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## Interval Edge-Colorings of Graphs

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INTERVAL EDGE-COLORINGS OF GRAPHS

by

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B.S. University of Central Florida, 2015

A thesis submitted in partial fulfilment of the requirements  
for the degree of Master of Science  
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Major Professor: Zixia Song

## ABSTRACT

A proper edge-coloring of a graph  $G$  by positive integers is called an interval edge-coloring if the colors assigned to the edges incident to any vertex in  $G$  are consecutive (i.e., those colors form an interval of integers). The notion of interval edge-colorings was first introduced by Asratian and Kamalian in 1987, motivated by the problem of finding compact school timetables. In 1992, Hansen described another scenario using interval edge-colorings to schedule parent-teacher conferences so that every person's conferences occur in consecutive slots. A solution exists if and only if the bipartite graph with vertices for parents and teachers, and edges for the required meetings, has an interval edge-coloring.

A well-known result of Vizing states that for any simple graph  $G$ ,  $\chi'(G) \leq \Delta(G) + 1$ , where  $\chi'(G)$  and  $\Delta(G)$  denote the edge-chromatic number and maximum degree of  $G$ , respectively. A graph  $G$  is called class 1 if  $\chi'(G) = \Delta(G)$ , and class 2 if  $\chi'(G) = \Delta(G) + 1$ . One can see that any graph admitting an interval edge-coloring must be of class 1, and thus every graph of class 2 does not have such a coloring.

Finding an interval edge-coloring of a given graph is hard. In fact, it has been shown that determining whether a bipartite graph has an interval edge-coloring is NP-complete. In this thesis, we survey known results on interval edge-colorings of graphs, with a focus on the progress of  $(a, b)$ -biregular bipartite graphs. Discussion of related topics and future work is included at the end. We also give a new proof of Theorem 3.15 on the existence of proper path factors of  $(3, 4)$ -biregular graphs. Finally, we obtain a new result, Theorem 3.18, which states that if a proper path factor of any  $(3, 4)$ -biregular graph has no path of length 8, then it contains paths of length 6 only. The new result we obtained and the method we developed in the proof of Theorem 3.15 might be helpful in attacking the open problems mentioned in the Future Work section of Chapter 5.

## **ACKNOWLEDGMENTS**

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# CHAPTER 1: INTRODUCTION

## Graph Terminology

A *graph*  $G$  consists of a set of *vertices*  $V(G)$  and a set of *edges*  $E(G)$ , where  $E(G)$  contains 2-element subsets of  $V(G)$ . In a drawing of a graph, vertices are points, and each edge is a line between two vertices which are known as its *endpoints*. When an edge  $e$  has vertices  $u$  and  $v$  as its endpoints, we say that  $e$  is *incident* to  $u$  and  $v$ , and write  $e = uv$ . On the other hand, since  $u$  and  $v$  are endpoints of the same edge  $e$ , we say those vertices are *adjacent*. We also say that  $u$  is a *neighbor* of  $v$ , and  $v$  is a neighbor of  $u$ . A *simple graph* is one with no *loops* (edges with the same vertex for both endpoints) or *parallel edges* (sets of edges which share the same endpoints). A *multigraph* is one which has loops or parallel edges. In this thesis, multigraphs have no loops unless specified otherwise.

A *subgraph*  $H$  of a graph  $G$ , denoted  $H \subseteq G$ , is a graph with  $V(H) \subseteq V(G)$  and  $E(H) \subseteq E(G)$ . We say  $H$  is a *spanning* subgraph of  $G$  if  $H \subseteq G$  with  $V(H) = V(G)$ . Given a vertex  $v \in V(G)$ , we write  $d_G(v)$  to represent the degree of the vertex  $v$  – that is, the number of edges incident to  $v$  – in  $G$ . We use  $\delta(G)$  and  $\Delta(G)$  to refer to the minimum and maximum degree of all vertices in  $G$ , respectively. The *neighborhood* of a vertex  $v$ , written  $N(v)$ , is the set of all vertices adjacent to  $v$ .

A *path* on  $n$  vertices, or an  $n$ -path,  $P_n$ , is a graph whose vertices can be ordered linearly as  $\{v_1, v_2, \dots, v_n\}$  such that  $v_i$  and  $v_j$  are adjacent if and only if they appear consecutively in the ordering. We define the number of edges in a path as its *length*. If a path  $P$  has vertices  $v_1, v_2, \dots, v_t$  in order, we write  $P = v_1v_2 \dots v_t$  and say that  $P$  is a path from  $v_1$  to  $v_t$ . We use  $v_1Pv_i$  (respectively,  $v_iPv_t$ ) to denote the subpath of  $P$  with vertices  $v_1, v_2, \dots, v_i$  (respectively,  $v_i, v_{i+1}, \dots, v_t$ ) in order. The vertices  $v_1$  and  $v_t$  are the *end vertices* or *ends* of  $P$ , and  $v_2, \dots, v_{t-1}$  are called *inter-*



nal vertices of  $P$ . The graph  $C_n := P_n + v_1v_n$  is called a *cycle* on  $n$  vertices. The *complete* graph on  $n$  vertices,  $K_n$ , is a graph in which every pair of distinct vertices are adjacent.

*Bipartite* graphs can be divided into two bipartitions, which are disjoint vertex sets covering all of the vertices in the graph, such that every edge has one endpoint in each bipartition. The *complete bipartite* graph  $K_{m,n}$  is a bipartite graph with bipartitions  $(X, Y)$  such that  $xy \in E(G)$  for all  $x \in X$  and  $y \in Y$ . Finally, an  $(a, b)$ -*biregular* graph is a bipartite graph with bipartition  $(X, Y)$  in which every vertex  $x \in X$  has degree  $a$  and every  $y \in Y$  has degree  $b$ .

We call a graph  $G$  *connected* if given any  $x, y \in V(G)$ , there exists an  $(x, y)$ -path  $P \subseteq G$  (a path with ends  $x$  and  $y$ ). The *distance* between two vertices  $x$  and  $y$  is defined to be the minimum length of an  $(x, y)$ -path, if one exists, and  $\infty$  otherwise. The *graph diameter*, denoted  $\text{diam}(G)$ , is the maximum distance between any pair of vertices in  $G$ . A *tree* is a connected graph which does not contain any cycles as subgraphs.

The *cartesian product* of two graphs,  $G$  and  $H$ , denoted  $G \times H$ , is the graph with vertex set  $V(G) \times V(H)$  such that  $(u, v)$  is adjacent to  $(x, y)$  when either (a)  $u = x$  and  $vy \in E(H)$ , or (b)  $v = y$  and  $ux \in E(G)$ .

A graph  $G$  is called *planar* if it has a planar embedding (a drawing of the graph on the plane without crossed edges). In a planar drawing of a graph, the vertices and edges of the graph divide the plane into *faces*, of which there is exactly one unbounded (infinite) face. *Outerplanar* graphs have the property that each vertex lies on the boundary of the unbounded face. Furthermore, the bounded faces of an *outerplanar triangulation* must be formed by triangles.

## Edge-Colorings

An *edge-coloring* of a graph  $G$  is a function  $c : E(G) \rightarrow \mathbb{N}$ , where the numbers represent colors. While vertex colorings are an interesting topic of study, all colorings in this paper are edge-colorings unless otherwise stated. We say a color is *present* at a vertex  $v$  under an edge-coloring  $c$  if  $c$  assigns it to an edge incident to  $v$ . Note that the colors present at a vertex  $v$  form a subset of integers, which we will denote  $c(v)$ . We call an edge-coloring a *proper* edge-coloring if no two edges incident to the same vertex are assigned the same color.

