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LINEAR QUADRATIC STOCHASTIC DIFFERENTIAL GAMES: OPEN-LOOP AND CLOSED-LOOP SADDLE POINTS*

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Abstract. In this paper, we consider a linear quadratic stochastic two-person zero-sum differential game. The controls for both players are allowed to appear in both drift and diffusion of the state equation. The weighting matrices in the performance functional are not assumed to be definite/nonsingular. The existence of an open-loop saddle point is characterized by the existence of an adapted solution to a linear forward-backward stochastic differential equation with constraints, together with a convexity-concavity condition, and the existence of a closed-loop saddle point is characterized by the existence of a *regular* solution to a Riccati differential equation. It turns out that there is a significant difference between open-loop and closed-loop saddle points. Also, it is found that there is an essential feature that prevents a linear quadratic optimal control problem from being a special case of linear quadratic two-person zero-sum differential games.

Key words. stochastic differential equation, linear quadratic differential game, two-person, zero-sum, saddle point, Riccati differential equation, closed-loop, open-loop

AMS subject classifications. 93E20, 91A23, 49N70

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1. Introduction. Let $(\Omega,\mathcal{F},\mathbb{F},\mathbb{P})$ be a given complete filtered probability space along with a one-dimensional standard Brownian motion $W=\{W(t),\mathcal{F}_t;0\leqslant t<\infty\}$, where $\mathbb{F}=\{\mathcal{F}_t\}_{t\geqslant 0}$ is the natural filtration of W augmented by all the $\mathbb{P}$-null sets in $\mathcal{F}$ [13, 25]. Consider the following controlled linear stochastic differential equation (SDE) on $[t,T]$:

$$\left\{\begin{aligned} dX(s)&=\big[A(s)X(s)+B_1(s)u_1(s)+B_2(s)u_2(s)+b(s)\big]ds\\ &\qquad+\big[C(s)X(s)+D_1(s)u_1(s)+D_2(s)u_2(s)+\sigma(s)\big]dW(s),\\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad s\in[t,T],\\ X(t)&=x.\end{aligned}\right.\tag{1.1}$$

In the above, $X(\cdot)$ is called the *state process* taking values in the n-dimensional Euclidean space $\mathbb{R}^n$ with the *initial state* x at the initial time t; for $i=1,2$, $u_i(\cdot)$ is called the *control process* of Player i taking values in $\mathbb{R}^{m_i}$, $m_i>0$. We assume that $A(\cdot),B_1(\cdot),B_2(\cdot),C(\cdot),D_1(\cdot),D_2(\cdot)$ are deterministic matrix-valued functions of proper dimensions and $b(\cdot),\sigma(\cdot)$ are vector-valued $\mathbb{F}$-progressively measurable processes. For any $t\in[0,T)$, we define

$$\mathcal{U}_i[t,T]=\Big\{u_i:[t,T]\times\Omega\to\mathbb{R}^{m_i}\;\big|\;u_i(\cdot)\text{ is }\mathbb{F}\text{-progressively measurable},\\ \mathbb{E}\int_t^T|u_i(s)|^2ds<\infty\Big\},\qquad i=1,2.$$

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Any element $u_i(\cdot)\in\mathcal{U}_i[t,T]$ is called an *admissible control* of Player i on $[t,T]$. Under some mild conditions on the coefficients, for any *initial pair* $(t,x)\in[0,T)\times\mathbb{R}^n$ and control pair $(u_1(\cdot),u_2(\cdot))\in\mathcal{U}_1[t,T]\times\mathcal{U}_2[t,T]$, state equation (1.1) admits a unique solution $X(\cdot)\equiv X(\cdot\,;t,x,u_1(\cdot),u_2(\cdot))$. To measure the performance of the controls $u_1(\cdot)$ and $u_2(\cdot)$, we introduce the following functional:

(1.2)

$$\begin{aligned}J(t,x;u_1(\cdot),u_2(\cdot))\triangleq\frac{1}{2}\mathbb{E}\Bigg\{&\langle GX(T),X(T)\rangle+2\langle g,X(T)\rangle\\&+\int_t^T\Bigg[\Bigg\langle\begin{pmatrix}Q(s)&S_1(s)^\top&S_2(s)^\top\\S_1(s)&R_{11}(s)&R_{12}(s)\\S_2(s)&R_{21}(s)&R_{22}(s)\end{pmatrix}\begin{pmatrix}X(s)\\u_1(s)\\u_2(s)\end{pmatrix},\begin{pmatrix}X(s)\\u_1(s)\\u_2(s)\end{pmatrix}\Bigg\rangle\\&+2\Bigg\langle\begin{pmatrix}q(s)\\\rho_1(s)\\\rho_2(s)\end{pmatrix},\begin{pmatrix}X(s)\\u_1(s)\\u_2(s)\end{pmatrix}\Bigg\rangle\Bigg]ds\Bigg\},\end{aligned}$$

where $Q(\cdot)$, $S_1(\cdot)$, $S_2(\cdot)$, $R_{11}(\cdot)$, $R_{12}(\cdot)$, $R_{21}(\cdot)$, $R_{22}(\cdot)$ are deterministic matrix-valued functions of proper dimensions with

$$Q(\cdot)^\top=Q(\cdot),\qquad G^\top=G,\qquad\begin{pmatrix}R_{11}(\cdot)&R_{12}(\cdot)\\R_{21}(\cdot)&R_{22}(\cdot)\end{pmatrix}^\top=\begin{pmatrix}R_{11}(\cdot)&R_{12}(\cdot)\\R_{21}(\cdot)&R_{22}(\cdot)\end{pmatrix},$$

and $q(\cdot)$, $\rho_1(\cdot)$, $\rho_2(\cdot)$ are allowed to be vector-valued $\mathbb{F}$-progressively measurable processes, g is allowed to be an $\mathcal{F}_T$-measurable random variable. We assume that (1.2) is a cost functional for Player 1 and a payoff functional for Player 2. Therefore, Player 1 wishes to minimize (1.2) by selecting a control process $u_1(\cdot)\in\mathcal{U}_1[t,T]$, while Player 2 wishes to maximize (1.2) by selecting a control process $u_2(\cdot)\in\mathcal{U}_2[t,T]$. The above described problem is referred to as a linear quadratic (LQ) stochastic *two-person zero-sum differential game*, denoted Problem (SG). When the diffusion is absent, the corresponding problem is called an LQ deterministic two-person zero-sum differential games, denoted Problem (DG). The study of Problem (DG) can be traced back to the work of Ho, Bryson, and Baron [10] in 1965. In 1970, Schmitendorf studied both open-loop and closed-loop strategies for Problem (DG) [21]; among other things, it was shown that the existence of a closed-loop saddle point may not imply that of an open-loop saddle point. In 1979, Bernhard carefully investigated Problem (DG) from a closed-loop point of view [5]; see also the book by Basar and Bernhard [2] in this aspect. In 2005, Zhang [26] proved that for a special Problem (DG), the existence of the open-loop value is equivalent to the finiteness of the corresponding open-loop lower and upper values, which is also equivalent to the existence of an open-loop saddle point. Along this line, a couple of follow-up works [8, 9] appeared afterward. In 2006, Mou and Yong studied Problem (SG) from an open-loop point of view by means of the Hilbert space method [18]. The main purpose of this paper is to study Problem (SG) from both open-loop and closed-loop points of view.

If we formally set $m_1=m$ (or equivalently, $m_2=0$), Problem (SG) becomes an LQ stochastic optimal control problem, denoted Problem (SLQ). Thus, formally, Problem (SLQ) can be regarded as a special case of Problem (SG). See [6, 1, 7, 11, 22, 17, 20] for some relevant results on Problem (SLQ). Further, when the stochastic part is absent, Problem (SLQ) is reduced to an LQ deterministic optimal control problem, denoted Problem (DLQ). Hence, Problem (DLQ) can be regarded as a special case of Problem (SLQ) and Problem (DG). The history of Problem (DLQ) can further be traced back to the work of Bellman, Glicksberg, and Gross [3] in 1958 and Kalman

[12] and Letov [14] in 1960. See [25] for some historic remarks on Problems (DLQ) and (SLQ).

For Problem (SG), one can introduce the notions of open-loop and closed-loop saddle points. The main results of this paper can be briefly summarized as follows: (i) The existence of an open-loop saddle point for Problem (SG) is characterized by the existence of an adapted solution to a forward-backward stochastic differential equation (FBSDE) with a constraint, plus a convexity-concavity condition for the performance functional. (ii) The existence of a closed-loop saddle point is characterized by the existence of a solution to a Riccati differential equation with certain regularity. We found several interesting facts.

Fact 1. For the case $m_1, m_2 > 0$, the convexity-concavity condition for the performance functional is necessary for the existence of an open-loop saddle point but not necessary for the existence of a closed-loop saddle point. Therefore, the existence of a closed-loop saddle point does not imply the existence of an open-loop saddle point (see Example 7.3), which extends a result of Schmitendorf [21]. On the other hand, because of the regularity requirement of the solution to the Riccati equation, we will present an example that the existence of an open-loop saddle point does not imply the existence of a closed-loop saddle point either (see Example 7.4).

Fact 2. Although Problems (DLQ) and (SLQ) are (formally) special cases of Problems (DG) and (SG), respectively, there is at least one essential difference: For the LQ optimal control problems, the existence of a closed-loop strategy implies the existence of an open-loop optimal control. However, Fact 1 above tells us that the existence of a closed-loop saddle point does not necessarily imply the existence of an open-loop saddle point. Hence, LQ optimal control problem can only remain a *formal* special case of LQ differential games.

Fact 3. The result of Zhang [26] on the equivalence of the existence of an open-loop saddle point and the finiteness of open-loop lower and upper value functions only holds for some special cases of LQ differential games. We will see that such a result does not hold in general (see Example 7.5).

The rest of the paper is organized as follows. Section 2 will collect some preliminary results. Among other things, we will recall/present some results on linear SDEs and backward stochastic differential equations (BSDEs) with unbounded coefficients. In section 3, we pose our differential game problem and carefully explain the open-loop and closed-loop saddle points. Section 4 is devoted to the study of open-loop saddle points by variational method. In section 5, we characterize closed-loop saddle points by means of the Riccati equation. In section 6, we look at a relation between the linear FBSDE used to characterize open-loop saddle points and the Riccati equation used to characterize closed-loop saddle points. Several examples are presented in section 7. Some concluding remarks are collected in section 8.

2. Preliminaries. We recall that $\mathbb{R}^n$ is the n-dimensional Euclidean space, $\mathbb{R}^{n\times m}$ is the space of all $(n\times m)$ matrices, endowed with the inner product $(M,N)\mapsto \operatorname{tr}[M^\top N]$, and $\mathbb{S}^n \subseteq \mathbb{R}^{n\times n}$ is the set of all $(n\times n)$ symmetric matrices. For any $M\in\mathbb{R}^{m\times n}$, there exists a unique matrix $M^\dagger\in\mathbb{R}^{n\times m}$, called the (Moore–Penrose) *pseudoinverse* of M, satisfying the following [19]:

$$MM^\dagger M = M, \quad M^\dagger M M^\dagger = M^\dagger, \quad (MM^\dagger)^\top = MM^\dagger, \quad (M^\dagger M)^\top = M^\dagger M.$$

In addition, if $M = M^\top \in \mathbb{S}^n$, then

$$M^\dagger = (M^\dagger)^\top, \quad MM^\dagger = M^\dagger M; \qquad \text{and} \qquad M \geqslant 0 \iff M^\dagger \geqslant 0.$$

By the way, for any $M\in\mathbb{R}^{m\times n}$, we let $\mathcal{R}(M)$ be the range of M.

Next, let $T > 0$ be a fixed time horizon. For any $t \in [0, T)$ and Euclidean space $\mathbb{H}$, let

$$
\begin{aligned}
C([t,T];\mathbb{H}) &= \Big\{\varphi : [t,T] \to \mathbb{H} \;\Big|\; \varphi(\cdot) \text{ is continuous}\Big\},\\
L^p(t,T;\mathbb{H}) &= \left\{\varphi : [t,T] \to \mathbb{H} \,\middle|\, \int_t^T |\varphi(s)|^p ds < \infty\right\}, \qquad 1 \leqslant p < \infty,\\
L^\infty(t,T;\mathbb{H}) &= \left\{\varphi : [t,T] \to \mathbb{H} \,\middle|\, \operatorname*{esssup}_{s\in[t,T]} |\varphi(s)| < \infty\right\}.
\end{aligned}
$$

We denote

$$
\begin{aligned}
L^2_{\mathcal{F}_T}(\Omega;\mathbb{H}) &= \Big\{\xi : \Omega \to \mathbb{H} \;\Big|\; \xi \text{ is } \mathcal{F}_T\text{-measurable}, \ \mathbb{E}|\xi|^2 < \infty\Big\},\\
L^2_{\mathbb{F}}(t,T;\mathbb{H}) &= \Bigg\{\varphi : [t,T]\times\Omega \to \mathbb{H} \;\Big|\; \varphi(\cdot) \text{ is } \mathbb{F}\text{-progressively measurable,}\\
&\qquad\qquad \mathbb{E}\int_t^T |\varphi(s)|^2 ds < \infty\Bigg\},\\
L^2_{\mathbb{F}}(\Omega;C([t,T];\mathbb{H})) &= \Bigg\{\varphi : [t,T]\times\Omega \to \mathbb{H} \;\Big|\; \varphi(\cdot) \text{ is } \mathbb{F}\text{-adapted, continuous,}\\
&\qquad\qquad \mathbb{E}\left[\sup_{s\in[t,T]} |\varphi(s)|^2\right] < \infty\Bigg\},\\
L^2_{\mathbb{F}}(\Omega;L^1(t,T;\mathbb{H})) &= \Bigg\{\varphi : [t,T]\times\Omega \to \mathbb{H} \;\Big|\; \varphi(\cdot) \text{ is } \mathbb{F}\text{-progressively measurable,}\\
&\qquad\qquad \mathbb{E}\left(\int_t^T |\varphi(s)| ds\right)^2 < \infty\Bigg\}.
\end{aligned}
$$

We now look at the linear SDE,

$$
\begin{cases} dX(s) = \big[A(s)X(s) + b(s)\big]ds + \big[C(s)X(s) + \sigma(s)\big]dW(s), \quad s \in [t,T],\\ X(t) = x \in \mathbb{R}^n, \end{cases} \tag{2.1}
$$

and the linear BSDE,

$$
\begin{cases} dY(s) = -\big[A(s)^\top Y(s) + C(s)^\top Z(s) + \varphi(s)\big]ds + Z(s)dW(s), \quad s \in [t,T],\\ Y(T) = \xi. \end{cases} \tag{2.2}
$$

We have the following result.

PROPOSITION 2.1. *Let*

$$
\begin{cases} A(\cdot) \in L^1(t,T;\mathbb{R}^{n\times n}), \quad C(\cdot) \in L^2(t,T;\mathbb{R}^{n\times n}),\\ b(\cdot), \varphi(\cdot) \in L^2_{\mathbb{F}}(\Omega; L^1(t,T;\mathbb{R}^n)), \quad \sigma(\cdot) \in L^2_{\mathbb{F}}(t,T;\mathbb{R}^n), \quad \xi \in L^2_{\mathcal{F}_T}(\Omega;\mathbb{R}^n). \end{cases} \tag{2.3}
$$

Then (2.1) *admits a unique strong solution* $X(\cdot) \in L^2_{\mathbb{F}}(\Omega;C([t,T];\mathbb{R}^n))$ *and* (2.2) *admits a unique adapted solution* $(Y(\cdot), Z(\cdot)) \in L^2_{\mathbb{F}}(\Omega;C([t,T];\mathbb{R}^n)) \times L^2_{\mathbb{F}}(t,T;\mathbb{R}^n)$.

Moreover, there exists a constant $K > 0$ such that

$$\mathbb{E}\left[\sup_{s\in[t,T]}|X(s)|^2\right] \leqslant K\mathbb{E}\left[|x|^2 + \left(\int_t^T |b(s)|ds\right)^2 + \int_t^T |\sigma(s)|^2 ds\right], \tag{2.4}$$

$$\mathbb{E}\left[\sup_{s\in[t,T]}|Y(s)|^2 + \int_t^T |Z(s)|^2 ds\right] \leqslant K\mathbb{E}\left[|\xi|^2 + \left(\int_t^T |\varphi(s)|ds\right)^2\right]. \tag{2.5}$$

Hereafter, $K > 0$ represents a generic constant which can be different from line to line.

Note that (2.3) allows the coefficients $A(\cdot)$ and $C(\cdot)$ to be unbounded, which is a little different from the standard case [25]. However, we believe that the above result is not new. Since we are not able to find an exact reference, for the reader's convenience we sketch a proof here.

Proof. For (2.1), we define

$$\big(\mathcal{S}\widetilde{X}(\cdot)\big)(s) = x + \int_t^s \big[A(r)\widetilde{X}(r) + b(r)\big]dr + \int_t^s \big[C(r)\widetilde{X}(r) + \sigma(r)\big]dW(r), \qquad \forall \widetilde{X}(\cdot) \in L^2_{\mathbb{F}}(\Omega; C([t,T];\mathbb{R}^n)).$$

By the Cauchy–Schwarz and Burkholder–Davis–Gundy inequalities, we have

$$\begin{aligned}
&\mathbb{E}\left[\sup_{s\in[t,\tau]}|\big(\mathcal{S}\widetilde{X}(\cdot)\big)(s)|^2\right] \\
&\quad\leqslant K\mathbb{E}\Bigg[|x|^2 + \left(\int_t^\tau |A(r)||\widetilde{X}(r)|dr\right)^2 + \left(\int_t^\tau |b(r)|dr\right)^2 \\
&\qquad\qquad + \mathbb{E}\int_t^\tau |C(r)\widetilde{X}(r)|^2 dr + \mathbb{E}\int_t^\tau |\sigma(r)|^2 dr\Bigg] \\
&\quad\leqslant K\left[|x|^2 + \mathbb{E}\left(\int_t^\tau |b(r)|dr\right)^2 + \mathbb{E}\int_t^\tau |\sigma(r)|^2 dr\right] \\
&\qquad + K\left[\left(\int_t^\tau |A(r)|dr\right)^2 + \int_t^\tau |C(r)|^2 dr\right]\mathbb{E}\left[\sup_{s\in[t,\tau]}|\widetilde{X}(s)|^2\right] \qquad \forall \tau\in[t,T].
\end{aligned} \tag{2.6}$$

And for any $\widetilde{X}_1(\cdot), \widetilde{X}_2(\cdot) \in L^2_{\mathbb{F}}(\Omega; C([t,T];\mathbb{R}^n))$, we have

$$\begin{aligned}
&\mathbb{E}\left[\sup_{s\in[t,\tau]}|\big(\mathcal{S}\widetilde{X}_1(\cdot)\big)(s) - \big(\mathcal{S}\widetilde{X}_2(\cdot)\big)(s)|^2\right] \\
&\quad\leqslant K\mathbb{E}\left[\left(\int_t^\tau |A(r)||\widetilde{X}_1(r) - \widetilde{X}_2(r)|dr\right)^2 + \int_t^\tau |C(r)|^2|\widetilde{X}_1(r) - \widetilde{X}_2(r)|^2 dr\right] \\
&\quad\leqslant K\left[\left(\int_t^\tau |A(r)|dr\right)^2 + \int_t^\tau |C(r)|^2 dr\right]\mathbb{E}\left[\sup_{s\in[t,\tau]}|\widetilde{X}_1(s) - \widetilde{X}_2(s)|^2\right] \qquad \forall \tau\in[t,T].
\end{aligned}$$

Hence, by our assumption, we may choose $\delta = \tau - t > 0$ small enough and use contraction mapping theorem to get a unique strong solution $X(\cdot)$ of (2.1) on $[t, t+\delta]$, and from (2.6), we see (2.4) holds on $[t, t+\delta]$. The well-posedness of (2.1) on $[t, T]$ follows from a usual continuation argument.

