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AN EXAMINATION OF THE EFFECTIVENESS
OF THE ADOMIAN DECOMPOSITION METHOD
IN FLUID DYNAMIC APPLICATIONS

by

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ABSTRACT

Since its introduction in the 1980s, the Adomian Decomposition Method (ADM) has proven to be an efficient and reliable method for solving many types of problems. Originally developed to solve nonlinear functional equations, the ADM has since been used for a wide range of equation types (like boundary value problems, integral equations, equations arising in flow of incompressible and compressible fluids etc...). This work is devoted to an evaluation of the effectiveness of this method when used for fluid dynamic applications. In particular, the ADM has been applied to the Blasius equation, the Falkner-Skan equation, and the Orr-Sommerfeld equation.

This study is divided into five Chapters and an Appendix. The first chapter is devoted to an introduction of the Adomian Decomposition method (ADM) with simple illustrations. The Second Chapter is devoted to the application of the ADM to generalized Blasius Equation and our result is compared to other published results when the parameter values are appropriately set. Chapter 3 presents the solution generated for the Falkner-Skan equation. Finally, the Orr-Sommerfeld equation is dealt with in the fourth Chapter. Chapter 5 is devoted to the findings and recommendations based on this study. The Appendix contains details of the solutions considered as well as an alternate solution for the generalized Blasius Equation using Bender's δ-perturbation method.

I would like to dedicate this achievement to my parents, Rudolf and Alice VonNiederhausern, who were my first math teachers. Thank you for helping me become who I am.

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CHAPTER ONE: INTRODUCTION

Fluid dynamics is an important aspect of applied physics and engineering. When one considers the amount of air and water in the surrounding environment, the large quantities of fluid operating in the human body, and the many devices which use fluids, one can begin to see the scope of the influence of fluid dynamics and understand the necessity for developing an understanding of this field.

Of particular interest in fluid dynamics is the study of the boundary layer. A boundary layer is a region "in which a rapid change occurs in the value of a variable" [26]. For instance, when considering the fluid flow near a solid surface, there exists a portion of the flow immediately adjacent to the surface where the velocity or some other related property of the fluid changes dramatically. This is usually called the boundary layer.

Fluid dynamic topics often give rise to nonlinear differential equations. These problems tend to be more difficult to solve, often with no known exact solution. As such, researchers are continually looking for ways to accurately and effeciently solve these problems. One newly developed method that shows potential in this application is the Adomian Decomposition Method.

1.1   Adomian Decomposition Method

In the 1980's, George Adomian introduced a new method to solve nonlinear functional equations [28]. This method has since been termed the Adomian decomposition method (ADM) and has been the subject of much investigation[13, 28, 47, 53, 60]. The ADM involves separating the equation under investigation into linear and nonlinear portions. The linear

operator representing the linear portion of the equation is inverted and the inverse operator is then applied to the equation. Any given conditions are taken into consideration. The non-linear portion is decomposed into a series of Adomian polynomials. This method generates a solution in the form of a series whose terms are determined by a recursive relationship using these Adomian polynomials. A brief outline of the method follows.

In reviewing the basic methodology involved, a general nonlinear differential equation will be used for simplicity. Consider

$$Fy = f$$

where F is a nonlinear differential operator and y and f are functions of t. Begin by rewriting the equation in operator form

$$Ly + Ry + Ny = f$$

where L is an operator representing the linear portion of F which is easily invertible, R is a linear operator for the remainder of the linear portion, and N is a nonlinear operator representing the nonlinear terms in F. Applying the inverse operator L^{-1}, the equation then becomes

$$L^{-1}Ly = L^{-1}f - L^{-1}Ry - L^{-1}Ny.$$

Since F was taken to be a differential operator and L is linear, L^{-1} would represent an integration and with any given initial or boundary conditions, $L^{-1}Ly$ will give an equation for y incorporating these conditions. This gives

$$y(t) = g(t) - L^{-1}Ry - L^{-1}Ny.$$

where g(t) represents the function generated by integrating f and using the initial/boundary conditions. Then assume that the unknown function can be written as an infinite series

$$y(t) = \sum_{n=0}^{\infty} y_n(t).$$

We set $y_0 = g(t)$ and the remaining terms are to be determined by a recursive relationship defined later. This is found by first decomposing the nonlinear term into a series of Adomian polynomials, A_n. The nonlinear term is written as

$$Ny = \sum_{n=0}^{\infty} A_n.$$

In order to determine the Adomian polynomials, a grouping parameter, λ, is introduced. It should be noted that λ is not a "smallness parameter" [28]. The series

$$y(\lambda) = \sum_{n=0}^{\infty} \lambda^n y_n$$

and

$$Ny(\lambda) = \sum_{n=0}^{\infty} \lambda^n A_n$$

are established. Then A_n can be determined by

$$A_n = \frac{1}{n!} \frac{d^n}{d\lambda^n} Ny(\lambda) \Big|_{\lambda=0}.$$

From

$$\sum_{n=0}^{\infty} y_n = y_0 - L^{-1} \sum_{n=0}^{\infty} Ry_n - L^{-1} \sum_{n=0}^{\infty} A_n,$$

the recursive relationship is found to be

$$y_0 = g(t)$$

$$y_{n+1} = L^{-1} Ry_n + L^{-1} A_n.$$

This method produces a convergent series solution [36] and the truncated series provides an approximate solution.

As stated above, the ADM produces a convergent series solution. The issue of convergence is addressed by several researchers [13, 28, 47]. According to Cherruault et al. [28], the

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series produced by the decomposition method is absolutely convergent as well as uniformly convergent. This is the case because the series "rearranges a strongly convergent Taylor series of the analytic functions u and f(u). The series converges uniformly (and absolutely and in norm), hence the sum is not changed by rearrangement of the terms" [28]. Babolian and Biazar [13] provide a definition from which the order of convergence for the method could be determined. Of course, having a higher order of convergence is desirable since then the series will converge more rapidly.

1.1.1 Example

As a simple example, consider the nonlinear, initial value problem

$$\frac{dy}{dx} = y^2 \tag{1.1}$$

with the initial condition

$$y(0) = 1. \tag{1.2}$$

This differential equation has the exact solution of $y(x) = \frac{1}{1-x}$.

Following the method described above, we define a linear operator

$$L = \frac{d}{dx}. \tag{1.3}$$

The inverse operator is then

$$L^{-1} = \int_0^x (\cdot) dx. \tag{1.4}$$

Rewriting the differential equation (1.1) in operator form, we have

$$Ly = Ny \tag{1.5}$$

where N is a nonlinear operator such that

$$Ny = y^2. \tag{1.6}$$

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Next we apply the inverse operator for L to the equation. On the left hand side of the equation, this gives

$$L^{-1}Ly = y(x) - y(0). \tag{1.7}$$

Using the initial condition, this becomes

$$L^{-1}Ly = y(x) - 1. \tag{1.8}$$

Returning this to equation (1.5), we now have

$$y(x) - 1 = L^{-1}(Ny) \tag{1.9}$$

or

$$y(x) = 1 + L^{-1}(Ny). \tag{1.10}$$

Next, we need to generate the Adomian polynomials, A_n. Let y be expanded as an infinite series $y(t) = \sum_{n=0}^{\infty} y_n(t)$ and define $Ny = \sum_{n=0}^{\infty} A_n$.
Then

$$\sum_{n=0}^{\infty} y_n(t) = 1 + L^{-1} \left(\sum_{n=0}^{\infty} A_n \right). \tag{1.11}$$

To find A_n, we introduce the scalar λ such that,

$$y(\lambda) = \sum_{n=0}^{\infty} \lambda^n y_n. \tag{1.12}$$

Then,

$$Ny(\lambda) = \sum_{n=0}^{\infty} \lambda^n \sum_{i=0}^{n} (y_i y_{n-i}). \tag{1.13}$$

From the definition of the Adomian polynomials,

$$A_n = \frac{1}{n!} \frac{d^n}{d\lambda^n} (Ny(\lambda)) \bigg|_{\lambda=0}, \tag{1.14}$$

we find the Adomian polynomials.

$$A_0 = y_0^2 \tag{1.15}$$

$$A_1 = 2y_0 y_1 \tag{1.16}$$

$$A_2 = 2y_0 y_2 + y_1^2 \tag{1.17}$$

$$A_3 = 2y_0 y_3 + 2y_1 y_2 \tag{1.18}$$

$$A_4 = 2y_0 y_4 + 2y_1 y_3 + y_2^2 \tag{1.19}$$

Returning the Adomian polynomials to equation (1.11), we can determine the recursive relationship that will be used to generate the solution.

$$y_0(x) = 1 \tag{1.20}$$

$$y_{n+1}(x) = L^{-1}(A_n) \tag{1.21}$$

Solving this yields

$$y_0 = 1 \tag{1.22}$$

$$y_1 = x \tag{1.23}$$

$$y_2 = x^2 \tag{1.24}$$

$$y_3 = x^3 \tag{1.25}$$

$$y_4 = x^4 \tag{1.26}$$

We can see that the series solution generated by this method is

$$y(x) = 1 + x + x^2 + x^3 + x^4 + \cdots = \sum_{n=0}^{\infty} x^n \tag{1.27}$$

which we recognize as the Taylor series for the exact solution

$$y(x) = \frac{1}{1-x}. \tag{1.28}$$

1.2 Advantages and Disadvantages of the Adomian Decomposition Method

Researchers who have used the ADM have frequently enumerated on the many advantages that it offers. Most often cited is the efficiency of the method. Many authors find that the ADM requires less computational work than traditional approaches [25, 53, 60]. Other advantages include the ability to solve nonlinear problems without linearization, the wide applicability to several types of problems and scientific fields, and the development of a reliable, analytic soltuion. According to Wang [60], this method does not linearize the problem nor use assumptions of weak nonlinearity and therefore can handle nonlinearities which are "quite general" and generates solutions that "may be more realisitic than those achieved by simplifying the model...to achieve conditions required for other techniques." Jiao et al.[44] state that the "ADM is quantitative rather than qualitative, analytic, requiring neither linearization nor perturbation, and continuous with no resort to discretization and consequent computer-intensive calculations".

The ADM does have some disadvantages, however. The first is that the method gives a series solution which must be truncated for practical applications. In addition, the rate and region of convergence are potential shortcomings. According to Jiao et al. [44], "although the series can be rapidly convergent in a very small region, it has very slow convergence rate in the wider region...and the truncated series solution is an inaccurate solution in that region, which will greatly restrict the application area of the method." An investigation into this claim would greatly benefit the scientific community.

Nonetheless, the ADM is proving to be a very useful tool with wide application. According to Wazwaz [61], "The main advantage of the method is that it can be applied directly for all types of differential and integral equations, linear or nonlinear, homogeneous or inhomogeneous, with constant coefficients or with variable coefficients. Another important advantage

is that the method is capable of greatly reducing the size of computational work while still maintaining high accuracy of the numerical solution." These advantages are presumably the basis for the wide-ranging applicability of the method.

1.3   Applications of the Adomian Decomposition Method

Adomian decomposition has been shown to provide solutions for a wide array of equations, including algebraic equations, ordinary and partial differential equations, integral equations, and integro-differential equations [5, 2, 1, 4, 38, 57, 60]. As such, this method has extensive applications in such fields as physics, biology, chemistry, and engineering [5, 3, 14, 24, 43, 37]. In fact, the ADM has recently been applied to "such diverse areas as chaos theory, heat and/or mass transfer, particle transport, nonlinear optics, and the fermentations process"[37].

In the field of fluid mechanics, Adomian decomposition has been applied to several problems already. A few examples of this are detailed below. To begin with, Bulut et al. [25] used Adomian decomposition to develop an analytic solution for "a steady flow problem of a viscous incompressible fluid through an orifice" as governed by the Navier-Stokes equations. By comparing the ADM results of a "simple problem of Poisson's equations" with results of a numerical solution, the authors found the decomposition method to be a reliable technique with "less computational work" and therefore, quite practical [25].

Adomian decomposition was also applied to "a time-fractional Navier-Stokes equation" for "unsteady flow of a viscous fluid in a tube" by Momani and Odibat [53]. The time-fractional Navier-Stokes equations are nonlinear and as such, "there is no known general method to solve these equations" and "very few cases where an exact solution can be obtained". The ADM allows the construction of an analytic solution in the form of a series through a reliable

technique with less work than traditional techniques. In this case, the authors found that the solution "continuously depends on the time-fractional derivative"[53].

Recently, Wang [60] applied the ADM to the classical Blasius equation. Although, the ADM easily provided an analytic solution to the classical problem, it was impossible to determine the value of the parameter $y''(0)$ with this solution. Therefore, the problem was transformed into a singular nonlinear boundary value problem to which the ADM was also applied. From this new solution, the parameter $y''(0)$ was easily determined. The 5-term approximate solution was comparable to the numerical solution. This showed that the ADM provided a reliable solution.

One final example of the applicability of this method was provided by Al-Hayani and Casús [7]. Their work applied the ADM to first order initial value problems with Heaviside functions and other discontinuities. The ADM worked well for this analysis and led to some interesting findings. To begin with, "the size of the jump does not affect the convergence of the method, which behaves equally well on both sides of the discontinuity"[7]. However, some cases required the inclusion of "more digits...in order to avoid unstable oscillations"[7]. Finally, the authors found that the error could be reduced with a slight modification to the ADM by including the term associated with the inverse operator applied to the source function in the first Adomian polynomial rather than the initial term in the series solution.

1.4   Modifications to the Adomian Decomposition Method

Since it was first presented in the 1980's, Adomian's decomposition method has led to several modifications on the method made by various researchers in an attempt to improve the accuarcy or expand the application of the original method. To begin with, Adomian and Rach [6] introduced modified Adomian polynomials which converge slightly faster than the

original or classical Adomian polynomials and are convenient for computer generation. The modified polynomials are defined using a differencing method. The first few terms of the modified Adomian polynomials generated are identical to the original Adomian polynomials, but higher order terms do exhibit differences. In addition to the classical and modified Adomian polynomials, Adomian also introduced accelerated Adomian polynomials [6, 28]. These Adomian polynomials provide faster convergence; however, they are "less convenient computationally" [6]. Despite the various types of Adomian polynomials available, the original Adomian polynomials are more generally used based on the advantage of a "convenient algorithm which is easily remembered" [28]. They are "easily generated without a computer and converge rapidly enough for most problems" [6].

Proposed modifications to the standard ADM can be as simple as the following. As stated previously, Jiao et al. [44] found that the ADM has a slow convergence rate in a wide region and has limited accuarcy. To improve on this, the authors introduced an aftertreatment technique to the original ADM when applied to nonlinear differential equations. The aftertreatment involves applying the Padé approximant to the truncated series generated by the ADM. Because the Padé approximant generally enlarges the "convergence domain of the truncated Tayolor series", its use tends to improve the convergence rate and accuracy of the ADM [44]. In the event of an oscillatory system, the Laplace transform is first applied to the truncated seris, then the Padé approximant is formed and the final solution is obtain by applying the inverse Laplace transformation. The effectiveness of this aftertreatment technique is supported by Hashim [36]. Hashim [36] compared the results of an original ADM to those with a Padé approximation of the truncated series and found that "the ADM with Padé approximation give more accurate results compared with the standard ADM without Padé approximation." Wazwaz [62] also used Padé approximants to the solution obtained using a modified decomposition method and found that not only does this improve the re-

sult, but that the "error decreases dramatically with the increase of the degree of the Padé approximants."

Another modification to the standard ADM was proposed by Wazwaz. Wazwaz [62] presented "a reliable modification of the Adomian decomposition method". In the standard ADM, the solution is defined as a series using a recursive relationship $y_0(x) = f(x)$, $y_{k+1} = L^{-1}(Nf)$. The modified decomposition method proposed by Wazwaz addresses this recursive relationship. It divides the original function into two parts, one assigned to the initial term of the series and the other to the second term. All remaining terms of the recursive relationship are defined as previously, but the modification results in a different series being generated. This method has been shown to be "computationally effecient"; however, it "does not always minimize the size of calculations needed and even requires much more calculations than the standard Adomian method" [50]. "The success of the modified method depends mainly on the proper choice" of the parts into which to divide the original function [50].

In 2005, Wazwaz [61] presented another type of modification to the ADM. The purpose of this new approach was to overcome the difficulties that arise when singular points are present. The modification arises in the initial definition of the operator when applying the ADM to the Emden-Fowler equation. According to Wazwaz [61], the "Adomian decomposition method usually starts by defining the equation in an operator form by considering the lowest-ordered derivative in the problem." However, by defining the differential operator in terms of both derivatives in the equation, the singularity behavior was easily overcome. "The most striking advantage of using this choice for the operator L is that it attacks the Emden-Fowler equation directly without any need for a transformation formula" [61]. This modification could prove useful for similar models with singularities.

Another modification was proposed by Luo [50]. This variation separates the ADM into two steps and therefore is termed the two-step Adomian decomposition method (TSAMD) [50, 66]. The purpose behind the proposed scheme is to identify the exact solution more readily and eliminate some calculations as such. The two steps proposed by Luo [50] are as follows:

(1) First, apply the inverse operator and the given conditions. Then, define a function, u_0, based on the resulting terms. If this satisfies the original equation and the conditions as checked by substiution, it is considered the exact solution and the calcuations terminated. Otherwise, continue on to step two.

(2) Continue with the standard Adomian recursive relationship.

As one can see, this modification involves "verifying that the zeroth component of the series solution includes the exact solution" [50]. As such, it offers the advantage of requiring less caluculations than the standard ADM.

Another recent modification is termed the restarted Adomian method [15, 16]. This method involves repeatedly updating the initial term of the series generated. Rather than calculating additional terms of the solution by determining "Adomian polynomials for large indexes" [17], a few Adomian polynomials are determined. After which, the series solution is generated and then using this result to reinitialize the initial term of the series, new Adomian polynomials and solution terms are generated. By repeating this only a few times, a more accurate approximation can be obtained [15, 16, 17] .

Several other researchers have developed modifications to the ADM [42, 41, 45]. The modifications arise from evaluating difficulties specific for the type of problem under consideration. Usually the modification involves only a slight change and is aimed at improving the con-

vergence or accuarcy of the series solution. This further demonstrates the wide applicablity that the ADM has, as well as its simplicity since it can be easily modified for the situation at hand.

1.5   Present Work

As one can see, much interest and research has been focused on the Adomian Decomposition Method. Researchers frequently laud the benefits of the ADM, whereas little mention is made of the disadvantages or drawbacks of the method. As such, a contextual evalutaion of the method is recommended. The present work represents an initial effort towards this evalution with respect to equations arising in fluid dynamic applications.

In order to provide cohesiveness in the work, three fluid dynamic problems were chosen relating to boundary layer theory and with some correlation to each other. The first problem analyzes the Blasius equation with generalized boundary conditions. This equation describes the velocity profile in the boundary layer along a semi-infinite flat plate. Next, the Falkner-Skan equation is examined, which also describes the velocity profile in the boundary layer. However, in this case the fluid flow is along a curved plate or wedge. Finally, an analysis of the Orr-Sommerfeld equation is developed. This equation is related to the hydrodynamic stability of the fluid flow when a small disturbance is introduced. The solutions developed by the ADM are compared to previously published results in order to examine the advantages and disadvantages arising from this method.

CHAPTER TWO: GENERALIZED BLASIUS EQUATION

2.1 The Blasius Equation

The Blasius Equation is a famous problem arising from boundary layer theory of fluid mechanics. This equation emerged when Blasius developed a method in which the boundary layer equations are reduced to ordinary differential equations [30]. This well-known equation is a third order, nonlinear differential equation,

$$f'''(\eta) + f''(\eta) \cdot f(\eta) = 0 \tag{2.1}$$

on $0 \leq \eta < \infty$ satisfying the boundary conditions

$$f(0) = 0, f'(0) = 0, f'(\infty) = 1. \tag{2.2}$$

This equation describes the velocity profile in the boundary layer when one considers the movement of an incompressible, viscous fluid along a semi-infinite plate [20, 22].

In this work, we generalize the boundary conditions as

$$f(0) = -\alpha, f'(0) = -\beta, f'(\infty) = 1 \tag{2.3}$$

where α and β are constants. According to Guedda [32], in the event of $f(0) = -\alpha$, α represents a suction/injection parameter where $-\alpha > 0$ represents suction and $-\alpha < 0$ corresponds to injection of the fluid. The initial condition, $f'(0) = -\beta$, indicates the slip condition at the the the wall [51]. The case where $\beta = 0$ represents no-slip.

Another parameter considered when evaluating the Blasius equation is the initial value of the second derivative, $f''(0)$. The value of $f''(0)$ is a significant parameter in the boundary layer theory which gave rise to the equation. According to Weyl [63], "the value $[f''(0)]$

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is the essential factor in the formula for the skin friction along the immersed plate". Due to its importance, a portion of this work is focused on accurately determining this parameter.

Much work has been done on the Blasius equation, although no exact solution is known. Solutions for the equation have been developed by many approaches. Blasius gave a power series solution [49]. Numerical methods, such as the Runge-Kutta method or the shooting mehtod, have also been used [29, 31]. Other techniques used include perturbation methods [22, 49], the homotopy analysis method [49], and the differential transformation method [65]. Recently, Wang [60] presented a solution utilizing the Adomian Decomposition Method to solve the classical Blasius equation. This method proves to be reliable and demonstrates many advantages.

In the Adomian decomposition method, the solution is expanded as an infinite series and is determined by a series of successive calculations. The partial sum of this series at any point provides an approximate solution, which can be improved by adding additional terms. Hashim [36] provided corrections to the numerical values in Wang's [60] article and also showed that the accuracy of the numercial solution can be improved by using Padé approximations. Despite the errors in the numerical values, the methodology in Wang [60] appears to be reliable.

In this section, we will apply Wang's [60] methodology to the generalized Blasius equation

$$f'''(\eta) + f''(\eta) \cdot f(\eta) = 0 \tag{2.4}$$

on $0 \leq \eta < \infty$ with

$$f(0) = -\alpha, f'(0) = -\beta, f'(\infty) = 1. \tag{2.5}$$

2.2 Adomian Decomposition of Blasius Equation

We begin by introducing a differential operator, L, as

$$L = \frac{d^3}{d\eta^3}. \tag{2.6}$$

Then the inverse operator is

$$L^{-1} = \int_0^\eta \int_0^\eta \int_0^\eta (\cdot) d\eta d\eta d\eta. \tag{2.7}$$

The Blasius equation is then written as $f''' = -f'' f$.

Therefore,

$$Lf = Nf \tag{2.8}$$

where

$$Nf = -f'' f. \tag{2.9}$$

Operating with L^{-1} yields

$$f(\eta) - f(0) - f'(0)\eta - \frac{1}{2}f'(0)\eta^2 = L^{-1}(-f'' f). \tag{2.10}$$

Using the boundary conditions and letting $f''(0) = k$,

$$f(\eta) = -\alpha - \beta\eta + \frac{1}{2}k\eta^2 + L^{-1}(-f'' f). \tag{2.11}$$

Let f be expanded as an infinite series $f(\eta) = \sum_{n=0}^\infty f_n(\eta)$.

Then

$$\sum_{n=0}^\infty f_n(\eta) = -\alpha - \beta\eta + \frac{1}{2}k\eta^2 + L^{-1}\left(\sum_{n=0}^\infty A_n\right). \tag{2.12}$$

Next, we need to determine the Adomian polynomials, A_n. To find A_n, we introduce the scalar λ,

$$f(\lambda) = \sum_{n=0}^\infty \lambda^n f_n \tag{2.13}$$

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such that

$$Nf(\lambda) = -\sum_{n=0}^{\infty} \lambda^n f_n'' \cdot \sum_{n=0}^{\infty} \lambda^n f_n = -\sum_{n=0}^{\infty} \lambda^n \cdot \left(\sum_{i=0}^{n} f_i f_{n-i}''\right). \tag{2.14}$$

From

$$A_n = \frac{1}{n!} \frac{d^n}{d\lambda^n} (Nf(\lambda))\bigg|_{\lambda=0}, \tag{2.15}$$

we find the Adomian polynomials.

$$A_0 = -f_0 f_0'' \tag{2.16}$$

$$A_1 = -f_1 f_0'' - f_0 f_1'' \tag{2.17}$$

$$A_2 = -f_0 f_2'' - f_1 f_1'' - f_2 f_0'' \tag{2.18}$$

$$A_3 = -f_0 f_3'' - f_1 f_2'' - f_2 f_1'' - f_3 f_0'' \tag{2.19}$$

$$A_4 = -f_0 f_4'' - f_1 f_3'' - f_2 f_2'' - f_3 f_1'' - f_4 f_0'' \tag{2.20}$$

Returning the Adomian polynomials to the equation

$$\sum_{n=0}^{\infty} f_n(t) = -\alpha - \beta\eta + \frac{1}{2}k\eta^2 + L^{-1}\left(\sum_{n=0}^{\infty} A_n\right), \tag{2.21}$$

we can determine the recursive relationship that will be used to generate the solution.

$$f_0(\eta) = -\alpha - \beta\eta + \frac{1}{2}k\eta^2 \tag{2.22}$$

$$f_{n+1}(\eta) = L^{-1}(A_n) \tag{2.23}$$

Solving this yields

$$f_0 = -\alpha - \beta\eta + \frac{1}{2}k\eta^2 \tag{2.24}$$

$$f_1 = -\frac{1}{120}k^2\eta^5 + \frac{1}{6}\alpha k\eta^3 + \frac{1}{24}\beta k\eta^4 \tag{2.25}$$

$$f_2 = 2.728175 \times 10^{-4}k^3\eta^8 + \frac{1}{24}\alpha^2 k\eta^4 + 0.025\alpha\beta k\eta^5 + \eta^6(-0.0069444\alpha k^2$$
$$+0.00416667\beta^2 k) - 0.00218254\beta k^2\eta^7 \tag{2.26}$$

$$f_3 = -9.394541 \times 10^{-6}k^4\eta^{11} + 0.008333\alpha^3 k\eta^5 + 0.008333\alpha^2\beta k\eta^6$$
$$+\eta^7(-0.00317460\alpha^2 k^2 + 0.00297619\alpha\beta^2 k) + \eta^8(-0.0020833\alpha\beta k^2$$
$$+3.720238 \times 10^{-4}\beta^3 k) + \eta^9(3.224206 \times 10^{-4}\alpha k^3$$
$$-3.554894 \times 10^{-4}\beta^2 k^2) + 1.033399 \times 10^{-4}\beta k^3\eta^{10} \tag{2.27}$$

$$f_4 = 3.199994 \times 10^{-7}k^5\eta^{14} + 0.00138889\alpha^4 k\eta^6 + 0.00198413\alpha^3\beta k\eta^7$$
$$+\eta^8(-0.001041667\alpha^3 k^2 + 0.00111607\alpha^2\beta^2 k) + \eta^9(-0.0010582\alpha^2\beta k^2$$
$$+2.89352 \times 10^{-4}\alpha\beta^3 k) + \eta^{10}(2.025463 \times 10^{-4}\alpha^2 k^3 - 3.69544 \times 10^{-4}\alpha\beta^2 k^2$$
$$+2.89352 \times 10^{-5}\beta^4 k) + \eta^{11}(1.322000 \times 10^{-4}\alpha\beta k^3 - 4.411677 \times 10^{-5}\beta^3 k^2)$$
$$+\eta^{12}(-1.41482 \times 10^{-5}\alpha k^4 + 2.204586 \times 10^{-5}\beta^2 k^3)$$
$$-4.47999 \times 10^{-6}\beta k^4\eta^{13}. \tag{2.28}$$

By truncating the solution, we have an approximate solution as

$$f = f_0 + f_1 + f_2 + f_3 + f_4. \tag{2.29}$$

The next step is to find k by the boundary condition $f'(\infty) = 1$. Unfortunately, the solution equations are polynomials in η, which has an indeterminable limit as $\eta \to \infty$. To accomodate for this, Wang [60] attempted using Padé approximants in order to find a limit; however, this approach failed for both the (2,2) and (3,3) Padé attempts. As an alternative, a transformed Blasius equation was used to determine $f''(0) = k$.

2.3 Transformed Blasius Equation

Let

$$x = f'(\eta) \tag{2.30}$$

$$y(x) = f''(\eta) \tag{2.31}$$

Then we see that

(1) $-\beta < x < 1$

(2) $y(x) = \frac{df'(\eta)}{d\eta} = \frac{dx}{d\eta}$

(3) Since $f' \to 1$ as $\eta \to \infty$ indicates f behaves as $g(\eta) = \eta$ for large η, we know that $f'' \to 0$ as $\eta \to \infty$ and therefore $y(x) = f''(\eta)$ means $y(1) = f''(\infty) = 0$.

Rearranging $f''' + f''f = 0$, we see that $f = -\frac{f'''}{f''}$. The derivative of the Blasius equation is then taken with respect to η and substitutions are made for f.

$$\frac{d}{d\eta}(f''' + f''f = 0) \tag{2.32}$$

$$f^{(4)} + f''f' + f'''f = 0 \tag{2.33}$$

$$f^{(4)} + f''f' - \frac{f'''^2}{f''} = 0 \tag{2.34}$$

Note that

$$y' = \frac{dy}{dx} = \frac{f'''}{f''} \tag{2.35}$$

$$y'' = \frac{d^2y}{dx^2} = \frac{f^{(4)}}{f''^2} - \frac{f'''^2}{f''^3} \tag{2.36}$$

From this we see that $y' = -f$ and therefore $f(0) = -\alpha$ gives us that $-y'(-\beta) = -\alpha$.

Substituting y' and y'' into equation (2.34), we now have

$$y''(x) \cdot (y(x))^2 + y(x) \cdot x = 0$$

$$or$$

$$y''(x)y(x) + x = 0 \tag{2.37}$$

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The transformed Blasius equation is then

$$y''(x) \cdot y(x) + x = 0 \tag{2.38}$$

on $-\beta < x < 1$ with the boundary conditions

$$y(1) = 0 \tag{2.39}$$

$$y'(-\beta) = \alpha \tag{2.40}$$

We now apply the Adomian Decomposition Method to the transformed equation. This will provide us with a solution from which we can determine our constant k.

2.4 Adomian Decomposition of Transformed Blasius Equation

Again, we introduce the operator L as

$$L = \frac{d^2}{dx^2} \tag{2.41}$$

Then

$$L^{-1} = \int_{-\beta}^{x} \int_{-\beta}^{x} (\cdot) dx dx \tag{2.42}$$

The transformed eqn is then $y'' = -\frac{-x}{y}$ or $Ly = -\frac{x}{y}$.

Operating with L^{-1} yields

$$y(x) - y(-\beta) - y'(-\beta)x - \beta y'(-\beta) = L^{-1}(-\frac{x}{y}). \tag{2.43}$$

Using the boundary conditions and letting $f''(0) = k$ which means $y(-\beta) = k$,

$$y(x) = k + \alpha x + \alpha \beta + L^{-1}(-\frac{x}{y}). \tag{2.44}$$

As before, we let f be expanded as an infinite series $y(x) = \sum_{n=0}^{\infty} y_n(x)$.

Then

$$\sum_{n=0}^{\infty} y_n(t) = k + \alpha x + \alpha \beta + L^{-1} \left(\sum_{n=0}^{\infty} A_n \right) \tag{2.45}$$

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Again, we need to find the Adomian polynomials, A_n, using

$$y(\lambda) = \sum_{n=0}^{\infty} \lambda^n y_n \tag{2.46}$$

and hence

$$Ly(\lambda) = -\frac{x}{\sum_{n=0}^{\infty} \lambda^n y_n}. \tag{2.47}$$

From the definition of the Adomian polynomials,

$$A_n = \frac{1}{n!} \frac{d^n}{d\lambda^n} (Ny(\lambda)) \Big|_{\lambda=0}, \tag{2.48}$$

we find

$$A_0 = -\frac{x}{y_0} \tag{2.49}$$

$$A_1 = \frac{x f_1}{y_0^2} \tag{2.50}$$

$$A_2 = -x \left[\frac{y_1^2}{y_0^3} - \frac{y_2}{y_0^2} \right] \tag{2.51}$$

$$A_3 = -x \left[-\frac{y_3}{y_0^2} + \frac{2y_1 y_2}{y_0^3} - \frac{y_1^3}{y_0^4} \right] \tag{2.52}$$

$$A_4 = -x \left[-\frac{y_4}{y_0^2} + \frac{y_2^2}{y_0^3} + \frac{2y_1 y_3}{y_0^3} - \frac{3y_1^2 y_2}{y_0^4} + \frac{y_1^4}{y_0^5} \right] \tag{2.53}$$

As before, we now have the recursive relationship

$$y_0(x) = k + \alpha x + \alpha \beta \tag{2.54}$$

$$y_{n+1}(x) = L^{-1}(A_n) \tag{2.55}$$

which is then solved. We find that, in this instance, the solution requires two cases: (1) for $\alpha = 0$ and (2) for $\alpha \neq 0$.

Case 1. $\alpha = 0$

$$y_0 = k \tag{2.56}$$

$$y_1 = -\frac{x^3}{6k} + \frac{\beta^2 x}{2k} + \frac{\beta^3}{3k} \tag{2.57}$$

$$y_2 = -\frac{x^6}{180k^3} + \frac{\beta^2 x^4}{24k^3} + \frac{\beta^3 x^3}{18k^3} - \frac{\beta^5 x}{30k^3} - \frac{\beta^6}{72k^3} \tag{2.58}$$

$$y_3 = -\frac{x^9}{2160k^5} + \frac{5\beta^2 x^7}{1008k^5} + \frac{\beta^3 x^6}{180k^5} - \frac{\beta^4 x^5}{80k^5} - \frac{11\beta^5 x^4}{360k^5} \tag{2.59}$$
$$-\frac{\beta^6 x^3}{48k^5} + \frac{\beta^8 x}{180k^5} + \frac{13\beta^9}{7560k^5}$$

$$y_4 = -\frac{x^{12}}{19008k^7} + \frac{37\beta^2 x^{10}}{50400k^7} + \frac{\beta^3 x^9}{1296k^7} - \frac{43\beta^4 x^8}{13440k^7} - \frac{\beta^5 x^7}{144k^7}$$
$$+\frac{\beta^6 x^6}{4320k^7} + \frac{17\beta^7 x^5}{1200k^7} + \frac{5\beta^8 x^4}{288k^7} + \frac{121\beta^9 x^3}{15120k^7}$$
$$-\frac{17\beta^{11} x}{13860k^7} - \frac{79\beta^{12}}{259200k^7} \tag{2.60}$$

Case 2. $\alpha \neq 0$

$$y_0 = k + \alpha\beta + \alpha x \tag{2.61}$$

$$y_1 = s_1 x^2 + s_2 x + s_3 \ln|f_0| + s_4 x \ln|f_0| + s_5 \tag{2.62}$$

$$y_2 = p_1 x^3 + p_2 x^2 + p_3 x + p_4 \ln|f_0| + p_5 x \ln|f_0| + p_6 \ln^2|f_0|$$
$$+p_7 x \ln^2|f_0| + p_8 x^2 \ln|f_0| + p_9 \tag{2.63}$$

$$y_3 = t_1 x^4 + t_2 x^3 + t_3 x^2 + t_4 x + \frac{t_5}{f_0} + t_6 \ln|f_0| + t_7 \ln^2|f_0| + t_8 \ln^3|f_0|$$
$$+t_9 x \ln|f_0| + t_{10} x \ln^2|f_0| + t_{11} x \ln^3|f_0| + t_{12} x^2 \ln|f_0| + t_{13} x^2 \ln^2|f_0|$$
$$+t_{14} x^3 \ln|f_0| + t_{15} \tag{2.64}$$

$$y_4 = z_1 x^5 + z_2 x^4 + z_3 x^3 + z_4 x^2 + z_5 x + \frac{z_6}{f_0} + \frac{z_7 \ln|f_0|}{f_0} + \frac{z_8}{f_0^2}$$
$$+\frac{z_9 \ln|f_0|}{f_0^2} + z_{10} \ln|f_0| + z_{11} \ln^2|f_0| + z_{12} \ln^3|f_0|$$
$$+z_{13} \ln^4|f_0| + z_{14} x \ln|f_0| + z_{15} x \ln^2|f_0| + z_{16} x \ln^3|f_0| + z_{17} x \ln^4|f_0|$$
$$+z_{18} x^2 \ln|f_0| + z_{19} x^2 \ln^2|f_0| + z_{20} x^2 \ln^3|f_0| + z_{21} x^3 \ln|f_0|$$
$$+z_{22} x^3 \ln^2|f_0| + z_{23} x^4 \ln|f_0| + z_{24} \tag{2.65}$$

where the coefficients, s_i, p_i, t_i, and z_i are functions of the initial conditions, α, β, and k, which are all constant values. Due to the size and complexity of these functions, they are given in Appendix A.

2.5 Adomian Decomposition Solution

The next step is to determine the value of $f''(0) = k$ which could then be returned to the original series solution, equation (2.29). This provides an approximate solution to the Blasius equation.

As a simple example, consider the case for $\alpha = \beta = 0$. This has the approximate solution from the truncated series,

$$y = y_0 + y_1 + y_2 + y_3 + y_4 = k - \frac{x^3}{6k} - \frac{x^6}{180k^3} - \frac{x^9}{2160k^5} - \frac{x^{12}}{19008k^7}. \tag{2.66}$$

Using the boundary condition from the transformed Blasisus equation, $y(1) = 0$, we are able to determine that

$k = 0.457674....$

The nonlinear shooting method was used to generate the numerical solution for the Blasius equation for comparison to the ADM solution. The numerical value for $k = f''(0) = 0.46960...$ for $\alpha = \beta = 0$. Therefore, the approximation generated by ADM has a relative error of 2.5% indicating that the Adomian decomposition provides an acceptable approximation.

Figure 2.1 shows the ADM solution and the numerical solution for $\alpha = \beta = 0$. This further demonstrates that the truncated ADM solution is a good approximation of the actual solution. We can also see that the solution is convergent in a small region ($\eta < 3.5$) after which it diverges quickly. However, it should be noted that the boundary layer is also a small region.
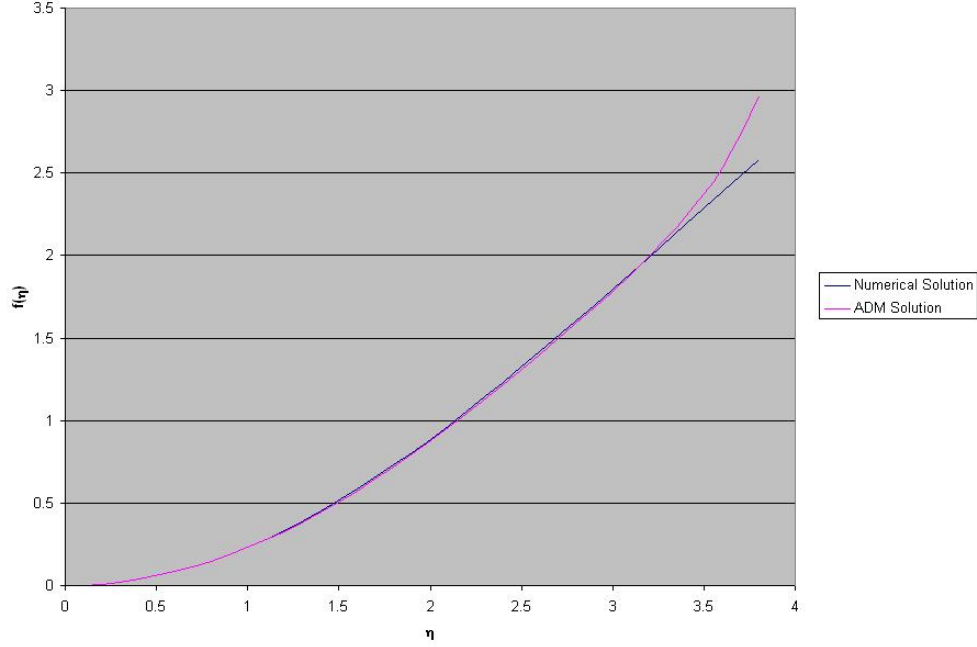
Figure 2.1: Solution Comparison, $\alpha = \beta = 0$

2.6 Results

In addition, the ADM solution was generated for various values of α and β. The values selected for β were based on the fact that $\beta = -1$ represents a straight line. This is a trivial solution to the Blasius equation (with $f''(0) = f'''(0) = 0$). Furthermore, the solutions are convex for $\beta \in (-1, 0]$ indicating $f''(0) > 0$; whereas, the solutions are concave, $f''(0) < 0$ for $\beta < -1$ [20].

The selection of the values for α were based on the following theorem defining existence and uniqueness of the solution presented by Hartman [35]. Slight modifications have been made to match the nomenclature used in the present work.

Theorem A [[35], pg. 531]. If $-1 < \beta < 0$, then equation (2.1) has one and only one solution for every $\alpha, -\infty < \alpha < \infty$. If $\beta = 0$, there exists a number $A \leq 0$ such that equation (2.1)

has a solution if and only if $-\alpha \geq A$; in this case, the solution is unique. In either case $0 < -\beta < 1$ or $\beta = 0$, the solution satisfies $f''(\eta) > 0$ for $0 \leq \eta\infty$.

Thus, values for α were chosen so that $-\alpha > 0$.
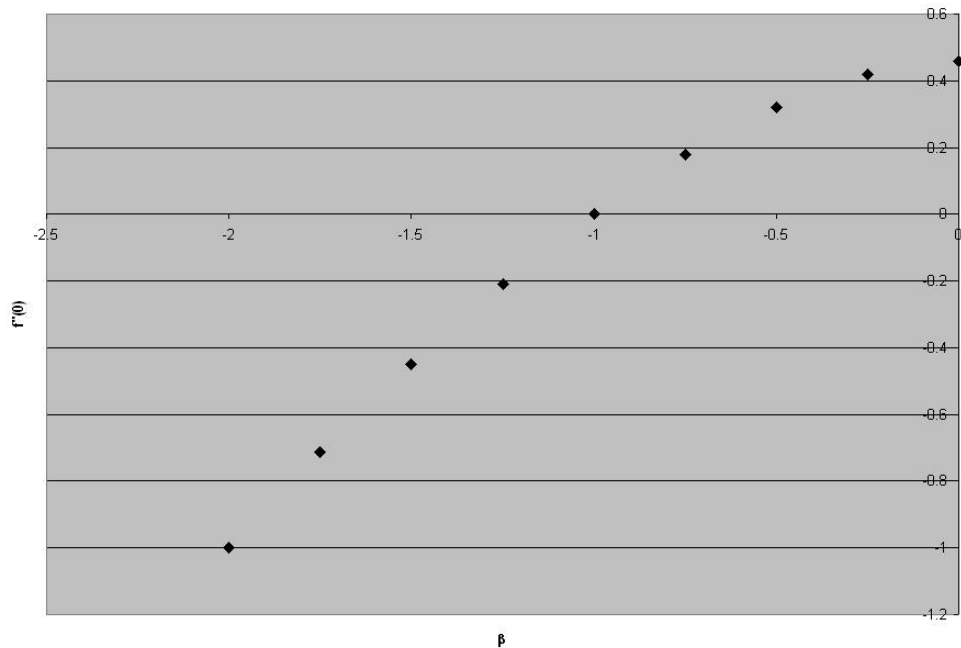
Figure 2.2: Solution Values for Case 1. $\alpha = 0$

Case 1. $\alpha = 0$.

The values for $f''(0)$ for various values of β are given in Table 2.1. The relative error of the ADM solution as compared to the numerical solution is also given. In figure 2.2, the results are shown graphically. The solution graphs for these values of $f''(0)$ are shown in figures 2.3 and 2.4.

Case 2. $\alpha \neq 0$.

The results for $\beta = 0$ are given in Table 2.2 and shown graphically in Figure 2.5. In addition,

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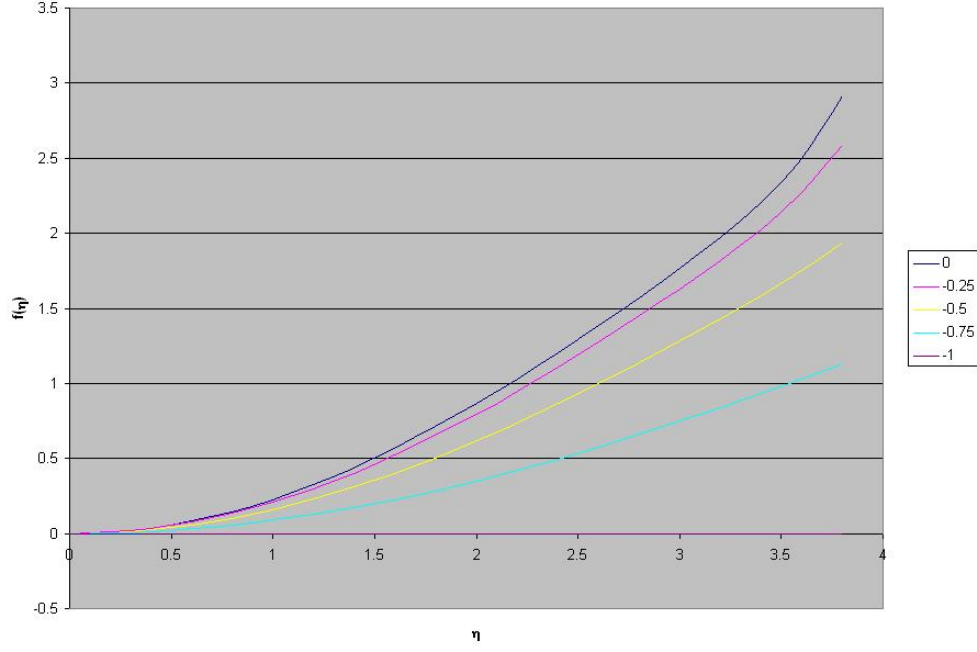
Figure 2.3: Solution Comparison for Case 1. $\alpha = 0$ with $\beta > 0$

Table 2.1: Solution values for $f''(0)$ for Case 1. $\alpha = 0$.

