CSCanada
Progress in Applied Mathematics
Vol. 5, No. 1, 2013, pp. [49–58]
DOI: 10.3968/j.pam.1925252820130501.5729

ISSN 1925-251X [Print]
ISSN 1925-2528 [Online]
www.cscanada.net
www.cscanada.org

On the Beta-Nakagami Distribution

Olanrewaju I. Shittu[a],* and Kazeem A. Adepoju[a]

[a] Department of Statistics, University of Ibadan, Ibadan, Nigeria.

* Corresponding author.
Address: Department of Statistics, University of Ibadan, Ibadan, Nigeria; E-Mail: oi.shittu@ui.edu.ng

Received: November 9, 2012/ Accepted: January 12, 2012/ Published: January 31, 2013

Abstract: This study is focused on combining Nakagami distribution and beta distribution with a view to obtaining a distribution that is better than each of them individually in terms of the estimate of their characteristics and parsimonious in their parameters using the logit of beta (the link function of the Beta generalized distribution by Jones (2004)). The resulting model, beta Nakagami distribution is better in terms of its flexibility and shape. The statistical properties of the proposed distribution such as moments, moment generating function, the asymptotic behavior among others were investigated. Our findings showed that beta Nakagami apart from being flexible, has better representation of data than Nakagami distribution. It therefore describes situations better than the Nakagami distribution.

Key words: Nakagami; Beta-nakagami; Moment generating function; Hazard rate

Shittu, O. I., & Adepoju, K. A. (2013). On the Beta-Nakagami Distribution. *Progress in Applied Mathematics*, *5*(1), 49–58. Available from http://www.cscanada.net/index.php/pam/article/view/j.pam.1925252820130501.5729 DOI: 10.3968/j.pam.1925252820130501.5729

1. INTRODUCTION

Nakagami distribution is used in measuring alternation of wireless signal traversing multiple paths, while the Beta distribution is one of the skewed distributions used in describing uncertainty or random variation in a system. It can be used also to rescale and shift to create distributions with a wide range of shapes.

The distribution was proposed by William C. Hoffman as used by Nakagami [1] as a function related to the gamma distribution with the first parameter being shape parameter, μ and second parameter, ω controls the spread.

The random variable X has the Nakagami distribution if its probability density function is defined as follows:

$$f(x) = \frac{2\mu^{\mu} x^{2\mu-1} e^{-\frac{\mu x^2}{\omega}}}{\Gamma(\mu)\omega^{\mu}}, \quad \mu \geq 0.5, \omega > 0, x > 0 \tag{1}$$

The graph of the probability density function of (1) with $\mu = 1.5, \omega = 2$ is given in Figure 1.

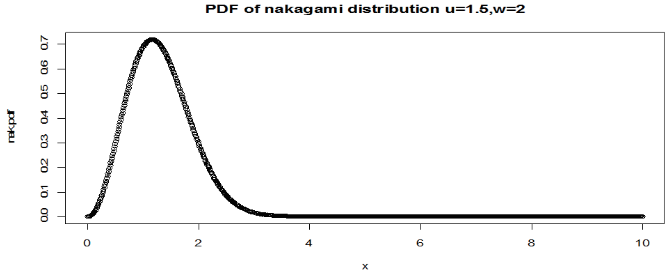

Figure 1
The Probability Density Function of Nakagami Distribution with $\mu = 1.5$, $\omega = 2$

The cumulative distribution of a random variable from (1) can be obtained as

$$P(X \leq x) = F(x) = \int_0^x \frac{2\mu^{\mu} t^{2\mu-1} e^{-\frac{\mu t^2}{\omega}} dt}{\Gamma(\mu)\omega^{\mu}} = \frac{2\mu^{\mu}}{\Gamma(\mu)\omega^{\mu}} \int_0^x t^{2\mu-1} e^{-\frac{\mu t^2}{\omega}} dt \tag{2}$$

Let

$$y = \frac{\mu x^2}{\omega}, \quad \text{then} \quad x = \left(\frac{\omega y}{\mu}\right)^{\frac{1}{2}} \tag{3}$$

