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Initial-value Technique For Singularly Perturbed Two Point Boundary Value Problems Via Cubic Spline

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INITIAL-VALUE TECHNIQUE FOR SINGULARLY PERTURBED TWO POINT BOUNDARY VALUE PROBLEMS VIA CUBIC SPLINE

by

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ABSTRACT

A recent method for solving singular perturbation problems is examined. It is designed for the applied mathematician or engineer who needs a convenient, useful tool that requires little preparation and can be readily implemented using little more than an industry-standard software package for spreadsheets. In this paper, we shall examine singularly perturbed two point boundary value problems with the boundary layer at one end point. An initial-value technique is used for its solution by replacing the problem with an asymptotically equivalent first order problem, which is, in turn, solved as an initial value problem by using cubic splines. Numerical examples are provided to show that the method presented provides a fine approximation of the exact solution.

The first chapter provides some background material to the cubic spline and boundary value problems. The works of several authors and a comparison of different solution methods are also discussed. Finally, some background into the specific singularly perturbed boundary value problems is introduced. The second chapter contains calculations and derivations necessary for the cubic spline and the initial value technique which are used in the solutions to the boundary value problems. The third chapter contains some worked numerical examples and the numerical data obtained along with most of the tables and figures that describe the solutions. The thesis concludes with some reflections on the results obtained and some discussion of the error bounds on the calculated approximations to the exact solutions for the numeric examples discussed.

This thesis is dedicated to my wife Katiuska and to my students, who have continuously reminded me of the value of patience and persistence. It is also dedicated to my family, who remind me that the most imposing of tasks can be accomplished one small step at a time.

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CHAPTER 1: INTRODUCTION

The cubic spline interpolation is based on the engineering tool used to draw smooth curves through a finite number of points. The engineer's spline consists of weights attached to a flat surface at the points to be connected, and a flexible strip is then bent across each of the weights, resulting in a smooth curve; the mathematical spline uses a similar principle. The points to be connected represent numerical data, and the "weights" are represented by numerical coefficients for a cubic polynomial that "bend" the line so that it passes continuously through each of the data points [5]. Since real-world numerical data is often difficult to analyze, finding a function that specifically relates the data is usually difficult to obtain and use. For example, in the study of heat transfer, problems of the deflection of plates and in a number of other scientific applications, we find a system of differential equations of different order with different boundary conditions. Many problems are formulated mathematically in boundary value problems for second order differential equations as in heat transfer and deflection in cables. Instead of trying to fit one function as the solution to the differential equations, we can use the series of unique cubic polynomials fitted between a set of data points. We will stipulate that the curve obtained must be continuous and smooth, and we can then use this cubic spline to interpolate data and rates of change over an interval.

A boundary value problem has conditions specified at the extremes of the independent variable. If the problem is dependent on both space and time, then instead of specifying the value of the problem at a given point for all time, the data could be given at a given time for all space. For example, the temperature of an iron bar with one end kept at absolute zero and the other end at the freezing point of water would be a boundary value problem. There are three types of boundary value problems, namely problems dealing with Dirichlet boundary conditions, Neumann boundary conditions, and Cauchy boundary condition

If the boundary gives a value to the problem then it is a Dirichlet boundary condition. For example if one end of an iron rod had one end held at absolute zero then the value of the problem would be known at that point in space. A Dirichlet boundary condition imposed on an ordinary differential equation or a partial differential equation specifies the values a solution is to take on the boundary of the domain. The question of finding solutions to such equations is known as the Dirichlet problem.

If the boundary gives a value to the normal derivative of the problem then it is a Neumann boundary condition. For example if one end of an iron rod had a heater at one end then energy would be added at a constant rate but the actual temperature would not be known. A Neumann boundary condition imposed on an ordinary differential equation or a partial differential equation specifies the values the derivative of a solution is to take on the boundary of the domain.

If the boundary has the form of a curve or surface that gives a value to the normal derivative and the problem itself then it is a Cauchy boundary condition. A Cauchy boundary condition imposed on an ordinary differential equation or a partial differential equation specifies both the values a solution of a differential equation is to take on the boundary of the domain and the normal derivative at the boundary. It basically corresponds to imposing both a Dirichlet and a Neumann boundary condition. Cauchy boundary conditions can be understood from the theory of second order ordinary differential equations, where to have a particular solution one has to specify the value of the function and the value of the derivative at a given initial or boundary point.

E.A. Al-Said has solved the system of second-order boundary value problems of the type

$$u'' = \begin{cases} f(x), & a \le x \le\le c \\ g(x)u(x) + f(x) + r, & c \le x \le d \\ f(x), & d \le x \le b \end{cases} \tag{i}$$

with the Dirichlet boundary conditions

$$u(a) = \alpha_1 \; and \; u(b) = \alpha_2 \tag{ii}$$

assuming the continuity conditions of u and u' at c and d, and where f and g are continuous functions on $[a,b]$ and $[c,d]$, respectively. The parameters r, α_1, α_2 are real finite constants. He used a cubic spline function to develop a numerical method for computing smooth approximations to the solution and its derivatives for a system of second-order boundary-value problems of the type (i) [2].

A. Khan and T. Aziz applied parametric cubic spline functions to develop a new numerical method for obtaining smooth approximations to the solution of the system of second-order boundary value problem of the type (i) having Dirichlet boundary conditions [9].