β	$f''(0)$	numerical $f''(0)$	error(%)
0	0.457673776	0.46960056	2.54
-0.25	0.418856959	0.42954	2.49
-0.5	0.32086020	0.32874079	2.40
-0.75	0.178633046	0.18284834	2.31
-1	-1.69E-14	0	
-1.25	-0.210116330	-0.21472	2.14
-1.5	-0.448284389	-0.45778062	2.07
-1.75	-0.711949665	-0.72657515	2.01
-2	-0.999121704	-1.0190742	1.96

the results for various values of α and β are given in Table 2.3. We see that the results for this case are not as accurate as the first. The reasons for this might be found in an examination of the existence and uniqueness of the solution. Figure 2.6 shows one example of y(1) as a

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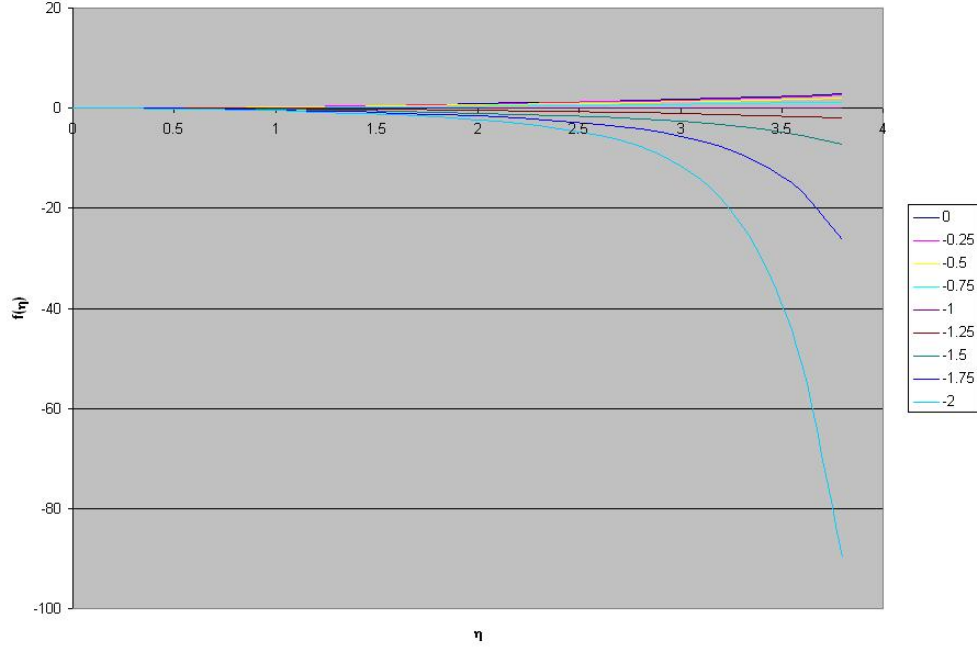
Figure 2.4: Solution Comparison for Case 1. $\alpha = 0$ with $\beta < 0$

function of k since the boundary conditions mandate the solution is a zero of this function. Ultimately, this case requires further examination.

Table 2.2: Solution values for $f''(0)$ for $\beta = 0$.

α	$f''(0)$	numerical $f''(0)$	error(%)
0	0.45767	0.46960	2.5
-0.25	0.63748	0.65774	3.1
-0.5	0.73001	0.85792	14.9
-0.75	0.92910	1.06725	12.9
-1	1.16151	1.28366	9.5
-1.5	2.50943	1.73199	44.9
-2	3.00028	2.19470	36.7
-2.5	1.49962	2.66758	43.8

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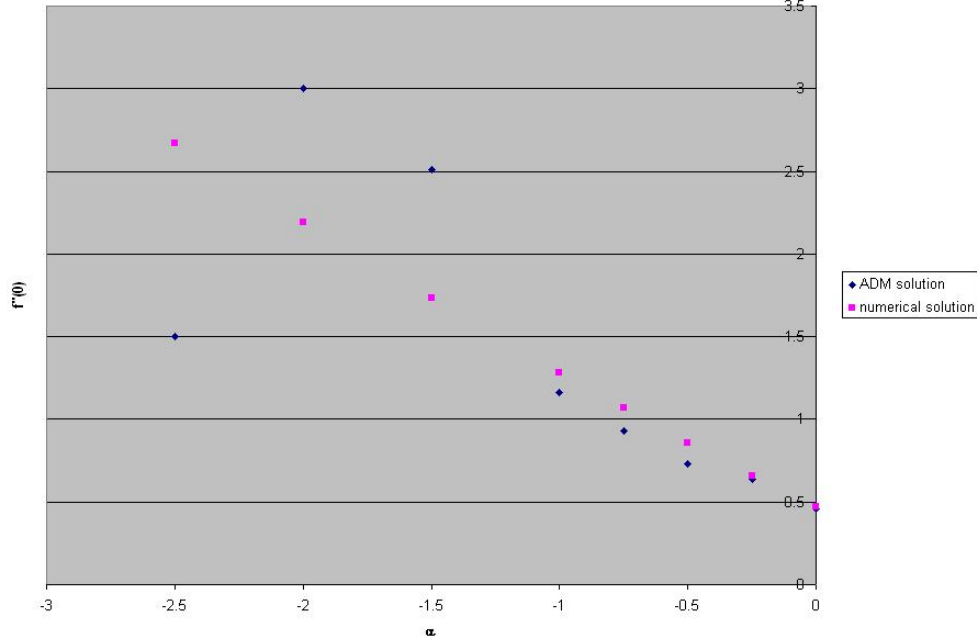
Figure 2.5: Solution values for Case 2 with $\beta = 0$

2.7 Aftertreatment Technique

An aftertreatment technique involving the application of Padé approximants to the truncated series generated by the ADM has been shown by many researchers to be an effective tool [36, 44, 62]. This process has been useful in improving the convergence rate and the accuracy of the method. As such, Padé approximants were applied to the solution generated above.

To begin with, a [6,6] Padé approximant was applied to equation (2.66), the approximate solution for $\alpha = \beta = 0$. This was chosen in accordance with the work presented by [36]. This yielded a solution of $k = 0.463257...$ which reduced the relative error to 1.4%. The new value was returned to the ADM solution and is graphed in Figure 2.7. Table 2.4 gives the results for Case 1. $\alpha = 0$ with a comparison to the relative error from the method without
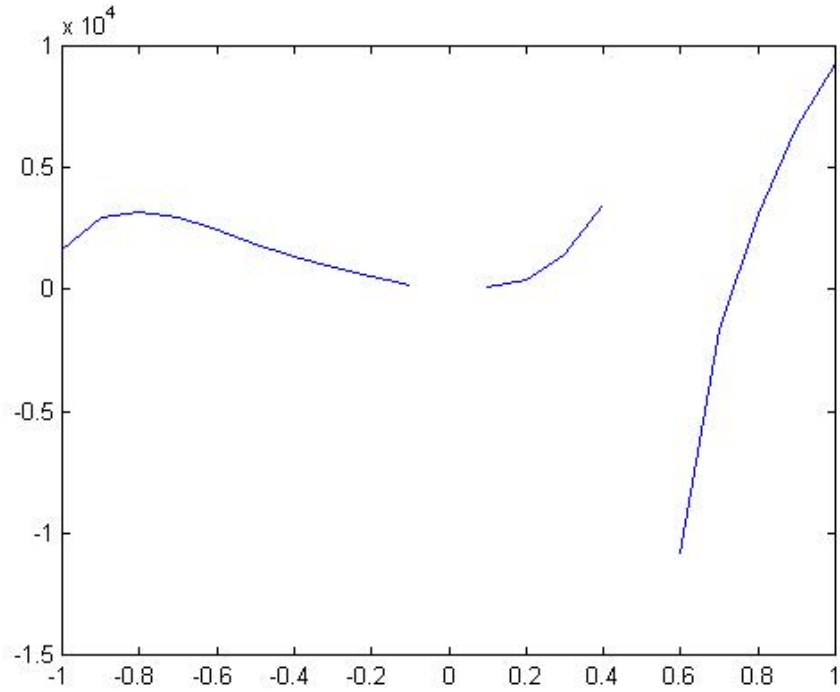
Figure 2.6: y(1) as a function of k for $\alpha = -0.5$, $\beta = 0$

the aftertreatment. This shows a limited effectiveness in applying the Padé approximants as an aftertreatment in this work. Possible reasons for this should be explored.

2.8   Conclusion

The use of the ADM demonstrates several advantages in this application. To begin with, it requires nominal computational work and does not require linearization or additional assumptions. The method generates fairly accurate results indicating that it is an effective method. As such, its application to fluid dynamic applications should be further investigated.

Table 2.3: Solution values for $f''(0)$ for $\alpha \neq 0$ and $\beta \neq 0$.

α	β	$f''(0)$	numerical $f''(0)$	error(%)
-0.25	-0.25	0.64366	0.56268	14.4
-0.5	-0.25	0.63531	0.70570	10.0
-0.75	-0.25	0.75549	0.85650	11.8
-1	-0.25	0.67637	1.01344	33.3
-0.25	-0.50	1.37114	0.41465	230.7
-0.50	-0.50	0.54653	0.50703	7.8
-0.75	-0.50	0.59692	0.60469	1.3
-1.00	-0.50	0.66743	0.70665	5.5
-0.25	-0.75	0.37305	0.22490	65.9
-0.50	-0.75	0.38392	0.27005	42.2
-0.75	-0.75	0.44351	0.31781	39.5
-1.00	-0.75	0.40011	0.36775	8.8
-0.25	-1.00	0.12926	0	12.9
-0.50	-1.00	0.16140	0	16.1
-0.75	-1.00	0.30740	0	30.7
-1.00	-1.00	0.91235	0	91.2

Table 2.4: Values for $f''(0)$ for Case 1. $\alpha = 0$ with aftertreatment.

β	$f''(0)$ Padé	numerical $f''(0)$	error(%)	original error (%)
-0.25	0.416625	0.429541	3.0	2.5
-0.5	0.271482	0.328741	17.4	2.4
-0.75	0.000000	0.182848	100.0	2.3
-1	0	0.000000		
-1.25	0.000000	-0.214718	100.0	2.1
-1.5	-0.447447	-0.457781	2.3	2.1
-1.75	-0.709768	-0.726575	2.3	2.0
-2	-0.990113	-1.019074	2.8	2.0

2.9   Comparison with Bender's Perturbation Method

Based on the small region of convergence, it is concluded that the ADM is only acceptable for a localized solution. If a global solution is desired, it is recommended that an alternate
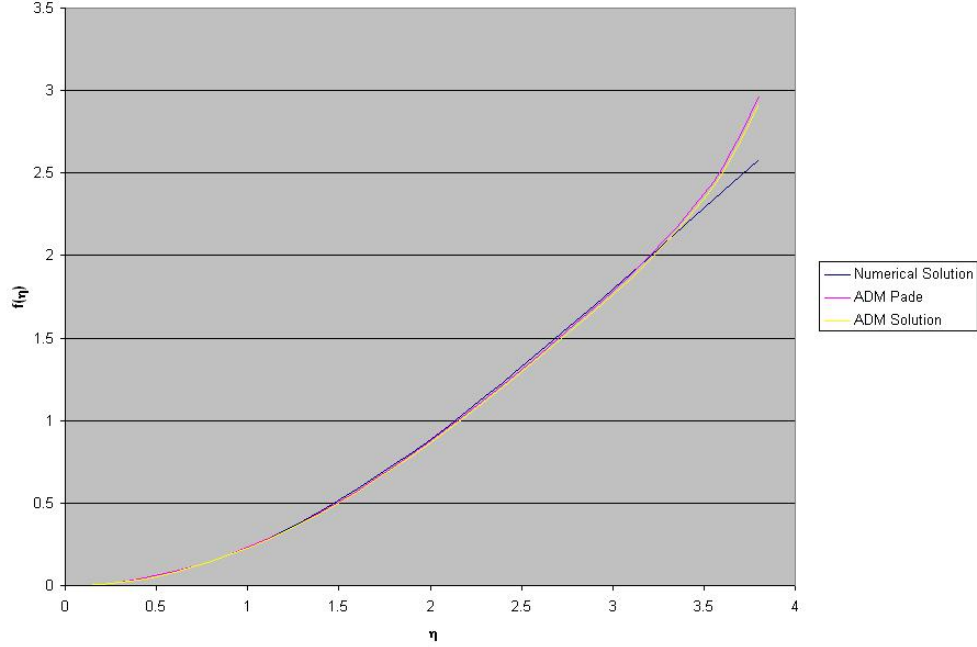
Figure 2.7: Solution Comparison with Padé Aftertreatment, $\alpha = \beta = 0$

method be employed. One method that has proved effective in this respect for the generalized Blasius equation (2.1) is the δ-perturbation method developed by Bender et al. [22]. This method is used to solve equation (2.1) in Appendix B of this work. In this section, we compare those results with the results of the present chapter.

To begin with, the case for $\alpha = \beta = 0$ is examined. In this instance, the δ-perturbation solution yields a skin friction value of

$k = 0.42871...$

This exhibits a relative error of 8.7% as compared to the numerical solution. This error is slightly larger than that of the ADM solution. However, it should be noted that the δ-perturbation method is only a second-order approximation; whereas, the ADM utilizes a fourth-order approximation. Figure 2.8 shows both the ADM solution and the δ-perturbation

solution in comparison to the numerical solution. In addition, Figure 2.9 illustrates the increase in accuracy of the solution as the degree of order for the approximation increases.
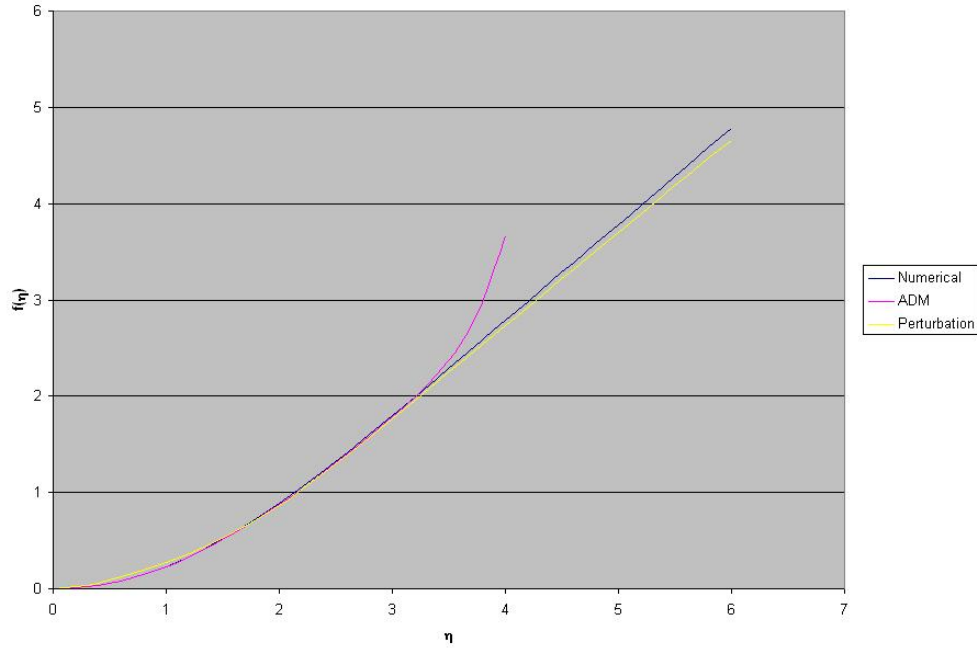
Figure 2.8: Solution Comparison for ADM and δ-Perturbation Method

Next, the case for $\alpha = 0$ was examined. Table 2.5 gives the results for the δ-perturbation method in conjunction with the ADM results. This is shown graphically in figure 2.10. The results for $\beta = 0$ are shown in table 2.6 and figure 2.11. Finally some results are displayed in table 2.7 for varying α and β.

Based on these findings, we note that while the ADM exhibited greater computational ease, the δ-perturbation solution yields a better quality solution. In addition to providing a global solution not hindered by a small region of convergence, it also maintains its accuracy as the variables α and β are changed.

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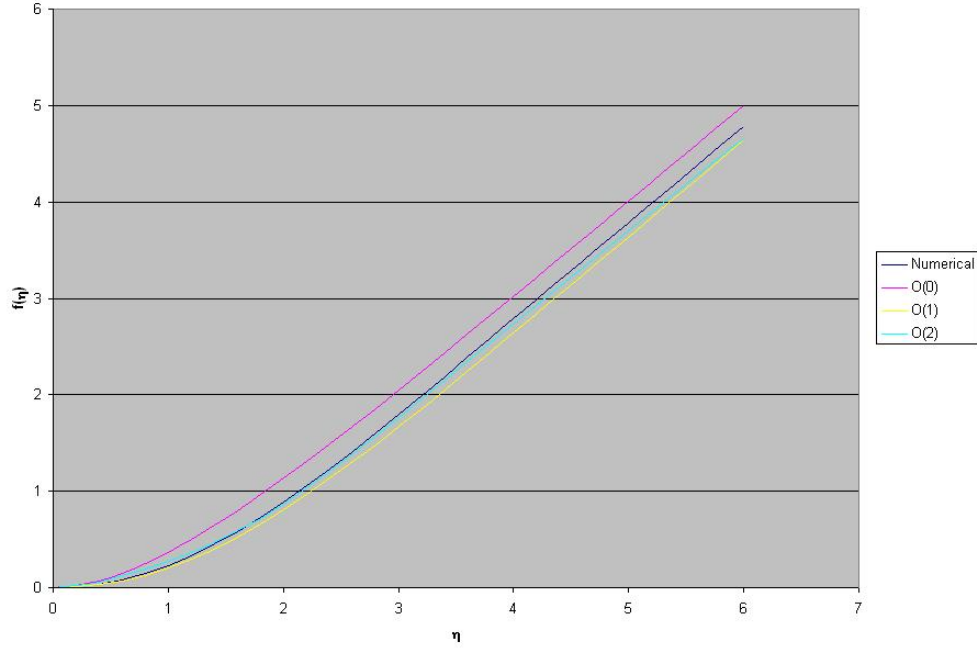
Figure 2.9: Solution Comparison for δ-Perturbation Method

Table 2.5: δ-Perturbation and ADM values for $f''(0)$ for Case 1. $\alpha = 0$.

β	$f''(0)\delta$	error(%)	$f''(0)$ ADM	error(%)
0	0.42871	8.7	0.45767	2.5
-0.25	0.40413	5.9	0.41886	2.5
-0.5	0.31607	3.9	0.32086	2.4
-0.75	0.17962	1.8	0.17863	2.3
-1	0.0	-	0.00000	-
-1.25	-0.20968	2.3	-0.21012	2.1
-1.5	-0.44708	2.3	-0.44828	2.1
-1.75	-0.70821	2.5	-0.71195	2.0
-2	-0.98652	3.2	-0.99912	2.0
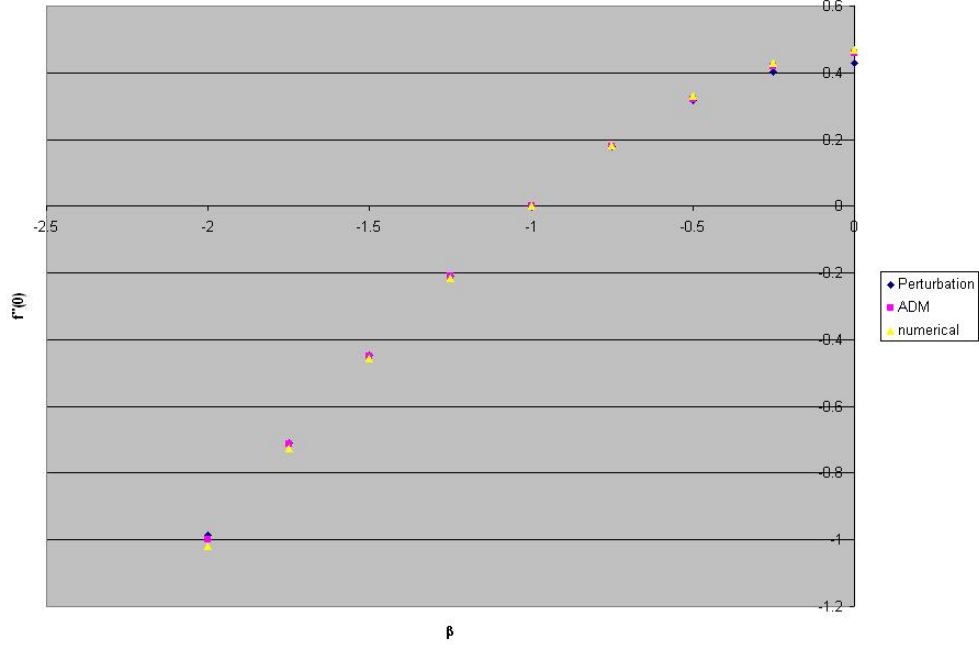
Figure 2.10: Solution Values for ADM and δ-Perturbation Method, Case 1. $\alpha = 0$

Table 2.6: δ-Perturbation and ADM values for $f''(0)$ for Case 2. $\alpha \neq 0$, $\beta = 0$.

α	$f''(0)$ δ	error(%)	$f''(0)$ ADM	error(%)
0	0.42871	8.7	0.45767	2.5
-0.25	0.63352	3.7	0.63748	3.1
-0.5	0.84616	1.4	0.73001	14.9
-0.75	1.06765	0.04	0.92910	12.9
-1	1.27500	0.7	1.16151	9.5
-1.5	1.71719	0.9	2.50943	44.9
-2	2.20905	0.7	3.00028	36.7
-2.5	2.77353	4.0	1.49962	43.8
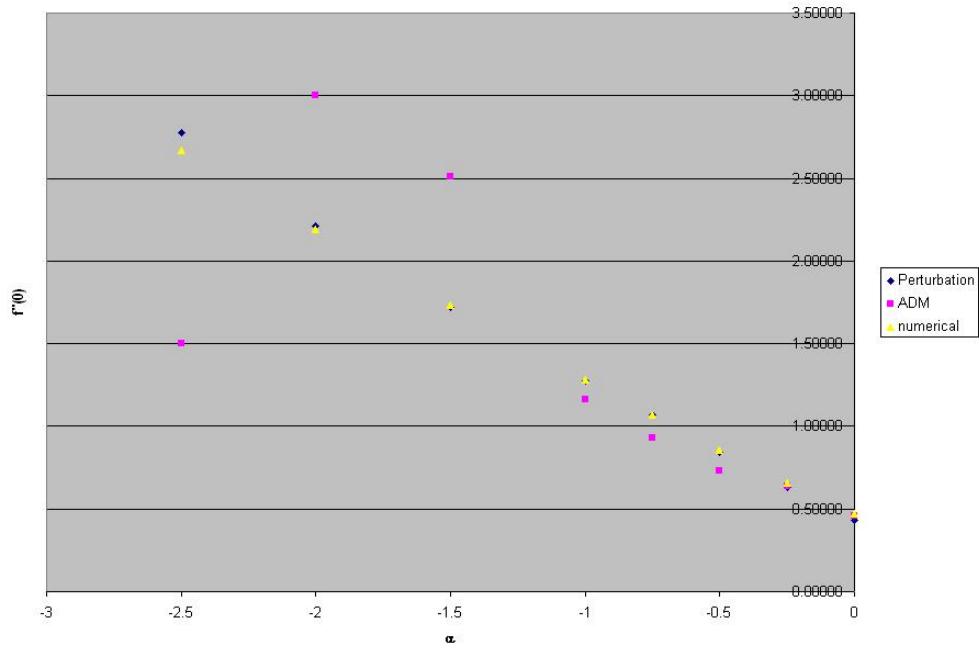
Figure 2.11: Solution Values for ADM and δ-Perturbation Method for $\beta = 0$

Table 2.7: δ-Perturbation and ADM values for $f''(0)$ for $\alpha \neq 0$ and $\beta \neq 0$.

α	β	$f''(0)\delta$	error (%)	$f''(0)$ ADM	error(%)
-0.25	-0.25	0.550297287	2.2	0.64366 1	4.4
-0.5	-0.25	0.672484886	4.7	0.63531	10.0
-0.75	-0.25	0.85454543	0.2	0.75549	11.8
-1	-0.25	1.005311119	0.8	0.67637	33.3
-0.25	-0.5	0.408016698	1.6	1.37114	230.7
-0.5	-0.5	0.510072545	0.6	0.54653	7.8
-0.75	-0.5	0.602349796	0.4	0.59692	1.3
-1	-0.5	0.700168194	0.9	0.66743	5.5
-0.25	-0.75	0.221528741	1.5	0.37305	65.9
-0.5	-0.75	0.270147278	0.04	0.38392	42.2
-0.75	-0.75	0.316182757	0.5	0.44351	39.5
-1	-0.75	0.364016828	1.0	0.40011	8.8

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CHAPTER THREE: FALKNER-SKAN EQUATION

A natural extension of the previous work on the Blasius Equation is to examine the Falkner-Skan equation since the Blasius equation can be viewed as a special case of the Falkner-Skan equation.

3.1 The Falkner-Skan Equation

The Falkner-Skan Equation is given by the third-order, nonlinear differential equation

$$f''' + ff'' + \beta(1 - f'^2) = 0 \tag{3.1}$$

with boundary conditions

$$f(0) = 0 \tag{3.2}$$

$$f'(0) = 0 \tag{3.3}$$

$$\lim_{\eta \to \infty} f'(\eta) = 1. \tag{3.4}$$

While the Blasius equation describes fluid flow past a flat plate, the Falkner-Skan equation would describe flow along a curved plate or wedge. As for the Blasius equation, the initial value of the second derivative, $f''(0)$, indicates the skin friction along the wall. A new parameter, β, is introduced with the Falkner-Skan equation, which is the pressure gradient parameter. Solutions for $\beta > 0$ corresponds to accelerating flows, $\beta = 0$ indicates constant flows, and $\beta < 0$ represents decelerating flows.

The Falkner-Skan is also frequently written in an alternate form utilizing m as the Falkner-Skan pressure parameter instead of β.

$$f''' + \frac{m+1}{2} ff'' + m(1 - f'^2) = 0 \tag{3.5}$$

The Falkner-Skan equation includes as special cases Blasius flow ($m = 0$) and Hiemenz stagnation point flow ($m = 1$)[34].

Like the Blasius equation, the Falkner-Skan equation has been the subject of much research [12, 21, 33, 64]. For instance, Xenos et al. [64] examined compressible turbulent boundary-layer flow over a wedge which has significant application in the field of aerodynamics. Their work involved transforming the governing equations, the Reynolds-Averaged Boundary Layer using the compressible Falkner-Skan transformation. In addition, two algebraic turbulence models were considered. The authors successfully developed a numerical solution using the Keller-box method, which they claim is unconditionally stable. Their results show that "an instant separation of the turbulent compressible boundary layer over the wedge occurs when $m \leq 0.1$"[64]. Additionally, "application of suction retains the boundary layer for larger values of the dimensionless pressure parameter, m"[64].

Furthermore, Guedda and Hammouch [33] examined the problem for similarity solutions based on the velocity distribution outside the boundary layer. The question of existence of solutions was addressed for "the case where the external velocity is an inverse-linear function" which occurs in sink flow [33]. The results indicate multiple solutions for lateral suction and no solutions for injection. In addition, they were able to define a sufficient condition for existence which "indicates that for the same positive value of the suction parameter the permeable wall stretching" with prescribed velocity has multiple boundary layer flows that are uniquely determined by the skin friction, $f''(0)$([33].

In addition, Belhachmi et al. [21] examined an equation similar to the Falkner-Skan equation that arises when considering a heated impermeable flat plate embedded in a porous medium. Solutions to this equation can give an approximation to the thermal boundary layer. Their

work included the derivation of the governing equations, established properties of the solutions, and examined the existence and uniqueness of solutions. The equation examined by their work includes a parameter, α that describes the temperature distribution on the wall. The problem only has physical meaning for $\alpha \in [-\frac{1}{3}, 1]$. They found that no solution exists for $\alpha < -\frac{1}{3}$, at least one solution for $\alpha \in (-\frac{1}{3}, 0)$, and one and only one solution for $\alpha = -\frac{1}{3}$ and $\alpha \geq 0$. It is interesting to find that solutions only exist where the problem has physical meaning. Due to the similarities of the equations, their work can provide a foundation for understanding the solutions of the Falkner-Skan equation.

While no closed-form solutions are known, the solutions to the Falkner-Skan equation are "similarity solutions of the two-dimensional incompressible laminar boundary layer equations" according to Asaithambi [12] who reported the numerical solutions for the Falkner-Skan based on the shooting and finite differences techniques. Asaithambi [12] presented one shooting method for solving the Falkner-Skan equation. The method utilizes the Taylor series method and was found to be efficient and successful. The solution was obtained by first beginning with a coordinate transformation, followed by a change of variables to convert the problem to a system of first-order problems. Then an algorithm was established utilizing a recursive evaluation of the Taylor coeffiecients. To evaluate the method, the author examined the cases of the Pohlhausen, Blasius, and Homann flows, accelerating, constant, and decelerating flows. The results were found to be "in excellent agreement" with previously reported solutions[12].

Another numerical solution presented by Asaithambi [11] also showed "excellent agreement" with previously published solutions. The method again started with a coordinate transformation and a change of variables technique. The new system of differential equations consists of a second-order equation which was "approximated using a Galerkin formulation

with piecewise linear elements" and a first-order equation which was "approximated using a centered-difference approximation"[11]. Asaithambi has repeatedly shown that by transforming the Falkner-Skan equation utilizing some coordinate transformation and a change of variables allows for easily generating solutions using various numerical methods, such as the finite-difference or the classical Runge-Kutta methods[10, 9, 8].

Padé [56] provided a proof of the existence and uniqueness for the solutions of the Falkner-Skan equation subject to "a physical set of boundary conditions", such as positive wall temperature, positive skin friction at the wall ($f''(0) > 0$), and favourable pressure gradients ($\beta > 0$). In addition, some properties of the solutions and "bounds on important quantities" were established. In relation to the present work, the Falkner-Skan equation has a unique solution satisfying $f'^2 < 1$, a unique solution satisfying $f'' > 0$. Furthermore, the solution to the Falkner-Skan equation, f, can be related to the solution to the Blasius equation, $f0$, such that $f \leq f0$. [56]

While much work has been done on the Falkner-Skan equation, the Adomian Decomposition Method potentially provides a way of quickly and accurately developing an analytical solution for the equation that would benefit other aspects of research. The Falkner-Skan equation is useful in many types of problems (see for example [34, 40]). Some instances of how it has already been used are mentioned below.

To begin with, the classical Falkner-Skan problem typically employs steady-state conditions. Recent research has explored cases involving unsteady conditions, particulary motion and temperature, which have increasing importance in applications such as aerodynamics and hydrodynamics [34]. For instance, Harris et al. [34] considered a transient Falkner-Skan problem with forced-convection, thermal boundary layer. A comprehensive solution was

generated utilizing a series solution for small time, steady state Falkner-Skan solution for large time, and a finite difference method for the transition range from the small time, unsteady state to large time, steady state. Their findings indicate that while the effect is "initially confined within a region close to the surface, as time progresses, diffusion effects eventually modify the solution at a great distance from the surface" [34]. In addition, the solution is in "very good agreement with all of the previously reported results" [34].

Butler et al. [27] developed a direct numerical method to generate the solution to the Prandtl problem. They utilized "the Falkner-Skan similarity solution of Prandtl's problem" to provide boundary conditions necessary for a direct numerical solution and also as a reference solution for determining the error of the numerical solution to the unknown exact solution. "Since the Falkner-Skan solution is known to converge Reynolds uniformly to the solution of Prandtl's problem, we can compute Reynolds uniform error bounds" [27]. Extensive numerical experiments were used to validate the performance of the direct numerical method developed and indicates the method is "Reynolds and uniform" [27].

Itoh [40] examined Gortler instability along a "concavely curved surface". Theoretical studies concerning this typically utlize the Blasius flow profile. However, Itoh [40] points out that Blasius boundary-layer profile "indicates no pressure gradient along the wall" and therefore utilized the Falkner-Skan boundary layer family to extend the stability analysis. In particular, Itoh [40] focused on stagnation point flow, $m = 1$, and generated a series solution for the eignevalue problem derived by reducing the disturbance equations governing Gortler instability. This approach yielded the "neutral stability curve with a minimum of the Gortler number" and provided the critical Gortler number and the critical wave number, which "could not be obtained from the classical parallel-flow theory".

As stated previously, the ADM provides a way to develop an analytical solution for the Falkner-Skan equation quickly and accurately. Based on its similarity to the Blasius equation, a solution for the Falkner-Skan equation will be generated utilizing the Adomian decomposition method following the techniques presented by Wang [60] and presented in the previous chapter.

3.2   Adomian Decomposition of the Falkner-Skan Equation

The linear differential operator, L, for the Falkner-Skan equation is the same as used previously for the Blasius equation

$$L = \frac{d^3}{d\eta^3}.$$
(3.6)

This then gives the inverse operator as

$$L^{-1} = \int_0^\eta \int_0^\eta \int_0^\eta (\cdot) d\eta d\eta d\eta.$$
(3.7)

The nonlinear operator is defined as

$$Nf = -ff'' + \beta f'^2.$$
(3.8)

The equation is then rewritten in operator form

$$Lf + \beta = Nf.$$
(3.9)

Operating with the inverse operator and incorporating the boundary conditions yields

$$f(\eta) = \frac{1}{2}k\eta^2 - \frac{1}{6}\beta\eta^3 + L^{-1}(Nf)$$
(3.10)

where k is used to represent the initial value of the second derivative, $f''(0)$, the skin friction. Next, the Adomian polynomials are generated such that

$$A_n = -(\sum_{i=0}^{n} f_i f''_{n-i}) + \beta(\sum_{i=0}^{n} f'_i f'_{n-1})$$
(3.11)

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hence

$$A_0 = -f_0 f_0'' + \beta f_0'^2 \tag{3.12}$$

$$A_1 = -(f_0 f_1'' + f_1 f_0'') + \beta(2 f_0' f_1') \tag{3.13}$$

$$A_2 = -(f_0 f_2'' + f_1 f_1'' + f_2 f_0'') + \beta(f_1'^2 + 2 f_0' f_2') \tag{3.14}$$

$$A_3 = -(f_0 f_3'' + f_1 f_2'' + f_2 f_1'' + f_3 f_0'') + \beta(2 f_0' f_3' + 2 f_1' f_2'). \tag{3.15}$$

From this, we can determine the recursive relationship that determines the solution.

$$f_0 = \frac{1}{2} k \eta^2 - \frac{1}{6} \beta \eta^3 \tag{3.16}$$

$$f_{n+1} = L^{-1}(A_n) \tag{3.17}$$

Utilizing this, the series solution to the Falkner-Skan is found to be

$$f_0 = \frac{1}{2}k\eta^2 - \frac{1}{6}\beta\eta^3 \tag{3.18}$$

$$f_1 = \left(-\frac{1}{1260}\beta^2 + \frac{1}{840}\beta^3\right)\eta^7 + \left(\frac{1}{180}k\beta - \frac{1}{120}k\beta^2\right)\eta^6 + \left(-\frac{1}{120}k^2\right.$$

$$\left. + \frac{1}{60}k^2\beta\right)\eta^5 \tag{3.19}$$

$$f_2 = \left(\frac{19}{1247400}\beta^4 - \frac{1}{155925}\beta^3 - \frac{1}{118800}\beta^5\right)\eta^{11} + \left(\frac{1}{14175}k\beta^2\right.$$

$$\left. + \frac{1}{10800}k\beta^4 - \frac{19}{113400}k\beta^3\right)\eta^{10} + \left(\frac{113}{181440}k^2\beta^2\right.$$

$$\left. - \frac{11}{30240}k^2\beta^3 - \frac{1}{4032}\beta k^2\right)\eta^9 + \left(\frac{11}{40320}k^3\right.$$

$$\left. - \frac{1}{1260}k^3\beta + \frac{1}{2016}k^3\beta^2\right)\eta^8 \tag{3.20}$$

$$f_3 = \left(\frac{1}{16848000}\beta^7 + \frac{3557}{20432412000}\beta^5 - \frac{563}{10216206000}\beta^4\right.$$

$$\left. - \frac{2407}{13621608000}\beta^6\right)\eta^{15} + \left(-\frac{1}{1123200}k\beta^6 - \frac{3557}{1362160800}k\beta^4\right.$$

$$\left. + \frac{563}{681080400}k\beta^3 + \frac{2407}{908107200}k\beta^5\right)\eta^{14} + \left(\frac{5723}{389188800}k^2\beta^3\right.$$

$$\left. - \frac{7081}{1556755200}k^2\beta^2 - \frac{3397}{222393600}k^2\beta^4 + \frac{1363}{259459200}k^2\beta^5\right)\eta^{13}$$

$$+ \left(\frac{577}{53222400}k^3\beta + \frac{1597}{39916800}k^3\beta^3 - \frac{19}{1330560}k^3\beta^4\right. \tag{3.21}$$

$$\left. - \frac{8849}{239500800}k^3\beta^2\right)\eta^{12} + \left(-\frac{19}{475200}k^4\beta^2 + \frac{233}{6652800}k^4\beta\right.$$

$$\left. - \frac{5}{532224}k^4 + \frac{1}{66528}k^4\beta^3\right)\eta^{11} \tag{3.22}$$

3.3 Transformed Falkner-Skan Equation

As before, it is necessary to transform the equation in order to determine the value of k.

We begin by letting

$$x = f'(\eta) \tag{3.23}$$

$$y(x) = f''(\eta) \tag{3.24}$$

This gives the Transformed Falkner-Skan Equation as

$$y^2 y'' - \beta(1 - x^2)y' + (1 - 2\beta)xy = 0 \tag{3.25}$$

on $0 \le x < 1$ subject to the boundary conditions

$$y(0) = k \tag{3.26}$$

$$y(1) = 0 \tag{3.27}$$

$$y'(0) = -\frac{\beta}{k} \tag{3.28}$$

This equation is rewritten as

$$y'' - \beta(1 - x^2)\frac{y'}{y^2} + (1 - 2\beta)\frac{x}{y} = 0 \tag{3.29}$$

for use with the ADM.

3.4   Adomian Decomposition of Transformed Falkner-Skan Equation

The Adomian Decomposition Method was then applied to the transformed Falkner-Skan equation. We begin by defining the linear operator as

$$L = \frac{d^2}{dx^2}. \tag{3.30}$$

This gives the inverse operator as

$$L^{-1} = \int_0^x \int_0^x (\cdot)\,dx\,dx. \tag{3.31}$$

Operating with this on Equation 3.29 and incorporating the boundary conditions yields

$$y(x) = k - \frac{\beta}{k}x + L^{-1}(\beta(1 - x^2)\frac{y'}{y^2} - (1 - 2\beta)\frac{x}{y}). \tag{3.32}$$

We take a linear combination of two nonlinear operators in order to decompose the nonlinear portion into the Adomian polynomials. We set

$$N_1 y = \frac{y'}{y^2} \tag{3.33}$$

$$N_2 y = \frac{1}{y}. \tag{3.34}$$

This gives the nonlinear portion as

$$L^{-1}\left(\beta(1-x^2)N_1y - (1-2\beta)xN_2y\right). \tag{3.35}$$

Using the definition for Adomian polynomials, we have

$$A_0 = \beta(1-x^2)\frac{y_0'}{y_0^2} - (1-2\beta)\frac{x}{y_0} \tag{3.36}$$

$$A_1 = \beta(1-x^2)\left(\frac{y_1'}{y_0^2} - \frac{2y_0'y_1}{y_0^3}\right) - (1-2\beta)x\left(-\frac{y_1}{y_0^2}\right) \tag{3.37}$$

$$A_2 = \beta(1-x^2)\left(\frac{y_2'}{y_0^2} - \frac{2y_1'y_1}{y_0^3} + \frac{3y_0'y_1^2}{y_0^4} - \frac{2y_0'y_2}{y_0^3}\right) - (1-2\beta)x\left(\frac{y_1^2}{y_0^3} - \frac{y_2}{y_0^2}\right) \tag{3.38}$$

$$A_3 = \beta(1-x^2)\left(\frac{y_3'}{y_0^2} - \frac{2y_2'y_1}{y_0^3} + \frac{3y_1'y_1^2}{y_0^4} - \frac{2y_1'y_2}{y_0^3} - \frac{4y_0'y_1^3}{y_0^5} + \frac{6y_0'y_1y_2}{y_0^4}\right.$$
$$\left. - \frac{2y_0'y_3}{y_0^3}\right) - (1-2\beta)x\left(-\frac{y_1^3}{y_0^4} + \frac{2y_1y_2}{y_0^3} - \frac{y_3}{y_0^2}\right). \tag{3.39}$$

Finally we define the recursive relationship as follows

$$y_0(x) = k - \frac{\beta}{k}x \tag{3.40}$$

$$y_{n+1}(x) = L^{-1}(A_n). \tag{3.41}$$

This yields the solution for transformed equation as

$$y_0 = k - \frac{\beta}{k}x \tag{3.42}$$

$$y_1 = p_1x^2 + p_2x + p_3\ln(y_0) + p_4x\ln(y_0) + p_5 \tag{3.43}$$

$$y_2 = q_1x^3 + q_2x^2 + q_3x + q_4\ln(y_0) + q_5x\ln(y_0) + q_6x^2\ln(y_0)$$
$$+ q_7\ln^2(y_0) + q_8x\ln^2(y_0) + \frac{q_9}{y_0} + \frac{q_{10}\ln(y_0)}{y_0} + q_{11} \tag{3.44}$$

$$y_3 = r_1x^4 + r_2x^3 + r_3x^2 + r_4x + r_5\ln(y_0) + r_6x\ln(y_0) + r_7x^2\ln(y_0) + r_8x^3\ln(y_0)$$
$$+ r_9\ln^2(y_0) + r_{10}x\ln^2(y_0) + r_{11}x^2\ln^2(y_0) + r_{12}\ln^3(y_0)$$
$$+ r_{13}x\ln^3(y_0) + \frac{r_{14}}{y_0} + \frac{r_{15}}{y_0^2} + \frac{r_{16}\ln(y_0)}{y_0} + \frac{r_{17}\ln(y_0)}{y_0^2} + \frac{r_{18}\ln^2(y_0)}{y_0}$$
$$+ \frac{r_{19}\ln^2(y_0)}{y_0^2} + r_{20}. \tag{3.45}$$

where the coefficients p_i, q_i, and r_i are functions of k and β. Again, these functions are given in Appendix A due to their size.

3.5   Solution

To determine a solution to the Falkner-Skan equation, the truncated solution to the transformed equation $y = y_0 + y_1 + y_2 + y_3$ could be evaluated for a particular value of β at the boundary condition $y(1) = 0$ to provide a value for the skin friction $f''(0)$. In addition to determining this important parameter, an analytical solution for $f(\eta)$ can be determined by returning k to the truncated solution of the original equation $f = f_0 + f_1 + f_2 + f_3$. From the coefficient functions for the solution to the transformed equation, we see that the special case, $\beta = 0$, describing the Blasius flow would have to be solved separately; however, since this was already done in the previous chapter, no further mention will be made. Therefore, in order to analyze the solution generated by this method, the special case for Hiemenz stagnation flow, $\beta = 1$, will be evaluated.

To find the solution, we fix β at 1. Then we apply the boundary condition $y(1) = 0$ in order to determine $f''(0) = k$. This yields $k = 1.092079498$. This value was compared to the numerical solution given by Asaithambi [11] which was obtained by using piecewise linear functions. The author utilized a coordinate transformation and change of variable to the original Falkner-Skan equation as mentioned previously. The value of the skin friction for Hiemenz stagnation was determined to be $k = 1.232589$ [11]. The results from the ADM has a relative error of 11.4% when compared to this value. While this is a small error, it is noted that it is only accurate to one significant digit. As such, the Padé aftertreatment was applied in attempt to improve the accuracy of the solution. This yielded a result of $k = 1.096566$, which has a relative error of 11.0%. This indicates that the Aftertreatment Technique was essentially ineffective in this result.

