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Ricci Solitons in f-Kenmotsu Manifolds and 3-Dimensional Trans-Sasakian Manifolds

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Abstract

In the Present paper we study Ricci solitons in trans-sasakian manifolds. In particular we consider Ricci solitons in f-Kenmotsu manifolds and we prove the conditions for the Ricci solitons to be shrinking, steady and expanding.

Key words

Ricci solitons; f-Kenmotsu; Trans-Sasakian; Shrinking; Steady; Expanding

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1. INTRODUCTION

In [10], Ramesh Sharma started the study of the Ricci solitons in contact geometry. Later Mukut Mani Tripathi [11], Cornelia Livia Bejan and Mircea Crasmareanu [3] and others extensively studied Ricci solitons in contact metric manifolds. A Ricci soliton is a generalization of an Einstein metric and is def ned on a Riemannian manifold (M, g) by

$$L_V g + 2Ric + 2\lambda g = 0, \tag{1.1}$$

where V is a complete vector f eld on M and λ is a constant. The Ricci soliton is said to be shrinking, steady and expanding according as λ is negative, zero and positive respectively. If the vector f eld V is the gradient of a potential function f then g is called a gradient Ricci soliton and (1.1) takes the form,

$$\nabla\nabla f = Ric + \lambda g.$$

Perelman [9] proved that a Ricci soliton on a compact n-manifold is a gradient Ricci soliton. In [11], Ramesh Sharma studied Ricci solitons in K-contact manifolds, where the structure f eld ξ is killing and he proved that a complete K-contact gradient soliton is compact Einstein and Sasakian. M. M. Tripathi [11] studied Ricci solitons in $N(K)$-contact metric and (k, μ) manifolds. Motivated by the above studies on Ricci solitons, in this paper, we study Ricci solitons in an important class of manifolds introduced by Kenmotsu in [6].

2. PRELIMINARIES

A (2n+1) dimensional smooth manifold M is said to be an almost contact metric manifold if it admits an almost contact metric structure (ϕ, ξ, η, g) consisting of a tensor f eld ϕ of type (1,1), a vector f eld ξ, a 1-form η and Riemannian metric g compatible with (ϕ, ξ, η) satisfying

$$\Phi^2 = -I + \eta \otimes \xi, \ \eta(\xi) = 1, \ \phi\xi = 0, \ \eta \circ \phi = 0 \tag{2.1}$$

and

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y). \tag{2.2}$$

An almost contact metric manifold is said to be an f-Kenmotsu manifold if

$$(\nabla_X \phi)Y = f[g(\phi X, Y)\xi - \phi(X)\eta(Y)], \tag{2.3}$$

where $f \in C^\infty(M)$ is strictly positive and $df \wedge \eta = 0$ holds.
From (2.3) we have

$$\nabla_X \xi = f(X - \eta(X)\xi). \tag{2.4}$$

An almost contact metric manifold is called a trans-Sasakian manifold [4] [8] if

$$(\nabla_X \phi)Y = \alpha(g(X, Y)\xi - \eta(Y)X) + \beta(g(\phi X, Y)\xi - \eta(Y)\phi X), \tag{2.5}$$

for some smooth functions α and β on M.

3. RICCI SOLITONS IN F-KENMOTSU MANIFOLDS

Let M be an n dimensional f-Kenmotsu manifold and let (g, V, λ) be a Ricci soliton in M. Let $\{e_i\}$, $1 \leq i \leq n$ be an orthonormal basis of $T_P M$ at $P \in M$. Then from (1.1), we have

$$S = -(\lambda g + \frac{1}{2} L_V g). \tag{3.1}$$

From (2.4), we have

$$(L_\xi g)(X, Y) = f[g(X, Y) - \eta(X)\eta(Y)]. \tag{3.2}$$

From (3.1) and (3.2), we have

$$S(X, Y) = -\lambda g(X, Y) - f[g(X, Y) - \eta(X)\eta(Y)]. \tag{3.3}$$

It is easy to verify from (3.3) that

$$S(\phi X, Y) = -S(X, \phi Y) \tag{3.4}$$

and

$$S(\xi, \xi) = -\lambda. \tag{3.5}$$

From (2.3) and (2.4), we f nd that

$$R(X, Y)\xi = f^2[\eta(X)Y - \eta(Y)X] + (Yf)\phi^2 X - (Xf)\phi^2 Y \tag{3.6}$$

