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CONDUCTIVITY IMAGING FROM ONE INTERIOR MEASUREMENT IN THE PRESENCE OF PERFECTLY CONDUCTING AND INSULATING INCLUSIONS*

AMIR MORADIFAM[†], ADRIAN NACHMAN[‡], AND ALEXANDRU TAMASAN[§]

Abstract. We consider the problem of recovering an isotropic conductivity outside some perfectly conducting inclusions or insulating inclusions from the interior measurement of the magnitude of one current density field $|J|$. We show that the conductivity outside the inclusions and the shape and position of the inclusions are uniquely determined (except in an exceptional case) by the magnitude of the current generated by imposing a given boundary voltage. Our results show that even when the minimizer of the least gradient problem $\min \int_{\Omega} a|\nabla u|$ with $u|_{\partial\Omega} = f$ exhibits flat regions (i.e., regions with $\nabla u = 0$) it can be identified as the voltage potential of a conductivity problem with perfectly conducting inclusions.

Key words. conductivity imaging, current density impedance imaging, minimal surfaces, 1-Laplacian

AMS subject classifications. 35R30, 35J60, 31A25, 62P10

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1. Introduction. This paper considers the inverse problem of determining an isotropic electrical conductivity σ from one measurement of the magnitude of the current density field $|J|$ generated inside the domain Ω while imposing the voltage f at the boundary. Extending the existing work, the problem here allows for some perfectly conducting and insulating inclusions to be embedded in Ω away from the boundary. The domain $\Omega \subset \mathbb{R}^n$, $n \geq 2$, is assumed to be bounded, open, and with a connected Lipschitz boundary.

The problem considered in this paper is modeled by two physical principles: the Maxwell model of the electromagnetic field at very low frequency and a magnetic resonance technique to image current densities pioneered in [23] and [54]. Employment of dual physical models is a fairly new trend in quantitative imaging which seeks better accuracy and resolution of the reconstructed images, compared to the methods based on just one physical principle. For recent progress in such hybrid imaging methods in conductivity imaging we refer to [13], [3], [16], [2], [5], [7], [58], [28], and the review articles [6] and [47].

Inspired by [23] and [54], two subclasses of conductivity imaging methods have been developed: the ones which use interior knowledge of the current density field and the ones that use the measurement of only one component of the magnetic field, known as magnetic resonance electric impedance tomography. (See [49], [51], [30], [35], [57], [36], [37] for work in this direction.) The problem considered here belongs to the former subclass. The idea of using the current density field to image electri-

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cal conductivity appeared first in [59]. In [21] a perturbation method recovered the conductivity in the linearized case. Using the fact that J is normal to equipotential lines, the method in [31] recovered two-dimensional isotropic conductivities. In [26] the problem is reduced to the Neumann problem for the 1-Laplacian, and the examples of nonuniqueness and nonexistence for this degenerate elliptic problem show that knowledge of the applied current at the boundary together with the magnitude of current density field inside is insufficient data to determine the conductivity. Instead, the “ J -substitution” algorithm based on knowledge of the magnitude of two current density fields has been proposed; see also [25] and [27]. The idea of using two currents goes back to [53]; in [50] the problem is reduced to a first order system of PDEs, and several numerical reconstructions based on solving this system are proposed. In independent work in [24] and, respectively [32], a simple formula recovers $\nabla \ln(\sigma)$ at each point in a region where two transversal current density vectors have been measured; see also [20] for careful experimental validation of this formula.

In [44] a reconstruction method which uses the interior knowledge of the magnitude of just one current density field $|J|$ has been proposed. This method relies on the fact that in the absence of singularities, equipotential sets are minimal surfaces in the metric $g = |J|^{2/(n-1)}I$ conformal to the Euclidean metric. In [46] it is shown that the equipotential surfaces are minimizers for the area functional

$$(1) \quad \mathcal{A}(\Sigma) = \int_{\Sigma} |J| dS,$$

where dS is the induced Euclidean surface measure. (Note that $\mathcal{A}(\Sigma)$ is the area of Σ in the Riemannian metric g described above.) Moreover, in [45] it is shown that the voltage potential u is a minimizer of the functional

$$(2) \quad \int_{\Omega} |J| \cdot |\nabla v| dx,$$

subject to $v \in W^{1,1}(\Omega)$ with $v = f$ at the boundary $\partial\Omega$, and that u is the unique minimizer among $v \in W^{1,1}(\Omega)$ with $|\nabla v| > 0$ a.e. in Ω and $v = f$ at the boundary. One can determine u and hence σ by a minimization algorithm. A structural stability result for the minimization of the functional in (2) can be found in [48]. Recently in [43] authors presented a convergent algorithm for finding the unique minimizer of (2). Formally, the Euler–Lagrange equation for the nonsmooth functional in (2) is the generalized 1-Laplacian. This is in contrast with the work in [3], [2], and [16], where the conductivity imaging from interior data leads to the generalized 0-Laplacian.

Partial reconstruction from incomplete data results are available for planar domains [46]: if $|J|$ is known throughout Ω but f is only known on parts of the boundary. More precisely, if some interval (α, β) of boundary voltages is twice contained in the known values of f , then one can recover the conductivity in the subregion

$$(3) \quad \Omega_{\alpha,\beta} := \{x \in \bar{\Omega} : \alpha < u(x) < \beta\}.$$

In fact $|J|$ need only be known in a subregion $\tilde{\Omega}$ which contains regions of type (3) for unknown values α 's and β 's. The method in [46] determines from the data if $\tilde{\Omega}$ contains regions of type (3) and if so recovers all the (maximal) intervals (α, β) , their corresponding $\Omega_{\alpha,\beta}$, and the conductivity therein.

In this paper we are interested in imaging an isotropic conductivity σ from the magnitude of one current density field in the presence of perfectly conducting and

insulating inclusions. Even though Ohm’s law is not valid in the classical sense inside perfectly conducting regions, we show that the conductivity outside the inclusions and the shape and position of the inclusions are uniquely determined (see Remark 2.2) by the magnitude of the current generated by imposing a given boundary voltage. We also establish a connection between the above problem and the uniqueness of the minimizers of the weighted least gradient problem $\min \int_{\Omega} a|\nabla u|dx$ with $u|_{\partial\Omega}$. Such minimizers often exhibit flat regions (i.e., regions where they are constant). We have found a new admissibility condition that allows such minimizers to be viewed as voltage potential of conductivity problems with perfectly conducting inclusions. Unlike the existing results (e.g., [44], [45], and [46]) we allow for $|J| \equiv 0$ and $|\nabla u| \equiv 0$ in open subsets of Ω . Such situations occur when insulating and perfectly conducting inclusions are embedded inside Ω . In dimensions $n \geq 3$ we allow any finite number of open connected insulating inclusions, but in dimension $n = 2$ we allow only one open connected insulating inclusion. In the following section we present and discuss our main results.

2. Main results. Let U be an open subset of Ω with $\overline{U} \subset \Omega$ to model the perfectly conducting inclusions, V be an open subset of Ω with $\overline{V} \subset \Omega$ to model the insulating inclusions, and χ_U and χ_V be their corresponding characteristic functions. We assume $\overline{U} \cap \overline{V} = \emptyset$, $\Omega \setminus \overline{U \cup V}$ is connected and the boundaries $\partial U, \partial V$ are piecewise $C^{1,\alpha}$. Let $\sigma_1 \in L^\infty(U)$ and $\sigma \in L^\infty(\Omega \setminus \overline{U \cup V})$ be bounded away from zero. For $k > 0$ consider the conductivity problem

$$(4) \quad \begin{cases} \nabla \cdot ((\chi_U(k\sigma_1 - \sigma) + \sigma)\nabla u) = 0 & \text{in } \Omega \setminus \overline{V}, \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial V, \\ u|_{\partial\Omega} = f. \end{cases}$$

The perfectly conducting inclusions occur in the limiting case $k \rightarrow \infty$. The limiting solution is the unique solution to the problem:

$$(5) \quad \begin{cases} \nabla \cdot \sigma \nabla u_0 = 0 & \text{in } \Omega \setminus \overline{U \cup V}, \\ \nabla u_0 = 0 & \text{in } U, \\ u_0|_+ = u_0|_- & \text{on } \partial(U \cup V), \\ \int_{\partial U_j} \sigma \frac{\partial u_0}{\partial \nu} |_+ ds = 0 & j = 1, 2, \dots, \\ \frac{\partial u_0}{\partial \nu} |_+ = 0 & \text{on } \partial V, \\ u_0|_{\partial\Omega} = f \end{cases}$$

(see the appendix for more details), where $U = \cup_{j=1}^\infty U_j$ is a partition of U into connected components.

For Lipschitz continuous conductivities in any dimension $n \geq 2$ or for essentially bounded conductivities in two dimensions, the solutions of the conductivity equation satisfy the unique continuation property (see [9] and references therein). Consequently the insulated (and possibly perfectly conducting) inclusions are the only open sets on which the interior data $|J|$ vanishes identically. However, in three dimensions or higher it is possible to have a Hölder continuous σ and boundary data f that yield $u \equiv \text{constant}$ in a proper open subset $W \subsetneq \Omega$; see [52], [41]. We call such regions W *singular inclusions*. On the other hand Ohm’s law does not hold in the classical sense inside perfect conductors: the current J inside perfectly conducting inclusions U is not necessarily zero while $\nabla u \equiv 0$ in U [4], [34].