One advantage of the ADM is that it provides an analytical solution for $f(\eta)$. The value of k was returned to the original solution which was then compared to the numerical solution graphically. This comparison is shown in figure 3.1. While the numerical solution can plot the solution, the benefits to having an equation to describe the solution are many.
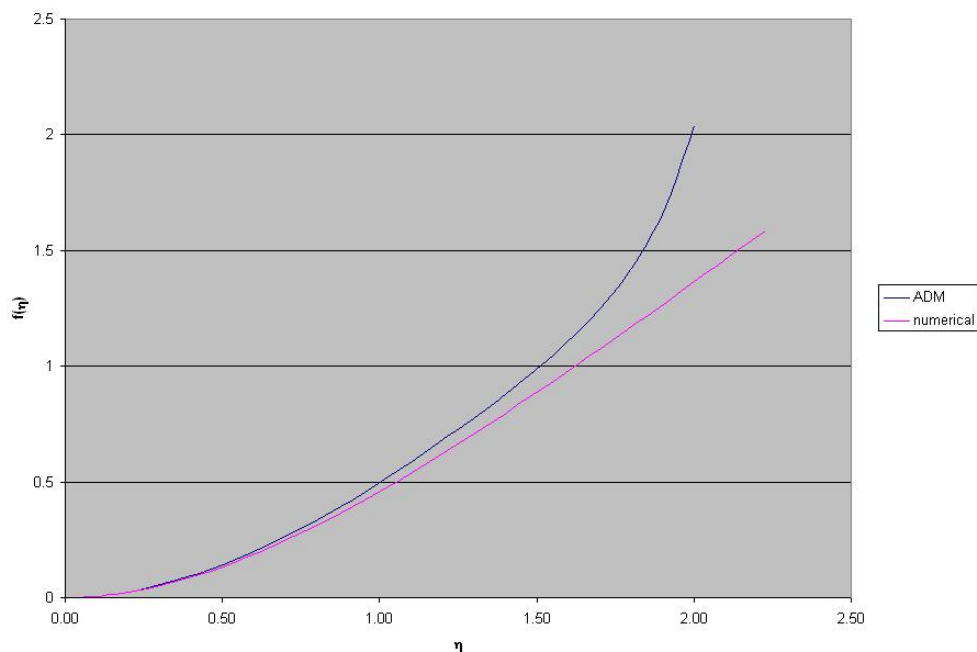
Figure 3.1: Solution Comparison to Numerical Solution

3.6 Results

To further examine the effectiveness of the solution, figure 3.2 shows the series solution utilizing the published value of k. This indicates that the original solution is quite accurate (exhibiting a relative error $< 0.1\%$ for $\eta \leq 1.25$). This seems to indicate that the transformed solution is less effective. Further analysis demonstrated that the best approximation of the transformed solution would be $O(2)$ solution. This yields $k = 1.187669477$ which has

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a relative error of 3.6%. Ordinarily, one would expect the accuracy to increase as more terms are included in the series. This prompted an examination of the solution term by term. This is demonstrated in figure 3.3, which shows the solution graph beginning with f_0 and then addiing successive terms. It is noted that for the small region of convergence f_0 is fairly accurate on its own, but when the region is widened it fails to remain so.
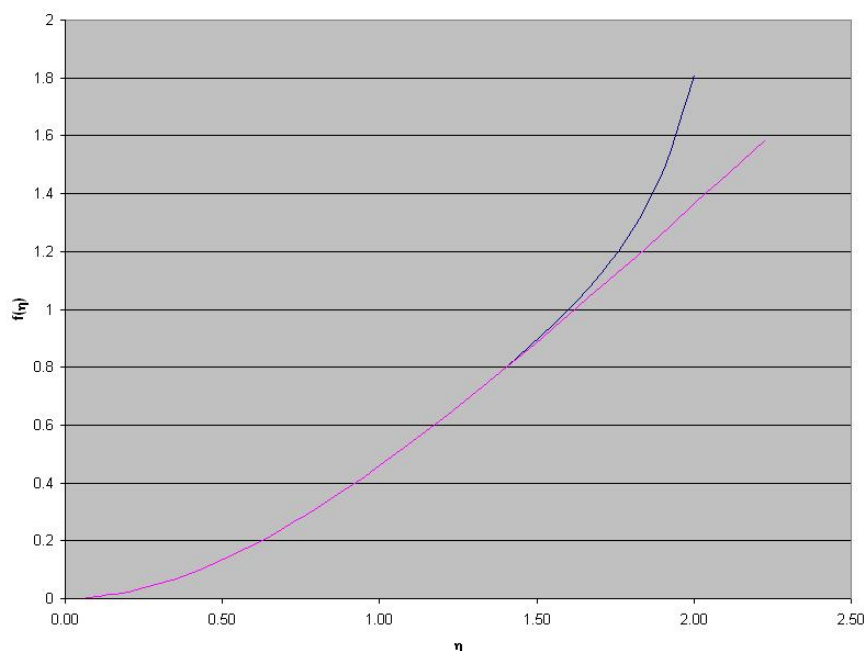
Figure 3.2: Examination of Original Solution

Solutions of the Falkner-Skan equation were also examined for various values of β. As stated previously, solutions for $\beta > 0$ corresponds to accelerating flows, $\beta = 0$, constant flows, and $\beta < 0$, decelerating flows. Since it has been determined that "physically relevant solutions exist only for $-0.19884 < \beta \leq 2$" [11], values were chosen in agreement with those bounds. Results are given in table 3.1 and displayed graphically in figure 3.4. These results seem to imply that the solution is only effective for $\beta > 0$. Figure 3.5 shows solutions for various β.
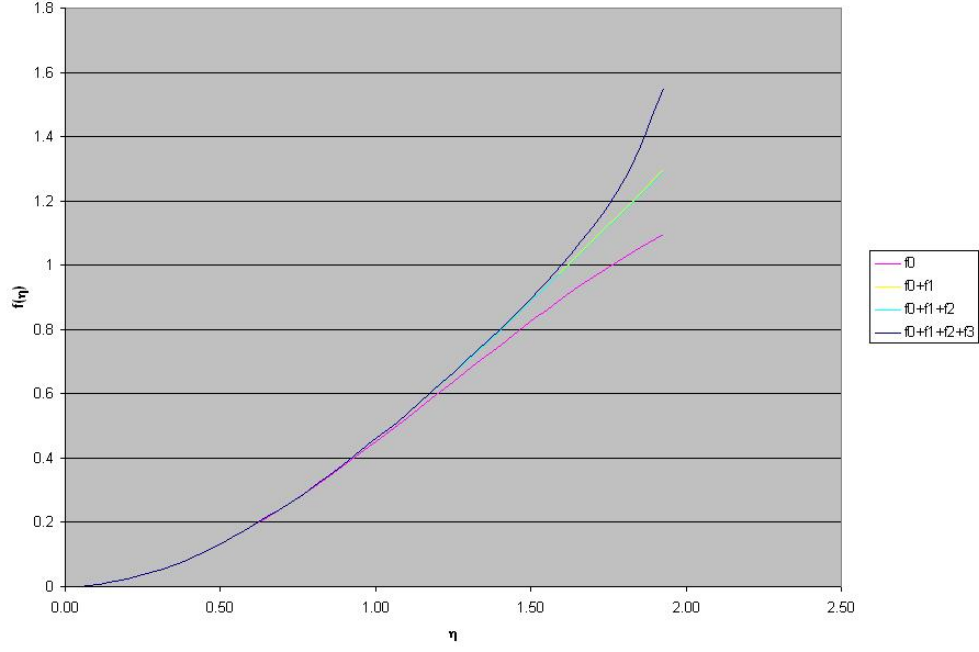
Figure 3.3: Solution Comparison for Increasing Series Terms

Table 3.1: Solution values for $f''(0)$ for various β.

β	$f''(0)$	numerical $f''(0)$	error(%)
2	1.8611	1.687222	10.3
1	1.1876	1.232589	3.6
0.5	0.8602	0.927682	7.3
-0.1	0.2180	0.31927	31.7
-0.12	0.2391	0.281762	15.2
-0.15	0.2678	0.21636	23.8
-0.18	0.2935	0.128637	128.1
-0.1988	0.3080	0.005217	5804.4

3.7   Conclusion

Again, several benefits can be seen in the use of the ADM for developing a solution. Primarily, the ADM yields an analytic expression for $f(\eta)$. Furthermore, it can be obtained by

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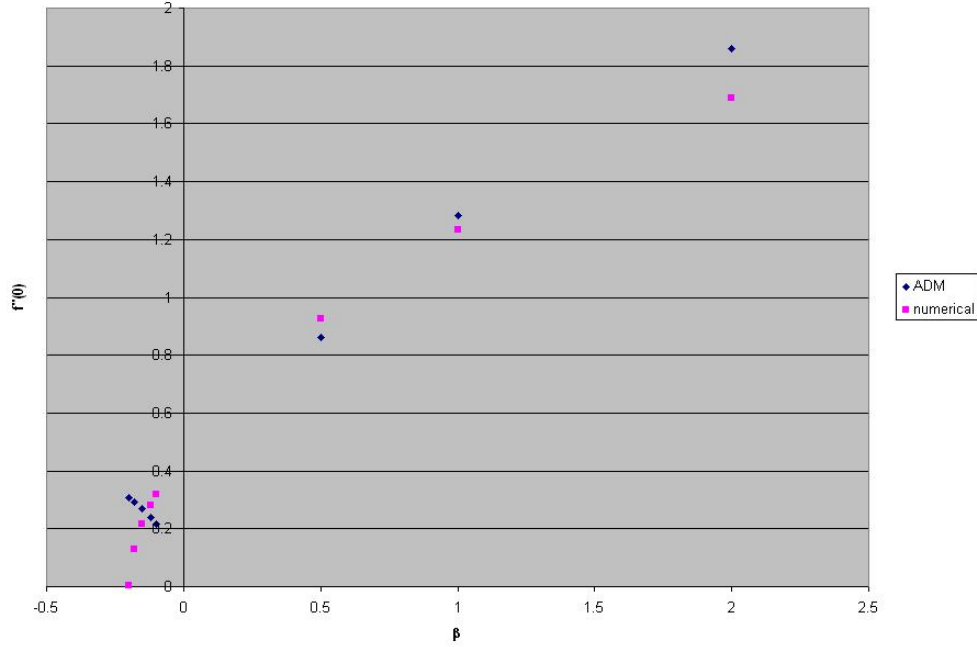
Figure 3.4: Solution Values for $f''(0)$ for Various β.

direct application to the Falkner-Skan equation. It is necessary to incorporate some modification in order to utilize the boundary condition at infinity to determine the skin friction; nevertheless, the solution developed by ADM of the original solution was shown to be quite accurate. Furthermore, we find the initial term of the series to be an adequate approximation on its own. In addition, we continue to see the ease of computation frequently remarked upon. We note that much of the work in this section can be performed without computer assistance.

Conversely, some negative aspects to utilizing this method were also noted. To begin with, we again see a small region of convergence for the solution. Likewise, the method exhibits slow convergence as higher-order approximations are needed for reasonably accurate results. This can be difficult to execute if the equations are complex and not conducive to successive iterations. In addition, the transformation seems to introduce some error. It is also noted
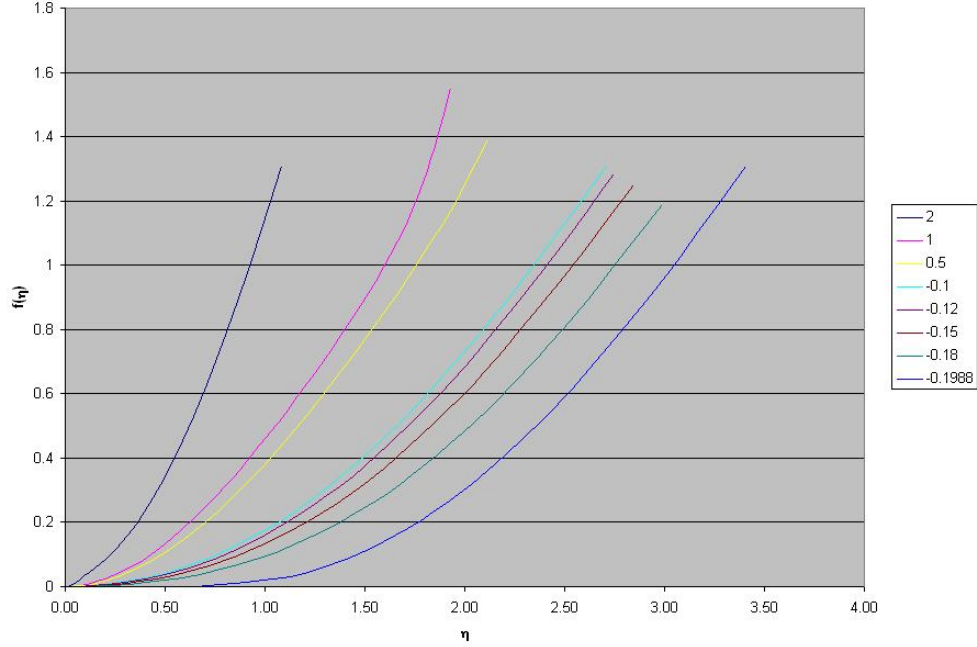
Figure 3.5: Solutions for Various β.

that when evaluating a highly symbolic equation as a means of increasing the generality of the problem, the series terms become increasingly large and unwieldy losing some of the computational ease. In addition, the type of nonlinearity seems to influence the quality of the solution. In the transformed equation where the nonlinearity is expressed as a quotient instead of a product, we begin to see unfavorable functions in certain terms viz. $ln(f_0)$ or $1/f_0$. This was also evidenced in the previous work on the Blasius equation.

Some of these difficulties have been overcome by a method similar to the ADM. Liao [48] developed a solution for the Falkner-Skan and Blasius equations using the newly developed Homotopy Analysis Method (HAM). This method incorporates the Adomian Decomposition Method and some perturbation methods as special cases. In the case of Hiemenz stagnation flow $\beta = 1$, Liao determined the skin friction to be 1.2308, 1.2327, and 1.2326 for the 10th,

20th, and 30th order approximations to $f''(0)$. While these are good approximations to the numerical result of 1.2326, it is difficult to objectively compare to the work herein since we only conducted a 3rd order approximation. Liao also noted slow convergence for $\beta < 0$ and attributes this to the question of multiple solutions existing in this instance. See [48] for details on the HAM and the solutions. The major benefit to the Homotopy Analysis Method is the ability to select the base functions for the series expansion. For instance, in the case of the original Falkner-Skan and Blasius equations, the ADM yields a series solution in terms of a power series of η (η^n). With the HAM, exponential terms (i.e. $e^{n\eta}$), which are more suited to the behavior of the solution, can be selected as the base functions and HAM could be used to avoid the special functions, such as $ln(f_0)$, that appear in the solutions for the transformed equations in this work.

3.8 Magnetohydrodynamic Boundary Layer Equations

A problem closely related to the Falkner-Skan equation are the magnetohydrodynamic boundary layer equations that were examined by Shivamoggi and Rollins [58]. The equations are a pair of coupled nonlinear ordinary differential equations that describe the fluid flow "past a semi-infinite flat plate in the presence of a magnetic field which is uniform at infinity and parallel to the stream" [58]. A brief examination into the usefulness of the ADM applied to these equations is conducted here.

The coupled equations are given by

$$g'' + \mu(fg' - f'g) = 0 \tag{3.46}$$

$$f''' + ff'' - \frac{1}{A^2}gg'' = 0 \tag{3.47}$$

$$\tag{3.48}$$

with boundary conditions

$$f(0) = 0, \qquad\qquad f'(0) = 0, \qquad\qquad g(0) = 0 \qquad\qquad (3.49)$$

As $\eta \Rightarrow \infty$,

$$f \approx 2\eta, \qquad\qquad\qquad g \approx 2\eta \qquad\qquad (3.50)$$

or equivalently

$$\lim_{\eta\to\infty} f'(\eta) = 2, \qquad\qquad \lim_{\eta\to\infty} g'(\eta) = 2. \qquad\qquad (3.51)$$

The ADM allows direct application to the equations without the need to decouple the equations. We start by defining two linear operators,

$$L_1 = \frac{d^2}{d\eta^2} \qquad\qquad (3.52)$$

$$L_2 = \frac{d^3}{d\eta^3} \qquad\qquad (3.53)$$

operating separately on equations (3.46) and (3.47), respectively. This gives the inverse operators as

$$L_1^{-1} = \int_0^\eta \int_0^\eta (\cdot)\,d\eta\,d\eta \qquad\qquad (3.54)$$

$$L_2^{-1} = \int_0^\eta \int_0^\eta \int_0^\eta (\cdot)\,d\eta\,d\eta\,d\eta. \qquad\qquad (3.55)$$

We then have two nonlinear operators,

$$N_1(fg) = (f'g - fg') \qquad\qquad (3.56)$$

$$N_2(fg) = \frac{1}{A^2}gg'' - ff'' \qquad\qquad (3.57)$$

We can rewrite equations (3.46) and (3.47) as

$$L_1 g = \mu N_1(fg) \qquad\qquad (3.58)$$

$$L_2 f = N_2(fg). \qquad\qquad (3.59)$$

Operating with the inverse operators and applying the initial conditions in (3.49), we have

$$g(\eta) = g'(0)\eta + L_1^{-1}(A_n) \tag{3.60}$$

$$f(\eta) = \frac{1}{2}f''(0)\eta^2 + L_2^{-1}(B_n). \tag{3.61}$$

where A_n and B_n are the Adomian polynomials based on N_1 and N_2, respectively. To determine the Adomian polynomials, we expand $g(\eta)$ and $f(\eta)$ as

$$g(\lambda) = \sum_{n=0}^{\infty} \lambda^n g_n \tag{3.62}$$

$$f(\lambda) = \sum_{n=0}^{\infty} \lambda^n f_n \tag{3.63}$$

such that

$$N_1(fg(\lambda)) = \sum_{n=0}^{\infty} \lambda^n \cdot \mu \left(\sum_{i=0}^{n} f_i' g_{n-i} - \sum_{i=0}^{n} f_i g_{n-i}' \right) \tag{3.64}$$

$$N_2(fg(\lambda)) = \sum_{n=0}^{\infty} \lambda^n \cdot \left(\frac{1}{A^2} \sum_{i=0}^{n} g_i g_{n-i}'' - \sum_{i=0}^{n} f_i f_{n-i}'' \right) \tag{3.65}$$

and utilize the definition

$$A_n = \frac{1}{n!} \frac{d^n}{d\lambda^n} (Nf(\lambda)) \bigg|_{\lambda=0}. \tag{3.66}$$

This yields the Adomian polynomials as

$$A_0 = \mu(f_0' g_0 - f_0 g_0')$$

$$A_1 = \mu(f_0' g_1 + f_1' g_0 - (f_0 g_1' + f_1 g_0'))$$

$$A_2 = \mu(f_0' g_2 + f_1' g_1 + f_2' g_0 - (f_0 g_2' + f_1 g_1' + f_2 g_0'))$$

$$A_3 = \mu(f_0' g_3 + f_1' g_2 + f_2' g_1 + f_3' g_0 - (f_0 g_3' + f_1 g_2' + f_2 g_1' + f_3 g_0'))$$

$$A_4 = \mu(f_0' g_4 + f_1' g_3 + f_2' g_2 + f_3' g_1 + f_4' g_0 - (f_0 g_4' + f_1 g_3' + f_2 g_2' + f_3 g_1' + f_4 g_0'))$$

and

$$B_0 = \frac{1}{A^2} g_0 g_0'' - f_0 f_0''$$

$$B_1 = \frac{1}{A^2} (g_0 g_1'' + g_1 g_0'') - (f_0 f_1'' + f_1 f_0'')$$

$$B_2 = \frac{1}{A^2} (g_0 g_2'' + g_1 g_1'' + g_2 g_0'') - (f_0 f_2'' + f_1 f_1'' + f_2 f_0'')$$

$$B_3 = \frac{1}{A^2} (g_0 g_3'' + g_1 g_2'' + g_2 g_1'' + g_3 g_0'') - (f_0 f_3'' + f_1 f_2'' + f_2 f_1'' + f_3 f_0'')$$

$$B_4 = \frac{1}{A^2} (g_0 g_4'' + g_1 g_3'' + g_2 g_2'' + g_3 g_1'' + g_4 g_0'') - (f_0 f_4'' + f_1 f_3'' + f_2 f_2'' + f_3 f_1'' + f_4 f_0'')$$

If we designate $g'(0) = \gamma_1$ and $f''(0) = \gamma_2$, we find the recursive relationship that defines the solution to be

$$g_0(\eta) = \gamma_1 \eta \qquad\qquad f_0(\eta) = \frac{1}{2}\gamma_2 \eta^2 \qquad\qquad (3.67)$$

$$g_{n+1}(\eta) = L_1^{-1}(A_n) \qquad\qquad f_{n+1}(\eta) = L_2^{-1}(B_n). \qquad\qquad (3.68)$$

Solving this yields the solution to the coupled magnetohydrodynamic boundary layer equations as

$$g_0 = \gamma_1 \eta \qquad\qquad (3.69)$$

$$g_1 = 1/24 \, \mu \, \gamma_2 \, \gamma_1 \, \eta^4 \qquad\qquad (3.70)$$

$$g_2 = \left(-\frac{1}{1008} \gamma_2{}^2 \mu^2 \gamma_1 - \frac{1}{1260} \gamma_2{}^2 \mu \, \gamma_1 \right) \eta^7 \qquad\qquad (3.71)$$

$$g_3 = \left(\frac{1}{36288} \gamma_2{}^3 \mu^3 \gamma_1 + \frac{11}{604800} \gamma_2{}^3 \mu^2 \gamma_1 \right.$$

$$\left. + \frac{11}{518400} \gamma_2{}^3 \mu \, \gamma_1 \right) \eta^{10} + \frac{1}{2688 A^2} \gamma_1^3 \mu^2 \gamma_2 \eta^8 \qquad\qquad (3.72)$$

$$g_4 = \left(-\frac{25}{41513472} \gamma_2^4 \mu \gamma_1 - \frac{1}{1747200} \gamma_2^4 \mu^3 \gamma_1 - \frac{1}{1415232} \gamma_2^4 \mu^4 \gamma_1 \right.$$

$$\left. - \frac{191}{566092800} \gamma_2^4 \mu^2 \gamma_1 \right) \eta^{13} + \left(-\frac{1}{69300 A^2} \gamma_2^2 \gamma_1^3 \mu^2 - \frac{19}{1900800 A^2} \gamma_2^2 \gamma_1^3 \mu^3 \right) \eta^{11} \qquad (3.73)$$

and

$$f_0 = \frac{1}{2}\gamma_2\eta^2 \tag{3.74}$$

$$f_1 = -\frac{1}{120}\,\gamma_2{}^2\eta^5 \tag{3.75}$$

$$f_2 = \frac{1}{240A^2}\gamma_1^2\mu\gamma_2\eta^6 + \frac{11}{40320}\,\gamma_2{}^3\eta^8 \tag{3.76}$$

$$f_3 = -\frac{5}{532224}\,\gamma_2{}^4\eta^{11} + \left(-\frac{1}{24192}\,\frac{\gamma_2{}^2\mu^2\gamma_1{}^2}{A^2} - \frac{1}{5040}\,\frac{\gamma_1{}^2\gamma_2{}^2\mu}{A^2}\right)\eta^9 \tag{3.77}$$

$$f_4 = \frac{9299}{29059430400}\,\gamma_2{}^5\eta^{14} + \left(\frac{1}{5322240}\,\frac{\gamma_1{}^2\gamma_2{}^3\mu^3}{A^2} + \frac{167}{159667200}\,\frac{\gamma_1{}^2\gamma_2{}^3\mu^2}{A^2}\right.$$
$$\left.+\frac{443}{53222400}\,\frac{\gamma_1{}^2\gamma_2{}^3\mu}{A^2}\right)\eta^{12} + \frac{1}{34560}\,\frac{\gamma_1{}^4\mu^2\gamma_2\,\eta^{10}}{A^4} \tag{3.78}$$
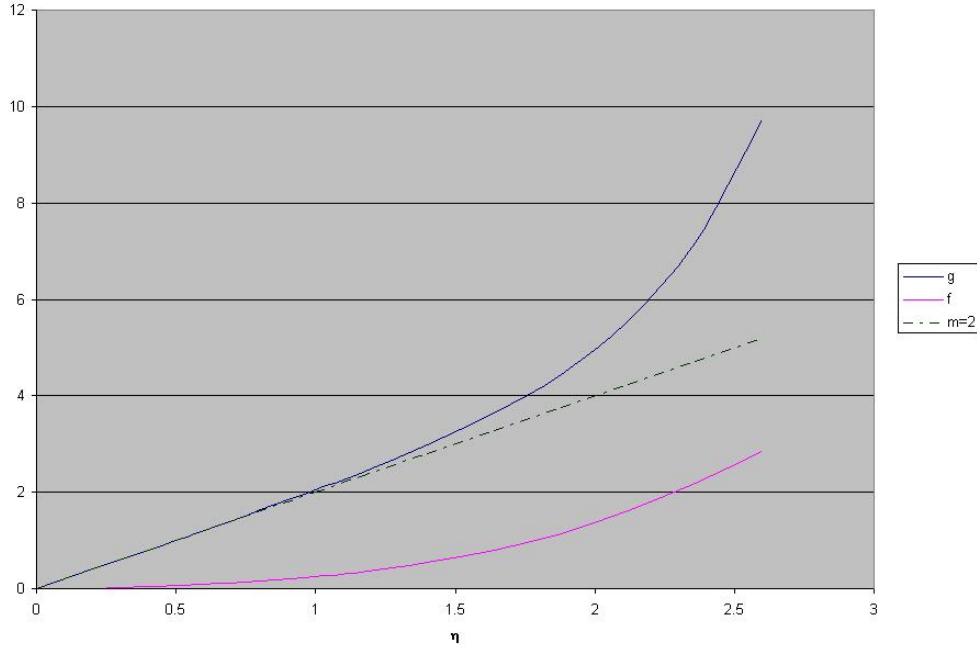
Figure 3.6: ADM Solution for $f(\eta)$ and $g(\eta)$ for magnetohydrodynamic equations.

While a solution needs to be acquired through a suitable modification that would allow incorporation of the boundary conditions, an initial glance at the behavior of this solution can be explored. By only considering the initial terms in equation (3.67), a rough estimate

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for γ_1 and γ_2 can be obtained. By setting

$$g_0(\eta) = 2\eta \hspace{5cm} f_0(\eta) = 2\eta, \hspace{3cm} (3.79)$$

we can use $\gamma_1 = 2$ as an approximation. This would also yield $\gamma_2 = \frac{4}{x} \Rightarrow 0$, which would give a null solution. Therefore, we can approximate γ_2 as small based on the projected region of convergence. If we approximate $\gamma_2 = 0.5$ for $\eta < 8$, we can graph the ADM solution in figure 3.6.

CHAPTER FOUR: ORR-SOMMERFELD EQUATION

The next equation chosen as a means of developing a cohesive work was the Orr-Sommerfeld equation. This equation arises when a small amplitude wavy disturbance is introduced into the flow. This disturbance is described by the Orr-Sommerfeld equation while the corresponding mean flow has been described using the Blasius flow by several researchers [23, 46].

4.1 The Orr-Sommerfeld Equation

The Orr-Sommerfeld Equation is given by

$$\frac{d^4v}{dy^4} - 2\alpha^2\frac{d^2v}{dy^2} + \alpha^4v - i\alpha R\left[(\overline{u} - \lambda)\left(\frac{d^2v}{dy^2} - \alpha^2v\right) - \frac{d^2\overline{u}}{dy^2}v\right] = 0 \tag{4.1}$$

with boundary conditions

$$v(-1) = v'(-1) = v(1) = v'(1) = 0. \tag{4.2}$$

According to Lahmann and Plum [46], the Orr-Sommerfeld equation is one of the central equations governing hydrodynamic stability of incompressible flows and constitutes a non-selfadjoint eigenvalue problem. It is "obtained by linearization of the Navier- Stokes equations for flat parallel flows between two fixed walls" [59].

The Orr-Sommerfeld equation deals with "the physical question of stability or instability of the underlying flow in response" to a disturbance and hence, the growth or decay of the disturbance in time [46, 54]. The undisturbed stream flow in the channel has Reynold's number R and a velocity profile $\overline{u}(y)$. The parameter α represents the wave number. Analysis of the equation involves λ, which represents the complex-valued wave velocity, specifically with $Re(\lambda)(= c_r)$ and $Im(\lambda)(= c_i)$ describing the phase velocity and the amplicfication factor,

respectively. As an eigenvalue problem, λ is treated as the eigenvalue parameter [19, 46, 54]. Subject to the boundary conditions above, the side walls are located at $y = \pm 1$. The disturbance can also be examined for flow along a single wall by utilizing a half-plane boundary $([0, \infty))$.

While the undisturbed velocity profile can be described in many ways depending on the physical conditions (i.e. Blasius flow, plane Pouseille flow, plane Couette flow), this work utilized plane Poiseuille flow, given as $\overline{u}(y) = 1 - y^2$, for simplicity and comparison to previously published results [55].

When considering stability, several aspects can be examined. A solution to the Orr-Sommerfeld equation with $Im(\lambda) > 0$ is considered "an unstable linear eigenmode, in the sense that the amplitude of the disturbance grows exponentially with time" [55]. For the case of plane Pouseille flow, the disturbance is unstable if $\alpha c_i > 0$ and stable for $\alpha c_i \leq 0$. If $\alpha c_i = 0$, disturbance is "marginally stable" for some α and R, if subject to the condition that $\alpha c_i > 0$ for neighboring α, R [19]. Furthermore, Lahmann and Plum indicate that flow is unstable if $c_r < 0$ [46].

Previously, researchers utilized numerical methods to determine the solutions to this equation. In 1971, Orszag utilized expansions in Chebyshev polynomials with the QR matrix eigenvalue algorithm to find the exact eigenvalue "for the most unstable mode of plane Poiseuille flow with $\alpha = 1, R = 10000$ [55]. This was reported as $0.23752649 + 0.00373967i$. He also determined the critical values of α, R to be $\alpha_c = 1.02056$ and $R_c = 5772.22$ and gave the 32 eigenvalues for the"least stable antisymmetric eigenmodes" [55].

Banerjee et al. established eigenvalue bounds for the Orr-Sommerfeld equation [18, 19]. From

this, they showed that for the case of plane Pouseille flow, the phase velocity c_r cannot be negative and thus, there cannot be any neutral backward perturbation wave, when $u_{min} = 0$ [18]. Their work shows that the disturbance wave velocity is bound by the maximum and minimum values of the base flow ($u_{min} < c_r < umax$). For the case of plane Pouseille flow, these values are specifically

$$\frac{-4}{\pi^2 + 4\alpha^2} < c_r < 1. \tag{4.3}$$

Recently, the main body of research on the Orr-Sommerfeld equation has focused on spectral analysis [46, 54, 59].

Lahmann and Plum [46] examined the "spectrum of the Orr-Sommerfeld equation". The aim of their research was to first obtain a mathematical instability proof for the Orr-Sommerfeld problem. The authors determined the essential spectrum and the eigenvalue enclosures in the complex plane [46].

Ng and Reid [54] also examined the spectrum of the Orr-Sommerfeld equation. They noted that there exist three distinct families of eigenvalues that "exhibit a Y-shaped pattern in the (c_r, c_i) plane" [54]. Their work involved successfully approximating these modes by an asymptotic formula, specifically for plane Pouseille flow with $\alpha = 0$.

Shkalikov and Tumanov [59] also examined the spectrum of the Orr-Sommerfeld equation, in particular for Couette and Pouseille flow. They show that the Orr-Sommerfeld problem can be reduced to a model problem. Further examination shows that the "limit spectral curves for the problem remain the same as for the model problem, and the aymptotic formulas are also preserved" [59].

Bera and Dey [23] used the Orr-Sommerfeld equation to examine the linear stability of

boundary layer flow subject to uniform shear. The equation was solved using a spectral collocation method based on Chebyshev polynomials. Their findings indicate that a free stream shear can stabilize the flow.

4.2   Solution for Plane Pouseille Flow

First, we define the linear differential operator

$$L = \frac{d^4}{dy^4}. \tag{4.4}$$

This gives the inverse operator as

$$L^{-1} = \int_{-1}^{y} \int_{-1}^{y} \int_{-1}^{y} \int_{-1}^{y} (\cdot) \, dy \, dy \, dy \, dy. \tag{4.5}$$

We apply these to equation 4.1 and designate the initial conditions $v''(-1) = k_1$ and $v'''(-1) = k_2$. We can then rewrite the equation as

$$v(y) = v_0 + L^{-1}(Rv) \tag{4.6}$$

where

$$
Rv = (-2\alpha^2 + \lambda i \alpha R)\frac{d^2 v}{dy^2} + (\alpha^4 - \lambda i \alpha^3 R)v - i\alpha R \overline{u}\frac{d^2 v}{dy^2} + \left(i\alpha^3 R\overline{u}\right.
$$
$$
\left. + i\alpha R\frac{d^2 \overline{u}}{dy^2}\right)v. \tag{4.7}
$$

From $L^{-1}Lv$, we can define

$$v_0 = \frac{k_2}{6}y^3 + \frac{k_1 + k_2}{2}y^2 + \left(k_1 + \frac{k_2}{2}\right)y + \frac{k_2}{6} + \frac{k_1}{2}. \tag{4.8}$$

In order to continue with the recursive relationship, we must designate the base flow utilizing plane Poiseuille flow for the velocity profile. Taking

$$\overline{u}(y) = 1 - y^2, \tag{4.9}$$

we substitute $\bar{u}(y)$ into equation 4.7 so that we can determine

$$v_{n+1}(y) = L^{-1}(Rv_n).\tag{4.10}$$

Utilizing this recursive relationship yields

$$v_1 = c_1 y^9 + c_2 y^8 + c_3 y^7 + c_4 y^6 + c_5 y^5 + c_6 y^4 + c_7 y^3 + c_8 y^2 + c_9 y + c_{10}\tag{4.11}$$

$$v_2 = g_1 y^{15} + g_2 y^{14} + g_3 y^{13} + g_4 y^{12} + g_5 y^{11} + g_6 y^{10} + g_7 y^9$$
$$+ g_8 y^8 + g_9 y^7 + g_{10} y^6 + g_{11} y^5 + g_{12} y^4 + g_{13} y^3 + g_{14} y^2 + g_{15} y + g_{16}\tag{4.12}$$

$$v_3 = h_1 y^{21} + h_2 y^{20} + h_3 y^{19} + h_4 y^{18} + h_5 y^{17} + h_6 y^{16} + h_7 y^{15}$$
$$+ h_8 y^{14} + h_9 y^{13} + h_{10} y^{12} + h_{11} y^{11} + h_{12} y^{10} + h_{13} y^9 + h_{14} y^8$$
$$+ h_{15} y^7 + h_{16} y^6 + h_{17} y^5 + h_{18} y^4 + h_{19} y^3 + h_{20} y^2 + h_{21} y + h_{22}.\tag{4.13}$$

The coefficients, c_i, g_i, and h_i, are functions of α, R, λ, and the initial conditions, k_1, and k_2. Again, these are given in Appendix A.

4.3   Results for Plane Pouseille Flow

To evaluate this solution, we examined the case with $\alpha = 1$, $R = 10,000$ in order to compare to the exact eigenvalue of $\lambda = 0.23752649 + 0.00373967i$ that was obtained by Orszag utilizing Chebysev polynomials [55].

To find the eigenvalues, we solved the truncated solution $v(y) = v_0 + v_1 + v_2 + v_3$ at $y = 1$ to utilize the boundary conditions $v(1) = v'(1) = 0$. This returns two equations as functions of λ, k_1, and k_2. By solving both equations for one of the unknown k_i and equating to each other, we can eliminate both k_i yielding an equation that can be solved for λ.

To demonstrate this method, we show the process for v approximated by $v_0 + v_1$. Evaluating

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$v(1) = 0$ yields

$$\frac{52}{63} k_2 + \frac{152000}{63} i\lambda k_2 + \frac{52000}{9} i\lambda k_1 + \frac{34}{45} k_1 - \frac{32000}{7} ik_1 - \frac{1312000}{567} ik_2 = 0 \qquad (4.14)$$

$$\frac{32000}{3} i\lambda k_1 + \frac{34}{45} k_2 + \frac{52000}{9} i\lambda k_2 - 2/5\, k_1 - \frac{232000}{21} ik_1 - \frac{400000}{63} ik_2 = 0 \qquad (4.15)$$

Solving both equations for k_1, we find

$$k_1 = -\frac{10}{9} \frac{k2\,(342000\,i\lambda + 117 - 328000\,i)}{910000\,i\lambda + 119 - 720000\,i} \qquad (4.16)$$

and

$$k_1 = -1/3\, \frac{k2\,(910000\,i\lambda + 119 - 1000000\,i)}{560000\,i\lambda - 21 - 580000\,i} \qquad (4.17)$$

Equating these two, we can eliminate k_2 and solve for λ.

This yields two eigenvalues:

$\lambda = 0.3966241512 - 0.00005191583577i$ and $\lambda = 1.141242656 + 0.0001685210018i$.

Both eigenvalues are stable based on c_r; however, the second eigenvalue lies outside the bounds given by equation 4.3. Comparing the first to Orszag's solution, we find a relative error of 67%.

If we expand the approximation to include v approximated by $v_0 + v_1 + v_2$, we find four eigenvalues:

$\lambda_1 = 0.7262427911 + 0.00002869052383i$

$\lambda_2 = 0.8686726277 + 0.0006556053226i$

$\lambda_3 = 0.3260290691 - 0.0001285417983i$

$\lambda_4 = 1.147543943 - 0.0001075476011i$

with the third eigenvalue exhibiting a relative error of 37.3%.

For v approximated by $v_0 + v_1 + v_2 + v_3$, we find six eigenvalues:

$\lambda_1 = 1.122077386 - 0.0002547450763i$

$\lambda_2 = .6429616572 + 0.0002092540891i$

$\lambda_3 = .9807599963 + 0.000499077057i$

$\lambda_4 = .7640821235 + 0.03234203810i$

$\lambda_5 = .7644742380 - 0.03150667053i$

$\lambda_6 = .2744327229 - 0.00008555253643i$

The sixth eigenvalue results in a relative error of 15.6%.

If we continue to expand the solution, we find $2n$ eigenvalues for the order of the approximation used. For the addition of the fourth and fifth terms, these are given in table 4.1. The eigenvalues are then plotted on the (c_r, c_i) plane in figure 4.1.

Table 4.1: Eigenvalues for Orr-Sommerfeld equation.

v_4	v_5
0.2391148096-0.00006589723323i	0.2132886117-0.00005199967033i
1.042083687+0.0004163308117i	0.5387369666+0.0004090920213i
0.5833654237+0.0003361058945i	1.071970411+0.01863580102i
0.8149149161+0.02993999132i	0.8430277474+0.023337901683i
0.7010801873+0.04112940677i	0.7590100501+0.03161103565i
0.7012920891-0.03996643045i	0.6531234661+0.04581859786i
0.8149521784-0.02898886403i	0.6533586912-0.04439381505i
1.093942316-0.0004282859236i	0.7589067198-0.016967363i
-	0.8427823736-0.02241511415i
-	1.072348165-0.01887326347i

We note that the first eigenvalue found for the fourth order approximation exhibits a relative error of 1.7%. However, we do not find any eigenvalues closely resembling the exact eigenvalue determined by Orszag [55] in the higher order approximations. We also note that none of the eigenvalues determined herein match any of the least stable eigenvalues reported
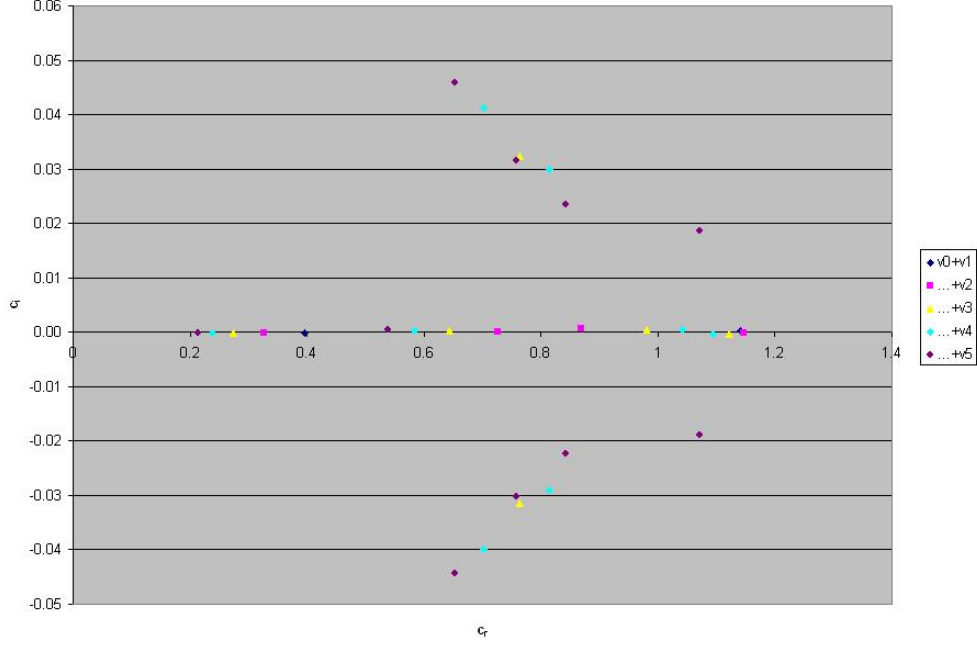
Figure 4.1: Eigenvalues for the Orr-Sommerfeld equation in the (c_r, c_i) plane by Orszag [55].

While some of the eigenvalues reported exceed the bounds established by Banerjee [18] ($-0.2884 < c_r < 1$ for the case being examined), all eigenvalues found lie within the spectrum established by Lahmann and Plum [46]. They report that eigenvalues are bound by the following.

$$\alpha^2 - \frac{R}{2}|u'|_{max} \leq Re(\lambda) \tag{4.18}$$

$$\alpha R u_{min} - \frac{R}{2}|u'|_{max} \leq Im(\lambda) \leq \alpha R u_{max} + \frac{R}{2}|u'|_{max} \tag{4.19}$$

For the case $u'' < 0$, Lahmann and Plum [46] improved the bounds by determining that all eigenvalues lie below the essential spectrum or $Im(\lambda) \leq \alpha R u_{max}$. This translates to the

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following bounds for the case being examined herein.

$$-9999 \leq Re(\lambda) \tag{4.20}$$

$$-10000 \leq Im(\lambda) \leq 10000 \tag{4.21}$$

Therefore, all the eigenvalues reported lie well within this spectrum.

One of the main advantages of utilizing this method is a means of expressing the solution or eigenfunction for the problem. However, the solution derived is expressed as a function of two unknown initial conditions, k_1 and k_2. Several attempts to determine these values have been so far unsuccessful.

4.4 Solution for Plane Couette Flow

In this case, we define the base flow as plane Couette flow instead of plane Pouseille flow. Therefore, we take

$$\bar{u}(y) = y. \tag{4.22}$$

Next, we substitute $\bar{u}(y)$ into equation (4.7) as before and use the recursive relationship based on equations (4.8) and (4.10). This yields the series solution which was truncated as $v(y) = v_0 + v_1 + v_2 + v_3 + v_4 + v_5$.

4.5 Results for Plane Couette Flow

Again, we examine the case with $\alpha = 1, R = 10,000$. An exact eigenvalue for comparison in this case is not available. The ten eigenvalues found for Couette flow in this case are as follows:

$\lambda_1 = 0.5100478269 + 0.00008734831670i$

$\lambda_2 = 0.2028383343 + 0.0003207142211i$

$$\lambda_3 = 0.01969431594 + 0.0005644943245i$$

$$\lambda_4 = -0.06543595313 + 0.1751337700i$$

$$\lambda_5 = -0.1864337287 + 0.08529065802i$$

$$\lambda_6 = -0.4213334371 + 0.1171511358i$$

$$\lambda_7 = -0.8459002026 - 0.0001280705563i$$

$$\lambda_8 = -0.4210798540 - 0.1168018015i$$

$$\lambda_9 = -0.1857557109 - 0.08414051006i$$

$$\lambda_{10} = -0.06712899028 - 0.1735280611i$$

The eigenvalues are plotted on the (c_r, c_i) plane in figure 4.2. It is interesting to note that a comparsion of the graphs reflects the difference in sign utilized by plane Couette flow versus plane Pouseille flows. The eigenvalue behavior generally appears reflected across a horizontal line in the plane.
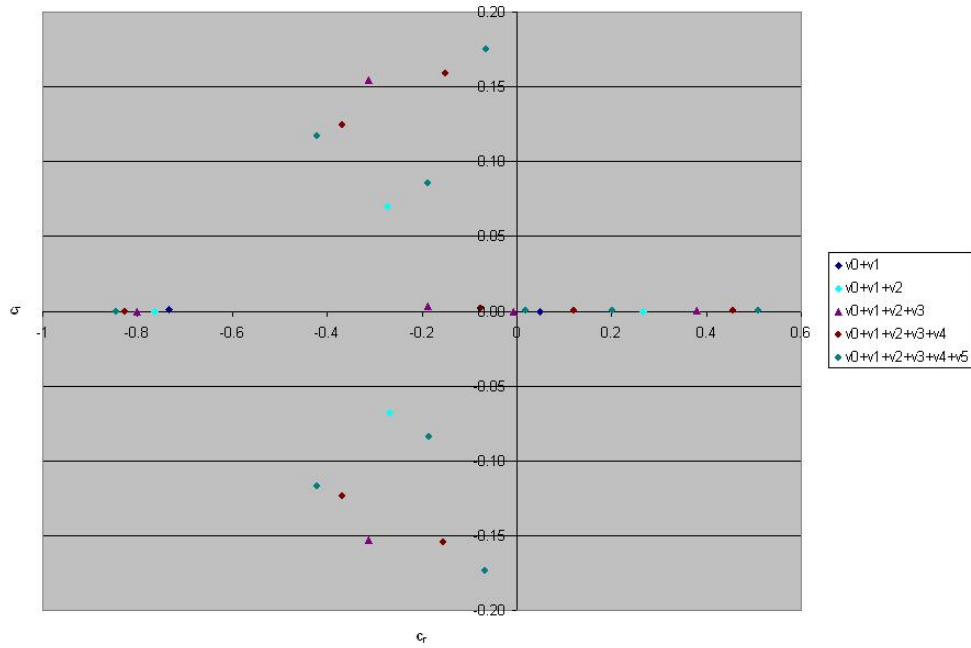
Figure 4.2: Eigenvalues for the Orr-Sommerfeld equation in the (c_r, c_i) plane

4.6   Conclusion

While it is difficult to say that the Adomian Decomposition Method was efficiently utilized for this problem since the equation exhibits no nonlinear terms and hence "decomposition" was unnecessary, the underlying approach established by the ADM was found useful in examining this equation. Again, many of the advantages previously expounded were noted. While examining this problem, it became evident that some of the difficulties encountered in this work could be due to the boundary conditions. All problems examined are boundary value problems and since the ADM has only a small region of convergence, utilizing the boundary conditions in the result would naturally lead to some error. Hence it is believed that the ADM is more useful for initial value problems than for boundary value problems.