Therefore,

$$F(x) = \frac{2\mu^{\mu}}{\Gamma(\mu)\omega^{\mu}} \int_0^x \left[\left(\frac{\omega y}{\mu}\right)^{\frac{1}{2}}\right]^{2\mu-1} \frac{e^{-y}\omega dy}{2\left(\frac{\omega y}{\mu}\right)^{\frac{1}{2}}\mu} \tag{4}$$

It is easy to show that

$$F(x) = \frac{1}{\Gamma(\mu)} \int_0^x y^{\mu-1} e^{-y} dy \tag{5}$$

It can be observed that (5) is an incomplete function which can be represented as

$$\frac{1}{\Gamma(\mu)}\gamma(\mu, y) \tag{6}$$

With $y = \frac{\mu x^2}{\omega}$, we have

$$F(x) = \frac{1}{\Gamma(\mu)}\gamma\left(\mu, \frac{\mu x^2}{\omega}\right) \tag{7}$$

The Literature on Generalized forms of the Nakagami distribution is very scanty. The aim of this paper is to develop a generalize form of Nakagami distribution using the logit of Beta as the link function.

2. THE PROPOSED BETA-NAKAGAMI DISTRIBUTION

A new class of probability distributions revealed lately involve compounded beta family of distribution which include beta-normal (Eugene & Famoye, 2002) [2]; beta-Gumbel (Nadarajah & Kotz, 2004) [3], beta-Weibull (Famoye, Lee & Olugbenga, 2005) [4], beta-exponential (Nadarajah & Kotz, 2006) [5]; beta-Rayleigh (Akinsete & Lowe, 2009) [6]; beta-Laplace (Kozubowski & Nadarajah, 2008) [7]; beta-Pareto (Akinsete, Famoye & Lee, 2008) [8]; beta-Gamma, beta-t, beta-f, beta-beta among others.

Now let X be a random variable form the distribution with parameters and defined in (1) using the logit of Beta defined by Jones [9] as:

$$g(x) = \frac{1}{\beta(a,b)}\left[F(x)\right]^{a-1}\left[1 - F(x)\right]^{b-1} f(x) \tag{8}$$

and the expression in (7) above: be in use, the Beta-Nakagami distribution is derived as follows

$$\begin{aligned} g(x) =& \frac{1}{\beta(a,b)}\left[\frac{1}{\Gamma(\mu)}\gamma\left(\mu, \frac{\mu x^2}{\omega}\right)\right]^{a-1} \\ &\times\left[1 - \frac{1}{\Gamma(\mu)}\gamma\left(\mu, \frac{\mu x^2}{\omega}\right)\right]^{b-1}\frac{2\mu^\mu x^{2\mu-1}e^{-\frac{\mu}{\omega}x^2}}{\Gamma(\mu)\omega^\mu} \end{aligned} \tag{9}$$

Let

$$M(x) = \frac{1}{\Gamma(\mu)}\gamma\left(\mu, \frac{\mu x^2}{\omega}\right) \tag{10}$$

Such that $\frac{dM}{dx} = \frac{2\mu^\mu x^{2\mu-1}e^{-\frac{\mu}{\omega}x^2}}{\Gamma(\mu)\omega^\mu}$

Equation (9) the pdf of Beta-Nakagami becomes

$$g(x) = \frac{1}{\beta\left(a,b\right)}M^{a-1}(1 - M)^{b-1}\frac{dM}{dx} \tag{11}$$

The pdf of Beta-Nakagami distribution for different values of the parameters is given in Figure 2.

This verifies that $g(x)$ is indeed a probability density function of a continuous distribution that is skewed to the right.