and

$$S(X, \xi) = -[(n-1)f^2 + \xi f]\eta(X) - (n-2)X(f). \tag{3.7}$$

From (3.7), we obtain

$$S(\xi, \xi) = -(n-1)[f^2 + \xi f]. \tag{3.8}$$

Comparing (3.5) and (3.8), we obtain

$$\lambda = (n-1)(f^2 + \xi f) \tag{3.9}$$

From (3.9), it is clear that λ is positive if f is a constant. Thus we have
Ricci soliton in a f-Kenmotsu manifold is expanding, provided f is a constant.
Suppose f is not a constant. If $\{e_i\}$ is an orthonormal basis of $T_P M$ at $P \in M$, then taking $X = Y = e_i$ in (3.3) and summing over $1 \leq i \leq n$, we get

$$r = -\lambda n - f(n-1), \tag{3.10}$$

where r is the scalar curvature.
Differentiating (3.10) covariantly w.r.to X, we get

$$X_r = -(n-1)X_f, \tag{3.11}$$

where

$$X_r = \nabla_X r, \quad X_f = \nabla_X f.$$

From (3.3), we have

$$QX = -\lambda X - f(\phi^2 X). \tag{3.12}$$

In view of (2.5), differentiation of (3.12) yields

$$(\nabla_Y Q)X = Yf(\phi^2 X) - f^2\eta(X)\phi^2 Y + f\Phi(X,Y)\xi.$$

Contracting the above equation with respect to Y, we get

$$(divQ)X = (\phi^2 X) + f^2(n-1)\eta(X). \tag{3.13}$$

Using (3.11) and the identity

$$(divQ)X = \frac{X_r}{2},$$

we obtain

$$(n-3)(Xf) = -2(\xi f + (n-1)f^2)\eta(X). \tag{3.14}$$

Putting $X = \xi$ in (3.14), we get

$$\xi f + 2f^2 = 0. \tag{3.15}$$

Using (3.15) in (3.9), we get

$$\lambda = -((n-1)f^2,$$

i.e. $\lambda < 0$ or the Ricci soliton g is shrinking. Thus we have
Theorem 3.1. *Ricci soliton in an f-Kenmotsu manifold,where f is a non-constant is shrinking.*
From (2.3), we have

$$\begin{aligned} R(X,Y)\phi Z = \phi(R(X,Y)Z) &+ Xf[g(\phi Y,Z)\xi - \phi(Y)\eta(Z)] \\ &+ f^2 g(\phi Y,Z)(X - \eta(X)\xi) - f^2 g(\phi X,Y)\eta(Z)\xi \\ &+ f^2\phi(X)\eta(Y)\eta(Z) - f^2\phi(Y)g(\phi X,\phi Z) \\ &+ fg(\phi X,\nabla_Y Z)\xi - (Yf)[g(\phi X,Z)\xi - \phi(X)\eta(Z)] \\ &- f^2 g(\phi X,Z)(Y - \eta(Y)\xi) + f^2 g(\phi Y,X)\eta(Z)\xi \\ &- f^2\phi(Y)\eta(X)\eta(Z) + f^2\phi(X)g(\phi Y,\phi Z) \\ &- fg(\phi Y,\nabla_X Z)\xi - fg(\phi(\nabla_X Y),Z)\xi + fg(\phi(\nabla_Y X),Z)\xi. \end{aligned} \tag{3.16}$$

