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The Study of the Fractional Brownian Motion Base 2-Index

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Abstract: In this paper, we use the wavelet transform to the Fractional Brownian motion of 2-index by Haar wavelet, we obtain some statistical properties about the stochastics processes and its density degree and wavelet expresses.

Key words: Fractional Brownian motion of 2-index; Wavelet analysis; Wavelet transform; Haar wavelet; Density degree

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1. INTRODUCTION

The Fractional Brownian motion is a class of important stochastic processes, about the processes, there are some study works [1,3,6–8]. In this paper, we study it use wavelet methods. With the rapid development of computerized scientific instruments comes a wide variety of interesting problems for data analysis and signal processing. In fields ranging from Extragalactic Astronomy to Molecular Spectroscopy to Medical Imaging to computer vision, one must recover a signal, curve, image, spectrum, or density from incomplete, indirect, and noisy data. Wavelets have contributed to this already intensely developed and rapidly advancing field.

Wavelet analysis is a remarkable tool for analyzing function of one or several variables that appear in mathematics or in signal and image processing. With

hindsight the wavelet transform can be viewed as diverse as mathematics, physics and electrical engineering. The basic idea is to use a family of building blocks to represent the object at hand in an efficient and insightful way, the building blocks come in different sizes, and are suitable for describing features with a resolution commensurate with their sizes. Some persons have studied wavelet problems of stochastic process or stochastic system [2–5,9–13]. In this paper, we study a class of random processes using wavelet analysis methodsand study its energy,the energy express by density degree.

2. BASIC DEFINITION

Definition 1 Fractional Brownian motion: Gauss processes $X_t(t \geq 0)$ be called ϕ-fractional Brownian motion, if $X_0 = 0, EX_t = 0$, and

$$E(X_s\ X_t) = \frac{1}{2}(s^\phi + t^\phi - |s-t|^\phi) \tag{1}$$

where, if we let $\phi = 2$, then we call it as Fractional Brownian motion of 2-index.
Definition 2 Let X_t is stochastic processes, then its wavelet alternative is [12]

$$W(s,x) = \frac{1}{s}\int_R X_t \psi(\frac{x-t}{s})dt \tag{2}$$

Then we have

$$E[W(s,x)] = \frac{1}{s}\int_R E[x(t)]\psi(\frac{x-t}{s})dt = 0$$

where ψ is continuous wavelet.
Definition 3 Let $\psi(x)$ is

$$\psi(x) = \begin{cases} 1, 0 \leq x < \frac{1}{2} \\ -1, \frac{1}{2} \leq x < 1 \\ 0, other \end{cases} \tag{3}$$

We call $\psi(x)$ haar wavelet [12].

3. STATISTICS PROPERTY AND ENERGY

We have

$$W(s,x) = \frac{1}{s}\left[\int_{x-\frac{s}{2}}^{x} x(t)dt - \int_{x-s}^{x-\frac{s}{2}} x(t)dt\right]$$

and $E\left[W(s,x)\right] = 0$, then relational function of $W(s,x)$ is

$$\begin{aligned} R(\tau) &= E[W(s,x)W(s,x+\tau)] \\ &= \frac{1}{s^2}\iint_{R^2} E[x(u)x(v)]\psi(\frac{x-u}{s})\psi(\frac{x+\tau-v}{s})dudv \\ &= \frac{1}{s^2}\Big(\int_{x-\frac{s}{2}}^{x}\int_{x-\frac{s}{2}+\tau}^{x+\tau} E[x(u)x(v)]dudv - \int_{x-\frac{s}{2}}^{x}\int_{x-s+\tau}^{x-\frac{s}{2}+\tau} E[x(u)x(v)]dudv \\ &- \int_{x-s}^{x-\frac{s}{2}}\int_{x-\frac{s}{2}+\tau}^{x+\tau} E[x(u)x(v)]dudv + \int_{x-s}^{x-\frac{s}{2}}\int_{x-s+\tau}^{x-\frac{s}{2}+\tau} E[x(u)x(v)]dudv\Big) \\ &= I_1 + I_2 + I_3 + I_4 \end{aligned} \tag{4}$$

4. THE WAVELET PROPERTIES OF SYSTEM

In Equation (1), $\phi = 2$, we do the research in the following.