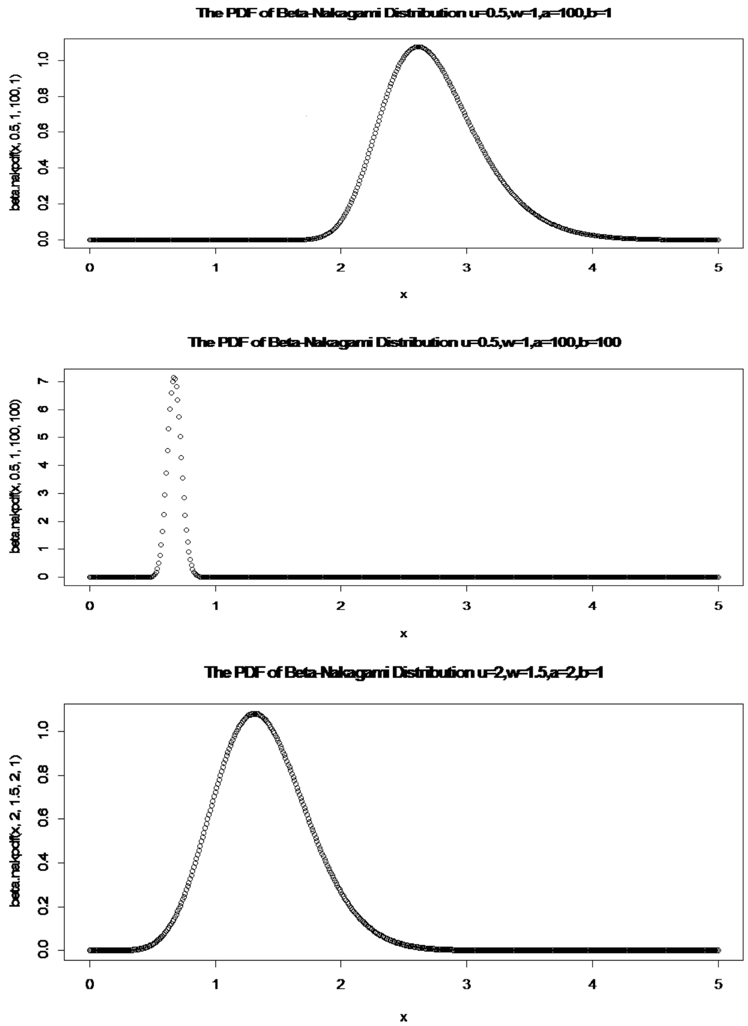

Figure 2
The Probability Density Function of Nakagami Distribution for Different Values of the Parameters

2.1. Cumulative Distribution Function

From Wolfram Math World and without loss of generality, the incomplete Gamma function is expressed as

$$\gamma(a,b) = b^a \Gamma(a) e^{-b} \sum_{k=0}^{\infty} \frac{b^k}{\Gamma(a+k+1)} \tag{12}$$

Then (10) can be expressed in terms of (12) as

$$\gamma\left(\mu, \frac{\mu}{\omega}x^2\right) = \left(\frac{\mu x^2}{\omega}\right)^{\mu} \Gamma(\mu) e^{-\frac{\mu x^2}{\omega}} \frac{\sum_{k=0}^{\infty}\left(\frac{\mu x^2}{\omega}\right)^k}{\Gamma(\mu+k+1)} \tag{13}$$

Given that X, be distributed as in (5), the cumulative distribution function may be expressed as

$$G(x) = P\left(X \le x\right)$$
$$= \int_0^x \frac{1}{\beta(a,b)}\left[\frac{1}{\Gamma(\mu)}\gamma\left(\mu, \frac{\mu t^2}{\omega}\right)\right]^{a-1}\left[1 - \frac{1}{\Gamma(\mu)}\gamma\left(\mu, \frac{\mu t^2}{\omega}\right)\right]^{b-1} \frac{2\mu^{\mu} t^{2\mu-1} e^{-\frac{\mu t^2}{\omega}}}{\Gamma(\mu)\omega^{\mu}} dt \tag{14}$$

Using (10) in (14), we have

$$G(x) = P(X \le x) = \int_0^x \frac{1}{\beta\left(a,b\right)} M^{a-1}(1-M)^{b-1} dM = \frac{\beta(M;a,b)}{\beta(a,b)} \tag{15}$$

where $\beta(M;a,b)$ is an incomplete Beta function.

According to Jones (2004) [9], the above expression can be written as:

$$G(x) = \frac{M^a}{\beta(a,b)}\left[\frac{1}{a} + \frac{1-b}{a+1}M + \dots + \frac{(1-b)(2-b)(n-b)M^n}{n!(a+n)}\right] \tag{16}$$

The graph of the cumulative distribution of Beta-Nakagami for various values of parameters is given in Figure 3.