We use  $\chi'(G)$  to denote the *chromatic index* of a graph  $G$ , which is the minimum number of colors that may be used to properly color the edges of  $G$ . Note that for any graph  $G$ ,  $\chi'(G) \geq \Delta(G)$ , as each vertex  $v \in V(G)$  must have  $d_G(v)$  distinct colors present. A well-known theorem of Vizing [35] states that  $\chi'(G) \leq \Delta(G) + 1$ ; as a result, graphs may be either of *class 1* (when  $\chi'(G) = \Delta(G)$ ), or *class 2* (when  $\chi'(G) = \Delta(G) + 1$ ).

A proper edge-coloring  $c : E(G) \rightarrow \mathbb{N}$  of a graph  $G$  is called an *interval edge-coloring* if for every vertex  $v \in V(G)$ , the colors present at  $v$  form an interval of positive integers. Such colorings have also been referred to as consecutive and compact edge-colorings. Asratian and Kamalian [3] introduced the topic of interval edge-colorings in 1987 (in English as [5]), and completed foundational work along with Hansen [17]. Recent work has addressed the existence of these colorings in certain classes of graphs, as well as the number of colors that may be used.

We take an *interval  $t$ -coloring* of a graph  $G$  to be an interval edge-coloring  $c : E(G) \rightarrow \{1, 2, \dots, t\}$ . Finally, we can characterize the existence of an interval edge-coloring in terms of *deficiency*. Given a graph  $G$ , the *deficiency* of a coloring  $c$  at a vertex  $v$  is the minimum number of colors which must be added to  $c(v)$  in order to form an interval of integers. Similarly, the *deficiency* of a coloring  $c$  is the sum of the deficiencies at each vertex in  $G$ . Lastly, the *deficiency* of a graph  $G$  is the min-

imum deficiency over all proper edge-colorings of  $G$ . By this definition, a graph has an interval edge-coloring if it has deficiency zero.

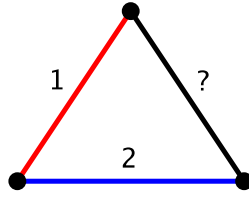
## **Applications**

Asratian and Kamalian [4] described the use of interval edge-colorings in the creation of compact school timetables, in which classes are scheduled so that neither lecturers nor groups of students experience gaps in the schedule. Hansen described a similar application in which conferences between parents and teachers are scheduled without waiting periods [10, 17]. More generally, Giaro, Kubale, and Malafiejski [14] frame the problem in terms of scheduling zero-one time operations in an “open shop.” Here “jobs” are scheduled for work among a collection of “processors,” at which each operation takes either zero or one unit of time. In each case, the two groups (lecturers and classes; parents and teachers; jobs and processors) are represented as bipartitions of a bipartite graph. A consecutive scheduling is represented by an interval edge-coloring of such a graph.

While bipartite graphs are a large focus of study, interval edge-colorings have applications related to non-bipartite graphs as well. Axenovich [7] described the situation in which a group of people wish to schedule conferences with others in the group. Given that all of the conferences are the same length of time, a consecutive schedule is once again represented by an interval edge-coloring of a graph representing the scheduling problem.

## **Topics of Study**

Not all graphs have interval edge-colorings – a small example is the triangle  $K_3$ , as shown in Figure 1.1 – and finding them is not simple. The existence of such colorings for various classes of general graphs is the focus of Chapter 2. Since bipartite graphs represent many scheduling



**Figure 1.1:** Triangle  $K_3$  with a partial edge-coloring.

applications, we explore this area separately in Chapter 3. Beyond existence, the maximum and minimum number of colors for which an interval edge-coloring exists is another interesting problem. In Chapter 4, we consider lower and upper bounds on the number of colors used in interval edge-colorings for several classes of graphs.

There are a number of generalizations and related topics. Graphs may have *near interval edge-colorings*, in which the set of colors present at any vertex would be an interval with the addition of a single color. They may also have *cyclic interval edge-colorings*, in which the colors at each vertex form an interval modulo the total number of colors. Both are discussed in Chapter 5, along with a brief note on the computational runtimes of algorithms related to interval edge-colorings.

## CHAPTER 2: GENERAL GRAPHS

In this chapter we consider interval edge-colorings of general and some special classes of graphs. A large amount of work on bipartite and biregular graphs is considered in Chapter 3.

As mentioned in Chapter 1, not all graphs have interval edge-colorings. In fact, there are many graphs which do not have such colorings. Asratian and Kamalian [5] proved the following:

**Proposition 2.1.** If a multigraph  $G$  has an interval edge-coloring, then  $\chi'(G) = \Delta(G)$ .

As a result, all class 2 graphs are known not to have interval colorings. The complete graph on three vertices,  $K_3$ , is such a graph (see Figure 1.1). In fact,  $K_{2n+1}$  has no interval edge-coloring for any  $n \in \mathbb{N}$ .

While a large number of graphs do not have interval edge-colorings, there are many graphs that do. Asratian and Kamalian [4] showed:

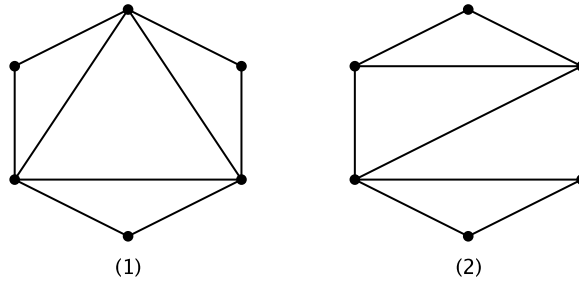
**Proposition 2.2.** A tree  $T$  has an interval  $\Delta(T)$ -coloring.

Such a coloring can be obtained by induction on the number of vertices of  $T$ , or by searching through the tree and assigning the next-lowest integer color which maintains an interval of colors on the source vertex. Asratian and Kamalian [5] proved:

**Corollary 2.3.** A  $k$ -regular multigraph  $G$  has an interval  $k$ -coloring if and only if  $\chi'(G) = k$ .

Casselgren and Toft [10] explain that in this case,  $E(G)$  consists of  $k$  perfect matchings, which may be colored accordingly. Giaro and Kubale [13] considered the cartesian product of graphs and proved the following:

**Theorem 2.4.** If a graph  $G$  has an interval  $\Delta(G)$ -coloring, then the graph cartesian products  $G \times P_k$  and  $G \times C_{2k}$  also have interval edge-colorings for any positive integer  $k$ .



**Figure 2.1:** Examples of outerplanar triangulations with (1) and without (2) separating triangles.

As a result, grids  $P_{n_1} \times P_{n_2} \times \cdots \times P_{n_i}$ , bipartite cylinders  $C_{2n} \times P_m$ , and bipartite torii  $C_{2n} \times C_{2k}$  (for  $m, n, n_1, n_2, \dots, n_i$  positive integers) all have interval edge-colorings.

## Planar Graphs

Several interesting results relate specifically to planar graphs. In this section, we review important terminology and survey results concerning the existence of interval edge-colorings in these graphs. Bounds on the number of colors used will be explored in Chapter 4.