The measured data for our inverse problem is the nonnegative function $a = |J(x)|$ in Ω , the magnitude of the current density field J induced by imposing a voltage f at the boundary $\partial\Omega$. We have $\nabla \cdot J = 0$. In the perfectly conducting inclusion U we will not rely on the Ohm's law; we will use condition (6) and the transmission condition $J_- \cdot \nu = J_+ \cdot \nu$ across the boundary of ∂U (see the appendix), where $J_- = J|_U$ and $J_+ = J|_{U^c}$. Indeed we have found an extension of the notion of admissibility of [45] which will be crucial in allowing us to treat the case of perfectly conducting and insulating inclusions considered here. In a different direction, this also makes it possible to extend results on uniqueness of minimizers of weighted least gradient problems as discussed later in this section.

To formulate our results, we first need to introduce a notion of admissibility.

DEFINITION 1. A pair of functions $(f, a) \in H^{1/2}(\partial\Omega) \times L^2(\Omega)$ is called admissible if the following conditions hold:

(i) There exist two disjoint open sets $U, V \subset \Omega$ (possibly empty) and a function $\sigma \in L^\infty(\Omega \setminus (U \cup V))$ bounded away from zero such that $\Omega \setminus (\overline{U \cup V})$ is connected, $\overline{U \cup V} \cap \partial\Omega = \emptyset$, and

$$\begin{cases} a = |\sigma \nabla u_\sigma| & \text{in } \Omega \setminus (\overline{U \cup V}), \\ a = 0 & \text{in } V, \end{cases}$$

where $u_\sigma \in H^1(\Omega)$ is the weak solution of (5).

(ii) The following holds:

$$(6) \quad \inf_{u \in W^{1,1}(U)} \left(\int_U a |\nabla u| dx - \int_{\partial U} \sigma \frac{\partial u_\sigma}{\partial \nu} \Big|_+ u ds \right) = 0,$$

where ν is the unit normal vector field on ∂U pointing outside U .

(iii) The set of zeroes of the function a outside \overline{U} can be partitioned as

$$(7) \quad \{x \in \Omega : a(x) = 0\} \cap (\Omega \setminus \overline{U}) = V \cup \overline{W} \cup \Gamma,$$

where W is an open set (possibly empty), Γ is a Lebesgue-negligible set, and \overline{W} has an empty interior.

We call σ a generating conductivity and u_σ the corresponding potential.

Since for $u = \text{constant}$,

$$\int_{U_j} a |\nabla u| dx - \int_{\partial U_j} \sigma \frac{\partial u_\sigma}{\partial \nu} \Big|_+ u ds = 0,$$

we have

$$\inf_{u \in W^{1,1}(U_j)} \left(\int_{U_j} a |\nabla u| dx - \int_{\partial U_j} \sigma \frac{\partial u_\sigma}{\partial \nu} \Big|_+ u ds \right) \leq 0.$$

Hence condition (6) holds if and only if

$$\inf_{u \in W^{1,1}(U_j)} \left(\int_{U_j} a |\nabla u| dx - \int_{\partial U_j} \sigma \frac{\partial u_\sigma}{\partial \nu} \Big|_+ u ds \right) = 0$$

for all connected components U_j of U .

We first note that any physical data (f, a) naturally satisfies the first two conditions (i) and (ii) in the above definition. Indeed if $a = |J|$ where $\nabla \cdot J = 0$ in Ω , then for any $u \in W^{1,1}(U)$ we have

$$\begin{aligned} \int_U a|\nabla u|dx - \int_{\partial U} \sigma \frac{\partial u_\sigma}{\partial \nu} u ds &= \int_U |J||\nabla u|dx - \int_{\partial U} \sigma \frac{\partial u_\sigma}{\partial \nu} u ds \\ &\geq \int_U J \cdot \nabla u dx - \int_{\partial U} \sigma \frac{\partial u_\sigma}{\partial \nu} u ds \\ &= \int_{\partial U} J \cdot \nu u dx - \int_{\partial U} \sigma \frac{\partial u_\sigma}{\partial \nu} u ds \\ &= \int_{\partial U} J \cdot \nu u ds - \int_{\partial U} J \cdot \nu u ds = 0. \end{aligned}$$

Also by the fourth equation in (5)

$$\int_U a|\nabla u|dx - \int_{\partial U} \sigma \frac{\partial u_\sigma}{\partial \nu} u ds = 0$$

for any constant function u in U . Hence (ii) holds for physical data (f, a) . The first condition (i) also obviously holds for physical data (f, a) . We have added condition (iii) for technical reasons. Even though it is not always satisfied, this condition is very general, at least for physical applications.

On the other hand if

$$\int_U \sigma \frac{\partial u_\sigma}{\partial \nu} dx \neq 0$$

then

$$E(u) = \int_U a|\nabla u|dx - \int_{\partial U} \sigma \frac{\partial u_\sigma}{\partial \nu} u ds$$

is not invariant under adding or subtracting constant and therefore

$$\inf_{u \in W^{1,1}(U)} \left(\int_U a|\nabla u|dx - \int_{\partial U} \sigma \frac{\partial u_\sigma}{\partial \nu} u ds \right) = \infty.$$

Thus we have the following proposition about condition (6).

PROPOSITION 2.1. *Let $a \in L^\infty(\Omega)$ and U be an open subset of Ω .*

- *If $a \geq |J|$ in U for some J with $\nabla \cdot J \equiv 0$ in U and $J_- = \sigma \frac{\partial u_\sigma}{\partial \nu}|_+$ on ∂U , then condition (6) in Definition 1 holds.*
- *If condition (6) in Definition 1 holds, then*

$$\int_U \sigma \frac{\partial u_\sigma}{\partial \nu} ds = 0.$$

DEFINITION 2. *We say that an open set $V \subset \mathbb{R}^n$, $n \geq 3$, is proper if V has a finite number of disjoint open connected components $\{V_1, V_2, \dots, V_m\}$ such that $\bar{V}_j \cap \bar{V}_i = \emptyset$ if $i \neq j$ and ∂V_i is a connected C^1 manifold homomorphic to S^{n-1} for all $1 \leq i \leq m$. An open set $V \subset \mathbb{R}^2$ is called proper if it is connected and ∂V is a connected C^1 manifold homomorphic to S^1 .*

We can now state one of our main uniqueness results.

THEOREM 2.1. *Let $\Omega \subset \mathbb{R}^n$, $n \geq 2$, be a domain with connected Lipschitz boundary and let $(f, |J|) \in C^{1,\alpha}(\partial\Omega) \times L^2(\Omega)$ be an admissible pair generated by some unknown conductivity $\sigma \in C^\alpha(\Omega \setminus (\overline{U \cup V}))$, where U and V are open sets as described in Definition 1. In addition assume V is proper. Then the potential u_σ is a minimizer of the problem*

$$(8) \quad u = \operatorname{argmin} \left\{ \int_{\Omega} |J| |\nabla v| dx : v \in W^{1,1}(\Omega) \cap C(\overline{\Omega}), \quad v|_{\partial\Omega} = f \right\},$$

and if u is another minimizer of the above problem, then $u = u_\sigma$ in

$$\Omega \setminus \{x \in \Omega : |J| = 0\}.$$

Moreover the set of zeros of $|J|$ and $|\nabla u_\sigma|$ can be decomposed as

$$\{x \in \Omega : |J| = 0\} \cup \{x \in \Omega : \nabla u_\sigma = 0\} =: Z \cup \Gamma,$$

where Z is an open set, Γ has measure zero, and

$$Z = U \cup V \cup W.$$

Consequently $\sigma = \frac{|J|}{|\nabla u_\sigma|} \in L^\infty(\Omega \setminus \overline{Z})$ is the unique $C^\alpha(\Omega \setminus \overline{Z})$ -conductivity outside Z for which $|J|$ is the magnitude of the current density corresponding to the voltage f at the boundary.

Remark 2.2. The above theorem allows us to identify the potential $u = u_\sigma$ and the conductivity σ outside the open set $Z = U \cup V \cup W$. There are a number of ways to determine if an open connected component O of Z is a perfectly conducting inclusion, an insulating inclusion, or a singular inclusion:

- If $\nabla u \equiv 0$ in O and $|J|(x) \neq 0$ for some $x \in O$, then O is a perfectly conducting inclusion.
- If $|J| \equiv 0$ in O and $u \not\equiv \text{constant}$ on ∂O , then O is an insulating inclusion.
- If $J \equiv 0$ in O , $u = \text{constant}$ on ∂O , and J is not C^α at x for some $x \in O$, then O is either an insulating inclusion or a perfectly conducting inclusion.
- If $J \equiv 0$, $u = \text{constant}$ on ∂O , and $J \in C^\alpha(\partial O)$, then the knowledge of the magnitude of the current $|J|$ (and even the full vector field J) is not enough to determine the type of the inclusion O .

Remark 2.3. One can compare the forward problem (5) with the minimization problem (8) to see that the second, third, fourth, and fifth condition in the forward problem (5) do not appear in the problem (8). This means that all of the information about the location and shape of the inclusions is encoded in $|J|$.

Now we introduce an interesting connection between Theorem 2.1 and the uniqueness of minimizers of weighted least gradient problems. Indeed, Theorem 2.1 can also be applied independently to prove uniqueness of the minimizers of the weighted least gradient problem

$$(9) \quad u_0 = \operatorname{argmin} \left\{ \int_{\Omega} a |\nabla u| dx, \quad u \in W^{1,1}(\Omega), \quad \text{and} \quad u|_{\partial\Omega} = f \right\}$$

in situations where the minimizer has flat regions (is constant on open sets).