Now, we consider BSDE (2.2). The following is based on a modification of the proof of [25, Theorem 7.3.2]. For any $\beta \in \mathbb{R}$, we define $\mathcal{M}_\beta[t,T]$ to be the Banach space

$$\mathcal{M}_\beta[t,T] = L^2_{\mathbb{F}}(\Omega; C[t,T]; \mathbb{R}^n) \times L^2_{\mathbb{F}}(t,T;\mathbb{R}^n) \left(\stackrel{\Delta}{=} \mathcal{M}[t,T]\right),$$

equipped with the norm

$$\|(Y(\cdot), Z(\cdot))\|_{\mathcal{M}_\beta[t,T]} \stackrel{\Delta}{=} \left\{ \mathbb{E}\left[\sup_{s\in[t,T]} |Y(s)|^2 e^{\beta h(s)}\right] + \mathbb{E}\int_t^T |Z(s)|^2 e^{\beta h(s)} ds \right\}^{\frac{1}{2}},$$

where

$$h(s) = \int_t^s \big[|A(r)| + |C(r)|^2\big] dr, \qquad s \in [t,T].$$

Since T is finite, all the norms $\|\cdot\|_{\mathcal{M}_\beta[t,T]}$ with different β are equivalent. For any $(y(\cdot), z(\cdot)) \in \mathcal{M}[t,T]$, let $(Y(\cdot), Z(\cdot))$ be the adapted solution to the following BSDE:

$$Y(s) = \xi + \int_s^T \big[A(r)^\top y(r) + C(r)^\top z(r) + \varphi(r)\big] dr - \int_s^T Z(r) dW(r), \qquad s \in [t,T],$$

and define a map $\mathcal{T}$ from $\mathcal{M}[t,T]$ to itself by

$$\mathcal{T}(y(\cdot), z(\cdot)) = (Y(\cdot), Z(\cdot)).$$

We are going to prove that for some $\beta > 0$,

$$\begin{aligned}\|\mathcal{T}(y_1(\cdot), z_1(\cdot)) - \mathcal{T}(y_2(\cdot), z_2(\cdot))\|_{\mathcal{M}_\beta[t,T]} &\leqslant \frac{1}{2}\|(y_1(\cdot), z_1(\cdot)) - (y_2(\cdot), z_2(\cdot))\|_{\mathcal{M}_\beta[t,T]},\\ &\qquad \forall (y_1(\cdot), z_1(\cdot)), (y_2(\cdot), z_2(\cdot)) \in \mathcal{M}_\beta[t,T].\end{aligned}$$

Then we use the contraction mapping theorem to obtain the well-posedness of (2.2).

For any $(y_i(\cdot), z_i(\cdot)) \in \mathcal{M}[t,T]$, $i = 1, 2$, let

$$\begin{cases} \mathcal{T}(y_i(\cdot), z_i(\cdot)) = (Y_i(\cdot), Z_i(\cdot)), \qquad i = 1,2, \\ \hat{y}(\cdot) = y_1(\cdot) - y_2(\cdot), \quad \hat{z}(\cdot) = z_1(\cdot) - z_2(\cdot), \\ \widehat{Y}(\cdot) = Y_1(\cdot) - Y_2(\cdot), \quad \widehat{Z}(\cdot) = Z_1(\cdot) - Z_2(\cdot). \end{cases}$$

Let $\beta > 0$ be undetermined. Applying Itô's formula to $r \mapsto |\widehat{Y}(r)|^2 e^{\beta h(r)}$, we have

$$\begin{aligned} &|\widehat{Y}(s)|^2 e^{\beta h(s)} + \int_s^T |\widehat{Z}(r)|^2 e^{\beta h(r)} dr \\ &\quad = -\int_s^T e^{\beta h(r)} \Big[\beta h'(r)|\widehat{Y}(r)|^2 - 2\,\langle\, \widehat{Y}(r), A(r)\hat{y}(r) + C(r)\hat{z}(r)\,\rangle\Big] dr \\ &\qquad - 2\int_s^T e^{\beta h(r)} \langle\, \widehat{Y}(r), \widehat{Z}(r)\,\rangle\, dW(r) \\ &\quad \leqslant \int_s^T e^{\beta h(r)} \Big\{ \big[-\beta h'(r) + \lambda^{-1}\big(|A(r)| + |C(r)|^2\big)\big]\, |\widehat{Y}(r)|^2 \\ &\qquad\qquad + \lambda \big[|A(r)||\hat{y}(r)|^2 + |\hat{z}(r)|^2\big] \Big\} dr \\ &\qquad - 2\int_s^T e^{\beta h(r)} \langle\, \widehat{Y}(r), \widehat{Z}(r)\,\rangle\, dW(r) \qquad\qquad \forall s \in [t,T] \quad \text{a.s.}, \end{aligned}$$

where we take $\lambda=\beta^{-1}>0$. Then the above implies

$$
\begin{aligned}
&|\widehat Y(s)|^2e^{\beta h(s)}+\int_s^T|\widehat Z(r)|^2e^{\beta h(r)}dr\\
&\quad\leqslant\lambda\left(\int_t^T|A(r)|dr+1\right)\left[\sup_{s\in[t,T]}\left(|\hat y(s)|^2e^{\beta h(s)}\right)+\int_t^T|\hat z(r)|^2e^{\beta h(r)}dr\right]\\
&\qquad-2\int_s^Te^{\beta h(r)}\langle\,\widehat Y(r),\widehat Z(r)\,\rangle\,dW(r).
\end{aligned}\tag{2.7}
$$

By taking expectation, one obtains

$$
\mathbb{E}\left[|\widehat Y(s)|^2e^{\beta h(s)}+\int_s^T|\widehat Z(r)|^2e^{\beta h(r)}dr\right]\leqslant\lambda\left(\int_t^T|A(r)|dr+1\right)\|(\hat y,\hat z)\|^2_{\mathcal{M}_\beta[t,T]}.\tag{2.8}
$$

On the other hand, by the Burkholder–Davis–Gundy inequality, we have (noting (2.8))

$$
\begin{aligned}
&\mathbb{E}\left\{\sup_{s\in[t,T]}\left|\int_s^Te^{\beta h(r)}\langle\,\widehat Y(r),\widehat Z(r)\,\rangle\,dW(r)\right|\right\}\\
&\leqslant K\mathbb{E}\left\{\int_t^Te^{2\beta h(r)}|\widehat Y(r)|^2|\widehat Z(r)|^2dr\right\}^{\frac12}\\
&\leqslant K\mathbb{E}\left\{\left(\sup_{s\in[t,T]}|\widehat Y(s)|^2e^{\beta h(s)}\right)^{\frac12}\left(\int_t^T|\widehat Z(r)|^2e^{\beta h(r)}dr\right)^{\frac12}\right\}\\
&\leqslant\frac14\mathbb{E}\left(\sup_{s\in[t,T]}|\widehat Y(s)|^2e^{\beta h(s)}\right)+K^2\mathbb{E}\left(\int_t^T|\widehat Z(r)|^2e^{\beta h(r)}dr\right)\\
&\leqslant\frac14\mathbb{E}\left(\sup_{s\in[t,T]}|\widehat Y(s)|^2e^{\beta h(s)}\right)+K^2\lambda\left(\int_t^T|A(r)|dr+1\right)\|(\hat y,\hat z)\|^2_{\mathcal{M}_\beta[t,T]}.
\end{aligned}\tag{2.9}
$$

Consequently, from (2.7), we have

$$
\begin{aligned}
\mathbb{E}\left[\sup_{s\in[t,T]}|\widehat Y(s)|^2e^{\beta h(s)}\right]&\leqslant\lambda\left(\textstyle\int_t^T|A(r)|dr+1\right)\|(\hat y,\hat z)\|^2_{\mathcal{M}_\beta[t,T]}\\
&\quad+2\mathbb{E}\left\{\sup_{s\in[t,T]}\left|\textstyle\int_s^Te^{\beta h(r)}\langle\,\widehat Y(r),\widehat Z(r)\,\rangle\,dW(r)\right|\right\}\\
&\leqslant(1+2K^2)\lambda\left(\int_t^T|A(r)|dr+1\right)\|(\hat y,\hat z)\|^2_{\mathcal{M}_\beta[t,T]}\\
&\quad+\frac12\mathbb{E}\left(\sup_{s\in[t,T]}|\widehat Y(s)|^2e^{\beta h(s)}\right).
\end{aligned}\tag{2.10}
$$

Combining (2.8) and (2.10) yields (noting $\lambda=\beta^{-1}$)

$$
\|(\widehat Y,\widehat Z)\|^2_{\mathcal{M}_\beta[t,T]}\leqslant\beta^{-1}(3+4K^2)\left(\int_t^T|A(r)|dr+1\right)\|(\hat y,\hat z)\|^2_{\mathcal{M}_\beta[t,T]}.\tag{2.11}
$$

Then we can choose $\beta > 0$ large enough to get the contractivity of the operator $\mathcal{T}$ on $\mathcal{M}_\beta[t,T]$.

To prove (2.5), let $(Y_0(\cdot), Z_0(\cdot))$ be the adapted solution to the following BSDE:

$$Y_0(s) = \xi + \int_s^T \varphi(r)dr - \int_s^T Z_0(r)dW(r), \qquad s \in [t,T].$$

It is well-known that $Z_0(\cdot)$ satisfies

$$\theta = \mathbb{E}[\theta] + \int_t^T Z_0(r)dW(r) \qquad \text{a.s.} \tag{2.12}$$

and $Y_0(\cdot)$ is given by

$$Y_0(s) = \mathbb{E}[\theta] - \int_t^s \varphi(r)dr + \int_t^s Z_0(r)dW(r), \qquad s \in [t,T], \tag{2.13}$$

where

$$\theta = \xi + \int_t^T \varphi(r)dr.$$

We have from (2.12) that

$$\mathbb{E}\int_t^T |Z_0(r)|^2 dr \leqslant 2\mathbb{E}|\theta|^2 \leqslant 4\mathbb{E}\left[|\xi|^2 + \left(\int_t^T |\varphi(r)|dr\right)^2\right], \tag{2.14}$$

and hence, from (2.13), we have

$$\begin{aligned}\mathbb{E}\left[\sup_{s\in[t,T]}|Y_0(s)|^2\right] &\leqslant K\left[\mathbb{E}|\theta|^2 + \mathbb{E}\left(\int_t^T |\varphi(r)|dr^2\right) + \mathbb{E}\int_t^T |Z_0(r)|^2 dr\right]\\ &\leqslant K\mathbb{E}\left[|\xi|^2 + \left(\int_t^T |\varphi(r)|dr\right)^2\right].\end{aligned} \tag{2.15}$$

Combining (2.14)–(2.15), we see that $(Y_0(\cdot), Z_0(\cdot))$ satisfies (2.5). By a routine iteration, we obtain estimate (2.5) for the adapted solution $(Y(\cdot), Z(\cdot))$ to BSDE (2.2). This completes the proof. $\square$

3. Stochastic differential games.

We now return to our Problem (SG). Recall the sets $\mathcal{U}_i[t,T] = L^2_{\mathbb{F}}(t,T;\mathbb{R}^{m_i})$ of all open-loop controls of Player i $(i = 1,2)$. For notational simplicity, we let $m = m_1 + m_2$ and denote

$$\begin{aligned}&B(\cdot) = (B_1(\cdot), B_2(\cdot)), \quad D(\cdot) = (D_1(\cdot), D_2(\cdot)),\\ &S(\cdot) = \begin{pmatrix} S_1(\cdot) \\ S_2(\cdot)\end{pmatrix}, \quad R(\cdot) = \begin{pmatrix} R_{11}(\cdot) & R_{12}(\cdot) \\ R_{21}(\cdot) & R_{22}(\cdot)\end{pmatrix} \equiv \begin{pmatrix} R_1(\cdot) \\ R_2(\cdot)\end{pmatrix},\\ &\rho(\cdot) = \begin{pmatrix} \rho_1(\cdot) \\ \rho_2(\cdot)\end{pmatrix}, \quad u(\cdot) = \begin{pmatrix} u_1(\cdot) \\ u_2(\cdot)\end{pmatrix}.\end{aligned}$$

Naturally, we identify $\mathcal{U}[t,T] = \mathcal{U}_1[t,T] \times \mathcal{U}_2[t,T]$. With such notation, the state equation becomes

$$\begin{cases} dX(s) = \big[A(s)X(s) + B(s)u(s) + b(s)\big]ds \\ \qquad\qquad + \big[C(s)X(s) + D(s)u(s) + \sigma(s)\big]dW(s), \qquad s \in [t,T], \\ X(t) = x, \end{cases} \tag{3.1}$$

and the performance functional becomes

$$
\begin{aligned}
&J(t,x;u_1(\cdot),u_2(\cdot))\\
&\quad = J(t,x;u(\cdot)) = \tfrac{1}{2}\,\mathbb{E}\bigg\{\langle GX(T),X(T)\rangle + 2\langle g,X(T)\rangle\\
&\qquad\qquad + \int_t^T\bigg[\bigg\langle\begin{pmatrix}Q(s) & S(s)^\top\\ S(s) & R(s)\end{pmatrix}\begin{pmatrix}X(s)\\ u(s)\end{pmatrix},\begin{pmatrix}X(s)\\ u(s)\end{pmatrix}\bigg\rangle\\
&\qquad\qquad\qquad + 2\bigg\langle\begin{pmatrix}q(s)\\ \rho(s)\end{pmatrix},\begin{pmatrix}X(s)\\ u(s)\end{pmatrix}\bigg\rangle\bigg]ds\bigg\}.
\end{aligned}
\tag{3.2}
$$

When $b(\cdot),\sigma(\cdot),q(\cdot),\rho(\cdot),g(\cdot)=0$, we denote the problem Problem $(\mathrm{SG})^0$, which is a special case of Problem (SG). With the above notation, we introduce the following standard assumptions:

(SG1) The coefficients of the state equation satisfy the following:

$$
\begin{cases}
A(\cdot)\in L^1(0,T;\mathbb{R}^{n\times n}), & B(\cdot)\in L^2(0,T;\mathbb{R}^{n\times m}), \quad b(\cdot)\in L^2_{\mathbb{F}}(\Omega;L^1(0,T;\mathbb{R}^n)),\\
C(\cdot)\in L^2(0,T;\mathbb{R}^{n\times n}), & D(\cdot)\in L^\infty(0,T;\mathbb{R}^{n\times m}), \quad \sigma(\cdot)\in L^2_{\mathbb{F}}(0,T;\mathbb{R}^n).
\end{cases}
$$

(SG2) The weighting coefficients in the performance functional satisfy the following:

$$
\begin{cases}
Q(\cdot)\in L^1(0,T;\mathbb{S}^n), \quad S(\cdot)\in L^2(0,T;\mathbb{R}^{m\times n}), \quad R(\cdot)\in L^\infty(0,T;\mathbb{S}^m),\\
q(\cdot)\in L^2_{\mathbb{F}}(\Omega;L^1(0,T;\mathbb{R}^n)), \quad \rho(\cdot)\in L^2_{\mathbb{F}}(0,T;\mathbb{R}^m), \quad G\in\mathbb{S}^n, \quad g\in L^2_{\mathcal{F}_T}(\Omega;\mathbb{R}^n).
\end{cases}
$$

Under (SG1), by Proposition 2.1, for any $(t,x)\in[0,T)\times\mathbb{R}^n$, and $u(\cdot)\in\mathcal{U}[t,T]$, (3.1) admits a unique strong solution

$$
X(\cdot)\equiv X(\cdot\,;t,x,u(\cdot))\in L^2_{\mathbb{F}}(\Omega;C([t,T];\mathbb{R}^n)).
$$

Moreover, the following estimate holds:

$$
\mathbb{E}\bigg[\sup_{s\in[t,T]}|X(s)|^2\bigg]\leqslant K\mathbb{E}\bigg[|x|^2+\bigg(\int_t^T|b(s)|ds\bigg)^2+\int_t^T|\sigma(s)|^2ds+\int_t^T|u(s)|^2ds\bigg].
$$

Therefore, under (SG1)–(SG2), the quadratic performance functional $J(t,x;u(\cdot))\equiv J(t,x;u_1(\cdot),u_2(\cdot))$ is well-defined for all $(t,x)\in[0,T)\times\mathbb{R}^n$ and $(u_1(\cdot),u_2(\cdot))\in\mathcal{U}_1[t,T]\times\mathcal{U}_2[t,T]$. Keeping in mind that when $m_1=m$, or, equivalently, $m_2=0$, Problem (SG) becomes Problem (SLQ). We now introduce the following definition.

DEFINITION 3.1.

(i) *For the case* $0<m_1,m_2<m$, *a pair* $(u_1^*(\cdot),u_2^*(\cdot))\in\mathcal{U}_1[t,T]\times\mathcal{U}_2[t,T]$ *is called an open-loop saddle point of Problem* (SG) *for the initial pair* $(t,x)\in[0,T)\times\mathbb{R}^n$ *if*

$$
\begin{aligned}
J(t,x;u_1^*(\cdot),u_2(\cdot))\leqslant J(t,x;u_1^*(\cdot),u_2^*(\cdot))\leqslant J(t,x;u_1(\cdot),u_2^*(\cdot))&\\
\forall(u_1(\cdot),u_2(\cdot))\in\mathcal{U}_1[t,T]\times\mathcal{U}_2[t,T].&
\end{aligned}
\tag{3.3}
$$

(ii) *For the case* $0<m_1,m_2<m$, *the open-loop upper value* $V^+(t,x)$ *and the open-loop lower value* $V^-(t,x)$ *of Problem* (SG) *at* $(t,x)\in[0,T)\times\mathbb{R}^n$ *are*

defined by the following:

$$
\begin{cases} V^+(t,x) = \inf_{u_1(\cdot)\in\mathcal{U}_1[t,T]} \sup_{u_2(\cdot)\in\mathcal{U}_2[t,T]} J(t,x;u_1(\cdot),u_2(\cdot)), \\ V^-(t,x) = \sup_{u_2(\cdot)\in\mathcal{U}_2[t,T]} \inf_{u_1(\cdot)\in\mathcal{U}_1[t,T]} J(t,x;u_1(\cdot),u_2(\cdot)), \end{cases} \tag{3.4}
$$

which automatically satisfy the following:

$$
V^-(t,x) \leqslant V^+(t,x), \qquad (t,x) \in [0,T) \times \mathbb{R}^n.
$$

In the case that

$$
V^-(t,x) = V^+(t,x) \equiv V(t,x), \tag{3.5}
$$

we say that Problem (SG) *admits an open-loop value* $V(t,x)$ *at* (t,x). *The maps* $(t,x) \mapsto V^{\pm}(t,x)$ *and* $(t,x) \mapsto V(t,x)$ *are called the open-loop upper value function, open-loop lower value function, and open-loop value function, respectively.*

(iii) *For the case* $m_1 = m$, *a* $\bar{u}(\cdot) \in \mathcal{U}[t,T]$ *is called an open-loop optimal control of Problem* (SLQ) *for the initial pair* $(t,x) \in [0,T) \times \mathbb{R}^n$ *if*

$$
J(t,x;\bar{u}(\cdot)) \leqslant J(t,x;u(\cdot)) \qquad \forall u(\cdot) \in \mathcal{U}[t,T], \tag{3.6}
$$

and

$$
V(t,x) = \inf_{u(\cdot)\in\mathcal{U}[t,T]} J(t,x;u(\cdot)), \qquad (t,x) \in [0,T] \times \mathbb{R}^n,
$$

is called the value function of Problem (SLQ).