A *separating triangle* is a bounded (triangular) face of an outerplanar triangulation which does not share any edges with the unbounded face. An edge  $e = xy$  is said to be assigned an *extremal color*, say  $\alpha$ , if  $\alpha$  is either the maximum or minimum of the set of all colors which are assigned to edges incident to  $x$  or  $y$ . A multigraph  $G$  is constructed from two multigraphs  $G_1$  and  $G_2$  by *attaching* them along an edge  $e = xy$  if

(a)  $V(G) = V(G_1) \cup V(G_2)$ ,

(b)  $E(G) = E(G_1) \cup E(G_2)$ ,

(c)  $V(G_1) \cap V(G_2) = \{x, y\}$ , and

(d)  $e \in E(G_1) \cap E(G_2)$ .

In this case, we write  $G = G_1 e G_2$ . If  $e$  does not have any parallel edges in  $G_1$  and  $G_2$ , we have  $E(G_1) \cap E(G_2) = \{e\}$ . Finally, an edge  $e$  in a graph  $G$  is called a *dividing edge* for an interval edge-coloring  $c$  if  $G = G_1 e G_2$  for some subgraphs  $G_1$  and  $G_2$ , and  $e$  receives extremal colors in  $G_1$  and  $G_2$  under  $c$ .

One simple example of a planar graph is the wheel  $W_n$ , constructed from a cycle  $C_n$  by adding a central vertex adjacent to each vertex on the cycle. Axenovich [7] showed:

**Theorem 2.5.** The wheel  $W_n$  has an interval edge-coloring if and only if  $n = 3, 6, 9$ .

Giario, Kubale, and Malafiejski [15] described wheels in terms of deficiency. They proved that the wheel  $W_n$  has deficiency zero for  $n = 3, 6, 9$ , one for  $n = 2, 4, 5, 7, 8, 10, 11$ , and two for  $n \geq 12$ . Interval edge-colorings for  $W_3, W_6, W_9$  are depicted in Figure 2.2.

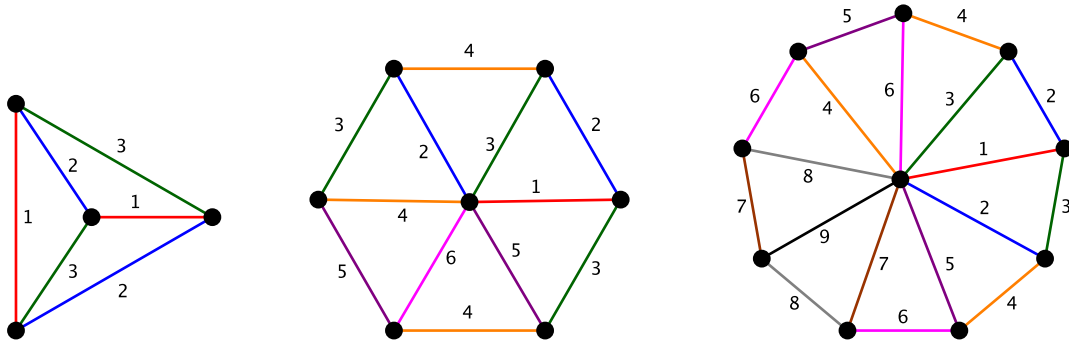
Petrosyan [30] worked with outerplanar graphs, and proved:

**Theorem 2.6.** If  $G$  is a 2-connected outerplanar graph with maximum degree at most 3, and is not an odd cycle, then it has an interval edge-coloring.

Furthermore, Petrosyan [30] demonstrated outerplanar graphs with maximum degree 4 which do not have interval edge-colorings. Related results concerning the number of colors used in such colorings are included in Chapter 4.

One particular subclass of planar graphs is the outerplanar triangulation. Note that  $K_3$  is an outerplanar triangulation, so not all such graphs have interval edge-colorings. Axenovich [7] proved:

**Theorem 2.7.** An outerplanar triangulation on at least four vertices, without a separating triangle, has an interval edge-coloring.



**Figure 2.2:** Interval edge-colorings of the wheels  $W_3$ ,  $W_6$ , and  $W_9$  [7].

As a corollary, all outerplanar bipartite graphs have interval edge-colorings. Petrosyan [30] proved that the condition of having no separating triangle is sufficient, but not necessary, for the existence of such a coloring. In order to prove the result about outerplanar triangulations, Axenovich [7] also showed the following two results:

**Theorem 2.8.** If two graphs  $G_1$  and  $G_2$  have interval edge-colorings which give an edge  $e$  extremal colors, then  $G_1eG_2$  has an interval edge-coloring.

**Lemma 2.9.** If a graph  $G$  has an interval edge-coloring, then the graph obtained by removing any number of dividing edges also has an interval edge-coloring.

Thus Theorem 2.7 can be extended to include outerplanar triangulations on at least four vertices, without separating triangles, which are missing one or more dividing edges.



## CHAPTER 3: BIPARTITE GRAPHS

In this chapter we focus on bipartite graphs. Since many scheduling problems relate to meetings between two separate groups (e.g. parents and teachers), bipartite graphs are a topic of particular interest. If the colors assigned to edges of a graph represent meeting times, an interval edge-coloring of such a graph represents a schedule in which each participant has consecutive meetings.

### General Bipartite Graphs

Not every bipartite graph has an interval edge-coloring; Sevastjanov [33] provided the first example of a bipartite graph without such a coloring. Since then, a number of smaller examples of bipartite graphs have been found [21]. Before giving one such example, we need some notation. A finite *projective plane* of order  $q$  consists of a set of  $q^2 + q + 1$  points and a set of  $q^2 + q + 1$  lines such that (a) each point belongs to  $q + 1$  lines, (b) each line contains  $q + 1$  points, and (c) each pair of lines meet at exactly one point.

**Example 3.1.** Jensen and Toft [21] related an example of a bipartite graph without an interval edge-coloring created by Paul Erdős. The construction of the graph  $G$  is based on a finite projective plane,  $P$ , of order  $q \geq 3$ . Let one bipartition,  $X$ , be the set of  $q^2 + q + 1$  points in  $P$ . Let the other bipartition,  $Y$ , be the set of  $q^2 + q + 1$  lines in  $P$ . Given any point  $x \in X$  and line  $y \in Y$ ,  $x$  is adjacent to  $y$  in  $G$  if  $x$  lies on  $y$  in  $P$ . At this point,  $d_G(v) = q + 1$  for all  $v \in V(G)$ . The final step is to add a new vertex,  $z$ , which is adjacent to all vertices in  $Y$ . Prior to the addition of the final vertex, this graph is  $(q + 1)$ -regular and bipartite. By Observation 3.3, it has an interval edge-coloring. The addition of the final vertex causes no interval edge-coloring to exist.

The smallest known example of a bipartite graph without an interval edge-coloring, which has 19 vertices, was reportedly found by Mirumyan in 1989 [24], but remained unpublished. Giaro, Kubale, and Malafiejski [16] discovered the graph independently. On the other hand, Giaro [11] previously demonstrated that bipartite graphs on at most 14 vertices have interval edge-colorings, and Khachatryan and Mamikonyan [24] showed that every bipartite graph on 15 vertices has an interval edge-coloring. Whether a bipartite graph  $G$  with  $16 \leq |V(G)| \leq 18$  has an interval edge-coloring remains open.

