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Algebraic Properties of the Category of Q-P Quantale Modules

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Abstract: In this paper, the definition of a Q-P quantale module and some relative concepts were introduced. Based on which, some properties of the Q-P quantale module, and the structure of the free Q-P quantale modules generated by a set were obtained. It was proved that the category of Q-P quantale modules is algebraic.

Key words: Q-P quantale quantale modules; Equalizer; Forgetful functor; Algebraic category

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1. INTRODUCTION

Quantale was proposed by Mulvey in 1986 for studying the foundations of quantum logic and for studying non-commutation C*-algebras. The term quantale was coined as a combination of "quantum logic" and "locale" by Mulvey in [1]. The systematic introduction of quantale theory came from the book [2], which written by Rosenthal in 1990.

Since quantale theory provides a powerful tool in studying noncommutative structures, it has a wide applications, especially in studying noncommutative C*-algebra theory [3], the ideal theory of commutative ring [4], linear logic [5] and so on. So, the quantale theory has aroused great interests of many scholar and experts, a great deal of new ideas and applications of quantale have been proposed in twenty years [6–18].

Since the ideal of quantale module was proposed by Abramsky and Vickers [19], the quantale module has attracted many scholars eyes. With the development of the quantale theory, the theory of quantale module was studied deeply in [20–25]. In this paper, some properties of the category of Q-P quantale modules was discussed, especially that the category of Q-P quantale modules is algebraic was proved.

2. PRELIMINARIES

Definition 2.1 [2] A quantale is a complete lattice Q with an associative binary operation "&" satisfying:

$a\&(\bigvee_{i\in I} b_i) = \bigvee_{i\in I}(a\&b_i)$ and $(\bigvee_{i\in I} b_i)\&a = \bigvee_{i\in I}(b_i\&a)$,

for all $a, b_i \in Q$, where I is a set, 0 and 1 denote the smallest element and the greatest element of Q respectively.

Definition 2.2 A nonzero element a in a quantale Q is said to be a *nonzero divisor* if for all nonzero element $b \in Q$ such that $a\&b \neq 0$, $b\&a \neq 0$. Q is *nonzero divisor* if every $a \in Q$ is a *nonzero divisor*.

Definition 2.3 Let Q, P be a quantale, a Q-P quantale module over Q, P (briefly, a Q-P-module) is a complete lattice M, together with a mapping $T : Q \times M \times P \longrightarrow M$ satisfies the following conditions:

(1) $T(\bigvee_{i\in I} a_i, m, \bigvee_{j\in J} b_j) = \bigvee_{i\in I}\bigvee_{j\in J} T(a_i, m, b_j)$;

(2) $T(a, (\bigvee_{k\in K} m_k), b) = \bigvee_{k\in K} T(a, m_k, b)$;

(3) $T(a\&b, m, c\&d) = T(a, T(b, m, c), d)$ for all $a_i, a, b \in Q$, $b_j, c, d \in P$, $m_k, m \in M$. We shall denote the Q-P quantale module M over Q, P by $(M,\ T)$.

If Q is unital quantale with unit e, we define $T(e, m, e) = m$ for all $m \in M$.

Example 2.4 (1) Let $Q = P = \{0, a, b, c, 1\}$ be a set, $M = \{0, d, e, 1\}$ is a complete lattice. The order relations of Q and M are given by the following figure 1 and 2, we give a binary operator "&" on Q satisfying the diagram 1.

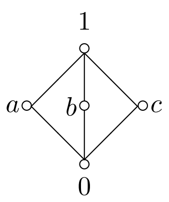

Figure 1

&	0	a	b	c	1
0	0	0	0	0	0
a	0	b	c	a	1
b	0	c	a	b	1
c	0	a	b	c	1
1	0	1	1	1	1

Diagram 1

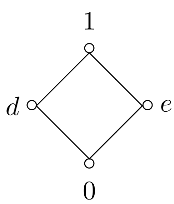

Figure 2

We can prove that Q is a quantale.

Now, define a mapping $T : Q \times M \times Q \longrightarrow M$ such that $T(x, m, y) = m$ for all $x, y \in Q$, $m \in M$. Then $(M,\ T)$ be a Q-P quantale module.