Inspired by [10, 21, 5, 9], we now consider closed-loop strategies of Problems (SG) and (SLQ), respectively. To this end, we let

$$
\mathscr{Q}_i[t,T] = L^2(t,T;\mathbb{R}^{m_i\times n}), \qquad i = 1,2.
$$

For any initial pair $(t,x) \in [0,T) \times \mathbb{R}^n$, $\Theta(\cdot) \equiv (\Theta_1(\cdot)^\top, \Theta_2(\cdot)^\top)^\top \in \mathscr{Q}_1[t,T] \times \mathscr{Q}_2[t,T]$ and $v(\cdot) \equiv (v_1(\cdot)^\top, v_2(\cdot)^\top)^\top \in \mathcal{U}_1[t,T] \times \mathcal{U}_2[t,T]$, consider the following system:

$$
\begin{cases} dX(s) = \Big\{ \big[A(s) + B(s)\Theta(s)\big]X(s) + B(s)v(s) + b(s) \Big\} ds \\ \qquad\qquad + \Big\{ \big[C(s) + D(s)\Theta(s)\big]X(s) + D(s)v(s) + \sigma(s) \Big\} dW(s), \quad s \in [t,T], \\ X(t) = x. \end{cases} \tag{3.7}
$$

Clearly, under (SG1), the above admits a unique solution $X(\cdot) \equiv X(\cdot\,;t,x,\Theta_1(\cdot),v_1(\cdot);\Theta_2(\cdot),v_2(\cdot))$. If we denote

$$
u_i(\cdot) = \Theta_i(\cdot)X(\cdot) + v_i(\cdot), \qquad i = 1,2,
$$

then the above (3.7) coincides with the original state equation (1.1). We refer to (3.7) as a *closed-loop system* of the original system. With the solution $X(\cdot)$ to (3.7), we denote

$$
\begin{aligned}
&J(t,x;\Theta_1(\cdot)X(\cdot)+v_1(\cdot),\Theta_2(\cdot)X(\cdot)+v_2(\cdot))\\
&\quad\equiv J(t,x;\Theta(\cdot)X(\cdot)+v(\cdot))\\
&\quad=\frac{1}{2}\mathbb{E}\Bigg\{\langle GX(T),X(T)\rangle+2\langle g,X(T)\rangle\\
&\qquad+\int_t^T\Bigg[\Bigg\langle\begin{pmatrix}Q(s)&S(s)^\top\\S(s)&R(s)\end{pmatrix}\begin{pmatrix}X(s)\\\Theta(s)X(s)+v(s)\end{pmatrix},\begin{pmatrix}X(s)\\\Theta(s)X(s)+v(s)\end{pmatrix}\Bigg\rangle\\
&\qquad\qquad+2\Bigg\langle\begin{pmatrix}q(s)\\\rho(s)\end{pmatrix},\begin{pmatrix}X(s)\\\Theta(s)X(s)+v(s)\end{pmatrix}\Bigg\rangle\Bigg]ds\Bigg\}\\
&\quad=\frac{1}{2}\mathbb{E}\Bigg\{\langle GX(T),X(T)\rangle+2\langle g,X(T)\rangle\\
&\qquad+\int_t^T\Bigg[\Bigg\langle\begin{pmatrix}Q+\Theta^\top S+S^\top\Theta+\Theta^\top R\Theta&S^\top+\Theta^\top R\\S+R\Theta&R\end{pmatrix}\begin{pmatrix}X\\v\end{pmatrix},\begin{pmatrix}X\\v\end{pmatrix}\Bigg\rangle\\
&\qquad\qquad+2\Bigg\langle\begin{pmatrix}q+\Theta^\top\rho\\\rho\end{pmatrix},\begin{pmatrix}X\\v\end{pmatrix}\Bigg\rangle\Bigg]ds\Bigg\}.
\end{aligned}
$$

Similarly, one can define $J(t,x;\Theta_1(\cdot)X(\cdot)+v_1(\cdot),v_2(\cdot))$, $J(t,x;v_1(\cdot),\Theta_2(\cdot)X(\cdot)+v_2(\cdot))$. Also, in the case that $m_1=m$, the meaning $J(t,x;\Theta(\cdot)X(\cdot)+v(\cdot))$ is similar. We now introduce the following definition.

Definition 3.2.

(i) *For the case* $0<m_1,m_2<m$, *a 4-tuple* $(\Theta_1^*(\cdot),v_1^*(\cdot);\Theta_2^*(\cdot),v_2^*(\cdot))\in\mathscr{Q}_1[t,T]\times\mathcal{U}_1[t,T]\times\mathscr{Q}_2[t,T]\times\mathcal{U}_2[t,T]$ *is called a closed-loop saddle point of Problem* (SG) *on* $[t,T]$ *if*

$$
\begin{aligned}
J(t,x;\Theta_1^*(\cdot)X(\cdot)+v_1^*(\cdot),u_2(\cdot))&\leqslant J(t,x;\Theta_1^*(\cdot)X^*(\cdot)+v_1^*(\cdot),\Theta_2^*(\cdot)X^*(\cdot)+v_2^*(\cdot))\\
&\leqslant J(t,x;u_1(\cdot),\Theta_2^*(\cdot)X(\cdot)+v_2^*(\cdot))\\
&\qquad\forall x\in\mathbb{R}^n,\ (u_1(\cdot),u_2(\cdot))\in\mathcal{U}_1[t,T]\times\mathcal{U}_2[t,T].
\end{aligned}\tag{3.8}
$$

(ii) *For the case* $m_1=m$ *(thus* $m_2=0$*), a pair* $(\bar\Theta(\cdot),\bar v(\cdot))\in\mathcal{Q}[t,T]\times\mathcal{U}[t,T]$ *is called a closed-loop optimal strategy of Problem* (SLQ) *on* $[t,T]$ *if*

$$
J(t,x;\bar\Theta(\cdot)\bar X(\cdot)+\bar v(\cdot))\leqslant J(t,x;u(\cdot))\qquad\forall x\in\mathbb{R}^n,\quad u(\cdot)\in\mathcal{U}[t,T].\tag{3.9}
$$

There are some important remarks to be made:

(i) An open-loop saddle point $(u_1^*(\cdot),u_2^*(\cdot))$ (and an open-loop optimal control $\bar u(\cdot)$ for the case $m_1=m$) usually depends on the initial state x, whereas a closed-loop saddle point $(\Theta_1^*(\cdot),v_1^*(\cdot);\Theta_2^*(\cdot),v_2^*(\cdot))$ (and a closed-loop optimal strategy $(\bar\Theta(\cdot),\bar v(\cdot))$ for the case $m=m_1$) is required to be independent of the initial state x.

(ii) For the case $m=m_1$, we have Problem (SLQ), and (3.9) implies that the outcome $\bar\Theta(\cdot)\bar X(\cdot)+\bar v(\cdot)$ of the closed-loop optimal strategy $(\bar\Theta(\cdot),\bar v(\cdot))$ is an open-loop optimal control of Problem (SLQ) for the initial pair $(t,\bar X(t))$. Hence, for Problem (SLQ), existence of a closed-loop optimal strategy implies the existence of open-loop optimal controls.

(iii) In (3.8), the state process $X(\cdot)$ appearing in $J(t,x;\Theta_1^*(\cdot)X(\cdot)+v_1^*(\cdot),u_2(\cdot))$ is different from that in $J(t,x;u_1(\cdot),\Theta_2^*(\cdot)X(\cdot)+v_2^*(\cdot))$, and both are different from $X^*(\cdot) \equiv X(\cdot\,; t, x, \Theta_1^*(\cdot), v_1^*(\cdot); \Theta_2^*(\cdot), v_2^*(\cdot))$, which is the solution of (3.7) corresponding to

$$(\Theta_1(\cdot), v_1(\cdot); \Theta_2(\cdot), v_2(\cdot)) = (\Theta_1^*(\cdot), v_1^*(\cdot); \Theta_2^*(\cdot), v_2^*(\cdot)).$$

Therefore, comparing with (3.3), we see that (3.8) does not imply that $(\Theta_1^*(\cdot) X^*(\cdot) + v_1^*(\cdot), \Theta_2^*(\cdot)X^*(\cdot) + v_2^*(\cdot))$ is an open-loop saddle point of Problem (SG) for the initial pair $(t, X^*(t))$. Hence, Problem (SG) and Problem (SLQ) are essentially different in a certain sense, and we can only say that Problem (SLQ) is a formal special case of Problem (SG).

Let us take a closer look at the issue on the open-loop and closed-loop saddle points mentioned in (iii) above. We compare the following two inequalities:

$$J(t,x;u_1^*(\cdot),u_2^*(\cdot)) \leqslant J(t,x;u_1(\cdot),u_2^*(\cdot)) \tag{3.10}$$

and

$$J(t,x;\Theta_1^*(\cdot)X^*(\cdot)+v_1^*(\cdot),\Theta_2^*(\cdot)X^*(\cdot)+v_2^*(\cdot)) \leqslant J(t,x;u_1(\cdot),\Theta_2^*(\cdot)X(\cdot)+v_2^*(\cdot)). \tag{3.11}$$

For (3.10), we look at the state equation

$$\begin{cases} dX(s) = \big[A(s)X(s) + B_1(s)u_1(s) + B_2(s)u_2^*(s) + b(s)\big]ds \\ \qquad\qquad + \big[C(s)X(s) + D_1(s)u_1(s) + D_2(s)u_2^*(s) + \sigma(s)\big]dW(s), \qquad s \in [t,T], \\ X(t) = x, \end{cases}$$

and the cost functional

$$\begin{aligned} &J_1(t,x;u_1(\cdot)) \\ &\quad \equiv J(t,x;u_1(\cdot),u_2^*(\cdot)) \\ &\quad = \frac{1}{2}\,\mathbb{E}\bigg\{ \langle GX(T),X(T)\rangle + 2\langle g,X(T)\rangle \\ &\qquad + \int_t^T \Big[\langle QX,X\rangle + 2\langle S_1X,u_1\rangle + \langle R_{11}u_1,u_1\rangle + \langle R_{22}u_2^*,u_2^*\rangle + 2\langle R_{12}u_2^*,u_1\rangle \\ &\qquad\qquad + 2\langle S_2X,u_2^*\rangle + 2\langle q,X\rangle + 2\langle \rho_1,u_1\rangle + 2\langle \rho_2,u_2^*\rangle \Big]ds \bigg\} \\ &\quad = \frac{1}{2}\mathbb{E}\bigg\{ \langle GX(T),X(T)\rangle + 2\langle g,X(T)\rangle \\ &\qquad + \int_t^T \Big[\langle QX,X\rangle + 2\langle S_1X,u_1\rangle + \langle R_{11}u_1,u_1\rangle + 2\langle q + S_2^\top u_2^*, X\rangle \\ &\qquad\qquad + 2\langle \rho_1 + R_{12}u_2^*, u_1\rangle + \langle R_{22}u_2^*,u_2^*\rangle + 2\langle \rho_2,u_2^*\rangle \Big]ds \bigg\}. \end{aligned}$$

Therefore, (3.10) holds if and only if $u_1^*(\cdot)$ is an open-loop optimal control of Problem (SLQ) with the corresponding coefficients (using tildes to distinguish them from the original ones)

$$\begin{cases} \widetilde{A} = A, \quad \widetilde{B} = B_1, \quad \widetilde{b} = b + B_2u_2^*, \\ \widetilde{C} = C, \quad \widetilde{D} = D_1, \quad \widetilde{\sigma} = \sigma + D_2u_2^*, \\ \widetilde{G} = G, \quad \widetilde{g} = g, \quad \widetilde{Q} = Q, \quad \widetilde{S} = S_1, \quad \widetilde{R} = R_{11}, \\ \widetilde{q} = q + S_2^\top u_2^*, \quad \widetilde{\rho} = \rho_1 + R_{12}u_2^*. \end{cases} \tag{3.12}$$

However, for (3.11), we look at the state equation

$$\begin{cases} dX_1(s) = \Big\{\big[A(s)+B_2(s)\Theta_2^*(s)\big]X_1(s)+B_1(s)u_1(s)+B_2(s)v_2^*(s)+b(s)\Big\}ds \\ \qquad\qquad + \Big\{\big[C(s)+D_2(s)\Theta_2^*(s)\big]X_1(s)+D_1(s)u_1(s)+D_2(s)v_2^*(s)+\sigma(s)\Big\}dW(s), \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad s\in[t,T], \\ X_1(t)=x, \end{cases}$$

and the cost functional

$$\begin{aligned}
&\bar J_1(t,x;u_1(\cdot)) \\
&\quad = J(t,x;u_1(\cdot),\Theta_2^*(\cdot)X_1(\cdot)+v_2^*(\cdot)) \\
&\quad = \frac{1}{2}\,\mathbb{E}\bigg\{\langle GX_1(T),X_1(T)\rangle+2\langle g,X_1(T)\rangle \\
&\qquad\quad +\int_t^T\Big[\langle QX_1,X_1\rangle+\langle R_{11}u_1,u_1\rangle+\langle R_{22}(\Theta_2^*X_1+v_2^*),\Theta_2^*X_1+v_2^*\rangle \\
&\qquad\qquad +2\langle S_1X_1,u_1\rangle+2\langle S_2X_1,\Theta_2^*X_1+v_2^*\rangle+2\langle R_{21}u_1,\Theta_2^*X_1+v_2^*\rangle \\
&\qquad\qquad +2\langle q,X_1\rangle+2\langle\rho_1,u_1\rangle+2\langle\rho_2,\Theta_2^*X_1+v_2^*\rangle\Big]ds\bigg\} \\
&\quad = \frac{1}{2}\mathbb{E}\bigg\{\langle GX_1(T),X_1(T)\rangle+2\langle g,X_1(T)\rangle \\
&\qquad\quad +\int_t^T\Big[\langle QX_1,X_1\rangle+\langle R_{11}u_1,u_1\rangle+\langle(\Theta_2^*)^\top R_{22}\Theta_2^*X_1,X_1\rangle \\
&\qquad\qquad +2\langle(\Theta_2^*)^\top R_{22}v_2^*,X_1\rangle+\langle R_{22}v_2^*,v_2^*\rangle+2\langle S_1X_1,u_1\rangle \\
&\qquad\qquad +\langle[S_2^\top\Theta_2^*+(\Theta_2^*)^\top S_2]X_1,X_1\rangle+2\langle S_2^\top v_2^*,X_1\rangle+2\langle R_{12}\Theta_2^*X_1,u_1\rangle \\
&\qquad\qquad +2\langle R_{12}v_2^*,u_1\rangle+2\langle q,X_1\rangle+2\langle\rho_1,u_1\rangle+2\langle(\Theta_2^*)^\top\rho_2,X_1\rangle \\
&\qquad\qquad +2\langle\rho_2,v_2^*\rangle\Big]ds\bigg\} \\
&\quad = \frac{1}{2}\mathbb{E}\bigg\{\langle GX_1(T),X_1(T)\rangle+2\langle g,X_1(T)\rangle \\
&\qquad\quad +\int_t^T\Big[\langle[Q+(\Theta_2^*)^\top R_{22}\Theta_2^*+(\Theta_2^*)^\top S_2+S_2^\top\Theta_2^*]X_1,X_1\rangle+\langle R_{11}u_1,u_1\rangle \\
&\qquad\qquad +2\langle(S_1+R_{12}\Theta_2^*)X_1,u_1\rangle+2\langle q+[S_2^\top+(\Theta_2^*)^\top R_{22}]v_2^*+(\Theta_2^*)^\top\rho_2,X_1\rangle \\
&\qquad\qquad +2\langle\rho_1+R_{12}v_2^*,u_1\rangle+\langle R_{22}v_2^*,v_2^*\rangle+2\langle\rho_2,v_2^*\rangle\Big]ds\bigg\}.
\end{aligned}$$

Then, (3.11) holds if and only if $(\Theta_1^*(\cdot),v_1^*(\cdot))$ is a closed-loop optimal strategy of

Problem (SLQ) with the corresponding coefficients

$$
\left\{\begin{array}{l}
\widetilde{A}=A+B_2\Theta_2^*, \quad \widetilde{B}=B_1, \quad \widetilde{b}=b+B_2v_2^*, \\
\widetilde{C}=C+D_2\Theta_2^*, \quad \widetilde{D}=D_1, \quad \widetilde{\sigma}=\sigma+D_2v_2^*, \\
\widetilde{Q}=Q+(\Theta_2^*)^\top R_{22}\Theta_2^*+(\Theta_2^*)^\top S_2+S_2^\top\Theta_2^*, \\
\widetilde{S}=S_1+R_{12}\Theta_2^*, \quad \widetilde{R}=R_{11}, \\
\widetilde{q}=q+[S_2^\top+(\Theta_2^*)^\top R_{22}]v_2^*+(\Theta_2^*)^\top\rho_2, \quad \widetilde{\rho}=\rho_1+R_{12}v_2^*, \\
\widetilde{G}=G, \quad \widetilde{g}=g.
\end{array}\right. \tag{3.13}
$$

Comparing (3.12) and (3.13), we see that one cannot say anything about whether the outcome $\Theta_1^*(\cdot)X^*(\cdot)+v_1^*(\cdot)$ of $(\Theta_1^*(\cdot),v_1^*(\cdot))$ for the initial pair (t,x) has anything to do with $u_1^*(\cdot)$.

On the other hand, the following result, which is similar to Berkovitz's equivalence lemma for Problem (DG) found in [4], tells us something a little different and will be useful below.

Proposition 3.3. *Let* (SG1)–(SG2) *hold. For* $(\Theta_i^*(\cdot),v_i^*(\cdot))\in\mathscr{Q}_i[t,T]\times\mathcal{U}_i[t,T]$, $i=1,2$, *the following statements are equivalent:*

(i) $(\Theta_1^*(\cdot),v_1^*(\cdot);\Theta_2^*(\cdot),v_2^*(\cdot))$ *is a closed-loop saddle point of Problem* (SG) *on* $[t,T]$.