We next state results on several classes of bipartite graphs which are known to have interval edge-colorings. Asratian and Kamalian [4] showed:

**Lemma 3.2.** The complete bipartite graph  $K_{a,b}$  has an interval  $(a + b - 1)$ -coloring.

To construct such a coloring, we may enumerate the vertices of each partite set as  $\{x_1, \dots, x_a\}$  and  $\{y_1, \dots, y_b\}$ . An edge  $x_i y_j$  receives color  $i + j - 1$  for  $1 \leq i \leq a$  and  $1 \leq j \leq b$ . As we will see in Chapter 4, many values of  $a$  and  $b$  may allow the use of fewer colors. Since every  $k$ -regular bipartite graph is of class 1, it follows from Corollary 2.3 that:

**Observation 3.3.** Every  $k$ -regular bipartite multigraph has an interval  $k$ -coloring.

In this case the graph can be decomposed into  $k$  perfect matchings, which are colored accordingly. Furthermore, Hansen [17] showed:

**Theorem 3.4.** All bipartite graphs with maximum degree at most 3 have interval 4-colorings.

Finally, a *doubly convex* bipartite graph  $G$  is one in which both bipartite sets can be enumerated such that for all  $v \in V(G)$ , the neighbors of  $v$  are consecutively numbered. Kamalian [22] showed that doubly convex bipartite graphs have interval edge-colorings.

## Biregular Graphs

Let  $a$  and  $b$  be positive integers. An  $(a, b)$ -biregular graph  $G$  is a bipartite graph with bipartition  $(X, Y)$  such that  $d_G(x) = a$  for all  $x \in X$  and  $d_G(y) = b$  for all  $y \in Y$ . Complete and regular bipartite graphs are common examples of biregular graphs. The following is a well-known conjecture on interval edge-colorings of biregular graphs [21, 34]:

**Conjecture 3.5.** Every  $(a, b)$ -biregular multigraph has an interval edge-coloring.

In the following, we review known results for several values of  $a$  and  $b$ ; the smallest values without a general solution are  $a = 3$  and  $b = 4$ , which will be explored separately.

Since  $(1, b)$ -biregular graphs are forests consisting of stars (which are trees), such graphs have interval edge-colorings. Hansen [17] showed that  $(2, b)$ -biregular multigraphs have such colorings when  $b$  is even. Furthermore, he noted that this is equivalent to a result by Petersen [28]: every  $2k$ -regular graph has a decomposition into 2-regular edge-disjoint subgraphs. Casselgren and Toft [10] noted that these graphs can always be colored such that the vertices of degree 2 receive colors  $2j - 1$  and  $2j$  for some positive integer  $j$ . In other words, the lesser of the two colors appearing at each vertex of degree 2 is odd. Hanson and Loten [19] later proved the existence of interval edge-colorings of  $(2, b)$ -biregular multigraphs for the case when  $b$  is odd (a result obtained independently by Kostochka [25]). Thus we have:

**Theorem 3.6.** Every  $(2, b)$ -biregular multigraph has an interval edge-coloring.

Additionally, Hanson and Loten [19] showed that such colorings exist with any given edge receiving a specified color. In application, this means that a schedule can be created with any one meeting time predetermined.

Casselgren and Toft [10] proved the following two results related to  $(3, 6)$ -biregular graphs:

**Theorem 3.7.** Every  $(3, 6)$ -biregular graph has an interval 7-coloring.

**Remark 3.8.** A  $(3, 6)$ -biregular graph has an interval 6-coloring if and only if it has a 3-regular subgraph covering all vertices of degree 6.

They also demonstrated a  $(3, 6)$ -biregular graph without such a 3-regular subgraph, and therefore no interval 6-coloring. Thus the use of 7 colors is a best-possible general result. Using this, they proceeded to show:

**Corollary 3.9.** Every  $(3, 9)$ -biregular graph containing a 3-regular subgraph which covers all vertices of degree 9 has an interval 10-coloring.

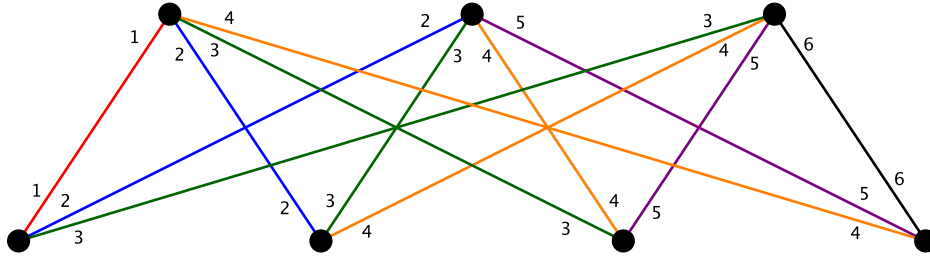
Moreover, Casselgren and Toft [10] showed that a  $(3, 9)$ -biregular graph has an interval 9-coloring if and only if it can be decomposed into three 3-regular edge-disjoint subgraphs. Note that this result does not preclude a  $(3, 9)$ -biregular graph without such a subgraph from having an interval edge-coloring with a greater number of colors.

Additionally, Casselgren and Toft [10] found conditions for the existence of an interval edge-coloring of  $(4, 6)$ - and  $(4, 8)$ -biregular graphs. They showed:

**Proposition 3.10.** A  $(4, 6)$ -biregular graph with bipartitions  $(X, Y)$  has an interval 8-coloring if  $X = X_1 \cup X_2 \cup X_3$  and  $Y = Y_1 \cup Y_2$  such that either

- (i)  $G[X_i \cup Y_j]$  is 2-regular for any  $i = 1, 2, 3$  or  $j = 1, 2$ , or
- (ii)  $G[X_2 \cup Y_i]$  is 2-regular for any  $i = 1, 2$  while  $G[X_1 \cup Y_1]$  and  $G[X_3 \cup Y_2]$  are both 4-regular.

Due to Theorem 4.14, the use of 8 colors is best-possible for all  $(4, 6)$ -biregular graphs. They further note that  $(4, 6)$ -biregular graphs not satisfying this criteria exist. Thus we expect that it is not a necessary condition for the existence of an interval edge-coloring. Finally, they demonstrated:



**Figure 3.1:** Interval 6-coloring of  $K_{3,4}$ .

**Proposition 3.11.** A  $(4, 8)$ -biregular graph has an interval 8-coloring if and only if it contains a 4-regular subgraph which covers all vertices of degree 8.

Results drawing from regular subgraphs which cover one of the bipartitions are noticeably common. We will see this again when we consider the  $(3, 4)$ -biregular case in the next section.

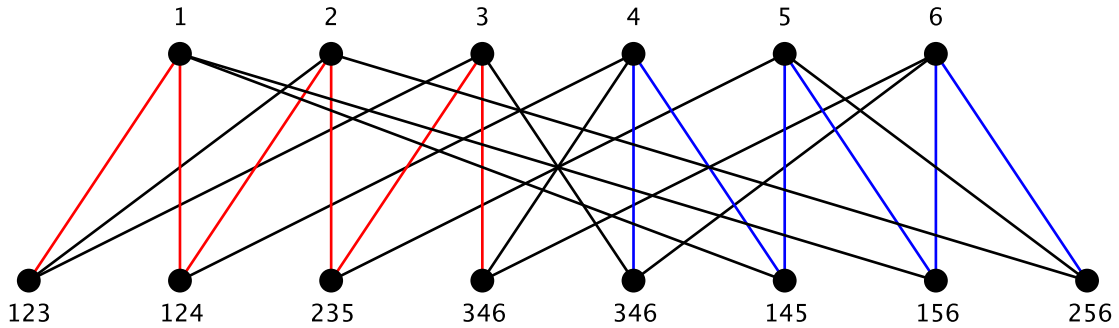
### (3, 4)-Biregular Graphs

As mentioned previously, the smallest unsolved case of Conjecture 3.5 is that of  $(3, 4)$ -biregular multigraphs. The simplest such graph is the complete bipartite graph  $K_{3,4}$ , which has an interval 6-coloring as given in Figure 3.1. As we will see in Chapter 4, the number 6 is best-possible.

