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A COMPARATIVE STUDY OF ANT COLONY OPTIMIZATION

by

MATTHEW BECKER
B.S., Kansas State University, 1987

A thesis submitted in partial fulfillment of the requirements
for the degree of Master of Science
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in the College of Sciences
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ABSTRACT

Ant Colony Optimization (ACO) belongs to a class of biologically-motivated approaches to computing that includes such metaheuristics as artificial neural networks, evolutionary algorithms, and artificial immune systems, among others. Emulating to varying degrees the particular biological phenomena from which their inspiration is drawn, these alternative computational systems have succeeded in finding solutions to complex problems that had heretofore eluded more traditional techniques. Often, the resulting algorithm bears little resemblance to its biological progenitor, evolving instead into a mathematical abstraction of a singularly useful quality of the phenomenon. In such cases, these abstract computational models may be termed biological metaphors. Mindful that a fine line separates metaphor from distortion, this paper outlines an attempt to better understand the potential consequences an insufficient understanding of the underlying biological phenomenon may have on its transformation into mathematical metaphor. To that end, the author independently develops a rudimentary ACO, remaining as faithful as possible to the behavioral qualities of an ant colony. Subsequently, the performance of this new ACO is compared with that of a more established ACO in three categories: (1) the hybridization of evolutionary computing and ACO, (2) the efficacy of daemon actions, and (3) theoretical properties and convergence proofs.

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LIST OF ACRONYMS

ACO	Ant Colony Optimization
ACS	Ant Colony System
AS	Ant System
EP	Evolutionary Programming
GA	Genetic Algorithms
MMAS	Max-Min Ant System
TSP	Traveling Salesman Problem

CHAPTER ONE: INTRODUCTION

This paper explores the effect that the hybridization of evolutionary computing and Ant Colony Optimization (ACO) methods, the inclusion of various daemon actions, and the exclusion of such phenomena as phase transitions in ant colony foraging behavior may have on the efficacy of the family of algorithms comprising the Ant Colony Optimization metaheuristic. The quality of Ant System (AS), the original ACO algorithm, that most distinguished it from such related metaheuristics as evolutionary computing, genetic algorithms, etc., was its faithful emulation of the highly coordinated behavior of actual ants during foraging, in particular. The eusocial insect phenomenon of order absent hierarchical control was the fundamental inspiration that set the ACO *ball* in motion [7]. And yet, paradoxically, the development of this particular category of computational methods has been marked by an inexorable abandonment of its founding precepts [10]. In fact, the first significant *improvement* made to AS was the incorporation of the so-called *elitist* strategy, the first of the daemon actions to be introduced into the algorithm.

While it is incontrovertible that the addition of various global controls, elitist strategy among them, has improved the performance of successive ACO incarnations, it is the author's suspicion that these enhancements are, nevertheless, in some measure at odds with the eminently distributed nature of an ant colony and their inclusion may come at a cost: namely, the resultant loss of autonomy among the ants and the inevitable diminution of their collective contribution to the algorithm's various applications.

This report is divided into five primary sections. Immediately following the introduction is a chapter in which the current literature on selected optimization techniques and ACO in

particular is review. The third section provides a concise history of the ACO metaheuristic. Section four focuses on the underlying principles and methodology of ACO. The fifth major section provides a conclusion. It also considers additional research and discusses the possibilities for future work based on this research.

CHAPTER TWO: LITERATURE REVIEW

The Ant Colony Optimization Metaheuristic

Artificial ants used in ACO are stochastic solution construction procedures that probabilistically build a solution by iteratively adding solution components to partial solutions while taking into account (i) heuristic information on the problem being solved, if available, and (ii) artificial pheromone trails that change dynamically at run-time to reflect the ants' acquired search experience.

The interpretation of ACO as an extension of construction heuristics is appealing for several reasons. First, a stochastic component in ACO allows the ants to build a wide variety of solutions and consequently to explore a much larger number of solutions than do conventional greedy heuristics. Second, the concurrent use of heuristic information, which is readily available for many problems, can direct the ants more quickly towards the most promising solutions. Third, the record of an ant's exploration of the solution space can be used to influence solution construction in future iterations of the algorithm in a manner reminiscent of reinforcement learning [18]. Fourth, given that a colony may comprise a multitude of individual ants, with built-in redundancy as far as the eye can see, the algorithm is profoundly robust and in many ACO applications, the collective interaction of a population of agents is required to efficiently solve a problem.

The range of applications which may be solved by ACO algorithms is vast. In principle, ACO can be applied to any discrete optimization problem for which some solution construction mechanism can be conceived. In the following section, a generic problem for which ACO might

be tasked to construct solutions is described, and the ants' behavior during a typical solution search is explored.

Problem Characterization

Consider a minimization problem (S, f, Ω) , in which S is the set of candidate solutions, f is the objective function which assigns to each candidate solution s an objective function, or cost $f(s, t)$, and Ω is a set of constraints. The goal is to find a globally optimal solution $s_{opt} \in S$ (i.e., a minimum cost solution that satisfies the constraints Ω).

A combinatorial optimization problem (S, f, Ω) such as this can be characterized as follows:

- A finite set $C = \{c_1, c_2, \dots, c_{N_c}\}$ of components.
- The states of the problem are defined in terms of all possible sequences $x = \langle c_i, c_j, \dots, c_h, \dots \rangle$ over the elements of C . The length of a sequence x is the number of components in the sequence and is expressed as $|x|$.
- The finite set of constraints Ω defines the set of feasible states \tilde{X} , with $\tilde{X} \subseteq X$.
- A set S^* of feasible solutions is given, with $S^* \subseteq \tilde{X}$ and $S^* \subseteq S$.
- A cost $f(s, t)$ is associated with each candidate solution $s \in S$.
- In some cases a cost, or the estimate of a cost, $J(x_i, t)$, can be associated with states other than solutions. If x_j can be obtained by adding solution components to a state x_i then $J(x_i, t) \leq J(x_j, t)$. Note that $J(s, t) \equiv f(s, t)$.

Given this translation of the problem into terminology understood by the algorithm's artificial ants, they set about building solutions by moving on the construction graph $G = (C, L)$,

where the vertices are the components C and the set L fully connects the components C (elements of L are called connections). The problem constraints Ω are implemented in directions given to the artificial ants, as will be explained in the next section. The choice of implementing the constraints in the construction *policy* of the artificial ants affords the algorithm a measure of flexibility. In fact, depending on the combinatorial optimization problem being considered, it may be more reasonable to implement more stringent constraints, allowing ants to build only feasible solutions. Conversely, the problem may be one that is better served by ants permitted to construct impracticable solutions (i.e., candidate solutions in $S \setminus S^*$) that will be penalized to a degree commensurate with their distance from viable solutions

Behavior of Virtual Ants

Virtual ants may be thought of as stochastic construction procedures which build solutions while moving on the construction graph $G = (C, L)$. Ants do not move arbitrarily on G , but rather follow a construction policy that is a function of the problem constraints Ω . Components $c_i \in C$ and connections $l_{ij} \in L$ may possess a corresponding pheromone trail τ , designated τ_i if associated with components and τ_{ij} if associated with connections. A heuristic value η (η_i and η_{ij} , as above) represents *a priori* information about either specific qualities of a problem or run-time information. In many cases, η is the cost, or an estimate of the cost, of extending the current state. These values define the initial parameters that govern the ants' stochastic motion on the graph.