We have

$$E[x(s)x(t)] = \frac{1}{2}(s^2 + t^2 - |s-t|^2) = st$$

Relational function of $W(s,x)$

$$\begin{aligned}
R_1(\tau) =& E[W(s,x)W(s,x+\tau)] \\
=& \frac{1}{s^2}\iint E[x(u)x(v)]\psi(\frac{x-u}{s})\psi(\frac{x+\tau-v}{s})dudv \\
=& \frac{1}{s^2}[\int_{x-s/2+\tau}^{x}\int_{x-s/2+\tau}^{x+\tau} uvdudv - \int_{x-s/2}^{x}\int_{x-s+\tau}^{x-s/2+\tau} uvdudv \\
& - \int_{x-s}^{x-s/2}\int_{x-s/2+\tau}^{x+\tau} uvdudv + \int_{x-s}^{x-s/2}\int_{x-s+\tau}^{x-s/2+\tau} uvdudv] \\
=& I_1 + I_2 + I_3 + I_4
\end{aligned}$$

$$I_1 = \frac{1}{s^2}\int_{x-s/2}^{x} udu \int_{x-s/2+\tau}^{x+\tau} vdv = \frac{1}{4s^2}(xs - \frac{s^2}{4})(s(x+\tau) - \frac{s^2}{4})$$

$$I_2 = -\frac{1}{s^2}\int_{x-s/2}^{x} udu \int_{x-s-\tau}^{x-s/2+\tau} vdv = -\frac{1}{4s^2}(xs - \frac{s^2}{4})(s(x+\tau) - \frac{3s^2}{4})$$

$$I_3 = -\frac{1}{s^2}\int_{x-s}^{x-s/2} udu \int_{x-s/2+\tau}^{x+\tau} vdv = -\frac{1}{4s^2}(xs - \frac{3}{4}s^2)(s(x+\tau) - \frac{s^2}{4})$$

$$I_4 = \frac{1}{s^2}\int_{x-s}^{x-s/2} udu \int_{x-s+\tau}^{x-s/2+\tau} vdv = \frac{1}{4s^2}(xs - \frac{3s^2}{4})(s(x+\tau) - \frac{3s^2}{4})$$

Then, we have $R_1(\tau) = \frac{1}{8}s(2x-s)$, $R_1'(\tau) = 0$.

Use above, we can obtain $R^{(4)}(0)$, then we can obtain the zero density of $W(s,x)\sqrt{\left|\frac{R''(0)}{\pi^2 R(0)}\right|}$ and the energy density $\sqrt{\left|\frac{R^{(4)}(0)}{\pi^2 R''(0)}\right|}$. Then, we have: The zero density of $W(s,x)$: $\sqrt{\left|\frac{R^{''}(0)}{\pi^2 R(0)}\right|} = 0$ and the energy density $\sqrt{\left|\frac{R^{(4)}(0)}{\pi^2 R''(0)}\right|}$ does not exist.

Also we have

$$\begin{aligned}
R_b(j,k;m,n) &= 2^{-\frac{j+k}{2}}\iint_{R^2} E[x(t)x(s)]\psi(2^j t - m)\psi(2^k s - n)dtds \\
&= 2^{-\frac{j+k}{2}}\iint_{R^2} ts\psi(2^j s - n)\psi(2^k s - m)dtds
\end{aligned}$$

$$\begin{aligned}
R_a(j,k;m,n) &= 2^{-\frac{j+k}{2}}\iint_{R^2} E[x(t)x(s)]\phi(2^j t - m)\phi(2^k s - n)dtds \\
&= 2^{-\frac{j+k}{2}}\iint_{R^2} ts\phi(2^j s - n)\phi(2^k s - m)dtds
\end{aligned}$$