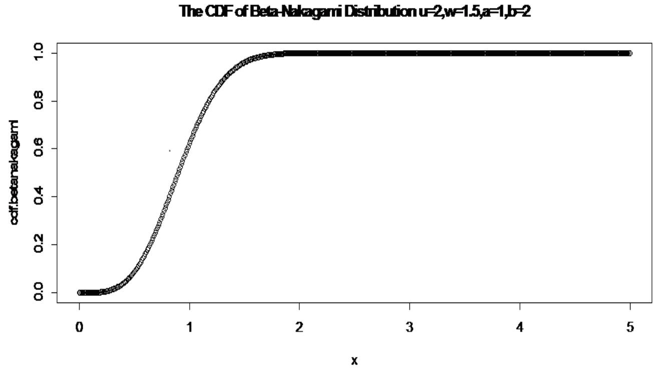

Figure 3
The Cumulative Distribution of Beta-Nakagami with $\mu = 2, \omega = 1.5$, $a = 1, b = 2$

Attempts to vary the values of the parameters produced graphs of the same shape and pattern.

2.2. The Asymptotic Properties

In this section we examine the asymptotic properties of Beta-Nakagami distribution with a view to determining its performance as $x \to \infty$. This is achieved by investigating the limiting behavior of the distribution.

$$\lim_{x\to\infty} g(x) = \lim_{x\to\infty} \frac{1}{\beta(a,b)} \left[\frac{1}{\Gamma(\mu)} \gamma\left(\mu, \frac{\mu x^2}{\omega}\right) \right]^{a-1} \times \left[1 - \frac{1}{\Gamma(\mu)} \gamma\left(\mu, \frac{\mu x^2}{\omega}\right) \right]^{b-1} \frac{2\mu^\mu x^{2\mu-1} e^{-\frac{\mu x^2}{\omega}}}{\Gamma(\mu)\omega^\mu}$$

This is because $\lim\limits_{x\to\infty} \dfrac{2\mu^\mu x^{2\mu-1} e^{-\frac{\mu x^2}{\omega}}}{\Gamma(\mu)\omega^\mu}$ is 0.

We further investigate the CDF, $G(x)$ as x tends zero, we see that $\lim G(x) \to 0$, since $\lim\limits_{x\to 0} \dfrac{2\mu^\mu x^{2\mu-1} e^{-\frac{\mu x^2}{\omega}}}{\Gamma(\mu)\omega^\mu} \to 0$.

In literature, if as x tends to zero, PDF tends to zero and as x tends to infinity, it tends to zero, it is an indication that at least one mode exists. Therefore the Beta Nakagami, distribution has a mode.

2.3. The Hazard Rate Function

The hazard rate function of a random variable X with the probability function $g(x)$ and a cumulative distribution function $G(x)$ is given by

$$h(x) = \frac{g(x)}{1 - G(x)}$$

For the Beta-Nakagami distribution with $g(x)$ and $G(x)$ respectively defined by (9) and (15),the hazard rate function is expressed as

$$h(x) = \frac{M^{a-1}(1-M)^{b-1}M^1}{\beta(a,b) - \beta(M;a,b)} \tag{17}$$

where M is distribution in (10).

It is straight forward to show from (17) that $\lim\limits_{x\to\infty} h(x) = 0$, and $\lim\limits_{x\to 0} h(x) = 0$.

The graph of the hazard functions of the distribution is given in Figure 4 for various values of parameters.

3. MOMENT & MOMENT GENERATING FUNCTIONS

Let X be a Beta-Nakagami random variable as given in (9). According to Hosking (1990), when a random variable X follows a generalized beta generated distribution i. e., $X \sim GBG(f,a,b,c)$, then $\mu_r^1 = E\left(F^{-1}U^{\frac{1}{c}}\right)^r$, where $U \sim B(a,b)$, c is a constant and $F^{-1}(x)$ is the inverse of the CDF of the Nakagami distribution, since