CHAPTER FIVE: CONCLUSION

The main objective of this study was to examine the effectiveness of the Adomian Decomposition Method. In particular, we focused on examining some well-known hydrodynamic equations arising in boundary layer theory, the Blasius equation, the Falkner-Skan equation, and the Orr-Sommerfeld equation. The ADM was used to find solutions for these equations and the results were then compared to previously published results as a means of evaluating the quality of these solutions.

Since its introduction in the 1980's, the ADM has been the subject of much research. As shown in the first chapter, many researchers find the method very convenient. It is also reported as efficient and accurate. However, an objective reporting of both advantages and disadvantages to the method was notably absent from the existing body of research. This omission prompted the current work.

To begin with, the ADM was applied to the Blasius equation with generalized boundary conditions in Chapter 2. From this work, the convenience of the method was observed in that it involves direct application to the problem and is easily performed. The results obtained were found to be quite accurate. The main disadvantage that was noted was the small region of convergence for the solution. Although several researchers have claimed to improve this with Padé approximants as an after-treatment, this work shows that while their use does improve accuracy, it does not appear to improve region of convergence (see section 2.7).

The ADM was next used for the Falkner-Skan equation in Chapter 3. The main advantage found from this was that the method generates an analytic expression for the solution. The computations are easily utilized and could be completed without computer assistance

if desired. This work indicates that an increase in symbolism results in an increase in the complexity of the solution, thereby losing the computational ease which has been a major advantage of this method (see section 3.4). Furthermore, it appears that the type or strength of nonlinearity also influences the qualitative properties of the solution (see section 3.2 and 3.4). The solutions generated in this work exhibit functions in the series terms that are unfavorable for use in recursive relationship.

Finally, the ADM was employed to develop a solution for the Orr-Sommerfeld equation in Chapter 5. This exhibits the wide applicability of the method. As Wazwaz indicated, it can be used for linear as well as nonlinear problems [61]. In utilizing the method for this problem, it became evident that the small region of convergence for this method could account for the difficulties in application to boundary value problems (see section 4.4).

Overall Observations

Based on our study, the advantages of the ADM can be enumerated as follows:

1. The ADM is a convenient tool. It can be used with direct application to the problem.

2. The ADM demonstrates computational ease. The process is simple and easily utilized and can be carried out by hand.

3. The ADM has wide applicability. It can be applied to many types of problems.

4. The ADM generates an analytic expression for the solution. While many solutions can be solved numerically, it is extremely beneficial to have an analytic solution.

5. The ADM produces an accurate solution in many situations.

The limitations exhibited by the ADM based on this work are as follows:

1. The ADM exhibits a small region of convergence. While the accuracy can be improved by the applied aftertreatment technique utilizing Padé approximants, it does note appear to

significantly improve the region of convergence. Hence, it is recommended that this method is acceptable only for localized solutions. If global solution is desired, it is suggested that alternate methods, such as HAM or Bender's δ-perturbation method, can be employedis utilized (see Appendix B for an alternate solution to the generalized Blasius equation).

2. The solution becomes cumbersome as symbolism for the equation increases. This study demonstrates that the successive equations for the series solution become large and awkward, losing some of the computational ease that is claimed as the main benefit of this method.

3. Occasionally, the method is unsuccessful and the problem needs to be modified in order to accomodate this (see section 2.2 and 3.2).

4. While it has been claimed that the ADM can be applied directly to nonlinear problems, this work seems to indicate difficulties based on the strength or type of nonlinearity. This was only briefly noted and requires further investigation before the claim can be asserted.

5. The convergence problems for the ADM lead to a hestinancy in recommending application to boundary value problems.

In general, the ADM is believed to be an easy and convenient tool with wide applicability. However, some discretion is recommended for its use as there are certain conditions which limit the quality of the solution generated.

CHAPTER SIX: FUTURE RESEARCH

The completion of this study has led to an awareness of several topics that require further investigation, such as the new homotopy analysis method (HAM). As such, a breif description of future research based on this work has been included.

To begin with, additional problems have already been identified for study. These include Ernigen's micropolar fluid equations [39] and the nano boundary layer equations examined by Matthews and Hill [52]. Ernigen's micropolar fluid equations considered by Ishak et al. [39] are a set of four coupled equations similar to those already examined in this work. The nano boundary layer equations examined by Matthews and Hill [52] are of particular interest because of the inclusion of the nonlinear boundary condition. This arises from the standard no-slip boundary condition being inapplicable at the micro and nano scale and therefore "must be replaced by a boundary condition that allows some degree of tangential slip".

Several topics for future research remain for the Falkner-Skan equation. To begin with, an effort should be made to improve the quality of the solution. Possible area that could be explored for this include the definition of the linear/nonlinear operators and utilizing an alternate transformation. In addition, the Falkner-Skan problem could include possible extensions to the Prandtl and the Görtler equations. Finally, a more thorough evaluation of the coupled magnetohydrodynamic equations should be conducted.

With respect to the Orr-Sommerfeld equation, future research should include an examination of alternate solution methods. This would hopefully lead to a better means of ascertaining the quality of the solution. The question as to the number and location of the eigenvalues is still undetermined in this work. In addition, this work can be expanded by establishing the

unknown values to give the eigenvalue functions.

The Blasius equation could also be examined further. As stated before, an alternate transformation or definition for the operators could be investigated. This problem could be used for examining the convergence issues and effectiveness of aftertreatment techniques.

In general, several topics specific to the ADM also require further examination. To begin with, an understanding of the convergence should be developed. This would hopefully lead to an understanding of the limited effectiveness of Padé aftertreatment. In addition, the degree to which the strength and type of nonlinearity affect the quality of the solution should also be investigated.

APPENDIX A: COEFFICIENT FUNCTIONS

A.1 Blasius Equation

The coeffiecient functions from the solution equations for Case 2, $\alpha \neq 0$ from Chapter 2 are given below.

$s_1 - s_5$ are the coefficients for f_1.

$$s_1 = -\frac{1}{2\alpha}$$

$$s_2 = -\frac{\beta}{\alpha} - \frac{k + \alpha\beta}{\alpha^2} - \frac{(k + \alpha\beta)\ln|k|}{\alpha^2}$$

$$s_3 = \frac{(k + \alpha\beta)^2}{\alpha^3}$$

$$s_4 = \frac{k + \alpha\beta}{\alpha^2}$$

$$s_5 = -\frac{\beta^2}{2\alpha} - \frac{\beta(k + \alpha\beta)}{\alpha^2} - \frac{(k + \alpha\beta)^2 \ln|k|}{\alpha^3}$$

$p_1 - p_9$ are the coefficients for f_2.

$$p_1 = -\frac{1}{12\alpha^3}$$

$$p_2 = -\frac{\beta}{2\alpha^3} - \frac{3(k + \alpha\beta)}{4\alpha^4} - \frac{(k + \alpha\beta)\ln|k|}{2\alpha^4}$$

$$p_3 = -\frac{2(k + \alpha\beta)^2}{\alpha^5} - \frac{\beta(k + \alpha\beta)}{\alpha^4} - \frac{5(k + \alpha\beta)^2 \ln|k|}{2\alpha^5} - \frac{\beta^2}{4\alpha^3}$$
$$- \frac{\beta(k + \alpha\beta)\ln|k|}{\alpha^4} - \frac{(k + \alpha\beta)^2 \ln^2|k|}{2\alpha^5} + \frac{\beta^2 \ln|k|}{2\alpha^3} - \frac{(k + \alpha\beta)^3}{2\alpha^5 k}$$
$$+ \frac{\beta^2(k + \alpha\beta)}{2\alpha^3 k}$$

$$p_4 = \frac{5(k + \alpha\beta)^3}{2\alpha^6} + \frac{\beta(k + \alpha\beta)^2}{\alpha^5} + \frac{(k + \alpha\beta)^3 \ln|k|}{\alpha^6} - \frac{\beta^2(k + \alpha\beta)}{\alpha^4}$$

$$p_5 = \frac{5(k+\alpha\beta)^2}{2\alpha^5} + \frac{\beta(k+\alpha\beta)}{\alpha^4} + \frac{(k+\alpha\beta)^2\ln|k|}{\alpha^5} - \frac{\beta^2}{2\alpha^3}$$

$$p_6 = -\frac{(k+\alpha\beta)^3}{2\alpha^6}$$

$$p_7 = -\frac{(k+\alpha\beta)^2}{2\alpha^5}$$

$$p_8 = \frac{k+\alpha\beta}{2\alpha^4}$$

$$p_9 = \frac{\beta^3}{6\alpha^3} - \frac{(k+\alpha\beta)^3\ln^2|k|}{2\alpha^6} - \frac{5(k+\alpha\beta)^3\ln|k|}{2\alpha^6} - \frac{\beta(k+\alpha\beta)^2\ln|k|}{\alpha^5}$$
$$- \frac{2\beta(k+\alpha\beta)^2}{\alpha^5} + \frac{\beta^2(k+\alpha\beta)\ln|k|}{\alpha^4} - \frac{5\beta^2(k+\alpha\beta)}{4\alpha^4} - \frac{\beta(k+\alpha\beta)^3}{2\alpha^5 k}$$
$$+ \frac{\beta^3(k+\alpha\beta)}{2\alpha^3 k}$$

$t_1 - t_{15}$ are the coefficients for f_3.

$$t_1 = -\frac{1}{36\alpha^5}$$

$$t_2 = -\frac{\beta}{4\alpha^5} - \frac{25(k+\alpha\beta)}{72\alpha^6} - \frac{(k+\alpha\beta)\ln|k|}{4\alpha^6}$$

$$t_3 = -\frac{3(k+\alpha\beta)^2}{8\alpha^7} - \frac{5\beta(k+\alpha\beta)}{4\alpha^6} - \frac{3(k+\alpha\beta)^2\ln|k|}{2\alpha^7} - \frac{3\beta^2}{4\alpha^5} - \frac{3(k+\alpha\beta)^2}{\alpha}$$
$$- \frac{3\beta(k+\alpha\beta)}{2} - \frac{3(k+\alpha\beta)^2\ln|k|}{2\alpha} + \frac{\beta^2\alpha}{4} - \frac{3\beta(k+\alpha\beta)\ln|k|}{2\alpha^6}$$
$$- \frac{(k+\alpha\beta)^3}{4\alpha^7 k} + \frac{\beta^2(k+\alpha\beta)}{4\alpha^5} - \frac{3(k+\alpha\beta)^2\ln^2|k|}{4\alpha^7} + \frac{\beta^2\ln|k|}{4\alpha^5}$$

$$t_4 = \frac{\beta^3 \alpha}{2} + \frac{\beta^4(k+\alpha\beta)}{8\alpha^4 k^2} - \frac{71(k+\alpha\beta)^3 \ln|k|}{6\alpha^8} - \frac{4(k+\alpha\beta)^3 \ln^2|k|}{\alpha^8}$$
$$- \frac{(k+\alpha\beta)^3 \ln^3|k|}{2\alpha^8} - \frac{3\beta(k+\alpha\beta)\ln|k|}{\alpha} + \frac{3\beta^2(k+\alpha\beta)\ln|k|}{4\alpha^6} - \frac{3\beta(k+\alpha\beta)^2 \ln^2|k|}{2\alpha^7}$$
$$+ \frac{5\beta^3 \ln|k|}{6\alpha^5} - \frac{9\beta(k+\alpha\beta)^2 \ln|k|}{2\alpha^7} - \frac{\beta^3(k+\alpha\beta)\ln|k|}{2\alpha^5 k}$$
$$+ \frac{\beta^2(k+\alpha\beta)\ln^2|k|}{2\alpha^6} - \frac{25(k+\alpha\beta)^4}{12\alpha^8 k} + \frac{3(k+\alpha\beta)^5 \ln|k|}{2\alpha^8 k^2}$$
$$- \frac{5\beta^3}{18\alpha^5} + \frac{7\beta^2(k+\alpha\beta)}{24\alpha^6} - \frac{4\beta(k+\alpha\beta)^2}{\alpha^7} - \frac{6\beta(k+\alpha\beta)^2}{\alpha}$$
$$- 3\beta^2(k+\alpha\beta) - \frac{109(k+\alpha\beta)^3}{12\alpha^8} + \frac{21(k+\alpha\beta)^5}{8\alpha^8 k^2} + \frac{2\beta^2(k+\alpha\beta)}{\alpha^6 k}$$
$$- \frac{\beta(k+\alpha\beta)^3}{2\alpha^7 k} - \frac{5\beta^3(k+\alpha\beta)}{6\alpha^5 k} + \frac{2\beta(k+\alpha\beta)^4}{\alpha^7 k^2} - \frac{\beta^3(k+\alpha\beta)^2}{2\alpha^5 k^2}$$
$$- \frac{\beta^4}{18\alpha^4 k} - \frac{(k+\alpha\beta)^4 \ln|k|}{\alpha^8 k} + \frac{\beta(k+\alpha\beta)^3 \ln|k|}{2\alpha^7 k}$$

$$t_5 = \frac{3(k+\alpha\beta)^5 \ln|k|}{2\alpha^9} + \frac{25(k+\alpha\beta)^5}{8\alpha^9} - \frac{\beta^2(k+\alpha\beta)^3}{2\alpha^7}$$
$$+ \frac{3\beta(k+\alpha\beta)^4}{2\alpha^8} + \frac{\beta^4(k+\alpha\beta)}{8\alpha^5}$$

$$t_6 = -\frac{3\beta^2(k+\alpha\beta)\ln|k|}{2\alpha^7} - \frac{3\beta^2(k+\alpha\beta)^3}{2\alpha^7 k} + \frac{7(k+\alpha\beta)^4 \ln|k|}{\alpha^9}$$
$$+ \frac{3(k+\alpha\beta)^4 \ln^2|k|}{2\alpha^9} + \frac{61(k+\alpha\beta)^4}{6\alpha^9} + \frac{3\beta(k+\alpha\beta)^3 \ln|k|}{\alpha^8} + \frac{3(k+\alpha\beta)^5}{2\alpha^9 k}$$
$$- \frac{3\beta^2(k+\alpha\beta)^2}{2\alpha^7} + \frac{13\beta(k+\alpha\beta)^3}{2\alpha^8} - \frac{5\beta^3(k+\alpha\beta)}{3\alpha^6} - \frac{\beta(k+\alpha\beta)^4}{\alpha^8 k}$$
$$+ \frac{\beta^3(k+\alpha\beta)^2}{\alpha^6 k} + \frac{\beta^4}{4\alpha^5}$$

$$t_7 = -\frac{7(k+\alpha\beta)^4}{2\alpha^9} + \frac{3\beta^2(k+\alpha\beta)^2}{4\alpha^7} - \frac{3\beta(k+\alpha\beta)^3}{2\alpha^8}$$
$$- \frac{3(k+\alpha\beta)^4 \ln|k|}{2\alpha^9}$$

$$t_8 = \frac{(k+\alpha\beta)^4}{2\alpha^9}$$

$$t_9 = \frac{71(k+\alpha\beta)^3}{6\alpha^8} - \frac{3\beta^2(k+\alpha\beta)}{4\alpha^6} + \frac{15\beta(k+\alpha\beta)^2}{2\alpha^7} + \frac{8(k+\alpha\beta)^3 \ln|k|}{\alpha^8}$$

$$+ \frac{3(k+\alpha\beta)^3 \ln^2|k|}{2\alpha^8} - \frac{\beta(k+\alpha\beta)^3}{2\alpha^7 k} + \frac{\beta^3(k+\alpha\beta)}{2\alpha^5 k} - \frac{\beta^2(k+\alpha\beta)\ln|k|}{\alpha^6}$$

$$+ \frac{3\beta(k+\alpha\beta)^2 \ln|k|}{\alpha^7} - \frac{5\beta^3}{6\alpha^5} - \frac{\beta^2(k+\alpha\beta)^2}{\alpha^6 k}$$

$$+ \frac{(k+\alpha\beta)^4}{\alpha^8 k}$$

$$t_{10} = -\frac{4(k+\alpha\beta)^3}{\alpha^8} + \frac{\beta^2(k+\alpha\beta)}{2\alpha^6} - \frac{3\beta(k+\alpha\beta)^2}{2\alpha^7} - \frac{3(k+\alpha\beta)^3 \ln|k|}{2\alpha^8}$$

$$t_{11} = \frac{(k+\alpha\beta)^3}{2\alpha^8}$$

$$t_{12} = \frac{3(k+\alpha\beta)^2}{2\alpha^7} + \frac{3\beta(k+\alpha\beta)}{2\alpha^6} + \frac{3(k+\alpha\beta)^2 \ln|k|}{2\alpha^7} - \frac{\beta^2}{4\alpha^5}$$

$$t_{13} = -\frac{3(k+\alpha\beta)^2}{4\alpha^7}$$

$$t_{14} = \frac{k+\alpha\beta}{4\alpha^6}$$

$$t_{15} = \frac{9\beta^2(k+\alpha\beta)^2 \ln|k|}{2\alpha^7} - \frac{3\beta^2(k+\alpha\beta)^2 \ln|k|}{2\alpha} + \frac{\beta(k+\alpha\beta)^4 \ln|k|}{\alpha^8 k} + \frac{\beta^4\alpha}{4}$$

$$+ \frac{\beta^5(k+\alpha\beta)}{8\alpha^4 k^2} + \frac{21\beta(k+\alpha\beta)^5}{8\alpha^8 k^2} + \frac{2\beta^2(k+\alpha\beta)^4}{\alpha^7 k^2}$$

$$- \frac{\beta^4(k+\alpha\beta)^2}{2\alpha^5 k^2} - \frac{\beta^5}{4\alpha^4 k} + \frac{11\beta^4(k+\alpha\beta)}{24\alpha^5 k}$$

$$- \frac{43\beta(k+\alpha\beta)^4}{12\alpha^8 k} - \frac{109\beta(k+\alpha\beta)^3}{12\alpha^8} - \frac{\beta^3(k+\alpha\beta)^2 \ln|k|}{\alpha^6 k}$$

$$+ \frac{3\beta^2(k+\alpha\beta)^3 \ln|k|}{2\alpha^7 k} - \frac{13\beta(k+\alpha\beta)^3 \ln|k|}{2\alpha^8} - \frac{3\beta(k+\alpha\beta)^3 \ln^2|k|}{2\alpha^8}$$

$$+ \frac{3\beta^2(k+\alpha\beta)^2 \ln^2|k|}{4\alpha^7} + \frac{5\beta^3(k+\alpha\beta)\ln|k|}{3\alpha^6} - \frac{7(k+\alpha\beta)^4 \ln^2|k|}{2\alpha^9}$$

$$+ \frac{3\beta^4(k+\alpha\beta)^5 \ln|k|}{2\alpha^8 k^2} + \frac{2\beta^3(k+\alpha\beta)^2}{\alpha^6 k} - \frac{\beta^4 \ln|k|}{4\alpha^5}$$

$$- \frac{25(k+\alpha\beta)^5}{8\alpha^9 k} + \frac{\beta^4}{4\alpha^5} + \frac{43\beta^3(k+\alpha\beta)}{36\alpha^6}$$

$$- \frac{29\beta^2(k+\alpha\beta)^2}{8\alpha^7} - \frac{3\beta^2(k+\alpha\beta)^2}{\alpha} - \frac{3\beta^3(k+\alpha\beta)}{2} + \frac{\beta^2(k+\alpha\beta)^3}{4\alpha^7 k}$$

$$- \frac{3(k+\alpha\beta)^5 \ln|k|}{\alpha^9 k} - \frac{(k+\alpha\beta)^4 \ln^3|k|}{2\alpha^9} - \frac{61(k+\alpha\beta)^4 \ln|k|}{6\alpha^9}$$

$z_1 - z_{24}$ are the coefficients for f_4.

$$z_1 = -\frac{17}{1440\alpha^7}$$

$$z_2 = -\frac{5\beta}{36\alpha^7} - \frac{25(k+\alpha\beta)}{96\alpha^8} - \frac{5(k+\alpha\beta)\ln|k|}{36\alpha^8} + \frac{k+\alpha\beta}{6\alpha^6}$$

$$\begin{aligned}
z_3 = &-\frac{575(k+\alpha\beta)^2}{432\alpha^9} - \frac{125\beta(k+\alpha\beta)}{72\alpha^8} - \frac{67(k+\alpha\beta)^2\ln|k|}{36\alpha^9} - \frac{5\beta^2}{9\alpha^7} \\
&-\frac{(k+\alpha\beta)^3}{8\alpha^9 k} - \frac{\beta(k+\alpha\beta)}{4\alpha^2} - \frac{(k+\alpha\beta)^2}{2\alpha^3} + \frac{\beta^2}{24\alpha} \\
&+\frac{\beta^2(k+\alpha\beta)}{8\alpha^7 k} - \frac{5\beta\ln|k|}{4\alpha^8} - \frac{5(k+\alpha\beta)^2\ln^2|k|}{8\alpha^9} - \frac{(k+\alpha\beta)^2\ln|k|}{4\alpha^3} \\
&+\frac{\beta^2\ln|k|}{8\alpha^7}
\end{aligned}$$

$$\begin{aligned}
z_4 = &-\frac{15(k+\alpha\beta)^3}{4\alpha^{11}} + \frac{3\beta^2(k+\alpha\beta)\ln|k|^2}{4\alpha^8} - \frac{3\beta(k+\alpha\beta)^2}{2\alpha^3} - \frac{7\beta^2(k+\alpha\beta)}{4\alpha^2} \\
&-\frac{(k+\alpha\beta)^4\ln|k|}{\alpha^{10}k} - \frac{5(k+\alpha\beta)^3\ln^3|k|}{4\alpha^{10}} - \frac{67(k+\alpha\beta)^3\ln^2|k|}{8\alpha^{10}} - \frac{71(k+\alpha\beta)^3\ln|k|}{3\alpha^{10}} \\
&+\frac{3(k+\alpha\beta)^3\ln|k|}{2\alpha^4} + \frac{11\beta^3 k^2\ln|k|}{12\alpha^7} - \frac{3595(k+\alpha\beta)^3}{144\alpha^{10}} - \frac{151\beta(k+\alpha\beta)^2}{8\alpha^9} \\
&-\frac{85(k+\alpha\beta)\beta^2}{24\alpha^8} + \frac{2\beta^2(k+\alpha\beta)^2}{\alpha^8 k} - \frac{5\beta(k+\alpha\beta)^3}{8\alpha^9 k} + \frac{19(k+\alpha\beta)\beta^3}{24\alpha^7 k} \\
&+\frac{21(k+\alpha\beta)^5}{16\alpha^{10}k^2} - \frac{49(k+\alpha\beta)^4}{24\alpha^{10}k} + \frac{3(k+\alpha\beta)^5\ln(k)}{4\alpha^{10}k^2} - \frac{3\beta(k+\alpha\beta)^2\ln(k)}{2\alpha^3} \\
&+\frac{3(k+\alpha\beta)^3}{\alpha^4} + \frac{\beta^3}{4\alpha} + \frac{(k+\alpha\beta)\beta^4}{16\alpha^6 k^2} - \frac{31\beta^3}{72\alpha^7} \\
&+\frac{\beta(k+\alpha\beta)^4}{\alpha^9 k^2} - \frac{\beta^3(k+\alpha\beta)^2}{4\alpha^7 k^2} - \frac{\beta^4}{8\alpha^6 k} - \frac{29\beta\ln(k)(k+\alpha\beta)^2}{2\alpha^9} \\
&+\frac{(k+\alpha\beta)^2\beta^2\ln(k)}{\alpha^8 k} + \frac{\beta\ln(k)(k+\alpha\beta)^3}{4\alpha^9 k} - \frac{\ln(k)\beta^3(k+\alpha\beta)}{4\alpha^7 k} - \frac{9(k+\alpha\beta)\beta^2\ln|k|}{8\alpha^8} \\
&-\frac{15\beta(k+\alpha\beta)^2\ln^2|k|}{4\alpha^9}
\end{aligned}$$

$$
\begin{aligned}
z_5 = {} & \frac{12(k+\alpha\beta)\beta^6}{288\alpha^5 k^3} + \frac{592(k+\alpha\beta)^7\ln(k)}{288\alpha^{11}k^3} - \frac{26836(k+\alpha\beta)^4}{288\alpha^{11}} + \frac{233\beta^4}{288\alpha^7} \\
& - \frac{108\beta^5(k+\alpha\beta)^2}{288\alpha^6 k^3} - \frac{684\beta^2(k+\alpha\beta)^5}{288\alpha^3 k^3} + \frac{216\beta^4(k+\alpha\beta)^3}{288\alpha^7 k^3} - \frac{252\beta(k+\alpha\beta)^6}{288\alpha^{10}k^3} \\
& - \frac{18\beta^6}{288\alpha^5 k^2} + \frac{2202(k+\alpha\beta)^6}{288\alpha^{11}k^2} - \frac{2324(k+\alpha\beta)^5}{288\alpha^{11}k} - \frac{168\beta^5}{288\alpha^6 k} + \frac{192\beta^3(k+\alpha\beta)}{288\alpha^6} \\
& - \frac{180(k+\alpha\beta)^4\ln(k)^4}{288\alpha^{11}} - \frac{4176\beta\ln(k)^2(k+\alpha\beta)^3}{288\alpha^{10}} - \frac{3024\beta(k+\alpha\beta)^3\ln(k)}{288\alpha^4} \\
& + \frac{372\beta^4\ln|k|}{288\alpha^7} + \frac{2592(k+\alpha\beta)^6\ln|k|}{288\alpha^{11}k^2} + \frac{3888(k+\alpha\beta)^4\ln|k|}{288\alpha^5} + \frac{396\beta(k+\alpha\beta)^5\ln|k|}{288\alpha^{10}k^2} \\
& + \frac{432\beta^2(k+\alpha\beta)^3\ln|k|}{288\alpha^3 k} + \frac{216(k+\alpha\beta)^2\beta^4\ln|k|}{288\alpha^7 k^2} - \frac{31052(k+\alpha\beta)^4\ln|k|}{288\alpha^{11}} \\
& - \frac{10284(k+\alpha\beta)^4\ln|k|^2}{288\alpha^{11}} - \frac{72\ln(k)\beta^4}{288\alpha} + \frac{864\ln(k)^2(k+\alpha\beta)^6}{288\alpha^{11}k^2} + \frac{72\ln(k)\beta^5}{288\alpha^6 k} \\
& - \frac{36\ln(k)^2\beta^4}{288\alpha^7} + \frac{1296\ln(k)^2(k+\alpha\beta)^4}{288\alpha^5} - \frac{432\beta(k+\alpha\beta)^6\ln(k)}{288\alpha^{10}k^3} + \frac{432(k+\alpha\beta)^5\ln(k)}{288\alpha^5 k} \\
& + \frac{72(k+\alpha\beta)^5\ln(k)^2}{288\alpha^{11}k} - \frac{516\ln(k)(k+\alpha\beta)^5}{288\alpha^{11}k} - \frac{2040(k+\alpha\beta)^4\ln(k)^3}{288\alpha^{11}} \\
& - \frac{360(k+\alpha\beta)^3\ln(k)}{288\alpha^{11}} + \frac{192(k+\alpha\beta)^2\ln(k)\beta^3}{288\alpha^8 k} - \frac{1296\beta^2(k+\alpha\beta)^2\ln(k)}{288\alpha^3} \\
& - \frac{720(k+\alpha\beta)^3\beta\ln(k)^3}{288\alpha^{10}} - \frac{324(k+\alpha\beta)^2\beta^2\ln(k)^2}{288\alpha^9} + \frac{984(k+\alpha\beta)\beta^3\ln(k)}{288\alpha^8} \\
& - \frac{576\beta^2\ln(k)(k+\alpha\beta)^4}{288\alpha^9 k^2} - \frac{864(k+\alpha\beta)^4\beta\ln(k)}{288\alpha^4 k} + \frac{1728(k+\alpha\beta)^3\ln(k)\beta^2}{288\alpha^9 k^2} \\
& + \frac{360(k+\alpha\beta)^3\ln(k)^2\beta^2}{288\alpha^9 k} - \frac{288(k+\alpha\beta)^3\ln(k)\beta^3}{288\alpha^8 k^2} + \frac{96(k+\alpha\beta)^2\beta^2\ln(k)}{288\alpha^9} \\
& - \frac{18168\beta\ln(k)(k+\alpha\beta)^3}{288\alpha^{10}} + \frac{216\ln(k)^3(k+\alpha\beta)^2\beta^2}{288\alpha^9} + \frac{528\ln(k)^2(k+\alpha\beta)\beta^3}{288\alpha^8} \\
& + \frac{432\ln(k)^2\beta^2(k+\alpha\beta)^2}{288\alpha^3} - \frac{144\ln(k)^2\beta^3(k+\alpha\beta)^2}{288\alpha^8 k} + \frac{144\ln(k)^2\beta(k+\alpha\beta)^4}{288\alpha^{10}k} \\
& - \frac{1728\ln(k)^2\beta(k+\alpha\beta)^3}{288\alpha^4} - \frac{36\ln(k)\beta^5(k+\alpha\beta)}{288\alpha^6 k^2} + \frac{720\ln(k)\beta^3(k+\alpha\beta)}{288\alpha^2} \\
& - \frac{432\ln(k)^2\beta(k+\alpha\beta)^5}{288\alpha^{10}k^2} + \frac{4320\beta\ln(k)(k+\alpha\beta)^3}{288\alpha^{11}} + \frac{1320(k+\alpha\beta)^4\ln(k)\beta}{288\alpha^{10}k} \\
& - \frac{708(k+\alpha\beta)\ln(k)\beta^4}{288\alpha^7 k} + \frac{864(k+\alpha\beta)^5}{288\alpha^5 k} + \frac{2592(k+\alpha\beta)^4}{288\alpha^5} + \frac{36\beta^4}{288\alpha} - \frac{416(k+\alpha\beta)^3}{288\alpha^{11}} \\
& + \frac{300(k+\alpha\beta)\beta^5}{288\alpha^6 k^2} + \frac{870(k+\alpha\beta)^4\beta^2}{288\alpha^9 k^2} - \frac{546\beta^4(k+\alpha\beta)^2}{288\alpha^7 k^2} - \frac{600\beta^3(k+\alpha\beta)^3}{288\alpha^8 k^2} \\
& + \frac{1572\beta(k+\alpha\beta)^5}{288\alpha^{10}k^2} - \frac{720(k+\alpha\beta)^6}{288\alpha^{11}k^2} - \frac{432\beta(k+\alpha\beta)^3}{288\alpha^4} - \frac{1512\beta^2(k+\alpha\beta)^2}{288\alpha^3} \\
& - \frac{72\beta^3(k+\alpha\beta)}{288\alpha^2} - \frac{5550\beta^2(k+\alpha\beta)^2}{288\alpha^9} + \frac{3060\beta^2(k+\alpha\beta)^3}{288\alpha^9 k} - \frac{21188\beta(k+\alpha\beta)^3}{288\alpha^{10}}
\end{aligned}
$$

z_5 continued

$$- \frac{80(k+\alpha\beta)\beta^3}{288\alpha^8} + \frac{984\beta(k+\alpha\beta)^4}{288\alpha^{10}k} + \frac{1280\beta^3(k+\alpha\beta)^2}{288\alpha^8 k} + \frac{48(k+\alpha\beta)\beta^4}{288\alpha^7 k}$$

$$- \frac{72\beta^2(k+\alpha\beta)^3}{288\alpha^3 k} + \frac{576\beta^3(k+\alpha\beta)^2}{288\alpha^2 k} - \frac{72(k+\alpha\beta)\beta^4}{288\alpha k^3} - \frac{1296(k+\alpha\beta)^4\beta}{288\alpha^4 k}$$

$$+ \frac{2160(k+\alpha\beta)^4}{288\alpha^{12}} - \frac{360(k+\alpha\beta)^2\beta}{288\alpha^{10}} + \frac{36(k+\alpha\beta)\beta^2}{288\alpha^9} + \frac{816(k+\alpha\beta)^7}{288\alpha^{11}k^3}$$

$$z_6 = \frac{24(k+\alpha\beta)^4\beta^3}{48\alpha^9 k} - \frac{12\beta^5(k+\alpha\beta)^2}{48\alpha^7 k} - \frac{24\beta^2(k+\alpha\beta)^5}{48\alpha^{10}k} + \frac{12\beta^4(k+\alpha\beta)^3}{48\alpha^8 k}$$

$$+ \frac{14(k+\alpha\beta)\beta^5}{48\alpha^7} - \frac{35(k+\alpha\beta)^4\beta^2}{48\alpha^{10}} + \frac{21\beta^4(k+\alpha\beta)^2}{48\alpha^8} - \frac{20\beta^3(k+\alpha\beta)^3}{48\alpha^9}$$

$$- \frac{30\beta(k+\alpha\beta)^5}{48\alpha^{11}} - \frac{12\beta(k+\alpha\beta)^6}{48\alpha^{11}k} + \frac{12(k+\alpha\beta)^7}{48\alpha^{12}k} + \frac{71(k+\alpha\beta)^6}{48\alpha^{12}}$$

$$+ \frac{6(k+\alpha\beta)^2\beta^4\ln(k)}{48\alpha^8} - \frac{3\beta^6}{48\alpha^6} - \frac{30(k+\alpha\beta)^6\ln(k)}{48\alpha^{12}} - \frac{12\beta^2\ln(k)(k+\alpha\beta)^4}{48\alpha^{10}}$$

$$- \frac{120(k+\alpha\beta)^5}{48\alpha^{12}}$$

$$z_7 = \frac{\beta^2(k+\alpha\beta)^4}{4\alpha^{10}} - \frac{(k+\alpha\beta)^6}{8\alpha^{12}} - \frac{\beta^4(k+\alpha\beta)^2}{8\alpha^8}$$

$$z_8 = \frac{\beta^2(k+\alpha\beta)^5}{16\alpha^{10}} - \frac{\beta^4(k+\alpha\beta)^3}{16\alpha^8} + \frac{17(k+\alpha\beta)^7}{144\alpha^{12}} + \frac{\beta^6(k+\alpha\beta)}{48\alpha^6}$$

$$z_9 = \frac{5(k+\alpha\beta)^7}{18\alpha^{12}}$$

$$z_{10} = -\frac{324(k+\alpha\beta)^7\ln(k)}{72\alpha^{12}k^2} + \frac{18\beta^5(k+\alpha\beta)^2}{72\alpha^7k^2} + \frac{288\beta^2(k+\alpha\beta)^5}{72\alpha^{10}k^2} - \frac{99\beta^4(k+\alpha\beta)^3}{72\alpha^8k^2}$$
$$- \frac{54\beta(k+\alpha\beta)^6}{72\alpha^{11}k^2} + \frac{216(k+\alpha\beta)^6}{72\alpha^{12}k} + \frac{6956(k+\alpha\beta)^5}{72\alpha^{12}} + \frac{42\beta^5}{72\alpha^7} + \frac{165(k+\alpha\beta)^2}{72\alpha^{12}}$$
$$+ \frac{108(k+\alpha\beta)^4\beta^3}{72\alpha^9k^2} - \frac{108\beta(k+\alpha\beta)^5\ln(k)}{72\alpha^{11}k} - \frac{216\beta^2(k+\alpha\beta)^3\ln(k)}{72\alpha^4}$$
$$+ \frac{216\beta(k+\alpha\beta)^6\ln(k)}{72\alpha^{11}k^2} - \frac{432(k+\alpha\beta)^5\ln(k)}{72\alpha^6} + \frac{1314(k+\alpha\beta)^5\ln(k)^2}{72\alpha^{12}}$$
$$+ \frac{4158\ln(k)(k+\alpha\beta)^5}{72\alpha^{12}} - \frac{396(k+\alpha\beta)^2\ln(k)\beta^3}{72\alpha^9} - \frac{216\beta^2\ln(k)(k+\alpha\beta)^4}{72\alpha^{10}k}$$
$$+ \frac{648(k+\alpha\beta)^4\beta\ln(k)}{72\alpha^5} + \frac{126(k+\alpha\beta)^3\ln(k)\beta^2}{72\alpha^{10}} - \frac{216(k+\alpha\beta)^3\ln(k)^2\beta^2}{72\alpha^{10}}$$
$$+ \frac{108(k+\alpha\beta)^3\ln(k)\beta^3}{72\alpha^9k} + \frac{540\ln(k)^2\beta(k+\alpha\beta)^4}{72\alpha^{11}} + \frac{1872(k+\alpha\beta)^4\ln(k)\beta}{72\alpha^{11}}$$
$$+ \frac{54(k+\alpha\beta)\ln(k)\beta^4}{72\alpha^8} - \frac{864(k+\alpha\beta)^5}{72\alpha^6} - \frac{72(k+\alpha\beta)\beta^5}{72\alpha^7k} - \frac{864(k+\alpha\beta)^4\beta^2}{72\alpha^{10}k}$$
$$+ \frac{372\beta^4(k+\alpha\beta)^2}{72\alpha^8k} - \frac{18\beta^3(k+\alpha\beta)^3}{72\alpha^9k} - \frac{246\beta(k+\alpha\beta)^5}{72\alpha^{11}k} - \frac{384\beta^2(k+\alpha\beta)^3}{72\alpha^{10}}$$
$$+ \frac{2460\beta(k+\alpha\beta)^4}{72\alpha^{11}} - \frac{218\beta^3(k+\alpha\beta)^2}{72\alpha^9} - \frac{177(k+\alpha\beta)\beta^4}{72\alpha^8} + \frac{288\beta^2(k+\alpha\beta)^3}{72\alpha^4}$$
$$- \frac{324\beta^3(k+\alpha\beta)^2}{72\alpha^3} + \frac{36(k+\alpha\beta)\beta^4}{72\alpha^2} + \frac{864(k+\alpha\beta)^4\beta}{72\alpha^5} - \frac{540(k+\alpha\beta)^5}{72\alpha^{13}}$$
$$+ \frac{180(k+\alpha\beta)^5\ln(k)^3}{72\alpha^{12}} - \frac{567(k+\alpha\beta)^7}{72\alpha^{12}k^2}$$

$$z_{11} = -\frac{12(k+\alpha\beta)^6}{8\alpha^{12}k} + \frac{17\beta^2(k+\alpha\beta)^3}{8\alpha^{10}} - \frac{60\beta\ln(k)(k+\alpha\beta)^4}{8\alpha^{11}}$$
$$+ \frac{24(k+\alpha\beta)^3\beta^2\ln(k)}{8\alpha^{10}} - \frac{146(k+\alpha\beta)^5\ln(k)}{8\alpha^{12}} - \frac{30(k+\alpha\beta)^5\ln(k)^2}{8\alpha^{12}}$$
$$- \frac{237(k+\alpha\beta)^5}{8\alpha^{12}} + \frac{12\beta^2(k+\alpha\beta)^4}{8\alpha^{10}k} - \frac{140\beta(k+\alpha\beta)^4}{8\alpha^{11}} + \frac{22(k+\alpha\beta)^2\beta^3}{8\alpha^9}$$
$$+ \frac{6\beta(k+\alpha\beta)^5}{8\alpha^{11}k} - \frac{6\beta^3(k+\alpha\beta)^3}{8\alpha^9k} - \frac{3\beta^4(k+\alpha\beta)}{8\alpha^8}$$

$$z_{12} = \frac{73(k+\alpha\beta)^5}{12\alpha^{12}} - \frac{\beta^2(k+\alpha\beta)^3}{\alpha^{10}} + \frac{5\beta(k+\alpha\beta)^4}{2\alpha^{11}} + \frac{5(k+\alpha\beta)^5\ln|k|}{2\alpha^{12}}$$

$$z_{13} = -\frac{5(k+\alpha\beta)^5}{8\alpha^{12}}$$

$$z_{14} = \frac{18\beta^4 \ln(k)}{72\alpha^7} + \frac{108\beta(k+\alpha\beta)^5 \ln(k)}{72\alpha^{10}k^2} - \frac{162(k+\alpha\beta)^2\beta^2 \ln(k)^2}{72\alpha^9}$$
$$- \frac{264(k+\alpha\beta)\beta^3 \ln(k)}{72\alpha^8} - \frac{54\beta^2 \ln(k)(k+\alpha\beta)^2}{72\alpha^9} + \frac{7853(k+\alpha\beta)^4}{72\alpha^{11}}$$
$$+ \frac{72\beta^3(k+\alpha\beta)^3}{72\alpha^8k^2} - \frac{648(k+\alpha\beta)^4}{72\alpha^5} + \frac{18\beta^4}{72\alpha} - \frac{108\beta^2(k+\alpha\beta)^2 \ln(k)}{72\alpha^3}$$
$$- \frac{180\beta^2 \ln(k)(k+\alpha\beta)^3}{72\alpha^9k} + \frac{72\beta^3(k+\alpha\beta)^2 \ln(k)}{72\alpha^8k} - \frac{72\beta(k+\alpha\beta)^4 \ln(k)}{72\alpha^{10}k}$$
$$+ \frac{2520\beta \ln(k)(k+\alpha\beta)^3}{72\alpha^{10}} + \frac{540\beta \ln(k)^2(k+\alpha\beta)^3}{72\alpha^{10}} - \frac{99\beta(k+\alpha\beta)^5}{72\alpha^{10}k^2}$$
$$+ \frac{9\beta^5(k+\alpha\beta)}{72\alpha^6k^2} + \frac{144\beta^2(k+\alpha\beta)^4}{72\alpha^9k^2} - \frac{54\beta^4(k+\alpha\beta)^2}{72\alpha^7k^2} - \frac{18\beta^5}{72\alpha^6k}$$
$$+ \frac{180(k+\alpha\beta)^4 \ln(k)^3}{72\alpha^{11}} + \frac{72(k+\alpha\beta)^5 \ln(k)}{72\alpha^{11}k} + \frac{1530 \ln(k)^2(k+\alpha\beta)^4}{72\alpha^{11}}$$
$$+ \frac{5142 \ln(k)(k+\alpha\beta)^4}{72\alpha^{11}} + \frac{345(k+\alpha\beta)^5}{72\alpha^{11}k} + \frac{432\beta(k+\alpha\beta)^3 \ln(k)}{72\alpha^4}$$
$$- \frac{324(k+\alpha\beta)^4 \ln(k)}{72\alpha^5} - \frac{378(k+\alpha\beta)^6}{72\alpha^{11}k^2} - \frac{216(k+\alpha\beta)^6 \ln(k)}{72\alpha^{11}k^2}$$
$$+ \frac{540\beta(k+\alpha\beta)^3}{72\alpha^4} - \frac{93\beta^4}{72\alpha^7} - \frac{246\beta^3(k+\alpha\beta)}{72\alpha^8} - \frac{24\beta^2(k+\alpha\beta)^2}{72\alpha^9}$$
$$+ \frac{270\beta^2(k+\alpha\beta)^2}{72\alpha^3} - \frac{180\beta^3(k+\alpha\beta)}{72\alpha^2} - \frac{648\beta^2(k+\alpha\beta)^3}{72\alpha^9k} + \frac{177\beta^4(k+\alpha\beta)}{72\alpha^7k}$$
$$+ \frac{3894\beta(k+\alpha\beta)^3}{72\alpha^{10}} - \frac{48\beta^3(k+\alpha\beta)^2}{72\alpha^8k} - \frac{114\beta(k+\alpha\beta)^4}{72\alpha^{10}k}$$