Use Equation (4), we have

$$\psi(2^j t - m) = \begin{cases} 1, m2^{-j} \le t < (1/2+m)2^j \\ -1, (1/2+m)2^{-j} \le t \le (1+m)2^{-j} \end{cases}$$

$$\psi(2^k s - n) = \begin{cases} 1, n2^{-k} \le s < (1/2+n)2^{-k} \\ -1, (1/2+n)2^{-k} \le s \le (1+n)2^{-k} \end{cases}$$

Then

$$\begin{aligned} & R_b(j,k;m,n) \\ =& 2^{-\frac{j+k}{2}} \Big[\int_{m2^{-j}}^{(\frac{1}{2}+m)2^{-j}} tdt \int_{n2^{-k}}^{(\frac{1}{2}+n)2^{-k}} sds - \int_{m2^{-j}}^{(\frac{1}{2}+m)2^{-j}} tdt \int_{(\frac{1}{2}+n)2^{-k}}^{(1+n)2^{-k}} sds \\ & - \int_{(\frac{1}{2}+m)2^{-j}}^{(1+m)2^{-j}} tdt \int_{n2^{-k}}^{(\frac{1}{2}+n)2^{-k}} sds + \int_{(\frac{1}{2}+m)2^{-j}}^{(1+m)2^{-j}} tdt \int_{(\frac{1}{2}+n)2^{-k}}^{(1+n)2^{-k}} sds \Big] \\ =& 2^{-\frac{j+k}{2}} \frac{1}{4} \big[2^{-2j}(\frac{1}{4}+m)(\frac{1}{4}+n)2^{-2k} - 2^{-2j}(\frac{1}{4}+m)(\frac{3}{4}+n)2^{-2k} \\ & - 2^{-2j}(m+\frac{3}{4})(\frac{1}{4}+n)2^{-2k} + 2^{-2j}(m+\frac{3}{4})(\frac{3}{4}+n)2^{-2k} \big] \\ =& -2^{-\frac{5(j+k)}{2}-2} \end{aligned} \tag{5}$$

Then, we express relational degree of $b(j,m)$. Let

$$\phi(t) = \begin{cases} 1, 0 \le t \le 1 \\ 0, other \end{cases}$$

Then, we have

$$\phi(2^j t - n) = \begin{cases} 1, n2^j \le t \le (n+1)2^j \\ 0, other \end{cases}$$

$$\phi(2^k s - m) = \begin{cases} 1, m2^k \le s \le (m+1)2^k \\ 0, other \end{cases}$$

then,

$$\begin{aligned} R_a(j,k;m,n) &= 2^{-\frac{j+k}{2}} \iint_{R^2} ts\phi(2^j s - n)\phi(2^k s - m)dtds \\ &= -\frac{j+k}{2} \int_{n2^j}^{(n+1)2^j} tdt \int_{m2^k}^{(m+1)2^k} sds \\ &= -\frac{j+k}{4} 2^{2(j+k)}(1+2n)(1+2m) \end{aligned}$$

5. WAVELET REPRESENTATION

Let real function ϕ is standard orthogonal element of multiresolution analysis $\{V_j\}, j \in Z$ [7], then exist $h_k \in l^2$, have

$$\phi(t) = \sqrt{2} \sum_k \phi(2t - k)$$

Let

$$\psi(t) = \sqrt{2}\sum_{k}(-1)^k h_{1-k}\phi(2t-k)$$

Then wavelet express of $X(t)$ in mean square is

$$X(t) = 2^{-\frac{J}{2}}\sum_{K} C_n^J \phi(2^{-J}t-n) + \sum_{j\le J} 2^{-\frac{j}{2}} \sum_{n\in Z} d_n^j \psi(2^{-j}t-n) \tag{6}$$