(2) Let $Q = P = \{0, a, b, 1\}$ be a complete lattice. The order relation on Q satisfies the following Figure 3 and the binary operation of Q satisfies the diagram 2:

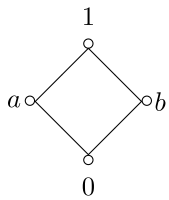

Figure 3

&	0	a	b	1
0	0	0	0	0
a	0	a	0	a
b	0	0	b	b
1	0	a	b	1

Diagram 2

It is easy to show that $(Q, \&)$ is a quantale. Let $M = \{0, a, 1\} \subseteq Q$, then M is a complete lattice with the inheriting order on Q. Now, we define $T : Q \times M \times Q \longrightarrow M$ satisfies $T(x, m, y) = x \& m \& y$ for all $x, y \in Q$, $m \in M$. Then $(M,\ T)$ is a Q-P quantale module.

Definition 2.5 Let Q, P be a quantale, (M_1, T_1) and (M_2, T_2) are Q-P quantale modules. A mapping f : $M_1 \longrightarrow M_2$ is said to be a $Q - P$ *quantale module homomorphism* if f satisfies the following conditions:

(1) $f(\bigvee\limits_{i \in I} m_i) = \bigvee\limits_{i \in I} f(m_i)$;

(2) $f(T_1(a, m, b)) = T_2(a, f(m), b)$ for all $a \in Q$,$b \in P$, $m_i, m \in M$.

Definition 2.6 Let (M, T_M) be a $Q - P$ quantale module over Q, P, N be a subset of M, N is said to be a *submodule* of M if N is closed under arbitrary join and $T_M(a, n, b) \in N$ for all $a \in Q$,$b \in P$, $n \in N$.

Definition 2.7 [26] A concrete category $(\mathcal{A}, U)$ is called *algebraic* provided that it satisfies the following conditions:

(1) $\mathcal{A}$ has coequalizers;

(2) U has a left adjoint;

(3) U preserves and reflects regular epimorphisms.

3. THE CATEGORY OF Q-P QUANTALE MODULES IS ALGEBRAIC

Definition 3.1. Let Q, P be a quantale, $_{\mathbf{Q}}\mathbf{Mod}_{\mathbf{P}}$ be the category whose objects are the Q-P quantale modules of Q, P, and morphisms are the Q-P quantale module homomorphisms, i.e.,

$\mathcal{O}b(_{\mathbf{Q}}\mathbf{Mod}_{\mathbf{P}})$={ M : M is Q-P quantale modules},

$\mathcal{M}or(_{\mathbf{Q}}\mathbf{Mod}_{\mathbf{P}})$={$f : M \longrightarrow N$ is the Q-P quantale modules homorphism}.

Hence, the category $_{\mathbf{Q}}\mathbf{Mod}_{\mathbf{P}}$ is a concrete category.

Definition 3.2. Let Q, P is a quantale, (M, T_M) is a Q-P quantale module, $R \subseteq M \times M$. The set R is said to be a *congruence* of Q-P quantale module on M if R satisfies:

(1) R is an equivalence relation on M;

(2) If $(m_i, n_i) \in R$ for all $i \in I$, then $(\vee_{i \in I} m_i, \vee_{i \in I} n_i) \in R$;

(3) If $(m, n) \in R$, then $(T_M(a, m, b), T_M(a, n, b)) \in R$ for all $a \in Q, b \in P$.

We denote the set of all congruence on M by $Con(_QM_P)$, then $Con(_QM_P)$ is a complete lattice with respect to the inclusion order.

Let Q, P be a quantale, M is a Q-P quantale module, R is a congrence of Q-P quantale module on M, define the order relation on M/R such that $[m] \leq [n]$ if and only if $[m \vee n] = [n]$ for all $[m], [n] \in M/R$.

Theorem 3.3. Let Q, P be a quantale, M be a Q-P quantale module, R be a congrence of double quantale module on M. Define $T_{M/R} : Q \times M/R \times P \longrightarrow M/R$ such that $T_{M/R}(a, [m], b) = [T_M(a, m, b)]$ for all $a \in Q, b \in P$, $[m] \in M/R$, then $(M/R, T_{M/R})$ is a Q-P quantale module and $\pi : m \mapsto [m] : M \longrightarrow M/R$ is a Q-P quantale module homomorphisms.