More precisely, each ant k of the colony has the following properties:

- It searches the graph $G = (C, L)$ for feasible solutions s of minimal cost (i.e., solutions such that $\hat{f}_s = \min_s f(s,t)$).
- It has memory M^k that it uses to store information regarding the path that it has negotiated. Memory can be used (i) to build feasible solutions (i.e., to implement constraints Ω), (ii) to evaluate a discovered solution, and (iii) to retrace the path backwards in order to deposit pheromone.
- It can be assigned an initial state x^k and one or more termination conditions e^k . Usually, the initial state is expressed either as a unit length sequence (i.e., a single component sequence) or an empty sequence.

- When in state $x_r = \langle x_{r-1}, i \rangle$ and if no termination condition has been satisfied, an ant moves to a node j in its neighborhood N_i^k and advances to a state $\langle x_r, j \rangle \in X$. Typically, motion in the direction of feasible states is favored, and may be attained through properly defined heuristic values η or by means of the ant's own memory.
- It selects each move according to a probabilistic decision rule. This rule is a function of (i) local pheromone trails and heuristic values that it has encountered, (ii) the ant's memory, and (iii) the relevant problem constraints.
- The construction procedure of ant k ceases when at least one of the termination conditions e^k is satisfied.
- When adding a component c_j to the current solution, an ant can update its pheromone trail or that of the corresponding connection. This action is referred to as online *step-by-step* pheromone update.
- Once a solution has been constructed, an ant can retrace its path backwards and update the pheromone trails of used components or connections. This action is called online *delayed* pheromone update.

It is important to note that each ant moves independently and is sufficiently complex to find its own (probably poor) solution to the problem under consideration. Usually, quality solutions emerge as a consequence of interaction among the ants, characterized by indirect communication that is mediated by the information ants read and write to the variables storing pheromone trail values. (This indirect means of communication via pheromone is called *stigmergy*.) One might consider this system a distributed learning process within which the

agents, or ants, are not adaptive themselves; rather, individual ants serve to modify the way in which the problem is represented and perceived by other ants.

The Metaheuristic

Informally, the behavior of ants in an ACO algorithm may be summarized as follows: A colony of ants concurrently and asynchronously move through adjacent states of a problem by building paths on the aforementioned solution space G . The ants move by applying a stochastic local decision-making policy that makes use of pheromone trails and heuristic information. As they move, ants incrementally build solutions to the optimization problem. Once an ant has built a solution, or as the solution is constructed, an ant evaluates its contribution to the solution and deposits pheromone upon trails representing the associated components or connections. This pheromone information provides guidance to future ants in their search for a solution.

In addition to the activity of ants, an ACO algorithm includes two more procedures: pheromone trail evaporation and daemon actions (this last component being optional). Pheromone evaporation is the process by which pheromone intensity decreases over time on trails traversed (i.e., partial trial solutions) by ants. The relatively ephemeral nature of pheromone reduces the likelihood of premature convergence by the algorithm towards a sub-optimal region (or to a local rather than a global minimum, for example)


```

procedure ACO metaheuristic
  ScheduleActivities
    ManageAntsActivity()
    EvaporatePheromone()
    DaemonActions() {Optional}
  end ScheduleActivities
end ACO metaheuristic

```

Figure 1. The ACO metaheuristic in pseudo-code. The procedure *DaemonActions()* is optional and refers to centralized actions executed by a daemon possessing global knowledge.

and consequently encourages the investigation of unexplored areas in the search space. Daemon actions can be used to implement centralized actions which cannot be performed by single ants. Examples of such actions include the activation of a local optimization procedure or the collection of global information to be used in determining whether additional pheromone should be deposited in order to bias the search process from a global perspective. As a practical example, the daemon is able to observe the paths traveled by each ant in the colony and subsequently to choose to deposit additional pheromone upon the elements used by the ant, thereby building the best available solution in a given iteration of the algorithm. Pheromone updates performed by the daemon are called *off-line* pheromone updates.

In Figure 1 the ACO metaheuristic behavior is described in pseudo-code. The main procedure of the ACO metaheuristic manages, via the *ScheduleActivities* construct, the scheduling of the three above discussed components of ACO algorithms: (i) management of the ants' activities, (ii) pheromone evaporation, and (iii) daemon actions. However, the *ScheduleActivities* construct does not specify the manner in which these three activities are to be

scheduled nor whether there should exist some coordination among them; it is the prerogative of the programmer to specify the manner in which these three procedures should interact.

CHAPTER THREE: A CONCISE HISTORY OF ACO

The first ACO algorithm proposed was Ant System (AS). AS was applied to some rather trivial examples of the Traveling Salesman Problem (TSP) with circuits of up to 75 cities. Although it was able to match the performance of other general-purpose heuristics like evolutionary computation [8, 13], for example, AS ultimately proved not to be competitive with state-of-the-art algorithms that had been specifically designed for the TSP, particularly as the numbers of cities were increased beyond its proverbial comfort zone. Consequently, a substantial amount of research was directed towards developing ACO algorithms demonstrating better performance than AS when applied to the TSP and similar challenging tasks.

Within this chapter, the biological metaphor upon which AS and ACO are based is introduced and discussed. An abridged history of the various algorithms comprising the intermediate developmental stages that mark the evolution of the ACO heuristic from the original AS incarnation to the most recent ACO algorithms is also included.

ACO as Biologically-Inspired Computing

In many species of ants, individual ants deposit pheromone—a chemical that may be sensed by other ants—as an indirect means of communication [9]. Through the deposition of pheromone, they create a trail that is used, for example, to mark a path from the nest to food sources and back. By sensing pheromone trails, foragers are able to find their way to food that

has been discovered by other ants. Moreover, they are capable of exploiting pheromone trails to choose the shortest among the available paths.

Deneubourg and colleagues [3, 15] used a double bridge connecting a nest of ants and a food source to study the phenomenon of pheromone trails, observing behavior in controlled experimental conditions. They ran a number of experiments in which they varied the ratio between the lengths of the two branches of the bridge. The most relevant of these experiments to the development of AS was the one in which the branches were of different lengths. In this experiment, a laboratory colony of Argentine ants (*Iridomyrmex humilis*) was initially allowed to move freely between the nest and the food source, and the shifting percentages of ants choosing one or the other of the two branches were recorded over time. Although in the initial phase, random oscillations in the relative amounts of ant *traffic* would occur, an overwhelming majority of the experiments found all the ants traveling via the shorter of the two branches.

This result can be explained as follows: Because there is no pheromone on the available paths initially, the ants have no preference and select with equal probability either of the two branches. Therefore, it can be expected that, on average, half of the ants will select the shorter branch and half the longer, although stochastic oscillations may occasionally favor one branch over the other. However, because one branch is shorter (and assuming all ants travel at the same speed), the ants choosing the shorter route will be the first to reach the food and, consequently, the first to begin their return trip. As they decide whether to return via the short or the long branch, the higher level of pheromone on the short branch biases their decision in its favor. Therefore, pheromone starts to accumulate faster on the short branch, which will eventually be used exclusively by a large percentage of the ants.