While a general result for  $(3, 4)$ -biregular multigraphs is not known, several special cases are known to have interval edge-colorings. Pyatkin [32] showed the following sufficient condition:

**Theorem 3.12.** Every  $(3, 4)$ -biregular graph which contains a “full” 3-regular subgraph (one which covers the vertices of degree 4) has an interval 6-coloring.

This sufficient condition is not necessary for the existence of an interval edge-coloring, as Asratian, Casselgren, Vandenbussche, and West [6] demonstrated a graph (Figure 3.2) with an interval edge-



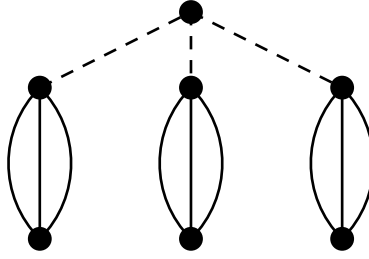
**Figure 3.2:** A  $(3, 4)$ -biregular graph which does not have a “full” 3-regular subgraph, but does have a proper path factor, given by the red and blue edges [6].

coloring that does not have such a subgraph. They also discovered an alternative sufficient condition for the existence of an interval edge-coloring. We define a *path factor* of a  $(3, 4)$ -biregular multigraph to be a spanning subgraph in which each component is a path with two endpoints in the bipartition of degree 3 vertices. A *proper path factor* is a path factor in which each path has length 2, 4, 6, or 8. An example of a proper path factor is given by the red and blue edges in Figure 3.2. Casselgren [9] proved the following result in his thesis, and later Asratian, Casselgren, Vandenbussche, and West [6] gave a shorter proof:

**Theorem 3.13.** Let  $G$  be a  $(3, 4)$ -biregular multigraph. If  $G$  has a proper path factor, then it has an interval 6-coloring.

Note that not every  $(3, 4)$ -biregular multigraph has a proper path factor [6], even if it has a “full” 3-regular subgraph as in Figure 3.3. Thus, for multigraphs, neither the presence of a proper path factor nor the presence of a “full” 3-regular subgraph imply the other. This changes when we consider simple graphs, however. In fact, Asratian, Casselgren, Vandenbussche, and West [6] made the following conjecture:

**Conjecture 3.14.** Every simple  $(3, 4)$ -biregular graph has a proper path factor.



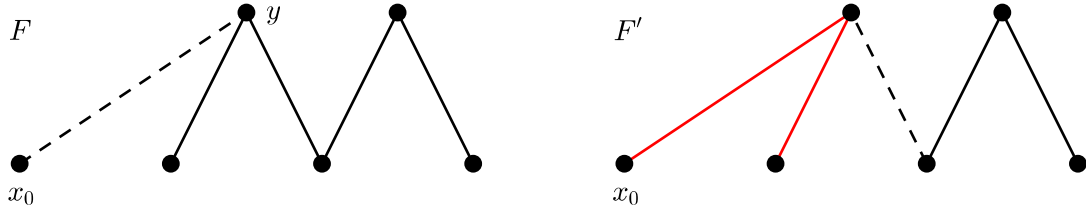
**Figure 3.3:** A  $(3, 4)$ -biregular multigraph which does not have a proper path factor, but does have a “full” 3-regular subgraph.

To state the progress on Conjecture 3.14, we need some definitions. Let  $G$  be a  $(3, 4)$ -biregular multigraph. A spanning subgraph  $F$  of  $G$  is called a *pseudo path factor* of  $G$  if each component of  $F$  is a path with two ends of degree 3 in  $G$ . Note that these paths may be single vertices of degree 3 in  $G$ . Pseudo path factors are easier to find than proper path factors; however, they remain useful. Below is a result due to Asratian and Casselgren [2]. The original proof is algorithmic; we give a new proof here.

**Theorem 3.15.** Let  $G$  be a  $(3, 4)$ -biregular graph with bipartition  $(X, Y)$ , so that each  $x \in X$  has degree 3 and each  $y \in Y$  has degree 4 in  $G$ . Let  $F$  be a pseudo path factor of  $G$ . Then there exists a path factor  $\mathcal{P}$  of  $G$  such that no path in  $\mathcal{P}$  is longer than the longest path in  $F$ .

*Proof.* Let  $G = (X, Y)$  and  $F$  be as given in the statement. We may assume that  $F$  is not a path factor, otherwise we are done. Given a pseudo path factor  $F$  of  $G$ , let  $\max F := \max\{|E(P)| : P \in F\}$  and  $V_F := \{x \in X : d_F(x) \geq 1\}$ . We shall proceed by contradiction.

Suppose that no such path factor  $\mathcal{P}$  exists. Let  $F^*$  be a pseudo path factor of  $G$ , with  $V_{F^*} \supseteq V_F$  and  $\max F^* \leq \max F$ , such that  $|X \setminus V_{F^*}|$  is minimum. Notice that  $X \setminus V_{F^*} \neq \emptyset$  (otherwise  $F^*$  would be a desired path factor) and each vertex in  $X \setminus V_{F^*}$  is a path of length zero in  $F^*$ . Since  $F^*$  is a spanning subgraph of  $G$  and each path in  $F^*$  has its ends in  $X$ , we see that each vertex in  $Y$  is



**Figure 3.4:** Transformation of  $F$  when  $x_0$  is adjacent to a path of length greater than 2 in  $F$ .

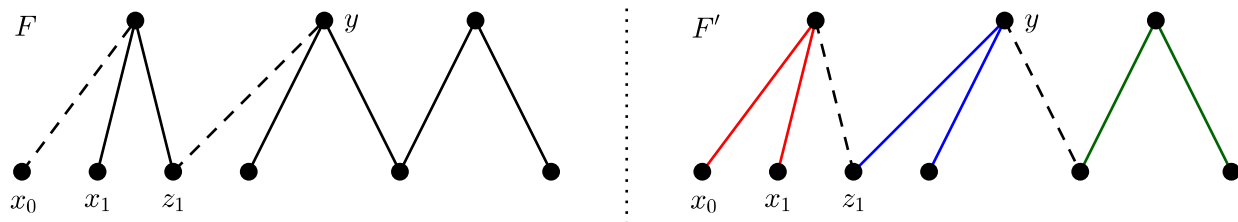
an internal vertex of some path in  $F^*$ . Let  $x_0 \in X \setminus V_{F^*}$ . Then  $N_G(x_0) \subseteq Y$ , and so each neighbor of  $x_0$  is an internal vertex of some path in  $F^*$ . We claim that each vertex in  $N_G(x_0)$  is the middle vertex of a 3-path in  $F^*$ .