Example 2.4. For instance, consider the following example [55]. Let $D = \{x \in \mathbb{R}^2 : x^2 + y^2 < 1\}$ be the unit disk and $f(x, y) = x^2 - y^2$. Consider the problem

$$(10) \quad u_0 = \operatorname{argmin} \left\{ \int_D |\nabla u| dx, \quad u \in W^{1,1}(D), \quad \text{and } u|_{\partial D} = f \right\},$$

which corresponds to $a \equiv |J| \equiv 1$ in D . We claim that $(1, x^2 - y^2)$ is an admissible pair according to Definition 1. To prove our claim we let $U = (-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}) \times (-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$ and $V = \emptyset$. Define

$$\sigma = \begin{cases} \frac{1}{4|x|} & \text{if } |x| \geq \frac{1}{\sqrt{2}}, \quad |y| \leq \frac{1}{\sqrt{2}}, \\ \frac{1}{4|y|} & \text{if } |x| \leq \frac{1}{\sqrt{2}}, \quad |y| \geq \frac{1}{\sqrt{2}}, \end{cases}$$

and

$$u_\sigma = \begin{cases} 2x^2 - 1 & \text{if } |x| \geq \frac{1}{\sqrt{2}}, \quad |y| \leq \frac{1}{\sqrt{2}}, \\ 0 & \text{if } (x, y) \in U, \\ 1 - 2y^2 & \text{if } |x| \leq \frac{1}{\sqrt{2}}, \quad |y| \geq \frac{1}{\sqrt{2}}. \end{cases}$$

It is easy to see that u_σ is the solution of (5) and $|J| \equiv 1 \equiv \sigma|\nabla u_\sigma|$ on $\Omega \setminus \overline{U}$. Hence (i) holds in the definition of admissibility, Definition 1. Condition (iii) also obviously holds. It remains to be shown that (6) holds. Define the vector field $J(x, y)$ in U as

$$J(x, y) = \begin{cases} -j & \text{if } y \geq |x|, \\ j & \text{if } -y \geq |x|, \\ i & \text{if } x > |y|, \\ -i & \text{if } -x > |y|. \end{cases}$$

Let

$$U_0 = \{(x, y) \in U \mid |x| \neq |y|\} = T_1 \cup T_2 \cup T_3 \cup T_4,$$

where $T_i, 1 \leq i \leq 4$, are the four disjoint triangles in Figure 1. Then $|J| = 1$ in U , $J \in C^\infty(U_0)$, and we have

$$\begin{aligned} \int_U |\nabla u| dx - \int_{\partial U} \sigma \frac{\partial u_\sigma}{\partial \nu} u ds &\geq \int_{U_0} |J| |\nabla u| dx - \int_{\partial U} \sigma \frac{\partial u_\sigma}{\partial \nu} u ds \\ &\geq \int_{U_0} J \cdot \nabla u dx - \int_{\partial U} \sigma \frac{\partial u_\sigma}{\partial \nu} u ds \\ &= \sum_{i=1}^4 \int_{T_i} J \cdot \nabla u dx - \int_{\partial U} \sigma \frac{\partial u_\sigma}{\partial \nu} u ds \\ &= \int_{\partial U} J \cdot \nu u ds - \int_{\partial U} \sigma \frac{\partial u_\sigma}{\partial \nu} u ds \\ &= 0, \end{aligned}$$

since $J \cdot \nu \equiv \sigma \frac{\partial u_\sigma}{\partial \nu}$ on ∂U . Thus condition (6) holds and $(1, x^2 - y^2)$ is admissible in the sense of Definition 1. It follows from Theorem 2.1 that u_σ is the unique minimizer of problem (10).

The following theorem shows that the equipotential sets contained entirely outside the conductive inclusions are area minimizers. We describe a surface as the level set

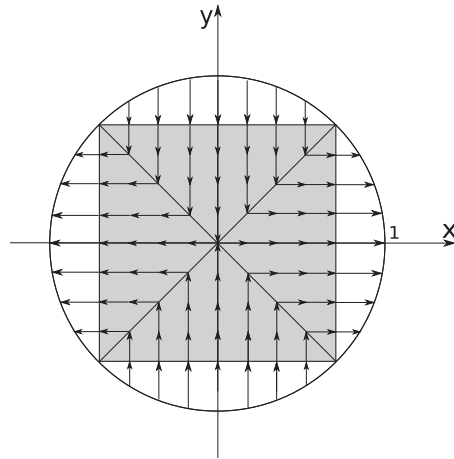


FIG. 1. Current density vector field for Example 2.4.

of a regular map u , while competitors are described by level sets of some compact perturbations of the regular map u .

THEOREM 2.5 (minimizing property of level sets). *Let $\Omega \subset \mathbb{R}^n$, $n \geq 2$, be a domain with connected Lipschitz boundary and let $(f, |J|) \in C^2(\partial\Omega) \times L^2(\overline{\Omega})$ be an admissible pair generated by some unknown C^1 conductivity. Then for every $v \in C^2(\overline{\Omega})$ with $v = f$ on $\partial\Omega$ such that*

$$\{x : |\nabla v| = 0\} = Z_v \cap L_v, \quad a(\overline{Z_v}) = \{0\},$$

where Z_v is open and L_v has Lebesgue measure zero, we have

$$(11) \quad \mathcal{A}(u^{-1}(\lambda)) \leq \mathcal{A}(v^{-1}(\lambda))$$

for a.e. $\lambda \in \mathbb{R}$, where \mathcal{A} is defined as (1).

The partial data result [46, Theorem 3.4] also recovers the conductivity in two-dimensional subregions of type (3) assuming that $|J| > 0$ almost everywhere. Below we show that under the assumption the full vector field J is known (not just its magnitude $|J|$), the partial reconstruction result is valid in three or higher dimensions. The result below can be viewed as the extension of the results in [31] to three or higher dimensional models.

THEOREM 2.6 (partial determination). *Let $\Omega \subset \mathbb{R}^n$ ($n \geq 2$) be simply connected. For $i = 1, 2$, let $\sigma^i \in C^\alpha(\Omega \setminus \overline{U^i \cup V^i})$ be bounded away from zero and u_i satisfy (5), where U^i and V^i are open sets of Ω , and let*

$$J_i = \begin{cases} \sigma^i \nabla u_i & \text{in } \Omega \setminus (U^i \cup V^i), \\ 0 & \text{in } V^i. \end{cases}$$

For $\alpha < \beta$ let

$$(12) \quad \Omega_{\alpha, \beta} := \{x \in \overline{\Omega} : \alpha < u_1 < \beta\} \quad \text{and} \quad \Gamma := \Omega_{\alpha, \beta} \cap \partial\Omega.$$

In addition assume that V is proper and

$$\{x \in \Omega \setminus \overline{U_1} : |J_1(x)| = 0\} = V^1 \cup W^1 \cup \Gamma^1,$$

where W^1 is open and Γ^1 has Lebesgue measure zero. Then the following hold:

1. If $u_1|_\Gamma = u_2|_\Gamma$ and $J_1 = J_2$ in Ω , then $U^1 \cap \Omega_{\alpha,\beta} = U^2 \cap \Omega_{\alpha,\beta}$, $(W^1 \cup V^1) \cap \Omega_{\alpha,\beta} = (W^2 \cup V^2) \cap \Omega_{\alpha,\beta}$

$$u_1 = u_2 \text{ in } \Omega_{\alpha,\beta} \setminus V^1 \text{ and } \sigma^1 = \sigma^2 \text{ in } \Omega_{\alpha,\beta} \setminus \overline{U^1 \cup V^1 \cup W^1}.$$

2. If $u_1|_\Gamma = u_2|_\Gamma$ and $J_1 = J_2$ in $\Omega_{\alpha,\beta}$, then

$$(13) \quad \{x \in \overline{\Omega} : \alpha < u_2(x) < \beta\} = \Omega_{\alpha,\beta},$$

$$U^1 \cap \Omega_{\alpha,\beta} = U^2 \cap \Omega_{\alpha,\beta}, (W^1 \cup V^1) \cap \Omega_{\alpha,\beta} = (W^2 \cup V^2) \cap \Omega_{\alpha,\beta}, \text{ and}$$

$$u_1 = u_2 \text{ in } \Omega_{\alpha,\beta} \setminus V^1 \text{ and } \sigma^1 = \sigma^2 \text{ in } \Omega_{\alpha,\beta} \setminus \overline{U^1 \cup V^1 \cup W^1}.$$

Similarly to Theorem 2.1 we may determine if an open connected component O of $U^1 \cup V^1 \cup W^1 = U^2 \cup V^2 \cup W^2$ is a perfectly conducting, insulating, or singular inclusion (see Remark 2.2).

3. Unique determination of the conductivity. In this section we prove Theorems 2.1 and 2.6. The arguments extend those in [45] and [46] by replacing the new admissibility condition. We start with the following proposition.

PROPOSITION 3.1. *Let $\Omega \subset \mathbb{R}^n$, $n \geq 2$, be a domain and $(f, |J|) \in H^{1/2}(\partial\Omega) \times L^2(\Omega)$. Then the following hold:*

1. Assume $(f, |J|)$ is admissible, say generated by some conductivity $\sigma \in L^\infty(\Omega \setminus (\overline{U \cup V}))$, where U and V are described in Definition 1 and u_0 is the corresponding voltage potential. Then u_0 is a minimizer for $F(u)$ in (2) over

$$(14) \quad A := \{u \in H^1(\Omega) : u|_\Omega = f\}.$$

Moreover, if $f \in C^{1,\alpha}(\partial\Omega)$ and if the generating conductivity $\sigma \in C^\alpha(\Omega \setminus \overline{U \cup V})$, then the corresponding potential $u_0 \in C^{1,\alpha}(\Omega \setminus \overline{U \cup V})$ is a minimizer of $F(u)$ over A .