Proof. (1) We will prove that " $\leq$ "is a partial order on M/R, and $T_{M/R}$ is well defined. In fact, for all $[m], [n], [l] \in M/R$,

(i) It's clearly that $[m] \leq [m]$;

(ii) Let $[m] \leq [n]$, $[n] \leq [m]$, then $[m \vee n] = [n] and [n \vee m] = [m]$, thus $[m] = [n]$;

(iii) Let $[m] \leq [n]$, $[n] \leq [l]$, then $[m \vee n] = [n] and [n \vee l] = [l]$, therefore $[m \vee l] = [m \vee (n \vee l)] = [(m \vee n) \vee e] = [n \vee l] = [l]$.

If $[m_1] = [m_2]$, then $(m_1, m_2) \in R$, $(T_M(a, m, b), T_M(a, n, b)) \in R$ for all $a, b \in Q$, i.e., $[T_M(a, m, b)] = [T_M(a, n, b)]$, thus $T_{M/R}$ is well defined.

(2) We will prove that $(M/R, \leq)$ is a complete lattice. Let $\{[m_i] \mid i \in I\} \subseteq M/R$, we have

(i) Since $[m_i \vee (\bigvee_{i \in I} m_i)] = [\bigvee_{i \in I} m_i]$ for all $i \in I$, then $[m_i] \leq [\bigvee_{i \in I} m_i]$;

(ii) Let $[m] \in M/R$ and $[m_i] \leq [m]$ for all $i \in I$, then $[m_i \vee m] = [m]$ for all $i \in I$, hence, $[(\bigvee_{i \in I} m_i) \vee m] = [\bigvee_{i \in I} (m_i \vee m)] = [m]$, i.e., $[\bigvee_{i \in I} m_i] \leq [m]$.

Thus $\overset{M/R}{\bigvee_{i \in I}} [m_i] = [\bigvee_{i \in I} m_i]$.

(3) For all $\{a_i \mid i \in I\} \subseteq Q$, $\{b_j \mid j \in J\} \subseteq Q$, $\{[m_l] \mid l \in H\} \subseteq M/R$, $a, b \in Q, c, d \in P$, $[m] \in M/R$, we have that

(i) $T_{M/R}(\bigvee_{i \in I} a_i, [m], \bigvee_{j \in J} b_j) = [T_M(\bigvee_{i \in I} a_i, m, \bigvee_{j \in J} b_j)] = [\bigvee_{i \in I} \bigvee_{j \in J} T_M(a_i, m, b_j)]$
$= \bigvee_{i \in I} \bigvee_{j \in J} T_M[a_i, m, b_j] = \bigvee_{i \in I} \bigvee_{j \in J} T_{M/R}(a_i, [m], b_j)$;

(ii) $T_{M/R}(a, (\bigvee_{j \in J} [m_j]), b) = T_{M/R}(a, [\bigvee_{j \in J} m_j], b) = [T_M(a, (\bigvee_{j \in J} m_j), b)] = [\bigvee_{j \in J} T_M(a, m_j, b)]$
$= \bigvee_{j \in J} [T_M(a, m_j, b)] = \bigvee_{j \in J} T_{M/R}(a, [m_j], b)$;

(iii) $T_{M/R}(a \& b, [m], c \& d) = [T_M(a \& b, m, c \& d)] = [T_M(a, T_M(b, m, c), d)]$
$= T_{M/R}(a, [T_M(b, m, c)], d) = T_{M/R}(a, T_{M/R}(b, [m], c), d)$.

Then $(M/R, T_{M/R})$ is a Q-P quantale module.

(4) For all $\{[m_i] \mid i \in I\} \subseteq M/R$, $a \in Q, b \in P$, $[m] \in M/R$,

$\pi(\bigvee_{i \in I} m_i) = [\bigvee_{i \in I} m_i] = \bigvee_{i \in I} [m_i] = \bigvee_{i \in I} \pi(m_i)$;

$\pi(T_M(a, m, b)) = [T_M(a, m, b)] = T_{M/R}(a, [m], b) = T_{M/R}(a, \pi(m), b)$.

So $\pi : m \mapsto [m] : M \longrightarrow M/R$ is a Q-P quantale module homomorphisms. $\square$

Theorem 3.4. Let Q, P be a quantale, M a double quantale module, then $\triangle = \{(x, x) \mid x \in M\}$ is a congrence of Q-P quantale module on M.