Suppose, to the contrary, that  $x_0$  is adjacent to an internal vertex  $y$  of a path  $P \in F^*$  with length greater than 2. Let  $d_P(u, v)$  denote the distance between two vertices  $u, v \in V(P)$  along  $P$ . Let  $a, b$  be the two ends of  $P$  with  $d_P(a, y) \leq d_P(y, b)$ , and let  $F' = (F^* \setminus P) \cup \{x_0yPa, dPb\}$ , where  $d$  is the neighbor of  $y$  on the subpath  $yPb$ . Then  $F'$  is a pseudo path factor of  $G$  with  $V_{F'} \supseteq V_{F^*} \cup \{x_0\}$  and  $\max F' \leq \max F^*$ , contrary to the choice of  $F^*$ . Thus each neighbor of  $x_0$  is the middle vertex of a 3-path in  $F^*$ , as claimed.

Let  $Q_1 \in F^*$  be a 3-path with vertices  $x_1, y_1, z_1$  in order such that  $x_0y_1 \in E(G)$ . We claim that neither  $x_1$  nor  $z_1$  can be adjacent to any internal vertex of a path with length greater than 2 in  $F^*$ . Suppose otherwise. Without loss of generality we may assume that  $z_1$  is adjacent to an internal vertex, say  $y$ , of a path  $P \in F^*$  with length greater than 2. In this case, we may construct a new pseudo path factor  $F'$ , with  $V_{F'} \supseteq V_{F^*} \cup \{x_0\}$  and  $\max F' \leq \max F^*$ , by splitting  $Q_1$  and  $P$  in the same manner as described previously (see Figure 3.5), again contrary to the choice of  $F^*$ .

Let  $Q_1, Q_2, \dots, Q_s$  be a maximal sequence of vertex-disjoint 3-paths  $Q_i = x_iy_iz_i$  in  $F^*$  such that  $x_0y_1 \in E(G)$ , and for each  $2 \leq i \leq s$ , the vertex  $y_i$  has a neighbor in  $\{x_0, x_1, z_1, \dots, x_{i-1}, z_{i-1}\}$ .





**Figure 3.5:** Transformation of  $F$  when  $x_1$  or  $z_1$  is adjacent to a path of length greater than 2 in  $F$ .

Using an argument similar to that used with  $Q_1$ , we see that the ends of each  $Q_i$  cannot be adjacent to an internal vertex of a path in  $F^*$  with length greater than 2. Let  $A = \{x_0, x_1, z_1, \dots, x_s, z_s\}$  and  $B = \{y_1, y_2, \dots, y_s\}$ . Then  $|A| = 2s + 1$  and  $|B| = s$ . Note that  $s \geq 3$  as  $x_0$  has three neighbors in  $G$ . One can see that  $N(A) \subseteq B$  by the maximality of  $s$ .

Let  $m$  be the number of edges between the sets  $A$  and  $B$ . We have  $3|A| \leq m$ , as each vertex in  $A$  has degree 3 and  $N(A) \subseteq B$ . Also,  $m \leq 4s$ , as each vertex in  $B$  has degree 4. As a result,

$$3(2s + 1) \leq m \leq 4s$$

and thus

$$s \leq -\frac{3}{2}.$$

This is a contradiction, as  $s \geq 3$ . We conclude that such a path factor  $\mathcal{P}$  must exist.  $\square$

Using this result, Casselgren [8] showed:

**Theorem 3.16.** Every simple  $(3, 4)$ -biregular graph has a path factor in which the maximum length of a path is at most 22.

Improving the maximum length from 22 to 8 would confirm Conjecture 3.14 and thus show that every  $(3, 4)$ -biregular graph has an interval 6-coloring. Asratian, Casselgren, Vandenbussche, and

West [6] gave sufficient conditions for the existence of a proper path factor. For  $(3, 4)$ -biregular graphs with a “full” 3-regular subgraph, they defined various types of *transversal* sets based on the neighborhoods of degree 3 vertices included in the subgraph. The existence of a *mixed transversal* is sufficient for the existence of a proper path factor. On the other hand, they showed:

**Theorem 3.17.** A  $(3, 4)$ -biregular graph  $G$  with bipartition  $(X, Y)$  has a proper path factor into 7-paths with ends in  $X$  if  $G$  has a  $(2, 4)$ -biregular subgraph covering  $X$ .

Asratian, Casselgren, Vandenbussche, and West [6] note that many, though not all, graphs have such a  $P_7$ -factor. This is not surprising when we consider the degree requirements of paths in  $(3, 4)$ -biregular graphs. We have the following new result.

**Theorem 3.18.** Let  $\mathcal{P}$  be any proper path factor of a  $(3, 4)$ -biregular graph  $G$  with  $t_2$  number of paths of length 2,  $t_4$  number of paths of length 4,  $t_6$  paths of length 6, and  $t_8$  paths of length 8. Then  $2t_2 + t_4 = t_8$ . In particular, if  $\mathcal{P}$  has no path of length 8, then  $\mathcal{P}$  has paths of length 6 only, namely,  $\mathcal{P}$  is a  $P_7$ -factor.

*Proof.* Let  $(X, Y)$  be the bipartition of  $G$  such that  $d_G(x) = 3$  for all  $x \in X$  and  $d_G(y) = 4$  for all  $y \in Y$ . By counting the number of edges in  $G$ , we obtain

$$3 \cdot |X| = |E(G)| = 4 \cdot |Y|.$$

We proceed by counting the number of vertices contributed to  $X$  and  $Y$  by each type of path.

Since the ends of each path lie in  $X$ , each copy of  $P_3$  contributes two vertices to  $X$  and one to  $Y$ . Similarly, each copy of  $P_5$  contributes three vertices to  $X$  and two to  $Y$ . Each copy of  $P_7$  contributes four to  $X$  and three to  $Y$ , while each copy of  $P_9$  contributes five to  $X$  and four to  $Y$ .

Since the proper path factor  $\mathcal{P}$  is a spanning path factor, we have

$$2t_2 + 3t_4 + 4t_6 + 5t_8 = |X|$$

and

$$t_2 + 2t_4 + 3t_6 + 4t_8 = |Y|.$$

Hence we find

$$6t_2 + 9t_4 + 12t_6 + 15t_8 = 3 \cdot |X| = 4 \cdot |Y| = 4t_2 + 8t_4 + 12t_6 + 16t_8$$

and, simplifying,

$$2t_2 + t_4 = t_8$$

for any proper path factor  $\mathcal{P}$ . Note that each  $t_2, t_4, t_6, t_8$  are nonnegative integers. We conclude if  $\mathcal{P}$  does not contain  $P_9$ , so that  $t_8 = 0$ , it must be the case that  $t_2 = t_4 = 0$ . Thus  $\mathcal{P}$  can only contain paths  $P_7$ .  $\square$

Note that the quantity  $t_6$ , the number of paths of length 6 in a proper path factor, cannot be controlled using this method. There is one final sufficient condition for the existence of an interval edge-coloring in a  $(3, 4)$ -biregular graph. Yang and Li [36] showed:

**Theorem 3.19.** A  $(3, 4)$ -biregular graph  $G$  with bipartition  $(X, Y)$  which can be decomposed into two edge-disjoint  $(2, 3)$ -biregular subgraphs  $G_1$  with bipartition  $(Y, X_1)$  and  $G_2$  with bipartition  $(Y, X_2)$ , where  $X_1 \cup X_2 = X$  and  $X_1 \cap X_2 = \emptyset$ , has an interval 6-coloring.

## CHAPTER 4: BOUNDS ON THE NUMBER OF COLORS

Beyond the question of whether an interval edge-coloring exists, we are also interested in the number of colors that may be used if such a coloring exists. In this chapter, we assume that a graph  $G$  of a particular type has an interval edge-coloring, and investigate the values of  $t$  for which an interval  $t$ -coloring of  $G$  exists. We may assume  $G$  is connected; otherwise we consider connected components of  $G$  separately.

