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Riemann Hypothesis Elementary Discussion

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Abstract: Areas of prime number theorem is proposed in this paper, and the area of prime number theorem. The basic theorem of prime number distribution are obtained. To prove the Rienann conjecture.

Key words: Prime number; Area; Guess; Theorem

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1. INTRODUCTION

In 1859, a German mathematician Riemann proposed: Riemann zeta function [1–3]:

$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s}, \quad \mathrm{Re}(s) > 1, \tag{1}$$

From Equation (1), we can get:

$$]\zeta(s) = 2\Gamma(1-s)(2\pi)^{s-1} \sin(\pi s/2)\zeta(1-s), \tag{2}$$

Surrounded by (2) of zero point is referred to as: zero. Riemann hypothesis: *Riemann zeta function all nontrivial zero point is in the critical line.*

2. RIEMANN PRIME DISTRIBUTION FORMULA

Set $\pi(x)$ is not greater than x number of prime Numbers, then:

$$\pi(x) = \sum_n \frac{\mu(n)}{n} J(x^{1/n}), \tag{3}$$

$$J(x) = Li(x) - \sum_{\operatorname{Im}\rho>0} [Li(x^\rho) + Li(x)^{1-\rho}] + \int_x^\infty \frac{dt}{t(t^2-1)\ln t} - \ln 2, \tag{4}$$

Here Equation (3) is Riemann prime distribution formula. The main conclusions of the Riemann was obtained in 1859. The function $\mu(x)$ is called: *M´obius function*, which is defined as follows [5]:

$$\mu(n) = \begin{cases} 1, & n = 1, \\ (-1)^k, & n = p_1, p_2, p_3, \cdots\cdots p_k, \\ 0, & \text{The rest.} \end{cases}$$

The p_k is not the same Prime number. Here Equation (4) refers to: *the step function*. $Li(x^\rho) + Li(x^{1-\rho})$ involves the Riemann content distribution of zero. Obviously, Riemann prime distribution formula of calculation is very complicated. In 1901, Swedish mathematician von Koch proved that if Riemann content was established, then [5,6]:

$$\pi(x) = Li(x) + O(x^{1/2}\ln x), \tag{5}$$

$$Li(x) = \int_2^x \frac{1}{\ln u} du$$

Here Equation (5) is a prime number theorem.

If Riemann hypothesis was established, we can also get the prime number theorem:

$$\pi(x) = Li(x) + O(x^{1/2+\varepsilon}),$$

In turn: if the prime number theorem Equation (5) was established, then the Riemann hypothesis was established. In fact for the x limited was set up. So as long as can prove sufficiently large x is set up, then Equation (5) is set up.

3. THEOREM OF PRIME NUMBER DISTRIBUTION SERIES

Set a large number x, parameter lambda $\lambda > 1$, prime number p, get:

$$\pi(x) \to s(x), \tag{6}$$

$$s(x) = \sum_{n=1}^{x/2-1} \frac{2}{\ln\lambda} \sum_{2n \leqslant p \leqslant 2n+2} \frac{1}{p}, \quad \lambda = \frac{n+1}{n},$$

Here Equation (6) is referred to: *Theorem of prime number distribution series*. $x/2$ is an integer. For example, set $x = 10$, the following can easily obtained by

Equation (6):

$$\begin{aligned}
s(10) &= \sum_{n=1}^{4} \frac{2}{\ln \frac{n+1}{n}} \sum_{2n \le p \le 2n+2} \frac{1}{p} \\
&= \frac{2}{\ln 2}\frac{1}{2} + \frac{2}{\ln 2}\frac{1}{3} + \frac{2}{\ln(3/2)}\frac{1}{5} + \frac{1}{\ln(4/3)}\frac{1}{7},
\end{aligned}$$

so $s(10) = 4 + 0.3841729$, actual $\pi(10) = 4$.

Proof. Set $2n < p < 2n+2$, if there is a prime number in the interval $(2n, 2n+2)$, it must be $2n+1$, which means

$$p = 2n+1, \quad p > 2,$$

and then

$$\lambda = \frac{n+1}{n} = \frac{2n+1+1}{2n+1-1} = \frac{p+1}{p-1},$$

By Equation (6), we can get:

$$\begin{aligned}
s(x) &= \sum_{n=1}^{x/2-1} \frac{2}{\ln \lambda} \sum_{2n \le p \le 2n+2} \frac{1}{p} \\
&= \frac{1}{\ln 2} + \sum_{n=1}^{x/2-1} \sum_{2n < p < 2n+2} \frac{2}{p \ln \lambda} \\
&= \frac{1}{\ln 2} + \sum_{2 < p < 2(x/2-1)+2} \frac{2}{p \ln \lambda} \\
&= \frac{1}{\ln 2} + \sum_{2 < p < x} \frac{2}{p \ln \lambda},
\end{aligned}$$