Theorem 3.5. Let Q, P be a quantale, M and N be Q-P quantale modules, $f : M \longrightarrow N$ a Q-P quantale module homphorism, R a Q-P quantale module congrence on N. Then $f^{-1}(R) = \{(x, y) \in M \times M \mid (f(x), f(y)) \in R\}$ is a Q-P quantale module congrence on M.

Theorem 3.6. Let Q, P be a quantale, M and N are Q-P quantale modules, $f : M \longrightarrow N$ be a Q-P quantale module homphorism. Then $f^{-1}(\triangle) = \{(x, y) \in M \times M \mid f(x) = f(y)\}$ be a Q-P quantale module congrence on M, where $\triangle = \{(a, a) \mid a \in N\}$.

Let Q, P be a quantale, M be a Q-P quantale module, $R \subseteq M \times M$, since $Con({}_QM_P)$ is a complete lattice, there exists a smallest Q-P quantale congrence containing R , which is the intersection all the Q-P quantale module congrence containing R on M. We said that this congrence is generated by R.

Theorem 3.7. The category ${}_{\mathbf{Q}}\mathbf{Mod}_{\mathbf{P}}$ has coequalizer.

$$\begin{tikzcd}
M \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & N \arrow[r, "h"] \arrow[d, "\pi"'] & E' \\
& E \arrow[ur, "\overline{h}"'] &
\end{tikzcd}$$

Proof. Let Q, P be a quantale, (M, T_M) and (N, T_N) be Q-P quantale modules, f and g be Q-P quantale module homomorphisms, Suppose R is the smallest congrence of the Q-P quantale modules on N, which contain $\{(f(x), g(x)) \mid x \in M\}$. Let $E = N/R$, $\pi : N \longrightarrow N/R$ is the canonical mapping, then $(N/R, T_{N/R})$ is a Q-P quantale module and π is a Q-P quantale module homomorphisms by theorem 3.3 . We will prove that (π, E) is the coequalier of f and g. In fact,

(1) $\pi \circ f = \pi \circ g$ is clear

(2) Let $(E', T_{E'})$ be a Q-P quantale module, $h : N \longrightarrow E'$ be a Q-P quantale module homomorphisms such that $h \circ f = h \circ g$. Let $R_1 = h^{-1}(\triangle)$, where $\triangle = \{(x, x) \mid x \in E'\}$. By theorem 3.5, we can see that R_1 is a congrence of Q-P quantale module on N. Since $h(f(x)) = h(g(x))$ for all $x \in M$, then $(f(x), g(x)) \in R_1$. Define $\overline{h} : N/R \longrightarrow E'$such that $\overline{h}([n]) = h(n)$ for all $[n] \in Q/R$. Let $n_1, n_2 \in N$ and $(n_1, n_2) \in R$, then $(n_1, n_2) \in R_1$, and we have that $h(n_1) = h(n_2)$. Therefore $\overline{h}$ is well defined.

For all $\{[n_i] \mid i \in I\} \subseteq N/R$, $a, b \in Q$, $[n] \in N/R$, we have that

$\overline{h}(\bigvee\limits_{i\in I} [n_i]) = \overline{h}([\bigvee\limits_{i\in I} n_i]) = h(\bigvee\limits_{i\in I} n_i) = \bigvee\limits_{i\in I} h(n_i) = \bigvee\limits_{i\in I} \overline{h}([n_i])$;

$\overline{h}(T_{N/R}(a, [n], b)) = \overline{h}([T(a, n, b)]) = h(T(a, n, b)) = T_{E'}(a, h(n), b) = T_{E'}(a, \overline{h}([n]), b)$.

Thus, $\overline{h}$ is a Q-P quantale module, and $\overline{h}$ is the unique homomorphism satisfy $\overline{h} \circ \pi = h$. Therefore (π, E) is the coequalizer of f and g. □

From now until the end of Section 3, we suppose Q be a unital quantale with unit e. Let X be a nonempty set, we consider the complete lattice $(Q^X, \bigvee^X)$, where Q^X is the set of all the function from X to Q and $(\bigvee\limits_{i\in I}^X f_i)(x) = \bigvee\limits_{i\in I} f_i(x)$ for all $x \in X$.