That this pheromone-based communication among ants inspired AS and subsequent algorithms is clear: the double bridge was substituted by a graph and pheromone trails by virtual pheromone trails. Also, because artificial ants are tasked with solving problems much more complicated than those solved by real ants, artificial ants are endowed with extra capacities such as superior memory (used to implement constraints and to allow the ants to retrace their path back to the nest without error) and the capacity to precisely deposit a quantity of pheromone proportional to the quality of the solution produced (a similar behavior is observed in some real ant species in which the quantity of pheromone deposited while returning to the nest from a food source is proportional to the quality of the food source found [3]).

Subsequent to the advent of AS, new algorithms have been developed that, though they retain some elements of the original biological inspiration, are less a near-faithful re-creation of the natural phenomena of ant colony foraging behavior but are instead increasingly motivated by the need to improve ACO algorithms in order that they might compete with state-of-the-art algorithms. Nevertheless, many aspects of the original Ant System remain:

- a requirement for a colony of ants.
- the role of autocatalysis (any process by which a decision taken at time t increases the probability of making the same decision at time $T > t$ by means of the implementation of positive feedback).
- cooperative behavior among ants managed by virtual pheromone trails. the probabilistic construction of solutions biased by pheromone and local heuristic information.
- pheromone updating apportioned according to solution quality.
- the evaporation of pheromone.

All of these elements survive as integral elements in more modern ACO algorithms. Ant algorithms are receiving increasing attention within the scientific community (consult for examples: [8, 9, 11]) and are developing a reputation as novel and effective approaches to solving problems of distributed control and optimization.

Historical Development

As mentioned previously, AS was the first example of an ACO algorithm to be proposed in the literature. In fact, AS was originally conceived as three versions of the same algorithm: ant-cycle, ant-density, and ant-quantity. These three algorithms were proposed in Marco Dorigo's doctoral dissertation [20] and first appeared in a technical report [5] that was published a few years later in the IEEE Transactions on Systems, Man, and Cybernetics [3].

While in the ant-density and ant-quantity components of the algorithm, the ants update the pheromone immediately following an excursion from one city to another, ant-cycle's pheromone update was performed only after all ants had constructed their tours, and the amount of pheromone deposited by each ant was determined as a function of the solution quality. Because ant-cycle performed more successfully than the other two variants, it later came to be called simply Ant System, while work on the other two algorithms was abandoned.

The primary merit of AS, the computational capabilities of which showed promise but were not competitive with more established approaches, was to stimulate a number of researchers, mostly in Europe, to develop extensions and improvements to its fundamental design.

Ant System and the Traveling Salesman Problem

The Traveling Salesman Problem (TSP) is a so-called \mathcal{NP} -hard combinatorial optimization problem that has attracted a considerable amount of research effort [4, 9, 12]. The TSP is an important problem also in the context of Ant Colony Optimization in that it is the problem to which the original AS was initially applied [5, 19], and it is frequently used as a benchmark to test new ideas and algorithmic variants.

In AS, each ant is initially situated on a randomly chosen city with a memory that will store the partial solution it devises with each iteration of the algorithm. Subsequently, an ant moves from city to city. While at city i , for example, an ant k will travel to an as yet unvisited city j with a probability given by

$$p_{ij}^k(t) = \frac{[\tau_{ij}(t)]^\alpha \cdot [\eta_{ij}]^\beta}{\sum_{l \in \mathcal{N}_i^k} [\tau_{il}(t)]^\alpha \cdot [\eta_{il}]^\beta} \quad \text{if } j \in \mathcal{N}_i^k \quad (1)$$

where $\eta_{ij} = 1/d_{ij}$ is heuristic information, α and β determine the relative contributions of pheromone and heuristic information, and \mathcal{N}_i^k is the plausible neighborhood of ant k (i.e., the collection of cities not yet visited by ant k). Parameters α and β exert the following influence on the algorithm's behavior: If $\alpha = 0$, the selection probabilities are proportional to $[\eta_{ij}]^\beta$, making the nearest cities more likely be selected. In this case, AS corresponds to a classical stochastic greedy algorithm (with multiple starting points since ants are initially randomly distributed on the cities). If $\beta = 0$, pheromone *amplification* acts alone on the population of artificial ants, leading to inevitable stagnation and the generation of partial solutions that are

most likely to be significantly suboptimal [16]. (*Search stagnation* is defined in [2] as a situation whereby all ants ultimately follow the same path and construct identical solutions.)

Solution construction ends after each ant has completed a tour, that is, after each ant has constructed a sequence of length n . The pheromone trails are then updated. In AS, this is achieved by initially reducing the amount of pheromone on all trails by a constant factor (i.e., pheromone evaporation) and then allowing each ant to deposit pheromone on the arcs that describe its own partial solution:

$$\tau_{ij}(t+1) = (1 - \rho) \cdot \tau_{ij}(t) + \sum_{k=1}^m \Delta\tau_{ij}^k(t) \quad \forall(i, j) \quad (2)$$

where $0 < \rho \leq 1$ is the pheromone evaporation rate and m is the number of ants. The parameter ρ prevents unlimited accumulation of pheromone, thus allowing the algorithm to dispense of poor solutions. Upon arcs which are not chosen by ants, the associated pheromone strength will decrease exponentially with the number of iterations. $\Delta\tau_{ij}^k(t)$ represents the amount of pheromone ant k deposits on the arcs and is defined as

$$\Delta\tau_{ij}^k(t) = \begin{cases} 1/L^k(t) & \text{if arc } (i, j) \text{ is used by ant } k. \\ 0 & \text{otherwise.} \end{cases} \quad (3)$$

where $L^k(t)$ is the length of the k th ant's tour. According to equation (3), the shorter an ant's tour, the more pheromone is received by its constituent arcs. In general, those arcs traversed by many ants and contained within shorter tours receive more pheromone and therefore are more likely to be chosen as potentially viable solutions in subsequent iterations of the algorithm.

Ant System and its Successors

Because AS was largely uncompetitive with state-of-the-art algorithms for TSP, researchers began to extend the algorithm in order to improve its performance. An initial improvement, deemed the *elitist* strategy, was introduced in [13, 22]. An algorithm employing elitist strategy awards the best tour since the algorithm's launch (called T^{gb} , where *gb* stands for global-best) a considerable additional weight. In practice, every time the pheromone trails are refreshed, those belonging to the edges of the global-best tour receive an additional amount of pheromone. For these edges, equation (3) becomes:

$$\Delta\tau_{ij}^{gb}(t) = \begin{cases} e/L^{gb}(t) & \text{if arc } (i, j) \in T^{gb} \\ 0 & \text{otherwise} \end{cases} \quad (3a)$$

The arcs of T^{gb} are therefore reinforced with a quantity of $e \cdot 1/L^{gb}$, where L^{gb} is the length of T^{gb} and e is a positive integer. Note that this version of pheromone replenishment is the first instance of daemon action to be implemented in the ACO family of algorithms.

Other improvements, described below, include the so-called rank-based version of Ant System (AS_{rank}), MAX-MIN Ant System (MMAS), and Ant Colony System (ACS). AS_{rank} [5] is in a sense a variation of the elitist strategy; it divides the ants into classes according to the lengths of the tours they have generated and, after each tour construction phase, only the $(\omega - 1)$ best ants and the global-best ant are permitted to deposit pheromone. The r th best ant of the colony contributes to the pheromone update with a weight given by $\max\{0, \omega - r\}$ while the global-best tour reinforces the pheromone trails with weight ω . Equation (2) becomes:

$$\tau_{ij}(t+1) = (1 - \rho) \cdot \tau_{ij}(t) + \sum_{r=1}^{w-1} (w - r) \cdot \Delta\tau_{ij}^r(t) + w \cdot \Delta\tau_{ij}^{gb}(t) \quad (2a)$$

where $\Delta\tau_{ij}^r(t) = 1/L^r(t)$ and $\Delta\tau_{ij}^{gb}(t) = 1/L^{gb}$.