(ii) *For any* $x\in\mathbb{R}^n$, $(\Theta_1(\cdot),\Theta_2(\cdot))\in\mathscr{Q}_1[t,T]\times\mathscr{Q}_2[t,T]$ *and* $(v_1(\cdot),v_2(\cdot))\in\mathcal{U}_1[t,T]\times\mathcal{U}_2[t,T]$, *the following holds:*

$$
\begin{aligned}
&J(t,x;\Theta_1^*(\cdot)X(\cdot)+v_1^*(\cdot),\Theta_2(\cdot)X(\cdot)+v_2(\cdot))\\
&\quad\leqslant J(t,x;\Theta_1^*(\cdot)X^*(\cdot)+v_1^*(\cdot),\Theta_2^*(\cdot)X^*(\cdot)+v_2^*(\cdot))\\
&\quad\leqslant J(t,x;\Theta_1(\cdot)X(\cdot)+v_1(\cdot),\Theta_2^*(\cdot)X(\cdot)+v_2^*(\cdot)).
\end{aligned} \tag{3.14}
$$

(iii) *For any* $x\in\mathbb{R}^n$ *and* $(v_1(\cdot),v_2(\cdot))\in\mathcal{U}_1[t,T]\times\mathcal{U}_2[t,T]$, *the following holds:*

$$
\begin{aligned}
&J(t,x;\Theta_1^*(\cdot)X(\cdot)+v_1^*(\cdot),\Theta_2^*(\cdot)X(\cdot)+v_2(\cdot))\\
&\quad\leqslant J(t,x;\Theta_1^*(\cdot)X^*(\cdot)+v_1^*(\cdot),\Theta_2^*(\cdot)X^*(\cdot)+v_2^*(\cdot))\\
&\quad\leqslant J(t,x;\Theta_1^*(\cdot)X(\cdot)+v_1(\cdot),\Theta_2^*(\cdot)X(\cdot)+v_2^*(\cdot)).
\end{aligned} \tag{3.15}
$$

Proof. (i) $\Rightarrow$ (ii) For any $\Theta_i(\cdot)\in\mathscr{Q}_i[t,T]$ and $v_i(\cdot)\in\mathcal{U}_i[t,T]$, $i=1,2$, let $X(\cdot)$ be the solution to the following SDE:

$$
\left\{\begin{aligned}
dX(s)&=\Big\{\big[A+B_1\Theta_1+B_2\Theta_2^*\big]X+B_1v_1+B_2v_2^*+b\Big\}ds\\
&\quad+\Big\{\big[C+D_1\Theta_1+D_2\Theta_2^*\big]X+D_1v_1+D_2v_2^*+\sigma\Big\}dW(s), \quad s\in[t,T],\\
X(t)&=x.
\end{aligned}\right. \tag{3.16}
$$

Set

$$
u_1(\cdot)\triangleq\Theta_1(\cdot)X(\cdot)+v_1(\cdot)\in\mathcal{U}_1[t,T].
$$

By uniqueness, $X(\cdot)$ also solves the following SDE:

$$
\left\{\begin{aligned}
dX(s)&=\Big\{\big[A+B_2\Theta_2^*\big]X+B_1u_1+B_2v_2^*+b\Big\}ds\\
&\quad+\Big\{\big[C+D_2\Theta_2^*\big]X+D_1u_1+D_2v_2^*+\sigma\Big\}dW(s), \qquad s\in[t,T],\\
X(t)&=x.
\end{aligned}\right. \tag{3.17}
$$

Therefore,

$$\begin{aligned}&J(t,x;\Theta_1^*(\cdot)X^*(\cdot)+v_1^*(\cdot),\Theta_2^*(\cdot)X^*(\cdot)+v_2^*(\cdot))\\&\quad\leqslant J(t,x;u_1(\cdot),\Theta_2^*(\cdot)X(\cdot)+v_2^*(\cdot))\\&\quad=J(t,x;\Theta_1(\cdot)X(\cdot)+v_1(\cdot),\Theta_2^*(\cdot)X(\cdot)+v_2^*(\cdot)).\end{aligned}$$

Similarly, we have

$$\begin{aligned}&J(t,x;\Theta_1^*(\cdot)X(\cdot)+v_1^*(\cdot),\Theta_2(\cdot)X(\cdot)+v_2(\cdot))\\&\quad\leqslant J(t,x;\Theta_1^*(\cdot)X^*(\cdot)+v_1^*(\cdot),\Theta_2^*(\cdot)X^*(\cdot)+v_2^*(\cdot)).\end{aligned}$$

(ii) $\Rightarrow$ (iii) This is trivial, by taking $\Theta_i(\cdot)=\Theta_i^*(\cdot)$, $i=1,2$.

(iii) $\Rightarrow$ (i) For any $x\in\mathbb{R}^n$, and any $u_1(\cdot)\in\mathcal{U}_1[t,T]$, let $X(\cdot)$ be the solution of the following SDE:

(3.18)
$$\begin{cases}dX(s)=\Big\{\big[A+B_2\Theta_2^*\big]X+B_1u_1+B_2v_2^*+b\Big\}ds\\\qquad\qquad+\Big\{\big[C+D_2\Theta_2^*\big]X+D_1u_1+D_2v_2^*+\sigma\Big\}dW(s),\qquad s\in[t,T],\\X(t)=x.\end{cases}$$

Set

$$v_1(\cdot)=u_1(\cdot)-\Theta_1^*(\cdot)X(\cdot)\in\mathcal{U}_1[t,T];$$

then $X(\cdot)$ is also the solution to the following SDE:

(3.19)
$$\begin{cases}dX(s)=\Big\{\big[A+B_1\Theta_1^*+B_2\Theta_2^*\big]X+B_1v_1+B_2v_2^*+b\Big\}ds\\\qquad\qquad+\Big\{\big[C+D_1\Theta_1^*+D_2\Theta_2^*\big]X+D_1v_1+D_2v_2^*+\sigma\Big\}dW(s),\quad s\in[t,T],\\X(t)=x.\end{cases}$$

Therefore,

$$\begin{aligned}&J(t,x;\Theta_1^*(\cdot)X^*(\cdot)+v_1^*(\cdot),\Theta_2^*(\cdot)X^*(\cdot)+v_2^*(\cdot))\\&\quad\leqslant J(t,x;\Theta_1^*(\cdot)X(\cdot)+v_1(\cdot),\Theta_2^*(\cdot)X^*(\cdot)+v_2^*(\cdot))\\&\quad=J(t,x;u_1(\cdot),\Theta_2^*(\cdot)X^*(\cdot)+v_2^*(\cdot)).\end{aligned}$$

Similarly, for any $x\in\mathbb{R}^n$, and any $u_2(\cdot)\in\mathcal{U}_2[t,T]$, we can show that

$$J(t,x;\Theta_1^*(\cdot)X(\cdot)+v_1^*(\cdot),u_2(\cdot))\leqslant J(t,x;\Theta_1^*(\cdot)X^*(\cdot)+v_1^*(\cdot),\Theta_2^*(\cdot)X^*(\cdot)+v_2^*(\cdot)).$$

Thus, (i) holds. ∎

We note that (iii) of Proposition 3.3 tells us that if $(\Theta^*(\cdot),v^*(\cdot))$ is a closed-loop saddle point of Problem (SG), by considering the state equation

$$\begin{cases}dX(s)=\big[(A+B\Theta^*)X+B_1v_1+B_2v_2^*+b\big]ds\\\qquad\qquad+\big[(C+D\Theta^*)X+D_1v_1+D_2v_2^*+\sigma\big]dW(s),\qquad s\in[t,T],\\X(t)=x,\end{cases}\tag{3.20}$$

with the cost functional

$$J_1(t,x;v_1(\cdot)) = J(t,x;\Theta_1^*(\cdot)X(\cdot)+v_1(\cdot),\Theta_2^*(\cdot)X(\cdot)+v_2^*(\cdot)), \tag{3.21}$$

we see that $v_1^*(\cdot)$ is an open-loop optimal control of the corresponding Problem (SLQ). Likewise, if we consider the state equation

$$\begin{cases} dX(s) = \big[(A+B\Theta^*)X + B_2v_2 + B_1v_1^* + b\big]ds \\ \qquad\qquad + \big[(C+D\Theta^*)X + D_2v_2 + D_1v_1^* + \sigma\big]dW(s), \qquad s\in[t,T], \\ X(t) = x, \end{cases} \tag{3.22}$$

with the cost functional

$$J_2(t,x;v_2(\cdot)) = -J(t,x;\Theta_1^*(\cdot)X(\cdot)+v_1^*(\cdot),\Theta_2^*(\cdot)X(\cdot)+v_2(\cdot)), \tag{3.23}$$

then $v_2^*(\cdot)$ is an open-loop optimal control of the corresponding Problem (SLQ).

4. Open-loop saddle points and FBSDEs. In this section, we present a characterization of open-loop saddle points of Problem (SG) in terms of FBSDEs. See [16] for some relevant results on FBSDEs. The main result of this section can be stated as follows.

THEOREM 4.1. *For $0 < m_1, m_2 < m$, let* (SG1)–(SG2) *hold and $(t,x) \in [t,T)\times\mathbb{R}^n$ be given. Let $u^*(\cdot) \equiv (u_1^*(\cdot)^\top, u_2^*(\cdot)^\top)^\top \in \mathcal{U}_1[t,T]\times\mathcal{U}_2[t,T]$ and $X^*(\cdot) \equiv X(\cdot\,;t,x,u^*(\cdot))$ be the corresponding state process. Then $u^*(\cdot)$ is an open-loop saddle point of Problem* (SG) *if and only if the following stationarity conditions hold:*

$$B(s)^\top Y^*(s) + D(s)^\top Z^*(s) + S(s)X^*(s) + R(s)u^*(s) + \rho(s) = 0, \qquad \text{a.e. } s\in[t,T], \text{ a.s.}, \tag{4.1}$$

where $(Y^*(\cdot), Z^*(\cdot))$ is the adapted solution to the following BSDE:

$$\begin{cases} dY^*(s) = -\big[A(s)^\top Y^*(s) + C(s)^\top Z^*(s) + Q(s)X^*(s) \\ \qquad\qquad + S(s)^\top u^*(s) + q(s)\big]ds + Z^*(s)dW(s), \qquad s\in[t,T], \\ Y^*(T) = GX^*(T) + g, \end{cases} \tag{4.2}$$

and the following convexity-concavity condition holds: For $i=1,2$,

$$\begin{aligned} &(-1)^{i-1}\mathbb{E}\bigg\{\langle GX_i(T), X_i(T)\rangle \\ &\qquad + \int_t^T\Big[\langle Q(s)X_i(s), X_i(s)\rangle + 2\langle S_i(s)X_i(s), u_i(s)\rangle \\ &\qquad\qquad + \langle R_{ii}(s)u_i(s), u_i(s)\rangle\Big]ds\bigg\} \geqslant 0 \quad \forall u_i(\cdot)\in\mathcal{U}_i[t,T], \end{aligned} \tag{4.3}$$

where $X_i(\cdot)$ solves

$$\begin{cases} dX_i(s) = \big[A(s)X_i(s) + B_i(s)u_i(s)\big]ds \\ \qquad\qquad + \big[C(s)X_i(s) + D_i(s)u_i(s)\big]dW(s), \qquad s\in[t,T], \\ X_i(t) = 0. \end{cases} \tag{4.4}$$

In the case that $m_1 = m$, $u^*(\cdot) \equiv u_1^*(\cdot)$ *is an open-loop optimal control of Problem* (SLQ) *if and only if* (4.1)–(4.2) *hold and the following convexity condition holds:*
(4.5)

$$\mathbb{E}\bigg\{ \langle GX_1(T), X_1(T) \rangle + \int_t^T \Big[\langle Q(s)X_1(s), X_1(s) \rangle + 2 \langle S_1(s)X_1(s), u_1(s) \rangle + \langle R(s)u_1(s), u_1(s) \rangle \Big] ds \bigg\} \geqslant 0 \quad \forall u_1(\cdot) \in \mathcal{U}_1[t,T],$$

where $X_1(\cdot)$ *solves*

$$\begin{cases} dX_1(s) = \big[A(s)X_1(s) + B_1(s)u_1(s)\big] ds \\ \qquad\qquad + \big[C(s)X_1(s) + D_1(s)u_1(s)\big] dW(s), \qquad s \in [t,T], \\ X_1(t) = 0. \end{cases} \tag{4.6}$$

Proof. We just prove the case $0 < m_1, m_2 < m$. The case $m_1 = m$ can be proved similarly. Let $u^*(\cdot) \equiv (u_1^*(\cdot)^\top, u_2^*(\cdot)^\top)^\top \in \mathcal{U}_1[t,T] \times \mathcal{U}_2[t,T]$ and $X^*(\cdot)$ be the corresponding state process. Further, let $(Y^*(\cdot), Z^*(\cdot))$ be the adapted solution to the BSDE (4.2). By definition, $u^*(\cdot)$ is an open-loop saddle point if and only if the following hold:

$$J(t,x;u_1^*(\cdot),u_2^*(\cdot)) \leqslant J(t,x;u_1^*(\cdot)+\varepsilon u_1(\cdot),u_2^*(\cdot)) \quad \forall u_1(\cdot) \in \mathcal{U}_1[t,T], \quad \varepsilon \in \mathbb{R}, \tag{4.7}$$

$$J(t,x;u_1^*(\cdot),u_2^*(\cdot)) \geqslant J(t,x;u_1^*(\cdot),u_2^*(\cdot)+\varepsilon u_2(\cdot)) \quad \forall u_2(\cdot) \in \mathcal{U}_2[t,T], \quad \varepsilon \in \mathbb{R}. \tag{4.8}$$

For any $u_1(\cdot) \in \mathcal{U}_1[t,T]$ and $\varepsilon \in \mathbb{R}$, let $X^\varepsilon(\cdot)$ be the solution to the following perturbed state equation on $[t,T]$:

$$\begin{cases} dX^\varepsilon(s) = \Big\{A(s)X^\varepsilon(s) + B_1(s)\big[u_1^*(s)+\varepsilon u_1(s)\big] + B_2(s)u_2^*(s) + b(s)\Big\} ds \\ \qquad\qquad + \Big\{C(s)X^\varepsilon(s) + D_1(s)\big[u_1^*(s)+\varepsilon u_1(s)\big] + D_2(s)u_2^*(s) + \sigma(s)\Big\} dW(s), \\ X^\varepsilon(t) = x. \end{cases}$$

Then $X_1(\cdot) = \frac{X^\varepsilon(\cdot) - X^*(\cdot)}{\varepsilon}$ is independent of ε satisfying (4.4) (with $i = 1$), and

$$\begin{aligned} &J(t,x;u_1^*(\cdot)+\varepsilon u_1(\cdot),u_2^*(\cdot)) - J(t,x;u_1^*(\cdot),u_2^*(\cdot)) \\ &\quad = \frac{\varepsilon}{2}\mathbb{E}\Bigg\{ \langle G\big[2X^*(T)+\varepsilon X_1(T)\big], X_1(T) \rangle + 2\langle g, X_1(T) \rangle \\ &\qquad + \int_t^T \Bigg[\Bigg\langle \begin{pmatrix} Q & S_1^\top & S_2^\top \\ S_1 & R_{11} & R_{12} \\ S_2 & R_{21} & R_{22} \end{pmatrix} \begin{pmatrix} 2X^*+\varepsilon X_1 \\ 2u_1^*+\varepsilon u_1 \\ 2u_2^* \end{pmatrix}, \begin{pmatrix} X_1 \\ u_1 \\ 0 \end{pmatrix} \Bigg\rangle + 2\Big\langle \begin{pmatrix} q \\ \rho_1 \end{pmatrix}, \begin{pmatrix} X_1 \\ u_1 \end{pmatrix} \Big\rangle \Bigg] ds \Bigg\} \\ &\quad = \varepsilon\mathbb{E}\bigg\{ \langle GX^*(T)+g, X_1(T) \rangle \\ &\qquad + \int_t^T \Big[\langle QX^* + S^\top u^* + q, X_1 \rangle + \langle S_1X^* + R_{11}u_1^* + R_{12}u_2^* + \rho_1, u_1 \rangle \Big] ds \bigg\} \\ &\qquad + \frac{\varepsilon^2}{2}\mathbb{E}\bigg\{ \langle GX_1(T), X_1(T) \rangle + \int_t^T \Big[\langle QX_1, X_1 \rangle + 2\langle S_1X_1, u_1 \rangle + \langle R_{11}u_1, u_1 \rangle \Big] ds \bigg\}. \end{aligned}$$

On the other hand, we have

$$
\begin{aligned}
&\mathbb{E}\bigg\{ \langle GX^*(T)+g, X_1(T) \rangle \\
&\qquad + \int_t^T \Big[\langle QX^* + S^\top u^* + q, X_1 \rangle + \langle S_1 X^* + R_{11} u_1^* + R_{12} u_2^* + \rho_1, u_1 \rangle \Big] ds \bigg\} \\
&\quad = \mathbb{E}\bigg\{ \int_t^T \Big[\langle -(A^\top Y^* + C^\top Z^* + QX^* + S^\top u^* + q), X_1 \rangle + \langle Y^*, AX_1 + B_1 u_1 \rangle \\
&\qquad\qquad + \langle Z^*, CX_1 + D_1 u_1 \rangle + \langle QX^* + S^\top u^* + q, X_1 \rangle \\
&\qquad\qquad + \langle S_1 X^* + R_{11} u_1^* + R_{12} u_2^* + \rho_1, u_1 \rangle \Big] ds \bigg\} \\
&\quad = \mathbb{E} \int_t^T \langle B_1^\top Y^* + D_1^\top Z^* + S_1 X^* + R_{11} u_1^* + R_{12} u_2^* + \rho_1, u_1 \rangle \, ds.
\end{aligned}
$$

Hence,

$$
\begin{aligned}
&J(t,x;u_1^*(\cdot)+\varepsilon u_1(\cdot), u_2^*(\cdot)) - J(t,x;u_1^*(\cdot), u_2^*(\cdot)) \\
&\quad = \varepsilon \mathbb{E} \int_t^T \langle B_1^\top Y^* + D_1^\top Z^* + S_1 X^* + R_{11} u_1^* + R_{12} u_2^* + \rho_1, u_1 \rangle \, ds \\
&\qquad + \frac{\varepsilon^2}{2} \mathbb{E} \bigg\{ \langle GX_1(T), X_1(T) \rangle + \int_t^T \Big[\langle QX_1, X_1 \rangle + 2\langle S_1 X_1, u_1 \rangle + \langle R_{11} u_1, u_1 \rangle \Big] ds \bigg\}.
\end{aligned}
$$

Similarly, for any $u_2(\cdot) \in \mathcal{U}_2[t,T]$ and $\varepsilon \in \mathbb{R}$,

$$
\begin{aligned}
&J(t,x;u_1^*(\cdot), u_2^*(\cdot)+\varepsilon u_2(\cdot)) - J(t,x;u_1^*(\cdot), u_2^*(\cdot)) \\
&\quad = \varepsilon \mathbb{E} \int_t^T \langle B_2^\top Y^* + D_2^\top Z^* + S_2 X^* + R_{22} u_2^* + R_{21} u_1^* + \rho_2, u_2 \rangle \, ds \\
&\qquad + \frac{\varepsilon^2}{2} \mathbb{E} \bigg\{ \langle GX_2(T), X_2(T) \rangle + \int_t^T \Big[\langle QX_2, X_2 \rangle + 2\langle S_2 X_2, u_2 \rangle + \langle R_{22} u_2, u_2 \rangle \Big] ds \bigg\},
\end{aligned}
$$

where $X_2(\cdot)$ is the solution of (4.4) with $i = 2$. Therefore, (4.7) holds if and only if (4.3) holds for $i = 1$, and

$$
B_1^\top Y^* + D_1^\top Z^* + S_1 X^* + R_{11} u_1^* + R_{12} u_2^* + \rho_1 = 0, \qquad \text{a.e. } s \in [t,T], \text{ a.s.} \tag{4.9}
$$

In the same way, one can show that (4.8) holds if and only if (4.3) holds for $i = 2$, and

$$
B_2^\top Y^* + D_2^\top Z^* + S_2 X^* + R_{21} u_1^* + R_{22} u_2^* + \rho_2 = 0, \qquad \text{a.e. } s \in [t,T], \text{ a.s.} \tag{4.10}
$$

Note that (4.1) is equivalent to (4.9) and (4.10). The proof is completed. ∎

From the above result, we see that if Problem (SG) admits an open-loop saddle point $u^*(\cdot) \equiv (u_1^*(\cdot)^\top, u_2^*(\cdot)^\top)^\top$, then the following FBSDE admits an adapted solution

$(X^*(\cdot), Y^*(\cdot), Z^*(\cdot))$:

$$
\begin{cases} dX^*(s) = \big[A(s)X^*(s) + B(s)u^*(s) + b(s)\big]ds \\ \qquad\qquad + \big[C(s)X^*(s) + D(s)u^*(s) + \sigma(s)\big]dW(s), \qquad s \in [t,T], \\ dY^*(s) = -\big[A(s)^\top Y^*(s) + C(s)^\top Z^*(s) + Q(s)X^*(s) \\ \qquad\qquad + S(s)^\top u^*(s) + q(s)\big]ds + Z^*(s)dW(s), \qquad s \in [t,T], \\ X^*(t) = x, \qquad Y^*(T) = GX^*(T) + g, \end{cases} \tag{4.11}
$$

and the following stationarity condition holds:

$$
B(s)^\top Y^*(s) + D(s)^\top Z^*(s) + S(s)X^*(s) + R(s)u^*(s) + \rho(s) = 0, \qquad \text{a.e. } s \in [t,T], \text{ a.s.} \tag{4.12}
$$

The following result is concerned with the uniqueness of open-loop saddle points.