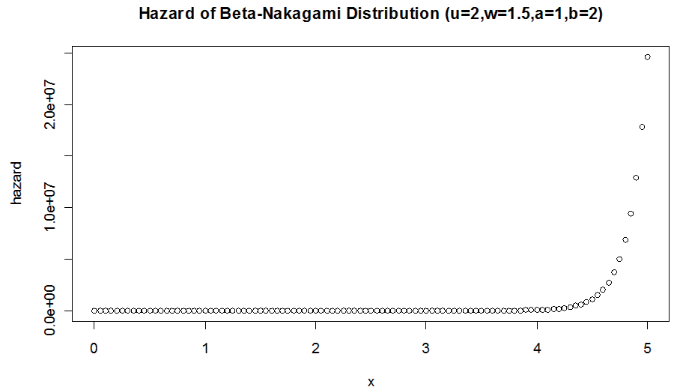

Figure 4
The Hazard Functions of Beta-Nakagami Distribution with $\mu = 2$, $\omega = 1.5$, $a = 1, b = 2$

Beta-Nakagami is a special form for $C = 1$, we have the general r^{th} moment of the Beta-generated distribution as

$$\mu_r^1 = \frac{1}{\beta(a,b)} \int_0^1 \left[F^{-1}(x)\right]^r x^{a-1}(1-x)^{b-1} dx \qquad (18)$$
$$u^1 = 1$$

The Taylor series expansion around the point $E(X_f) = \mu_f$ to obtain

$$\mu_r^1 \cong \sum_{k=0}^{r} \binom{r}{k} \left[F^{-1}(\mu_f)\right]^{r-k} \left[F^{-1(1)}\mu_f\right]^k \sum_{k=0}^{r} (-1)^i \binom{k}{i} \qquad (19)$$

Cordeiro and de Castro (2011) [10] gave alternative series expansion for μ_r^1 in terms of $r(r,m) = E\left\lfloor Y^r F(Y)^m \right\rfloor$ where Y follows the parent distribution then for $m = 0, 1, ...$

$$\mu_r^1 = \frac{1}{\beta(a,b+1)} \sum_{i=0}^{\infty} (-1)^i \binom{b}{i} r\,(r, a, i-1) \qquad (20)$$

Now using another moment generating function of X for generalized Beta-Distribution given also by Cordeiro and de Castro (2011) [10] as:

$$M_{(t)} = \frac{1}{\beta(a,b+1)} \sum_{i=0}^{\infty} (-1)^i \binom{b}{i} \rho\,(t, ai-1) \qquad (21)$$

where

$$\rho\,(t,r) = \int_{-\infty}^{\infty} e^{tx} [F\,(x)]^r f\,(x)\, dx$$

Then

$$M_{(t)} = \frac{1}{\beta(a,b+1)} \sum_{i=0}^{\infty} (-1) \binom{b}{i} \int_{-\infty}^{\infty} e^{tx} \left[G_{(x)}\right]^{ai-1} f\,(x)\, dx \qquad (22)$$

The moment generating function of Beta-Nakagami distribution i. e. obtained as

$$M_{(t)} = \frac{1}{\beta(a,b+1)}\sum_{i=0}^{\infty} -1^{i}\begin{pmatrix} b \\ i \end{pmatrix}\int e^{tx}\left[\frac{1}{\Gamma(\mu)}\gamma\left(\mu, \frac{\mu}{\omega}x^2\right)\right]^{ai-1}\frac{2\mu^{\mu}x^{2\mu-1}e^{-\frac{\mu}{\omega}x^2}}{\Gamma(\mu)\omega^{\mu}} \tag{23}$$

By setting $a = b = i = 1$ in (23) gives the moment generating function of the Nakagami distribution.

4. ESTIMATION OF PARAMETERS

In this section attempt will be made to derive the maximum likelihood estimates of the parameters of the Beta Nakagami distribution.

Let θ be a vector of parameters, Cordeiro and de Castro (2011) [10] gave the log-likelihood function for $\theta = (a, b, c, \tau)$ where $\tau = (\mu, \omega)$ as

$$\begin{aligned} L(\theta) =& n\log C - n\log[\beta(a,b)] + \sum_{i=1}^{n}\log f(x_i,\tau) \\ &+ (a-1)\sum_{i=1}^{n}\log F(x_i,\tau) + (b-1)\sum_{i=1}^{n}\log[1 - F^{c}(x_i,\tau)] \end{aligned} \tag{24}$$