Therefore, the following equation is obtained:

$$s(x) = \frac{1}{\ln 2} + \sum_{2<p<x} \frac{2}{p \ln \lambda}, \quad \lambda = \frac{p+1}{p-1}. \tag{7}$$

Set $x > y$, form Equation (7),

$$s(x) - s(y) = \sum_{y<p<x} \frac{2}{p \ln \lambda}, \quad \lambda = \frac{p+1}{p-1},$$

If x is a large number, and $y = 2[x^{1/2}/2]$, then:

$$\begin{aligned}
\ln \lambda &= \ln\left(1 + \frac{2}{p-1}\right) = \frac{2}{p}, \\
s(x) - s(y) &= \sum_{y<p<x} 1 = \pi(x) - \pi(y), \\
\pi(x) &= s(x) - s(y) + \pi(y),
\end{aligned}$$

Obviously,

$$\pi(x) \to s(x).$$

The theorem (6) was proved. ∎

For example,

x	$\pi(x)$	$s(x)$
10	4	$4 + 0.3841729$
10^2	25	$25 + 0.37$
10^3	168	$168 + 0.37$
10^4	1229	$1229 + 0.37$
10^5	9592	$9592 + 0.37$
10^6	78498	$78498 + 0.37$
10^7	664579	$664579 + 0.37$
10^8	5761455	$5761455 + 0.37$

4. INTERVAL PRIME NUMBER THEOREM

Set $x > y$, ignore the reminder term and table as an integer, we can get the following from Equation (6):

$$\begin{aligned} &\pi(x) - \pi(y) = s(x) - s(y), \\ &s(x) - s(y) = \sum_{n=y/2}^{x/2-1} \frac{2}{\ln \lambda} \sum_{2n \le p \le 2n+2} \frac{1}{p}, \quad \lambda = \frac{n+1}{n}, \end{aligned} \tag{8}$$

Here Equation (8) is: *interval prime number theorem.* Where $x/2$ and $y/2$ are integers.

Proof. The same as that prove Theorem (6). □

By Equation (8), we can get: $\pi(x) = s(x) - s(y) + \pi(y)$.

For example, set $y = 2[x^{1/2}/2]$, by Equation (8), the following can be obtained:

x	$\pi(x)$	$s(x) - s(y) + \pi(y)$
10	4	$4 + 0.3841729$
10^2	25	$25 - 0.0096482$
10^3	168	$168 - 0.0023702$
10^4	1229	$1229 - 0.0006029$
10^5	9592	$9592 - 0.0001529$
10^6	78498	$78498 - 0.000042$
10^7	664579	$664579 - 0.0000118$
10^8	5761455	$5761455 + 0.000000142$

Ignore the remainder term and table as an integer, $\pi(x) = s(x) - s(y) + \pi(y)$.

5. TRANSFORM THEOREM OF PRIME NUMBER DISTRIBUTION

From Equation (6), we have

$$2\pi(y) > s(y), \tag{9}$$

Substitute Equation (9) into Equation (8), we have

$$\pi(x) = s(x) - s(y) + \pi(y) > s(x) - 2\pi(y) + \pi(y),$$

and

$$\pi(x) > s(x) - 2\pi(y), \tag{10}$$

From Equation (8), we have

$$\pi(x) = s(x) - s(y) + \pi(y) < s(x) + 2\pi(y), \tag{11}$$

Combining Equation (10) and Equation (11):

$$s(x) - 2\pi(y) < \pi(x) < s(x) + 2\pi(x), \tag{12}$$

We can prove a new prime number theorem by Equation (12).
In 1874, mathematician Mertens proved [4]:

$$\sum_{p \leqslant x} \frac{1}{p} = \ln\ln x + A_1 + O\left(\frac{1}{\ln x}\right), \tag{13}$$

Here Equation (13) is called: *Mertens theory*. Where A_1 is a constant.
Set $\ln x \to \infty$, by Equation (13) [7,8]:

$$\sum_{p \le x} \frac{1}{p} = \ln\ln x + A_1,$$

$$\sum_{2n \le p \le 2n+2} \frac{1}{p} = \sum_{n=y/2}^{x/2-1} \ln \frac{\ln(2n+2)}{\ln(2n)} = \sum_{n=y/2}^{x/2-1} \ln\left(1 + \frac{\ln\lambda}{\ln(2n)}\right),$$

and

$$\begin{aligned} \sum_{n=y/2}^{x/2-1} \ln\left(1 + \frac{\ln\lambda}{\ln(2n)}\right) &= \sum_{n=y/2}^{x/2-1} \frac{\ln\lambda}{\ln(2n)}, \\ \sum_{2n \leqslant p \leqslant 2n+2} \frac{1}{p} &= \sum_{n=y/2}^{x/2-1} \frac{\ln\lambda}{\ln(2n)}, \quad \lambda = \frac{n+1}{n}, \end{aligned} \tag{14}$$

Here Equation (14) is called *interval prime number theorem*.