2. Assume that the set of zeros of $a = |J|$ can be decomposed as

$$\{x \in \Omega : a(x) = 0\} = V \cup \Gamma_1,$$

where V is an open set and Γ_1 has measure zero. Suppose u_0 is a minimizer for $F(u)$ in (2) over A and the set of zeroes of $|\nabla u_0|$ can be decomposed as

$$\{x \in \Omega \setminus V : |\nabla u_0| = 0\} = \overline{U} \cup \Gamma_2,$$

where U is an open set, $\overline{U \cup V} \subset \Omega$, and Γ_2 has measure zero. If $U \cap V = \emptyset$ and $|J|/|\nabla u_0| \in L^\infty(\Omega \setminus (\overline{U \cup V}))$, then $(f, |J|)$ is admissible.

Proof. Assume $(f, |J|)$ is admissible and generated by some conductivity $\sigma \in L^\infty_+(\Omega \setminus (U \cup V))$. For any $u \in A$ we have

$$\begin{aligned} F(u) &= \int_{\Omega \setminus (\overline{U \cup V})} \sigma |\nabla u_0| |\nabla u| dx + \int_U |J| |\nabla u| dx \\ &\geq \int_{\Omega \setminus (\overline{U \cup V})} \sigma \nabla u_0 \cdot \nabla u dx + \int_U |J| |\nabla u| dx \\ &= \int_{\partial\Omega} \sigma \frac{\partial u_0}{\partial \nu} u ds - \int_{\partial V} \sigma \frac{\partial u_0}{\partial \nu} u ds - \int_{\partial U} \sigma \frac{\partial u_0}{\partial \nu} u ds + \int_U |J| |\nabla u| dx \\ &= \int_{\partial\Omega} \sigma \frac{\partial u_0}{\partial \nu} u ds - \int_{\partial U} \sigma \frac{\partial u_0}{\partial \nu} u ds + \int_U |J| |\nabla u| dx \\ &\geq \int_{\partial\Omega} \sigma \frac{\partial u_0}{\partial \nu} f ds = F(u_0), \end{aligned}$$

where we have used the admissibility condition (6) and ν is the outer normal to the boundary of Ω , U , and V . Hence u_0 is a minimizer of $F(u)$.

To prove (2) we note that by the Lebesgue dominated convergence theorem, the functional F is Gateaux-differentiable at $u \in H^1(\Omega)$ with $\frac{|J|}{|\nabla u|} \in L_+^\infty(\Omega \setminus (\overline{U \cup V}))$. Since

$$F(u_0) = \int_{\Omega} |J| |\nabla u_0| dx = \int_{\Omega \setminus \overline{U \cup V}} |J| |\nabla u_0| dx,$$

at a minimizer u_0 we have

$$F'(u_0)(\varphi) = \int_{\Omega \setminus \overline{U \cup V}} \frac{|J|}{|\nabla u_0|} \nabla u_0 \cdot \nabla \varphi dx = 0$$

for all $\varphi \in W_0^{1,1}(\Omega \setminus \overline{U})$. Now let $\sigma = \frac{|J|}{|\nabla u_0|}$; then $\nabla \cdot (\sigma \nabla u_0) = 0$ in $\Omega \setminus \overline{V \cup U}$. On the other hand we have

$$\int_{\Omega \setminus \overline{U \cup V}} \frac{|J|}{|\nabla u_0|} \nabla u_0 \cdot \nabla \varphi dx = \int_{\partial(U \cup V)} \sigma \frac{\partial u_0}{\partial \nu} \varphi ds = \int_{\partial V} \sigma \frac{\partial u_0}{\partial \nu} \varphi ds = 0$$

for all $\varphi \in W_0^{1,1}(\Omega \setminus \overline{U})$. Therefore $\frac{\partial u_0}{\partial \nu} = 0$ on ∂V . Now let O be a connected component of U . Then for all $\varphi \in W_0^{1,1}(\Omega \setminus \overline{U \setminus O})$ with $\varphi \equiv 1$ in O we have

$$\int_{\Omega \setminus \overline{U \cup V}} \frac{|J|}{|\nabla u|} \nabla u_0 \cdot \nabla \varphi dx = \int_{\partial(U \cup V)} \sigma \frac{\partial u_0}{\partial \nu} \varphi ds = \int_{\partial O} \sigma \frac{\partial u_0}{\partial \nu} ds = 0.$$

This implies that u_0 is a solution of (5). (See the appendix for more details.)

Moreover for every $u \in W_0^{1,1}(\Omega)$ with $u|_{\partial\Omega} = f$

$$\begin{aligned} \int_{\Omega} |J| |\nabla u_0| dx &\leq \int_{\Omega \setminus \overline{V}} |J| |\nabla u| dx \\ &= \int_U |J| |\nabla u| dx + \int_{\Omega \setminus \overline{U \cup V}} |J| |\nabla u| dx \\ &= \int_U |J| |\nabla u| dx + \int_{\Omega \setminus \overline{U \cup V}} \sigma |\nabla u_0| |\nabla u| dx \\ &= \int_U |J| |\nabla u| dx + \int_{\Omega \setminus \overline{U \cup V}} \sigma \nabla u_0 \cdot \nabla u dx \\ &= \int_U |J| |\nabla u| dx - \int_{\partial U} \sigma \frac{\partial u_0}{\partial \nu} u ds + \int_{\partial\Omega} \sigma \frac{\partial u_0}{\partial \nu} f ds. \end{aligned}$$

Since

$$\int_{\Omega} |J| |\nabla u_0| dx = \int_{\partial\Omega} \sigma \frac{\partial u_0}{\partial \nu} f ds,$$

the admissibility condition (6) follows from the above inequality. Thus $(|J|, f)$ is an admissible pair. \square

Now we are ready to prove Theorem 2.1.

Proof of Theorem 2.1. Assume u_0 is a solution of (5) that corresponds to the admissible pair $(f, |J|)$. It is a direct consequence of the admissibility assumption that

$$\{x \in \Omega : |J| = 0\} \cup \{x \in \Omega : \nabla u_0 = 0\} =: Z \cup \Gamma,$$

where Z is an open set, Γ has measure zero, and

$$Z = U \cup V \cup W.$$

Since $\partial(U \cup V)$ is piecewise $C^{1,\alpha}$,

$$u_0 \in C^{1,\alpha}(\Omega \setminus \overline{U \cup V}) \cap C(\Omega \setminus \overline{U \cup V} \cup \partial\Omega) \cap C^{1,\alpha}(\Omega \setminus \overline{U \cup V} \cup T)$$

for every $C^{1,\alpha}$ component of $\partial(U \cup V)$.

By our assumptions $|J| > 0$ a.e. in $\Omega \setminus \overline{U \cup V \cup W}$. Hence, equality in (6) yields $|\nabla u_0| > 0$ a.e. on $\Omega \setminus \overline{U \cup V \cup W}$. Since $U \cup W$ is a disjoint union of countably many connected open sets and u_0 is constant on every connected open subset of $U \cup W$, the set

$$\Theta := \{u_0(x) : x \in \overline{U \cup W}\}$$

is countable.

Now suppose u_1 is another minimizer. Then we have

$$\nabla u_0 = 0 \text{ in } U \text{ and } \frac{\partial u_0}{\partial \nu} = 0 \text{ on } \partial(V \cup W).$$

Without loss of generality we can assume $u_0 \geq 0$ in $\overline{\Omega}$. Then

$$\begin{aligned} F(u_1) &= \int_{\Omega \setminus \overline{U \cup V \cup W}} \sigma |\nabla u_0| \cdot |\nabla u_1| dx \geq \int_{\Omega \setminus \overline{U \cup V \cup W}} \sigma |\nabla u_0 \cdot \nabla u_1| dx \\ (15) \quad &\geq \int_{\Omega \setminus \overline{U \cup V \cup W}} \sigma \nabla u_0 \cdot \nabla u_1 dx = \int_{\partial\Omega} \sigma_0 \frac{\partial u_0}{\partial \nu} u_1 ds = \int_{\partial\Omega} \sigma_0 \frac{\partial u_0}{\partial \nu} f ds \\ &= F(u_0), \end{aligned}$$

where ν is the outer normal to the boundary of Ω . Since u_0 and u_1 both minimize the functional $F(u)$, equality holds in (15). On the other hand the equality in Cauchy inequality can only hold for parallel vectors, so we have that

$$(16) \quad \nabla u_1(x) = \lambda(x) \nabla u_0(x) \quad \text{a.e. } x \in \Omega \setminus \overline{U \cup V \cup W}$$

for some Lebesgue-measurable $\lambda(x)$. In particular,

$$(17) \quad \frac{\nabla u_0(x)}{|\nabla u_0(x)|} = \frac{\nabla u_1(x)}{|\nabla u_1(x)|}$$

a.e. on

$$(\Omega \setminus \overline{U \cup V \cup W}) \cap \{x \in \Omega : |\nabla u_1| \neq 0\}.$$

Let $E_t = \{x \in \Omega \setminus \overline{U \cup V \cup W} : u_0(x) > t\}$. Since Θ is countable, for a.e. $t > 0$, $\partial E_t \cap \overline{(U \cup W)} = \emptyset$. (Otherwise u_0 must be a constant.) We claim that the sets $\partial E_t \cap (\Omega \setminus \overline{V})$ are smooth C^1 manifolds in $\Omega \setminus \overline{V}$ for almost all $t > 0$ with $\partial E_t \cap \overline{U \cup W} = \emptyset$. To prove this note that since $u_0 \in C^1(\Omega \setminus \overline{U \cup V})$, from equality (17) we have that the measure theoretical normal $\nu_t(x) = -\frac{\nabla u_0}{|\nabla u_0|}$ extends continuously from $\partial^* E_t \cap (\Omega \setminus \overline{V})$ to the topological boundary $\partial E_t \cap (\Omega \setminus \overline{V})$, where $\partial^* E_t$ is the measure theoretical boundary of E_t . By the regularity result of De Giorgi (see, e.g.,

Theorem 4.11 in [18]), we conclude that $\partial E_t \cap \Omega \setminus \overline{V}$ is a C^1 -hypersurface for almost all $t > 0$.