The class of Generalized Beta distribution reduces to the class of Beta generated distribution when $c = 1$. We have $\theta = (a, b, 1, \tau)$ as

$$\begin{aligned} L(\theta) =& -n\log[\beta(a,b)] + \sum_{i=1}^{n}\log f(x_i,\tau) \\ &+ (a-1)\sum_{i=1}^{n}\log F(x_i,\tau) + (b-1)\sum_{i=1}^{n}\log[1 - F^{c}(x_i,\tau)] \end{aligned}$$

where $f(x,\tau) = \dfrac{2\mu^{\mu}x^{2\mu-1}e^{-\frac{\mu}{\omega}x^2}}{\Gamma(\mu)\omega^{\mu}}$, $F(x,\tau) = \dfrac{1}{\Gamma(\mu)}\gamma\left(\mu, \dfrac{\mu x^2}{\omega}\right)$.

$$\begin{aligned} L(\theta) =& -n\log[\beta(a,b)] + \sum_{i=1}^{n}\frac{\log 2\mu^{\mu}x^{2\mu-1}e^{-\frac{\mu}{\omega}x^2}}{\Gamma(\mu)\omega^{\mu}} \\ &+ (a-1)\sum_{i=1}^{n}\log\frac{1}{\Gamma(\mu)}\gamma\left(\mu, \frac{\mu x^2}{\omega}\right) + (b-1)\sum_{i=1}^{n}\left[1 - \frac{1}{\Gamma(\mu)}\gamma\left(\mu, \frac{\mu x^2}{\omega}\right)\right] \end{aligned} \tag{25}$$

Equation (25) is differentiated a, b, μ and ω.

Noting that $\beta(a,b) = \dfrac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}$.

$$\frac{dL(\theta)}{d(a)} = -a\frac{\Gamma'(a)}{\Gamma(a)} + n\frac{\Gamma'(a)}{\Gamma(a)} + \sum_{i=1}^{n}\log\frac{\gamma(\mu, \frac{\mu x^2}{\omega})}{\Gamma(\mu)} \tag{26}$$

$$\frac{\partial L(\theta)}{\partial b} = -n\frac{\Gamma'(b)}{\Gamma(b)} + n\frac{\Gamma'(a+b)}{\Gamma(a+b)} + \sum_{i=1}^{n}\left[1 - \frac{\gamma(\mu, \frac{\mu x^2}{\omega})}{\Gamma(\mu)}\right] \tag{27}$$

$$\begin{aligned}\frac{\partial L(\theta)}{\partial \mu} =& \sum_{i=1}^{n}\Gamma(\mu)\omega^{\mu}\frac{1}{2\mu^{\mu}x^{2\mu-1}e^{-\mu x^2/\omega}}\left[2\frac{\mu^{\mu}x^{2\mu-1}e^{-\mu x^2/\omega}}{\Gamma(\mu)\omega^{\mu}}\right]' \\ &+ (a-1)\sum_{i=1}^{n}\frac{\gamma(\mu, \frac{\mu x^2}{\omega})}{\gamma'(\mu, \frac{\mu x^2}{\omega})} + \frac{(b-1)}{\Gamma(\mu)}\sum_{i=1}^{n}\left[1 - \gamma(\mu, \frac{\mu x^2}{\omega})\right]\end{aligned} \tag{28}$$

$$\begin{aligned}\frac{\partial L(\theta)}{\partial \omega} =& \frac{(a-1)}{\Gamma(\mu)}\sum_{i=1}^{n}\frac{\gamma(\mu, \frac{\mu x^2}{\omega})'}{\gamma(\mu, \frac{\mu x^2}{\omega})} + (b-1)\sum_{i=1}^{n}\left[1 - \frac{\gamma(\mu, \frac{\mu x^2}{\omega})}{\Gamma(\mu)}\right]' \\ &+ \sum\left[\log 2\frac{\mu^{\mu}x^{2\mu-1}}{\Gamma(\mu)\omega^{\mu}}e^{-\frac{\mu x^2}{\omega}}\right]'\end{aligned} \tag{29}$$

The above equations can be solved using numerical method to obtain the $\hat{a}, \hat{b}, \hat{\mu}, \hat{\omega}$, the MLE of (a, b, μ, ω) respectively.

For interval estimation and hypothesis tests on the model parameters, the information matrix needed can be derived by differentiating 26, 27, 28 and 29 with respect to the parameters mentioned earlier.