Increasing the significance of the *exploitation* of data collected by previous ants with respect to the *exploration* of the search space affords ACS [19, 21] a marked advantage over AS. This improvement may be attributed to two particular features of ACS. First, an elitist strategy is emphasized in the updating of pheromone on the trails. Second, ants now act according to a so-called pseudo-random proportional rule [10], requiring that they travel to city j with probability q_0 , for which the product between pheromone trail and heuristic information is maximum:

$j = \arg \max_{j \in N_i^k} \{\tau_{ij}^k(t) \cdot \eta_{ij}^\beta\}$, while with probability $1 - q_0$, they operate a biased exploration in

which the probability $p_{ij}^k(t)$ is the same as in AS (see equation (1)). The value q_0 is a parameter: when it is set to a value close to 1, as it is the case of most ACS applications, exploitation is favored over exploration. Obviously, when $q_0 = 0$, the probabilistic decision rule becomes the same as that of AS.

Again, when pheromone is updated using an elitist strategy, only the trails of the ant delivering the best solution are updated; and they are supplemented according to the following pheromone update rule:

$$\tau_{ij}(t+1) = (1 - \rho) \cdot \tau_{ij}(t) + \rho \cdot \Delta t_{ij}^{\text{best}}(t) \quad (4)$$

The best ant may be alternately designated to be the *iteration-best* ant (i.e., the best in the current iteration, or the *global-best* ant (i.e., the ant that having the best tour since the algorithm began). Lastly, ACS differs from its ACO algorithm ancestry in that ants update the pheromone trails and construct solutions concurrently (as was done in the ant-quantity and in ant-density

algorithms). Incidentally, another measure by which ACS encourages exploration as a means of compensation for the effect of the two modifications described above is to allow *erosion* of pheromone upon heavily traveled trails in order to preclude the ant algorithm equivalent of a traffic jam from occurring. Furthermore, ACS has been augmented with local search routines which receive the solutions generated by ants and calculate their local optimum values preceding pheromone update.

MMAS [11] introduces both a constraint of upper and lower bounds to the values of pheromone available to the ants and a different initialization of these values. Also, MMAS restricts the range of the pheromone strength to within the interval $[\tau_{min}, \tau_{max}]$, and pheromone trails are initialized to their greatest allowable value, resulting in exploration over a broader range of potential solutions near the outset of the algorithm. As in ACS, MMAS allows only the best ant to add pheromone following each iteration of the algorithm. Results indicate that superior results occur when the global-best ant (as opposed to the iteration-best ant) alternative is employed with increasing frequency over the course of the algorithm's execution. Lastly, MMAS, like ACS, makes use of a local search subroutine in order to optimize its performance in these smaller regions, or subsets, of the solution space.

CHAPTER FOUR: THEORETICAL CONSIDERATIONS

The brief history of the ACO metaheuristic is almost exclusively a history of experimental research. The process of trial and error has directed most early researchers and still guides the preponderance of ongoing research efforts. This is the typical situation for virtually all existing metaheuristics: it is only after experimental work has shown the practical applications of a new metaheuristic that researchers attempt to extend their understanding of a metaheuristic's functioning not only through increasingly complicated experiments but also by means of an exploration of relevant theory. Often, the initial theoretical problem considered is one concerning a metaheuristic's capacity to converge to an optimal solution, followed by inquiries regarding such topics as the speed of convergence, the effects of the metaheuristic's parameters, and the identification of problem characteristics that make success more likely. In short, given the novelty of the approach and the mathematical intractability of certain applications, the body of theory surrounding ACO is rather limited. Notwithstanding these limitations, we discuss the convergence of certain ACO algorithms to optimal solutions and investigate the relationship between ACO and other well-known techniques. The convergence proofs presented in the following sections do not apply generally to the metaheuristic, but instead to those ACO algorithms that are more amenable to theoretical analysis, such as the MAX-MIN Ant System or the Ant Colony System.

The first theoretical aspect of ACO to be considered in this chapter is the convergence problem: Will the algorithm find an optimal solution? After all, ACO algorithms are stochastic search procedures in which bias introduced by the pheromone trails could prevent them from converging to the optimum. A stochastic optimization algorithm may be assessed according to

two kinds of convergence: convergence in value and convergence in solution. Informally, convergence in value considers the probability that an algorithm will generate an optimal solution at least once. Convergence in solution, on the other hand, entails evaluating the probability that an algorithm will attain a state that will continue generating the same optimal solution. Note that while convergence in solution is more difficult to prove than convergence in value, from a practical point of view, finding the optimal solution just once will suffice. Therefore, in this case, convergence in value is all that is needed.

In the following, we define two ACO algorithms called $ACO_{bs, \tau_{min}}$ and $ACO_{bs, \tau_{min}(\Theta)}$, and we prove convergence results for both of them: convergence in value for ACO algorithms in $ACO_{bs, \tau_{min}(\Theta)}$ and convergence in solution for ACO algorithms in $ACO_{bs, \tau_{min}(\Theta)}$. Here, Θ is the iteration counter of the ACO algorithm and $\tau_{min}(\Theta)$ indicates that the t_{min} parameter may change during a run of the algorithm. We then show that these proofs continue to hold when typical elements of ACO, such as local search and heuristic information, are introduced. Finally, we discuss the significance of these results and demonstrate that the proof of convergence in value applies directly to two of the most successful ACO algorithms: MMAS and ACS.

Unfortunately, no results are currently available on the speed of convergence of any ACO algorithm. Therefore, the only way to measure algorithmic performance is to run extensive experimental tests.

Convergence Proofs

In this section, we study the convergence properties of some important subsets of ACO algorithms. First, we define the $ACO_{bs, \tau_{min}}$ algorithm and prove its convergence in value. Next,

we define the $\text{ACO}_{\text{bs}, \tau_{\min}}(\Theta)$ algorithm and prove its convergence in solution. After showing that both proofs continue to hold when local search and heuristic information are added, we discuss the meaning of the proofs and determine to which of the ACO algorithms the convergence in value proof applies.