The function u_1 is constant on every C^1 connected components of $\partial E_t \cap (\Omega \setminus \overline{V})$. Indeed, let $\gamma : (-\epsilon, +\epsilon) \rightarrow \partial E_t \cap (\Omega \setminus \overline{V})$ be an arbitrary C^1 curve in $\partial E_t \cap (\Omega \setminus \overline{V})$. Then we have

$$\frac{d}{dt}u_1(\gamma(s)) = |\nabla u_1(\gamma(s))|\nu(\gamma(s)) \cdot \gamma'(s) = 0,$$

because either $|\nabla u_1(\gamma(s))| = 0$ or $\nu(\gamma(s)) \cdot \gamma'(s) = 0$ on $\partial E_t \cap (\Omega \setminus \overline{V})$. So u_1 is constant along γ .

Let t be one of the values for which $\partial E_t \cap (\Omega \setminus \overline{V})$ is a hypersurface and $\partial E_t \cap \overline{U \cup W} = \emptyset$ (which is the case for almost every $t > 0$). We show next that each connected component of ∂E_t intersects the boundary $\partial\Omega$.

Arguing by contradiction, assume that Σ_t is a connected component of ∂E_t such that $\Sigma_t \cap \partial\Omega = \emptyset$. We consider two cases:

- (I) $\overline{\Sigma}_t$ is a manifold without boundary in $\Omega \setminus V$.
- (II) $\overline{\Sigma}_t$ is not a manifold without boundary in $\Omega \setminus V$.

Case I. Assume that $\overline{\Sigma}_t$ is a manifold without boundary in Ω . Then $\partial\Omega \cup \Sigma_t$ is a compact manifold with two connected components. By the Alexander duality theorem for $\partial\Omega \cup \Sigma_t$ (see, e.g., Theorem 27.10 in [19]) we have that $\mathbb{R}^n \setminus (\partial\Omega \cup \Sigma_t)$ is partitioned into three open connected components: $\mathbb{R}^n = (\mathbb{R}^n \setminus \overline{\Omega} \cup O_1 \cup O_2)$. Since $\Sigma_t \subset \Omega$ we have $O_1 \cup O_2 = \Omega \setminus \Sigma_t$ and then $\partial O_i \subset \partial\Omega \cup \Sigma_t$ for $i = 1, 2$.

We claim that at least one of ∂O_1 or ∂O_2 is in Σ_t . Assume not, i.e., for $i = 1, 2$, $\partial O_i \cap \partial\Omega \neq \emptyset$. Since $\partial\Omega$ is connected (by assumption) we have that $O_1 \cup O_2 \cup \partial\Omega$ is connected which implies that $O_1 \cup O_2 \cup (\mathbb{R}^n \setminus \Omega)$ is also connected. Again by applying the Alexander duality theorem for $\Sigma_t \subset \mathbb{R}^n$, we have that $\mathbb{R}^n \setminus \Sigma_t$ has exactly two open connected components, one of which is unbounded: $\mathbb{R}^n \setminus \Sigma_t = O_\infty \cup O_0$. Since $O_1 \cup O_2 \cup (\mathbb{R}^n \setminus \Omega)$ is connected and unbounded, we have that $O_1 \cup O_2 \cup (\mathbb{R}^n \setminus \Omega) \subset O_\infty$, which leaves $O_0 \subset \mathbb{R}^n \setminus (O_1 \cup O_2 \cup (\mathbb{R}^n \setminus \Omega)) \subset \Sigma_t$. This is impossible since O_0 is open and Σ_t is a hypersurface. Therefore either O_1 or O_2 or both have the boundary in Σ_t .

Assume $\partial O_1 \subset \Sigma_t$. We claim that $u_0 = t$ in O_1 . Indeed, since O_1 is an extension domain ($\partial\Omega_1$ has a unit normal everywhere) the new map \tilde{u}_0 defined by

$$\tilde{u}_0 := \begin{cases} u_0, & x \in \Omega \setminus O_1, \\ t, & x \in \overline{O_1}, \end{cases}$$

is in $W^{1,1}(\Omega) \cap C(\overline{\Omega})$ and decreases the functional, which contradicts the minimality of u_0 . Therefore $u_0 = t$ in O_1 , which makes $|\nabla u_0| = 0$ in O_1 . This is a contradiction since we have assumed $\partial E_t \cap \overline{U \cup W} = \emptyset$.

Case II. Assume $\overline{\Sigma}_t$ is not a manifold without boundary in $\Omega \setminus V$. First assume $n \geq 3$ and for any $0 < \epsilon < \epsilon^* := \min\{\min_{i \neq j} \text{dist}(V_i, V_j), \min_i \text{dist}(V_i, \partial\Omega)\}$ define

$$V^\epsilon = V \cup \{x \in \Omega : \text{dist}(x, V) < \epsilon\}.$$

Then V^ϵ is an open set with the same number of disjoint open connected components as V . Now let $\Sigma_t^\epsilon = \Sigma_t \setminus V^\epsilon$. Since $\partial\Sigma_t^\epsilon \subset \partial V^\epsilon$ and $\partial V^\epsilon \setminus \Sigma_t^\epsilon$ is open, each connected component of $\partial\Sigma_t^\epsilon$ is the boundary of an open set in ∂V^ϵ with connected boundary. Suppose M is a connected component of $\partial\Sigma_t^\epsilon$. Then $M \subset \partial V_i^\epsilon$ for some $1 \leq i \leq m$. Now if $x_0 \in \Sigma_t^\epsilon$, then for $\rho > 0$ sufficiently small $B(x_0, \rho) \setminus \Sigma_t^\epsilon = B_1 \cup B_2$,

where B_1, B_2 are disjoint open sets with $u_0(B_1) \subset (t, \infty)$ and $u_0(B_2) \subset (-\infty, t)$. Therefore

$$\partial V_i^\epsilon \setminus M = \pi_1 \cup \pi_2,$$

where π_1, π_2 are disjoint open connected (with respect to the topology of ∂V^ϵ) sets. Since Σ_t^ϵ can be extended inside $V^\epsilon \setminus V$, we can extend Σ_t^ϵ inside V_i^ϵ to obtain a C^1 hypersurface Σ such that

$$\Sigma \cap (\Omega \setminus V^\epsilon) = \Sigma_t^\epsilon \cap (\Omega \setminus V^\epsilon)$$

and $\partial(\Sigma \cap V^\epsilon) = M$. Repeating this procedure for other connected components of $\partial \Sigma_t^\epsilon$ leads to a C^1 orientable hypersurface S^ϵ with no boundary and $\Omega \cap S^\epsilon = \emptyset$ and $S^\epsilon \cap (\Omega \setminus V^\epsilon) = \Sigma_t^\epsilon$. Now apply Alexander’s duality theorem to get the partition

$$\mathbb{R}^n \setminus S^\epsilon = O^\epsilon \cup O_\infty^\epsilon,$$

where O^ϵ and O_∞^ϵ are open subsets of \mathbb{R}^n and O_∞^ϵ is unbounded. Notice that $\Sigma_t^\epsilon \subset \partial O^\epsilon \subset \Sigma_t^\epsilon \cup V^\epsilon$ and consequently $\Sigma_t^\epsilon \subset \partial(O^\epsilon \setminus \bar{V}^\epsilon) \subset \partial V^\epsilon \cup \Sigma_t^\epsilon$. If $\epsilon' < \epsilon$, then $\Sigma_t^\epsilon \subset \Sigma_t^{\epsilon'}$ and $V^{\epsilon'} \subset V^\epsilon$. Therefore

$$O^\epsilon \setminus \bar{V}^\epsilon \subset O^{\epsilon'} \setminus \bar{V}^{\epsilon'}.$$

Now let

$$O = \cup_{0 < \epsilon < \epsilon^*} (O^\epsilon \setminus \bar{V}^\epsilon).$$

Then O is open and $\partial O \subset \Sigma_t \cup V$. We claim that $u_0 = t$ in O . Indeed the new map defined by

$$(18) \quad \tilde{u}_0 := \begin{cases} u_0, & x \in \Omega \setminus (V \cup O), \\ t, & x \in \bar{O}, \end{cases}$$

can be extended to a function in $W^{1,1}(\Omega) \cap C(\bar{\Omega})$ which decreases the functional and contradicts the minimality of u_0 . Hence $u_0 = t$ in O which is a contradiction because we have assumed $E_t \cap \overline{U \cup W} = \emptyset$.

Now assume $n = 2$. Since $\Sigma_t \cap \partial \Omega = \emptyset$ and V has only one connected component, there exists two distinct point $a, b \in \bar{\Sigma}_t \cap \partial V$ such that

$$\partial V \setminus \{a, b\} = \pi_1 \cup \pi_2.$$

Now notice that $\Sigma_t \cup \pi_1$ is a continuous closed curve in \mathbb{R}^2 . By the Jordan curve theorem there exists a bounded open set O_1 such that $\partial O_1 = \Sigma_t \cup \pi_1$. Define $O = O_1 \setminus \bar{V} \neq \emptyset$. Then $\partial O \subset \Sigma_t \cup \partial V$ which is a contradiction in view of (18).