6. PROOF A PRIME NUMBER THEOREM

Substitute Equation (14) into Equation (8), we have

$$\begin{aligned} s(x) - s(y) &= \sum_{n=y/2}^{x/2-1} \frac{2}{\ln(2n)} = \sum_{n=1}^{x/2-1} \frac{2}{\ln(2n)} - \sum_{n=1}^{y/2-1} \frac{2}{\ln(2n)}, \\ s(x) &= \sum_{n=1}^{x/2-1} \frac{2}{\ln(2n)} = \sum_{n=2}^{x} \frac{1}{\ln(n)}, \end{aligned} \tag{15}$$

If x is a large number and $y = x^{1/2}$, then the following can be obtained from Equation (15) and Equation (12):

$$\begin{aligned} & s(x) - 2\pi(x^{1/2}) < \pi(x) < s(x) + 2\pi(x^{1/2}), \quad (x \to \infty), \\ & s(x) = \sum_{n=2}^{x} \frac{1}{\ln(n)}, \end{aligned} \tag{16}$$

Here Equation (16) is a new prime number theorem.

7. PROVE RIEMANN HYPOTHESIS

In integral sense, Equation (5) and Equation (16) are the same, therefore

$$Li(x) = \int_2^x \frac{1}{\ln u} du = \sum_{n=2}^{x} \frac{1}{\ln(n)},$$
$$Li(x) = s(x),$$

For example:

x	$\pi(x)$	$Li(x)$	$s(x)$
10	4	6	6
10^2	25	30	30
10^3	168	177	177
10^4	1229	1246	1246
10^5	9592	9530	9530
10^6	78498	78627	78627
10^7	664579	664918	664918
10^8	5761455	5762209	5762209

Substitute $Li(x) = s(x)$ into Equation (16), we have

$$Li(x) - 2\pi(x^{1/2}) < \pi(x) < Li(x) + 2\pi(x^{1/2}), \quad (x \to \infty). \tag{17}$$

Here Equation (17) Shows that Riemann hypothesis is established when x tends to infinity.

By Mertens theore (13), if of the small x, can is $2\pi(x^{1/2})$, then all the x, can is $2\pi(x^{1/2})$.

For example: set $\pi(x) = Li(x) - 2\pi(x^{1/2})c(x)$, then:

x	$\pi(x)$	$Li(x)$	$c(x)$
10	4	6	0.6
10^2	25	30	0.62
10^3	168	177	0.41
10^4	1229	1246	0.34
10^5	9592	9530	0.29
10^6	78498	78627	0.38
10^7	664579	664918	0.38
10^8	5761455	5762209	0.31

Clearly all the x, can is $\pi(x) > Li(x) - 2\pi(x^{1/2})$.

And from Equation (17), table as an integer, we have:

$$Li(x) - 2\pi(x^{1/2}) \leq \pi(x) \leq Li(x) + 2\pi(x^{1/2}),$$
$$\pi(x) = Li(x) + O(x^{1/2+\epsilon}), \quad x \geq 2.$$

So the theorem (5) was established. And Riemann hypothesis was established.

8. DISCUSSION

This is the focus of the prove Theorem of prime number distribution series. So Proving a and Riemann hypothesi The prime number theorem of equivalence. It is proved that: Riemann zeta function all nontrivial zero point is in the critical line.

In addition, Theorem of prime number distribution series can There are different forms.

Set Positive number $s \geq 2$, has:

$$\begin{aligned} &\pi(x) = s(x), \\ &s(x) = \sum_{n=1}^{x/2-1} \frac{x+y}{2} \sum_{y \leqslant p \leqslant x} \frac{1}{p}, \end{aligned} \tag{18}$$

From (18), we have:

$$\begin{aligned} &\pi(x) = Li(x) + O(x^{1/s}), \quad x \geq c, \\ &y = x^{1/s}, \end{aligned}$$

where c is an arbitrary large number.

This problem is beyond over This subject, should not be discussed in detail.

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