Theorem 3.8. Let X be a nonempty set, and Q is idempotent and unital quantale with unit e, define $T_X : Q \times Q^X \times Q \longrightarrow Q^X$ such that $T_X(a, f, b)(x) = a\&f(x)\&b$, for all $a, b \in Q, f \in Q^X$, $x \in X$. Then (Q^X, T_X) is the free double quantale module generated by X, equipped with the map $\varphi : x \in X \longmapsto \varphi_x \in Q^X$, where φ_x is defined by $\varphi_x(y) = \begin{cases} 0, & y \neq x, \\ e\,, & y = x. \end{cases}$ for all $y \in X$.

Proof. It's easy to prove that (Q^X, T_X) is a double quantale module. Let (M, T_M) be any double quantale module and $g : X \longrightarrow M$ be an arbitrary map. First observe that for all $f \in Q^X$, Q be a unital quantale with unit e, hence $f = T_X(e, f, e)$ by definition 2.2. So every elments of Q^X could denote by $T_X(c, f, d)$ for some $c, d \in Q, f \in Q^X$. Define map $h_g : Q^X \longrightarrow M$ such that $h_g(T_X(c, f, d)) = \bigvee\limits_{x \in X} T_M(c, T_M(f(x), g(x), f(x)), d)$, for all $T_X(c, f, d) \in Q^X$, $c, d \in Q$.

For all $x' \in Z$, $(h_g \circ \varphi)(x') = h_g(\varphi_{x'}) = \bigvee\limits_{x \in X} T_M(\varphi_{x'}(x), g(x), \varphi_{x'}(x)) = T_M(e, g(x), e) = f(x)$, hence $h_g \circ \varphi = f$. This implies that the following diagram commute.

$$\begin{tikzcd} X \arrow[r, "\varphi"] \arrow[dr, "g"'] & Q^X \arrow[d, "h_g"] \\ & M \end{tikzcd}$$

We will prove that h_g is a Q-P quantale module homomorphism.

For all $\{f_i\}_{i \in I}$, $a, b \in Q$, $f \in Q^X$, we have

$$\begin{aligned}
\text{(i)} h_g(\bigvee_{i \in I} f_i) &= h_g(T_X(e, \bigvee_{i \in I} f_i, e) \\
&= \bigvee_{x \in X} T_M(e, T_M(\bigvee_{i \in I} f_i, g(x), \bigvee_{i \in I} f_i), e) \\
&= \bigvee_{x \in X} T_M(\bigvee_{i \in I} f_i, g(x), \bigvee_{i \in I} f_i) \\
&= \bigvee_{i \in I} \bigvee_{x \in X} T_M(f_i, g(x), f_i) \\
&= \bigvee_{i \in I} h_g(f_i);
\end{aligned}$$

$$\begin{aligned}
\text{(ii)} h_g(T_X(a, f, b)) &= \bigvee_{x \in X} T_M(a, T_M(f(x), g(x), f(x)), b) \\
&= T_M(a, \bigvee_{x \in X} T_M(f(x), g(x), f(x)), b) \\
&= T_M(a, h_g(f), b).
\end{aligned}$$

Therefore, h_g is a Q-P quantale module homomorphism.

Next, we will prove that h_g is an unique Q-P quantale module homomorphism satisfying the above conditions.

Now, let $h'_g : Q^X \longrightarrow M$ be another unique Q-P quantale module homomorphism such that $h'_g \circ \varphi = g$. For all $T_X(c,f,d) \in Q^X$, we have

$$\begin{aligned} h_g(T_X(c,f,d)) &= \bigvee_{x\in X} T_M(c, T_M(f(x), g(x), f(x)), d) \\ &= \bigvee_{x\in X} T_M(c, T_M(f(x), (h'_g \circ \varphi)(x), f(x)), d) \\ &= T_M(c, h'_g(\bigvee_{x\in X} T_X(f(x), \varphi_x, f(x))), d) \\ &= T_M(c, h'_g(f), d) \qquad\qquad (\bigvee_{x\in X} T_X(f(x), \varphi_x, f(x)) = f) \\ &= h'_g(T_X(c,f,d)). \end{aligned}$$

Therefore, (Q^X, T_X) is the free Q-P quantale module generated by X, equipped with the map φ. □

Definition 3.9. Let X be a nonempty set, Q, P is unital quantale , (Q^X, T_X) is called *free Q-P quantale module* generated by X.