Consider that, in the ant solution construction procedure, the initial location of each ant is chosen in a problem-specific way so that $F_{ij}(\tau_{ij}) \equiv F(\tau_{ij})$ [i.e., we remove the dependence of the function F on the arc (i, j) to which it is applied; this is tantamount to removing the dependence on the heuristic η]. Additionally, in order to facilitate the following derivations, we assume $F(\tau_{ij})$ to be of the form used in almost all ACO algorithms: $F(\tau_{ij}) = \tau_{ij}^\alpha$, where $0 < \alpha < +\infty$ is a parameter. The probabilistic construction rule applied by the ants to build solutions yields

$$P_{\mathcal{T}}(c_{h+1} = j | x_h) = \begin{cases} \frac{\tau_{ij}^\alpha}{\sum_{(i,l) \in \mathcal{N}_i^k} \tau_{il}^\alpha}, & \text{if } (i, j) \in \mathcal{N}_i^k; \\ 0, & \text{otherwise.} \end{cases} \quad (5)$$

Second, the pheromone update procedure is implemented by choosing $\hat{S}_\theta = s^{bs}$ (i.e., the reference set contains only the best-so-far solution) and, additionally, a lower limit $\tau_{\min} > 0$ is imposed upon the value of pheromone trails. In practice, the **ACO metaheuristic** procedure of figure 1 becomes the $\text{ACO}_{\text{bs}, \tau_{\min}}$ **PheromoneUpdate** procedure shown in figure 2.

```

procedure  $\text{ACO}_{bs, \tau_{min}}$  PheromoneUpdate
  foreach  $(i, j) \in L$  do
     $\tau_{ij} \leftarrow (1 - \rho)\tau_{ij}$ 
  end-foreach
  if  $f(s^\theta) < f(s^{bs})$  then
     $s^{bs} \leftarrow s^\theta$ 
  end-if
  foreach  $(i, j) \in s^{bs}$  do
     $\tau_{ij} \leftarrow \tau_{ij} + q_f(s^{bs})$ 
  end-foreach
  foreach  $(i, j)$  do
     $\tau_{ij} \leftarrow \max\{\tau_{min}, \tau_{ij}\}$ 
  end-foreach
end-procedure

```

Figure 2. High-level pseudo-code for the procedure PheromoneUpdate.

The value τ_{min} is a parameter of $\text{ACO}_{bs, \tau_{min}}$; in the following, we assume that $\tau_{min} < q_f(s^*)$. This can be achieved by setting, for example, $\tau_0 < q_f(\acute{s})/2$, where \acute{s} is a solution used to initialize $\text{ACO}_{bs, \tau_{min}}$.

The choice of the name $\text{ACO}_{bs, \tau_{min}}$ for this algorithm derives from the fact that the best-so-far solution is used to update pheromones and that a lower limit τ_{min} on the range of feasible pheromone trails is introduced.

Convergence in Value

In this subsection we prove that $\text{ACO}_{bs, \tau_{min}}$ is guaranteed to find an optimal solution with a probability that can be made arbitrarily close to 1 if given sufficient time. However, as was previously mentioned, we cannot prove convergence in solution for $\text{ACO}_{bs, \tau_{min}}$.

Before proving the first theorem, it is convenient to show that, due to pheromone evaporation, the maximum possible pheromone level τ_{\max} is asymptotically bounded.

Proposition 1 *For any τ_{ij} it holds that*

$$\lim_{\Theta \rightarrow \infty} \tau_{ij}(\Theta) \leq \tau_{\max} = \frac{q_f(s^*)}{\rho}.$$

Proof The maximum possible amount of pheromone added to any arc (i, j) after any iteration is $q_f(s^*)$. Clearly, at iteration 1 the maximum possible pheromone trail is $(1 - \rho)\tau_0 + q_f(s^*)$, at iteration 2 it is $(1 - \rho)^2\tau_0 + (1 - \rho)q_f(s^*) + q_f(s^*)$, and so on. Hence, due to pheromone evaporation, the pheromone trail at iteration Θ is bounded by

$$\tau_{ij}^{\max}(\Theta) = (1 - \rho)^\Theta \tau_0 + \sum_{i=1}^{\Theta} (1 - \rho)^{\Theta-i} q_f(s^*).$$

As $0 < \rho \leq 1$, this sum converges asymptotically to

$$\tau_{\max} = \frac{q_f(s^*)}{\rho}.$$

Proposition 2 *Once an optimal solution s^* has been found, it holds that*

$$\forall (i, j) \in s^* : \lim_{\Theta \rightarrow \infty} \tau_{ij}^*(\Theta) = \tau_{\max} = \frac{q_f(s^*)}{\rho},$$

where τ_{ij}^* is the pheromone trail value on connections $(i, j) \in s^*$.

Proof Once an optimal solution has been found, recalling that $\forall \Theta \geq 1, \tau_{ij}^*(\Theta) \geq \tau_{\min}$ and that the best-so-far update rule is used, we have that $\tau_{ij}^*(\Theta)$ monotonically increases. The proof of proposition 2 is basically a repetition of the proof of proposition 1, restricted to the connections

of the optimal solution (τ_0 is replaced by $\tau_{ij}^*(\Theta^*)$ in the proof of proposition 1, where Θ^* is the iteration in which the first optimal solution was found).

Proposition 1 implies that, for the following proof of theorem 1, the only essential point is that $\tau_{\min} > 0$, because τ_{\max} will be bounded by pheromone evaporation. Proposition 2 additionally states that, once an optimal solution has been found, the value of the pheromone trails on all connections of s^* converges to $\tau_{\max} = \frac{qf(s^*)}{\rho}$.

We can now prove the following theorem:

Theorem 1 *Let $P^*(\Theta)$ be the probability that the algorithm finds an optimal solution at least once within the first Θ iterations. Then, for an arbitrarily small $\varepsilon > 0$ and for a sufficiently large Θ , it holds that*

$$P^*(\Theta) \geq 1 - \varepsilon$$

and, by definition, $\lim_{\Theta \rightarrow \infty} P^*(\Theta) = 1$.

Proof Due to the pheromone trail limits τ_{\min} and τ_{\max} we can guarantee that any feasible choice in equation (5) for any partial solution x_h is made with a probability $p_{\min} > 0$. A trivial lower bound for p_{\min} is given by

$$p_{\min} \geq \hat{p}_{\min} = \frac{\tau_{\min}^\alpha}{(N_C - 1)\tau_{\max}^\alpha + \tau_{\min}^\alpha},$$

where N_c is the cardinality of the set C of components. (For the derivation of this bound we consider the following ‘‘worst-case’’ situation: the pheromone trail associated with the desired decision is τ_{\min} , while all the other feasible choices —there are

at most $N_c - I$ —have an associated pheromone trail of τ_{\max} .) Then, any generic solution s , including any optimal solution $s^* \in S^*$, can be generated with a probability $\hat{p}_{ij}^*(\Theta) > 0$, where $n < +\infty$ is the maximum length of a sequence. Because only one ant need find an optimal solution, a lower bound for $P^*(\Theta)$ is given by

$$\hat{P}^*(\Theta) = 1 - (1 - \hat{p})^\Theta.$$

By choosing a sufficiently large Θ , this probability can be made larger than any value $1 - \varepsilon$. Hence, we have that $\lim_{\Theta \rightarrow \infty} \hat{P}^*(\Theta) = 1$.

Convergence in Solution

In this subsection we prove convergence in solution for $\text{ACO}_{\text{bs},\tau_{\min}(\Theta)}$, which differs from $\text{ACO}_{\text{bs},\tau_{\min}}$ in that it allows a change in value for τ_{\min} while solving a problem. That is, we prove that, in the limit, any arbitrary ant of the colony will construct the optimal solution with probability 1. This cannot be proved if we impose, as was done in the case of $\text{ACO}_{\text{bs},\tau_{\min}}$, a small, positive lower bound on the lower pheromone trail limits because at any iteration Θ , each ant can construct any solution with a nonzero probability. The key to the proof is therefore to allow the lower pheromone trail limits to decrease over time toward zero, but making this decrement occur at a sufficiently slow rate to guarantee that the optimal solution is eventually found. We call $\text{ACO}_{\text{bs},\tau_{\min}(\Theta)}$ the modification of $\text{ACO}_{\text{bs},\tau_{\min}}$ obtained in this way, where $\tau_{\min}(\Theta)$ indicates the dependence of the lower pheromone trail limits on the iteration counter.