For $f = 1$, the equation (3.16) yields

$$\begin{aligned} R(X,Y)\phi Z =& \phi(R(X,Y)Z) - g(\phi Y,Z)\phi^2 X - 2g(\phi X,Y)\eta(Z)\xi - g(X,Z)\phi Y \\ &+ g(\phi X,\nabla_Y Z)\xi + g(\phi X,Z)\phi^2 Y + g(Y,Z)\phi X \\ &- g(\phi Y,\nabla_X Z)\xi - g(\phi(\nabla_X Y,Z)\xi + g(\phi(\nabla_Y X),Z)\xi. \end{aligned}$$

Contracting the above with respect to W, we get

$$\begin{aligned} 'R(X,Y,\phi Z,W) =& g(R(X,Y)\phi Z,W) \\ =& g(\phi(R(X,Y)Z),W) - g(\phi Y,Z)g(\phi^2 X,W) - 2g(\phi X,Y)\eta(Z)\eta(W) \\ &- g(X,Z)g(\phi Y,W) + g(\phi X,\nabla_Y Z)\eta(W) + g(\phi X,Z)g(\phi^2 Y,W) + g(Y,Z)g(\phi X,W) \\ &- g(\phi Y,\nabla_X Z)\eta(W) - g(\phi(\nabla_X Y),Z)\eta(W) + g(\phi(\nabla_Y X),Z)\eta(W). \end{aligned}$$

Taking $X = W = e_i$ and summing over $1 \leq i \leq n$ in the above equation, we get

$$S(Y,\phi Z) =' C(\bar{R}(Y,Z)) + (f + n - 2)g(\phi Y,Z) + g(\phi Z,\nabla_\xi Y) - g(\phi Y,\nabla_\xi Z), \tag{3.17}$$

where

$$'C(\bar{R}(Y,Z)) = g(\phi('R(e_i,Y)Z)e_i)).$$

From (3.4) and (3.17), it is easy to see that

$$'C(\bar{R}(Y,Z)) = -'C(\bar{R}(Z,Y)).$$

From (3.3) and (3.17), we obtain

$$'C(\bar{R}(Y,Z)) = (\lambda - (n-2))g(\phi Y,Z) - g(\phi Z,\nabla_\xi Y) + g(\phi Y,\nabla_\xi Z). \tag{3.18}$$

Thus we have

Theorem 3.2. *In a Kenmotsu manifold* (M^n, g)*, where* g *is a Ricci soliton,* $'C(\bar{R}(Y,Z))$ *is given by* (3.18).

Lie derivation of (3.3) yields

$$(L_\xi S)(Y,Z) = -2f(\lambda + f)g(\phi Y,\phi Z) + f[\eta(\nabla_\xi Y)\eta(Z) + \eta(\nabla_\xi Z)\eta(Y)]. \tag{3.19}$$

Taking $Y = Z = e_i$ in (3.19), and summing over $1 \leq i \leq n$, we obtain

$$-\xi r + 2fr - 2f(n-1)(f^2 + \xi f) = -2f(\lambda + f)(n-1).$$

Now for $f = 1$, this yields

$$\lambda = \frac{\frac{1}{2}\xi r - r}{n-1}.$$

As it is well known that for a Kenmotsu manif ld the curvature r is negative. Hence λ is positive for constant r. Thus we have,

Theorem 3.3. *A Ricci soliton in a Kenmotsu manifold of constant curvature is expanding.*

4. RICCI SOLITONS IN 3-DIMENSIONAL TRANS-SASAKIAN MANIFOLDS

Suppose (M^n, g) is a 3-dimensional trans-Sasakian manifold and (g, V, λ) is a Ricci soliton in (M^n, g). If V is a conformal killing vector f eld, then

$$L_V g = \rho g, \tag{4.1}$$

for some scalar function ρ.