Theorem 3.10. The forgetfull functor $U : {}_{\mathbf{Q}}\mathbf{Mod}_{\mathbf{P}} \longrightarrow \mathbf{Set}$ have a left adjoint.

Proof. Let X and Y be nonempty sets, (Q^X, T_X) and (Q^Y, T_Y) be the free Q-p quantale module generated by X and Y respectively.

Corresponding map $f : X \longrightarrow Y$ defines $M(f) : Q^X \longrightarrow Q^Y$ such that $M(f)(g)(y) = \bigvee\{g(x) \mid f(x) = y, x \in X\}$, for all g in Q^X, $y \in Y$. Obiviously, $M(f)$ is well defined.

We check $M(f)$ is a Q-p quantale module homomorphism.

For all $g_i, g \in Q^X, a \in Q, b \in P, y \in Y$ we have

$$\begin{aligned} \text{(i)}M(f)(\bigvee_{i\in I} g_i) &= \bigvee\{\bigvee_{i\in I} g_i(x) \mid f(x) = y, x \in X\} \\ &= \bigvee_{i\in I}(\bigvee\{g_i(x) \mid f(x) = y, x \in X\}) \\ &= \bigvee_{i\in I} M(f)(g_i)(y). \end{aligned}$$

Thus $M(f)$ preserves arbitrary joins.

$$\begin{aligned} \text{(ii)}M(f)(T_X(a,g,b))(y) &= \bigvee\{T_X(a,g,b)(x) \mid f(x) = y, x \in X\} \\ &= \bigvee\{a\&g(x)\&b \mid f(x) = y, x \in X\} \\ &= a\&(\bigvee\{g(x) \mid f(x) = y, x \in X\})\&b \\ &= a\&(M(f)(g)(y))\&b \\ &= T_Y(a, M(f)(g), b)(y). \end{aligned}$$

Thus $M(f)(T_X(a,g,b))(y) = T_Y(a, M(f)(g), b)(y)$. It is readily verified that $M(f)$ is a Q-P quantale module homomorphism.

Next, we will check that $M : \mathbf{Set} \longrightarrow {}_{\mathbf{Q}}\mathbf{Mod}_{\mathbf{P}}$ is a functor.
Let $f : X \longrightarrow Y$, $g : Y \longrightarrow Z$, id_X is the identity function on X. For all $h \in Q^X$, $x \in X$, $z \in Z$, we have

(i) $M(id_X)(h)(x) = \bigvee\{h(x) \mid id_X(x) = x\} = h(x) = id_{Q^X}(h)(x)$, it shows that M preserves identity function.

$$\begin{aligned}(\mathrm{ii})(M(g)\circ M(f))(h)(z) &= \bigvee\{M(f)(h)(y) \mid g(y) = z, y \in Y\}\\ &= \bigvee\{\bigvee\{h(x) \mid f(x) = y, x \in X\} \mid g(y) = z, y \in Y\}\\ &= \bigvee\{h(x) \mid f(x) = y, g(y) = z, x \in X, y \in Y\}\\ &= \bigvee\{h(x) \mid g(f(x)) = z, x \in X\}\\ &= M(g\circ f)(h)(z),\end{aligned}$$

then M preserves composition.

Finally, we will prove that M is the left adjoint of U.

By theorem 3.8, we have (Q^X, T_X) is the free Q-P quantale module generated by X, equipped with the map φ, therefore, M is the left adjoint of U.

□

Theorem 3.11. The forgetful functor U : ${}_{\mathbf{Q}}\mathbf{Mod}_{\mathbf{P}} \longrightarrow \mathbf{Set}$ preserves and reflects regular epimorphisms.

Proof. It is easy to be verified that the forgetful functor U preserves regular epimorphisms. We will check the forgetful functor U reflects regular epimorphisms.

At first, every regular epimorphisms is a surjective homomorphism in ${}_{\mathbf{Q}}\mathbf{Mod}_{\mathbf{P}}$ by Theorem 3.7.

Next, we prove that every surjective homomorphism is a regular epimorphisms in ${}_{\mathbf{Q}}\mathbf{Mod}_{\mathbf{P}}$.

Let $h : M_1 \longrightarrow M_2$ be a surjective Q-P quantale module homomorphism. Since the surjective morphism is an regular epimorphism in **Set**. Then h is a regular epimorphism in **Set**, there exists a set X and maps f, g such that (h, M_2) is a coequalizer of f and g.