COROLLARY 4.2. *For $0 < m_1, m_2 < m$, let* (SG1)–(SG2) *hold, and let $(t,x) \in [0,T) \times \mathbb{R}^n$ be given. Suppose Problem* (SG) *admits a unique open-loop saddle point $u^*(\cdot)$ at (t,x). Then the unique adapted solution $(X^*(\cdot), Y^*(\cdot), Z^*(\cdot))$ of the decoupled FBSDE* (4.11) *together with $u^*(\cdot)$ is the unique* 4*-tuple of $\mathbb{F}$-progressively measurable processes that satisfy* (4.11)–(4.12). *Conversely, if the convexity-concavity condition* (4.3)–(4.4) *holds and there exists a unique $\mathbb{F}$-progressively measurable process $(X^*(\cdot), Y^*(\cdot), Z^*(\cdot), u^*(\cdot))$ satisfying* (4.11)–(4.12), *then $u^*(\cdot)$ is the unique open-loop saddle point of Problem* (SG).

Proof. Suppose $u^*(\cdot) \in \mathcal{U}[t,T]$ is the unique open-loop saddle point of Problem (SG) at (t,x). By Theorem 4.1, the unique adapted solution $(X^*(\cdot), Y^*(\cdot), Z^*(\cdot))$ of the decoupled FBSDE (4.11), together with $u^*(\cdot)$, satisfies the stationarity condition (4.12), and the convexity-concavity condition stated in Theorem 4.1 holds. Now, if there is another different 4-tuple $(\widehat{X}(\cdot), \widehat{Y}(\cdot), \widehat{Z}(\cdot), \widehat{u}(\cdot))$ satisfying (4.11)–(4.12), then it is necessary that $\widehat{u}(\cdot) \neq u^*(\cdot)$; otherwise, $(\widehat{X}(\cdot), \widehat{Y}(\cdot), \widehat{Z}(\cdot)) = (X^*(\cdot), Y^*(\cdot), Z^*(\cdot))$ by the uniqueness of the adapted solutions to the decoupled FBSDE (4.11). Hence, by the sufficiency of Theorem 4.1, $\widehat{u}(\cdot)$ has to be another different open-loop saddle point, a contradiction.

Conversely, if Problem (SG) has two different open-loop saddle points, then by the necessity of Theorem 4.1, the process $(X^*(\cdot), Y^*(\cdot), Z^*(\cdot), u^*(\cdot))$ satisfying FBSDE (4.11) and stationarity condition (4.12) will not be unique. ∎

A result similar to Corollary 4.2 for the case $m = m_1$ can be stated and proved. We omit the details here.

Clearly, if Problem (SG) admits an open-loop saddle point at (t,x), then both the open-loop lower value $V^-(t,x)$ and the open-loop upper value $V^+(t,x)$ are finite. In 2005, Zhang [26] proved that for Problem (DG) with $R_{11} > 0, R_{22} < 0$, and $R_{12} = R_{21}^\top = 0$, the finiteness of the open-loop lower and upper values is equivalent to the existence of an open-loop saddle point. However, such a result does not hold in general (see Example 7.5). Instead, comparing with Theorem 4.1 concerning open-loop saddle points, we have the following general weaker conclusion under weaker conditions.

PROPOSITION 4.3. *For $0 < m_1, m_2 < m$, let* (SG1)–(SG2) *hold, and let $(t,x) \in [0,T) \times \mathbb{R}^n$ be given. If $V^\pm(t,x)$ are finite, then the convexity-concavity condition* (4.3)–(4.4) *holds. Likewise, for $m_1 = m$, if $V(t,x)$ is finite, then the convexity condition* (4.5)–(4.6) *holds.*

Proof. Let $0 < m_1, m_2 < m$. Since $V^-(t,x)$ is finite, there exists a $u_2(\cdot) \in \mathcal{U}_2[t,T]$ such that

$$J(t,x;\lambda u_1(\cdot),u_2(\cdot)) > -\infty \qquad \forall u_1(\cdot) \in \mathcal{U}_1[t,T], \quad \lambda \in \mathbb{R}. \tag{4.13}$$

For any $u_1(\cdot) \in \mathcal{U}_1[t,T]$ and $\lambda \in \mathbb{R}$, let $X^\lambda(\cdot)$ be the solution to the following SDE:

$$\begin{cases} dX^\lambda(s) = \big[A(s)X^\lambda(s) + \lambda B_1(s)u_1(s) + B_2(s)u_2(s) + b(s)\big]ds \\ \qquad\qquad + \big[C(s)X^\lambda(s) + \lambda D_1(s)u_1(s) + D_2(s)u_2(s) + \sigma(s)\big]dW(s), \quad s \in [t,T], \\ X^\lambda(t) = x. \end{cases}$$

Then $X_1(\cdot) = \frac{X^\lambda(\cdot)-X^0(\cdot)}{\lambda}$ is independent of λ satisfying (4.4) (with $i=1$), and by a similar computation as in the proof of Theorem 4.1, we obtain

$$\begin{aligned} &J(t,x;\lambda u_1(\cdot),u_2(\cdot)) - J(t,x;0,u_2(\cdot)) \\ &\quad = \lambda\mathbb{E}\bigg\{ \langle GX^0(T)+g, X_1(T)\rangle \\ &\qquad\qquad + \int_t^T \big[\langle QX^0 + S_2^\top u_2 + q, X_1\rangle + \langle S_1X^0 + R_{12}u_2 + \rho_1, u_1\rangle\big]ds\bigg\} \\ &\qquad + \frac{\lambda^2}{2}\mathbb{E}\bigg\{\langle GX_1(T),X_1(T)\rangle + \int_t^T \big[\langle QX_1,X_1\rangle + 2\langle S_1X_1,u_1\rangle + \langle R_{11}u_1,u_1\rangle\big]ds\bigg\}. \end{aligned}$$

Then, if (4.13) holds, it is necessary that

$$\mathbb{E}\bigg\{\langle GX_1(T),X_1(T)\rangle + \int_t^T \big[\langle QX_1,X_1\rangle + 2\langle S_1X_1,u_1\rangle + \langle R_{11}u_1,u_1\rangle\big]ds\bigg\} \geqslant 0 \\ \forall u_1(\cdot) \in \mathcal{U}_1[t,T].$$

The rest can be proved similarly. ∎

5. Closed-loop saddle points and Riccati equations. We now look at closed-loop saddle points for Problem (SG). First, we present the following result, which is a consequence of Theorem 4.1.

PROPOSITION 5.1. *Let* (SG1)–(SG2) *hold and* $t \in [0,T)$. *Let* $(\Theta^*(\cdot),v^*(\cdot)) \in \mathscr{Q}[t,T]\times\mathcal{U}[t,T]$ *be a closed-loop saddle point of Problem* (SG). *Then for any* $x \in \mathbb{R}^n$, *the following FBSDE admits an adapted solution* $(X^*(\cdot),Y^*(\cdot),Z^*(\cdot))$*:*

$$\begin{cases} dX^*(s) = \big[(A+B\Theta^*)X^* + Bv^* + b\big]ds \\ \qquad\qquad + \big[(C+D\Theta^*)X^* + Dv^* + \sigma\big]dW(s), \qquad s \in [t,T], \\ dY^*(s) = -\big[A^\top Y^* + C^\top Z^* + (Q+S^\top\Theta^*)X^* + S^\top v^* + q\big]ds \\ \qquad\qquad + Z^*dW(s), \qquad s \in [t,T], \\ X^*(t) = x, \qquad Y^*(T) = GX^*(T) + g, \end{cases} \tag{5.1}$$

and the following stationarity condition holds:

$$Rv^* + B^\top Y^* + D^\top Z^* + (S+R\Theta^*)X^* + \rho = 0 \qquad \text{a.e. a.s.} \tag{5.2}$$

Proof. Let $(\Theta^*(\cdot), v^*(\cdot)) \in \mathscr{Q}[t,T] \times \mathcal{U}[t,T]$ be a closed-loop saddle point of Problem (SG) with $\Theta^*(\cdot) = (\Theta_1^*(\cdot)^\top, \Theta_2^*(\cdot)^\top)^\top$ and $v^*(\cdot) = (v_1^*(\cdot)^\top, v_2^*(\cdot)^\top)^\top$. We consider state equation (3.20) with the cost functional (3.21) for which we carry out some computation: (denoting $\widetilde{v} = (v_1^\top, (v_2^*)^\top)^\top$)

$$
\begin{aligned}
&J_1(t,x;v_1(\cdot))\\
&\quad \equiv J(t,x;\Theta^* X(\cdot) + \widetilde{v}(\cdot))\\
&\quad = \frac{1}{2}\mathbb{E}\bigg\{ \langle GX(T), X(T)\rangle + 2\langle g, X(T)\rangle \\
&\qquad\qquad + \int_t^T \Big[\langle QX, X\rangle + 2\langle SX, \Theta^* X + \widetilde{v}\rangle \\
&\qquad\qquad\qquad + \langle R(\Theta^* X + \widetilde{v}), \Theta^* X + \widetilde{v}\rangle + 2\langle q, X\rangle + 2\langle \rho, \Theta^* X + \widetilde{v}\rangle \Big] ds \bigg\}\\
&\quad = \frac{1}{2}\mathbb{E}\bigg\{ \langle GX(T), X(T)\rangle + 2\langle g, X(T)\rangle \\
&\qquad\qquad + \int_t^T \Big[\langle [Q + (\Theta^*)^\top S + S^\top \Theta^* + (\Theta^*)^\top R \Theta^*]X, X\rangle \\
&\qquad\qquad\qquad + 2\left\langle \begin{pmatrix} (S_1 + R_1\Theta^*)X \\ (S_2 + R_2\Theta^*)X \end{pmatrix}, \begin{pmatrix} v_1 \\ v_2^* \end{pmatrix} \right\rangle + \left\langle \begin{pmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{pmatrix} \begin{pmatrix} v_1 \\ v_2^* \end{pmatrix}, \begin{pmatrix} v_1 \\ v_2^* \end{pmatrix} \right\rangle \\
&\qquad\qquad\qquad + 2\langle q + (\Theta^*)^\top \rho, X\rangle + 2\left\langle \begin{pmatrix} \rho_1 \\ \rho_2 \end{pmatrix}, \begin{pmatrix} v_1 \\ v_2^* \end{pmatrix} \right\rangle \Big] ds \bigg\}\\
&\quad = \frac{1}{2}\mathbb{E}\bigg\{ \langle GX(T), X(T)\rangle + 2\langle g, X(T)\rangle \\
&\qquad\qquad + \int_t^T \Big[\langle [Q + (\Theta^*)^\top S + S^\top \Theta^* + (\Theta^*)^\top R \Theta^*]X, X\rangle \\
&\qquad\qquad\qquad + 2\langle (S_1 + R_1\Theta^*)X, v_1\rangle + 2\langle q + (\Theta^*)^\top \rho + (S_2 + R_2\Theta^*)^\top v_2^*, X\rangle \\
&\qquad\qquad\qquad + \langle R_{11} v_1, v_1\rangle + 2\langle \rho_1 + R_{12} v_2^*, v_1\rangle + \langle R_{22} v_2^*, v_2^*\rangle + 2\langle \rho_2, v_2^*\rangle \Big] ds \bigg\}.
\end{aligned}
$$

We know that $v_1^*(\cdot)$ is an open-loop optimal control for the problem with state equation (3.20) and the above cost functional. Thus, according to Theorem 4.1, we have

$$
0 = B_1^\top Y^* + D_1^\top Z^* + (S_1 + R_1\Theta^*)X^* + R_{11} v_1^* + \rho_1 + R_{12} v_2^* \qquad \text{a.e.} \quad \text{a.s.}
$$

with $(Y^*(\cdot), Z^*(\cdot))$ being the adapted solution to the following BSDE on $[t,T]$:

$$
\begin{cases}
dY^* = -\big\{(A + B\Theta^*)^\top Y^* + (C + D\Theta^*)^\top Z^* \\
\qquad\qquad + [Q + (\Theta^*)^\top S + S^\top \Theta^* + (\Theta^*)^\top R \Theta^*]X^* \\
\qquad\qquad + (S_1 + R_1\Theta^*)^\top v_1^* + q + (\Theta^*)^\top \rho + (S_2 + R_2\Theta^*)^\top v_2^* \big\} ds + Z^* dW \\
\qquad = -\big\{A^\top Y^* + C^\top Z^* + QX^* + S^\top(\Theta^* X^* + v^*) + q \\
\qquad\qquad + (\Theta^*)^\top [B^\top Y^* + D^\top Z^* + SX^* + R(\Theta^* X^* + v^*) + \rho]\big\} ds + Z^* dW, \\
Y^*(T) = GX^*(T) + g.
\end{cases}
$$

Likewise, by considering state equation (3.22) and cost functional (3.23), we can obtain

$$0 = B_2^\top Y^* + D_2^\top Z^* + (S_2 + R_2\Theta^*)X^* + R_{21}v_1^* + \rho_2 + R_{22}v_2^* \qquad \text{a.e. a.s.}$$

with $(Y^*(\cdot), Z^*(\cdot))$ being the adapted solution to the same BSDE as above. Thus,

$$0 = B^\top Y^* + D^\top Z^* + (S + R\Theta^*)X^* + Rv^* + \rho \qquad \text{a.e. a.s.}$$

Then the above BSDE is reduced to that in (5.1). ▯

We point out that unlike the open-loop saddle point case, the convexity-concavity condition (4.3)–(4.4) is not claimed to be necessary for the existence of a closed-loop saddle point of Problem (SG). From this, one sees the essential difference between the open-loop and closed-loop saddle points. The following result gives a characterization for closed-loop saddle points of Problem (SG).

THEOREM 5.2. *Let* $0 < m_1, m_2 < m$ *and* (SG1)–(SG2) *hold. Then Problem* (SG) *admits a closed-loop saddle point* $(\Theta^*(\cdot), v^*(\cdot)) \in \mathscr{Q}[t,T] \times \mathcal{U}[t,T]$ *with* $\Theta^*(\cdot) \equiv (\Theta_1^*(\cdot)^\top, \Theta_2^*(\cdot)^\top)^\top$ *and* $v^*(\cdot) \equiv (v_1^*(\cdot)^\top, v_2^*(\cdot)^\top)^\top$ *if and only if the Riccati equation*

$$\begin{cases} \dot P(s) + P(s)A(s) + A(s)^\top P(s) + C(s)^\top P(s)C(s) + Q(s) \\ \quad - \big[P(s)B(s) + C(s)^\top P(s)D(s) + S(s)^\top\big]\big[R(s) + D(s)^\top P(s)D(s)\big]^\dagger \\ \qquad \cdot \big[B(s)^\top P(s) + D(s)^\top P(s)C(s) + S(s)\big] = 0 \qquad \text{a.e. } s \in [t,T], \\ P(T) = G, \end{cases} \tag{5.3}$$

admits a solution $P(\cdot) \in C([t,T];\mathbb{S}^n)$ *such that*

$$\mathcal{R}\big(B(s)^\top P(s) + D(s)^\top P(s)C(s) + S(s)\big) \subseteq \mathcal{R}\big(R(s) + D(s)^\top P(s)D(s)\big) \qquad \text{a.e. } s \in [t,T], \tag{5.4}$$

$$\big[R(\cdot)+D(\cdot)^\top P(\cdot)D(\cdot)\big]^\dagger\big[B(\cdot)^\top P(\cdot)+D(\cdot)^\top P(\cdot)C(\cdot)+S(\cdot)\big] \in L^2(t,T;\mathbb{R}^{m\times n}), \tag{5.5}$$

$$R_{11}(s) + D_1(s)^\top P(s)D_1(s) \geqslant 0, \qquad R_{22}(s) + D_2(s)^\top P(s)D_2(s) \leqslant 0, \qquad \text{a.e. } s \in [t,T], \tag{5.6}$$

and the adapted solution $(\eta(\cdot), \zeta(\cdot))$ *of the BSDE*

$$\begin{cases} d\eta(s) = -\Big\{\big[A^\top - (PB + C^\top PD + S^\top)(R + D^\top PD)^\dagger B^\top\big]\eta \\ \qquad + \big[C^\top - (PB + C^\top PD + S^\top)(R + D^\top PD)^\dagger D^\top\big]\zeta \\ \qquad + \big[C^\top - (PB + C^\top PD + S^\top)(R + D^\top PD)^\dagger D^\top\big]P\sigma \\ \qquad - (PB + C^\top PD + S^\top)(R + D^\top PD)^\dagger \rho + Pb + q\Big\}ds \\ \qquad\qquad + \zeta dW(s), \qquad s \in [t,T], \\ \eta(T) = g, \end{cases} \tag{5.7}$$

satisfies

$$B(s)^\top\eta(s) + D(s)^\top\zeta(s) + D(s)^\top P(s)\sigma(s) + \rho(s) \in \mathcal{R}\big(R(s) + D(s)^\top P(s)D(s)\big) \qquad \text{a.e. } s \in [t,T], \text{ a.s.} \tag{5.8}$$

$$\big[R(\cdot)+D(\cdot)^\top P(\cdot)D(\cdot)\big]^\dagger\big[B(\cdot)^\top\eta(\cdot)+D(\cdot)^\top\zeta(\cdot)+D(\cdot)^\top P(\cdot)\sigma(\cdot)+\rho(\cdot)\big] \in L^2_{\mathbb{F}}(t,T;\mathbb{R}^m). \tag{5.9}$$

In this case, the closed-loop saddle point $(\Theta^*(\cdot), v^*(\cdot))$ *admits the representation*
(5.10)
$$\begin{cases} \Theta^*(\cdot) = -\big[R(\cdot) + D(\cdot)^\top P(\cdot)D(\cdot)\big]^\dagger \big[B(\cdot)^\top P(\cdot) + D(\cdot)^\top P(\cdot)C(\cdot) + S(\cdot)\big] \\ \qquad\qquad + \big\{I - \big[R(\cdot) + D(\cdot)^\top P(\cdot)D(\cdot)\big]^\dagger \big[R(\cdot) + D(\cdot)^\top P(\cdot)D(\cdot)\big]\big\}\Pi(\cdot), \\ v^*(\cdot) = -\big[R(\cdot)+D(\cdot)^\top P(\cdot)D(\cdot)\big]^\dagger \big[B(\cdot)^\top \eta(\cdot) + D(\cdot)^\top \zeta(\cdot) + D(\cdot)^\top P(\cdot)\sigma(\cdot) + \rho(\cdot)\big] \\ \qquad\qquad + \big\{I - \big[R(\cdot) + D(\cdot)^\top P(\cdot)D(\cdot)\big]^\dagger \big[R(\cdot) + D(\cdot)^\top P(\cdot)D(\cdot)\big]\big\}\nu(\cdot) \end{cases}$$

for some $\Pi(\cdot) \in L^2(t, T; \mathbb{R}^{m\times n})$ *and* $\nu(\cdot) \in L^2_{\mathbb{F}}(t, T; \mathbb{R}^m)$*, and the value function admits the following representation:*
(5.11)
$$\begin{aligned} V(t, x) = \frac{1}{2}\mathbb{E}\bigg\{ &\langle P(t)x, x\rangle + 2\langle \eta(t), x\rangle \\ &+ \int_t^T \big[\langle P\sigma, \sigma\rangle + 2\langle \eta, b\rangle + 2\langle \zeta, \sigma\rangle - \langle (R+D^\top PD)^\dagger \\ &\qquad (B^\top\eta+D^\top\zeta+D^\top P\sigma+\rho), B^\top\eta+D^\top\zeta+D^\top P\sigma+\rho\rangle\big]ds\bigg\}. \end{aligned}$$

Before proving the above result, let us point out that the closed-loop saddle point $(\Theta^*(\cdot), v^*(\cdot))$ given by (5.10) only depends on the coefficients of Problem (SG), and it is independent of the initial state x. Also, we see that the convexity-concavity condition (4.3)–(4.4) is not even mentioned in the above.