The proof of convergence in solution is organized in two theorems. First, in theorem 2 (in a way analogous to what was done in the proof of theorem 1), we prove that it can still be

guaranteed that an optimal solution is found with a probability converging to 1 when lower pheromone trail limits of the $ACO_{bs, \tau_{min}(\Theta)}$ algorithm decrease toward 0 at not more than logarithmic speed (in other words, we prove that $ACO_{bs, \tau_{min}(\Theta)}$ converges in value). Next, in theorem 3 we prove, under the same conditions, convergence in solution of $ACO_{bs, \tau_{min}(\Theta)}$.

Theorem 2 *Let the lower pheromone trail limits in $ACO_{bs, \tau_{min}(\Theta)}$ be*

$$\forall \Theta \geq 1, \tau_{min}(\Theta) = \frac{d}{\ln(\Theta + 1)},$$

with d being a constant, and let $\hat{p}(\Theta) \geq (\hat{p}'_{min}(\Theta))^n$ be the probability that the algorithm finds an optimal solution at least once within the first Θ iterations. Then it holds that

$$\lim_{\Theta \rightarrow \infty} P^*(\Theta) = 1.$$

Proof In a manner distinct from what was done in the proof of theorem 1, we prove here that an upper bound on the probability of not constructing the optimal solution is 0 in the limit (i.e., the optimal solution is found in the limit with probability 1). Let the event E_Θ denote that iteration Θ is the iteration in which an optimal solution is found for the first time. The event $\bigwedge_{\Theta=1}^{\infty} \neg E_\Theta$ that no optimal solution is ever found, implies that also one arbitrary, but fixed, optimal solution s^* is never found. Therefore, an upper bound to the probability $P(\bigwedge_{\Theta=1}^{\infty} \neg E_\Theta)$ is given by $P(s^*$ is never traversed):

$$P(\bigwedge_{\Theta=1}^{\infty} \neg E_\Theta) \leq P(s^* \text{ is never traversed}), \quad (6)$$

Now, in a way similar to what was done in the proof of theorem 1, we can guarantee that at a generic iteration Θ any feasible choice according to equation (5) can be made with probability p_{min} bounded as follows:

$$\begin{aligned}
p_{min} &\geq \hat{p}_{min}(\theta) = \frac{\tau_{min}^\alpha(\theta)}{(N_C - 1)\tau_{max}^\alpha + \tau_{min}^\alpha(\theta)} \\
&\geq \frac{\tau_{min}^\alpha(\theta)}{N_C \tau_{max}^\alpha} = \hat{p}'_{min}(\theta).
\end{aligned}$$

Then, a lower bound on the probability that a fixed ant k is constructing the optimal solution s^* is given by $\hat{p}(\Theta) \geq (\hat{p}'_{min}(\Theta))^n$, where $n < +\infty$ is the maximum length of a sequence. This bound is independent of what happened before iteration Θ . Therefore, we can give the following upper bound on the right side of equation (6):

$$\begin{aligned}
P(s^* \text{ is never traversed}) &\leq \prod_{\theta=1}^{\infty} (1 - (\hat{p}'_{min}(\theta))^n) \\
&= \prod_{\theta=1}^{\infty} \left(1 - \left(\frac{\tau_{min}^\alpha(\theta)}{N_C \tau_{max}^\alpha} \right)^n \right).
\end{aligned} \tag{7}$$

We now must prove that this product is equal to 0. To do so, we consider its logarithm

$$\sum_{\theta=1}^{\infty} \ln \left[1 - \left(\frac{\tau_{min}^\alpha(\theta)}{N_C \tau_{max}^\alpha} \right)^n \right],$$

and we show that the resulting series, starting from some finite number l , grows more quickly than the harmonic series, so that it diverges to $-\infty$, which implies that the original product is equal to 0. First, recall that $\tau_{min}(\Theta) = d/\ln(\Theta + 1)$. Then

$$\begin{aligned}
\sum_{\theta=1}^{\infty} \ln \left[1 - \left(\frac{\tau_{min}^{\alpha}(\theta)}{N_c \tau_{max}^{\alpha}} \right)^n \right] &= \sum_{\theta=1}^{\infty} \ln \left[1 - \left(\frac{\left(\frac{d}{\ln(\theta+1)} \right)^{\alpha}}{N_c \tau_{max}^{\alpha}} \right)^n \right] \\
&= \sum_{\theta=1}^{\infty} \ln \left[1 - \left(\frac{d_1}{(\ln(\theta+1))^{\alpha}} \right)^n \right] \\
&\leq -d_1^n \sum_{\theta=1}^{\infty} \left(\frac{1}{\ln(\theta+1)} \right)^{\alpha n} = -\infty,
\end{aligned}$$

where $d_1 = d^{\alpha} / N_c \tau_{max}^{\alpha}$.

The inequality holds because for any $x < 1$, $\ln(1-x) \leq -x$. The equality holds because

$\sum_x (\ln x)^{-i}$ is a diverging series. To see the latter, note that for each positive constant $\delta > 0$ and for sufficiently large x , $(\ln x)^i \leq \delta * x$, and therefore $\delta / (\ln x)^i \geq 1/x$. It then suffices to remember that $\sum_x 1/x$ is the harmonic series, which is known to diverge to ∞ .

These derivations assert that an upper bound exists for the logarithm of the product given in equation (7) and, hence, the logarithm on the right side of equation (6) is $-\infty$; therefore, the products given in equation (7) and on the right side of equation (6) must both be 0; in other words, the probability of never finding the optimal solution $P(\bigwedge_{\theta=1}^{\infty} \neg E_{\theta})$ is 0. Thus, an optimal solution will be found with probability 1.

In the limiting case, once the optimal solution has been found, we can estimate an ant's probability of constructing an optimal solution when following the stochastic policy of the algorithm. In fact, it can be proved that any ant will in the limit construct the optimal solution with probability 1—that is, we can prove convergence in solution. Before the

proof of this assertion, it is convenient to show that the pheromone trails of connections that do not belong to the optimal solution asymptotically converge to 0.

Proposition 3 *Once an optimal solution has been found and for any $\tau_{ij}(\Theta)$ such that $(i, j) \notin s^*$, it holds that*

$$\lim_{\Theta \rightarrow \infty} P(s^*, \Theta, k) = 1.$$

Proof After the optimal solution has been found, connections not belonging to the optimal solution no longer receive pheromone. Thus, their value can only decrease. In particular, after one iteration, $\tau_{ij}(\Theta^* + 1) = \max\{\tau_{\min}(\Theta^*), (1 - \rho)\tau_{ij}(\Theta^*)\}$, after two iterations, $\tau_{ij}(\Theta^* + 2) = \max\{\tau_{\min}(\Theta^*), (1 - \rho)^2\tau_{ij}(\Theta^*)\}$, and so on (Θ^* is the iteration in which s^* was first found). Additionally, we have that $\lim_{\Theta \rightarrow \infty} d/\ln(\Theta^* + \Theta + 1) = 0$ and $\lim_{\Theta \rightarrow \infty} (1 - \rho)^\Theta \tau_{ij}(\Theta^* + \Theta) = 0$.

Therefore, $\lim_{\Theta \rightarrow \infty} \tau_{ij}(\Theta^* + \Theta) = 0$.