In both cases (I) and (II) the contradiction follows from the assumption that $\Sigma_t \cap \partial \Omega = \emptyset$. We conclude that each connected component of ∂E_t reaches the boundary $\partial \Omega_t$. Since u_0 and u_1 coincide on the boundary $\partial \Omega$, we have shown that $u_0|_{\partial E_t} = u_1|_{\partial E_t} = t$ for almost every t . Therefore $u_0 = u_1$ a.e. in $\Omega \setminus \overline{U \cup W}$.

Now note that $u_0 = u_1$ on the boundary of each connected component of $U \cup W$. Since, u_0 and u_1 are constant on each connected component of $U \cup W$, u_0 and u_1 should also agree on $U \cup W$. Hence $u_0 = u_1$ on $\Omega \setminus V$ and the proof is complete. \square

Proof of Theorem 2.6. To prove the theorem we shall prove the stronger statement (2). It is enough to prove the theorem for each connected component of $\Omega_{\alpha,\beta}$.

Hence without loss of generality we may assume that $\Omega_{\alpha,\beta}$ is connected. By the definition of $\Omega_{\alpha,\beta}$ we have

$$(19) \quad u_1(\partial\Omega_{\alpha,\beta} \setminus \Gamma) \subset \{\alpha, \beta\}.$$

Let $J(x) := J_1(x) = J_2(x)$ for $x \in \Omega_{\alpha,\beta}$. By our assumptions $|J| > 0$ a.e. in $\Omega \setminus_{\alpha,\beta} \overline{U^1 \cup V^1 \cup W^1}$. Hence, $|\nabla u_1| > 0$ a.e. on $\Omega_{\alpha,\beta} \setminus \overline{U^1 \cup V^1 \cup W^1}$. Since $U^1 \cup W^1$ is a disjoint union of countably many connected open sets and u_1 is constant on every connected open subset of $U^1 \cup W^1$, the set

$$\Theta := \{u_1(x) : x \in \overline{U^1 \cup W^1}\}$$

is countable. Without loss of generality we can assume $u_1 \geq 0$ in $\Omega_{\alpha,\beta}$.

Since $J_1 = J_2$ in $\Omega_{\alpha,\beta}$, we have that

$$(20) \quad \nabla u_1(x) = \lambda(x)\nabla u_2(x) \quad \text{a.e. } x \in \Omega_{\alpha,\beta} \setminus \overline{U^1 \cup V^1 \cup W^1}$$

for some nonnegative Lebesgue-measurable function $\lambda(x)$. In particular, for a.e. $x \in \Omega_{\alpha,\beta} \setminus \overline{U^1 \cup V^1 \cup W^1}$ we must have

$$(21) \quad \frac{\nabla u_1(x)}{|\nabla u_1(x)|} = \frac{\nabla u_2(x)}{|\nabla u_2(x)|}.$$

Let $E_t = \{x \in \Omega_{\alpha,\beta} \setminus \overline{U^1 \cup V^1 \cup W^1} : u_1(x) > t\}$. Since Θ is countable, for a.e. $t > 0$, $\partial E_t \cap \overline{U^1 \cup W^1} = \emptyset$. (Otherwise u_1 must be a constant.) With an argument similar to that of Theorem 2.1, one can show that the sets $\partial E_t \cap (\Omega_{\alpha,\beta} \setminus \overline{V^1})$ are smooth C^1 manifolds in $\Omega_{\alpha,\beta}$ for almost all $t > 0$ with $\partial E_t \cap \overline{U^1 \cup W^1} = \emptyset$ and the function u_2 is constant on each connected component of $\partial E_t \cap (\Omega_{\alpha,\beta} \setminus \overline{V^1})$.

Now let $t \neq \alpha, \beta$ be one of the values for which $\partial E_t \cap (\Omega_{\alpha,\beta} \setminus \overline{V^1})$ is a hypersurface and $\partial E_t \cap \overline{U^1 \cup W^1} = \emptyset$ (which is the case for almost every $t > 0$). We next show that each connected component of ∂E_t intersects Γ .

Arguing by contradiction, assume that $\Sigma_t \subset \Omega_{\alpha,\beta}$ is a connected component of ∂E_t such that $\Sigma_t \cap \partial\Omega = \emptyset$. We consider two cases:

- (I) $\overline{\Sigma}_t$ is a manifold without boundary in $\Omega \setminus V$.
- (II) $\overline{\Sigma}_t$ is not a manifold without boundary in $\Omega \setminus V$.

Case I. Assume that $\overline{\Sigma}_t$ is a manifold without boundary in $\Omega \setminus V$. Then $\partial\Omega \cup \Sigma_t$ is a compact manifold with two connected components. By the Alexander duality theorem we have that $\mathbb{R}^n \setminus (\partial\Omega \cup \Sigma_t)$ is partitioned into three open connected components: $\mathbb{R}^n = ((\mathbb{R}^n \setminus \overline{\Omega}) \cup O_1 \cup O_2)$. Since $\Sigma_t \subset \Omega$ we have $O_1 \cup O_2 = \Omega \setminus \Sigma_t$, and then $\partial O_i \subset \partial\Omega \cup \Sigma_t$ for $i = 1, 2$. With an argument similar to the one provided for the proof of Theorem 2.1, we can show that at least one of ∂O_1 or ∂O_2 is in Σ_t . Assume $\partial O_1 \subset \Sigma_t$. Since u_1 satisfies the elliptic equation

$$\nabla \cdot (\sigma_1 \nabla u_1) = 0 \quad \text{in } O_1$$

and $u_1 = t$ on ∂O_1 , $u_1 = t$ in O_1 and therefore $|J| = 0$ on O_1 . This is a contradiction since we have assumed $\partial E_t \cap \overline{U^1 \cup W^1} = \emptyset$.

Case II. If $\overline{\Sigma}_t$ is not a manifold without boundary in $\Omega \setminus V$, then with an argument similar to the one used in the proof of Theorem 2.1 we can show that there exists an open set O such that the new map defined by

$$\tilde{u}_0 := \begin{cases} u_0, & x \in \Omega \setminus (V^1 \cup O), \\ t, & x \in O, \end{cases}$$

belongs to $W^{1,1}(\Omega) \cap C(\bar{\Omega})$ and solves (5). Since (5) has a unique solution $u = \tilde{u}$, $u_0 = t$ in O which is a contradiction since we have assumed $\partial E_t \cap \overline{U^1 \cup W^1} = \emptyset$.

In both cases the contradiction follows from the assumption $\Sigma_t \cap \partial\Omega_{\alpha,\beta} = \emptyset$. Since $t \neq \alpha, \beta$ and

$$u_1(\partial\Omega_{\alpha,\beta} \setminus \Gamma) \subset \{\alpha, \beta\},$$

E_t intersects Γ for almost every $t \geq 0$.

Since u_0 and u_1 coincide on Γ , we have shown that $u_1|_{\partial E_t} = u_2|_{\partial E_t} = t$ for almost every t . Therefore $u_0 = u_1$ a.e. in $\Omega_{\alpha,\beta} \setminus \overline{U^1 \cup W^1}$. Now note that $u_1 = u_2$ on the boundary of each connected component of the set $U^1 \cup W$. Since u_1 and u_2 are constant on each connected component of $U^1 \cup W^1$, u_1 and u_2 should also agree on $U^1 \cup W$. Hence $u_1 = u_2$ on $\Omega_{\alpha,\beta} \setminus \overline{V^1}$. The proof is complete. \square

4. Equipotential surfaces are area minimizing in the conformal metric.

In this section we present the proof of Theorem 2.5. We prove that the equipotential sets are global minimizers of $E(\Sigma)$. This is a consequence of the minimizing property of the voltage potential for the functional $F(u)$. First we recall the co-area formula.

THEOREM 4.1 (co-area formula). *Let $u \in Lip(\Omega)$ and a be integrable in $\Omega \subset \mathbb{R}^n$. Then, for a.e. $t \in \mathbb{R}$, $H^{n-1}(u^{-1}(t) \cap \Omega) < \infty$ and*

$$(22) \quad \int_{\Omega} a|\nabla u(x)|dx = \int_{-\infty}^{\infty} \int_{u^{-1}(t)} adH^{n-1}(x)dt,$$

where H^{n-1} is the $(n - 1)$ -dimensional Hausdorff measure.