5. APPLICATION TO REAL DATA

In this section we compare the results of fitting the Beta-Nakagami and Nakagami distribution to the data set studied by Choulakian and Stephens *et al.* (2001) [11] and Akinsete *et al.* (2008) [8].

The data consists of 72 exceedances for the years 1958-1984, rounded to one decimal point. The distribution of the data is highly skewed to the left. Using the *R* codes the maximum likelihood estimates and the maximized log-likelihood for the Beta-Nakagami distribution are: $\hat{a} = 36.67079$, $\hat{b} = 0.000000089$, $\hat{\omega} = 110.8301$, $\hat{\mu} = 0.5$, and $l_{BN} = -8.86189e{+}12$ while the maximum likelihood estimates and the maximized log-likelihood for the Nakagami distribution are: $\hat{\mu} = 1.894254e - 06$, $\hat{\omega} = 5.524395e - 07$, and $\hat{l}_N = -74089.74$ (l_{BN} and $\hat{l}_N$ denote the negative log-likelihood of Beta-Nakagami and Nakagami distribution.)

The likelihood ratio statistics for testing the hypothesis that $a = b = 1$ for Nakagami versus Beta-Nakagami is $\omega = 1.1124e{+}13$ (the Wald statistic) which is an indication that Nakagami distribution should be rejected, in favor Beta-Nakagami distribution in addition to the fact that, distribution with lower negative log-likelihood gives a better fit.

The asymptotic covariance matrix of the maximum likelihood estimates for the Beta-Nakagami distribution, which is from the inverse of the information matrix, is given by

$$\begin{pmatrix} -1.452411e+02 & 2.414463e+15 & 0.000000 \\ 2.414463e+15 & -8.850098e+20 & -3.209603e+02 \\ 0.000000 & -3.209603e+02 & 0.000000 \end{pmatrix}$$

6. CONCLUSION

The existing two parameter Nakagami distribution is extended with the introduction of two extra shape parameters, thus defining the Beta-Nakagami distribution which has a better shape, broader tails and a class of hazard rate functions depending on the parameters.

Detailed studies of the statistical properties of the proposed distribution which include moments, moment generating function among others have been presented. The parameters of the model were estimated by method of maximum likelihood, to pave way for the derivation of fisher information matrix. Real life application indicates that Beta-Nakagami distribution apart from being more flexible has better representation of data than Nakagami distribution.

REFERENCES

[1] Nakagami, M. (1960). The m-distribution a general formulation of intensity distribution of rapid fading. In W. C. Hoffman (Ed.), *Statistical Method in Radio Wave Propagation* (pp. 3-36). Oxford: Pergamon.

[2] Eugene, N. C., & Famoye, F. (2002). Beta-normal distribution and its applications. *Communication in Statistics: Theory and Methods, 31*, 497-512.

[3] Nadarajah, S., & Kotz, S. (2004). The beta-gumbel distribution. *Mathematical Problems in Engineering, 2004*(1), 323-332.

[4] Famoye, F., Lee, C., & Olugbenga, O. (2005). The beta-weibull distribution. *Journal of Statistical Theory and Applications, 4*(2), 121-138.

[5] Nadarajah, S., & Kotz, S. (2006). The beta-exponential distribution. *Reliability Engineering and System Safety, 91*(1), 689-697.

[6] Akinsete, A., & Lowe, C. (2009, January). Beta-rayleigh distribution in reliability measure. Section on Physical and Engineering Sciences, In *Proceedings of the American Statistical Association* (pp. 3103-3107).

[7] Kozubowski, T., & Nadarajah, S. (2008). The beta-Laplace distribution. *Journal of Computational Analysis and Applications, 10*(3), 305-318.

[8] Akinsete, A., Famoye, F., & Lee, C. (2008). The beta-pareto distribution. *Statistics, 42*(6), 547-563.

[9] Jones, M. C. (2004). Families of distributions arising from distributions of order statistics. *Sociedad de Estadistica e Investigacion Operativa Test, 13*(1), 1-43.

[10] Cordeiro, G. M., & de Castro, M. (2011). A new family of generalized distributions. *Journal of Statistical Computation and Simulation, 81*(7), 883-898.

[11] Choulakian, V., & Stephens, M. A. (2001). Goodness-of-fit tests for the generalized pareto distribution. *Technometrics, 43*(4), 478-484.