### General Graphs

There are several results concerning the number of colors used in an interval edge-coloring of a general graph. Asratian and Kamalian [4] showed that if a graph  $G$  has an interval  $t$ -coloring, then  $t \leq 2|V(G)| - 1$ . Their proof uses a bound for bipartite graphs (see Theorem 4.10) by constructing a bipartite graph from  $G$ . Kamalian [22] improved their bound by showing that if a graph  $G$  on at least two vertices has an interval  $t$ -coloring, then  $t \leq 2|V(G)| - 3$ . Giaro, Kubale, and Malafiejski [15] improved the previous bound one step further as follows:

**Theorem 4.1.** If a graph  $G$  on at least 3 vertices has an interval  $t$ -coloring, then  $t \leq 2|V(G)| - 4$ .

Kamalian and Petrosyan [23] obtained a better bound for  $k$ -regular graphs:

**Theorem 4.2.** If  $G$  is a  $k$ -regular graph on at least  $2k + 2$  vertices with an interval  $t$ -coloring, then  $t \leq 2|V(G)| - 5$ .

Bounds of this form,  $2|V(G)| - a$ , may improve in the constant term  $a$  due to further study. However, Petrosyan [29] showed below that the linear term in  $|V(G)|$  cannot be improved.

**Theorem 4.3.** For any small parameter  $\varepsilon > 0$ , there exists a graph  $G$  such that the maximum number of colors which can be used in an interval edge-coloring of  $G$  is at least  $(2 - \varepsilon)|V(G)|$ .

Note that graphs satisfying the condition of Theorem 4.3 for any particular  $\varepsilon$  may require a large number of vertices. Thus far, each of the results has been an upper bound related to the number of vertices in the graph. We can also express bounds in terms of other properties. For example, Giaro, Kubale, and Malafiejski [15] used the set of simple paths (ones without repeated vertices) in  $G$  to obtain an upper bound for the number of colors used in an interval edge-coloring.

**Proposition 4.4.** If a graph  $G$  has an interval  $t$ -coloring, then

$$t \leq 1 + \max_{P \in \mathcal{P}} \sum_{v \in V(P)} (d_G(v) - 1),$$

where  $\mathcal{P}$  is the set of all simple paths in  $G$ .

They further mentioned the following corollary to Proposition 4.4 in terms of the maximum degree and diameter of the graph:

**Proposition 4.5.** If a graph  $G$  has an interval  $t$ -coloring, then  $t \leq (\text{diam}(G) + 1)(\Delta(G) - 1) + 1$ .

To describe the relationship between vertex degrees and the number of colors required, Asratian and Kamalian [4] noted the following corollary as well:

**Corollary 4.6.** If a graph  $G$  has the property that  $d_G(x) + d_G(y) \geq |V(G)| - 1$  for every pair of nonadjacent vertices  $x, y \in V(G)$ , and  $G$  has an interval  $t$ -coloring, then  $t \leq 3\Delta(G) - 2$ .

We have seen upper bounds related to the number of colors used in an interval edge-coloring of a general graph if such a coloring exists. Axenovich [7] investigated bounds for planar graphs, and obtained the following:

**Theorem 4.7.** If  $G$  is a planar graph with an interval  $t$ -coloring, then  $t \leq \frac{11}{6}|V(G)|$ .

Furthermore, based on the methods used in the proof of Theorem 4.7, Axenovich [7] conjectured:

**Conjecture 4.8.** If  $G$  is a planar graph with an interval  $t$ -coloring, then  $t \leq \frac{3}{2}|V(G)|$ .

Theorem 2.6 concerns 2-connected outerplanar graphs. Petrosyan [30] also noted:

**Theorem 4.9.** Let  $G$  be a 2-connected outerplanar graph with maximum degree at most 3 which is not an odd cycle. If  $|V(G)|$  is even, then the minimum number of colors used is 3. If  $|V(G)|$  is odd, the minimum number of colors used is 4.

As a final note, for any interval  $t$ -coloring of a graph  $G$ ,  $t \geq \Delta(G)$ . This follows directly from the definition of an interval edge-coloring.

## Bipartite Graphs

Asratian and Kamalian [4] showed the following bound for triangle-free graphs:

**Theorem 4.10.** If  $G$  is a triangle-free graph with an interval  $t$ -coloring, then  $t \leq |V(G)| - 1$ .

Since bipartite graphs do not contain any odd cycles, including triangles, this result serves as an upper bound for interval edge-colorings of bipartite graphs. Asratian and Kamalian [4] pointed out that this bound is tight for complete bipartite graphs  $K_{m,n}$ . They also obtained a bound based on the maximum degree and diameter of the graph:

**Corollary 4.11.** If  $G$  is a bipartite graph with an interval  $t$ -coloring, then

$$t \leq \text{diam}(G)(\Delta(G) - 1) + 1.$$

Giario, Kubale, and Malafiejski [15] pointed out that this bound is tight in the case of complete regular bipartite graphs  $K_{n,n}$ .

A lower bound based strictly upon the maximum degree of a graph would be useful, however Asratian and Kamalian [4] demonstrated that such a bound is not possible for bipartite graphs:

**Proposition 4.12.** For any positive integer  $p$ , there exists a bipartite graph  $G$  such that the minimum number of colors required in an interval edge-coloring is at least  $\Delta(G) + p$ .

Thus we can always find a bipartite graph which violates a proposed lower bound on the number of colors used in an interval edge-coloring based on the maximum degree alone.

### Biregular Graphs

Several results relate specifically to biregular bipartite graphs. In this section,  $\gcd(x, y)$  is the greatest common divisor of integers  $x$  and  $y$ . The special structure of biregular graphs give us an improvement to the upper bound for triangle-free graphs (see Theorem 4.10). Asratian and Casselgren [1] showed the following:

**Theorem 4.13.** If  $G$  is an  $(a, b)$ -biregular graph on at least  $2(a + b)$  vertices with an interval  $t$ -coloring, then  $t \leq |V(G)| - 3$ .

They further note that any  $(a, b)$ -biregular graph with  $\gcd(a, b) = 1$  which is not  $K_{a,b}$  satisfies the conditions of Theorem 4.13. Moreover, they show that the bound in Theorem 4.13 is tight.

Kamalian [22] previously showed that a complete bipartite graph  $K_{a,b}$  with an interval  $t$ -coloring must have  $t \geq a + b - \gcd(a, b)$ . Hanson and Loten [18] extended this result for biregular graphs:

**Theorem 4.14.** If  $G$  is an  $(a, b)$ -biregular graph with an interval  $t$ -coloring, then

$$t \geq a + b - \gcd(a, b).$$

Note that Theorem 4.14 shows the interval 6-coloring of  $K_{3,4}$  demonstrated in Figure 3.1 is best-possible in terms of the number of colors. Hanson and Loten [18] note that this lower bound cannot always be achieved in a particular  $(a, b)$ -biregular graph.



## CHAPTER 5: RELATED TOPICS

We have explored the existence of interval edge-colorings, as well as the number of colors which can be used, with scheduling applications in mind. Since not all graphs have such colorings, a number of generalizations have been developed. In this chapter we examine those generalizations, study the computational complexity of interval edge-coloring problems, and consider future work.