PROPOSITION 4.1. *Let $a \geq 0$ be integrable in Ω , U be an open subset of Ω , and*

$$u \in \operatorname{argmin} \left\{ \int_{\Omega} a|\nabla v|dx : v \in Lip(\Omega) \text{ and } v|_{\Omega} = f \right\}.$$

For $\lambda \in \mathbb{R}$ arbitrary fixed, let $u_+ = \max\{u - \lambda, 0\}$ and $u_- = \max\{u, \lambda\}$ be defined in Ω and $f_+ = \max\{f - \lambda, 0\}$, respectively, $f_- = \min\{f, \lambda\}$, be defined on the boundary $\partial\Omega$. Then

$$u_+ \in \operatorname{argmin} \left\{ \int_{\Omega} a|\nabla v|dx : v \in Lip(\Omega) \text{ and } v|_{\Omega} = f_+ \right\},$$

and

$$u_- \in \operatorname{argmin} \left\{ \int_{\Omega} a|\nabla v|dx : v \in Lip(\Omega) \text{ and } v|_{\Omega} = f_- \right\}.$$

Proof. The proof is similar to the proof of Proposition 2.2 [46], and we omit it. \square

COROLLARY 4.2. *Let $a \geq 0$ be integrable in Ω , U be an open subset of Ω , and*

$$u \in \operatorname{argmin} \left\{ \int_{\Omega} a|\nabla v|dx : v \in Lip(\Omega) \text{ and } v|_{\Omega} = f \right\}.$$

For every $\lambda \in \mathbb{R}$ and $\epsilon > 0$ define

$$(23) \quad u_{\lambda,\epsilon} := \frac{1}{\epsilon} \min\{\epsilon, \max\{u - \lambda, 0\}\},$$

and let $f_{\lambda,\epsilon}$ be its trace on the boundary $\partial\Omega$. Then $u_{\lambda,\epsilon} \in Lip(\Omega)$ and

$$u_{\lambda,\epsilon} \in \operatorname{argmin} \left\{ \int_{\Omega} a|\nabla v|dx : v \in Lip(\Omega) \text{ and } v|_{\Omega} = f_{\lambda,\epsilon} \right\}.$$

Proof. The proof follows directly from Proposition 4.1 applied twice. \square

LEMMA 4.3. Let $a, u \in Lip(\Omega)$ such that

$$\{x : |\nabla u(x)| = 0\} = Z \cup L,$$

where Z is open and L has Lebesgue measure zero, $a(\overline{Z}) = \{0\}$, and

$$(24) \quad a \frac{\nabla u}{|\nabla u|} \in W^{1,1}(\Omega \setminus \overline{Z}).$$

Then for almost every $\lambda \in \mathbb{R}$,

$$(25) \quad \lim_{\epsilon \rightarrow 0} \int_{\Omega} a|\nabla u_{\lambda,\epsilon}|dx = \int_{u^{-1}(\lambda)} adH^{n-1}(x),$$

where $u_{\lambda,\epsilon}$ is defined by (23).

Proof. The proof is similar to the proof of Lemma 2.4 in [46]. From Theorem 4.1, we have $H^{n-1}(u^{-1}(\lambda) \cap \Omega) < \infty$ a.e. $\lambda \in \mathbb{R}$. In particular

$$(26) \quad H^n(u^{-1}(\lambda) \cap \Omega) = 0.$$

Since $H^{n-1}(\partial\Omega) < \infty$, from the disjoint partition $\partial\Omega = \bigcup_{\lambda \in \mathbb{R}} (u^{-1}(\lambda) \cap \partial\Omega)$ we have

$$H^{n-1}(u^{-1}(\lambda) \cap \partial\Omega) > 0$$

for at most countable many λ . In particular, for almost every $\lambda \in \mathbb{R}$

$$(27) \quad H^{n-1}(u^{-1}(\lambda) \cap \partial\Omega) = 0.$$

Let $\lambda \in \operatorname{Range}(u)$ be such that both (26) and (27) hold and $\epsilon > 0$. Recall

$$u_{\lambda,\epsilon} = \begin{cases} 0 & \text{if } u(x) < \lambda, \\ (u(x) - \lambda)/\epsilon & \text{if } \lambda \leq u(x) \leq \lambda + \epsilon, \\ 0 & \text{if } u(x) > \lambda + \epsilon. \end{cases}$$

From the co-area formula we have

$$(28) \quad \begin{aligned} \int_{\Omega} a|\nabla u_{\lambda,\epsilon}|dx &= \int_{-\infty}^{+\infty} \int_{(u_{\lambda,\epsilon})^{-1}(t)} adH^{n-1}(x)dt \\ &= \int_0^1 \int_{\{x: u(x)=\lambda+t\epsilon\}} adH^{n-1}(x). \end{aligned}$$

To complete the proof it is enough to prove that

$$(29) \quad \lim_{\epsilon \rightarrow 0} \int_{\{x: u(x)=\lambda+\epsilon\}} adH^{n-1}(x) = \int_{\{x: u(x)=\lambda\}} adH^{n-1}(x)$$

holds uniformly for almost every $t \in [0, 1]$. The domain

$$\Omega_{t,\epsilon} := \{x \in \Omega : \lambda < u(x) < \lambda + t\epsilon\}$$

is Lipschitz. Since $a \in Lip(\Omega)$, it extends continuously to the boundary. The $a\nabla u/|\nabla u| \in W^{1,1}(\Omega \setminus \bar{Z})$ also extends to the boundary $\partial(\Omega \setminus Z)$ as a bounded function. Now notice that $u(\bar{Z})$ is at most countable. Therefore, for a.e. $\lambda \in \text{Range}(u)$ and a.e. $t \in [0, 1]$ the outer unit normal ν to the boundary $\partial\Omega_{t,\epsilon}$ exists. Then Green's formula in $\Omega_{t,\epsilon}$ yields

$$\begin{aligned} & \left| \int_{u^{-1}(\lambda+t\epsilon)} adH^{n-1} - \int_{u^{-1}(\lambda)} adH^{n-1} \right| \\ &= \left| \int_{u^{-1}(\lambda+t\epsilon) \cap \Omega \setminus \bar{Z}} adH^{n-1} - \int_{u^{-1}(\lambda) \cap \Omega \setminus \bar{Z}} adH^{n-1} \right| \\ &\leq \left| \int_{\{x \in \partial\Omega : \lambda < u(x) < \lambda + \epsilon\}} a \frac{\nabla u}{|\nabla u|} \nu dH^{n-1}(x) \right| \\ &\quad + \left| \int_{\{x \in \Omega : \lambda < u(x) < \lambda + \epsilon\}} \nabla \cdot a \frac{\nabla u}{|\nabla u|} dx \right|. \end{aligned}$$

Using (26) we have

$$\begin{aligned} \lim_{\epsilon \rightarrow 0} H^n(\{x \in \Omega : \lambda < u(x) < \lambda + \epsilon\}) &= H^n\left(\bigcap_{\epsilon > 0} \{x \in \Omega : \lambda < u(x) < \lambda + \epsilon\}\right) \\ &\leq H^n\left(\bigcap_{\epsilon > 0} \{x \in \Omega : \lambda \leq u(x) < \lambda + \epsilon\}\right) \\ &= H^n(u^{-1}(\lambda) \cap \Omega) = 0. \end{aligned}$$

Similarly by (27) we obtain

$$\lim_{\epsilon \rightarrow 0} H^{n-1}(\{x \in \partial\Omega : \lambda < u(x) < \lambda + \epsilon\}) = 0.$$

This proves (29). By taking the limit $\epsilon \rightarrow 0$ in (28) and using (29) we obtain (25). \square

Proof of Theorem 2.5. For $\lambda \notin \text{Range}(u)$, the left-hand side of (11) is zero and the inequality trivially holds. Since u obeys the maximum principle and $u = v$ on $\partial\Omega$, $\text{Range}(u) \subset \text{Range}(v)$.

Now let $\lambda \in \text{Range}(u) \setminus (u(\bar{Z}) \cup v(\bar{Z}_v))$ and recall that $u(\bar{Z})$ and $u(\bar{Z}_v)$ are both countable. Since $|\nabla u| \neq 0$ a.e. in $\Omega \setminus \bar{Z}$ and $|\nabla v| \neq 0$ a.e. in $\Omega \setminus \bar{Z}_v$, for almost every $\lambda \in \text{Range}(u)$ the corresponding λ -level set is a C^1 -smooth oriented surface. In particular the H^{n-1} -measure coincides with the induced Lebesgue measure on the respective surface. Moreover, u and v satisfy (26) and (27) for a.e. $\lambda \in \mathbb{R}$.

For $\epsilon > 0$ arbitrary fixed, let $u_{\lambda,\epsilon}$ be defined by (23) and define similarly

$$v_{\lambda,\epsilon} := \min\{\epsilon, \max\{v - \lambda, 0\}/\epsilon\}.$$

Since $u = v$ on the boundary $\partial\Omega$, we also have $u_{\lambda,\epsilon} = v_{\lambda,\epsilon}$ on $\partial\Omega$. From Corollary 4.2 we have

$$(30) \quad \int_{\Omega} a|\nabla u_{\lambda,\epsilon}| dx \leq \int_{\Omega} a|\nabla v_{\lambda,\epsilon}| dx.$$

Letting $\epsilon \rightarrow 0$, and applying Lemma 4.3 we obtain (11). \square

5. Appendix: Perfectly conductive and insulating inclusions. The results in this appendix formalize the definition of perfectly conducting as the infinity limit of conductivity. They are slight generalization of the ones in [8] to include both perfectly conductive and insulating inclusions.

Let $U = \cup_{j=1}^{\infty} U_j$ be an open subset of Ω with $\overline{U} \subset \Omega$ to model the union of the connected components U_j ($j = 1, 2, \dots$) of *perfectly conductive inclusions* and V be an open subset of Ω with $\overline{V} \subset \Omega$ to model the union of all connected *insulating inclusions*. Let χ_U and χ_V be their corresponding characteristic function. We assume that $\overline{U} \cap \overline{V} = \emptyset$, $\Omega \setminus \overline{U \cup V}$ is connected and that the boundaries ∂U , ∂V are piecewise $C^{1,\alpha}$. Let $\sigma_1 \in L^\infty(U)$ and $\sigma \in L^\infty(\Omega \setminus \overline{U \cup V})$ be such that

$$(31) \quad 0 < \lambda \leq \sigma_1, \sigma \leq \Lambda < \infty$$

for some positive constants λ and Λ .