$$z_{15} = -\frac{30(k+\alpha\beta)^5}{24\alpha^{11}k} - \frac{510(k+\alpha\beta)^4 \ln(k)}{24\alpha^{11}} - \frac{90(k+\alpha\beta)^4 \ln(k)^2}{24\alpha^{11}} + \frac{45\beta^2(k+\alpha\beta)^2}{24\alpha^9}$$
$$- \frac{857(k+\alpha\beta)^4}{24\alpha^{11}} + \frac{30\beta^2(k+\alpha\beta)^3}{24\alpha^9k} - \frac{492\beta(k+\alpha\beta)^3}{24\alpha^{10}} + \frac{44(k+\alpha\beta)\beta^3}{24\alpha^8}$$
$$+ \frac{12\beta(k+\alpha\beta)^4}{24\alpha^{10}k} - \frac{12\beta^3(k+\alpha\beta)^2}{24\alpha^8k} - \frac{3\beta^4}{24\alpha^7} + \frac{54(k+\alpha\beta)^2\beta^2 \ln(k)}{24\alpha^9}$$
$$- \frac{180\beta \ln(k)(k+\alpha\beta)^3}{24\alpha^{10}}$$

$$z_{16} = \frac{85(k+\alpha\beta)^4}{12\alpha^{11}} - \frac{3\beta^2(k+\alpha\beta)^2}{4\alpha^9} + \frac{5\beta(k+\alpha\beta)^3}{2\alpha^{10}} + \frac{5(k+\alpha\beta)^4 \ln|k|}{2\alpha^{11}}$$

$$z_{17} = -\frac{5(k+\alpha\beta)^4}{8\alpha^{11}}$$

$$z_{18} = -\frac{3\beta^2(k+\alpha\beta)\ln|k|}{2\alpha^8} + \frac{15\beta(k+\alpha\beta)^2\ln|k|}{2\alpha^9} + \frac{97(k+\alpha\beta)^3}{6\alpha^{10}} - \frac{11\beta^3}{12\alpha^7}$$

$$-\frac{\beta(k+\alpha\beta)^3}{4\alpha^9 k} + \frac{\beta^3(k+\alpha\beta)}{4\alpha^7 k} - \frac{\beta^2(k+\alpha\beta)^2}{\alpha^8 k} + \frac{9\beta^2(k+\alpha\beta)}{8\alpha^8}$$

$$+\frac{16\beta(k+\alpha\beta)^2}{\alpha^9} + \frac{67(k+\alpha\beta)^3\ln|k|}{4\alpha^{10}} + \frac{15(k+\alpha\beta)^3\ln^2|k|}{4\alpha^{10}}$$

$$+\frac{(k+\alpha\beta)^4}{\alpha^{10}k} + \frac{15(k+\alpha\beta)^3}{2\alpha^{11}}$$

$$z_{19} = -\frac{67(k+\alpha\beta)^2}{8\alpha^{10}} + \frac{3\beta^2}{4\alpha^8} - \frac{15\beta(k+\alpha\beta)}{4\alpha^9} - \frac{15(k+\alpha\beta)^2\ln|k|}{4\alpha^{10}}$$

$$z_{20} = \frac{5(k+\alpha\beta)^3}{4\alpha^{10}}$$

$$z_{21} = \frac{67(k+\alpha\beta)^2}{36\alpha^9} + \frac{5\beta(k+\alpha\beta)}{4\alpha^8} + \frac{5(k+\alpha\beta)^2\ln|k|}{4\alpha^9} - \frac{\beta^2}{8\alpha^7}$$

$$z_{22} = -\frac{5(k+\alpha\beta)^2}{8\alpha^9}$$

$$z_{23} = \frac{5(k+\alpha\beta)}{36\alpha^8}$$

$$z_{24} = -\frac{9810\beta^4(k+\alpha\beta)^3}{4320\alpha^8 k^2} - \frac{5400\beta\ln(k)(k+\alpha\beta)^3}{4320\alpha^{11}} + \frac{16200\beta^2(k+\alpha\beta)^3}{4320\alpha^{11}}$$
$$- \frac{209970\beta^2(k+\alpha\beta)^3}{4320\alpha^{10}} - \frac{7460\beta^3(k+\alpha\beta)^2}{4320\alpha^9} - \frac{1710\beta^6}{4320\alpha^6 k} - \frac{360\beta^5}{4320\alpha}$$
$$+ \frac{19800\beta^2(k+\alpha\beta)^5}{4320\alpha^{10}k^2} + \frac{26730\beta^2(k+\alpha\beta)^4}{4320\alpha^{10}k} - \frac{19440\beta^2(k+\alpha\beta)^3}{4320\alpha^4}$$
$$+ \frac{49860\beta^3(k+\alpha\beta)^3}{4320\alpha^9 k} - \frac{3420\beta^5(k+\alpha\beta)}{4320\alpha^7 k} + \frac{8670\beta^4(k+\alpha\beta)^2}{4320\alpha^8 k}$$
$$- \frac{18360\beta^3(k+\alpha\beta)^2}{4320\alpha^3} + \frac{5400\beta^4(k+\alpha\beta)}{4320\alpha^2} + \frac{4140\beta^6(k+\alpha\beta)}{4320\alpha^6 k^2}$$
$$+ \frac{6570\beta^3(k+\alpha\beta)^4}{4320\alpha^9 k^2} - \frac{6030\beta^5(k+\alpha\beta)^2}{4320\alpha^7 k^2} - \frac{402540\beta(k+\alpha\beta)^4}{4320\alpha^{11}}$$
$$+ \frac{34110\beta(k+\alpha\beta)^6}{4320\alpha^{11}k^2} + \frac{10800(k+\alpha\beta)^5}{4320\alpha^{12}k} + \frac{180\beta^7(k+\alpha\beta)}{4320\alpha^5 k^3}$$
$$- \frac{1620\beta^6(k+\alpha\beta)^2}{4320\alpha^6 k^3} - \frac{10260\beta^3(k+\alpha\beta)^5}{4320\alpha^9 k^3} + \frac{3240\beta^5(k+\alpha\beta)^3}{4320\alpha^7 k^3}$$
$$- \frac{3780\beta^2(k+\alpha\beta)^6}{4320\alpha^{10}k^3} - \frac{270\beta^7}{4320\alpha^5 k^2} + \frac{12240\beta(k+\alpha\beta)^7}{4320\alpha^{11}k^3}$$
$$- \frac{1590(k+\alpha\beta)^7}{4320\alpha^{12}k^2} - \frac{6390(k+\alpha\beta)^6}{4320\alpha^{12}k} - \frac{5400\beta^2(k+\alpha\beta)^2}{4320\alpha^{10}}$$
$$- \frac{6240\beta(k+\alpha\beta)^3}{4320\alpha^{11}} + \frac{540(k+\alpha\beta)\beta^3}{4320\alpha^9} - \frac{32160\beta(k+\alpha\beta)^5}{4320\alpha^{11}k}$$
$$+ \frac{12960\beta(k+\alpha\beta)^5}{4320\alpha^5 k} + \frac{38880\beta(k+\alpha\beta)^4}{4320\alpha^5} - \frac{26280(k+\alpha\beta)^5\ln(k)^3}{4320\alpha^{12}}$$
$$- \frac{9720(k+\alpha\beta)^6\ln(k)}{4320\alpha^{12}k} - \frac{2520\beta^5\ln(k)}{4320\alpha^7} - \frac{2700(k+\alpha\beta)^5\ln(k)^4}{4320\alpha^{12}}$$
$$+ \frac{32820(k+\alpha\beta)^7\ln(k)}{4320\alpha^{12}k^2} - \frac{417360\ln(k)(k+\alpha\beta)^5}{4320\alpha^{12}} - \frac{9900\ln(k)(k+\alpha\beta)^2}{4320\alpha^{12}}$$
$$+ \frac{51840\ln(k)(k+\alpha\beta)^5}{4320\alpha^6} + \frac{19440\ln(k)^2(k+\alpha\beta)^7}{4320\alpha^{12}k^2} + \frac{25920\ln(k)^2(k+\alpha\beta)^5}{4320\alpha^6}$$
$$- \frac{121500\ln(k)^2(k+\alpha\beta)^5}{4320\alpha^{12}} + \frac{6480(k+\alpha\beta)^6\ln(k)^2}{4320\alpha^{12}k} + \frac{8880\beta(k+\alpha\beta)^7\ln(k)}{4320\alpha^{11}k^3}$$
$$- \frac{6480\beta^2(k+\alpha\beta)^6\ln(k)}{4320\alpha^{10}k^3} + \frac{6480\beta^2\ln(k)^2(k+\alpha\beta)^4}{4320\alpha^{10}k} + \frac{12960\beta^2\ln(k)^2(k+\alpha\beta)^3}{4320\alpha^4}$$
$$- \frac{2160\beta^4\ln(k)(k+\alpha\beta)}{4320\alpha^2} + \frac{32400\beta^2\ln(k)(k+\alpha\beta)^3}{4320\alpha^{11}} + \frac{19440\beta(k+\alpha\beta)^6\ln(k)}{4320\alpha^{11}k^2}$$
$$- \frac{32400\beta(k+\alpha\beta)^4\ln(k)}{4320\alpha^5} - \frac{142200\beta(k+\alpha\beta)^4\ln(k)}{4320\alpha^{11}} - \frac{10800(k+\alpha\beta)^4\beta\ln(k)^3}{4320\alpha^{11}}$$
$$- \frac{36720\beta(k+\alpha\beta)^4\ln(k)^2}{4320\alpha^{11}} - \frac{12960\beta\ln(k)^2(k+\alpha\beta)^6}{4320\alpha^{11}k^2} + \frac{6480\beta^3(k+\alpha\beta)^3\ln(k)}{4320\alpha^3 k}$$
$$- \frac{1080\beta^5(k+\alpha\beta)^2\ln(k)}{4320\alpha^7 k^2} - \frac{38880\beta\ln(k)^2(k+\alpha\beta)^4}{4320\alpha^5} + \frac{6480\beta(k+\alpha\beta)^5\ln(k)}{4320\alpha^5 k}$$

z_{24} continued

$$+ \frac{3240\beta(k+\alpha\beta)^5 \ln(k)^2}{4320\alpha^{11}k} + \frac{27720\beta \ln(k)(k+\alpha\beta)^5}{4320\alpha^{11}k} - \frac{16740(k+\alpha\beta)^3\beta^2 \ln(k)^2}{4320\alpha^{10}}$$

$$- \frac{1620(k+\alpha\beta)\beta^4 \ln(k)^2}{4320\alpha^8} + \frac{6600\beta^3 \ln(k)(k+\alpha\beta)^2}{4320\alpha^9} + \frac{21600\beta^3(k+\alpha\beta)^2 \ln(k)}{4320\alpha^3}$$

$$- \frac{22320\beta^4(k+\alpha\beta)^2 \ln(k)}{4320\alpha^8 k} + \frac{64800\beta^2(k+\alpha\beta)^4 \ln(k)}{4320\alpha^{10}k} + \frac{16560\beta^2(k+\alpha\beta)^3 \ln(k)}{4320\alpha^{10}}$$

$$+ \frac{4320(k+\alpha\beta)^3\beta^2 \ln(k)^3}{4320\alpha^{10}} + \frac{11880(k+\alpha\beta)^2\beta^3 \ln(k)^2}{4320\alpha^9} + \frac{10620(k+\alpha\beta)\beta^4 \ln(k)}{4320\alpha^8}$$

$$+ \frac{32400 \ln(k)(k+\alpha\beta)^5}{4320\alpha^{13}} - \frac{36720\beta^2(k+\alpha\beta)^3 \ln(k)}{4320\alpha^4} - \frac{20520\beta^2(k+\alpha\beta)^5 \ln(k)}{4320\alpha^{10}k^2}$$

$$- \frac{11880\beta^3 \ln(k)(k+\alpha\beta)^3}{4320\alpha^9 k} + \frac{4320\beta^5 \ln(k)(k+\alpha\beta)}{4320\alpha^7 k} - \frac{6480\beta^3 \ln(k)(k+\alpha\beta)^4}{4320\alpha^9 k^2}$$

$$- \frac{12960\beta^2(k+\alpha\beta)^4 \ln(k)}{4320\alpha^4 k} - \frac{3240\beta^3(k+\alpha\beta)^3 \ln(k)^2}{4320\alpha^9 k} + \frac{5940\beta^4(k+\alpha\beta)^3 \ln(k)}{4320\alpha^8 k^2}$$

$$- \frac{10800\beta(k+\alpha\beta)^6}{4320\alpha^{11}k^2} - \frac{1080\beta^3(k+\alpha\beta)^3}{4320\alpha^3 k} + \frac{8640\beta^4(k+\alpha\beta)^2}{4320\alpha^2 k}$$

$$- \frac{1080\beta^5(k+\alpha\beta)}{4320\alpha k} - \frac{19440\beta^2(k+\alpha\beta)^4}{4320\alpha^4 k} + \frac{32400\beta(k+\alpha\beta)^4}{4320\alpha^{12}} + \frac{3504\beta^5}{4320\alpha^7}$$

$$+ \frac{2160\beta^4(k+\alpha\beta)}{4320\alpha^6} + \frac{7725\beta^4(k+\alpha\beta)}{4320\alpha^8}$$

A.2   Falkner-Skan Equation

The functions for the coeffiecients of the solution equations for the transformed Falkner-Skan equation considered in Chapter 3 are given below.

$p_1 - p_5$ are the coefficients for y_1.

$$p_1 = -1/2 \frac{\beta^3 (\beta - 1)}{k^3}$$

$$p_2 = -\frac{k^4 + k^4\beta + \beta^3 + k^4 \ln(k)}{\beta^2 k}$$

$$p_3 = -\frac{k (\beta^3 + k^4 + k^4\beta)}{\beta^3}$$

$$p_4 = \frac{k^3}{\beta^2}$$

$$p_5 = \frac{k (\beta^3 + k^4 + k^4\beta) \ln(k)}{\beta^3}$$

$q_1 - q_{11}$ are the coefficients for y_2.

$$q_1 = \frac{1}{12}\beta (\beta - 1)(2\beta - 1)$$

$$q_2 = \frac{1}{4\beta^4}k \left(k^4\beta - 5 k^4 + 2\beta^6 - 4\beta^5 + 2 k^4\beta^2 + 4\beta^4 + 2\beta k^4 \ln(k) - 2\beta^3\right.$$
$$\left. -2 k^4 \ln(k)\right)$$

$$q_3 = -\frac{1}{2\beta^5 k} \left(-6 k^4\beta^3 + \beta^8 - 8 k^8 \ln(k) - \beta^7 + 2\beta^4 k^4 + k^8 (\ln(k))^2 \beta\right.$$
$$+k^4 (\ln(k))^2 \beta^3 - 3\beta^5 k^4 \ln(k) - 4 k^8 \ln(k) \beta + 5\beta^4 k^4 \ln(k) + 2\beta^2 k^8 \ln(k)$$
$$\left.-2 k^4 \ln(k) \beta^3 - 9 k^8 - 3 k^4\beta^6 - k^4\beta^5 - 13 k^8\beta - 2 k^8\beta^2 - k^8 (\ln(k))^2\right)$$

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$$q_4 = -\frac{1}{2\beta^6}k\left(2\beta^6 - 4k^4\beta^3 + 2\beta^8 - 2k^8\ln(k) - 2\beta^7 + 2\beta^4k^4 + 2k^8\ln(k)\beta\right.$$
$$\left. + 4\beta^4k^4\ln(k) + 4\beta^2k^8\ln(k) + 6k^4\ln(k)\beta^3 - 9k^8 - 4k^4\beta^6 - 15k^8\beta - 4k^8\beta^2\right)$$

$$q_5 = -\frac{1}{2\beta^5}k^3\left(2\beta^3 + 4k^4\beta - 2k^4\beta^2 - 5\beta^4 - 2\beta k^4\ln(k) + 2k^4\ln(k) - 2\ln(k)\beta^3\right.$$
$$\left. + 8k^4 + 3\beta^5\right)$$

$$q_6 = -\frac{1}{2\beta^4}k^5\left(\beta - 1\right)$$

$$q_7 = \frac{1}{2\beta^6}k^5\left(-k^4 + k^4\beta + 2\beta^4 + 2k^4\beta^2 + 3\beta^3\right)$$

$$q_8 = -\frac{1}{2\beta^5}k^3\left(-k^4 + \beta^3 + k^4\beta\right)$$

$$q_9 = -\frac{1}{2\beta^5}k^2\left(4\beta^5 + 2k^4\beta^2 + \beta^7 - \beta^6 + 2k^8\ln(k)\beta - 2\ln(k)\beta^5 - k^4\beta^5 + \beta^4k^4\right.$$
$$\left. - 2k^8 - 4k^8\beta\right)$$

$$q_{10} = -\frac{k^2}{\beta^4}\left(-k^8 + \beta^4\right)$$

$$q_{11} = \frac{1}{2\beta^6}k\left(4\beta^6 + 2k^4\beta^3 + 2\beta^4k^4\left(\ln(k)\right)^2 + \beta^8 - 9k^8\ln(k) - \beta^7 + k^8\left(\ln(k)\right)^2\beta\right.$$
$$+ 3k^4\left(\ln(k)\right)^2\beta^3 - 4\beta^6k^4\ln(k) - 15k^8\ln(k)\beta + 2\beta^4k^4\ln(k) - 4\beta^2k^8\ln(k)$$
$$- 4k^4\ln(k)\beta^3 + 2\ln(k)\beta^6 + 2\ln(k)\beta^8 - 2\ln(k)\beta^7 - k^4\beta^6 + k^4\beta^5 - 2k^8\beta$$
$$\left. - 4k^8\beta^2 - k^8\left(\ln(k)\right)^2 + 2\beta^2k^8\left(\ln(k)\right)^2\right)$$

$r_1 - r_{25}$ are the coefficients for y_3.

$$r_1 = -\frac{1}{144\beta\,k^3}\left(\beta - 1\right)\left(-12\beta^5 + 3\beta^4 + 6k^4\beta^2 - 5k^4\beta + k^4 + 9\beta^6\right)$$

$$r_2 = -\frac{1}{72\beta^6 k}\left(44\,k^4\beta^7 + 12\,k^8\beta^3 - 18\,k^8\ln(k)\,\beta + 12\,\beta^4 k^4\ln(k) + 12\,\beta^2 k^8\ln(k)\right.$$

$$+6\,k^8\ln(k) + 6\,k^4\beta^3 - 36\,\beta^5 k^4\ln(k) + 60\,\beta^9 - 45\,\beta^{10} + 18\,\beta^{11} - 45\,\beta^8 + 12\,\beta^7$$

$$\left. -4\,\beta^4 k^4 + 20\,k^8 + 24\,\beta^6 k^4\ln(k) - 50\,k^4\beta^6 + 4\,k^4\beta^5 - 42\,k^8\beta + 4\,k^8\beta^2\right)$$

$$r_3 = \frac{1}{24\beta^7 k^3}\left(108\,k^{12}\beta - 23\,k^4\beta^7 - 24\,\beta^8 k^4\ln(k) + 12\,\beta^7 k^4\ln(k) - 6\,k^8\left(\ln(k)\right)^2\beta^3\right.$$

$$+18\,k^8\ln(k)\,\beta^3 + 81\,k^8\beta^3 + 12\,\beta^7 k^8\ln(k) + 6\,k^8\beta^4\left(\ln(k)\right)^2 - 24\,k^{12}\beta\left(\ln(k)\right)^2$$

$$+6\,k^{12}\beta^2\left(\ln(k)\right)^2 - 66\,k^{12}\beta^2\ln(k) - 84\,k^8\beta^4\ln(k) - 66\,k^8\beta^6\ln(k) + 108\,k^8\beta^5\ln(k)$$

$$+12\,k^{12}\beta^3\ln(k) + 12\,\beta^9 k^4\ln(k) - 36\,k^{12}\beta\ln(k) + 90\,k^8\beta^5 + 3\,\beta^{11} + 3\,\beta^{13} - 6\,\beta^{12}$$

$$+185\,k^{12} - 30\,k^{12}\beta^3 - 119\,k^{12}\beta^2 + 114\,k^{12}\ln(k) - 149\,k^8\beta^4 + 18\,k^{12}\left(\ln(k)\right)^2$$

$$\left. -66\,k^8\beta^7 - 76\,k^4\beta^9 + 51\,k^4\beta^8 + 50\,k^8\beta^6 - 18\,\beta^{11}k^4 + 12\,k^4\beta^6 + 54\,\beta^{10}k^4\right)$$

$$r_4 = \frac{1}{24\beta^8 k}\left(-1206\,k^{12}\beta + 88\,k^4\beta^7 - 144\,\beta^8 k^4\ln(k) + 60\,\beta^7 k^4\ln(k)\right.$$

$$+24\,k^8\left(\ln(k)\right)^2\beta^3 - 300\,k^8\ln(k)\,\beta^3 - 540\,k^8\beta^3 + 120\,\beta^7 k^8\ln(k) + 160\,k^8\beta^4\left(\ln(k)\right)^2$$

$$+44\,k^{12}\beta\left(\ln(k)\right)^2 + 124\,k^{12}\beta^2\left(\ln(k)\right)^2 + 144\,k^{12}\beta^2\ln(k) + 352\,k^8\beta^4\ln(k)$$

$$-328\,k^8\beta^6\ln(k) - 60\,k^8\beta^5\ln(k) + 96\,k^{12}\beta^3\ln(k) + 168\,\beta^9 k^4\ln(k) - 672\,k^{12}\beta\ln(k)$$

$$+166\,k^8\beta^5 - 8\,k^8\left(\ln(k)\right)^3\beta^4 + 20\,k^{12}\left(\ln(k)\right)^3\beta - 8\,k^{12}\left(\ln(k)\right)^3\beta^2 + 12\,k^8\left(\ln(k)\right)^3\beta^3$$

$$+24\,k^4\left(\ln(k)\right)^2\beta^8 + 60\,k^8\left(\ln(k)\right)^2\beta^6 - 60\,k^4\ln(k)\,\beta^{10} - 16\left(\ln(k)\right)^2\beta^7 k^4$$

$$-12\,k^{12}\left(\ln(k)\right)^3 - 24\,\beta^{10} + 42\,\beta^{11} + 18\,\beta^{13} - 36\,\beta^{12} - 638\,k^{12} - 628\,k^{12}\beta^2$$

$$-636\,k^{12}\ln(k) - 404\,k^8\beta^4 - 144\,k^{12}\left(\ln(k)\right)^2 - 124\,k^8\beta^7 + 86\,k^4\beta^9 - 231\,k^4\beta^8$$

$$-96\,\beta^6 k^4\ln(k) - 574\,k^8\beta^6 - 60\,\beta^{11}k^4 - 360\,k^4\beta^6 - 116\,k^8\beta^5\left(\ln(k)\right)^2$$

$$\left. -16\,k^{12}\beta^3\left(\ln(k)\right)^2 + 12\,\beta^6 k^4\left(\ln(k)\right)^2 + 33\,\beta^{10}k^4\right)$$

$$r_5 = \frac{1}{12\beta^9} k \left(-690\,k^{12}\beta - 27\,k^4\beta^7 + 54\,\beta^8 k^4 \ln(k) - 42\,\beta^7 k^4 \ln(k) + 48\,k^8 (\ln(k))^2 \beta^3\right.$$

$$+90\,k^8 \ln(k)\,\beta^3 - 243\,k^8\beta^3 + 120\,\beta^7 k^8 \ln(k) + 48\,k^{12}\beta\,(\ln(k))^2 + 42\,k^{12}\beta^2\,(\ln(k))^2$$

$$+318\,k^{12}\beta^2 \ln(k) + 420\,k^8\beta^4 \ln(k) - 54\,k^8\beta^6 \ln(k) + 48\,k^8\beta^5 \ln(k) + 120\,k^{12}\beta^3 \ln(k)$$

$$+36\,\beta^9 k^4 \ln(k) - 36\,k^{12}\beta \ln(k) - 12\ln(k)\,\beta^{11} + 12\ln(k)\,\beta^{10} + 88\,k^8\beta^5 + 12\,\beta^9$$

$$-30\,\beta^{10} + 48\,\beta^{11} + 18\,\beta^{13} - 36\,\beta^{12} - 325\,k^{12} - 6\,k^{12}\beta^3 - 443\,k^{12}\beta^2 - 138\,k^{12}\ln(k)$$

$$-247\,k^8\beta^4 - 18\,k^{12}(\ln(k))^2 - 124\,k^8\beta^7 + 27\,k^4\beta^9 - 84\,k^4\beta^8 + 120\,\beta^6 k^4 \ln(k)$$

$$-296\,k^8\beta^6 - 45\,\beta^{11}k^4 - 156\,k^4\beta^6 - 24\,k^8\beta^5 (\ln(k))^2 - 24\,k^{12}\beta^3 (\ln(k))^2$$

$$\left.-18\,\beta^6 k^4 (\ln(k))^2 + 45\,\beta^{10}k^4\right)$$

$$r_6 = \frac{k^3}{6\beta^8} \left(-30\,k^4\beta^7 + 12\,k^8 \ln(k)\,\beta^3 - 24\,k^8\beta^3 - 6\ln(k)\,\beta^6 - 12\ln(k)\,\beta^8 + 12\ln(k)\,\beta^7\right.$$

$$+24\,\beta^6 + 75\,k^4\beta^3 - 42\,\beta^9 + 15\,\beta^{10} + 36\,\beta^8 - 15\,\beta^7 - 88\,\beta^4 k^4 + 72\,k^8 \ln(k)$$

$$-18\,k^8 \ln(k)\,\beta - 72\,\beta^4 k^4 \ln(k) - 54\,\beta^2 k^8 \ln(k) - 30\,\beta^6 k^4 \ln(k) + 6\,\beta^2 k^8 (\ln(k))^2$$

$$+159\,k^8 + 9\,k^8 (\ln(k))^2 + 82\,k^4\beta^6 + 15\,k^4\beta^5 + 168\,k^8\beta - 36\,k^8\beta^2 - 15\,k^8 (\ln(k))^2 \beta$$

$$\left.-9\,k^4 (\ln(k))^2 \beta^3 + 66\,\beta^5 k^4 \ln(k) - 12\,k^4 \ln(k)\,\beta^3 + 6\,\beta^4 k^4 (\ln(k))^2\right)$$

$$r_7 = -\frac{k}{4\beta^7} \left(2\,k^4\beta^7 + 2\,\beta^2 k^8 \ln(k) + 2\,\beta^4 k^4 \ln(k) + 2\,k^8\beta^3 + 3\,k^4\beta^3 + 2\,\beta^9 + 6\,k^8 \ln(k)\right.$$

$$-4\,\beta^8 + 2\,\beta^7 - 14\,\beta^4 k^4 + 19\,k^8 - 11\,k^4\beta^6 + 18\,k^4\beta^5 - 6\,k^8\beta - 11\,k^8\beta^2$$

$$\left.-8\,k^8 \ln(k)\,\beta - 2\,k^4 \ln(k)\,\beta^3\right)$$

$$r_8 = \frac{k^3}{12\beta^6} (\beta - 1) \left(-k^4 - 2\,\beta^4 + 4\,\beta^5 + 2\,k^4\beta\right)$$

$$r_9 = -\frac{k^5}{4\beta^9} \left(12\,k^4\beta^7 - 4\,\beta^2 k^8 \ln(k) - 4\,\beta^4 k^4 \ln(k) + 8\,k^8\beta^3 + 14\,\beta^6 + 17\,k^4\beta^3 + 12\,\beta^9\right.$$

$$+12\,k^8 \ln(k) - 8\,\beta^8 + 25\,\beta^4 k^4 + 7\,k^8 - 10\,k^4\beta^6 + 12\,k^4\beta^5 - 12\,k^8\beta + 5\,k^8\beta^2$$

$$\left.-8\,k^8 \ln(k)\,\beta + 12\,k^4 \ln(k)\,\beta^3\right)$$

$$r_{10} = \frac{k^3}{2\beta^8} \left(-2\,\beta^2 k^8 \ln(k) - 2\,\beta^4 k^4 \ln(k) - 2\,k^8 \beta^3 + \beta^6 + 2\,k^4 \beta^3 - 3\,k^8 \ln(k) + 2\,\beta^8 \right.$$

$$-2\,\beta^7 + 12\,\beta^4 k^4 - 12\,k^8 + 5\,k^4 \beta^6 - 11\,k^4 \beta^5 + 3\,k^8 \beta + 9\,k^8 \beta^2 + 5\,k^8 \ln(k)\,\beta$$

$$\left. +3\,k^4 \ln(k)\,\beta^3 \right)$$

$$r_{11} = \frac{k^5}{4\beta^7} \left(3\,k^4 - \beta^3 - 4\,k^4 \beta + \beta^4 + k^4 \beta^2 \right)$$

$$r_{12} = -\frac{k^5}{6\beta^9} \left(3\,k^8 - 8\,k^4 \beta^3 - 6\,k^8 \beta + 4\,\beta^4 k^4 - 3\,k^8 \beta^2 + 6\,k^8 \beta^3 + 3\,\beta^6 + 2\,\beta^7 + 8\,k^4 \beta^5 \right)$$

$$r_{13} = \frac{k^7}{6\beta^8} \left(3\,k^4 - 3\,\beta^3 - 5\,k^4 \beta + 2\,\beta^4 + 2\,k^4 \beta^2 \right)$$

$$r_{14} = \frac{k^2}{24\beta^9} \left(-198\,k^{12}\beta - 238\,k^4 \beta^7 + 48\,\beta^8 k^4 \ln(k) + 48\,\beta^7 k^4 \ln(k) + 48\,k^8 \ln(k)\,\beta^3 \right.$$

$$+30\,k^8 \beta^3 + 144\,\beta^7 k^8 \ln(k) + 24\,k^{12}\beta\,(\ln(k))^2 + 84\,k^{12}\beta^2\,(\ln(k))^2$$

$$+168\,k^{12}\beta^2 \ln(k) + 168\,k^8 \beta^4 \ln(k) - 144\,k^8 \beta^6 \ln(k) + 408\,k^8 \beta^5 \ln(k) + 312\,k^{12}\beta^3 \ln(k)$$

$$-24\,\beta^9 k^4 \ln(k) - 24\,\ln(k)\,\beta^{11} + 24\,\ln(k)\,\beta^{10} - 446\,k^8 \beta^5 + 120\,\beta^9 - 96\,\beta^{10} + 120\,\beta^{11}$$

$$+24\,\beta^{13} - 48\,\beta^{12} - 12\,k^{12} - 300\,k^{12}\beta^3 - 24\,\beta^9 \ln(k) - 450\,k^{12}\beta^2 - 306\,k^8 \beta^4 - 280\,k^8 \beta^7$$

$$-38\,k^4 \beta^9 + 63\,k^4 \beta^8 + 96\,\beta^6 k^4 \ln(k) + 114\,k^8 \beta^6 - 36\,\beta^{11} k^4 + 24\,k^4 \beta^6 - 48\,k^8 \beta^5\,(\ln(k))^2$$

$$\left. -48\,k^{12}\beta^3\,(\ln(k))^2 - 36\,\beta^6 k^4\,(\ln(k))^2 + 69\,\beta^{10} k^4 \right)$$

$$r_{15} = -\frac{k^3}{24\beta^8} \left(-24\,k^{12}\beta + 24\,k^4 \beta^7 + 8\,k^8\,(\ln(k))^2\,\beta^3 - 24\,k^8 \beta^3 - 48\,\ln(k)\,\beta^8 \right.$$

$$-4\,k^8 \beta^4\,(\ln(k))^2 + 8\,k^{12}\beta\,(\ln(k))^2 - 8\,k^{12}\beta^2\,(\ln(k))^2 + 24\,k^{12}\beta^2 \ln(k)$$

$$+48\,k^8 \beta^4 \ln(k) + 12\,k^8 \beta^6 \ln(k) - 12\,k^8 \beta^5 \ln(k) + 24\,k^{12}\beta \ln(k) - 12\,\ln(k)\,\beta^{10}$$

$$-24\,\beta^9 + 27\,\beta^{10} - 6\,\beta^{11} + 3\,\beta^{12} - 12\,k^{12} + 12\,\beta^9 \ln(k) - 48\,k^8 \beta^4 + 60\,\beta^8 + 4\,k^{12}\,(\ln(k))^2$$

$$+6\,k^4 \beta^9 - 15\,k^4 \beta^8 - 24\,\beta^6 k^4 \ln(k) - 12\,k^8 \beta^6 - 12\,k^4 \beta^6 + 48\,k^4 \beta^5 + 12\,k^8 \beta^2$$

$$\left. +16\,\beta^6 k^4\,(\ln(k))^2 - 3\,\beta^{10} k^4 - 24\,\beta^5 k^4 \ln(k) + 12\,(\ln(k))^2\,\beta^8 \right)$$

$$r_{16} = \frac{k^2}{\beta^7} \left(-2\,k^4\beta^5 - 2\,k^8\beta + 2\,\beta^5 k^4 \ln(k) + 3\,\beta^4 k^4 \ln(k) - 3\,k^{12}\ln(k) + 8\,k^8\ln(k)\,\beta^3 \right.$$

$$+ 4\,\beta^2 k^8 \ln(k) + 6\,k^{12}\beta\,\ln(k) - 17\,k^8\beta^3 + \beta^7 - \beta^8 + \beta^9 - 13\,k^{12}\beta - 7\,k^{12} - 7\,k^8\beta^2$$

$$\left. - 6\,k^8\beta^5 + k^4\beta^7 - 2\,k^4\beta^6 + 6\,k^8\beta^4 - 4\,\beta^4 k^4 \right)$$

$$r_{17} = -\frac{k^3}{2\beta^7} \left(2\,k^4\beta^5 - 4\,k^8\beta^3 + 4\,\beta^7 - \beta^8 + \beta^9 - 2\,k^{12}\beta - 2\,k^{12} - k^8\beta^5 - 2\ln(k)\,\beta^7 \right.$$

$$\left. + k^8\beta^4 + 2\,\beta^4 k^4 - 2\,\beta^5 k^4 \ln(k) + 2\,k^8\ln(k)\,\beta^3 + 2\,k^{12}\beta\,\ln(k) \right)$$

$$r_{18} = -\frac{k^6}{2\beta^9} \left(-6\,k^8 + 8\,k^8\beta + 2\,\beta^4 k^4 - 5\,k^8\beta^2 + 2\,k^4\beta^3 + 6\,k^8\beta^3 + 3\,\beta^6 + 2\,\beta^7 + 8\,k^4\beta^5 \right)$$

$$r_{19} = -\frac{k^3}{2\beta^6} \left(\beta^4 k^4 - k^8\beta^2 + \beta^6 - k^{12} \right)$$

$$r_{20} = -\frac{k}{24\beta^9} \left(-186\,k^{12}\beta - 226\,k^4\beta^7 - 168\,\beta^8 k^4 \ln(k) - 54\,\beta^7 k^4 \ln(k) + 54\,k^8 \left(\ln(k)\right)^2 \beta^3 \right.$$

$$- 486\,k^8\ln(k)\,\beta^3 + 18\,k^8\beta^3 - 248\,\beta^7 k^8 \ln(k) + 754\,k^8\beta^4 \left(\ln(k)\right)^2 - 76\,k^{12}\beta \left(\ln(k)\right)^2$$

$$+ 670\,k^{12}\beta^2 \left(\ln(k)\right)^2 - 886\,k^{12}\beta^2 \ln(k) - 494\,k^8\beta^4 \ln(k) - 592\,k^8\beta^6 \ln(k)$$

$$+ 176\,k^8\beta^5 \ln(k) - 12\,k^{12}\beta^3 \ln(k) + 54\,\beta^9 k^4 \ln(k) - 1380\,k^{12}\beta\,\ln(k)$$

$$+ 168\,\beta^7 k^8 \left(\ln(k)\right)^2 - 24 \left(\ln(k)\right)^2 \beta^{11} + 36\ln(k)\,\beta^{13} - 72\ln(k)\,\beta^{12} + 96\ln(k)\,\beta^{11}$$

$$- 60\ln(k)\,\beta^{10} - 398\,k^8\beta^5 - 48\,\beta^6 k^4 \left(\ln(k)\right)^3 - 72\,k^{12}\beta^3 \left(\ln(k)\right)^3 - 8\,k^4 \left(\ln(k)\right)^3 \beta^7$$

$$- 80\,k^8\beta^5 \left(\ln(k)\right)^3 - 90\ln(k)\,\beta^{11} k^4 + 24 \left(\ln(k)\right)^2 \beta^{10} + 8\,k^8 \left(\ln(k)\right)^3 \beta^4$$

$$+ 168\,k^{12} \left(\ln(k)\right)^3 \beta + 120\,k^{12} \left(\ln(k)\right)^3 \beta^2 + 56\,k^8 \left(\ln(k)\right)^3 \beta^3 + 156\,k^4 \left(\ln(k)\right)^2 \beta^8$$

$$- 48\,k^8 \left(\ln(k)\right)^2 \beta^6 + 90\,k^4 \ln(k)\,\beta^{10} - 64 \left(\ln(k)\right)^2 \beta^7 k^4 - 120\,k^{12} \left(\ln(k)\right)^3 + 60\,\beta^9$$

$$- 72\,\beta^{10} + 93\,\beta^{11} + 21\,\beta^{13} - 42\,\beta^{12} - 12\,k^{12} - 300\,k^{12}\beta^3 + 24\,\beta^9 \ln(k) - 426\,k^{12}\beta^2$$

$$- 650\,k^{12}\ln(k) - 282\,k^8\beta^4 - 246\,k^{12} \left(\ln(k)\right)^2 - 268\,k^8\beta^7 - 23\,k^4\beta^9 + 39\,k^4\beta^8$$

$$- 312\,\beta^6 k^4 \ln(k) + 114\,k^8\beta^6 - 33\,\beta^{11} k^4 - 24\,k^4\beta^6 + 64\,k^8\beta^5 \left(\ln(k)\right)^2$$

$$\left. + 212\,k^{12}\beta^3 \left(\ln(k)\right)^2 + 156\,\beta^6 k^4 \left(\ln(k)\right)^2 + 63\,\beta^{10} k^4 \right)$$

A.3   Orr-Sommerfeld Equation

The coefficients from the solution equations for the Orr-Sommerfeld equation considered in Chapter 4 are given below.

$c_1 - c_{10}$ are the coefficients for v_1.

$$c_1 = -\frac{1}{18144}i\alpha^3 Rk2$$

$$c_2 = -\frac{1}{3360}i\alpha^3 R(k1 + k2)$$

$$c_3 = -\frac{1}{2520}i\alpha^3 Rk2 + \frac{1}{5040}\alpha^4 k2 - \frac{1}{5040}i\lambda\alpha^3 Rk2 - \frac{1}{840}i\alpha^3 Rk1$$
$$+ \frac{1}{1260}i\alpha Rk2$$

$$c_4 = \frac{1}{1080}i\alpha^3 Rk2 + \frac{1}{720}\alpha^4 k1 + \frac{1}{720}\alpha^4 k2 - \frac{1}{720}i\lambda\alpha^3 Rk1 - \frac{1}{720}i\lambda\alpha^3 Rk2$$

$$c_5 = -\frac{1}{60}i\alpha Rk2 - \frac{1}{60}i\alpha Rk1 + \frac{1}{240}i\alpha^3 Rk2 + \frac{1}{120}i\lambda\alpha Rk2 + \frac{1}{120}\alpha^4 k1$$
$$+ \frac{1}{240}\alpha^4 k2 - \frac{1}{240}i\lambda\alpha^3 Rk2 + \frac{1}{120}i\alpha^3 Rk1 - \frac{1}{120}i\lambda\alpha^3 Rk1 - \frac{2}{120}\alpha^2 k2$$

$$c_6 = -\frac{1}{144}i\lambda\alpha^3 Rk2 + \frac{1}{144}i\alpha^3 Rk2 - \frac{1}{18}i\alpha Rk2 - \frac{1}{48}i\lambda\alpha^3 Rk1 + \frac{1}{144}\alpha^4 k2$$
$$+ \frac{1}{48}i\alpha^3 Rk1 - \frac{1}{12}i\alpha Rk1 + \frac{1}{24}i\lambda\alpha Rk1 - \frac{2}{24}\alpha^2 k1 + \frac{1}{24}i\lambda\alpha Rk2$$
$$- \frac{2}{24}\alpha^2 k2 + \frac{1}{48}\alpha^4 k1$$

$$c_7 = -\frac{1}{36}i\lambda\alpha^3 Rk1 + \frac{7}{1080}i\alpha^3 Rk2 - \frac{2}{12}\alpha^2 k2 + \frac{1}{40}i\alpha^3 Rk1 - \frac{1}{6}i\alpha Rk1$$
$$+ \frac{1}{36}\alpha^4 k1 - \frac{1}{144}i\lambda\alpha^3 Rk2 - \frac{1}{12}i\alpha Rk2 + \frac{1}{12}i\lambda\alpha Rk2 + \frac{1}{6}i\lambda\alpha Rk1$$
$$- \frac{2}{6}\alpha^2 k1 + \frac{1}{144}\alpha^4 k2$$

$$c_8 = \frac{1}{48}\alpha^4 k1 - \frac{1}{240}i\lambda\alpha^3 Rk2 + \frac{1}{60}i\alpha^3 Rk1 - \frac{1}{48}i\lambda\alpha^3 Rk1 - \frac{2}{4}\alpha^2 k1$$
$$+ \frac{1}{240}\alpha^4 k2 - \frac{1}{6}i\alpha Rk1 - \frac{1}{15}i\alpha Rk2 + \frac{1}{280}i\alpha^3 Rk2 + \frac{1}{4}i\lambda\alpha Rk1$$
$$+ \frac{1}{12}i\lambda\alpha Rk2 - \frac{2}{12}\alpha^2 k2$$

$$c_9 = \frac{1}{120}\alpha^4 k1 + \frac{11}{10080}i\alpha^3 Rk2 - \frac{1}{720}i\lambda\alpha^3 Rk2 + \frac{1}{720}\alpha^4 k2 + \frac{1}{24}i\lambda\alpha Rk2$$
$$- \frac{1}{120}i\lambda\alpha^3 Rk1 - \frac{2}{24}\alpha^2 k2 - \frac{1}{36}i\alpha Rk2 + \frac{1}{168}i\alpha^3 Rk1 + \frac{1}{6}i\lambda\alpha Rk1$$
$$- \frac{1}{12}i\alpha Rk1 - \frac{2}{6}\alpha^2 k1$$

$$c_{10} = -\frac{1}{5040}i\lambda\alpha^3 Rk2 - \frac{1}{720}i\lambda\alpha^3 Rk1 - \frac{1}{60}i\alpha Rk1 - \frac{2}{24}\alpha^2 k1 - \frac{2}{120}\alpha^2 k2$$
$$+ \frac{1}{1120}i\alpha^3 Rk1 + \frac{1}{24}i\lambda\alpha Rk1 + \frac{1}{120}i\lambda\alpha Rk2 - \frac{1}{210}i\alpha Rk2$$
$$+ \frac{13}{90720}i\alpha^3 Rk2 + \frac{1}{5040}\alpha^4 k2 + \frac{1}{720}\alpha^4 k1$$

$g_1 - g_{10}$ are the coefficients for v_2.