Theorem 3 Let Θ^* be the iteration in which the first optimal solution has been found and $P(s^*, \Theta, k)$ be the probability that an arbitrary ant k constructs s^* in the Θ th iteration, with $\Theta > \Theta^*$. Then it holds that

$$\lim_{\Theta \rightarrow \infty} P(s^*, \Theta, k) = 1.$$

Proof Let ant k be located on component i and (i, j) be a connection of s^* . A lower bound $\hat{p}_{ij}^*(\Theta)$ for the probability $p_{ij}^*(\Theta)$ that ant k makes the ‘‘correct choice’’ (i, j) is given by the term

$$\hat{p}_{ij}^*(\theta) = \frac{(\tau_{ij}^*(\theta))^\alpha}{(\tau_{ij}^*(\theta))^\alpha + \sum_{(i,h) \notin s^*} (\tau_{ih}(\theta))^\alpha}.$$

Because of propositions 2 and 3 we have

$$\hat{p}_{ij}^* = \lim_{\theta \rightarrow \infty} \hat{p}_{ij}^*(\theta) = \frac{\lim_{\theta \rightarrow \infty} (\tau_{ij}^*(\theta))^\alpha}{\lim_{\theta \rightarrow \infty} (\tau_{ij}^*(\theta))^\alpha + \sum_{(i,h) \notin s^*} \lim_{\theta \rightarrow \infty} (\tau_{ih}(\theta))^\alpha}$$

$$= \frac{\tau_{max}^\alpha}{\tau_{max}^\alpha + \sum_{(i,h) \notin s^*} 0^\alpha} = 1.$$

Hence, in the limit, any fixed ant will construct the optimal solution with probability 1 because at each construction step, it makes the correct decision with probability 1.

Further Considerations

Many ACO algorithms include some features that are present neither in $ACO_{bs,\tau_{min}}$ nor in $ACO_{bs,\tau_{min}(\Theta)}$. The most important among them are the use of local search algorithms to improve the solutions constructed by the ants and the use of heuristic information in the choice of the next component. Therefore, a natural question to consider is how these features affect the convergence proof for $ACO_{bs,\tau_{min}}$. Note that here and in the following, because remarks made about $ACO_{bs,\tau_{min}}$ typically also apply to $ACO_{bs,\tau_{min}(\Theta)}$, we often refer only to $ACO_{bs,\tau_{min}}$.

Let us first consider the use of local search. Local search tries to improve an ant's solution s by iteratively applying small, local changes to it. Typically, the best solution s' found by the local search is returned and used to update the pheromone trails. It is rather easy to see that the use of local search neither affects the convergence properties of $ACO_{bs,\tau_{min}}$, nor those of $ACO_{bs,\tau_{min}(\Theta)}$. In fact, the validity of both convergence proofs depends only on the way solutions are constructed and not on the fact that the solutions are taken or not to their local optima by a local search routine.

Available a priori information concerning the problem can be used to derive heuristic information that biases the probabilistic decisions to be made by the ants. When incorporating such heuristic information into $ACO_{bs, \tau_{min}}$, the most common choice is $F_{ij}(\tau_{ij}) = [\tau_{ij}]^\alpha [\eta_{ij}]^\beta$. In this case equation (5), becomes

$$P_T(c_{h+1} = j | x_h) = \begin{cases} \frac{[\tau_{ij}]^\alpha [\eta_{ij}]^\beta}{\sum_{(i,l) \in N_i^k} [\tau_{il}]^\alpha [\eta_{il}]^\beta}, & \text{if } (i, j) \in N_i^k; \\ 0, & \text{otherwise} \end{cases}$$

where η_{ij} measures the heuristic desirability of adding solution component j . In fact, neither theorem 1 nor theorems 2 and 3 are affected by the heuristic information if we have $0 < \eta_{ij} < +\infty$ for each $(i, j) \in L$ and $\beta < \infty$. Given these assumptions, η is limited to some (problem-specific) interval $[\eta_{min}, \eta_{max}]$, with $\eta_{min} > 0$ and $\eta_{max} < +\infty$. Then, the heuristic information simply has the effect of changing the lower bound on the probability p_{min} of making a specific decision.

Summary

It is instructive to reflect upon what theorems 1 through 3 really tell us. First, theorem 1 expresses that when using a fixed positive lower bound on the pheromone trails, $ACO_{bs, \tau_{min}}$ is guaranteed to find the optimal solution. Theorem 2 extends this result by saying that we essentially can keep this property for $ACO_{bs, \tau_{min}(\Theta)}$ algorithms, if we decrease the bound τ_{min} to 0 slowly enough. (Unfortunately, theorem 2 cannot be proved for the exponentially fast decrement of the pheromone trails obtained by a constant pheromone evaporation rate, which most ACO algorithms use.) However, the proofs say nothing about the time required to find an optimal

solution, which can be prohibitively long. A similar limitation applies to other well-known convergence proofs, such as those formulated for simulated annealing by Hajek (1988) and by Romeo & Sangiovanni-Vincentelli (1991). Finally, theorem 3 shows that a sufficiently slow decrement of the lower pheromone trail limits leads to the effect that the algorithm converges to a state in which the ants repeatedly construct the optimal solution. In fact, for this latter result it is essential that in the limit the pheromone trails go to 0. If, as is done in $ACO_{bs, \tau_{min}}$, a fixed lower bound τ_{min} is set, it can only be proved that the probability of constructing an optimal solution is larger than $1 - \hat{\varepsilon}(\tau_{min}, \tau_{max})$, where $\hat{\varepsilon}$ is a function of τ_{min} and τ_{max} . Because in practice we are more interested in finding an optimal solution at least once than in generating it again and again, we should take a closer look at the role played by τ_{min} and τ_{max} in the proof of theorem 1: the smaller the ratio τ_{max} / τ_{min} , the larger the lower bound \hat{p}_{min} given in the proof. This is important because the larger the \hat{p}_{min} , the smaller the worst-case estimate of the number of iterations Θ needed to assure that an optimal solution is found with a probability larger than the quantity $1 - \varepsilon$. In fact, the most restrictive bound is obtained if all pheromone trails are the same; that is, for the case of uniformly random solution construction. In this case, we would have $\hat{p}_{min} = 1 / N_c$ (note that this fact is independent of the proximity of the lower bound used in theorem 1). This somewhat counterintuitive result is due to the fact that our proof is based on a worst-case analysis; we need to consider the worst-case situation in which the bias in the solution construction introduced by the pheromone trails is counterproductive and leads to suboptimal solutions. In other words, we have to assume that the pheromone level associated with the connection an ant needs to pass in order to construct an optimal solution is τ_{min} , while on the other connections, it is much higher—in the worst case corresponding to τ_{max} . In practice,

however, as shown by the results of many published experimental works, this does not happen, and the bias introduced by the pheromone trails does indeed help to hasten convergence to an optimal solution.

CHAPTER FIVE: EVOLUTIONARY COMPUTING/ACO HYBRIDS

Having introduced the ACO metaheuristic and reviewed its operation from both a theoretical and a practical perspective, we may now address the first of the issues enumerated in the introduction, which will be presented here as a critique of the appropriateness of the hybridization of evolutionary computing (as well as the closely related family of genetic algorithms) and ACO from the perspective of biological theory. Unfortunately, in light of the mathematically intractable nature of problem, the argument for a closer review of the suitability of combining these two metaheuristics is conducted as follows: First, the mechanism of natural selection as it is broadly implemented in evolutionary computing is contrasted with the most modern theories of ant ecology. Second, an assessment of two ACO algorithm's, one being a hybridization of a conventional evolutionary algorithm and ACO and the other a hybridization that more accurately reflects the relevant biological principles, will be conducted. Specifically, each will be tasked with identical Steiner tree construction problems in order that their relative performance may be evaluated. A prerequisite to the preceding goals, however, is a better understanding of certain evolutionary computing principles.