Proof. Necessity. Let $(\Theta^*(\cdot), v^*(\cdot))$ be a closed-loop saddle point of Problem (SG) over $[t, T]$, where $\Theta^*(\cdot) \equiv (\Theta_1^*(\cdot)^\top, \Theta_2^*(\cdot)^\top)^\top \in \mathscr{Q}_1[t, T] \times \mathscr{Q}_2[t, T]$ and $v^*(\cdot) \equiv (v_1^*(\cdot)^\top, v_2^*(\cdot)^\top)^\top \in \mathcal{U}_1[t, T] \times \mathcal{U}_2[t, T]$. Then, by Proposition 5.1, for any $x \in \mathbb{R}^n$, the following FBSDE admits an adapted solution $(X^*(\cdot), Y^*(\cdot), Z^*(\cdot))$:

$$\begin{cases} dX^*(s) = \big[(A + B\Theta^*)X^* + Bv^* + b\big]ds \\ \qquad\qquad + \big[(C + D\Theta^*)X^* + Dv^* + \sigma\big]dW(s), \qquad s \in [t, T], \\ dY^*(s) = -\big[A^\top Y^* + C^\top Z^* + (Q + S^\top\Theta^*)X^* + S^\top v^* + q\big]ds \\ \qquad\qquad + Z^* dW(s), \qquad s \in [t, T], \\ X^*(t) = x, \qquad Y^*(T) = GX^*(T) + g; \end{cases} \tag{5.12}$$

the following stationarity condition holds:

$$B^\top Y^* + D^\top Z^* + (S + R\Theta^*)X^* + Rv^* + \rho = 0 \qquad \text{a.e. a.s.} \tag{5.13}$$

Since the above admits a solution for each $x \in \mathbb{R}^n$, and $(\Theta^*(\cdot), v^*(\cdot))$ is independent of x, by subtracting solutions corresponding to x and 0, the later from the former, we see that for any $x \in \mathbb{R}^n$, as long as $(X(\cdot), Y(\cdot), Z(\cdot))$ is the adapted solution to the FBSDE

$$\begin{cases} dX(s) = (A + B\Theta^*)Xds + (C + D\Theta^*)XdW(s), \qquad s \in [t, T], \\ dY(s) = -\big[A^\top Y + C^\top Z + (Q + S^\top\Theta^*)X\big]ds + ZdW(s), \qquad s \in [t, T], \\ X(t) = x, \qquad Y(T) = GX(T), \end{cases} \tag{5.14}$$

one must have the following stationarity condition:

$$B^\top Y + D^\top Z + (S + R\Theta^*)X = 0 \qquad \text{a.e. a.s.} \tag{5.15}$$

Now, we let

$$\begin{cases} d\mathbb{X}(s) = \big[A(s) + B(s)\Theta^*(s)\big]\mathbb{X}(s)ds \\ \qquad\qquad + \big[C(s) + D(s)\Theta^*(s)\big]\mathbb{X}(s)dW(s), \qquad s \in [t,T], \\ \mathbb{X}(t) = I, \end{cases} \tag{5.16}$$

and let

$$\begin{cases} d\mathbb{Y}(s) = -\big\{A(s)^\top\mathbb{Y}(s) + C(s)^\top\mathbb{Z}(s) + \big[Q(s) + S(s)^\top\Theta^*(s)\big]\mathbb{X}(s)\big\}ds \\ \qquad\qquad\qquad\qquad\qquad\qquad + \mathbb{Z}(s)dW(s), \qquad s \in [t,T], \\ \mathbb{Y}(T) = G\mathbb{X}(T). \end{cases} \tag{5.17}$$

Obviously, $\mathbb{X}(\cdot)$, $\mathbb{Y}(\cdot)$, and $\mathbb{Z}(\cdot)$ are all well-defined $\mathbb{R}^{n\times n}$-valued processes. Further, (5.15) implies

$$B^\top\mathbb{Y} + D^\top\mathbb{Z} + (S + R\Theta^*)\mathbb{X} = 0 \qquad \text{a.e. a.s.} \tag{5.18}$$

Clearly, $\mathbb{X}(\cdot)^{-1}$ exists and satisfies the following:

$$\begin{cases} d\big[\mathbb{X}(s)^{-1}\big] = \mathbb{X}(s)^{-1}\big\{\big[C(s) + D(s)\Theta^*(s)\big]^2 - A(s) - B(s)\Theta^*(s)\big\}ds \\ \qquad\qquad\qquad - \mathbb{X}(s)^{-1}\big[C(s) + D(s)\Theta^*(s)\big]dW(s), \qquad s \in [t,T], \\ \mathbb{X}(t)^{-1} = I. \end{cases} \tag{5.19}$$

We define

$$P(\cdot) = \mathbb{Y}(\cdot)\mathbb{X}(\cdot)^{-1}, \qquad \Delta(\cdot) = \mathbb{Z}(\cdot)\mathbb{X}(\cdot)^{-1}.$$

Then (5.18) implies

$$B^\top P + D^\top\Delta + (S + R\Theta^*) = 0 \qquad \text{a.e. a.s.} \tag{5.20}$$

Also, by Itô's formula,

$$\begin{aligned} dP &= \Big\{ - \big[A^\top\mathbb{Y} + C^\top\mathbb{Z} + (Q + S^\top\Theta^*)\mathbb{X}\big]\mathbb{X}^{-1} + \mathbb{Y}\mathbb{X}^{-1}\big[(C + D\Theta^*)^2 - A - B\Theta^*\big] \\ &\qquad - \mathbb{Z}\mathbb{X}^{-1}(C + D\Theta^*)\Big\}ds + \Big\{\mathbb{Z}\mathbb{X}^{-1} - \mathbb{Y}\mathbb{X}^{-1}(C + D\Theta^*)\Big\}dW(s) \\ &= \Big\{ - A^\top P - C^\top\Delta - Q - S^\top\Theta^* + P\big[(C + D\Theta^*)^2 - A - B\Theta^*\big] \\ &\qquad - \Delta(C + D\Theta^*)\Big\}ds + \Big\{\Delta - P(C + D\Theta^*)\Big\}dW(s). \end{aligned}$$

Let

$$\Lambda = \Delta - P(C + D\Theta^*).$$

Then

$$\begin{aligned} dP &= \Big\{ -A^\top P - C^\top\big[\Lambda + P(C + D\Theta^*)\big] - Q - S^\top\Theta^* + P\big[(C + D\Theta^*)^2 - A - B\Theta^*\big] \\ &\qquad - \big[\Lambda + P(C + D\Theta^*)\big](C + D\Theta^*)\Big\}ds + \Lambda dW(s) \\ &= \Big\{ -PA - A^\top P - \Lambda C - C^\top\Lambda - C^\top PC - (PB + C^\top PD + S^\top + \Lambda D)\Theta^* - Q\Big\}ds \\ &\qquad\qquad\qquad\qquad\qquad\qquad + \Lambda dW(s), \qquad s \in [t,T], \end{aligned}$$

and $P(T)=G$. Thus, $(P(\cdot),\Lambda(\cdot))$ is the adapted solution of a BSDE with deterministic coefficients. Hence, $P(\cdot)$ is deterministic and $\Lambda(\cdot)=0$, which means

$$\Delta=\mathbb{Z}\mathbb{X}^{-1}=P(C+D\Theta^*). \tag{5.21}$$

Therefore,

$$\dot P+PA+A^\top P+C^\top PC+(PB+C^\top PD+S^\top)\Theta^*+Q=0 \qquad \text{a.e.} \tag{5.22}$$

and (5.20) becomes

$$\begin{aligned}0&=B^\top P+D^\top P(C+D\Theta^*)+S+R\Theta^*\\ &=B^\top P+D^\top PC+S+(R+D^\top PD)\Theta^* \qquad \text{a.e.}\end{aligned} \tag{5.23}$$

This implies

$$\mathcal{R}\big(B^\top P+D^\top PC+S\big)\subseteq\mathcal{R}\big(R+D^\top PD\big) \qquad \text{a.e.}$$

Using (5.23), (5.22) can be written as

$$\begin{aligned}0=\dot P&+P(A+B\Theta^*)+(A+B\Theta^*)^\top P+(C+D\Theta^*)^\top P(C+D\Theta^*)\\ &+(\Theta^*)^\top R\Theta^*+S^\top\Theta^*+(\Theta^*)^\top S+Q \qquad \text{a.e.}\end{aligned}$$

Since $P(T)=G\in\mathbb{S}^n$ and $Q(\cdot),R(\cdot)$ are symmetric, by uniqueness, we must have $P(\cdot)\in C([t,T];\mathbb{S}^n)$. Denoting $\widehat R=R+D^\top PD$, since

$$\widehat R^\dagger(B^\top P+D^\top PC+S)=-\widehat R^\dagger\widehat R\Theta^*,$$

and $\widehat R^\dagger\widehat R$ is an orthogonal projection, we see that (5.5) holds and

$$\Theta^*=-\widehat R^\dagger(B^\top P+D^\top PC+S)+\big(I-\widehat R^\dagger\widehat R\big)\Pi$$

for some $\Pi(\cdot)\in L^2(t,T;\mathbb{R}^{m\times n})$. Consequently,

$$\begin{aligned}(PB+C^\top PD+S^\top)\Theta^*&=(\Theta^*)^\top\widehat R\widehat R^\dagger(B^\top P+D^\top PC+S)\\ &=-(PB+C^\top PD+S^\top)\widehat R^\dagger(B^\top P+D^\top PC+S).\end{aligned} \tag{5.24}$$

Plugging the above into (5.22), we obtain Riccati equation (5.3). To determine $v^*(\cdot)$, we define

$$\begin{cases}\eta(s)=Y^*(s)-P(s)X^*(s),\\ \zeta(s)=Z^*(s)-P(s)\big[C(s)+D(s)\Theta^*(s)\big]X^*(s) \qquad\qquad s\in[t,T].\\ \qquad\qquad -P(s)D(s)v^*(s)-P(s)\sigma(s),\end{cases}$$

Then

$$
\begin{aligned}
d\eta &= dY^* - \dot{P}X^* ds - PdX^* \\
&= -\big[A^\top Y^* + C^\top Z^* + (Q + S^\top\Theta^*)X^* + S^\top v^* + q\big]ds \\
&\quad + Z^* dW + \Big\{\big[PA + A^\top P + C^\top PC + Q \\
&\qquad\qquad - (PB + C^\top PD + S^\top)(R + D^\top PD)^\dagger(B^\top P + D^\top PC + S)\big]X^* \\
&\qquad\qquad - P\big[(A + B\Theta^*)X^* + Bv^* + b\big]\Big\}ds \\
&\quad -P\big[(C + D\Theta^*)X^* + Dv^* + \sigma\big]dW \\
&= -\Big\{A^\top(\eta + PX^*) + C^\top\big[\zeta + P(C + D\Theta^*)X^* + PDv^* + P\sigma\big] \\
&\qquad + (Q + S^\top\Theta^*)X^* + S^\top v^* + q - \big[PA + A^\top P + C^\top PC + Q \\
&\qquad - (PB + C^\top PD + S^\top)(R + D^\top PD)^\dagger(B^\top P + D^\top PC + S)\big]X^* \\
&\qquad + P\big[(A + B\Theta^*)X^* + Bv^* + b\big]\Big\}ds + \zeta dW \\
&= \Big\{ - A^\top\eta - C^\top\zeta - (PB + C^\top PD + S^\top)\Theta^* X^* \\
&\quad - (PB + C^\top PD + S^\top)v^* - C^\top P\sigma - Pb - q \\
&\quad - \big[(PB + C^\top PD + S^\top)(R + D^\top PD)^\dagger(B^\top P + D^\top PC + S)\big]X^*\Big\}ds + \zeta dW \\
&= -\big[A^\top\eta + C^\top\zeta + (PB + C^\top PD + S^\top)v^* + C^\top P\sigma + Pb + q\big]ds \\
&\quad + {}'\zeta dW, \qquad s \in [t, T].
\end{aligned}
$$

According to (5.13), we have

$$
\begin{aligned}
0 &= B^\top Y^* + D^\top Z^* + (S + R\Theta^*)X^* + Rv^* + \rho \\
&= B^\top(\eta + PX^*) + D^\top\big[\zeta + P(C + D\Theta^*)X^* + PDv^* + P\sigma\big] \\
&\quad + (S + R\Theta^*)X^* + Rv^* + \rho \\
&= \big[B^\top P + D^\top PC + S + (R + D^\top PD)\Theta^*\big]X^* \\
&\quad + B^\top\eta + D^\top\zeta + D^\top P\sigma + \rho + (R + D^\top PD)v^* \\
&= B^\top\eta + D^\top\zeta + D^\top P\sigma + \rho + (R + D^\top PD)v^*.
\end{aligned}
$$

Hence, the following is true:

$$B^\top\eta + D^\top\zeta + D^\top P\sigma + \rho \in \mathcal{R}(R + D^\top PD) \qquad \text{a.e.} \quad \text{a.s.}$$

On the other hand, since

$$\widehat{R}^\dagger(B^\top\eta + D^\top\zeta + D^\top P\sigma + \rho) = -\widehat{R}^\dagger\widehat{R}v^*,$$

and $\widehat{R}^\dagger\widehat{R}$ is an orthogonal projection, we see that (5.9) holds and

$$v^* = -\widehat{R}^\dagger(B^\top\eta + D^\top\zeta + D^\top P\sigma + \rho) + \big[I - \widehat{R}^\dagger\widehat{R}\big]\nu$$

for some $\nu(\cdot) \in L^2_{\mathbb{F}}(t, T; \mathbb{R}^m)$. Consequently,

$$
\begin{aligned}
&(PB + C^\top PD + S^\top)v^* \\
&\quad = -(PB + C^\top PD + S^\top)(R + D^\top PD)^\dagger(B^\top\eta + D^\top\zeta + D^\top P\sigma + \rho) \\
&\qquad + (PB + C^\top PD + S^\top)\big[I - (R + D^\top PD)^\dagger(R + D^\top PD)\big]\nu \\
&\quad = -(PB + C^\top PD + S^\top)(R + D^\top PD)^\dagger(B^\top\eta + D^\top\zeta + D^\top P\sigma + \rho).
\end{aligned}
$$

Then

$$
\begin{aligned}
&A^\top\eta + C^\top\zeta + (PB + C^\top PD + S^\top)v^* + C^\top P\sigma + Pb + q\\
&\quad = A^\top\eta + C^\top\zeta + C^\top P\sigma + Pb + q\\
&\qquad - (PB + C^\top PD + S^\top)(R + D^\top PD)^\dagger(B^\top\eta + D^\top\zeta + D^\top P\sigma + \rho)\\
&\quad = \big[A^\top - (PB + C^\top PD + S^\top)(R + D^\top PD)^\dagger B^\top\big]\eta\\
&\qquad + \big[C^\top - (PB + C^\top PD + S^\top)(R + D^\top PD)^\dagger D^\top\big]\zeta\\
&\qquad + \big[C^\top - (PB + C^\top PD + S^\top)(R + D^\top PD)^\dagger D^\top\big]P\sigma\\
&\qquad - (PB + C^\top PD + S^\top)(R + D^\top PD)^\dagger\rho + Pb + q.
\end{aligned}
$$

Therefore, $(\eta(\cdot), \zeta(\cdot))$ is the adapted solution to BSDE (5.7). This proves the necessity, except (5.6), whose proof is contained in the proof of sufficiency below.

Sufficiency. We take any $u(\cdot) = (u_1(\cdot)^\top, u_2(\cdot)^\top)^\top \in \mathcal{U}_1[t,T] \times \mathcal{U}_2[t,T]$, and let $X(\cdot) \equiv X(\cdot\,; t, x, u(\cdot))$ be the corresponding state process. Then

$$
\begin{aligned}
J(t,x;u(\cdot)) &= \frac{1}{2}\mathbb{E}\bigg\{\langle GX(T), X(T)\rangle + 2\langle g, X(T)\rangle\\
&\qquad + \int_t^T \big[\langle QX, X\rangle + 2\langle SX, u\rangle + \langle Ru, u\rangle + 2\langle q, X\rangle + 2\langle \rho, u\rangle\big]ds\bigg\}\\
&= \frac{1}{2}\mathbb{E}\bigg\{\langle P(t)x, x\rangle + 2\langle \eta(t), x\rangle\\
&\qquad + \int_t^T \big[\langle\big[-PA - A^\top P - C^\top PC - Q + (PB + C^\top PD + S^\top)\\
&\qquad\qquad \times (R + D^\top PD)^\dagger(B^\top P + D^\top PC + S)\big]X, X\rangle\\
&\qquad + \langle P(AX + Bu + b), X\rangle + \langle PX, AX + Bu + b\rangle\\
&\qquad + \langle P(CX + Du + \sigma), CX + Du + \sigma\rangle\\
&\qquad + 2\langle\big[-A^\top + (PB + C^\top PD + S^\top)(R + D^\top PD)^\dagger B^\top\big]\eta, X\rangle\\
&\qquad + 2\langle\big[-C^\top + (PB + C^\top PD + S^\top)(R + D^\top PD)^\dagger D^\top\big]P\sigma, X\rangle\\
&\qquad + 2\langle\big[-C^\top + (PB + C^\top PD + S^\top)(R + D^\top PD)^\dagger D^\top\big]\zeta, X\rangle\\
&\qquad + 2\langle(PB + C^\top PD + S^\top)(R + D^\top PD)^\dagger\rho - Pb - q, X\rangle\\
&\qquad + 2\langle\zeta, CX + Du + \sigma\rangle + 2\langle\eta, AX + Bu + b\rangle + \langle QX, X\rangle\\
&\qquad + 2\langle SX, u\rangle + \langle Ru, u\rangle + 2\langle q, X\rangle + 2\langle\rho, u\rangle\big]ds\bigg\}\\
&= \frac{1}{2}\mathbb{E}\bigg\{\langle P(t)x, x\rangle + 2\langle\eta(t), x\rangle + \int_t^T\Big[\langle P\sigma, \sigma\rangle + 2\langle\eta, b\rangle + 2\langle\zeta, \sigma\rangle\\
&\qquad + \langle(PB + C^\top PD + S^\top)(R + D^\top PD)^\dagger(B^\top P + D^\top PC + S)X, X\rangle\\
&\qquad + 2\langle(B^\top P + D^\top PC + S)X + B^\top\eta + D^\top\zeta + D^\top P\sigma + \rho, u\rangle\\
&\qquad + \langle(R + D^\top PD)u, u\rangle + 2\langle(PB + C^\top PD + S^\top)(R + D^\top PD)^\dagger\\
&\qquad\qquad \cdot(B^\top\eta + D^\top\zeta + D^\top P\sigma + \rho), X\rangle\Big]ds\bigg\}.
\end{aligned}
$$

Let $(\Theta^*(\cdot), v^*(\cdot))$ be defined by (5.10). Then

$$\begin{cases} B^\top P + D^\top PC + S = -(R + D^\top PD)\Theta^* \equiv -\widehat{R}\Theta^*, \\ B^\top \eta + D^\top \zeta + D^\top P\sigma + \rho = -(R + D^\top PD)v^* \equiv -\widehat{R}v^*. \end{cases}$$