$$g_1 = -\frac{1}{594397440}\alpha^6 R^2 k2$$

$$g_2 = -\frac{1}{80720640}\alpha^6 R^2 (k1 + k2)$$

$$g_3 = \frac{211}{778377600}\alpha^4 R^2 k2 - \frac{31}{1556755200}\alpha^6 R^2 k2 - \frac{1}{14414400}\alpha^6 R^2 k1$$
$$- \frac{23}{1556755200}\lambda\alpha^6 R^2 k2 - \frac{23}{1556755200}i\alpha^7 Rk2$$

$$g_4 = \frac{1}{39916800}\alpha^6 R^2 k1 - \frac{17}{119750400}i\alpha^7 Rk2 - \frac{17}{119750400}i\alpha^7 Rk1$$
$$+ \frac{1}{739200}\alpha^4 R^2 k2 - \frac{17}{119750400}\lambda\alpha^6 R^2 k1 - \frac{17}{119750400}\lambda\alpha^6 R^2 k2$$
$$+ \frac{37}{359251200}\alpha^6 R^2 k2 + \frac{1}{739200}\alpha^4 R^2 k1$$

$$g_5 = -\frac{1}{1425600}\alpha^4 R^2 k2 + \frac{13}{3326400}\alpha^4 R^2 k1 - \frac{1}{249480}\alpha^2 R^2 k2$$

$$+ \frac{1}{907200}\lambda\alpha^4 R^2 k2 + \frac{31}{19958400}\lambda\alpha^4 R^2 k2 - \frac{1}{19958400}i\lambda\alpha^7 R k2$$

$$- \frac{1}{831600}\lambda\alpha^6 R^2 k1 + \frac{62}{19958400}i\alpha^5 R k2 + \frac{1}{907200}i\alpha^5 R k2$$

$$- \frac{1}{1814400}i\alpha^7 R k2 + \frac{1}{831600}\alpha^6 R^2 k1 - \frac{1}{1814400}\lambda\alpha^6 R^2 k2$$

$$+ \frac{1}{39916800}\alpha^8 k2 + \frac{23}{39916800}\alpha^6 R^2 k2 - \frac{1}{39916800}\lambda^2\alpha^6 R^2 k2$$

$$- \frac{1}{831600}i\alpha^7 R k1$$

$$g_6 = -\frac{1}{259200}\lambda\alpha^6 R^2 k1 + \frac{2}{86400}i\alpha^5 R k1 - \frac{1}{1814400}i\lambda\alpha^7 R k2$$

$$- \frac{1}{1814400}i\lambda\alpha^7 R k1 - \frac{53}{2721600}\alpha^4 R^2 k2 + \frac{1}{86400}\lambda\alpha^4 R^2 k1$$

$$- \frac{1}{3628800}\lambda^2\alpha^6 R^2 k1 + \frac{1}{241920}\alpha^6 R^2 k1 + \frac{1}{86400}\lambda\alpha^4 R^2 k2$$

$$- \frac{1}{1088640}i\alpha^7 R k2 - \frac{1}{1088640}\lambda\alpha^6 R^2 k2 + \frac{1}{129600}\lambda\alpha^4 R^2 k1$$

$$+ \frac{1}{3628800}\alpha^8 k2 - \frac{1}{50400}\alpha^4 R^2 k1 + \frac{13}{10886400}\alpha^6 R^2 k2$$

$$+ \frac{1}{129600}\lambda\alpha^4 R^2 k2 + \frac{1}{129600}i\alpha^5 R k2 + \frac{1}{3628800}\alpha^8 k1$$

$$+ \frac{1}{129600}i\alpha^5 R k1 + \frac{2}{86400}i\alpha^5 R k2 - \frac{1}{3628800}\lambda^2\alpha^6 R^2 k2$$

$$- \frac{1}{259200}i\alpha^7 R k1$$

$$g_7 = \frac{1}{1306368}\alpha^6 R^2 k2 - \frac{2}{181440}\alpha^6 k2 + \frac{1}{181440}\lambda^2\alpha^4 R^2 k2$$

$$- \frac{11}{181440}\lambda\alpha^2 R^2 k2 + \frac{11}{362880}\lambda\alpha^4 R^2 k2 + \frac{13}{181440}\lambda\alpha^4 R^2 k1$$

$$+ \frac{1}{181440}i\lambda\alpha^5 Rk2 + \frac{2}{181440}i\lambda\alpha^5 Rk2 - \frac{22}{181440}i\alpha^3 Rk2$$

$$+ \frac{22}{362880}i\alpha^5 Rk2 + \frac{26}{181440}i\alpha^5 Rk1 + \frac{1}{22680}\lambda\alpha^4 R^2 k1$$

$$+ \frac{1}{60480}\lambda\alpha^4 R^2 k2 + \frac{1}{2177280}\lambda\alpha^6 R^2 k2 - \frac{1}{272160}\lambda\alpha^6 R^2 k1$$

$$- \frac{1}{725760}\lambda^2\alpha^6 R^2 k2 - \frac{1}{362880}\lambda^2\alpha^6 R^2 k1 + \frac{1}{10080}\alpha^2 R^2 k1$$

$$+ \frac{1}{9072}\alpha^2 R^2 k2 + \frac{1}{362880}\alpha^8 k1 + \frac{1}{725760}\alpha^8 k2 - \frac{1}{362880}i\lambda\alpha^7 Rk2$$

$$- \frac{1}{181440}i\lambda\alpha^7 Rk1 + \frac{1}{22680}i\alpha^5 Rk1 + \frac{1}{60480}i\alpha^5 Rk2$$

$$+ \frac{1}{2177280}i\alpha^7 Rk2 - \frac{1}{272160}i\alpha^7 Rk1 - \frac{19}{362880}\alpha^4 R^2 k2$$

$$- \frac{1}{8640}\alpha^4 R^2 k1 + \frac{1}{181440}\alpha^6 R^2 k1$$

$$g_8 = - \frac{17}{8467200}\alpha^6 R^2 k2 - \frac{2}{20160}\alpha^6 k2 - \frac{2}{20160}\alpha^6 k1 + \frac{1}{20160}\lambda^2\alpha^4 R^2 k2$$

$$+ \frac{1}{20160}\lambda^2\alpha^4 R^2 k1 - \frac{1}{4032}\lambda\alpha^2 R^2 k1 - \frac{1}{4032}\lambda\alpha^2 R^2 k2$$

$$+ \frac{1}{120960}\lambda\alpha^4 R^2 k2 + \frac{1}{8064}\lambda\alpha^4 R^2 k1 + \frac{1}{20160}i\lambda\alpha^5 Rk2$$

$$+ \frac{1}{20160}i\lambda\alpha^5 Rk1 + \frac{2}{20160}i\lambda\alpha^5 Rk2 + \frac{2}{20160}i\lambda\alpha^5 Rk1 - \frac{2}{4032}i\alpha^3 Rk1$$

$$- \frac{2}{4032}i\alpha^3 Rk2 + \frac{2}{120960}i\alpha^5 Rk2 + \frac{2}{8064}i\alpha^5 Rk1 + \frac{1}{20160}\lambda\alpha^4 R^2 k1$$

$$- \frac{1}{60480}\lambda\alpha^4 R^2 k2 + \frac{1}{172800}\lambda\alpha^6 R^2 k2 + \frac{1}{80640}\lambda\alpha^6 R^2 k1$$

$$- \frac{1}{241920}\lambda^2\alpha^6 R^2 k2 - \frac{1}{80640}\lambda^2\alpha^6 R^2 k1 + \frac{1}{2016}\alpha^2 R^2 k1$$

$$+ \frac{1}{3024}\alpha^2 R^2 k2 + \frac{1}{80640}\alpha^8 k1 + \frac{1}{241920}\alpha^8 k2 - \frac{1}{120960}i\lambda\alpha^7 Rk2$$

$$- \frac{1}{40320}i\lambda\alpha^7 Rk1 + \frac{1}{20160}i\alpha^5 Rk1 - \frac{1}{60480}i\alpha^5 Rk2 + \frac{1}{172800}i\alpha^7 Rk2$$

$$+ \frac{1}{80640}i\alpha^7 Rk1 - \frac{19}{604800}\alpha^4 R^2 k2 - \frac{1}{5760}\alpha^4 R^2 k1 - \frac{1}{403200}\alpha^6 R^2 k1$$

$$g_9 = -\frac{163}{25401600}\alpha^6 R^2 k2 - \frac{2}{5040}\alpha^6 k2 - \frac{2}{2520}\alpha^6 k1 + \frac{1}{5040}\lambda^2\alpha^4 R^2 k2$$

$$+ \frac{1}{2520}\lambda^2\alpha^4 R^2 k1 - \frac{1}{2520}\lambda\alpha^2 R^2 k1 + \frac{1}{5040}\lambda\alpha^2 R^2 k2 - \frac{1}{6720}\lambda\alpha^4 R^2 k2$$

$$- \frac{1}{5040}\lambda\alpha^4 R^2 k1 + \frac{1}{5040}i\lambda\alpha^5 Rk2 + \frac{1}{2520}i\lambda\alpha^5 Rk1 - \frac{2}{2520}i\lambda\alpha^2 Rk2$$

$$+ \frac{2}{5040}i\lambda\alpha^4 Rk2 + \frac{2}{2520}i\lambda\alpha^4 Rk1 - \frac{2}{2520}i\alpha^3 Rk1 + \frac{2}{5040}i\alpha^3 Rk2$$

$$- \frac{2}{6720}i\alpha^5 Rk2 - \frac{2}{5040}i\alpha^5 Rk1 - \frac{1}{3780}\lambda\alpha^4 R^2 k1 - \frac{1}{6048}\lambda\alpha^4 R^2 k2$$

$$+ \frac{13}{907200}\lambda\alpha^6 R^2 k2 + \frac{1}{18900}\lambda\alpha^6 R^2 k1 - \frac{1}{5040}\lambda^2\alpha^2 R^2 k2$$

$$- \frac{1}{120960}\lambda^2\alpha^6 R^2 k2 - \frac{1}{30240}\lambda^2\alpha^6 R^2 k1 + \frac{1}{2520}\alpha^2 R^2 k1 + \frac{1}{30240}\alpha^8 k1$$

$$+ \frac{1}{120960}\alpha^8 k2 + \frac{4}{5040}\alpha^4 k2 - \frac{1}{60480}i\lambda\alpha^7 Rk2 - \frac{1}{15120}i\lambda\alpha^7 Rk1$$

$$- \frac{1}{3780}i\alpha^5 Rk1 - \frac{1}{6048}i\alpha^5 Rk2 + \frac{13}{907200}i\alpha^7 Rk2 + \frac{1}{18900}i\alpha^7 Rk1$$

$$+ \frac{61}{453600}\alpha^4 R^2 k2 + \frac{1}{5600}\alpha^4 R^2 k1 - \frac{1}{44100}\alpha^6 R^2 k1$$

$$g_{10} = -\frac{311}{32659200}\alpha^6 R^2 k2 - \frac{2}{2160}\alpha^6 k2 - \frac{2}{720}\alpha^6 k1 + \frac{1}{2160}\lambda^2\alpha^4 R^2 k2$$

$$+ \frac{1}{720}\lambda^2\alpha^6 R^2 k1 + \frac{1}{240}\lambda\alpha^2 R^2 k1 + \frac{7}{2160}\lambda\alpha^2 R^2 k2$$

$$- \frac{19}{43200}\lambda\alpha^4 R^2 k2 - \frac{11}{8640}\lambda\alpha^4 R^2 k1 + \frac{1}{2160}i\lambda\alpha^5 Rk2$$

$$+ \frac{1}{720}i\lambda\alpha^5 Rk1 - \frac{2}{360}i\lambda\alpha^3 Rk1 - \frac{2}{360}i\lambda\alpha^3 Rk2 + \frac{2}{2160}i\lambda\alpha^5 Rk2$$

$$+ \frac{2}{720}i\lambda\alpha^5 Rk1 + \frac{2}{240}i\alpha^3 Rk1 + \frac{14}{2160}i\alpha^3 Rk2 - \frac{38}{43200}i\alpha^5 Rk2$$

$$- \frac{22}{8640}i\alpha^5 Rk1 - \frac{1}{864}\lambda\alpha^4 R^2 k1 - \frac{1}{2400}\lambda\alpha^4 R^2 k2 + \frac{19}{907200}\lambda\alpha^6 R^2 k2$$

$$+ \frac{13}{129600}\lambda\alpha^6 R^2 k1 - \frac{1}{720}\lambda^2\alpha^2 R^2 k1 - \frac{1}{720}\lambda^2\alpha^2 R^2 k2$$

$$- \frac{1}{86400}\lambda^2\alpha^6 R^2 k2 - \frac{1}{17280}\lambda^2\alpha^6 R^2 k1 - \frac{1}{360}\alpha^2 R^2 k1 - \frac{1}{540}\alpha^2 R^2 k2$$

$$+ \frac{1}{17280}\alpha^8 k1 + \frac{1}{86400}\alpha^8 k2 + \frac{4}{720}\alpha^4 k1 + \frac{4}{720}\alpha^4 k2 - \frac{1}{43200}i\lambda\alpha^7 Rk2$$

$$- \frac{1}{8640}i\lambda\alpha^7 Rk1 - \frac{1}{864}i\alpha^5 Rk1 - \frac{1}{2400}i\alpha^5 Rk2 + \frac{19}{907200}i\alpha^7 Rk2$$

$$+ \frac{13}{129600}i\alpha^7 Rk1 + \frac{61}{151200}\alpha^4 R^2 k2 + \frac{1}{900}\alpha^4 R^2 k1 - \frac{53}{1209600}\alpha^6 R^2 k1$$

$$
\begin{aligned}
g_{11} ={}& -\frac{11}{1209600}\alpha^6 R^2 k2 - \frac{2}{1440}\alpha^6 k2 - \frac{2}{360}\alpha^6 k1 + \frac{1}{1440}\lambda^2\alpha^4 R^2 k2 \\
&+ \frac{1}{360}\lambda^2\alpha^4 R^2 k1 + \frac{7}{360}\lambda\alpha^2 R^2 k1 + \frac{13}{1440}\lambda\alpha^2 R^2 k2 - \frac{29}{43200}\lambda\alpha^4 R^2 k2 \\
&- \frac{19}{7200}\lambda\alpha^4 R^2 k1 + \frac{1}{1440}i\lambda\alpha^5 Rk2 + \frac{1}{360}i\lambda\alpha^5 Rk1 - \frac{2}{60}i\lambda\alpha^3 Rk1 \\
&- \frac{2}{120}i\lambda\alpha^3 Rk2 + \frac{2}{1440}i\lambda\alpha^5 Rk2 + \frac{2}{360}i\lambda\alpha^5 Rk1 + \frac{14}{360}i\alpha^3 Rk1 \\
&+ \frac{26}{1440}i\alpha^3 Rk2 - \frac{58}{43200}i\alpha^5 Rk2 - \frac{38}{7200}i\alpha^5 Rk1 - \frac{1}{450}\lambda\alpha^4 R^2 k1 \\
&- \frac{13}{21600}\lambda\alpha^4 R^2 k2 + \frac{1}{48384}\lambda\alpha^6 R^2 k2 + \frac{1}{8400}\lambda\alpha^6 R^2 k1 - \frac{1}{120}\lambda^2\alpha^2 R^2 k1 \\
&- \frac{1}{240}\lambda^2\alpha^2 R^2 k2 - \frac{1}{86400}\lambda^2\alpha^6 R^2 k2 - \frac{1}{14400}\lambda^2\alpha^6 R^2 k1 - \frac{7}{720}\alpha^2 R^2 k1 \\
&- \frac{1}{216}\alpha^2 R^2 k2 + \frac{1}{14400}\alpha^8 k1 + \frac{1}{86400}\alpha^8 k2 + \frac{4}{120}\alpha^4 k1 + \frac{4}{240}\alpha^4 k2 \\
&- \frac{1}{43200}i\lambda\alpha^7 Rk2 - \frac{1}{7200}i\lambda\alpha^7 Rk1 - \frac{1}{450}i\alpha^5 Rk1 - \frac{13}{21600}i\alpha^5 Rk2 \\
&+ \frac{1}{48384}i\alpha^7 Rk2 + \frac{1}{8400}i\alpha^7 Rk1 + \frac{347}{604800}\alpha^4 R^2 k2 + \frac{103}{50400}\alpha^4 R^2 k1 \\
&- \frac{1}{20160}\alpha^6 R^2 k1
\end{aligned}
$$

$$
\begin{aligned}
g_{12} = {}& -\frac{13}{2177280}\alpha^6 R^2 k2 - \frac{2}{1440}\alpha^6 k2 - \frac{2}{288}\alpha^6 k1 + \frac{1}{1440}\lambda^2\alpha^4 R^2 k2 \\
& + \frac{1}{288}\lambda^2\alpha^4 R^2 k1 + \frac{11}{288}\lambda\alpha^2 R^2 k1 + \frac{19}{1440}\lambda\alpha^2 R^2 k2 - \frac{13}{20160}\lambda\alpha^4 R^2 k2 \\
& - \frac{1}{320}\lambda\alpha^4 R^2 k1 + \frac{1}{1440}i\lambda\alpha^5 Rk2 + \frac{1}{288}i\lambda\alpha^5 Rk1 - \frac{2}{24}i\lambda\alpha^3 Rk1 \\
& - \frac{2}{72}i\lambda\alpha^3 Rk2 + \frac{2}{1440}i\lambda\alpha^5 Rk2 + \frac{2}{288}i\lambda\alpha^5 Rk1 + \frac{22}{288}i\alpha^3 Rk1 \\
& + \frac{28}{1440}i\alpha^3 Rk2 - \frac{26}{20160}i\alpha^5 Rk2 - \frac{2}{320}i\alpha^5 Rk1 - \frac{11}{4320}\lambda\alpha^4 R^2 k1 \\
& - \frac{17}{30240}\lambda\alpha^4 R^2 k2 + \frac{31}{2177280}\lambda\alpha^6 R^2 k2 + \frac{23}{241920}\lambda\alpha^6 R^2 k1 \\
& - \frac{1}{48}\lambda^2\alpha^2 R^2 k1 - \frac{1}{144}\lambda^2\alpha^2 R^2 k2 - \frac{1}{120960}\lambda^2\alpha^6 R^2 k2 \\
& - \frac{1}{17280}\lambda^2\alpha^6 R^2 k1 - \frac{11}{720}\alpha^2 R^2 k1 - \frac{1}{168}\alpha^2 R^2 k2 + \frac{1}{17280}\alpha^8 k1 \\
& + \frac{1}{120960}\alpha^8 k2 + \frac{4}{48}\alpha^4 k1 + \frac{4}{144}\alpha^4 k2 - \frac{1}{60480}i\lambda\alpha^7 Rk2 \\
& - \frac{1}{8640}i\lambda\alpha^7 Rk1 - \frac{11}{4320}i\alpha^5 Rk1 - \frac{17}{30240}i\alpha^5 Rk2 + \frac{31}{2177280}i\alpha^7 Rk2 \\
& + \frac{23}{241920}i\alpha^7 Rk1 + \frac{79}{155520}\alpha^4 R^2 k2 + \frac{29}{13440}\alpha^4 R^2 k1 - \frac{1}{26880}\alpha^6 R^2 k1
\end{aligned}
$$

$$
\begin{aligned}
g_{13} = {}& -\frac{971}{359251200}\alpha^6 R^2 k2 - \frac{2}{2160}\alpha^6 k2 - \frac{2}{360}\alpha^6 k1 + \frac{1}{2160}\lambda^2\alpha^4 R^2 k2 \\
& + \frac{1}{360}\lambda^2\alpha^4 R^2 k1 + \frac{1}{24}\lambda\alpha^2 R^2 k1 + \frac{5}{432}\lambda\alpha^2 R^2 k2 - \frac{7}{17280}\lambda\alpha^4 R^2 k2 \\
& - \frac{1}{432}\lambda\alpha^4 R^2 k1 + \frac{1}{2160}i\lambda\alpha^5 Rk2 + \frac{1}{360}i\lambda\alpha^5 Rk1 - \frac{2}{18}i\lambda\alpha^3 Rk1 \\
& - \frac{2}{72}i\lambda\alpha^3 Rk2 + \frac{2}{2160}i\lambda\alpha^5 Rk2 + \frac{2}{360}i\lambda\alpha^5 Rk1 + \frac{2}{24}i\alpha^3 Rk1 \\
& + \frac{10}{432}i\alpha^3 Rk2 - \frac{14}{17280}i\alpha^5 Rk2 - \frac{2}{432}i\alpha^5 Rk1 - \frac{1}{540}\lambda\alpha^4 R^2 k1 \\
& - \frac{1}{2880}\lambda\alpha^4 R^2 k2 + \frac{37}{5443200}\lambda\alpha^6 R^2 k2 + \frac{1}{19440}\lambda\alpha^6 R^2 k1 \\
& - \frac{1}{36}\lambda^2\alpha^2 R^2 k1 - \frac{1}{144}\lambda^2\alpha^2 R^2 k2 - \frac{1}{241920}\lambda^2\alpha^6 R^2 k2 \\
& - \frac{1}{30240}\lambda^2\alpha^6 R^2 k1 - \frac{1}{72}\alpha^2 R^2 k1 - \frac{1}{216}\alpha^2 R^2 k2 + \frac{1}{30240}\alpha^8 k1 \\
& + \frac{1}{241920}\alpha^8 k2 + \frac{4}{36}\alpha^4 k1 + \frac{4}{144}\alpha^4 k2 - \frac{1}{120960}i\lambda\alpha^7 Rk2 \\
& - \frac{1}{15120}i\lambda\alpha^7 Rk1 - \frac{1}{540}i\alpha^5 Rk1 - \frac{1}{2880}i\alpha^5 Rk2 + \frac{37}{5443200}i\alpha^7 Rk2 \\
& + \frac{1}{19440}i\alpha^7 Rk1 + \frac{89}{302400}\alpha^4 R^2 k2 + \frac{29}{20160}\alpha^4 R^2 k1 - \frac{47}{2494800}\alpha^6 R^2 k1
\end{aligned}
$$

$$
\begin{aligned}
g_{14} = {}& -\frac{421}{518918400}\alpha^6 R^2 k2 - \frac{2}{5040}\alpha^6 k2 - \frac{2}{720}\alpha^6 k1 + \frac{1}{5040}\lambda^2\alpha^4 R^2 k2 \\
& + \frac{1}{720}\lambda^2\alpha^4 R^2 k1 + \frac{19}{720}\lambda\alpha^2 R^2 k1 + \frac{31}{5040}\lambda\alpha^2 R^2 k2 - \frac{59}{362880}\lambda\alpha^4 R^2 k2 \\
& - \frac{43}{40320}\lambda\alpha^4 R^2 k1 + \frac{1}{5040}i\lambda\alpha^5 Rk2 + \frac{1}{720}i\lambda\alpha^5 Rk1 - \frac{2}{24}i\lambda\alpha^3 Rk1 \\
& - \frac{2}{120}i\lambda\alpha^3 Rk2 + \frac{2}{5040}i\lambda\alpha^5 Rk2 + \frac{2}{720}i\lambda\alpha^5 Rk1 + \frac{38}{720}i\alpha^3 Rk1 \\
& + \frac{62}{5040}i\alpha^3 Rk2 - \frac{118}{362880}i\alpha^5 Rk2 - \frac{86}{40320}i\alpha^5 Rk1 \\
& - \frac{17}{20160}\lambda\alpha^4 R^2 k1 - \frac{5}{36288}\lambda\alpha^4 R^2 k2 + \frac{43}{19958400}\lambda\alpha^6 R^2 k2 \\
& + \frac{11}{604800}\lambda\alpha^6 R^2 k1 - \frac{1}{48}\lambda^2\alpha^2 R^2 k1 - \frac{1}{240}\lambda^2\alpha^2 R^2 k2 \\
& - \frac{1}{725760}\lambda^2\alpha^6 R^2 k2 - \frac{1}{80640}\lambda^2\alpha^6 R^2 k1 - \frac{19}{2520}\alpha^2 R^2 k1 - \frac{5}{2268}\alpha^2 R^2 k2 \\
& + \frac{1}{80640}\alpha^8 k1 + \frac{1}{725760}\alpha^8 k2 + \frac{4}{48}\alpha^4 k1 \frac{4}{240}\alpha^4 k2 - \frac{1}{362880}i\lambda\alpha^7 Rk2 \\
& - \frac{1}{40320}i\lambda\alpha^7 Rk1 - \frac{17}{20160}i\alpha^5 Rk1 - \frac{5}{36288}i\alpha^5 Rk2 + \frac{43}{19958400}i\alpha^7 Rk2 \\
& + \frac{11}{604800}i\alpha^7 Rk1 + \frac{1091}{9979200}\alpha^4 R^2 k2 + \frac{13}{21600}\alpha^4 R^2 k1 - \frac{83}{13305600}\alpha^6 R^2 k1
\end{aligned}
$$

$$\begin{aligned}
g_{15} = {} & -\frac{1591}{10897286400}\alpha^6 R^2 k2 - \frac{2}{20160}\alpha^6 k2 - \frac{2}{2520}\alpha^6 k1 + \frac{1}{20160}\lambda^2\alpha^4 R^2 k2 \\
& + \frac{1}{2520}\lambda^2\alpha^4 R^2 k1 + \frac{23}{2520}\lambda\alpha^2 R^2 k1 + \frac{37}{20160}\lambda\alpha^2 R^2 k2 - \frac{23}{604800}\lambda\alpha^4 R^2 k2 \\
& - \frac{17}{60480}\lambda\alpha^4 R^2 k1 + \frac{1}{20160}i\lambda\alpha^5 Rk2 + \frac{1}{2520}i\lambda\alpha^5 Rk1 - \frac{2}{60}i\lambda\alpha^3 Rk1 \\
& - \frac{2}{360}i\lambda\alpha^3 Rk2 + \frac{2}{20160}i\lambda\alpha^5 Rk2 + \frac{2}{2520}i\lambda\alpha^5 Rk1 + \frac{46}{2520}i\alpha^3 Rk1 \\
& + \frac{74}{20160}i\alpha^3 Rk2 - \frac{46}{604800}i\alpha^5 Rk2 - \frac{34}{60480}i\alpha^5 Rk1 - \frac{1}{4536}\lambda\alpha^4 R^2 k1 \\
& - \frac{29}{907200}\lambda\alpha^4 R^2 k2 + \frac{7}{17107200}\lambda\alpha^6 R^2 k2 + \frac{19}{4989600}\lambda\alpha^6 R^2 k1 \\
& - \frac{1}{120}\lambda^2\alpha^2 R^2 k1 - \frac{1}{720}\lambda^2\alpha^2 R^2 k2 - \frac{1}{3628800}\lambda^2\alpha^6 R^2 k2 \\
& - \frac{1}{362880}\lambda^2\alpha^6 R^2 k1 - \frac{23}{10080}\alpha^2 R^2 k1 - \frac{1}{1680}\alpha^2 R^2 k2 + \frac{1}{362880}\alpha^8 k1 \\
& + \frac{1}{3628800}\alpha^8 k2 + \frac{4}{120}\alpha^4 k1 + \frac{4}{720}\alpha^4 k2 - \frac{1}{1814400}i\lambda\alpha^7 Rk2 \\
& - \frac{1}{181440}i\lambda\alpha^7 Rk1 - \frac{1}{4536}i\alpha^5 Rk1 - \frac{29}{907200}i\alpha^5 Rk2 \\
& + \frac{7}{17107200}i\alpha^7 Rk2 + \frac{19}{4989600}i\alpha^7 Rk1 + \frac{1423}{59875200}\alpha^4 R^2 k2 \\
& + \frac{23}{158400}\alpha^4 R^2 k1 - \frac{53}{43243200}\alpha^6 R^2 k1
\end{aligned}$$

$$g_{16} = -\frac{391}{32691859200}\alpha^6 R^2 k2 - \frac{2}{181440}\alpha^6 k2 - \frac{2}{20160}\alpha^6 k1$$

$$+ \frac{1}{181440}\lambda^2\alpha^4 R^2 k2 + \frac{1}{20160}\lambda^2\alpha^4 R^2 k1 + \frac{3}{2240}\lambda\alpha^2 R^2 k1$$

$$+ \frac{43}{181440}\lambda\alpha^2 R^2 k2 - \frac{79}{19958400}\lambda\alpha^4 R^2 k2 - \frac{59}{1814400}\lambda\alpha^4 R^2 k1$$

$$+ \frac{1}{181440}i\lambda\alpha^5 Rk2 + \frac{1}{20160}i\lambda\alpha^5 Rk1 - \frac{2}{360}i\lambda\alpha^3 Rk1 - \frac{2}{2520}i\lambda\alpha^3 Rk2$$

$$+ \frac{2}{181440}i\lambda\alpha^5 Rk2 + \frac{2}{20160}i\lambda\alpha^5 Rk1 + \frac{6}{2240}i\alpha^3 Rk1 + \frac{86}{181440}i\alpha^3 Rk2$$

$$- \frac{158}{19958400}i\alpha^5 Rk2 - \frac{118}{1814400}i\alpha^5 Rk1 - \frac{23}{907200}\lambda\alpha^4 R^2 k1$$

$$- \frac{1}{302400}\lambda\alpha^4 R^2 k2 + \frac{1}{28304640}\lambda\alpha^6 R^2 k2 + \frac{43}{119750400}\lambda\alpha^6 R^2 k1$$

$$- \frac{1}{720}\lambda^2\alpha^2 R^2 k1 - \frac{1}{5040}\lambda^2\alpha^2 R^2 k2 - \frac{1}{39916800}\lambda^2\alpha^6 R^2 k2$$

$$- \frac{1}{3628800}\lambda^2\alpha^6 R^2 k1 - \frac{1}{3360}\alpha^2 R^2 k1 - \frac{1}{14256}\alpha^2 R^2 k2 + \frac{1}{3628800}\alpha^8 k1$$

$$+ \frac{1}{39916800}\alpha^8 k2 + \frac{4}{720}\alpha^4 k1 + \frac{4}{5040}\alpha^4 k2 - \frac{1}{19958400}i\lambda\alpha^7 Rk2$$

$$- \frac{1}{1814400}i\lambda\alpha^7 Rk1 - \frac{23}{907200}i\alpha^5 Rk1 - \frac{1}{302400}i\alpha^5 Rk2$$

$$+ \frac{1}{28304640}i\alpha^7 Rk2 + \frac{43}{119750400}i\alpha^7 Rk1 + \frac{599}{259459200}\alpha^4 R^2 k2$$

$$+ \frac{103}{6652800}\alpha^4 R^2 k1 - \frac{79}{726485760}\alpha^6 R^2 k1$$

$h_1 - h_{22}$ are the coefficients for v_3.

$$h_1 = \frac{1}{85379248281600}i\alpha^9 R^3 k2$$

$$h_2 = \frac{1}{9386196019200}i\alpha^9 R^3 k2 + \frac{1}{9386196019200}i\alpha^9 R^3 k1$$

$$h_3 = \frac{269}{1520563755110400}i\lambda\alpha^9 R^3 k2 - \frac{10151}{1520563755110400}i\alpha^7 R^3 k2$$

$$- \frac{269}{1520563755110400}\alpha^{10}R^2 k2 + \frac{1}{1340885145600}i\alpha^9 R^3 k1$$

$$+ \frac{149}{760281877555200}i\alpha^9 R^3 k2$$

$$h_4 = -\frac{1}{1961511552000}i\alpha^9 R^3 k1 - \frac{943}{600222534912000}i\alpha^9 R^3 k2$$

$$+ \frac{841}{400148356608000}i\lambda\alpha^9 R^3 k2 - \frac{841}{400148356608000}\alpha^{10} R^2 k2$$

$$- \frac{241}{4940103168000}i\alpha^7 R^3 k1 - \frac{841}{400148356608000}\alpha^{10} R^2 k1$$

$$+ \frac{841}{400148356608000}i\lambda\alpha^9 R^3 k1 - \frac{241}{4940103168000}i\alpha^7 R^3 k2$$

$$h_5 = \frac{106}{1587890304000}\alpha^8 R^2 k2 + \frac{1}{44910028800}i\lambda\alpha^9 R^3 k1$$

$$- \frac{1}{44910028800}\alpha^{10} R^2 k1 + \frac{71}{1111523212800}\alpha^8 R^2 k2$$

$$- \frac{433}{44460928512000}\alpha^{10} R^2 k2 + \frac{433}{44460928512000}i\lambda\alpha^9 R^3 k2$$

$$+ \frac{17807}{22230464256000}i\alpha^7 R^3 k2 - \frac{631}{2470051584000}i\alpha^7 R^3 k1$$

$$- \frac{31}{22230464256000}\lambda\alpha^{10} R^2 k2 + \frac{31}{44460928512000}i\lambda^2\alpha^9 R^3 k2$$

$$- \frac{1}{44910028800}i\alpha^9 R^3 k1 - \frac{53}{1587890304000}i\lambda\alpha^7 R^3 k2$$

$$- \frac{31}{44460928512000}i\alpha^{11} R^2 k2 - \frac{71}{1111523212800}i\lambda\alpha^7 R^3 k2$$

$$- \frac{29}{2778808032000}i\alpha^9 R^3 k2 - \frac{271}{8892185702400}i\alpha^7 R^3 k2$$

$$h_6 = -\frac{17}{201180672000}\alpha^{10}R^2 k1 - \frac{1}{52306974720}\lambda\alpha^{10}R^2 k2 + \frac{1}{248371200}i\alpha^5 R^3 k1$$

$$+ \frac{1}{104613949440}i\lambda^2\alpha^9 R^3 k1 - \frac{23}{72648576000}i\lambda\alpha^7 R^3 k1$$

$$- \frac{23}{72648576000}i\lambda\alpha^7 R^3 k2 + \frac{41}{67060224000}i\alpha^7 R^3 k1$$

$$- \frac{1}{104613949440}i\alpha^{11}R k2 - \frac{11}{713276928000}\alpha^{10}R^2 k2$$

$$+ \frac{103}{163459296000}\alpha^8 R^2 k2 - \frac{41}{435891456000}i\alpha^9 R^3 k1$$

$$- \frac{1}{52306974720}\lambda\alpha^{10}R^2 k1 + \frac{6551}{7846046208000}i\alpha^7 R^3 k2$$

$$+ \frac{1}{104613949440}i\lambda^2\alpha^9 R^3 k2 - \frac{103}{163459296000}i\lambda\alpha^7 R^3 k1$$

$$+ frac{4}{672648576000}\alpha^8 R^2 k2 - \frac{1}{104613949440}i\alpha^{11}R k1$$

$$- \frac{46}{72648576000}\alpha^8 R^2 k1 - \frac{1}{40030848000}i\alpha^9 R^3 k2$$

$$- \frac{103}{163459296000}i\lambda\alpha^7 R^3 k2 + \frac{11}{713276928000}i\lambda\alpha^9 R^3 k2$$

$$+ \frac{17}{201180672000}i\alpha^9 R^3 k1 + \frac{103}{163459296000}\alpha^8 R^2 k1$$

$$+ \frac{1}{248371200}i\alpha^5 R^3 k2$$

$$h_7 = -\frac{19}{163459296000}\,\alpha^{10}\lambda\,R^2 k2 - \frac{79}{326918592000}\,\alpha^{10}\lambda\,R^2 k1$$

$$-\frac{17}{838252800}\,\alpha^6 R^2 k2 + \frac{433}{54486432000}\,i\alpha^7 R^3 k1$$

$$+\frac{467}{130767436800}\,i\alpha^7 R^3 k2 - \frac{37}{11675664000}\,i\lambda\,\alpha^7 R^3 k2$$

$$-\frac{433}{54486432000}\,i\lambda\,\alpha^7 R^3 k1 + \frac{1}{1307674368000}\,i\lambda^3\alpha^9 R^3 k2$$

$$+\frac{677}{163459296000}\,\alpha^8 R^2 k2 + \frac{311}{29719872000}\,\alpha^8 R^2 k1$$

$$-\frac{19}{490377888000}\,\alpha^{10} R^2 k1 - \frac{43}{326918592000}\,i\alpha^9 R^3 k1$$

$$-\frac{1}{75675600}\,i\alpha^3 R^3 k2 - \frac{17}{2942267328000}\,i\alpha^9 R^3 k2$$

$$+\frac{1}{1307674368000}\,\alpha^{12} k2 + \frac{19}{392302310400}\,\alpha^{10} R^2 k2$$

$$-\frac{19}{326918592000}\,\alpha^{11} iR k2 - \frac{79}{653837184000}\,\alpha^{11} iR k1$$

$$+\frac{79}{653837184000}\,i\lambda^2\alpha^9 R^3 k1 - \frac{263}{653837184000}\,i\lambda^2\alpha^7 R^3 k2$$

$$+\frac{1}{1452971520}\,\alpha^9 iR k2 + \frac{19}{326918592000}\,i\lambda^2\alpha^9 R^3 k2$$

$$+\frac{179}{18162144000}\,i\alpha^5 R^3 k1 - \frac{1}{435891456000}\,\alpha^{11} i\lambda\,R k2$$

$$-\frac{1159}{163459296000}\,i\alpha^5 R^3 k2 - \frac{1}{435891456000}\,\alpha^{10}\lambda^2 R^2 k2$$

$$+\frac{19}{490377888000}\,i\lambda\,\alpha^9 R^3 k1 + \frac{3929}{326918592000}\,i\lambda\,\alpha^5 R^3 k2$$

$$+\frac{713}{653837184000}\,\alpha^8\lambda\,R^2 k2 - \frac{19}{392302310400}\,i\lambda\,\alpha^9 R^3 k2$$

$$h_8 = -\frac{1}{2514758400}\,\alpha^{10}\lambda\,R^2 k2 - \frac{29}{21794572800}\,\alpha^{10}\lambda\,R^2 k1$$

$$-\frac{1619}{10897286400}\,\alpha^6 R^2 k2 + \frac{311}{14529715200}\,i\alpha^7 R^3 k1$$

$$+\frac{563}{130767436800}\,i\alpha^7 R^3 k2 - \frac{1}{16345929600}\,i\lambda\,\alpha^7 R^3 k2$$

$$-\frac{41}{2179457280}\,i\lambda\,\alpha^7 R^3 k1 + \frac{1}{87178291200}\,i\lambda^3\alpha^9 R^3 k2$$

$$-\frac{1}{1556755200}\,\alpha^8 R^2 k2 + \frac{509}{21794572800}\,\alpha^8 R^2 k1$$

$$+\frac{37}{43589145600}\,\alpha^{10} R^2 k1 + \frac{1}{3632428800}\,i\alpha^9 R^3 k1$$

$$+\frac{61}{457686028800}\,i\alpha^9 R^3 k2 + \frac{1}{87178291200}\,\alpha^{12} k2$$

$$-\frac{1619}{10897286400}\,\alpha^6 R^2 k1 + \frac{29}{3632428800}\,\alpha^9 iRk1$$

$$-\frac{1}{29059430400}\,\alpha^{11} i\lambda\,Rk1 - \frac{29}{6227020800}\,\lambda^2\alpha^7 R^3 k1$$

$$+\frac{43}{130767436800}\,\alpha^{10} R^2 k2 - \frac{1}{5029516800}\,\alpha^{11} iRk2$$

$$-\frac{29}{43589145600}\,\alpha^{11} iRk1 + \frac{29}{43589145600}\,i\lambda^2\alpha^9 R^3 k1$$

$$-\frac{29}{6227020800}\,i\lambda^2\alpha^7 R^3 k2 + \frac{29}{3632428800}\,\alpha^9 iRk2$$

$$+\frac{1}{5029516800}\,i\lambda^2\alpha^9 R^3 k2 - \frac{61}{605404800}\,i\alpha^5 R^3 k1$$

$$-\frac{1}{29059430400}\,\alpha^{11} i\lambda\,Rk2 - \frac{11}{118879488}\,i\alpha^5 R^3 k2$$

$$-\frac{1}{29059430400}\,\alpha^{10}\lambda^2 R^2 k2 - \frac{37}{43589145600}\,i\lambda\,\alpha^9 R^3 k1$$

$$+\frac{1927}{21794572800}\,i\lambda\,\alpha^5 R^3 k2 + \frac{551}{43589145600}\,\alpha^8\lambda\,R^2 k2$$

$$-\frac{43}{130767436800}\,i\lambda\,\alpha^9 R^3 k2 + \frac{1}{87178291200}\,i\lambda^3\alpha^9 R^3 k1$$

$$+\frac{551}{43589145600}\,\alpha^8\lambda\,R^2 k1 + \frac{1927}{21794572800}\,i\lambda\,\alpha^5 R^3 k1$$

$$-\frac{1}{29059430400}\,\alpha^{10}\lambda^2 R^2 k1 + \frac{1}{87178291200}\,\alpha^{12} k1$$

$$
\begin{aligned}
h_9 = {} & -\frac{1}{1334361600}\,\alpha^{10}\lambda\,R^2 k2 - \frac{37}{9340531200}\,\alpha^{10}\lambda\,R^2 k1 \\
& -\frac{29}{119750400}\,\alpha^6 R^2 k2 - \frac{37}{15567552000}\,i\alpha^7 R^3 k1 \\
& -\frac{247}{21555072000}\,i\alpha^7 R^3 k2 + \frac{31}{934053120}\,i\lambda\,\alpha^7 R^3 k2 \\
& +\frac{191}{4670265600}\,i\lambda\,\alpha^7 R^3 k1 + \frac{1}{12454041600}\,i\lambda^3\alpha^9 R^3 k2 \\
& -\frac{17}{359251200}\,\alpha^8 R^2 k2 - \frac{601}{9340531200}\,\alpha^8 R^2 k1 \\
& -\frac{17}{62270208}\,i\lambda\,\alpha^3 R^3 k2 + \frac{1}{518918400}\,\lambda^2\alpha^8 R^2 k2 \\
& -\frac{1}{2075673600}\,i\lambda^3\alpha^7 R^3 k2 + \frac{13}{3592512000}\,\alpha^{10} R^2 k1 \\
& -\frac{1}{1037836800}\,\alpha^{10} k2 + \frac{179}{108972864000}\,i\alpha^9 R^3 k1 \\
& +\frac{37}{77837760}\,i\alpha^3 R^3 k2 + \frac{821}{1961511552000}\,i\alpha^9 R^3 k2 \\
& +\frac{17}{31135104}\,\alpha^4 R^2 k2 + \frac{1}{2471040}\,i\alpha^3 R^3 k1 \\
& -\frac{79}{518918400}\,\alpha^7 iRk2 + \frac{1}{12454041600}\,\alpha^{12} k2 \\
& +\frac{17}{311351040}\,i\lambda^2\alpha^5 R^3 k2 + \frac{1}{415134720}\,i\lambda\,\alpha^9 Rk2 \\
& -\frac{577}{3113510400}\,\alpha^6\lambda\,R^2 k2 - \frac{121}{141523200}\,\alpha^6 R^2 k1 \\
& +\frac{29}{345945600}\,\alpha^9 iRk1 - \frac{1}{2075673600}\,\alpha^{11} i\lambda\,Rk1 \\
& +\frac{19}{389188800}\,i\lambda^2\alpha^7 R^3 k1 + \frac{239}{280215936000}\,\alpha^{10} R^2 k2 \\
& -\frac{1}{2668723200}\,\alpha^{11} iRk2 - \frac{37}{18681062400}\,\alpha^{11} iRk1 \\
& +\frac{37}{18681062400}\,i\lambda^2\alpha^9 R^3 k1 - \frac{1}{43545600}\,i\lambda^2\alpha^7 R^3 k2 \\
& +\frac{41}{1037836800}\,\alpha^9 iRk2 + \frac{1}{2668723200}\,i\lambda^2\alpha^9 R^3 k2 \\
& -\frac{89}{172972800}\,i\alpha^5 R^3 k1 - \frac{1}{4151347200}\,\alpha^{11} i\lambda\,Rk2 \\
& -\frac{631}{3113510400}\,i\alpha^5 R^3 k2 - \frac{1}{4151347200}\,\alpha^{10}\lambda^2 R^2 k2 \\
& -\frac{13}{3592512000}\,i\lambda\,\alpha^9 R^3 k1 + \frac{461}{3113510400}\,i\lambda\,\alpha^5 R^3 k2
\end{aligned}
$$

h_9 continued

$$+ \frac{389}{6227020800}\,\alpha^8\lambda\,R^2 k2 - \frac{239}{280215936000}\,i\lambda\,\alpha^9 R^3 k2$$
$$+ \frac{1}{6227020800}\,i\lambda^3\alpha^9 R^3 k1 + \frac{59}{444787200}\,\alpha^8\lambda\,R^2 k1$$
$$+ \frac{89}{172972800}\,i\lambda\,\alpha^5 R^3 k1 - \frac{1}{2075673600}\,\alpha^{10}\lambda^2 R^2 k1$$
$$+ \frac{1}{6227020800}\,\alpha^{12}k1$$

$$
\begin{aligned}
h_{10} = {} & -\frac{1}{3592512000}\,\alpha^{10}\lambda\,R^2 k2 - \frac{1}{179625600}\,\alpha^{10}\lambda\,R^2 k1 \\
& + \frac{137}{119750400}\,\alpha^6 R^2 k2 - \frac{361}{2395008000}\,i\alpha^7 R^3 k1 \\
& - \frac{551}{10059033600}\,i\alpha^7 R^3 k2 + \frac{19}{163296000}\,i\lambda\,\alpha^7 R^3 k2 \\
& + \frac{1}{2851200}\,i\lambda\,\alpha^7 R^3 k1 + \frac{1}{2874009600}\,i\lambda^3\alpha^9 R^3 k2 \\
& - \frac{293}{1796256000}\,\alpha^8 R^2 k2 - \frac{359}{718502400}\,\alpha^8 R^2 k1 \\
& - \frac{1}{887040}\,i\lambda\,\alpha^3 R^3 k2 + \frac{1}{39916800}\,\lambda^2\alpha^8 R^2 k2 \\
& - \frac{1}{159667200}\,i\lambda^3\alpha^7 R^3 k1 - \frac{1}{159667200}\,i\lambda^3\alpha^7 R^3 k2 \\
& + \frac{31}{4311014400}\,\alpha^{10} R^2 k1 - \frac{1}{79833600}\,\alpha^{10} k2 \\
& - \frac{1}{79833600}\,\alpha^{10} k1 + \frac{1}{287400960}\,i\alpha^9 R^3 k1 \\
& + \frac{1}{665280}\,i\alpha^3 R^3 k2 + \frac{859}{1357969536000}\,i\alpha^9 R^3 k2 \\
& + \frac{1}{443520}\,\alpha^4 R^2 k2 + \frac{1}{443520}\,i\alpha^3 R^3 k1 \\
& - \frac{19}{13305600}\,\alpha^7 i R k1 - \frac{19}{13305600}\,\alpha^7 i R k2 \\
& + \frac{1}{443520}\,\alpha^4 R^2 k1 + \frac{1}{2874009600}\,\alpha^{12} k2 \\
& - \frac{1}{887040}\,i\lambda\,\alpha^3 R^3 k1 + \frac{61}{119750400}\,i\lambda^2\alpha^5 R^3 k1 \\
& + \frac{61}{119750400}\,i\lambda^2\alpha^5 R^3 k2 + \frac{1}{31933440}\,i\lambda\,\alpha^9 R k2 \\
& - \frac{83}{47900160}\,\alpha^6\lambda\,R^2 k2 - \frac{83}{47900160}\,\alpha^6\lambda\,R^2 k1 \\
& + \frac{1}{31933440}\,i\lambda\,\alpha^9 R k1 + \frac{1}{39916800}\,\alpha^8\lambda^2 R^2 k1 \\
& - \frac{1}{29937600}\,\alpha^6 R^2 k1 + \frac{29}{79833600}\,\alpha^9 i R k1 \\
& - \frac{1}{319334400}\,\alpha^{11} i\lambda\,R k1 - \frac{101}{479001600}\,i\lambda^2\alpha^7 R^3 k1 \\
& + \frac{167}{150885504000}\,\alpha^{10} R^2 k2 - \frac{1}{7185024000}\,\alpha^{11} i R k2
\end{aligned}
$$

