-\gamma_1)}{\alpha\beta[(1-\gamma_3)\delta - (1-\gamma_2)\Delta] + \alpha(\gamma_2 - \gamma_3)(1-\gamma_1) + (1-\gamma_1)(1-\gamma_2)}, \\ H_2 &= \frac{(1-\gamma_1)(1-\gamma_2)}{\alpha\beta[(1-\gamma_3)\delta - (1-\gamma_2)\Delta] + \alpha(\gamma_2 - \gamma_3)(1-\gamma_1) + (1-\gamma_1)(1-\gamma_2)}. \end{aligned}$$

Theorem 5. If $\rho < 1$, $\theta, \beta > 0$, the LST of stationary waiting time W is

$$\begin{aligned} \tilde{W}(s) &= H_1 \frac{\beta}{\beta+s} \frac{(\mu+s)(1-\gamma_2)}{\mu(1-\gamma_2)+s} \frac{\mu(1-\gamma_3)}{\mu(1-\gamma_3)+s} + H_2 \frac{\theta}{\theta+s} \frac{\mu(1-\gamma_3)}{\mu(1-\gamma_3)+s} \frac{\beta}{\beta+s} \\ &+ H_0 \left[\mu + \frac{(\delta-\Delta)s}{\delta(1-\gamma_3) - \Delta(1-\gamma_2)} \right] \frac{(\mu+s)(1-\gamma_1)}{\mu(1-\gamma_1)+s} \frac{(1-\gamma_2)}{\mu(1-\gamma_2)+s} \frac{\mu(1-\gamma_3)}{\mu(1-\gamma_3)+s} \end{aligned}$$

Proof. When a new customer arrives, if there are k customers and the server is in the service period, the waiting time equals k service times by the rate μ . Then, we have

$$\begin{aligned} \sum_{k=1}^{\infty} \pi_{k0} \tilde{W}_{k0}(s) &= K \alpha \beta \sum_{k=1}^{\infty} \left(\frac{\gamma_1^k - \gamma_2^k}{\gamma_1 - \gamma_2} \delta - \frac{\gamma_1^k - \gamma_3^k}{\gamma_1 - \gamma_3} \Delta \right) \left(\frac{\mu}{\mu + s} \right)^k \\ &= H_0 \left[\mu + \frac{(\delta - \Delta)s}{\delta(1 - \gamma_3) - \Delta(1 - \gamma_2)} \right] \frac{(\mu + s)(1 - \gamma_1)}{\mu(1 - \gamma_1) + s} \frac{(1 - \gamma_2)}{\mu(1 - \gamma_2) + s} \frac{\mu(1 - \gamma_3)}{\mu(1 - \gamma_3) + s} \end{aligned} \quad (5)$$

When a new customer arrives, if there are k customers and the server is in the set-up period, the waiting time is the sum of the residual set-up time and k service times by the rate μ . Then, we have

$$\begin{aligned} \sum_{k=1}^{\infty} \pi_{k1} \tilde{W}_{k1}(s) &= K \alpha \frac{\beta}{\beta + s} \sum_{k=1}^{\infty} (\gamma_2^k - \gamma_3^k) \left(\frac{\mu}{\mu + s} \right)^k \\ &= H_1 \frac{\beta}{\beta + s} \frac{(\mu + s)(1 - \gamma_2)}{\mu(1 - \gamma_2) + s} \frac{\mu(1 - \gamma_3)}{\mu(1 - \gamma_3) + s}. \end{aligned} \quad (6)$$

Similarly,

$$\sum_{k=1}^{\infty} \pi_{k2} \tilde{W}_{k2}(s) = K \sum_{k=1}^{\infty} \left(\frac{\mu \gamma_3}{\mu + s} \right)^k \frac{\beta}{\beta + s} \frac{\theta}{\theta + s} = H_2 \frac{\theta}{\theta + s} \frac{\mu(1 - \gamma_3)}{\mu(1 - \gamma_3) + s} \frac{\beta}{\beta + s}. \quad (7)$$

From (5)-(7), we have the result in Theorem 4.

With the structure in Theorem 4, we can get the expected waiting time

$$\begin{aligned} E(W) &= H_2 \left[\frac{1}{\theta} + \frac{1}{\mu(1 - \gamma_3)} + \frac{1}{\beta} \right] + H_1 \left[\frac{\gamma_2}{\mu(1 - \gamma_2)} + \frac{1}{\mu(1 - \gamma_3)} + \frac{1}{\beta} \right] \\ &\quad + H_0 \left[\frac{\gamma_1}{\mu(1 - \gamma_1)} - \frac{\delta - \Delta}{\mu [\delta(1 - \gamma_3) - \Delta(1 - \gamma_2)]} + \frac{1}{\mu(1 - \gamma_2)} + \frac{1}{\mu(1 - \gamma_3)} \right]. \end{aligned}$$

REFERENCES

- Doshi, B.T.(1986). Queuing systems with vacations-a survey. *Queueing Syst*, 1, 29-66
- Qin, X. (2006). GI/Geo/1 discrete-time queue with set-up period and multiple vacations. *Operation Research*, 15, 52-57.
- Tian, N.S. (2006). *Vacation Queueing Models: Theory and Application*. New York: Springer-Verlag.
- Tian, N.S. (1989). The GI/M/1 queue with exponential vacations. *Queueing Syst*, 5 331-344.
- Tian, N.S.(2002). The discrete time GI/Geo/1 queue with multiple vacations. *Queueing Syst*, 40 283-294.
- Neuts, M. (1981). *Matrix-Geometric Solutions in Stochastic models*. Baltimore: Johns Hopkins University Press.