Now from (3.3), we have

$$S(X,Y) = (-\lambda + \frac{\rho}{2})g(X,Y), \tag{4.2}$$

$$QX = (-\lambda + \frac{\rho}{2})X \tag{4.3}$$

and

$$r = 3(-\lambda + \frac{\rho}{2}). \tag{4.4}$$

As it is well that in a 3-dimensional trans-Sasakian manifold, the curvature tensor R is given by

$$\begin{aligned} R(X,Y)Z =& [S(Y,Z)X - S(X,Z)Y + g(Y,Z)QX - g(X,Z)QY] \\ & - \frac{r}{2}[g(Y,Z)X - g(X,Z)Y]. \end{aligned} \tag{4.5}$$

Using (4.2), (4.3), (4.4) in (4.5), we get

$$R(X,Y)Z = ((-2\lambda + \rho) - \frac{r}{2})[g(Y,Z)X - g(X,Z)Y]. \tag{4.6}$$

In a trans-Sasakian manifold, $R(X,Y)\xi$ is given by

$$\begin{aligned} R(X,Y)\xi =& (\alpha^2 - \beta^2)(\eta(Y)X - \eta(X)Y) + 2\alpha\beta(\eta(Y)\phi X - \eta(X)\phi Y) \\ & - (X\alpha)\phi Y + (Y\alpha)\phi X - (X\beta)\phi^2 Y + (Y\beta)\phi^2 X. \end{aligned} \tag{4.7}$$

Taking $X = Z = \xi$ in (4.6) and comparing it with (4.7) with $X = \xi$, we get

$$((\alpha^2 - \beta^2) - \xi\beta + \frac{r}{2})[\eta(Y)\eta(W) - g(Y,W)] = 0.$$

This implies

$$r = 2\xi\beta - 2(\alpha^2 - \beta^2) \tag{4.8}$$

From (4.4) and (4.8), we have

$$6\lambda = \rho - 4[\xi\beta - (\alpha^2 - \beta^2)]. \tag{4.9}$$

From (4.9), we have

Theorem 4.1. *In a 3-dimensional trans-Sasakian manifold, a Ricci Soliton (g, V, λ), where V is conformal killing is*
i) expanding for $\rho > 4(\xi\beta - (\alpha^2 - \beta^2))$
ii) shrinking for $\rho < 4(\xi\beta - (\alpha^2 - \beta^2))$
and iii) is steady for $\rho = 4(\xi\beta - (\alpha^2 - \beta^2))$

Taking $\beta = 0$ in (4.9), we get $\rho = -4\alpha^2$ if and only if $\lambda = 0$.
Since ρ is positive, λ cannot be zero. Thus we have

Theorem 4.2. *A Ricci soliton (g, V, λ) in an α-Sasakian manifold, where V is conformal killing cannot be steady.*

Let (M^n, g) be a f-Kenmotsu manifold. Then from (4.2), we have

$$\begin{aligned} R.S &= S(R(X,Y)Z, W) + S(Z, R(X,Y)W) \\ &= (-\lambda + \frac{\rho}{2})[g(R(X,Y)Z, W) + g(R(X,Y)W, Z) \\ &= (-\lambda + \frac{\rho}{2})[{}'R(X,Y,Z,W) +{}' R(X,Y,W,Z)] = 0, \end{aligned}$$

i.e (M^n, g) is Ricci semi-symmetric.

Conversely suppose $R.S = 0$, i.e

$$S(R(X,Y)Z,W) + S(Z,R(X,Y)W) = 0. \tag{4.10}$$

Taking $f = 1$ in (3.6) and (3.7), we get

$$R(X,Y)\xi = \eta(X)Y - \eta(Y)X, \tag{4.11}$$

$$S(X,\xi) = -(n-1)\eta(X). \tag{4.12}$$

Taking $W = \xi$ in (4.10) and using (4.11) and (4.12), we obtain

$$S(Y,Z) = -(n-1)g(Y,Z).$$

Substituting this in (3.1), we get

$$(L_V g)(Y,Z) = \rho g(Y,Z)$$

where $\rho = 2((n-1) - \lambda)$. i.e V is conf rmal killing. Thus we have

Theorem 4.3. *Let (g,V,λ) be a Ricci soliton in a Kenmotsu manifold (M^n,g). Then (M^n,g) is Ricci-semi symmetric if and only if V is conformal killing.*

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