A Primer on Evolutionary Programming and Genetic Algorithms

Evolutionary Programming (EP), as the name implies, mimics evolutionary biology. Using natural selection as a metaphor, EP looks upon generations of a species and its competitors within a particular ecological niche as *solutions* for occupying that niche. EP submits the notion that solutions, which in this case would be analogous to parents, may be mutated to produce new more viable solutions as their *offspring*. Within Genetic Algorithms (GA), the fundamental

agents are best compared to individual organisms, whereas in EP, that role would fall to the algorithms equivalent of a species of animal. The evolution of a solution (i.e., a species) into progressively improved solutions is realized by the EP model according to the following process:

1. A set of finite automata is randomly generated to represent a population of solutions. This may be regarded as the first group of parents—a veritable Adam and Eve, if you will.
2. Each of the solutions is copied into a new population of offspring, wherein a mutation operator that alters the behavior of the individuals in the new population is applied. The behavior of each of the offspring is compared to that of its parent in order to generate a distribution of changes that best addresses the problem at hand.
3. Each individual is then assessed for fitness. Fitness may be evaluated against a variety of criteria (the ability to recognize a particular input sequence, for example). A percentage of the individuals deemed most fit are selected to serve as the next generation of parents.

Steps 2 and 3 continue until a set of finite automata demonstrate the capacity to attain the prescribed goal.

Evolutionary Programming is conceptually similar to Genetic Algorithms, with the following notable exceptions: First, EP takes note of the ways in which the behavioral qualities of parents and their offspring differ, whereas GA's usually represent individual solutions as vectors or strings. Interjecting the biological metaphor momentarily, EP functions in the domain of the phenotype, while GA operates in the domain of the genotype. In the case of EP, the

structure of the genes is not considered; rather, it is their impact on the behavior of the proverbial creature, or automaton, that is of interest.

Second, EP represents mutation differently than GA. An EP mutation operator is defined in such a way that it favors mutations that generate behavioral distributions of small rather than large variance. Additionally, as the algorithm progresses and good solutions begin to evolve, the mutation operator creates distributions of progressively smaller variances.

Both EP and GA are useful methods for problems of optimization. EP is particularly useful for combinatorial optimization, especially when there are a multitude of potential solutions as opposed to a single global solution. The matter of their utility when combined with ACO is addressed in the following section.

Evolution and the Ant

None other than perhaps the most preeminent of entomologists (or more specifically, myrmecologists), Edward O. Wilson, has shown that natural selection operates upon the social ants at the *colony* level [16]. The colony is selected as a whole, and its members contribute to colony fitness rather than to individual fitness. Indeed, Wilson characterizes the extremely rare and inconsequential occurrence of individual selection among such ants as a dissolutive force. Consequently, insofar as a computing discipline appropriates natural selection as a metaphor with respect to the behavior of ants, the particular means by which the phenomenon of natural selection imposes itself in the case of the ant would seem an integral (and perhaps, essential) component in any model thusly derived. In that both Evolutionary Computing and Genetic

Algorithms employ selection at the level of the *individual*, any combination of either of these two metaheuristics with ACO raises the question of their compatibility.

It is this question of compatibility that is addressed in the following section. Regrettably, it is not a question that lends itself to rigorous mathematical analysis. Such an analysis, if possible, would likely shed light upon the matter. Instead, this matter of compatibility is addressed in a qualitative fashion, using the aforementioned task of Steiner tree construction as an admittedly imprecise measure of the relative qualities of two solutions: one from a hybridized EC/ACO algorithm and the other from an ACO algorithm that implements a procedure analogous to colony-level selection. The latter is a program developed by the author in Matlab. Broadly speaking, it pits colonies of ants in competition against one another, the most fit of which (with a colony's fitness assessed by the collective quality of its solutions) survive to send some number of *queens* (each bearing a uniquely mutated version of the *genetic* character of the original colony's queen) to locations in the solution space that have previously shown promise.

Results

Each of the aforementioned programs was presented with identical tasks. A constellation of random points was generated, with a subset of these points selected (again, randomly) to be the nodes of a Steiner tree. The cost upon which both algorithms based the relative worth of potential solutions was simply the sum of the lengths of the *branches*, or segments connecting the nodes of the Steiner tree. Both programs proposed reasonable solutions that appear comparable in quality. A more precise comparison of their performance would offer little more in the way of conclusions as there are significant differences between them other than their

disparate implementations of the evolution metaphor. However, the viability of the algorithm incorporating selection at the colony-level appears to have been confirmed. Figure 2 displays a typical result.

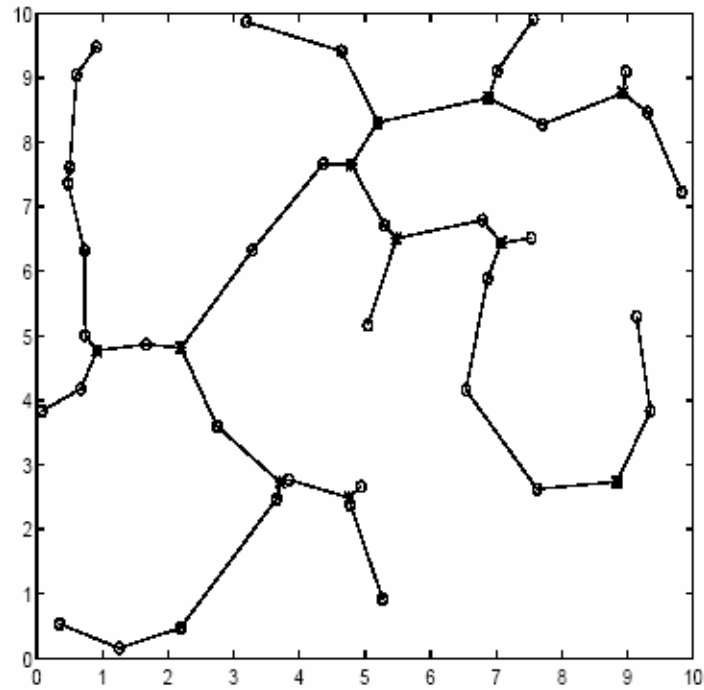


Figure 3. Steiner tree solution generated by Matlab ACO with colony-level selection.

CHAPTER SIX: CONCLUSIONS

Despite the critical tenor of this review, there is no denying ACO's successful exploitation of stigmergy (i.e., the term describing the ants' indirect, pheromone-based mode of communication) as a computing tool of significant potential. However, the possibility that daemon actions may exert a dissolutive influence (similar to that of the potential misapplication of evolutionary theory considered in the last chapter) upon a paradigm that is fundamentally one of decentralized control warrants further investigation. Though inclusion of such global controls in ACO has undoubtedly yielded net improvement, a cursory examination admits the possibility that they may suppress in some measure the densely heterarchical dynamic observed among ants in a colony. A definitive answer is more likely to be found analytically. The latter of these two approaches had been the primary goal of this exercise, though it is one that will have to be addressed at some future date for reasons stated previously.

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