Also, one has

$$\begin{aligned} &\langle (R + D^\top PD)v^*, v^* \rangle \\ &\quad = \langle \widehat{R}\widehat{R}^\dagger(B^\top \eta + D^\top \zeta + D^\top P\sigma + \rho), \widehat{R}^\dagger(B^\top \eta + D^\top \zeta + D^\top P\sigma + \rho) \rangle \\ &\quad = \langle (R + D^\top PD)^\dagger(B^\top \eta + D^\top \zeta + D^\top P\sigma + \rho), B^\top \eta + D^\top \zeta + D^\top P\sigma + \rho \rangle . \end{aligned}$$

Thus,

$$\begin{aligned} J(t, x; u(\cdot)) &= \tfrac{1}{2}\,\mathbb{E}\bigg\{ \langle P(t)x, x \rangle + 2 \langle \eta(t), x \rangle + \textstyle\int_t^T \Big[\langle P\sigma, \sigma \rangle + 2 \langle \eta, b \rangle + 2 \langle \zeta, \sigma \rangle \\ &\qquad + \langle (PB + C^\top PD + S^\top)(R + D^\top PD)^\dagger(B^\top P + D^\top PC + S)X, X \rangle \\ &\qquad + 2 \langle (B^\top P + D^\top PC + S)X + B^\top \eta + D^\top \zeta + D^\top P\sigma + \rho, u \rangle \\ &\qquad + \langle (R + D^\top PD)u, u \rangle + 2 \langle (PB + C^\top PD + S^\top)(R + D^\top PD)^\dagger \\ &\qquad \cdot (B^\top \eta + D^\top \zeta + D^\top P\sigma + \rho), X \rangle \Big] ds \bigg\} \\ &= \frac{1}{2}\mathbb{E}\bigg\{ \langle P(t)x, x \rangle + 2 \langle \eta(t), x \rangle \\ &\qquad + \int_t^T \Big[\langle P\sigma, \sigma \rangle + 2 \langle \eta, b \rangle + 2 \langle \zeta, \sigma \rangle + \langle (\Theta^*)^\top \widehat{R}\widehat{R}^\dagger\widehat{R}\Theta^* X, X \rangle \\ &\qquad -2 \langle \widehat{R}(\Theta^* X + v^*), u \rangle + \langle \widehat{R}u, u \rangle + 2 \langle (\Theta^*)^\top \widehat{R}\widehat{R}^\dagger\widehat{R}v^*, X \rangle \Big] ds \bigg\} \\ &= \frac{1}{2}\mathbb{E}\bigg\{ \langle P(t)x, x \rangle + 2 \langle \eta(t), x \rangle \\ &\qquad + \int_t^T \Big[\langle P\sigma, \sigma \rangle + 2 \langle \eta, b \rangle + 2 \langle \zeta, \sigma \rangle + \langle \widehat{R}\Theta^* X, \Theta^* X \rangle \\ &\qquad -2 \langle \widehat{R}(\Theta^* X + v^*), u \rangle + \langle \widehat{R}u, u \rangle + 2 \langle \widehat{R}\Theta^* X, v^* \rangle \Big] ds \bigg\} \\ &= \frac{1}{2}\mathbb{E}\bigg\{ \langle P(t)x, x \rangle + 2 \langle \eta(t), x \rangle \\ &\qquad + \int_t^T \Big[\langle P\sigma, \sigma \rangle + 2 \langle \eta, b \rangle + 2 \langle \zeta, \sigma \rangle \\ &\qquad - \langle (R + D^\top PD)^\dagger(B^\top \eta + D^\top \zeta + D^\top P\sigma + \rho), \\ &\qquad\quad B^\top \eta + D^\top \zeta + D^\top P\sigma + \rho \rangle \\ &\qquad + \langle (R + D^\top PD)(u - \Theta^* X - v^*), u - \Theta^* X - v^* \rangle \Big] ds \bigg\} \\ &= J(t, x; \Theta^*(\cdot)X^*(\cdot) + v^*(\cdot)) \\ &\quad + \frac{1}{2}\mathbb{E}\int_t^T \langle (R + D^\top PD)(u - \Theta^* X - v^*), u - \Theta^* X - v^* \rangle \, ds. \end{aligned}$$

Consequently,

$$\begin{aligned}&J(t,x;\Theta_1^*(\cdot)X(\cdot)+v_1(\cdot),\Theta_2^*(\cdot)X(\cdot)+v_2^*(\cdot))\\&\quad=J(t,x;\Theta^*(\cdot)X^*(\cdot)+v^*(\cdot))+\frac{1}{2}\mathbb{E}\int_t^T\langle(R_{11}+D_1^\top PD_1)(v_1-v_1^*),v_1-v_1^*\rangle\,ds.\end{aligned}$$

Hence,

$$\begin{aligned}J(t,x;\Theta^*(\cdot)X^*(\cdot)+v^*(\cdot))\leqslant J(t,x;\Theta_1^*(\cdot)X(\cdot)+v_1(\cdot),\Theta_2^*(\cdot)X(\cdot)+v_2^*(\cdot))\\ \forall v_1(\cdot)\in\mathcal{U}_1[t,T]\end{aligned}$$

if and only if

$$R_{11}+D_1^\top PD_1\geqslant 0\qquad\text{a.e. } s\in[t,T].$$

Similarly,

$$\begin{aligned}&J(t,x;\Theta_1^*(\cdot)X(\cdot)+v_1^*(\cdot),\Theta_2^*(\cdot)X(\cdot)+v_2(\cdot))\\&\quad=J(t,x;\Theta^*(\cdot)X^*(\cdot)+v^*(\cdot))+\frac{1}{2}\mathbb{E}\int_t^T\langle(R_{22}+D_2^\top PD_2)(v_2-v_2^*),v_2-v_2^*\rangle\,ds.\end{aligned}$$

Hence,

$$\begin{aligned}J(t,x;\Theta^*(\cdot)X^*(\cdot)+v^*(\cdot))\geqslant J(t,x;\Theta_1^*(\cdot)X(\cdot)+v_1^*(\cdot),\Theta_2^*(\cdot)X(\cdot)+v_2(\cdot))\\ \forall v_2(\cdot)\in\mathcal{U}_2[t,T]\end{aligned}$$

if and only if

$$R_{22}+D_2^\top PD_2\leqslant 0\qquad\text{a.e. } s\in[t,T].$$

That is, $(\Theta^*(\cdot),v^*(\cdot))$ is a closed-loop saddle point of Problem (SG). This proves the sufficiency as well as (5.6). ∎

Since the convexity-concavity condition (4.3)–(4.4) is necessary for the existence of an open-loop saddle point but is not necessary for the existence of a closed-loop saddle point, we expect that there must be a case for which a closed-loop saddle point exists but no open-loop saddle point exists. We will see such an example in section 7 for Problem (SG). This then tells us that the open-loop and closed-loop saddle points are different.

Note that by letting $m_1=m$ (or equivalently, $m_2=0$), using the same arguments, we can obtain a characterization of closed-loop optimal strategy for Problem (SLQ). Unlike the case $0<m_1,m_2<m$, as we pointed out in section 3, the existence of a closed-loop optimal strategy implies the existence of open-loop optimal controls. Hence, the existence of a closed-loop optimal strategy implies the convexity condition (4.5)–(4.6). Consequently, such a feature really prevents Problem (SLQ) from being a special case of Problem (SG).

We point out that the solution of the Riccati equation (5.3) may be nonunique. Such an example will be presented in section 7. A solution $P(\cdot)$ of (5.3) satisfying (5.4)–(5.6) is called a *regular* solution of (5.3). The following result shows that the regular solution of (5.3) is unique.

COROLLARY 5.3. *Let* (SG1)–(SG2) *hold. Then the Riccati equation* (5.3) *admits at most one regular solution* $P(\cdot)\in C([t,T];\mathbb{S}^n)$.

Proof. Consider Problem $(\mathrm{SG})^0$. Suppose $P(\cdot)$ is a solution of Riccati equation (5.3) satisfying (5.4)–(5.6). Then the adapted solution $(\eta(\cdot),\zeta(\cdot))$ of (5.7) is $(\eta(\cdot),\zeta(\cdot))\equiv(0,0)$. By Theorem 5.2, we have

$$2V(t,x)=\langle\, P(t)x,x\,\rangle \qquad \forall x\in\mathbb{R}.$$

Now, if $\bar P(\cdot)$ is another solution of Riccati equation (5.3) satisfying (5.4)–(5.6), for the same reason, we have

$$2V(t,x)=\langle\, \bar P(t)x,x\,\rangle \qquad \forall x\in\mathbb{R}.$$

Hence, $P(t)=\bar P(t)$. By considering Problem $(\mathrm{SG})^0$ on $[s,T]$, $t<s<T$, we obtain

$$P(s)=\bar P(s) \qquad \forall s\in[t,T].$$

This proves our claim. $\square$

6. Linear FBSDEs and Riccati equations. We have seen that linear FBSDE (4.11)–(4.12) and Riccati equation (5.3)–(5.6) together with BSDE (5.7)–(5.9) have played central roles in characterization of the existence of an open-loop saddle point and a closed-loop saddle point of Problem (SG), respectively. Inspired by the four-step scheme introduced in [15] (see also [16]), in this section, we will establish a relation between the linear FBSDE and the Riccati equation. More precisely, we have the following result (for simplicity of notation, we will suppress the time variable s below).

THEOREM 6.1. *Let* (SG1)–(SG2) *hold and* $t\in[0,T)$. *Suppose that* $P(\cdot)\in C([t,T];\mathbb{S}^n)$ *is a solution to the Riccati equation*

$$\begin{cases}\dot P+PA+A^\top P+C^\top PC+Q\\ \quad-(PB+C^\top PD+S^\top)(R+D^\top PD)^\dagger(B^\top P+D^\top PC+S)=0\\ \hfill \text{a.e. } s\in[t,T],\\ P(T)=G,\end{cases}\tag{6.1}$$

such that

$$\begin{cases}\mathcal{R}(B^\top P+D^\top PC+S)\subseteq\mathcal{R}(R+D^\top PD) \qquad \text{a.e. } s\in[t,T],\\ (R+D^\top PD)^\dagger(B^\top P+D^\top PC+S)\in L^2(t,T;\mathbb{R}^{m\times n})\end{cases}\tag{6.2}$$

hold and that the adapted solution $(\eta(\cdot),\zeta(\cdot))$ *to the BSDE*

$$\begin{cases}d\eta(s)=-\Big\{\big[A^\top-(PB+C^\top PD+S^\top)(R+D^\top PD)^\dagger B^\top\big]\eta\\ \qquad\qquad +\big[C^\top-(PB+C^\top PD+S^\top)(R+D^\top PD)^\dagger D^\top\big]\zeta\\ \qquad\qquad +\big[C^\top-(PB+C^\top PD+S^\top)(R+D^\top PD)^\dagger D^\top\big]P\sigma\\ \qquad\qquad -(PB+C^\top PD+S^\top)(R+D^\top PD)^\dagger\rho+Pb+q\Big\}+\ \zeta dW(s),ds\\ \hfill s\in[t,T],\\ \eta(T)=g,\end{cases}\tag{6.3}$$

satisfies

$$\begin{cases}B^\top\eta+D^\top\zeta+D^\top P\sigma+\rho\in\mathcal{R}(R+D^\top PD) \qquad \text{a.e. } s\in[t,T],\ \text{a.s.},\\ (R+D^\top PD)^\dagger(B^\top\eta+D^\top\zeta+D^\top P\sigma+\rho)\in L^2_{\mathbb{F}}(t,T;\mathbb{R}^m).\end{cases}\tag{6.4}$$

Then for any $x \in \mathbb{R}^n$, there exists a 4-tuple of adapted processes $(X(\cdot), Y(\cdot), Z(\cdot), u(\cdot)) \in L^2_{\mathbb{F}}(\Omega; C([t,T];\mathbb{R}^n)) \times L^2_{\mathbb{F}}(\Omega; C([t,T];\mathbb{R}^n)) \times L^2_{\mathbb{F}}(t,T;\mathbb{R}^n) \times L^2_{\mathbb{F}}(t,T;\mathbb{R}^m)$ satisfying

$$
\begin{cases} dX(s) = (AX + Bu + b)ds + (CX + Du + \sigma)dW(s), \qquad s \in [t,T], \\ dY(s) = -(A^\top Y + C^\top Z + QX + S^\top u + q)ds + ZdW(s), \qquad s \in [t,T], \\ X(t) = x, \qquad Y(T) = GX(T) + g, \end{cases} \tag{6.5}
$$

and the following constraint holds:

$$
B^\top Y + D^\top Z + SX + Ru + \rho = 0 \qquad \text{a.e. } s \in [t,T], \text{ a.s.} \tag{6.6}
$$

Proof. Denote

$$
\begin{aligned}
\alpha = -\Big\{ & \big[A^\top - (PB + C^\top PD + S^\top)(R + D^\top PD)^\dagger B^\top\big]\eta \\
& + \big[C^\top - (PB + C^\top PD + S^\top)(R + D^\top PD)^\dagger D^\top\big]\zeta \\
& + \big[C^\top - (PB + C^\top PD + S^\top)(R + D^\top PD)^\dagger D^\top\big]P\sigma \\
& - (PB + C^\top PD + S^\top)(R + D^\top PD)^\dagger \rho + Pb + q\Big\}.
\end{aligned}
$$

Then

$$
\begin{cases} d\eta(s) = \alpha ds + \zeta dW(s), \qquad s \in [t,T], \\ \eta(T) = g, \end{cases}
$$

and

$$
\begin{aligned}
& \alpha + A^\top \eta + C^\top \zeta + C^\top P\sigma + Pb \\
& \quad = (PB + C^\top PD + S^\top)(R + D^\top PD)^\dagger (B^\top \eta + D^\top \zeta + D^\top P\sigma + \rho) - q.
\end{aligned}
$$

Let

$$
\begin{cases} \Theta = -(R + D^\top PD)^\dagger (B^\top P + D^\top PC + S) \in L^2(t,T;\mathbb{R}^{m\times n}), \\ v = -(R + D^\top PD)^\dagger (B^\top \eta + D^\top \zeta + D^\top P\sigma + \rho) \in L^2_{\mathbb{F}}(t,T;\mathbb{R}^m). \end{cases}
$$

Then for the given initial pair $(t,x) \in [0,T) \times \mathbb{R}^n$, the following SDE admits a unique strong solution $X(\cdot)$:

$$
\begin{cases} dX(s) = \big[(A + B\Theta)X + Bv + b\big]ds + \big[(C + D\Theta)X + Dv + \sigma\big]dW(s), \quad s \in [t,T], \\ X(t) = x. \end{cases}
$$

By defining

$$
u \triangleq \Theta X + v = -(R+D^\top PD)^\dagger \big[(B^\top P + D^\top PC + S)X + (B^\top \eta + D^\top \zeta + D^\top P\sigma + \rho)\big],
$$

we see that $(X(\cdot), u(\cdot))$ is a state-control pair. Since

$$
(B^\top P + D^\top PC + S)X + (B^\top \eta + D^\top \zeta + D^\top P\sigma + \rho) \in \mathcal{R}(R + D^\top PD),
$$

one has

$$
(R + D^\top PD)u + (B^\top P + D^\top PC + S)X + (B^\top \eta + D^\top \zeta + D^\top P\sigma + \rho) = 0.
$$

Now, set

$$Y = PX + \eta, \qquad Z = PCX + PDu + P\sigma + \zeta.$$

Then

$$Y(T) = P(T)X(T) + \eta(T) = GX(T) + g$$

and

$$\begin{aligned}
&B^\top Y + D^\top Z + SX + Ru + \rho \\
&\quad = B^\top(PX+\eta) + D^\top(PCX + PDu + P\sigma + \zeta) + SX + Ru + \rho \\
&\quad = (B^\top P + D^\top PC + S)X + (R + D^\top PD)u + B^\top\eta + D^\top\zeta + D^\top P\sigma + \rho = 0.
\end{aligned}$$

Thus, the constraint (6.6) holds. Also,

$$\begin{aligned}
dY &= \big[\dot PX + P(AX + Bu + b) + \alpha\big]ds + \big[P(CX + Du + \sigma) + \zeta\big]dW \\
&= \big\{\big[-A^\top P - C^\top PC - Q + (PB + C^\top PD + S^\top)(R + D^\top PD)^\dagger \\
&\qquad \cdot (B^\top P + D^\top PC + S)\big]X + PBu + Pb + \alpha\big\}ds + ZdW \\
&= \big[-A^\top(Y-\eta) - C^\top(Z - PDu - P\sigma - \zeta) - QX + PBu + Pb + \alpha \\
&\qquad + (PB + C^\top PD + S^\top)(R + D^\top PD)^\dagger(B^\top P + D^\top PC + S)X\big]ds + ZdW \\
&= \big[-A^\top Y - C^\top Z - QX + A^\top\eta + (PB + C^\top PD)u + C^\top P\sigma + C^\top\zeta + Pb + \alpha \\
&\qquad + (PB + C^\top PD + S^\top)(R + D^\top PD)^\dagger(B^\top P + D^\top PC + S)X\big]ds + ZdW \\
&= \big\{-A^\top Y - C^\top Z - QX - S^\top u - q \\
&\qquad + (PB + C^\top PD + S^\top)\big[u + (R + D^\top PD)^\dagger(B^\top\eta + D^\top\zeta + D^\top P\sigma + \rho) \\
&\qquad + (R + D^\top PD)^\dagger(B^\top P + D^\top PC + S)X\big]\big\}ds + ZdW \\
&= (-A^\top Y - C^\top Z - QX - S^\top u - q)ds + ZdW.
\end{aligned}$$

This proves our conclusion. □

From the above, we have the following corollary.

COROLLARY 6.2. *Let* (SG1)–(SG2) *hold. Suppose the convexity-concavity condition* (4.3)–(4.4) *holds and there exists a closed-loop saddle point for Problem* (SG) *on* $[t,T]$. *Then for any* $x \in \mathbb{R}^n$, *Problem* (SG) *admits an open-loop saddle point for* (t,x).

The proof follows immediately from Theorems 6.1 and 4.1.

We conjecture that if for any $(t,x) \in [0,T) \times \mathbb{R}^n$, there exists a unique 4-tuple of adapted processes $(X(\cdot), Y(\cdot), Z(\cdot), u(\cdot))$ satisfying FBSDE (6.5) with constraint (6.6), then the Riccati equation (6.1) admits a unique solution $P(\cdot)$ satisfying (6.2), and the BSDE (6.3) admits a unique adapted solution $(\eta(\cdot), \zeta(\cdot))$ satisfying (6.4). Such a result is true for Problem (DLQ) [25]. However, at the moment, we could not overcome some technical difficulties in proving such a result for Problem (SG).

7. Some examples. In this section, we present some examples. The first two examples are concerned with Problem (SLQ) and the rest are for Problem (SG).

The first example shows that the solvability of the Riccati differential equation is not sufficient enough for the existence of a closed-loop optimal strategy. So the regularity conditions (5.4)–(5.6) are necessary.