Let (Q^X, T_X) be a Q-P quantale module generated by X. Since Q be a unital quantale with unit e, hence $s = T_X(e, s, e)$ for all $s \in Q^X$.

Define map $h_f, h_g : Q^X \longrightarrow M$ such that

$$h_f(T_X(a, s, b)) = \bigvee_{x\in X} T_{M_1}(a, T_{M_1}(s(x), f(x), s(x)), b).$$

$$h_g(T_X(a, s, b)) = \bigvee_{x\in X} T_{M_1}(a, T_{M_1}(s(x), g(x), s(x)), b),$$

for all $T_X(a, s, b) \in Q^X$, $s \in Q^X$, $a, b \in Q$.

We know that h_f and h_g are Q-P quantale module homomorphisms by theorem 3.8.

Since h_f is a Q-P quantale module homomorphism, and $h\circ f = h\circ g$, then $h\circ h_f = h\circ h_g$. Suppose there is a Q-P quantale module homomorphism h' : $M_1 \longrightarrow M_2$ with $h'\circ h_f = h'\circ h_g$, then we have $h'\circ f = h'\circ g$.

Because (h, M_2) is the coequalizer of f and g, there is a unique Q-P quantale module homomorphism $\overline{h} : M_2 \longrightarrow M_3$ such that $h' = \overline{h}\circ h$. Since h is a surjective of Q-P quantale module homomorphism, then there exists $x', y' \in M_1$ and $\{x'_i\}_{i\in I} \subseteq M_1$ such that $h(x_1) = x, h(y_1) = y, h(x'_i) = x_i$.

We check that $\overline{h}$ be a Q-P quantale module homomorphism in the following.

(i) $\overline{h}(\bigvee\limits_{i\in I} x_i) = \overline{h}(\bigvee\limits_{i\in I} h(x'_i)) = \overline{h}h(\bigvee\limits_{i\in I} x'_i) = h'(\bigvee\limits_{i\in I} x'_i) = \bigvee\limits_{i\in I} h(x'_i) = \bigvee\limits_{i\in I} \overline{h}h(x'_i) = \bigvee\limits_{i\in I} \overline{h}(x_i)$,

(ii) For any $a \in Q, b \in P,\ m \in M_2$, since h is a surjective of double quantale module homomorphism, there exists m' in M such that $h(m') = m$.

So we have $T_3(a, \overline{h}(m), b) = T_3(a, \overline{h}(h(m')), b) = T_3(a, h'(m'), b) = h'(T_1(a, m', b))$ $= \overline{h}h(T_1(a, m', b)) = \overline{h}(T_2(a, h(m'), b) = \overline{h}(T_2(a, m, b))$.

Hence, (h, M_2) is an coequalizer of h_f and h_g in ${}_{\mathbf{Q}}\mathbf{Mod}_{\mathbf{P}}$, so h is a regular epimorphism in ${}_{\mathbf{Q}}\mathbf{Mod}_{\mathbf{P}}$. Therefore, the regular epimorphisms are precisely surjective homomorphisms in ${}_{\mathbf{Q}}\mathbf{Mod}_{\mathbf{P}}$. Since the forgetfull functor $U : {}_{\mathbf{Q}}\mathbf{Mod}_{\mathbf{P}} \longrightarrow \mathbf{Set}$ reflects surjective homomorphisms, hence $U : {}_{\mathbf{Q}}\mathbf{Mod}_{\mathbf{P}} \longrightarrow \mathbf{Set}$ reflects regular epimorphisms.

$$\begin{tikzcd} X \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & M_1 \arrow[r, "h'"] \arrow[d, "h"] & M_3 \\ & M_2 \arrow[ur, "\overline{h}"'] & \end{tikzcd} \qquad \begin{tikzcd} Q^X \arrow[r, shift left, "h_f"] \arrow[r, shift right, "h_g"'] & M_1 \arrow[r, "h'"] \arrow[d, "h"'] & M_3 \\ & M_2 \arrow[ur, "\overline{h}"'] & \end{tikzcd}$$

□

The combination of theorem 3.7, theorem 3.10 and theorem 3.11, we can obtain the main result of this paper.

Theorem 3.12. The category ${}_{\mathbf{Q}}\mathbf{Mod}_{\mathbf{P}}$ is algebraic.