For each $0 < k < 1$ consider the conductivity problem

$$(32) \quad \nabla \cdot \left(\chi_U \left(\frac{1}{k} \sigma_1 - \sigma \right) + \sigma \right) \nabla u = 0, \quad \frac{\partial u}{\partial \nu} = 0 \text{ on } \partial V, \text{ and } u|_{\partial \Omega} = f.$$

The condition on ∂V ensures that V is insulating. It is well known that the problem (32) has a unique solution $u_k \in H^1(\Omega)$ which also solves

$$(33) \quad \begin{cases} \nabla \cdot \sigma \nabla u_k = 0 & \text{in } \Omega \setminus \overline{U \cup V}, \\ \nabla \cdot \sigma_1 \nabla u_k = 0 & \text{in } U, \\ u_k|_+ = u_k|_- & \text{on } \partial U, \\ \frac{1}{k} \sigma_1 \frac{\partial u_k}{\partial \nu} \Big|_- = \sigma \frac{\partial u_k}{\partial \nu} \Big|_+ & \text{on } \partial U, \\ \frac{\partial u_k}{\partial \nu} \Big|_+ = 0 & \text{on } \partial V, \\ u_k|_{\partial \Omega} = f. \end{cases}$$

Moreover, the energy functional

$$(34) \quad I_k[v] = \frac{1}{2k} \int_U \sigma_1 |\nabla v|^2 dx + \frac{1}{2} \int_{\Omega \setminus \overline{U \cup V}} \sigma |\nabla v|^2 dx$$

has a unique minimizer over the maps in $H^1(\Omega)$ with trace f at $\partial \Omega$ which is the unique solution u_k of (33).

We shall show below why the limiting solution (with $k \rightarrow 0$) solves

$$(35) \quad \begin{cases} \nabla \cdot \sigma \nabla u_0 = 0 & \text{in } \Omega \setminus \overline{U \cup V}, \\ \nabla u_0 = 0 & \text{in } U, \\ u_0|_+ = u_0|_- & \text{on } \partial U, \\ \int_{\partial U_j} \sigma \frac{\partial u_0}{\partial \nu} \Big|_+ ds = 0 & j = 1, 2, \dots, \\ \frac{\partial u_0}{\partial \nu} \Big|_+ = 0 & \text{on } \partial V, \\ u_0|_{\partial \Omega} = f. \end{cases}$$

By elliptic regularity $u_0 \in C^{1,\alpha}(\Omega \setminus U \cup V)$ and for any $C^{1,\alpha}$ boundary portion T of $\partial(U \cup V)$, $u_0 \in C^{1,\alpha}((\Omega \setminus (U \cup V)) \cup T)$.

PROPOSITION 5.1. *The problem (35) has a unique solution in $H^1(\Omega)$ which is the unique minimizer of the functional*

$$(36) \quad I_0[v] = \frac{1}{2} \int_{\Omega \setminus \overline{U \cup V}} \sigma |\nabla v|^2 dx$$

over the set $A_0 := \{u \in H^1(\Omega \setminus \overline{V}); u|_{\partial \Omega} = f, \nabla u = 0 \text{ in } U\}$.

Proof. Note that A_0 is weakly closed in $H^1(\Omega \setminus \bar{V})$. The functional I_0 is lower semicontinuous, strictly convex, and thus has a unique minimizer u_0^* in A_0 .

First we show that u_0^* is a solution of (35). Since u_0^* minimizes (36), we have

$$(37) \quad 0 = \int_{\Omega \setminus \bar{U} \cup \bar{V}} \sigma \nabla u_0^* \cdot \nabla \varphi dx$$

for all $\varphi \in H^1(\Omega \setminus \bar{V})$ with $\varphi|_{\partial\Omega} = 0$ and $\nabla\varphi = 0$ in U . In particular, if $\varphi \in H_0^1(\Omega \setminus \bar{V})$, we get $\int_{\Omega \setminus \bar{U} \cup \bar{V}} (\nabla \cdot \sigma \nabla u_0^*) \varphi dx = 0$ and thus u_0^* solves the conductivity equation in (35). If we choose $\varphi \in H^1(\Omega \setminus \bar{V})$ with $\varphi|_{\partial\Omega} = 0$ and $\varphi \equiv 0$ in U , from Green's formula applied to (37) we get $\int_{\partial V} \sigma \frac{\partial u_0^*}{\partial \nu} |_{+} \varphi = 0$, for all $\varphi|_{\partial V} \in H^{1/2}(\partial V)$, or, equivalently, $\sigma \frac{\partial u_0^*}{\partial \nu} |_{\partial V} = 0$. If we choose $\varphi_j \in H_0^1(\Omega \setminus \bar{V})$ with $\varphi_j \equiv 1$ in the connected component U_j of U and $\varphi_j \equiv 0$ in $U \setminus U_j$, from Green's formula applied to (37) we obtain $\int_{\partial U_j} \sigma \frac{\partial u_0^*}{\partial \nu} = 0$.

Next we show that (35) has a unique solution, and consequently $u_0^* = u_0|_{\Omega \setminus \bar{V}}$. Assume that u^1 and u^2 are two solutions and let $u = u^2 - u^1$; then $u|_{\partial\Omega} = 0$ and

$$(38) \quad 0 = - \int_{\Omega \setminus \bar{U} \cup \bar{V}} (\nabla \cdot \sigma \nabla u) u dx = - \int_{\partial\Omega} \sigma \frac{\partial u}{\partial \nu} u ds + \int_{\partial V} \sigma \frac{\partial u}{\partial \nu} |_{+} u ds$$

$$(39) \quad + \int_{\partial U} \sigma \frac{\partial u}{\partial \nu} |_{+} u ds + \int_{\Omega \setminus \bar{U} \cup \bar{V}} \sigma |\nabla u|^2 dx = \int_{\Omega \setminus \bar{U} \cup \bar{V}} \sigma |\nabla u|^2 dx.$$

Since $\sigma \geq \lambda > 0$, we get $|\nabla u| = 0$ in $\Omega \setminus \bar{V}$. Since $\Omega \setminus \bar{V}$ is connected and $u = 0$ at the boundary, we conclude uniqueness of the solution of (35). \square

THEOREM 5.1. *Let u_k and u_0 be the unique solution of (33), respectively, (35), in $H^1(\Omega)$. Then $u_k \rightharpoonup u_0$, and consequently $I_k[u_k] \rightarrow I_0[u_0]$ as $k \rightarrow 0^+$.*

Proof. We show first that $\{u_k\}$ is bounded in $H^1(\Omega)$ uniformly in $k \in (0, 1)$. Since $1/k > 1$, we have

$$\begin{aligned} \frac{\lambda}{2} \|\nabla u_k\|_{L^2(\Omega \setminus \bar{V})}^2 &\leq \frac{1}{2} \int_{\Omega \setminus \bar{U} \cup \bar{V}} \sigma |\nabla u_k|^2 dx + \frac{1}{2k} \int_U \sigma_2 |\nabla u_k|^2 dx \\ &\leq I_k[u_k] \leq I_k[u_0] \leq \frac{\Lambda}{2} \|\nabla u_0\|_{L^2(\Omega \setminus \bar{V})}^2 \end{aligned}$$

or

$$(40) \quad \|\nabla u_k\|_{L^2(\Omega \setminus \bar{V})}^2 \leq \frac{\Lambda}{\lambda} \|\nabla u_0\|_{L^2(\Omega \setminus \bar{V})}^2.$$

From (40) and the fact that $u_k|_{\partial\Omega} = f$, we see that $\{u_k\}$ is uniformly bounded in $H^1(\Omega \setminus V)$ and hence weakly compact, so on a subsequence $u_k \rightharpoonup u_0^*$ in $H^1(\Omega \setminus V)$ for some u_0^* with trace f at $\partial\Omega$.

We will show next that u_0^* satisfies (35), and therefore $u_0^* = u_0$ on Ω . By the uniqueness of solutions of (35) we also conclude that the whole sequence converges to u_0 .

Since $u_k \rightharpoonup u_0^*$ we have that $0 = \int_{\Omega \setminus \bar{U} \cup \bar{V}} \sigma \nabla u_k \cdot \nabla \varphi dx \rightarrow \int_{\Omega \setminus \bar{U} \cup \bar{V}} \sigma_2 \nabla u_0^* \cdot \nabla \varphi dx$ for all $\varphi \in C_0^\infty(\Omega \setminus \bar{U} \cup \bar{V})$. Therefore $\nabla \cdot \sigma \nabla u_0^* = 0$ in $\Omega \setminus \bar{U} \cup \bar{V}$. Also because u_k is a minimizer of $I[u_k]$ we must have $\nabla u_0^* = 0$ in U . To check the boundary conditions, note that for all $\varphi \in C_0^\infty(\Omega)$ with $\varphi \equiv 0$ in U , we have $\int_{\partial V} \sigma \frac{\partial u_k}{\partial \nu} |_{+} \varphi ds = 0$. Using

the fact that φ were arbitrary, by taking the weak limit in $k \rightarrow 0$ we get $\frac{\partial u_0^*}{\partial \nu}|_+ = 0$ on ∂V . A similar argument applied to $\varphi \in C_0^\infty(\Omega)$ with $\varphi \equiv 0$ in V , $\varphi \equiv 1$ in U_j , and $\varphi \equiv 0$ in $U \setminus U_j$ also shows that $\int_{\partial U_j} \sigma \frac{\partial u_0^*}{\partial \nu}|_+ \varphi ds = 0$. Hence u_0^* is the unique solution of (35) on $\Omega \setminus \bar{V}$. Thus u_k converges weakly to the solution u_0 of (35) in $\Omega \setminus \bar{V}$. \square

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