Example 7.1. Consider the following optimal control problem (one-player game):

$$\begin{cases} dX(s) = u(s)ds + u(s)dW(s), \qquad s \in [0,1], \\ X(0) = x, \end{cases}$$

with cost functional

$$J(x;u(\cdot)) = \frac{1}{2}\mathbb{E}\left[X(1)^2 + \int_0^1 \left(\frac{1}{2}s^3 - s^2\right) u(s)^2 ds\right].$$

In this example,

$$\begin{cases} A = 0, \quad B = 1, \quad b = 0, \quad C = 0, \quad D = 1, \quad \sigma = 0, \\ G = 1, \quad g = 0, \quad Q = 0, \quad S = 0, \quad R(s) = \dfrac{1}{2}s^3 - s^2, \quad q = 0, \quad \rho = 0. \end{cases}$$

The corresponding Riccati equation reads

$$\begin{cases} \dot{P}(s) = \dfrac{2P(s)^2}{s^3 - 2s^2 + 2P(s)} \qquad \text{a.e. } s \in [0,1], \\ P(1) = 1. \end{cases} \tag{7.1}$$

It is easy to see that $P(s) = s^2$ is the unique solution of (7.1), and

$$\begin{cases} B(s)^\top P(s) + D(s)^\top P(s)C(s) + S(s) = s^2, \\ R(s) + D(s)^\top P(s)D(s) = \dfrac{1}{2}s^3 \geqslant 0, \end{cases} \qquad s \in [0,1].$$

Thus, (5.4) holds. Now, if the problem has a closed-loop optimal strategy, then we should have

$$\Theta^*(s) = -\big[R(s) + P(s)\big]^{-1} P(s) = -\frac{2}{s}, \qquad s \in (0,1],$$

which is not in $L^2(0,1;\mathbb{R})$. This means that the problem does not have a closed-loop optimal strategy.

From the next example, we can see that the solution of the Riccati equation may be nonunique, and only the regular solution can be used to construct a closed-loop optimal strategy.

Example 7.2. Consider the following one-dimensional controlled system:

$$\begin{cases} dX(s) = \big[A(s)X(s) + B(s)u(s)\big]ds + u(s)dW(s), \qquad s \in [0,1], \\ X(0) = x, \end{cases}$$

with cost functional

$$J(x;u(\cdot)) = \frac{1}{2}\mathbb{E}\left\{-X(1)^2 + \int_0^1 \left[Q(s)X(s)^2 + R(s)u(s)^2\right] ds\right\},$$

where

$$A = \frac{1}{2}\left[\frac{(R-1)^2}{R^2} - 1\right], \quad B = \frac{R-1}{R}, \quad Q = -\frac{1}{R}, \quad R(s) = \left(s - \frac{3}{2}\right)^2 + \frac{3}{4} > 0.$$

The corresponding Riccati equation reads

$$\begin{cases} \dot{P} + 2AP + Q - \dfrac{B^2P^2}{R+P} = 0 \qquad \text{a.e. } s \in [0,1], \\ P(1) = -1. \end{cases} \tag{7.2}$$

Note that

$$B^2 - 2A = 1, \qquad Q + 2AR = -2, \qquad QR = -1.$$

Then,

$$\frac{B^2P^2}{R+P} - 2AP - Q = \frac{(B^2-2A)P^2 - (Q+2AR)P - QR}{R+P} = \frac{P^2+2P+1}{R+P},$$

and (7.2) becomes

$$\begin{cases} \dot{P}(s) = \dfrac{P(s)^2 + 2P(s) + 1}{R(s) + P(s)} \qquad \text{a.e. } s \in [0,1], \\ P(1) = -1, \end{cases} \tag{7.3}$$

which has two solutions:

$$P_1(s) = -1, \qquad s \in [0,1],$$

and

$$P_2(s) = s - 2, \qquad s \in [0,1].$$

We have

$$R(s) + P_1(s) = s^2 - 3s + 2 = (s-1)(s-2) \geqslant 0, \qquad s \in [0,1],$$

and

$$R(s) + P_2(s) = s^2 - 2s + 1 = (s-1)^2 \geqslant 0, \qquad s \in [0,1].$$

Now, we have

$$\begin{aligned} 2J(x;u(\cdot)) &= \mathbb{E}\left\{-X(1)^2 + \int_0^1 \big[Q(s)X(s)^2 + R(s)u(s)^2\big]ds\right\} \\ &= P(0)x^2 + \mathbb{E}\int_0^1 \Big\{\big[\dot{P}(s) + 2A(s)P(s) + Q(s)\big]X(s)^2 \\ &\qquad + 2P(s)B(s)X(s)u(s) + \big[R(s)+P(s)\big]u(s)^2\Big\}ds \\ &= P(0)x^2 + \mathbb{E}\int_0^1 \big[R(s)+P(s)\big]\left|u(s) + \frac{B(s)P(s)}{R(s)+P(s)}X(s)\right|^2 ds \\ &= P(0)x^2 + \mathbb{E}\int_0^1 \big[R(s)+P(s)\big]\left|u(s) + \frac{\big[R(s)-1\big]P(s)}{R(s)\big[R(s)+P(s)\big]}X(s)\right|^2 ds. \end{aligned}$$

Note that

$$\frac{(R-1)P_1}{R(R+P_1)} = \frac{(R-1)(-1)}{R(R-1)} = -\frac{1}{R}$$

and

$$\frac{(R-1)P_2}{R(R+P_2)} = \frac{(s^2-3s+2)(s-2)}{(s^2-3s+3)(s^2-2s+1)} = \frac{(s-2)^2}{(s^2-3s+3)(s-1)}.$$

Thus,

$$\begin{aligned} 2J(x;u(\cdot)) &= -x^2 + \mathbb{E}\int_0^1 (s-1)(s-2)\left|u(s) - \frac{X(s)}{(s-1)(s-2)+1}\right|^2 ds \\ &\geqslant -x^2 = 2J(x;u^*(\cdot)) \end{aligned}$$

with

$$u^*(s) = \frac{X(s)}{(s-1)(s-2)+1} \equiv \frac{X(s)}{R(s)} = -\frac{B(s)P_1(s)}{R(s)+P_1(s)}X(s), \qquad s\in[0,1],$$

which is an optimal control. The closed-loop system reads

$$\begin{cases} dX(s) = \left[\frac{1}{2}\left(\frac{(R-1)^2}{R^2} - 1\right) + \left(\frac{R-1}{R^2}\right)\right] X ds + \frac{1}{R} X dW(s), \qquad s\in[0,1], \\ X(0) = x, \end{cases}$$

which is well-posed. Thus, optimal control exists, but Riccati equation (7.3) has more than one solution.

On the other hand, by taking $P(s) = P_2(s) = s-2$, we have

$$J(x;u(\cdot)) = -x^2 + \frac{1}{2}\mathbb{E}\int_0^1 (s-1)^2 \left|u(s) + \frac{(s-2)^2}{(s^2-3s+3)(s-1)}X(s)\right|^2 ds.$$

If

$$\bar u(s) = -\frac{(s-2)^2}{(s^2-3s+3)(s-1)}X(s)$$

is an optimal control, the closed-loop system reads

$$\begin{cases} dX(s) = \left[\frac{1}{2}\left(\frac{(R-1)^2}{R^2} - 1\right) - \left(\frac{R-1}{R}\right)\frac{(s-2)^2}{(s^2-3s+3)(s-1)}\right] X ds \\ \qquad\qquad - \left[\frac{(s-2)^2}{(s^2-3s+3)(s-1)}\right] X dW(s), \qquad s\in[0,1], \\ X(0) = x, \end{cases}$$

which is *not* well-posed, since

$$\bar\Theta(s) \equiv -\frac{(s-2)^2}{(s^2-3s+3)(s-1)} \notin L^2(0,1;\mathbb{R}).$$

Thus, $\bar u(\cdot)$ cannot be an optimal control, a contradiction.

Concerning differential games, we present the following example, which shows that the existence of a closed-loop saddle point does not imply the existence of an open-loop saddle point. This gives a stochastic version of a similar example for the deterministic case given by Schmitendorf [21].

Example 7.3. Consider the one-dimensional state equation

$$\begin{cases} dX(s) = \big[u_1(s) - u_2(s)\big]ds + \big[u_1(s) - u_2(s)\big]dW(s), \qquad s\in[t,1], \\ X(t) = x, \end{cases} \tag{7.4}$$

and the performance functional:

$$J(t,x;u_1(\cdot),u_2(\cdot))=\frac{1}{2}\mathbb{E}\left\{X(1)^2+\int_t^1\big[u_1(s)^2-u_2(s)^2\big]ds\right\}.$$

The corresponding Riccati equation reads

$$\begin{cases}\dot P=P(1,-1)\begin{pmatrix}1+P & -P\\ -P & -1+P\end{pmatrix}^{-1}\begin{pmatrix}1\\-1\end{pmatrix}P=0, & s\in[t,1],\\ P(1)=1.\end{cases}$$

We can check that $P(s)\equiv 1$ is the unique solution. Since $R(s)+D(s)^\top P(s)D(s)=\left(\begin{smallmatrix}2 & -1\\ -1 & 0\end{smallmatrix}\right)$ is nonsingular, the range inclusion condition automatically holds. Also (noting $C(\cdot)=0$, $S(\cdot)=0$),

$$\big[R(s)+D(s)^\top P(s)D(s)\big]^\dagger B(s)^\top P(s)=\begin{pmatrix}2 & -1\\ -1 & 0\end{pmatrix}^{-1}\begin{pmatrix}1\\-1\end{pmatrix}=\begin{pmatrix}1\\1\end{pmatrix}\in L^2(t,1;\mathbb{R}^{2\times 1}),$$

$$R_{11}(s)+D_1(s)^\top P(s)D_1(s)=2>0,\qquad R_{22}(s)+D_2(s)^\top P(s)D_2(s)=0.$$

Hence, by Theorem 5.2, the game admits a unique closed-loop saddle point $(\Theta^*(\cdot), v^*(\cdot))$ given by the following:

$$\Theta^*(s)=-\big[R(s)+D(s)^\top P(s)D(s)\big]^{-1}B(s)^\top P(s)=-\begin{pmatrix}1\\1\end{pmatrix},\quad v^*(s)=\begin{pmatrix}0\\0\end{pmatrix}.$$

On the other hand, for any $u_1(\cdot)\in L^2_{\mathbb{F}}(t,1;\mathbb{R})$, taking $u_2(\cdot)=u_1(\cdot)-\lambda,\ \lambda\in\mathbb{R}$, the corresponding solution of (7.4) is given by

$$X(s)=x+\lambda(s-t)+\lambda\big(W(s)-W(t)\big),\qquad s\in[t,1].$$

Hence,

$$\begin{aligned}J(t,x;u_1(\cdot),u_2(\cdot))&=\frac{1}{2}\mathbb{E}\left\{X(1)^2+\int_t^1\big[u_1(s)^2-u_2(s)^2\big]ds\right\}\\&=\frac{1}{2}\left\{\big[x+\lambda(1-t)\big]^2+2\lambda\mathbb{E}\int_t^1u_1(s)ds\right\}.\end{aligned}\tag{7.5}$$

This leads to

$$\begin{aligned}V^+(t,x)&=\inf_{u_1(\cdot)\in L^2_{\mathbb{F}}(t,1;\mathbb{R})}\ \sup_{u_2(\cdot)\in L^2_{\mathbb{F}}(t,1;\mathbb{R})}J(t,x;u_1(\cdot),u_2(\cdot))\\&\geqslant\sup_{u_2(\cdot)\in L^2_{\mathbb{F}}(t,1;\mathbb{R})}J(t,x;0,u_2(\cdot))\geqslant\frac{1}{2}\big[x+\lambda(1-t)\big]^2\to\infty\qquad\text{as }\lambda\to\infty.\end{aligned}$$

So the open-loop saddle point does not exist. Note that for this example, from (7.5), we see that

$$J(t,0;0,u_2(\cdot))=\frac{1}{2}\lambda^2(1-t)^2\geqslant 0.$$

Hence, the convexity-concavity condition (4.3)–(4.4) fails.

The following example shows that the existence of an open-loop saddle point does not necessarily imply the existence of a closed-loop saddle point.

Example 7.4. Consider the following two-dimensional controlled state equation:

$$\begin{cases} d\begin{pmatrix} X_1(s) \\ X_2(s) \end{pmatrix} = \begin{pmatrix} u_1(s) \\ u_2(s) \end{pmatrix} ds, \qquad s \in [t,T], \\ \begin{pmatrix} X_1(t) \\ X_2(t) \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \end{cases}$$

with performance functional

$$J(t,x;u_1(\cdot),u_2(\cdot)) = \frac{1}{2}\mathbb{E}\big[X_1(T)^2 - X_2(T)^2\big].$$

Let $(t,x) \in [0,T) \times \mathbb{R}^2$ with $x = (x_1,x_2)^\top$. For any $\lambda_i \geqslant \frac{1}{T-t}$ $(i=1,2)$, define

$$u_i^{\lambda_i}(s) = -\lambda_i x_i \mathbf{1}_{[t,t+\frac{1}{\lambda_i}]}(s), \qquad s \in [t,T], \quad i=1,2.$$

Then, for any $(u_1(\cdot),u_2(\cdot)) \in L^2_{\mathbb{F}}(t,T;\mathbb{R}) \times L^2_{\mathbb{F}}(t,T;\mathbb{R})$, we have

$$J(t,x;u_1^{\lambda_1}(\cdot),u_2(\cdot)) \leqslant J(t,x;u_1^{\lambda_1}(\cdot),u_2^{\lambda_2}(\cdot)) = 0 \leqslant J(t,x;u_1(\cdot),u_2^{\lambda_2}(\cdot)).$$

Thus, $(u_1^{\lambda_1}(\cdot),u_2^{\lambda_2}(\cdot))$ is an open-loop saddle point. In the current case,

$$\begin{aligned} &A = C = D = Q = R = S = 0, \quad B = I, \quad G = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \\ &b = \sigma = q = \rho = g = 0. \end{aligned}$$

Hence, the Riccati equation reads

$$\begin{cases} \dot{P}(s) = 0 \qquad \text{a.e. } s \in [t,T], \\ P(T) = G, \end{cases}$$

whose solution is $P(s) \equiv G \neq 0$. Then the range condition

$$\mathcal{R}(P) \subseteq \mathcal{R}(R) = \{0\}$$

cannot be true. Consequently, there is no closed-loop saddle point for this Problem (SG).

Finally, we will present an example showing that the result of Zhang [26] on the equivalence of the existence of an open-loop saddle point and the finiteness of open-loop lower and upper value functions does not hold in general.

Example 7.5. Consider the one-dimensional state equation

$$\begin{cases} dX(s) = u_1(s)ds + u_2(s)dW(s), \qquad s \in [t,1], \\ X(t) = x, \end{cases} \tag{7.6}$$

and the performance functional:

$$J(t,x;u_1(\cdot),u_2(\cdot)) = \frac{1}{2}\mathbb{E}\int_t^1 \big[X(s)^2 - u_2(s)^2\big]ds. \tag{7.7}$$

The open-loop lower value function satisfies

$$\begin{aligned} V^-(t,x) &= \sup_{u_2(\cdot)\in L^2_{\mathbb{F}}(t,1;\mathbb{R})} \inf_{u_1(\cdot)\in L^2_{\mathbb{F}}(t,1;\mathbb{R})} J(t,x;u_1(\cdot),u_2(\cdot)) \\ &\geqslant \inf_{u_1(\cdot)\in L^2_{\mathbb{F}}(t,1;\mathbb{R})} J(t,x;u_1(\cdot),0) = \begin{cases} 0, & t<1, \\ \dfrac{1}{2}x^2, & t=1. \end{cases} \end{aligned}$$

On the other hand, for any $u_2(\cdot)\in L^2_{\mathbb{F}}(t,1;\mathbb{R})$ and $u_1(\cdot)=0$, one has

$$X(s) = x + \int_t^s u_2(r)dW(r), \qquad s\in[t,1].$$

Hence,

$$\mathbb{E}X(s)^2 = x^2 + \mathbb{E}\int_t^s u_2(r)^2 dr.$$

Consequently,

$$\begin{aligned} V^+(t,x) &= \inf_{u_1(\cdot)\in L^2_{\mathbb{F}}(t,1;\mathbb{R})} \sup_{u_2(\cdot)\in L^2_{\mathbb{F}}(t,1;\mathbb{R})} J(t,x;u_1(\cdot),u_2(\cdot)) \\ &\leqslant \sup_{u_2(\cdot)\in L^2_{\mathbb{F}}(t,1;\mathbb{R})} J(t,x;0,u_2(\cdot)) \\ &= \sup_{u_2(\cdot)\in L^2_{\mathbb{F}}(t,1;\mathbb{R})} \frac{1}{2}\mathbb{E}\int_t^1 \left(x^2 + \int_t^s u_2(r)^2 dr - u_2(s)^2\right) ds \\ &= \frac{1}{2}(1-t)x^2 + \sup_{u_2(\cdot)\in L^2_{\mathbb{F}}(t,1;\mathbb{R})} \frac{1}{2}\mathbb{E}\int_t^1 \big(-su_2(s)^2\big)ds = \frac{1}{2}(1-t)x^2. \end{aligned}$$

Thus, both the open-loop lower and upper value functions are finite. Now, suppose $(u_1^*(\cdot),u_2^*(\cdot))\in\mathcal{U}_1[t,1]\times\mathcal{U}_2[t,1]$ is an open-loop saddle point of the above problem for the initial pair $(t,x)\in[0,1)\times(\mathbb{R}\setminus\{0\})$; then by Theorem 4.1, we have

$$\begin{pmatrix}1\\0\end{pmatrix}Y^*(s) + \begin{pmatrix}0\\1\end{pmatrix}Z^*(s) + \begin{pmatrix}0 & 0\\ 0 & -1\end{pmatrix}\begin{pmatrix}u_1^*(s)\\u_2^*(s)\end{pmatrix} = 0 \qquad \text{a.e. } s\in[t,1], \text{ a.s.} \tag{7.8}$$

where $(X^*(\cdot),Y^*(\cdot),Z^*(\cdot))$ is the adapted solution of the following FBSDE:

$$\begin{cases} dX^*(s) = u_1^*(s)ds + u_2^*(s)dW(s), & s\in[t,1], \\ dY^*(s) = -X^*(s)ds + Z^*(s)dW(s), & s\in[t,1], \\ X^*(t)=x, \qquad Y^*(1)=0. \end{cases} \tag{7.9}$$

From (7.8), we have

$$Y^*(s)=0, \qquad Z^*(s)-u_2^*(s)=0 \qquad \text{a.e. } s\in[t,1], \text{ a.s.}$$

Hence, it is necessary that

$$\begin{cases} X^*(s)=Z^*(s)=0 & \text{a.e. } s\in[t,1], \text{ a.s.}, \\ u_1^*(s)=u_2^*(s)=0 & \text{a.e. } s\in[t,1], \text{ a.s.} \end{cases}$$

This leads to a contradiction since $X^*(t)=x\neq 0$. Therefore, the corresponding differential game does not have an open-loop saddle point for $(t,x)\in[0,1)\times(\mathbb{R}\setminus\{0\})$, although both open-loop lower and upper value functions are finite.

8. Concluding remarks. In this paper, we present characterizations of the existence (and uniqueness) of open-loop saddle points and closed-loop saddle points of linear quadratic two-person zero-sum stochastic differential games, respectively, in terms of the existence of an adapted solution to a linear FBSDE, and a differential Riccati equation, with certain regularity. There are some challenging problems still left open: (i) The solvability of the Riccati equation with pseudoinverse involved. We mention here that some relevant results can be found in [1] and [17], but more complete results are desirable. (ii) The solvability of the linear FBSDE (4.11) with a constraint (4.12) without the help of the Riccati equation. Note that due to the constraint (4.12), the FBSDE (4.11) is coupled. Some extension of the results found in [23, 24] might be helpful in studying such FBSDEs. (iii) We conjecture that under proper conditions, if for any initial pair $(t, x) \in [0, T) \times \mathbb{R}^n$, Problem (SG) admits a unique open-loop saddle point, then the game admits a closed-loop saddle point. Such a result holds for Problem (DLQ). However, at the moment, we still could not overcome some technical difficulties. (iv) The random coefficients case. This will lead to more involved issues, for example, the corresponding Riccati equation should be a BSDE, as indicated in [6, 7] for LQ stochastic optimal control problems with random coefficients. We hope to report some results relevant to the above-mentioned problems in our future publications.

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