REFERENCES

[1] Mulvey, C. J. (1986). &. *Suppl. Rend. Circ. Mat. Palermo Ser., 12*, 99–104.

[2] Rosenthal, K. I. (1990). *Quantales and their applications*. London: Longman Scientific and Technical.

[3] Nawaz, M. (1985). *Quantales: quantale sets* (Ph. D. Thesis). UK: University of Sussex.

[4] Niefield, S., & Rosenthal, K. I. (1985). Strong De Morgan's law and the spectrum of a commutative ring. *Journal of Algebra, 93*, 169–181.

[5] Girard, J. Y. (1987). Linear logic. *Theoretical Computer Science, 50*, 1–102.

[6] Rosenthal, K. I. (1992). A general approach to Gabriel filters on quantales. *Communications in Algebra, 20*(11), 3393–3409.

[7] Brown, C., & Gurr, D. (1993). A representation theorem for quantales. *Journal of Pure and Applied Algebra, 85*, 27–42.

[8] Resende, P. (2001). Quantales, finite observations and strong bisimulation. *Theoretical Computer Science, 254*, 95–149.

[9] Berni-Canani, U., Borceux, F., & Succi-Cruciani, R. (1989). A theory of quantale sets. *Journal of Pure and Applied Algebra, 62*, 123–136.

[10] Sun, S. H. (1990). Remarks on quantic nuclei. *Math. Proc. Camb. Phil. Soc., 108*, 257–260.

[11] Vermeulen, J. J. C. (1994). Proper maps of locales. *Journal of Pure and Applied Algebra, 92*, 79–107.

[12] Miraglia, F., & Solitro, U. (1998). Sheaves over right sided idempotent quantales. *Logic J. IGPL, 6*(4), 545–600.

[13] Coniglio, M. E., & Miraglia, F. (2001). Modules in the category of sheaves over quantales. *Annals of Pure and Applied Logic, 108*, 103–136.

[14] Zhao, B., & Li, J. (2005). On nilpotent matrices over idempotent and right-sided quantale. *Fuzzy Systems and Mathematics, 19*(4), 1–6 (In Chinese).

[15] Wang, S. Q., & Zhao B. (2005). Prequantale congruence and its properties. *Advances in Mathematics, 4*, 746–752 (In Chinese).

[16] Liu, Z. B., & Zhao, B. (2006). Algebraic properties of category of quantale. *Acta Mathematica Sinica, Chinese Series, 49*(6), 1253–1258.

[17] Zhou, Y. H., & Zhao, B. (2006). The free objects in the category of involutive quantales and its property of well-powered. *Chinese Journal of Engineering Mathematics, 23*(2), 216–224 (In Chinese).

[18] Zhao, B., & Han, S. W. (2007). Researches on pre-Girard quantale and its properties. *Journal of Shaanxi Normal University (Natural Science Edition), 35*(4), 1–4 (In Chinese).

[19] Abramsky, S., & Vickers, S. (1993). Quantales,observational logic and process semantics. *Math. Struct. Comput. Sci., 3*, 161–227.

[20] Kruml, D. (2002). Spatial quantales. *Applied Categorial Structures, 10*, 49–62.

[21] Resende, P. (2004). Sup-lattice 2-forms and quantales. *Journal of Algebra, 276*, 143–167.

[22] Resende, P. (2002). Tropological systems are points of quantales. *Journal of Pure and Applied Algebra, 173*, 87–120.

[23] Paseka, J. (2000). A note on Girard bimodules. *International Journal of Theoretical Physics, 39*(3), 805–812.

[24] Paseka, J. (1999). Hilbert Q-modules and nuclear ideals in the category of $\vee$-semilattices with a duality. In *CTCS'99: Conference on Category Theory and Computer Science (Edinburgh), Elsevier, Amsterdam*, 1999, (Paper No. 29019), 19 pages. (electronic). MR 2001k: 0314

[25] Paseka, J. (2001). Morita equivalence in the context of hilbert modules. In *Proceedings of the Ninth Prague Topological Symposium Contribution papers from the symposium held in Prague, CzechRepublic*, 19-25 August 2001 (pp. 223–251).

[26] Herrilich, H., & Strecker, E. (1978). *Category theory.* Berlin: Heldermann Verlag.