where,

$$C_n^j = 2^{-\frac{j}{2}}\int_R X(t)\phi(2^{-j}t-n)dt$$

$$d_n^j = 2^{-\frac{j}{2}}\int_R X(t)\psi(2^{-j}t-n)dt$$

Then we have

$$\begin{aligned} E\left[C_n^j C_m^k\right] &= 2^{-\frac{j+k}{2}}\iint_{R^2} E[X(t)X(s)]\phi(2^{-j}t-n)\phi(2^{-k}s-m)dsdt \\ E\left[d_n^j d_m^k\right] &= 2^{-\frac{j+k}{2}}\iint_{R^2} E[X(t)X(s)]\psi(2^{-j}t-n)\psi(2^{-k}s-m)dsdt \end{aligned} \tag{7}$$

If we let

$$\phi(t) = \begin{cases} 1, 0\le t\le 1 \\ 0, other \end{cases}$$

We can obtain the value of $E(C_n^j C_m^k)$ according to value of (5) on above. Now we consider function $\psi(t)$ that exist compact support set on $[-k_1, k_2]$,$k_1, k_2 \ge 0$, and exist enough large M, have $\int_R t^m\psi(t)dt = 0, 0\le m\le M-1$, then ϕ exist compact support set on $[-k_3, k_4]$ satisfy $k_1+k_2 = k_3+k_4, k_3, k_4 \ge 0$.

Let $b(j,k) =< X(t), \psi_{jk} >$, $a(j,k) =< X(t), \phi_{jk} >$ and J is a constant, then

$$\left\{2^{\frac{J}{2}}\phi(2^J x-k), k\in Z\right\} \cup \left\{2^{\frac{j}{2}}\psi(2^j t-k), k\in Z\right\}_{j\ge J}$$

are a standard orthonormal basis of space $L^2(R)$, then have

$$X(t) = 2^{\frac{J}{2}}\sum_{K\in Z} a(J,K)\phi(2^J t-K) + \sum_{j\ge J}\sum_{K\in Z} 2^{\frac{j}{2}} b(j,K)\psi(2^j t-K)$$

Therefore, the self-correlation function of $b(j,m)$

$$\begin{aligned} R_b(j,K;m,n) &= E\left[b(j,m)b(k,n)\right] \\ &= 2^{-\frac{j+K}{2}}\iint_{R^2} E\left[X(t)X(s)\right]\psi(2^j t-m)\psi(2^K s-n)dtds \end{aligned} \tag{8}$$

And also we have

$$\begin{aligned} R_a(j,K;m,n) &= E\left[a(j,m)a(k,n)\right] \\ &= 2^{-\frac{j+K}{2}}\iint_{R^2} E\left[X(t)X(s)\right]\phi(2^j t-m)\phi(2^K s-n)dtds \end{aligned}$$

Let

$$F(2^{j-K},t)=\int_R \psi(2^{j-K}s-t)\psi(s)ds$$

If $\psi(t)$ has $(M-1)$-order waning moments, then $F(2^{j-K},t)$ has $(2M-1)$-order waning moments. Actually,

$$\begin{aligned}&\iint_{R^2} t^m\psi(2^{j-K}s-t)\psi(s)dsdt\\&=-\iint_{R^2}(2^{j-K}s-t)^m\psi(t)\psi(s)dsdt\\&=-\iint_{R^2}\sum_n C_m^n(2^{j-K}s)^{m-n}(-t)^n\psi(t)\psi(s)dsdt=0, m<2M\end{aligned}$$

Therefore, we have

Theorem Let $X(t)$ is solution process of system (1), $\psi(t)$ has compact supported set on $[-K_1,K_2]$, $K_1,K_2>0$, and $\psi(t)$ has $(M-1)$-order waning moments, and $\psi(t)$ is standard orthonormal wavelet function. Then stochastic process $b(J,m)$ are stationary process.

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