h_{10} continued

$$- \frac{1}{359251200} \alpha^{11} iRk1 + \frac{1}{359251200} i\lambda^2 \alpha^9 R^3 k1$$

$$- \frac{83}{1437004800} i\lambda^2 \alpha^7 R^3 k2 + \frac{1}{9979200} \alpha^9 iRk2$$

$$+ \frac{1}{7185024000} i\lambda^2 \alpha^9 R^3 k2 - \frac{29}{79833600} i\alpha^5 R^3 k1$$

$$- \frac{1}{958003200} \alpha^{11} i\lambda Rk2 + \frac{677}{3592512000} i\alpha^5 R^3 k2$$

$$- \frac{1}{958003200} \alpha^{10} \lambda^2 R^2 k2 - \frac{31}{4311014400} i\lambda \alpha^9 R^3 k1$$

$$- \frac{67}{102643200} i\lambda \alpha^5 R^3 k2 + \frac{227}{1437004800} \alpha^8 \lambda R^2 k2$$

$$- \frac{167}{150885504000} i\lambda \alpha^9 R^3 k2 + \frac{1}{958003200} i\lambda^3 \alpha^9 R^3 k1$$

$$+ \frac{1}{1741824} \alpha^8 \lambda R^2 k1 + \frac{19}{239500800} i\lambda \alpha^5 R^3 k1$$

$$- \frac{1}{319334400} \alpha^{10} \lambda^2 R^2 k1 + \frac{1}{958003200} \alpha^{12} k1$$

$$
\begin{aligned}
h_{11} = {} & \frac{1}{359251200}\,\alpha^{10}\lambda\,R^2 k2 + \frac{1}{239500800}\,\alpha^{10}\lambda\,R^2 k1 \\
& + \frac{149}{23950080}\,\alpha^6 R^2 k2 - \frac{559}{1397088000}\,i\alpha^7 R^3 k1 \\
& - \frac{9539}{100590336000}\,i\alpha^7 R^3 k2 + \frac{23}{128304000}\,i\lambda\,\alpha^7 R^3 k2 \\
& + \frac{529}{598752000}\,i\lambda\,\alpha^7 R^3 k1 + \frac{1}{958003200}\,i\lambda^3\alpha^9 R^3 k2 \\
& - \frac{151}{598752000}\,\alpha^8 R^2 k2 - \frac{1487}{1197504000}\,\alpha^8 R^2 k1 \\
& + \frac{17}{6652800}\,i\lambda\,\alpha^3 R^3 k2 + \frac{1}{6652800}\,\lambda^2\alpha^8 R^2 k2 \\
& - \frac{1}{13305600}\,i\lambda^3\alpha^7 R^3 k1 - \frac{1}{26611200}\,i\lambda^3\alpha^7 R^3 k2 \\
& - \frac{31}{19958400}\,i\lambda^2\alpha^3 R^3 k2 + \frac{1}{13305600}\,i\lambda^3\alpha^5 R^3 k2 \\
& + \frac{23}{4191264000}\,\alpha^{10} R^2 k1 - \frac{1}{13305600}\,\alpha^{10} k2 \\
& - \frac{1}{6652800}\,\alpha^{10} k1 + \frac{19}{5588352000}\,i\alpha^9 R^3 k1 \\
& - \frac{1}{997920}\,i\alpha^3 R^3 k2 + \frac{17}{50295168000}\,i\alpha^9 R^3 k2 \\
& - \frac{17}{3326400}\,\alpha^4 R^2 k2 + \frac{1}{907200}\,i\alpha^3 R^3 k1 \\
& - \frac{13}{1108800}\,\alpha^7 i R k1 - \frac{1}{190080}\,\alpha^7 i R k2 \\
& + \frac{1}{453600}\,\alpha^4 R^2 k1 + \frac{31}{4989600}\,\alpha^5 i R k2 \\
& + \frac{1}{958003200}\,\alpha^{12} k2 - \frac{1}{907200}\,i\lambda\,\alpha^3 R^3 k1 \\
& + \frac{83}{19958400}\,i\lambda^2\alpha^5 R^3 k1 + \frac{37}{19958400}\,i\lambda^2\alpha^5 R^3 k2 \\
& - \frac{1}{1663200}\,\alpha^7 i\lambda\,R k2 - \frac{1}{2661120}\,\alpha^6\lambda^2 R^2 k2 \\
& + \frac{1}{5322240}\,i\lambda\,\alpha^9 R k2 + \frac{31}{4989600}\,\alpha^4\lambda\,R^2 k2 \\
& - \frac{23}{3628800}\,\alpha^6\lambda\,R^2 k2 - \frac{283}{19958400}\,\alpha^6\lambda\,R^2 k1 \\
& + \frac{1}{2661120}\,i\lambda\,\alpha^9 R k1 + \frac{1}{3326400}\,\alpha^8\lambda^2 R^2 k1
\end{aligned}
$$

h_{11} continued

$$+ \frac{1}{3326400}\,\alpha^8 k2 + \frac{731}{59875200}\,\alpha^6 R^2 k1$$

$$+ \frac{29}{39916800}\,\alpha^9 iRk1 - \frac{1}{79833600}\,\alpha^{11} i\lambda\,Rk1$$

$$- \frac{1}{2395008}\,i\lambda^2\alpha^7 R^3 k1 - \frac{1}{100590336000}\,\alpha^{10} R^2 k2$$

$$+ \frac{1}{718502400}\,\alpha^{11} iRk2 + \frac{1}{479001600}\,\alpha^{11} iRk1$$

$$- \frac{1}{479001600}\,i\lambda^2\alpha^9 R^3 k1 - \frac{23}{479001600}\,i\lambda^2\alpha^7 R^3 k2$$

$$+ \frac{1}{11404800}\,\alpha^9 iRk2 - \frac{1}{718502400}\,i\lambda^2\alpha^9 R^3 k2$$

$$+ \frac{43}{13305600}\,i\alpha^5 R^3 k1 - \frac{1}{319334400}\,\alpha^{11} i\lambda\,Rk2$$

$$+ \frac{25}{14370048}\,i\alpha^5 R^3 k2 - \frac{1}{319334400}\,\alpha^{10}\lambda^2 R^2 k2$$

$$- \frac{23}{4191264000}\,i\lambda\,\alpha^9 R^3 k1 - \frac{863}{239500800}\,i\lambda\,\alpha^5 R^3 k2$$

$$+ \frac{13}{95800320}\,\alpha^8\lambda\,R^2 k2 + \frac{1}{100590336000}\,i\lambda\,\alpha^9 R^3 k2$$

$$+ \frac{1}{239500800}\,i\lambda^3\alpha^9 R^3 k1 + \frac{137}{119750400}\,\alpha^8\lambda\,R^2 k1$$

$$- \frac{419}{59875200}\,i\lambda\,\alpha^5 R^3 k1 - \frac{1}{79833600}\,\alpha^{10}\lambda^2 R^2 k1$$

$$+ \frac{1}{239500800}\,\alpha^{12} k1$$

$$
\begin{aligned}
h_{12} ={}& \frac{11}{1143072000}\,\alpha^{10}\lambda\,R^2 k2 + \frac{13}{326592000}\,\alpha^{10}\lambda\,R^2 k1 \\
&+ \frac{5}{435456}\,\alpha^6 R^2 k2 - \frac{1291}{3048192000}\,i\alpha^7 R^3 k1 \\
&- \frac{169}{3292047360}\,i\alpha^7 R^3 k2 + \frac{11}{571536000}\,i\lambda\,\alpha^7 R^3 k2 \\
&+ \frac{19}{23328000}\,i\lambda\,\alpha^7 R^3 k1 + \frac{1}{435456000}\,i\lambda^3\alpha^9 R^3 k2 \\
&- \frac{43}{1143072000}\,\alpha^8 R^2 k2 - \frac{377}{326592000}\,\alpha^8 R^2 k1 \\
&- \frac{19}{1814400}\,i\lambda^2\alpha^3 R^3 k1 + \frac{1}{1209600}\,i\lambda^3\alpha^5 R^3 k1 \\
&- \frac{1}{151200}\,\alpha^7 i\lambda\,R k1 - \frac{1}{241920}\,\alpha^6\lambda^2 R^2 k1 \\
&+ \frac{19}{453600}\,\alpha^4\lambda\,R^2 k1 + \frac{19}{777600}\,i\lambda\,\alpha^3 R^3 k2 \\
&+ \frac{1}{1814400}\,\lambda^2\alpha^8 R^2 k2 - \frac{1}{2419200}\,i\lambda^3\alpha^7 R^3 k1 \\
&- \frac{1}{7257600}\,i\lambda^3\alpha^7 R^3 k2 - \frac{19}{1814400}\,i\lambda^2\alpha^3 R^3 k2 \\
&+ \frac{1}{1209600}\,i\lambda^3\alpha^5 R^3 k2 - \frac{89}{9144576000}\,\alpha^{10} R^2 k1 \\
&- \frac{1}{3628800}\,\alpha^{10} k2 - \frac{1}{1209600}\,\alpha^{10} k1 \\
&+ \frac{1}{302400}\,\alpha^8 k1 - \frac{1}{762048000}\,i\alpha^9 R^3 k1 \\
&- \frac{19}{1360800}\,i\alpha^3 R^3 k2 - \frac{29}{41150592000}\,i\alpha^9 R^3 k2 \\
&+ \frac{19}{453600}\,\alpha^5 i R k1 - \frac{19}{388800}\,\alpha^4 R^2 k2 \\
&- \frac{19}{907200}\,i\alpha^3 R^3 k1 - \frac{1}{28800}\,\alpha^7 i R k1 \\
&- \frac{13}{1814400}\,\alpha^7 i R k2 - \frac{19}{302400}\,\alpha^4 R^2 k1 \\
&+ \frac{19}{453600}\,\alpha^5 i R k2 + \frac{1}{435456000}\,\alpha^{12} k2 \\
&+ \frac{19}{604800}\,i\lambda\,\alpha^3 R^3 k1 + \frac{11}{907200}\,i\lambda^2\alpha^5 R^3 k1 \\
&- \frac{13}{5443200}\,i\lambda^2\alpha^5 R^3 k2 - \frac{1}{151200}\,\alpha^7 i\lambda\,R k2
\end{aligned}
$$

h_{12} continued

$$- \frac{1}{241920}\, \alpha^6 \lambda^2 R^2 k2 + \frac{1}{1451520}\, i\lambda\, \alpha^9 Rk2$$

$$+ \frac{19}{453600}\, \alpha^4 i\lambda\, R^2 k2 - \frac{13}{1555200}\, \alpha^6 \lambda\, R^2 k2$$

$$- \frac{151}{3628800}\, \alpha^6 \lambda\, R^2 k1 + \frac{1}{483840}\, i\lambda\, \alpha^9 Rk1$$

$$+ \frac{1}{604800}\, \alpha^8 \lambda^2 R^2 k1 + \frac{1}{302400}\, \alpha^8 k2$$

$$+ \frac{443}{10886400}\, \alpha^6 R^2 k1 - \frac{1}{29030400}\, \alpha^{11} i\lambda\, Rk1$$

$$+ \frac{1}{43545600}\, i\lambda^2 \alpha^7 R^3 k1 - \frac{53}{16460236800}\, \alpha^{10} R^2 k2$$

$$+ \frac{11}{2286144000}\, \alpha^{11} iRk2 + \frac{13}{653184000}\, \alpha^{11} iRk1$$

$$- \frac{13}{653184000}\, i\lambda^2 \alpha^9 R^3 k1 + \frac{37}{217728000}\, i\lambda^2 \alpha^7 R^3 k2$$

$$- \frac{1}{3628800}\, \alpha^9 iRk2 - \frac{11}{2286144000}\, i\lambda^2 \alpha^9 R^3 k2$$

$$+ \frac{1}{94500}\, i\alpha^5 R^3 k1 - \frac{1}{145152000}\, \alpha^{11} i\lambda\, Rk2$$

$$+ \frac{1297}{381024000}\, i\alpha^5 R^3 k2 - \frac{1}{145152000}\, \alpha^{10} \lambda^2 R^2 k2$$

$$+ \frac{89}{9144576000}\, i\lambda\, \alpha^9 R^3 k1 - \frac{103}{15552000}\, i\lambda\, \alpha^5 R^3 k2$$

$$- \frac{97}{217728000}\, \alpha^8 \lambda\, R^2 k2 + \frac{53}{16460236800}\, i\lambda\, \alpha^9 R^3 k2$$

$$+ \frac{1}{87091200}\, i\lambda^3 \alpha^9 R^3 k1 - \frac{1}{43545600}\, \alpha^8 \lambda\, R^2 k1$$

$$- \frac{19}{806400}\, i\lambda\, \alpha^5 R^3 k1 - \frac{1}{29030400}\, \alpha^{10} \lambda^2 R^2 k1$$

$$+ \frac{1}{87091200}\, \alpha^{12} k1$$

$$h_{13} = \frac{17}{914457600} \, \alpha^{10} \lambda \, R^2 k2 + \frac{47}{457228800} \, \alpha^{10} \lambda \, R^2 k1$$

$$+ \frac{17}{10886400} \, \alpha^6 R^2 k2 + \frac{67}{304819200} \, i\alpha^7 R^3 k1$$

$$+ \frac{233}{1828915200} \, i\alpha^7 R^3 k2 - \frac{241}{457228800} \, i\lambda \, \alpha^7 R^3 k2$$

$$- \frac{19}{15240960} \, i\lambda \, \alpha^7 R^3 k1 + \frac{1}{261273600} \, i\lambda^3 \alpha^9 R^3 k2$$

$$+ \frac{23}{32659200} \, \alpha^8 R^2 k2 + \frac{109}{65318400} \, \alpha^8 R^2 k1$$

$$+ \frac{1}{18144} \, i\lambda^2 \alpha^3 R^3 k1 + \frac{1}{120960} \, i\lambda^3 \alpha^5 R^3 k1$$

$$+ \frac{1}{30240} \, \alpha^5 i\lambda \, Rk2 + \frac{1}{60480} \, \alpha^4 \lambda^2 R^2 k2$$

$$- \frac{1}{15120} \, \alpha^7 i\lambda \, Rk1 - \frac{1}{24192} \, \alpha^6 \lambda^2 R^2 k1$$

$$+ \frac{1}{4536} \, \alpha^4 \lambda \, R^2 k1 + \frac{37}{725760} \, i\lambda \, \alpha^3 R^3 k2$$

$$+ \frac{1}{725760} \, \lambda^2 \alpha^8 R^2 k2 - \frac{1}{725760} \, \lambda^3 \alpha^7 R^3 k1$$

$$- \frac{1}{2903040} \, \lambda^3 \alpha^7 R^3 k2 - \frac{1}{51840} \, i\lambda^2 \alpha^3 R^3 k2$$

$$+ \frac{1}{241920} \, i\lambda^3 \alpha^5 R^3 k2 - \frac{319}{8230118400} \, \alpha^{10} R^2 k1$$

$$- \frac{1}{1451520} \, \alpha^{10} k2 - \frac{1}{362880} \, \alpha^{10} k1$$

$$+ \frac{1}{30240} \, \alpha^8 k1 - \frac{1}{45360} \, \alpha^6 k2$$

$$- \frac{307}{30177100800} \, i\alpha^9 R^3 k1 - \frac{1}{36288} \, i\alpha^3 R^3 k2$$

$$- \frac{41}{19399564800} \, i\alpha^9 R^3 k2 + \frac{1}{4536} \, \alpha^5 iRk1$$

$$- \frac{37}{362880} \, \alpha^4 R^2 k2 - \frac{23}{362880} \, i\alpha^3 R^3 k1$$

$$- \frac{1}{60480} \, \alpha^7 iRk1 + \frac{1}{80640} \, \alpha^7 iRk2$$

$$- \frac{23}{90720} \, \alpha^4 R^2 k1 + \frac{1}{12960} \, \alpha^5 iRk2$$

$$+ \frac{1}{261273600} \, \alpha^{12} k2 - \frac{1}{362880} \, i\lambda^3 \alpha^3 R^3 k2$$

h_{13} continued

$$+ \frac{23}{181440} i\lambda\, \alpha^3 R^3 k1 + \frac{1}{217728} i\lambda^2 \alpha^5 R^3 k1$$

$$- \frac{11}{2177280} i\lambda^2 \alpha^5 R^3 k2 - \frac{1}{30240} \alpha^7 i\lambda\, Rk2$$

$$- \frac{1}{48384} \alpha^6 \lambda^2 R^2 k2 + \frac{1}{580608} i\lambda\, \alpha^9 Rk2$$

$$+ \frac{1}{12960} \alpha^4 \lambda\, R^2 k2 + \frac{71}{4354560} \alpha^6 \lambda\, R^2 k2$$

$$- \frac{19}{1088640} \alpha^6 \lambda\, R^2 k1 + \frac{1}{145152} i\lambda\, \alpha^9 Rk1$$

$$+ \frac{1}{181440} \alpha^8 \lambda^2 R^2 k1 + \frac{1}{60480} \alpha^8 k2$$

$$+ \frac{457}{10886400} \alpha^6 R^2 k1 - \frac{29}{7257600} \alpha^9 iRk1$$

$$- \frac{1}{14515200} \alpha^{11} i\lambda\, Rk1 + \frac{13}{5443200} i\lambda^2 \alpha^7 R^3 k1$$

$$- \frac{5}{658409472} \alpha^{10} R^2 k2 + \frac{17}{1828915200} \alpha^{11} iRk2$$

$$+ \frac{47}{914457600} \alpha^{11} iRk1 - \frac{47}{914457600} i\lambda^2 \alpha^9 R^3 k1$$

$$- \frac{97}{130636800} i\lambda^2 \alpha^7 R^3 k2 - \frac{1}{806400} \alpha^9 iRk2$$

$$- \frac{17}{1828915200} i\lambda^2 \alpha^9 R^3 k2 + \frac{281}{25401600} i\alpha^5 R^3 k1$$

$$- \frac{1}{87091200} \alpha^{11} i\lambda\, Rk2 + \frac{283}{182891520} i\alpha^5 R^3 k2$$

$$- \frac{1}{87091200} \alpha^{10} \lambda^2 R^2 k2 + \frac{319}{8230118400} i\lambda\, \alpha^9 R^3 k1$$

$$- \frac{43}{65318400} i\lambda\, \alpha^5 R^3 k2 - \frac{37}{18662400} \alpha^8 \lambda\, R^2 k2$$

$$+ \frac{5}{658409472} i\lambda\, \alpha^9 R^3 k2 + \frac{1}{43545600} i\lambda^3 \alpha^9 R^3 k1$$

$$- \frac{139}{21772800} \alpha^8 \lambda\, R^2 k1 - \frac{263}{10886400} i\lambda\, \alpha^5 R^3 k1$$

$$- \frac{1}{14515200} \alpha^{10} \lambda^2 R^2 k1 + \frac{1}{43545600} \alpha^{12} k1$$

$$h_{14} = \frac{23}{914457600} \, \alpha^{10} \lambda \, R^2 k2 + \frac{17}{101606400} \, \alpha^{10} \lambda \, R^2 k1$$

$$- \frac{977}{25401600} \, \alpha^6 R^2 k2 + \frac{1511}{1016064000} \, i\alpha^7 R^3 k1$$

$$+ \frac{37399}{100590336000} \, i\alpha^7 R^3 k2 - \frac{17}{13063680} \, i\lambda \, \alpha^7 R^3 k2$$

$$- \frac{281}{50803200} \, i\lambda \, \alpha^7 R^3 k1 + \frac{1}{203212800} \, i\lambda^3 \alpha^9 R^3 k2$$

$$+ \frac{67}{38102400} \, \alpha^8 R^2 k2 + \frac{383}{50803200} \, \alpha^8 R^2 k1$$

$$- \frac{1}{40320} \, i\lambda^2 \alpha^3 R^3 k1 + \frac{1}{26880} \, i\lambda^3 \alpha^5 R^3 k1$$

$$+ \frac{1}{3360} \, \alpha^5 i\lambda \, R k2 + \frac{1}{6720} \, \alpha^4 \lambda^2 R^2 k2$$

$$- \frac{1}{3360} \, \alpha^7 i\lambda \, R k1 - \frac{1}{5376} \, \alpha^6 \lambda^2 R^2 k1$$

$$+ \frac{1}{10080} \, \alpha^4 \lambda \, R^2 k1 - \frac{1}{80640} \, i\lambda \, \alpha^3 R^3 k2$$

$$+ \frac{1}{403200} \, \lambda^2 \alpha^8 R^2 k2 - \frac{1}{322560} \, i\lambda^3 \alpha^7 R^3 k1$$

$$+ \frac{1}{1612800} \, i\lambda^3 \alpha^7 R^3 k2 + \frac{1}{24192} \, i\lambda^2 \alpha^3 R^3 k2$$

$$+ \frac{1}{80640} \, i\lambda^3 \alpha^5 R^3 k2 - \frac{23}{338688000} \, \alpha^{10} R^2 k1$$

$$- \frac{1}{806400} \, \alpha^{10} k2 - \frac{1}{161280} \, \alpha^{10} k1$$

$$+ \frac{1}{6720} \, \alpha^8 k1 - \frac{1}{5040} \, \alpha^6 k2$$

$$+ \frac{1}{3360} \, \alpha^5 i\lambda \, R k1 - \frac{103}{5588352000} \, i\alpha^9 R^3 k1$$

$$- \frac{1}{423360} \, i\alpha^3 R^3 k2 - \frac{251}{81729648000} \, i\alpha^9 R^3 k2$$

$$+ \frac{1}{10080} \, \alpha^5 iR k1 - \frac{1}{5040} \, \alpha^6 k1$$

$$- \frac{1}{40320} \, i\lambda^3 \alpha^3 R^3 k1 + \frac{1}{6720} \, \alpha^4 \lambda^2 R^2 k1$$

$$+ \frac{1}{40320} \, \alpha^4 R^2 k2 - \frac{1}{24192} \, i\alpha^3 R^3 k1$$

$$+ \frac{1}{5376} \, \alpha^7 iR k1 + \frac{31}{403200} \, \alpha^7 iR k2$$

h_{14} continued

$$- \frac{5}{24192} \, \alpha^4 R^2 k1 - \frac{1}{6048} \, \alpha^5 i R k2$$

$$+ \frac{1}{203212800} \, \alpha^{12} k2 - \frac{1}{40320} \, i\lambda^3 \alpha^3 R^3 k2$$

$$+ \frac{5}{48384} \, i\lambda \, \alpha^3 R^3 k1 - \frac{17}{241920} \, i\lambda^2 \alpha^5 R^3 k1$$

$$- \frac{1}{34560} \, i\lambda^2 \alpha^5 R^3 k2 - \frac{1}{10080} \, \alpha^7 i\lambda \, R k2$$

$$- \frac{1}{16128} \, \alpha^6 \lambda^2 R^2 k2 + \frac{1}{322560} \, i\lambda \, \alpha^9 R k2$$

$$- \frac{1}{6048} \, \alpha^4 \lambda \, R^2 k2 + \frac{233}{2419200} \, \alpha^6 \lambda \, R^2 k2$$

$$+ \frac{113}{483840} \, \alpha^6 \lambda \, R^2 k1 + \frac{1}{64512} \, i\lambda \, \alpha^9 R k1$$

$$+ \frac{1}{80640} \, \alpha^8 \lambda^2 R^2 k1 + \frac{1}{20160} \, \alpha^8 k2$$

$$- \frac{139}{1814400} \, \alpha^6 R^2 k1 - \frac{29}{2419200} \, \alpha^9 i R k1$$

$$- \frac{1}{9676800} \, \alpha^{11} i\lambda \, R k1 + \frac{103}{14515200} \, i\lambda^2 \alpha^7 R^3 k1$$

$$- \frac{1081}{100590336000} \, \alpha^{10} R^2 k2 + \frac{23}{1828915200} \, \alpha^{11} i R k2$$

$$+ \frac{17}{203212800} \, \alpha^{11} i R k1 - \frac{17}{203212800} \, i\lambda^2 \alpha^9 R^3 k1$$

$$+ \frac{157}{101606400} \, i\lambda^2 \alpha^7 R^3 k2 - \frac{11}{4233600} \, \alpha^9 i R k2$$

$$- \frac{23}{1828915200} \, i\lambda^2 \alpha^9 R^3 k2 - \frac{131}{11289600} \, i\alpha^5 R^3 k1$$

$$- \frac{1}{67737600} \, \alpha^{11} i\lambda \, R k2 - \frac{5863}{914457600} \, i\alpha^5 R^3 k2$$

$$- \frac{1}{67737600} \, \alpha^{10} \lambda^2 R^2 k2 + \frac{23}{338688000} \, i\lambda \, \alpha^9 R^3 k1$$

$$+ \frac{1171}{50803200} \, i\lambda \, \alpha^5 R^3 k2 - \frac{421}{101606400} \, \alpha^8 \lambda \, R^2 k2$$

$$+ \frac{1081}{100590336000} \, i\lambda \, \alpha^9 R^3 k2 + \frac{1}{29030400} \, i\lambda^3 \alpha^9 R^3 k1$$

$$- \frac{277}{14515200} \, \alpha^8 \lambda \, R^2 k1 + \frac{331}{7257600} \, i\lambda \, \alpha^5 R^3 k1$$

$$- \frac{1}{9676800} \, \alpha^{10} \lambda^2 R^2 k1 + \frac{1}{29030400} \, \alpha^{12} k1$$

$$
\begin{aligned}
h_{15} = {} & \frac{29}{1143072000}\,\alpha^{10}\lambda\,R^2k2 + \frac{89}{457228800}\,\alpha^{10}\lambda\,R^2k1 \\[2mm]
& - \frac{2459}{25401600}\,\alpha^6 R^2k2 + \frac{11027}{4191264000}\,i\alpha^7 R^3 k1 \\[2mm]
& + \frac{3173}{6035420160}\,i\alpha^7 R^3 k2 - \frac{1051}{571536000}\,i\lambda\,\alpha^7 R^3 k2 \\[2mm]
& - \frac{2251}{228614400}\,i\lambda\,\alpha^7 R^3 k1 + \frac{1}{203212800}\,i\lambda^3\alpha^9 R^3 k2 \\[2mm]
& + \frac{407}{163296000}\,\alpha^8 R^2 k2 + \frac{6149}{457228800}\,\alpha^8 R^2 k1 \\[2mm]
& + \frac{1}{1890}\,i\lambda^2\alpha^3 R^3 k1 + \frac{1}{10080}\,i\lambda^3\alpha^5 R^3 k1 \\[2mm]
& + \frac{1}{840}\,\alpha^5 i\lambda\,Rk2 + \frac{1}{1680}\,\alpha^4\lambda^2 R^2 k2 \\[2mm]
& - \frac{1}{1260}\,\alpha^7 i\lambda\,Rk1 - \frac{1}{2016}\,\alpha^6\lambda^2 R^2 k1 \\[2mm]
& - \frac{2}{945}\,\alpha^4\lambda\,R^2 k1 - \frac{7}{25920}\,i\lambda\,\alpha^3 R^3 k2 \\[2mm]
& + \frac{1}{302400}\,\lambda^2\alpha^8 R^2 k2 - \frac{1}{201600}\,i\lambda^3\alpha^7 R^3 k1 \\[2mm]
& - \frac{1}{1209600}\,i\lambda^3\alpha^7 R^3 k2 + \frac{17}{60480}\,i\lambda^2\alpha^3 R^3 k2 \\[2mm]
& + \frac{1}{40320}\,i\lambda^3\alpha^5 R^3 k2 - \frac{199}{2514758400}\,\alpha^{10} R^2 k1 \\[2mm]
& - \frac{1}{604800}\,\alpha^{10}k2 - \frac{1}{100800}\,\alpha^{10}k1 \\[2mm]
& + \frac{1}{2520}\,\alpha^8 k1 - \frac{1}{1260}\,\alpha^6 k2 \\[2mm]
& + \frac{1}{420}\,\alpha^5 i\lambda\,Rk1 - \frac{457}{21794572800}\,i\alpha^9 R^3 k1 \\[2mm]
& + \frac{1}{11340}\,i\alpha^3 R^3 k2 - \frac{20897}{6865290432000}\,i\alpha^9 R^3 k2 \\[2mm]
& - \frac{2}{945}\,\alpha^5 iRk1 - \frac{1}{630}\,\alpha^6 k1 \\[2mm]
& - \frac{1}{5040}\,i\lambda^3\alpha^3 R^3 k1 + \frac{1}{840}\,\alpha^4\lambda^2 R^2 k1 \\[2mm]
& + \frac{7}{12960}\,\alpha^4 R^2 k2 + \frac{1}{6048}\,i\alpha^3 R^3 k1 \\[2mm]
& + \frac{11}{16800}\,\alpha^7 iRk1 + \frac{53}{302400}\,\alpha^7 iRk2
\end{aligned}
$$

h_{15} continued

$$+ \frac{1}{1008}\,\alpha^4 R^2 k1 - \frac{17}{15120}\,\alpha^5 iRk2$$

$$+ \frac{1}{203212800}\,\alpha^{12} k2 - \frac{1}{10080}\,i\lambda^3 \alpha^3 R^3 k2$$

$$- \frac{1}{2016}\,i\lambda\,\alpha^3 R^3 k1 - \frac{73}{302400}\,i\lambda^2 \alpha^5 R^3 k1$$

$$- \frac{59}{907200}\,i\lambda^2 \alpha^5 R^3 k2 - \frac{1}{5040}\,\alpha^7 i\lambda\,Rk2$$

$$- \frac{1}{8064}\,\alpha^6 \lambda^2 R^2 k2 + \frac{1}{241920}\,i\lambda\,\alpha^9 Rk2$$

$$- \frac{17}{15120}\,\alpha^4 \lambda\,R^2 k2 + \frac{79}{362880}\,\alpha^6 \lambda\,R^2 k2$$

$$+ \frac{7}{8640}\,\alpha^6 \lambda\,R^2 k1 + \frac{1}{40320}\,i\lambda\,\alpha^9 Rk1$$

$$+ \frac{1}{50400}\,\alpha^8 \lambda^2 R^2 k1 + \frac{1}{10080}\,\alpha^8 k2$$

$$- \frac{2153}{6350400}\,\alpha^6 R^2 k1 - \frac{29}{1411200}\,\alpha^9 iRk1$$

$$- \frac{1}{8467200}\,\alpha^{11} i\lambda\,Rk1 + \frac{11}{907200}\,i\lambda^2 \alpha^7 R^3 k1$$

$$- \frac{1633}{150885504000}\,\alpha^{10} R^2 k2 + \frac{29}{2286144000}\,\alpha^{11} iRk2$$

$$+ \frac{89}{914457600}\,\alpha^{11} iRk1 - \frac{89}{914457600}\,i\lambda^2 \alpha^9 R^3 k1$$

$$+ \frac{31}{14515200}\,i\lambda^2 \alpha^7 R^3 k2 - \frac{61}{16934400}\,\alpha^9 iRk2$$

$$- \frac{29}{2286144000}\,i\lambda^2 \alpha^9 R^3 k2 - \frac{157}{2822400}\,i\alpha^5 R^3 k1$$

$$- \frac{1}{67737600}\,\alpha^{11} i\lambda\,Rk2 - \frac{2167}{127008000}\,i\alpha^5 R^3 k2$$

$$- \frac{1}{67737600}\,\alpha^{10} \lambda^2 R^2 k2 + \frac{199}{2514758400}\,i\lambda\,\alpha^9 R^3 k1$$

$$+ \frac{2921}{50803200}\,i\lambda\,\alpha^5 R^3 k2 - \frac{583}{101606400}\,\alpha^8 \lambda\,R^2 k2$$

$$+ \frac{1633}{150885504000}\,i\lambda\,\alpha^9 R^3 k2 + \frac{1}{25401600}\,i\lambda^3 \alpha^9 R^3 k1$$

$$- \frac{83}{2540160}\,\alpha^8 \lambda\,R^2 k1 + \frac{47}{235200}\,i\lambda\,\alpha^5 R^3 k1$$

$$- \frac{1}{8467200}\,\alpha^{10} \lambda^2 R^2 k1 + \frac{1}{25401600}\,\alpha^{12} k1$$

$$
\begin{aligned}
h_{16} =\ & \frac{1}{51321600}\,\alpha^{10}\lambda\,R^2 k2 + \frac{11}{65318400}\,\alpha^{10}\lambda\,R^2 k1 \\
& - \frac{293}{2177280}\,\alpha^6 R^2 k2 + \frac{6871}{2395008000}\,i\alpha^7 R^3 k1 \\
& + \frac{46357}{93405312000}\,i\alpha^7 R^3 k2 - \frac{1609}{898128000}\,i\lambda\,\alpha^7 R^3 k2 \\
& - \frac{611}{54432000}\,i\lambda\,\alpha^7 R^3 k1 + \frac{1}{261273600}\,i\lambda^3\alpha^9 R^3 k2 \\
& + \frac{397}{163296000}\,\alpha^8 R^2 k2 + \frac{5009}{326592000}\,\alpha^8 R^2 k1 \\
& + \frac{17}{8640}\,i\lambda^2\alpha^3 R^3 k1 + \frac{1}{5760}\,i\lambda^3\alpha^5 R^3 k1 \\
& + \frac{1}{360}\,\alpha^5 i\lambda\,R k2 + \frac{1}{720}\,\alpha^4\lambda^2 R^2 k2 \\
& - \frac{1}{720}\,\alpha^7 i\lambda\,R k1 - \frac{1}{1152}\,\alpha^6\lambda^2 R^2 k1 \\
& - \frac{17}{2160}\,\alpha^4\lambda\,R^2 k1 - \frac{193}{302400}\,i\lambda\,\alpha^3 R^3 k2 \\
& + \frac{1}{302400}\,\lambda^2\alpha^8 R^2 k2 - \frac{1}{172800}\,i\lambda^3\alpha^7 R^3 k1 \\
& - \frac{1}{1209600}\,i\lambda^3\alpha^7 R^3 k2 + \frac{29}{43200}\,i\lambda^2\alpha^3 R^3 k2 \\
& + \frac{1}{28800}\,i\lambda^3\alpha^5 R^3 k2 - \frac{131}{1959552000}\,\alpha^{10} R^2 k1 \\
& - \frac{1}{604800}\,\alpha^{10} k2 - \frac{1}{86400}\,\alpha^{10} k1 \\
& + \frac{1}{1440}\,\alpha^8 k1 - \frac{1}{540}\,\alpha^6 k2 \\
& + \frac{1}{120}\,\alpha^5 i\lambda\,R k1 - \frac{253}{14859936000}\,i\alpha^9 R^3 k1 \\
& + \frac{1}{5040}\,i\alpha^3 R^3 k2 - \frac{6533}{2942267328000}\,i\alpha^9 R^3 k2 \\
& - \frac{17}{2160}\,\alpha^5 i R k1 - \frac{1}{180}\,\alpha^6 k1 \\
& - \frac{1}{1440}\,i\lambda^3\alpha^3 R^3 k1 + \frac{1}{240}\,\alpha^4\lambda^2 R^2 k1 \\
& + \frac{193}{151200}\,\alpha^4 R^2 k2 + \frac{11}{21600}\,i\alpha^3 R^3 k1 \\
& + \frac{17}{14400}\,\alpha^7 i R k1 + \frac{1}{4032}\,\alpha^7 i R k2
\end{aligned}
$$

h_{16} continued

$$+ \frac{77}{21600}\,\alpha^4 R^2 k1 - \frac{29}{10800}\,\alpha^5 iRk2$$

$$+ \frac{1}{261273600}\,\alpha^{12} k2 - \frac{1}{4320}\,i\lambda^3\alpha^3 R^3 k2$$

$$- \frac{77}{43200}\,i\lambda\,\alpha^3 R^3 k1 - \frac{7}{16200}\,i\lambda^2\alpha^5 R^3 k1$$

$$- \frac{83}{907200}\,i\lambda^2\alpha^5 R^3 k2 - \frac{1}{3600}\,\alpha^7 i\lambda\,Rk2$$

$$- \frac{1}{5760}\,\alpha^6\lambda^2 R^2 k2 + \frac{1}{241920}\,i\lambda\,\alpha^9 Rk2$$

$$- \frac{29}{10800}\,\alpha^4\lambda\,R^2 k2 + \frac{557}{1814400}\,\alpha^6\lambda\,R^2 k2$$

$$+ \frac{377}{259200}\,\alpha^6\lambda\,R^2 k1 + \frac{1}{34560}\,i\lambda\,\alpha^9 Rk1$$

$$+ \frac{1}{43200}\,\alpha^8\lambda^2 R^2 k1 + \frac{1}{7200}\,\alpha^8 k2$$

$$- \frac{2167}{3628800}\,\alpha^6 R^2 k1 - \frac{29}{1209600}\,\alpha^9 iRk1$$

$$- \frac{1}{9676800}\,\alpha^{11} i\lambda\,Rk1 + \frac{41}{2903040}\,i\lambda^2\alpha^7 R^3 k1$$

$$- \frac{2281}{280215936000}\,\alpha^{10} R^2 k2 + \frac{1}{102643200}\,\alpha^{11} iRk2$$

$$+ \frac{11}{130636800}\,\alpha^{11} iRk1 - \frac{11}{130636800}\,i\lambda^2\alpha^9 R^3 k1$$

$$+ \frac{277}{130636800}\,i\lambda^2\alpha^7 R^3 k2 - \frac{13}{3628800}\,\alpha^9 iRk2$$

$$- \frac{1}{102643200}\,i\lambda^2\alpha^9 R^3 k2 - \frac{167}{1814400}\,i\alpha^5 R^3 k1$$

$$- \frac{1}{87091200}\,\alpha^{11} i\lambda\,Rk2 - \frac{8213}{359251200}\,i\alpha^5 R^3 k2$$

$$- \frac{1}{87091200}\,\alpha^{10}\lambda^2 R^2 k2 + \frac{131}{1959552000}\,i\lambda\,\alpha^9 R^3 k1$$

$$+ \frac{5207}{65318400}\,i\lambda\,\alpha^5 R^3 k2 - \frac{149}{26127360}\,\alpha^8\lambda\,R^2 k2$$

$$+ \frac{2281}{280215936000}\,i\lambda\,\alpha^9 R^3 k2 + \frac{1}{29030400}\,i\lambda^3\alpha^9 R^3 k1$$

$$- \frac{79}{2073600}\,\alpha^8\lambda\,R^2 k1 + \frac{2551}{7257600}\,i\lambda\,\alpha^5 R^3 k1$$

$$- \frac{1}{9676800}\,\alpha^{10}\lambda^2 R^2 k1 + \frac{1}{29030400}\,\alpha^{12} k1$$

$$
\begin{aligned}
h_{17} = {} & \frac{41}{3592512000}\,\alpha^{10}\lambda\,R^2 k2 + \frac{131}{1197504000}\,\alpha^{10}\lambda\,R^2 k1 \\
& - \frac{1357}{10886400}\,\alpha^6 R^2 k2 + \frac{11273}{5189184000}\,i\alpha^7 R^3 k1 \\
& + \frac{43889}{130767436800}\,i\alpha^7 R^3 k2 - \frac{2269}{1796256000}\,i\lambda\,\alpha^7 R^3 k2 \\
& - \frac{767}{85536000}\,i\lambda\,\alpha^7 R^3 k1 + \frac{1}{435456000}\,i\lambda^3\alpha^9 R^3 k2 \\
& + \frac{1027}{598752000}\,\alpha^8 R^2 k2 + \frac{14671}{1197504000}\,\alpha^8 R^2 k1 \\
& + \frac{13}{3600}\,i\lambda^2\alpha^3 R^3 k1 + \frac{1}{4800}\,i\lambda^3\alpha^5 R^3 k1 \\
& + \frac{1}{240}\,\alpha^5 i\lambda\,R k2 + \frac{1}{480}\,\alpha^4\lambda^2 R^2 k2 \\
& - \frac{1}{600}\,\alpha^7 i\lambda\,R k1 - \frac{1}{960}\,\alpha^6\lambda^2 R^2 k1 \\
& - \frac{13}{900}\,\alpha^4\lambda\,R^2 k1 - \frac{113}{134400}\,i\lambda\,\alpha^3 R^3 k2 \\
& + \frac{1}{403200}\,i\lambda^2\alpha^8 R^2 k2 - \frac{1}{201600}\,i\lambda^3\alpha^7 R^3 k1 \\
& - \frac{1}{1612800}\,i\lambda^3\alpha^7 R^3 k2 + \frac{41}{43200}\,i\lambda^2\alpha^3 R^3 k2 \\
& + \frac{1}{28800}\,i\lambda^3\alpha^5 R^3 k2 - \frac{653}{15567552000}\,\alpha^{10} R^2 k1 \\
& - \frac{1}{806400}\,\alpha^{10} k2 - \frac{1}{100800}\,\alpha^{10} k1 \\
& + \frac{1}{1200}\,\alpha^8 k1 - \frac{1}{360}\,\alpha^6 k2 \\
& + \frac{1}{60}\,\alpha^5 i\lambda\,R k1 - \frac{53}{5189184000}\,i\alpha^9 R^3 k1 \\
& + \frac{73}{302400}\,i\alpha^3 R^3 k2 - \frac{1591}{1307674368000}\,i\alpha^9 R^3 k2 \\
& - \frac{13}{900}\,\alpha^5 i R k1 - \frac{1}{90}\,\alpha^6 k1 \\
& - \frac{1}{720}\,i\lambda^3\alpha^3 R^3 k1 + \frac{1}{120}\,\alpha^4\lambda^2 R^2 k1 \\
& + \frac{113}{67200}\,\alpha^4 R^2 k2 + \frac{443}{604800}\,i\alpha^3 R^3 k1 \\
& + \frac{23}{16800}\,\alpha^7 i R k1 + \frac{97}{403200}\,\alpha^7 i R k2
\end{aligned}
$$

h_{17} continued

$$+ \frac{443}{75600}\, \alpha^4 R^2 k1 - \frac{41}{10800}\, \alpha^5 iRk2$$

$$+ \frac{1}{435456000}\, \alpha^{12} k2 - \frac{1}{2880}\, i\lambda^3 \alpha^3 R^3 k2$$

$$- \frac{443}{151200}\, i\lambda\, \alpha^3 R^3 k1 - \frac{151}{302400}\, i\lambda^2 \alpha^5 R^3 k1$$

$$- \frac{107}{1209600}\, i\lambda^2 \alpha^5 R^3 k2 - \frac{1}{3600}\, \alpha^7 i\lambda\, Rk2$$

$$- \frac{1}{5760}\, \alpha^6 \lambda^2 R^2 k2 + \frac{1}{322560}\, i\lambda\, \alpha^9 Rk2$$

$$- \frac{41}{10800}\, \alpha^4 \lambda\, R^2 k2 + \frac{719}{2419200}\, \alpha^6 \lambda\, R^2 k2$$

$$+ \frac{509}{302400}\, \alpha^6 \lambda\, R^2 k1 + \frac{1}{40320}\, i\lambda\, \alpha^9 Rk1$$

$$+ \frac{1}{50400}\, \alpha^8 \lambda^2 R^2 k1 + \frac{1}{7200}\, \alpha^8 k2$$

$$- \frac{7099}{10886400}\, \alpha^6 R^2 k1 - \frac{29}{1451520}\, \alpha^9 iRk1$$

$$- \frac{1}{14515200}\, \alpha^{11} i\lambda\, Rk1 + \frac{1}{85050}\, i\lambda^2 \alpha^7 R^3 k1$$

$$- \frac{11}{2377589760}\, \alpha^{10} R^2 k2 + \frac{41}{7185024000}\, \alpha^{11} iRk2$$

$$+ \frac{131}{2395008000}\, \alpha^{11} iRk1 - \frac{131}{2395008000}\, i\lambda^2 \alpha^9 R^3 k1$$

$$+ \frac{337}{217728000}\, i\lambda^2 \alpha^7 R^3 k2 - \frac{19}{7257600}\, \alpha^9 iRk2$$

$$- \frac{41}{7185024000}\, i\lambda^2 \alpha^9 R^3 k2 - \frac{6211}{66528000}\, i\alpha^5 R^3 k1$$

$$- \frac{1}{145152000}\, \alpha^{11} i\lambda\, Rk2 - \frac{72109}{3592512000}\, i\alpha^5 R^3 k2$$

$$- \frac{1}{145152000}\, \alpha^{10} \lambda^2 R^2 k2 + \frac{653}{15567552000}\, i\lambda\, \alpha^9 R^3 k1$$