### Near Interval Edge-Colorings

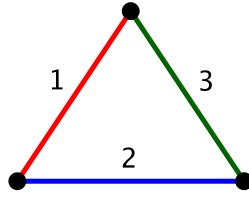
As mentioned previously, the existence of an interval edge-coloring of a graph  $G$  is equivalent to having a coloring with zero deficiency. Thus a natural generalization is to look at graphs with low deficiency. A *near interval edge-coloring* is one in which the deficiency at any vertex is at most one. Petrosyan, Arakelyan, and Baghdasaryan [31] studied the topic under the name *interval  $(t, 1)$  coloring*, and provided examples of graphs which do not have such colorings. In application, a near interval edge-coloring might represent a schedule in which each individual has at most one “gap” between consecutive meetings. One can easily see that every  $(3, 4)$ -biregular graph has a near-interval 4-coloring. Casselgren and Toft [10] proved the following:

**Proposition 5.1.** If  $G$  is a bipartite graph with  $\delta(G) = n - 1$  and  $\Delta(G) = n$  for some  $n$ , then  $G$  has a near interval  $n$ -coloring.

This notably covers many cases of  $(a, b)$ -biregular graphs directly (i.e. when  $b = a + 1$ ). Working with this result allowed Casselgren and Toft [10] to show the following cases as well:

**Corollary 5.2.** Every  $(3, 5)$ -biregular graph has a near interval 6-coloring.

**Theorem 5.3.** Every  $(4, 6)$ -biregular graph has a near interval 7-coloring.



**Figure 5.1:** Triangle  $K_3$  with a cyclic interval edge-coloring.

They note that both of these results are best-possible in terms of the number of colors used.

Note that all interval edge-colorings are also near interval edge-colorings. As a result, Conjecture 3.5 can be weakened to state that every  $(a, b)$ -biregular multigraph has a near interval edge-coloring. It is unknown whether this weakened conjecture is true.

### Cyclic Interval Edge-Colorings

Another generalization is the *cyclic interval edge-coloring*, in which the colors present at each vertex form an interval modulo the total number of colors used. For example,  $K_3$  has a cyclic interval edge-coloring as depicted in Figure 5.1; the sets  $\{1, 2\}$ ,  $\{2, 3\}$ , and  $\{3, 1\}$  each form an interval modulo 3. Nadolski [27] demonstrated graphs which do not have such a coloring.

Casselgren and Toft [10] found a result that is similar to Proposition 5.1:

**Proposition 5.4.** If  $G$  is a bipartite graph with  $\delta(G) = n - 1$  and  $\Delta(G) = n$ , then  $G$  has a cyclic interval  $n$ -coloring.

Once again, this covers many cases of  $(a, b)$ -biregular graphs (i.e. when  $b = a + 1$ ). They also showed the following:

**Theorem 5.5.** Every  $(4, 8)$ -biregular graph has a cyclic interval 8-coloring.

Since all interval edge-colorings are also cyclic interval edge-colorings, we can again weaken Conjecture 3.5 to state that every  $(a, b)$ -biregular multigraph has a cyclic interval edge-coloring. Furthermore, Casselgren and Toft [10] note that every interval edge-coloring of a graph  $G$  can be transformed into a cyclic interval  $\Delta(G)$ -coloring by taking colors modulo  $\Delta(G)$ . Thus the following conjecture is also a consequence of Conjecture 3.5:

**Conjecture 5.6.** Every  $(a, b)$ -biregular multigraph has a cyclic interval  $\max\{a, b\}$ -coloring.

Casselgren and Toft [10] note that the smallest unsolved case of Conjecture 5.6 is that of  $(3, 5)$ -biregular multigraphs. They also cite sufficient conditions for the existence of such a coloring in preparation for future work.

### Computational Complexity

In this section we address the computational difficulty of answering various questions which are posed throughout this thesis.

Proposition 2.1 states that all multigraphs  $G$  with interval edge-colorings have  $\chi'(G) = \Delta(G)$ . Holyer [20] showed that deciding whether a graph is of class 1 is NP-complete. Giaro, Kubale, and Malafiejski [15] conclude that the general problem of deciding whether an interval edge-coloring exists for a given graph is NP-complete. Similarly, Asratian and Kamalian [4] proved that the problem of determining whether a  $k$ -regular graph has an interval edge-coloring is also NP-complete.

Sevastjanov [33] proved that determining whether a bipartite graph has an interval edge-coloring is NP-complete. Giaro [12] showed that deciding whether a bipartite graph  $G$  with  $\Delta(G) \geq 4$  has an interval 4-coloring occurs in polynomial time; however, it is NP-complete to decide whether a bipartite graph  $G$  with  $\Delta(G) \geq 5$  has an interval 5-coloring. Kubale and Nadolski [26] showed that determining whether a bipartite graph has a cyclic interval edge-coloring is also NP-complete.

Proofs of the results obtained in [10] yield polynomial-time algorithms for finding such colorings: interval 7-colorings of  $(3, 6)$ -biregular graphs, interval 10-colorings of  $(3, 9)$ -biregular graphs with 3-regular subgraphs covering the vertices of degree 9, near interval 6-colorings of  $(3, 5)$ -biregular graphs, near interval 7-colorings of  $(4, 6)$ -biregular graphs, and finally cyclic interval 8-colorings of  $(4, 8)$ -biregular graphs.

On the other hand, Asratian and Casselgren [1] showed that the problem of determining whether a  $(3, 6)$ -biregular graph has a 3-regular subgraph covering all vertices of degree 6 is NP-complete. Therefore, deciding whether a  $(3, 6)$ -biregular graph has an interval 6-coloring is NP-complete [10]. This implies that determining the existence of a 3-regular subgraph covering all vertices of degree 9 (and therefore, the existence of an interval 9-coloring) of a  $(3, 9)$ -biregular graph is also NP-complete. Finally, the problem of deciding whether a  $(4, 8)$ -biregular graph has a 4-regular subgraph covering the vertices of degree 8 is NP-complete, and therefore so is determining the existence of an interval 8-coloring.

Concerning  $(3, 4)$ -biregular graphs, Pyatkin [32] showed that the problem of determining whether such a graph has a “full” 3-regular subgraph is NP-complete. Finally, Asratian and Casselgren [1] proved that the problem of determining the existence of an interval  $t$ -coloring of an  $(a, b)$ -biregular graph with  $b > a \geq 3$  in which  $a$  is a divisor of  $b$  is NP-complete.

## Future Work

In this thesis, we have mentioned several conjectures. The most immediate topic for future work involves Conjecture 3.14, that every simple  $(3, 4)$ -biregular graph has a proper path factor (and therefore an interval 6-coloring). Reducing the maximum length of a path in Theorem 3.16 from 22 to 8 is sufficient to confirm the conjecture, and a topic of particular interest for future work.

Partial results for graphs of certain order (e.g. those on at most 50 vertices) might be obtained using Theorem 3.18, which we obtained in this thesis.

Notably, this is not sufficient to confirm the existence of an interval edge-coloring for all  $(3, 4)$ -biregular graphs, particularly multigraphs. Another topic of future study is whether every  $(3, 4)$ -biregular graph has either a proper path factor or a “full” 3-regular subgraph. The method we developed in the new proof of Theorem 3.15 might be helpful in solving this problem.

Of course, the case of  $(3, 4)$ -biregular graphs is only the smallest unsolved case of Conjecture 3.5. Further work may confirm or disprove Conjecture 3.5 for other cases.

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