$$+ \frac{1147}{15552000}\, i\lambda\, \alpha^5 R^3 k2 - \frac{907}{217728000}\, \alpha^8 \lambda\, R^2 k2$$

$$+ \frac{11}{2377589760}\, i\lambda\, \alpha^9 R^3 k2 + \frac{1}{43545600}\, i\lambda^3 \alpha^9 R^3 k1$$

$$- \frac{691}{21772800}\, \alpha^8 \lambda\, R^2 k1 + \frac{4177}{10886400}\, i\lambda\, \alpha^5 R^3 k1$$

$$- \frac{1}{14515200}\, \alpha^{10} \lambda^2 R^2 k1 + \frac{1}{43545600}\, \alpha^{12} k1$$

$$
\begin{aligned}
h_{18} = {}& \frac{47}{9340531200}\,\alpha^{10}\lambda\,R^2 k2 + \frac{19}{359251200}\,\alpha^{10}\lambda\,R^2 k1 \\
& - \frac{9629}{119750400}\,\alpha^6 R^2 k2 + \frac{51173}{43589145600}\,i\alpha^7 R^3 k1 \\
& + \frac{64651}{392302310400}\,i\alpha^7 R^3 k2 - \frac{433}{667180800}\,i\lambda\,\alpha^7 R^3 k2 \\
& - \frac{23}{4490640}\,i\lambda\,\alpha^7 R^3 k1 + \frac{1}{958003200}\,i\lambda^3 \alpha^9 R^3 k2 \\
& + \frac{4117}{4670265600}\,\alpha^8 R^2 k2 + \frac{457}{65318400}\,\alpha^8 R^2 k1 \\
& + \frac{7}{1728}\,i\lambda^2 \alpha^3 R^3 k1 + \frac{1}{5760}\,i\lambda^3 \alpha^5 R^3 k1 \\
& + \frac{1}{240}\,\alpha^5 i\lambda\,R k2 + \frac{1}{480}\,\alpha^4 \lambda^2 R^2 k2 \\
& - \frac{1}{720}\,\alpha^7 i\lambda\,R k1 - \frac{1}{1152}\,\alpha^6 \lambda^2 R^2 k1 \\
& - \frac{7}{432}\,\alpha^4 \lambda\,R^2 k1 - \frac{1559}{2177280}\,i\lambda\,\alpha^3 R^3 k2 \\
& + \frac{1}{725760}\,\lambda^2 \alpha^8 R^2 k2 - \frac{1}{322560}\,i\lambda^3 \alpha^7 R^3 k1 \\
& - \frac{1}{2903040}\,i\lambda^3 \alpha^7 R^3 k2 + \frac{53}{60480}\,i\lambda^2 \alpha^3 R^3 k2 \\
& + \frac{1}{40320}\,i\lambda^3 \alpha^5 R^3 k2 - \frac{2549}{130767436800}\,\alpha^{10} R^2 k1 \\
& - \frac{1}{1451520}\,\alpha^{10} k2 - \frac{1}{161280}\,\alpha^{10} k1 \\
& + \frac{1}{1440}\,\alpha^8 k1 - \frac{1}{360}\,\alpha^6 k2 \\
& + 1/48\,\alpha^5 i\lambda\,R k1 - \frac{79}{17435658240}\,i\alpha^9 R^3 k1 \\
& + \frac{227}{1197504}\,i\alpha^3 R^3 k2 - \frac{391}{784604620800}\,i\alpha^9 R^3 k2 \\
& - \frac{7}{432}\,\alpha^5 i R k1 - \frac{1}{72}\,\alpha^6 k1 \\
& - \frac{1}{576}\,i\lambda^3 \alpha^3 R^3 k1 + \frac{1}{96}\,\alpha^4 \lambda^2 R^2 k1 \\
& + \frac{1559}{1088640}\,\alpha^4 R^2 k2 + \frac{79}{120960}\,i\alpha^3 R^3 k1 \\
& + \frac{29}{26880}\,\alpha^7 i R k1 + \frac{17}{103680}\,\alpha^7 i R k2
\end{aligned}
$$

h_{18} continued

$$+ \frac{79}{13440} \alpha^4 R^2 k1 - \frac{53}{15120} \alpha^5 iRk2$$

$$+ \frac{1}{958003200} \alpha^{12} k2 - \frac{1}{2880} i\lambda^3 \alpha^3 R^3 k2$$

$$- \frac{79}{26880} i\lambda \alpha^3 R^3 k1 - \frac{19}{48384} i\lambda^2 \alpha^5 R^3 k1$$

$$- \frac{131}{2177280} i\lambda^2 \alpha^5 R^3 k2 - \frac{1}{5040} \alpha^7 i\lambda Rk2$$

$$- \frac{1}{8064} \alpha^6 \lambda^2 R^2 k2 + \frac{1}{580608} i\lambda \alpha^9 Rk2$$

$$- \frac{53}{15120} \alpha^4 \lambda R^2 k2 + \frac{881}{4354560} \alpha^6 \lambda R^2 k2$$

$$+ \frac{641}{483840} \alpha^6 \lambda R^2 k1 + \frac{1}{64512} i\lambda \alpha^9 Rk1$$

$$+ \frac{1}{80640} \alpha^8 \lambda^2 R^2 k1 + \frac{1}{10080} \alpha^8 k2$$

$$- \frac{653}{1360800} \alpha^6 R^2 k1$$

$$- \frac{29}{2419200} \alpha^9 iRk1 - \frac{1}{29030400} \alpha^{11} i\lambda Rk1$$

$$+ \frac{307}{43545600} i\lambda^2 \alpha^7 R^3 k1 - \frac{773}{392302310400} \alpha^{10} R^2 k2$$

$$+ \frac{47}{18681062400} \alpha^{11} iRk2 + \frac{19}{718502400} \alpha^{11} iRk1$$

$$- \frac{19}{718502400} i\lambda^2 \alpha^9 R^3 k1 + \frac{397}{479001600} i\lambda^2 \alpha^7 R^3 k2$$

$$- \frac{1}{712800} \alpha^9 iRk2 - \frac{47}{18681062400} i\lambda^2 \alpha^9 R^3 k2$$

$$- \frac{1699}{26611200} i\alpha^5 R^3 k1 - \frac{1}{319334400} \alpha^{11} i\lambda Rk2$$

$$- \frac{7613}{622702080} i\alpha^5 R^3 k2 - \frac{1}{319334400} \alpha^{10} \lambda^2 R^2 k2$$

$$+ \frac{2549}{130767436800} i\lambda \alpha^9 R^3 k1 + \frac{11387}{239500800} i\lambda \alpha^5 R^3 k2$$

$$- \frac{1069}{479001600} \alpha^8 \lambda R^2 k2 + \frac{773}{392302310400} i\lambda \alpha^9 R^3 k2$$

$$+ \frac{1}{87091200} i\lambda^3 \alpha^9 R^3 k1 - \frac{829}{43545600} \alpha^8 \lambda R^2 k1$$

$$+ \frac{683}{2419200} i\lambda \alpha^5 R^3 k1 - \frac{1}{29030400} \alpha^{10} \lambda^2 R^2 k1$$

$$+ \frac{1}{87091200} \alpha^{12} k1$$

$$
\begin{aligned}
h_{19} = {} & \frac{53}{32691859200}\, \alpha^{10} \lambda\, R^2 k2 + \frac{173}{9340531200}\, \alpha^{10} \lambda\, R^2 k1 \\
& - \frac{4309}{119750400}\, \alpha^6 R^2 k2 + \frac{1877}{4191264000}\, i\alpha^7 R^3 k1 \\
& + \frac{151103}{2615348736000}\, i\alpha^7 R^3 k2 - \frac{779}{3269185920}\, i\lambda\, \alpha^7 R^3 k2 \\
& - \frac{17}{8236800}\, i\lambda\, \alpha^7 R^3 k1 + \frac{1}{2874009600}\, i\lambda^3 \alpha^9 R^3 k2 \\
& + \frac{10583}{32691859200}\, \alpha^8 R^2 k2 + \frac{26329}{9340531200}\, \alpha^8 R^2 k1 \\
& + \frac{11}{3780}\, i\lambda^2 \alpha^3 R^3 k1 + \frac{1}{10080}\, i\lambda^3 \alpha^5 R^3 k1 \\
& + \frac{1}{360}\, \alpha^5 i\lambda\, R k2 + \frac{1}{720}\, \alpha^4 \lambda^2 R^2 k2 \\
& - \frac{1}{1260}\, \alpha^7 i\lambda\, R k1 - \frac{1}{2016}\, \alpha^6 \lambda^2 R^2 k1 \\
& - \frac{11}{945}\, \alpha^4 \lambda\, R^2 k1 - \frac{7}{17280}\, i\lambda\, \alpha^3 R^3 k2 \\
& + \frac{1}{1814400}\, \lambda^2 \alpha^8 R^2 k2 - \frac{1}{725760}\, i\lambda^3 \alpha^7 R^3 k1 \\
& - \frac{1}{7257600}\, i\lambda^3 \alpha^7 R^3 k2 + \frac{13}{24192}\, i\lambda^2 \alpha^3 R^3 k2 \\
& + \frac{1}{80640}\, i\lambda^3 \alpha^5 R^3 k2 - \frac{247}{37721376000}\, \alpha^{10} R^2 k1 \\
& - \frac{1}{3628800}\, \alpha^{10} k2 - \frac{1}{362880}\, \alpha^{10} k1 \\
& + \frac{1}{2520}\, \alpha^8 k1 - \frac{1}{540}\, \alpha^6 k2 \\
& + \frac{1}{60}\, \alpha^5 i\lambda\, R k1 - \frac{32303}{22230464256000}\, i\alpha^9 R^3 k1 \\
& + \frac{1}{10080}\, i\alpha^3 R^3 k2 - \frac{89263}{600222534912000}\, i\alpha^9 R^3 k2 \\
& - \frac{11}{945}\, \alpha^5 i R k1 - \frac{1}{90}\, \alpha^6 k1 \\
& - \frac{1}{720}\, i\lambda^3 \alpha^3 R^3 k1 + \frac{1}{120}\, \alpha^4 \lambda^2 R^2 k1 \\
& + \frac{7}{8640}\, \alpha^4 R^2 k2 + \frac{23}{60480}\, i\alpha^3 R^3 k1 \\
& + \frac{1}{1728}\, \alpha^7 i R k1 + \frac{47}{604800}\, \alpha^7 i R k2
\end{aligned}
$$

h_{19} continued

$$+ \frac{23}{6048}\, \alpha^4 R^2 k1 - \frac{13}{6048}\, \alpha^5 i R k2$$

$$+ \frac{1}{2874009600}\, \alpha^{12} k2 - \frac{1}{4320}\, i\lambda^3 \alpha^3 R^3 k2$$

$$- \frac{23}{12096}\, i\lambda\, \alpha^3 R^3 k1 - \frac{229}{1088640}\, i\lambda^2 \alpha^5 R^3 k1$$

$$- \frac{31}{1088640}\, i\lambda^2 \alpha^5 R^3 k2 - \frac{1}{10080}\, \alpha^7 i\lambda\, R k2$$

$$- \frac{1}{16128}\, \alpha^6 \lambda^2 R^2 k2 + \frac{1}{1451520}\, i\lambda\, \alpha^9 R k2$$

$$- \frac{13}{6048}\, \alpha^4 \lambda\, R^2 k2 + \frac{149}{1555200}\, \alpha^6 \lambda\, R^2 k2$$

$$+ \frac{773}{1088640}\, \alpha^6 \lambda\, R^2 k1 + \frac{1}{145152}\, i\lambda\, \alpha^9 R k1$$

$$+ \frac{1}{181440}\, \alpha^8 \lambda^2 R^2 k1 + \frac{1}{20160}\, \alpha^8 k2$$

$$- \frac{14381}{59875200}\, \alpha^6 R^2 k1 - \frac{29}{5702400}\, \alpha^9 i R k1$$

$$- \frac{1}{79833600}\, \alpha^{11} i\lambda\, R k1 + \frac{179}{59875200}\, i\lambda^2 \alpha^7 R^3 k1$$

$$- \frac{4801}{7846046208000}\, \alpha^{10} R^2 k2 + \frac{53}{65383718400}\, \alpha^{11} i R k2$$

$$+ \frac{173}{18681062400}\, \alpha^{11} i R k1 - \frac{173}{18681062400}\, i\lambda^2 \alpha^9 R^3 k1$$

$$+ \frac{457}{1437004800}\, i\lambda^2 \alpha^7 R^3 k2 - \frac{43}{79833600}\, \alpha^9 i R k2$$

$$- \frac{53}{65383718400}\, i\lambda^2 \alpha^9 R^3 k2 - \frac{1189}{39916800}\, i\alpha^5 R^3 k1$$

$$- \frac{1}{958003200}\, \alpha^{11} i\lambda\, R k2 - \frac{1181}{228614400}\, i\alpha^5 R^3 k2$$

$$- \frac{1}{958003200}\, \alpha^{10} \lambda^2 R^2 k2 + \frac{247}{37721376000}\, i\lambda\, \alpha^9 R^3 k1$$

$$+ \frac{2183}{102643200}\, i\lambda\, \alpha^5 R^3 k2 - \frac{1231}{1437004800}\, \alpha^8 \lambda\, R^2 k2$$

$$+ \frac{4801}{7846046208000}\, i\lambda\, \alpha^9 R^3 k2 + \frac{1}{239500800}\, i\lambda^3 \alpha^9 R^3 k1$$

$$- \frac{967}{119750400}\, \alpha^8 \lambda\, R^2 k1 + \frac{8461}{59875200}\, i\lambda\, \alpha^5 R^3 k1$$

$$- \frac{1}{79833600}\, \alpha^{10} \lambda^2 R^2 k1 + \frac{1}{239500800}\, \alpha^{12} k1$$

$$
\begin{aligned}
h_{20} =\ & \frac{59}{163459296000}\,\alpha^{10}\lambda\,R^2 k2 + \frac{97}{21794572800}\,\alpha^{10}\lambda\,R^2 k1 \\
& - \frac{1283}{119750400}\,\alpha^6 R^2 k2 + \frac{33479}{290594304000}\,i\alpha^7 R^3 k1 \\
& + \frac{122483}{8892185702400}\,i\alpha^7 R^3 k2 - \frac{4861}{81729648000}\,i\lambda\,\alpha^7 R^3 k2 \\
& - \frac{6103}{10897286400}\,i\lambda\,\alpha^7 R^3 k1 + \frac{1}{12454041600}\,i\lambda^3\alpha^9 R^3 k2 \\
& + \frac{629}{7783776000}\,\alpha^8 R^2 k2 + \frac{2381}{3113510400}\,\alpha^8 R^2 k1 \\
& + \frac{53}{40320}\,i\lambda^2\alpha^3 R^3 k1 + \frac{1}{26880}\,i\lambda^3\alpha^5 R^3 k1 \\
& + \frac{1}{840}\,\alpha^5 i\lambda\,R k2 + \frac{1}{1680}\,\alpha^4\lambda^2 R^2 k2 \\
& - \frac{1}{3360}\,\alpha^7 i\lambda\,R k1 - \frac{1}{5376}\,\alpha^6\lambda^2 R^2 k1 \\
& - \frac{53}{10080}\,\alpha^4\lambda\,R^2 k1 - \frac{197}{1330560}\,i\lambda\,\alpha^3 R^3 k2 \\
& + \frac{1}{6652800}\,\lambda^2\alpha^8 R^2 k2 - \frac{1}{2419200}\,i\lambda^3\alpha^7 R^3 k1 \\
& - \frac{1}{26611200}\,i\lambda^3\alpha^7 R^3 k2 + \frac{11}{51840}\,i\lambda^2\alpha^3 R^3 k2 \\
& + \frac{1}{241920}\,i\lambda^3\alpha^5 R^3 k2 - \frac{263}{174356582400}\,\alpha^{10} R^2 k1 \\
& - \frac{1}{13305600}\,\alpha^{10} k2 - \frac{1}{1209600}\,\alpha^{10} k1 \\
& + \frac{1}{6720}\,\alpha^8 k1 - \frac{1}{1260}\,\alpha^6 k2 \\
& + \frac{1}{120}\,\alpha^5 i\lambda\,R k1 - \frac{79}{247005158400}\,i\alpha^9 R^3 k1 \\
& + \frac{437}{12972960}\,i\alpha^3 R^3 k2 - \frac{1849}{60339831552000}\,i\alpha^9 R^3 k2 \\
& - \frac{53}{10080}\,\alpha^5 iR k1 - \frac{1}{180}\,\alpha^6 k1 \\
& - \frac{1}{1440}\,i\lambda^3\alpha^3 R^3 k1 \\
& + \frac{1}{240}\,\alpha^4\lambda^2 R^2 k1 + \frac{197}{665280}\,\alpha^4 R^2 k2 \\
& + \frac{283}{1995840}\,i\alpha^3 R^3 k1 + \frac{41}{201600}\,\alpha^7 iR k1
\end{aligned}
$$

h_{20} continued

$$+ \frac{163}{6652800} \, \alpha^7 i R k2 + \frac{283}{181440} \, \alpha^4 R^2 k1$$

$$- \frac{11}{12960} \, \alpha^5 i R k2 + \frac{1}{12454041600} \, \alpha^{12} k2$$

$$- \frac{1}{10080} \, i \lambda^3 \alpha^3 R^3 k2 - \frac{283}{362880} \, i \lambda \, \alpha^3 R^3 k1$$

$$- \frac{67}{907200} \, i \lambda^2 \alpha^5 R^3 k1 - \frac{179}{19958400} \, i \lambda^2 \alpha^5 R^3 k2$$

$$- \frac{1}{30240} \, \alpha^7 i \lambda \, R k2 - \frac{1}{48384} \, \alpha^6 \lambda^2 R^2 k2$$

$$+ \frac{1}{5322240} \, i \lambda \, \alpha^9 R k2 - \frac{11}{12960} \, \alpha^4 \lambda \, R^2 k2$$

$$+ \frac{241}{7983360} \, \alpha^6 \lambda \, R^2 k2 + \frac{181}{725760} \, \alpha^6 \lambda \, R^2 k1$$

$$+ \frac{1}{483840} \, i \lambda \, \alpha^9 R k1 + \frac{1}{604800} \, \alpha^8 \lambda^2 R^2 k1$$

$$+ \frac{1}{60480} \, \alpha^8 k2 - \frac{859}{10886400} \, \alpha^6 R^2 k1$$

$$- \frac{29}{19958400} \, \alpha^9 i R k1 - \frac{1}{319334400} \, \alpha^{11} i \lambda \, R k1$$

$$+ \frac{409}{479001600} \, i \lambda^2 \alpha^7 R^3 k1 - \frac{5833}{44460928512000} \, \alpha^{10} R^2 k2$$

$$+ \frac{59}{326918592000} \, \alpha^{11} i R k2 + \frac{97}{43589145600} \, \alpha^{11} i R k1$$

$$- \frac{97}{43589145600} \, i \lambda^2 \alpha^9 R^3 k1 + \frac{47}{566092800} \, i \lambda^2 \alpha^7 R^3 k2$$

$$- \frac{73}{518918400} \, \alpha^9 i R k2 - \frac{59}{326918592000} \, i \lambda^2 \alpha^9 R^3 k2$$

$$- \frac{2771}{302702400} \, i \alpha^5 R^3 k1 - \frac{1}{4151347200} \, \alpha^{11} i \lambda \, R k2$$

$$- \frac{237733}{163459296000} \, i \alpha^5 R^3 k2 - \frac{1}{4151347200} \, \alpha^{10} \lambda^2 R^2 k2$$

$$+ \frac{263}{174356582400} \, i \lambda \, \alpha^9 R^3 k1 + \frac{19711}{3113510400} \, i \lambda \, \alpha^5 R^3 k2$$

$$- \frac{199}{889574400} \, \alpha^8 \lambda \, R^2 k2 + \frac{5833}{44460928512000} \, i \lambda \, \alpha^9 R^3 k2$$

$$+ \frac{1}{958003200} \, i \lambda^3 \alpha^9 R^3 k1 - \frac{221}{95800320} \, \alpha^8 \lambda \, R^2 k1$$

$$+ \frac{11119}{239500800} \, i \lambda \, \alpha^5 R^3 k1 - \frac{1}{319334400} \, \alpha^{10} \lambda^2 R^2 k1$$

$$+ \frac{1}{958003200} \, \alpha^{12} k1$$

$$
\begin{aligned}
h_{21} =\ & \frac{1}{20118067200}\,\alpha^{10}\lambda\,R^2 k2 + \frac{43}{65383718400}\,\alpha^{10}\lambda\,R^2 k1 \\
& - \frac{4177}{2179457280}\,\alpha^6 R^2 k2 + \frac{133403}{7410154752000}\,i\alpha^7 R^3 k1 \\
& + \frac{803381}{400148356608000}\,i\alpha^7 R^3 k2 - \frac{11}{1213056000}\,i\lambda\,\alpha^7 R^3 k2 \\
& - \frac{15061}{163459296000}\,i\lambda\,\alpha^7 R^3 k1 + \frac{1}{87178291200}\,i\lambda^3\alpha^9 R^3 k2 \\
& + \frac{1007}{81729648000}\,\alpha^8 R^2 k2 + \frac{41123}{326918592000}\,\alpha^8 R^2 k1 \\
& + \frac{31}{90720}\,i\lambda^2\alpha^3 R^3 k1 + \frac{1}{120960}\,i\lambda^3\alpha^5 R^3 k1 \\
& + \frac{1}{3360}\,\alpha^5 i\lambda\,R k2 + \frac{1}{6720}\,\alpha^4\lambda^2 R^2 k2 \\
& - \frac{1}{15120}\,\alpha^7 i\lambda\,R k1 - \frac{1}{24192}\,\alpha^6\lambda^2 R^2 k1 \\
& - \frac{31}{22680}\,\alpha^4\lambda\,R^2 k1 - \frac{3809}{119750400}\,i\lambda\,\alpha^3 R^3 k2 \\
& + \frac{1}{39916800}\,\lambda^2\alpha^8 R^2 k2 - \frac{1}{13305600}\,i\lambda^3\alpha^7 R^3 k1 \\
& - \frac{1}{159667200}\,i\lambda^3\alpha^7 R^3 k2 + \frac{89}{1814400}\,i\lambda^2\alpha^3 R^3 k2 \\
& + \frac{1}{1209600}\,i\lambda^3\alpha^5 R^3 k2 - \frac{4751}{22230464256000}\,\alpha^{10} R^2 k1 \\
& - \frac{1}{79833600}\,\alpha^{10} k2 - \frac{1}{6652800}\,\alpha^{10} k1 \\
& + \frac{1}{30240}\,\alpha^8 k1 - \frac{1}{5040}\,\alpha^6 k2 \\
& + \frac{1}{420}\,\alpha^5 i\lambda\,R k1 - \frac{359}{8281937664000}\,i\alpha^9 R^3 k1 \\
& + \frac{283}{41912640}\,i\alpha^3 R^3 k2 - \frac{3713}{950352346944000}\,i\alpha^9 R^3 k2 \\
& - \frac{31}{22680}\,\alpha^5 i R k1 - \frac{1}{630}\,\alpha^6 k1 \\
& - \frac{1}{5040}\,i\lambda^3\alpha^3 R^3 k1 + \frac{1}{840}\,\alpha^4\lambda^2 R^2 k1 \\
& + \frac{3809}{59875200}\,\alpha^4 R^2 k2 + \frac{617}{19958400}\,i\alpha^3 R^3 k1 \\
& + \frac{47}{1108800}\,\alpha^7 i R k1 + \frac{37}{7983360}\,\alpha^7 i R k2
\end{aligned}
$$

h_{21} continued

$$+ \frac{617}{1663200}\,\alpha^4 R^2 k1 - \frac{89}{453600}\,\alpha^5 iRk2$$

$$+ \frac{1}{87178291200}\,\alpha^{12} k2 - \frac{1}{40320}\,i\lambda^3\alpha^3 R^3 k2$$

$$- \frac{617}{3326400}\,i\lambda\,\alpha^3 R^3 k1 - \frac{307}{19958400}\,i\lambda^2\alpha^5 R^3 k1$$

$$- \frac{29}{17107200}\,i\lambda^2\alpha^5 R^3 k2 - \frac{1}{151200}\,\alpha^7 i\lambda\,Rk2$$

$$- \frac{1}{241920}\,\alpha^6\lambda^2 R^2 k2 + \frac{1}{31933440}\,i\lambda\,\alpha^9 Rk2$$

$$- \frac{89}{453600}\,\alpha^4\lambda\,R^2 k2 + \frac{1367}{239500800}\,\alpha^6\lambda\,R^2 k2$$

$$+ \frac{1037}{19958400}\,\alpha^6\lambda\,R^2 k1 + \frac{1}{2661120}\,i\lambda\,\alpha^9 Rk1$$

$$+ \frac{1}{3326400}\,\alpha^8\lambda^2 R^2 k1 + \frac{1}{302400}\,\alpha^8 k2$$

$$- \frac{23999}{1556755200}\,\alpha^6 R^2 k1 - \frac{29}{115315200}\,\alpha^9 iRk1$$

$$- \frac{1}{2075673600}\,\alpha^{11} i\lambda\,Rk1 + \frac{23}{155675520}\,i\lambda^2\alpha^7 R^3 k1$$

$$- \frac{6961}{400148356608000}\,\alpha^{10} R^2 k2 + \frac{1}{40236134400}\,\alpha^{11} iRk2$$

$$+ \frac{43}{130767436800}\,\alpha^{11} iRk1 - \frac{43}{130767436800}\,i\lambda^2\alpha^9 R^3 k1$$

$$+ \frac{577}{43589145600}\,i\lambda^2\alpha^7 R^3 k2 - \frac{163}{7264857600}\,\alpha^9 iRk2$$

$$- \frac{1}{40236134400}\,i\lambda^2\alpha^9 R^3 k2 - \frac{2033}{1210809600}\,i\alpha^5 R^3 k1$$

$$- \frac{1}{29059430400}\,\alpha^{11} i\lambda\,Rk2 - \frac{21487}{87178291200}\,i\alpha^5 R^3 k2$$

$$- \frac{1}{29059430400}\,\alpha^{10}\lambda^2 R^2 k2 + \frac{4751}{22230464256000}\,i\lambda\,\alpha^9 R^3 k1$$

$$+ \frac{24677}{21794572800}\,i\lambda\,\alpha^5 R^3 k2 - \frac{311}{8717829120}\,\alpha^8\lambda\,R^2 k2$$

$$+ \frac{6961}{400148356608000}\,i\lambda\,\alpha^9 R^3 k2 + \frac{1}{6227020800}\,i\lambda^3\alpha^9 R^3 k1$$

$$- \frac{113}{283046400}\,\alpha^8\lambda\,R^2 k1 + \frac{523}{57657600}\,i\lambda\,\alpha^5 R^3 k1$$

$$- \frac{1}{2075673600}\,\alpha^{10}\lambda^2 R^2 k1 + \frac{1}{6227020800}\,\alpha^{12} k1$$

$$
\begin{aligned}
h_{22} ={}& \frac{71}{22230464256000}\,\alpha^{10}\lambda\,R^2 k2 + \frac{59}{1307674368000}\,\alpha^{10}\lambda\,R^2 k1 \\
& - \frac{131}{838252800}\,\alpha^6 R^2 k2 + \frac{19181}{14820309504000}\,i\alpha^7 R^3 k1 \\
& + \frac{1759}{12996271411200}\,i\alpha^7 R^3 k2 - \frac{7099}{11115232128000}\,i\lambda\,\alpha^7 R^3 k2 \\
& - \frac{1517}{217945728000}\,i\lambda\,\alpha^7 R^3 k1 + \frac{1}{1307674368000}\,i\lambda^3\alpha^9 R^3 k2 \\
& + \frac{53}{61072704000}\,\alpha^8 R^2 k2 + \frac{1553}{163459296000}\,\alpha^8 R^2 k1 \\
& + \frac{71}{1814400}\,i\lambda^2\alpha^3 R^3 k1 + \frac{1}{1209600}\,i\lambda^3\alpha^5 R^3 k1 \\
& + \frac{1}{30240}\,\alpha^5 i\lambda\,R k2 + \frac{1}{60480}\,\alpha^4\lambda^2 R^2 k2 \\
& - \frac{1}{151200}\,\alpha^7 i\lambda\,R k1 - \frac{1}{241920}\,\alpha^6\lambda^2 R^2 k1 \\
& - \frac{71}{453600}\,\alpha^4\lambda\,R^2 k1 - \frac{227}{74131200}\,i\lambda\,\alpha^3 R^3 k2 \\
& + \frac{1}{518918400}\,\lambda^2\alpha^8 R^2 k2 - \frac{1}{159667200}\,i\lambda^3\alpha^7 R^3 k1 \\
& - \frac{1}{2075673600}\,i\lambda^3\alpha^7 R^3 k2 + \frac{101}{19958400}\,i\lambda^2\alpha^3 R^3 k2 \\
& + \frac{1}{13305600}\,i\lambda^3\alpha^5 R^3 k2 - \frac{433}{30780642816000}\,\alpha^{10} R^2 k1 \\
& - \frac{1}{1037836800}\,\alpha^{10} k2 - \frac{1}{79833600}\,\alpha^{10} k1 \\
& + \frac{1}{302400}\,\alpha^8 k1 - \frac{1}{45360}\,\alpha^6 k2 \\
& + \frac{1}{3360}\,\alpha^5 i\lambda\,R k1 - \frac{3463}{1267136462592000}\,i\alpha^9 R^3 k1 \\
& + \frac{79}{129729600}\,i\alpha^3 R^3 k2 - \frac{9287}{39914798571648000}\,i\alpha^9 R^3 k2 \\
& - \frac{71}{453600}\,\alpha^5 i R k1 - \frac{1}{5040}\,\alpha^6 k1 \\
& - \frac{1}{40320}\,i\lambda^3\alpha^3 R^3 k1 + \frac{1}{6720}\,\alpha^4\lambda^2 R^2 k1 \\
& + \frac{227}{37065600}\,\alpha^4 R^2 k2 + \frac{71}{23587200}\,i\alpha^3 R^3 k1 \\
& + \frac{53}{13305600}\,\alpha^7 i R k1 + \frac{23}{57657600}\,\alpha^7 i R k2
\end{aligned}
$$

h_{22} continued

$$+ \frac{71}{1814400}\, \alpha^4 R^2 k1 - \frac{101}{4989600}\, \alpha^5 i R k2$$

$$+ \frac{1}{1307674368000}\, \alpha^{12} k2 - \frac{1}{362880}\, i\lambda^3 \alpha^3 R^3 k2$$

$$- \frac{71}{3628800}\, i\lambda\, \alpha^3 R^3 k1 - \frac{173}{119750400}\, i\lambda^2 \alpha^5 R^3 k1$$

$$- \frac{227}{1556755200}\, i\lambda^2 \alpha^5 R^3 k2 - \frac{1}{1663200}\, \alpha^7 i\lambda\, R k2$$

$$- \frac{1}{2661120}\, \alpha^6 \lambda^2 R^2 k2 + \frac{1}{415134720}\, i\lambda\, \alpha^9 R k2$$

$$- \frac{101}{4989600}\, \alpha^4 \lambda\, R^2 k2 + \frac{139}{283046400}\, \alpha^6 \lambda\, R^2 k2$$

$$+ \frac{167}{34214400}\, \alpha^6 \lambda\, R^2 k1 + \frac{1}{31933440}\, i\lambda\, \alpha^9 R k1$$

$$+ \frac{1}{39916800}\, \alpha^8 \lambda^2 R^2 k1 + \frac{1}{3326400}\, \alpha^8 k2$$

$$- \frac{7421}{5448643200}\, \alpha^6 R^2 k1 - \frac{29}{1452971520}\, \alpha^9 i R k1$$

$$- \frac{1}{29059430400}\, \alpha^{11} i\lambda\, R k1 + \frac{73}{6227020800}\, i\lambda^2 \alpha^7 R^3 k1$$

$$- \frac{1637}{1520563755110400}\, \alpha^{10} R^2 k2 + \frac{71}{44460928512000}\, \alpha^{11} i R k2$$

$$+ \frac{59}{2615348736000}\, \alpha^{11} i R k1 - \frac{59}{2615348736000}\, i\lambda^2 \alpha^9 R^3 k1$$

$$+ \frac{1}{1026432000}\, i\lambda^2 \alpha^7 R^3 k2 - \frac{1}{605404800}\, \alpha^9 i R k2$$

$$- \frac{71}{44460928512000}\, i\lambda^2 \alpha^9 R^3 k2 - \frac{20303}{145297152000}\, i\alpha^5 R^3 k1$$

$$- \frac{1}{435891456000}\, \alpha^{11} i\lambda\, R k2 - \frac{424159}{22230464256000}\, i\alpha^5 R^3 k2$$

$$- \frac{1}{435891456000}\, \alpha^{10} \lambda^2 R^2 k2 + \frac{433}{30780642816000}\, i\lambda\, \alpha^9 R^3 k1$$

$$+ \frac{30179}{326918592000}\, i\lambda\, \alpha^5 R^3 k2 - \frac{1717}{653837184000}\, \alpha^8 \lambda\, R^2 k2$$

$$+ \frac{1637}{1520563755110400}\, i\lambda\, \alpha^9 R^3 k2 + \frac{1}{87178291200}\, i\lambda^3 \alpha^9 R^3 k1$$

$$- \frac{1381}{43589145600}\, \alpha^8 \lambda\, R^2 k1 + \frac{17467}{21794572800}\, i\lambda\, \alpha^5 R^3 k1$$

$$- \frac{1}{29059430400}\, \alpha^{10} \lambda^2 R^2 k1 + \frac{1}{87178291200}\, \alpha^{12} k1$$

APPENDIX B: PERTURBATION SOLUTION FOR BLASIUS EQUATION

In 1989, Bender et al.[22] used a δ-perturbation expansion method to obtain an acceptable approximation for $y''(0)$ for the case $\alpha = \beta = 0$. In this section, we use the same technique presented by Bender et al. to obtain a perturbative solution that approximates the solution to the Blasius equation and in particular, the value of $y''(0)$.

We begin with the generalized Blasius equation as shown below with the accompanying boundary conditions.

$$y'''(t) + y''(t) \cdot y(t) = 0 \tag{B.1}$$

on $0 \leq t < \infty$ satisfying the boundary conditions

$$y(0) = -\alpha, y'(0) = -\beta, y'(\infty) = 1 \tag{B.2}$$

where α and β are constants. We next introduce a new parameter into the equation, δ. The boundary-value problem then becomes

$$y'''(t) + y''(t) \cdot y(t)^\delta = 0 \tag{B.3}$$

with the original boundary conditions, B.2. Next, we consider y(t) as a series expansion in terms of δ,

$$y(t) = y_0(t) + \delta \cdot y_1(t) + \delta^2 \cdot y_2(t) + \cdots \tag{B.4}$$

We substitute this expansion into equation B.1 above and take

$$y_0(t)^\delta = e^{\delta \ln y_0(t)} = 1 + \delta \cdot \ln y_0(t) + \frac{\delta^2}{2} \cdot \ln^2 y_0(t) + \cdots \tag{B.5}$$

and

$$
\begin{aligned}
\left(1 + \frac{\delta \cdot y_1}{y_0} + \frac{\delta^2 \cdot y_2}{y_0} + \cdots\right)^\delta &= 1 + \delta \left(\frac{\delta \cdot y_1}{y_0} + \frac{\delta^2 \cdot y_2}{y_0} + \cdots\right) \\
&+ \frac{\delta^2 - \delta}{2}\left(\frac{\delta \cdot y_1}{y_0} + \frac{\delta^2 \cdot y_2}{y_0} + \cdots\right)^2 + \cdots
\end{aligned}
\tag{B.6}
$$

After all substitutions and by comparing similar powers of δ, we derive a series of linear differential equations whose solutions will be combined into a δ series for $y(t)$ and for $y''(0)$ as follows.

$$y(t) = y_0(t) + \delta \cdot y_1(t) + \delta^2 \cdot y_2(t) + \cdots \tag{B.7}$$

and

$$y''(0) = y_0''(0) + \delta \cdot y_1''(0) + \delta^2 \cdot y_2''(0) + \cdots \tag{B.8}$$

The set of quations derived by this method are

O(1):

$$y_0'''(t) + y_0''(t) = 0 \tag{B.9}$$

$$y_0(0) = -\alpha, y_0'(0) = -\beta, y_0'(\infty) = 1$$

O(δ):

$$y_1'''(t) + y_1''(t) = -y_0''(t) \cdot \ln(y_0(t)) \tag{B.10}$$

$$y_1(0) = 0, y_1'(0) = 0, y_1'(\infty) = 0$$

O(δ^2):

$$y_2'''(t) + y_2''(t) = -\frac{y_0''(t)}{2} \cdot \ln(y_0(t))^2 - y_1''(t) \cdot \ln(y_0(t)) - \frac{y_0''(t) \cdot y_1(t)}{y_0(t)} \tag{B.11}$$

$$y_2(0) = 0, y_2'(0) = 0, y_2'(\infty) = 0$$

The zero-order equation can be solved easily since it is a linear, homogeneous equation. However, it is noted that all the equations can be written in the form

$$u'''(t) + u''(t) = g(t) \tag{B.12}$$

where $g(t)$ is a function of the solution to the previous orders. If we define the vectors

$$\vec{u}(t) = \begin{bmatrix} u(t) \\ u'(t) \\ u''(t) \end{bmatrix} \tag{B.13}$$

and

$$\vec{f}(t) = \begin{bmatrix} 0 \\ 0 \\ g(t) \end{bmatrix},$$

(B.14)

we can convert equation B.12 into a first-order linear system

$$\vec{u}'(t) = A(t) \cdot \vec{u}(t) + \vec{f}(t)$$

(B.15)

where

$$A(t) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & -1 \end{bmatrix}$$

(B.16)

This has a general solution of

$$\vec{u}(t) = X(t) \cdot \vec{c} + X(t) \cdot \int_0^t X^{-1}(s)\vec{f}(s)ds$$

(B.17)

where $X(t)$ is the fundamental matrix and $\vec{c}$ is a constant vector. The eigenvalues of $A(t)$, $\lambda = -1$ and $\lambda = 0$ with multiplicity of 2, leads to the fundamental matrix

$$X(t) = \begin{bmatrix} e^{-t} & 1 & t \\ -e^{-t} & 0 & 1 \\ e^{-t} & 0 & 0 \end{bmatrix}$$

(B.18)

The determinant of $X(t)$ is e^{-t}; therefore, $X(t)$ has an inverse,

$$X^{-1}(t) = \begin{bmatrix} 0 & 0 & e^t \\ 1 & -t & -1-t \\ 0 & 1 & 1 \end{bmatrix}$$

(B.19)

Using these matrices in the general solution, we can now solve the series of equations generated by the δ-perturbation expansion.

B.1   Zero-Order Solution

Since the zero-order equation is linear and homogeneous, it is easily solved to give

$$y_0(t) = (1 + \beta)e^{-t} + t - (1 + \alpha + \beta). \tag{B.20}$$

Using the general solution from the linear system above and letting $g(t) = 0$, we can recreate the zero-order solution and thus verify both the solution and our methodology. Since $g(t) = 0$ gives $\vec{f}(t) = \vec{0}$, we begin with

$$\vec{y}_0(t) = X(t) \cdot \vec{c}.$$

Using the boundary conditions, we can find the constant vector,

$$\vec{c}_0 = \begin{bmatrix} 1 + \beta \\ -(1 + \beta + \alpha) \\ 1 \end{bmatrix}.$$

This gives the solution of

$$\vec{y}_0(t) = \begin{bmatrix} (1 + \beta)e^{-t} + t - (1 + \beta + \alpha) \\ -(1 + \beta)e^{-t} + 1 \\ (1 + \beta)e^{-t} \end{bmatrix}$$

or

$$y_0(t) = (1 + \beta)e^{-t} + t - (1 + \alpha + \beta) \tag{B.21}$$

$$y_0'(t) = -(1 + \beta)e^{-t} + 1 \tag{B.22}$$

$$y_0''(t) = (1 + \beta)e^{-t}. \tag{B.23}$$

This then gives

$$y_0''(0) = 1 + \beta. \tag{B.24}$$

B.2   First-Order Solution

For the first-order equation, we have $g(t) = -y_0''(t) \cdot \ln(y_0(t))$. First, we note that for $\ln(y_0(t))$ to exist, we need to have $y_0(t) > 0$ on $t \geq 0$. This is satisfied when $\alpha < 0$ and $\beta < 0$. Using the first-order boundary conditions in the general solution from above, we can find the first-order constant vector

$$
\vec{c}_1 = \begin{bmatrix} (1+\beta) \int_0^\infty e^{-s} \ln(y_0(s)) \, ds \\ -(1+\beta) \int_0^\infty e^{-s} \ln(y_0(s)) \, ds \\ (1+\beta) \int_0^\infty e^{-s} \ln(y_0(s)) \, ds \end{bmatrix}.
$$

Returning this constant to the general solution, we can find the first-order solution. First we note that each entry in the first-order constant vector has the same value. We will designate this as the constant c_1 to simplify the following equations.

$$
y_1(t) = c_1 e^{-t} - c_1 + c_1 t - (1+\beta)e^{-t} \int_0^t \ln(y_0(s)) \, ds
$$
$$
+ (1+\beta) \int_0^t ((1+s)e^{-s} \ln(y_0(s)) \, ds - (1+\beta)t \int_0^t e^{-s} \ln(y_0(s)) \, ds \tag{B.25}
$$

$$
y_1'(t) = -c_1 e^{-t} + c_1 + (1+\beta)e^{-t} \int_0^t \ln(y_0(s)) \, ds - (1+\beta) \int_0^t e^{-s} \ln(y_0(s)) \, ds \tag{B.26}
$$

$$
y_1''(t) = c_1 e^{-t} - (1+\beta)e^{-t} \int_0^t \ln(y_0(s)) \, ds \tag{B.27}
$$

This then gives

$$
y_1''(0) = (1+\beta) \int_0^\infty e^{-s} \ln(y_0(s)) \, ds \tag{B.28}
$$

which interestingly is c_1, the value for the constant vector.

B.3   Second-Order Solution

For this case, $g(t) = -\frac{y_0''(t)}{2} \cdot \ln(y_0(t))^2 - y_1''(t) \cdot \ln(y_0(t)) - \frac{y_0''(t) \cdot y_1(t)}{y_0(t)}$. Using the same method as previously, the second order constant vector is

$$
\vec{c}_2 = \begin{bmatrix} \int_0^\infty \left[\frac{y_0''}{2} (\ln(y_0))^2 + y_1'' \ln(y_0) + \frac{y_0'' \cdot y_1}{y_0} \right] ds \\ -\int_0^\infty \left[\frac{y_0''}{2} (\ln(y_0))^2 + y_1'' \ln(y_0) + \frac{y_0'' \cdot y_1}{y_0} \right] ds \\ \int_0^\infty \left[\frac{y_0''}{2} (\ln(y_0))^2 + y_1'' \ln(y_0) + \frac{y_0'' \cdot y_1}{y_0} \right] ds \end{bmatrix}
$$

and the second-order solution is

$$
\begin{aligned}
y_2(t) &= c_2 e^{-t} - c_2 + c_2 t + e^{-t} \int_0^t e^{-s} \left[-\frac{y_0''}{2} (\ln(y_0))^2 - y_1'' \ln(y_0) - \frac{y_0'' \cdot y_1}{y_0} \right] ds \\
&+ \int_0^t (-1 - s) \left[-\frac{y_0''}{2} (\ln(y_0))^2 - y_1'' \ln(y_0) - \frac{y_0'' \cdot y_1}{y_0} \right] ds \\
&+ t \int_0^t \left[-\frac{y_0''}{2} (\ln(y_0))^2 - y_1'' \ln(y_0) - \frac{y_0'' \cdot y_1}{y_0} \right] ds
\end{aligned} \tag{B.29}
$$

$$
\begin{aligned}
y_2'(t) &= -c_2 e^{-t} + c_2 - e^{-t} \int_0^t e^{-s} \left[-\frac{y_0''}{2} (\ln(y_0))^2 - y_1'' \ln(y_0) - \frac{y_0'' \cdot y_1}{y_0} \right] ds \\
&+ \int_0^t \left[-\frac{y_0''}{2} (\ln(y_0))^2 - y_1'' \ln(y_0) - \frac{y_0'' \cdot y_1}{y_0} \right] ds
\end{aligned} \tag{B.30}
$$

$$
y_2''(t) = c_2 e^{-t} + e^{-t} \int_0^t e^{-s} \left[-\frac{y_0''}{2} (\ln(y_0))^2 - y_1'' \ln(y_0) - \frac{y_0'' \cdot y_1}{y_0} \right] ds \tag{B.31}
$$

where c_2 is the value of the components of $\vec{c}_2$. This gives

$$
y_2''(0) = \int_0^\infty \left[\frac{y_0''}{2} (\ln(y_0))^2 + y_1'' \ln(y_0) + \frac{y_0'' \cdot y_1}{y_0} \right] ds \tag{B.32}
$$

where again, $y_2''(0) = c_2$ and figures significantly in the constant acquired.

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