Siraj-ul-Islam and Ikram A. Tirmizi have applied non-polynomial spline functions that have a polynomial and trigonometric parts to develop a new numerical method for obtaining smooth approximations to the solution of the system of second-order boundary value problem of the type (i) having Dirichlet boundary conditions [16].

Arshad Khan has derived a uniformly convergent uniform mesh difference scheme using parametric cubic spline for the solution of the two-point boundary value problem with Dirichlet boundary conditions of the type

$$\begin{aligned} &y''(x) = f(x)y(x) + g(x), a \le x \le b \\ &y(a)=\alpha_0, \quad y(b)=\alpha_1, \end{aligned} \tag{iii}$$

where $f(x)$ and $g(x)$ are continuous functions on $[a,b]$ and a,b,α_0,α_1 are arbitrary real finite constants. Such problems arise plate deflection theory and a number of other scientific applications [10]. In general it is difficult to obtain the analytical solution of (iii) for arbitrary choices of $f(x)$ and $g(x)$. The standard numerical methods for the numerical treatment of (iii) consist of finite difference methods discussed by many authors.

The literature of numerical analysis contains little on the solution of second order two-point boundary value problems subjected to Neumann boundary conditions. M.A. Ramadan and I.F. Lashien have used both polynomial and non-polynomial spline functions to develop numerical methods for obtaining smooth approximations for the solution of the linear second order two-point boundary value problem subjected to Neumann boundary conditions [14].

Albasiny and Raghavarao solved linear second order two-point boundary problem (i) subjected to Dirichlet boundary conditions using cubic polynomial spline [1] [13]. Blue solved this problem using quintic polynomial spline [4], while, Caglar et al. solved this problem using cubic B-spline [6].

Singular perturbation problems occur commonly in many branches of mathematics. The governing equations of various mathematical models in physical, biological, economic, or engineering applications often involve characteristics that make it difficult to obtain an exact solution to a problem. Some solutions may have a closed form, but result in a complicated integral solution, while solutions to other models are more easily obtained but result in an infinite series solution.

When a large parameter or small parameter occurs within the mathematical model in one of these processes, perturbation methods are used to construct a series of simpler equations which can be used to approximate the solution to the problem. In general, many techniques used to solve singularly perturbed problems consist of dividing the problem into inner and outer regions, expressing the inner and outer solutions as asymptotic expansions, equating terms in the expressions developed to determine the constants in the expressions, and uniting the inner and outer solutions to obtain the final valid solution to the problem. However, the difficulties in using these techniques often arise when matching the coefficients in the inner and outer expansions in order to yield the final solution. Recently, non-asymptotic methods have been used to solve certain classes of singularly perturbed problems, replacing singularly perturbed two-point boundary value problems by initial-value techniques.

We shall investigate an initial-value technique for singularly perturbed two-point boundary value problems via cubic splines introduced by Manoj Kumar, Pitam Singh, and Hradyesh Kumar Mishra. The initial-value technique will be examined and tested an alternate method to approximate solutions. The approximate solutions will then be compared to the exact solutions to see if the method is successful for solving ordinary differential equations. Then the method will be applied to approximate solutions to linear and nonlinear singularly perturbed boundary value problems.

CHAPTER 2: INTRODUCTORY DISCUSSION FOR THE METHOD

The Cubic Spline

We want to fit a piecewise function of the form

$$S(x) = \begin{cases} s_1(x), & x_1 \le x \le x_2 \\ s_2(x), & x_2 \le x \le x_3 \\ & \vdots \\ s_{n-1}(x), & x_{n-1} \le x \le x_n \end{cases} \tag{1}$$

where s_i is a third-degree polynomial defined by

$$s_i(x) = a_i(x - x_i)^3 + b_i(x - x_i)^2 + c_i(x - x_i) + d_i \tag{2}$$

for $i = 1, 2, \ldots, n-1$.

The first and second derivatives are also necessary to the process and are given by

$$s_i'(x) = 3a_i(x - x_i)^2 + 2b_i(x - x_i) + c_i \tag{3}$$

$$s_i''(x) = 6a_i(x - x_i) + 2b_i \tag{4}$$

for $i = 1, 2, \ldots, n-1$.

The cubic spline will need to conform to the following four properties:

1. The piecewise function $S(x)$ will interpolate all the data points; that is,

$$S(x_i) = y_i \tag{5}$$

for $i = 1, 2, \ldots, n-1$. Since each $x_i \in [x_i, x_{i+1}]$, $S(x_i) = s(x_i)$, and from (2) we have

$$\begin{aligned} y_i &= s(x_i) \\ y_i &= a_i(x_i - x_i)^3 + b_i(x_i - x_i)^2 + c_i(x_i - x_i) + d_i \\ y_i &= d_i \end{aligned} \tag{6}$$

for $i = 1, 2, \ldots, n-1$.

2. The function $S(x)$ will be continuous on the interval $[x_1, x_n]$. Then we must ensure that each sub-function must join at the data points, so for $i = 2, \ldots, n$, we have

$$s_{i-1}(x_i) = s_i(x_i) \tag{7}$$

Then using equation (2), we have

$$s_i(x_i) = d_i$$

and

$$s_{i-1}(x_i) = a_{i-1}(x_i - x_{i-1})^3 + b_{i-1}(x_i - x_{i-1})^2 + c_{i-1}(x_i - x_{i-1}) + d_{i-1} \tag{8}$$

so

$$d_i = a_{i-1}(x_i - x_{i-1})^3 + b_{i-1}(x_i - x_{i-1})^2 + c_{i-1}(x_i - x_{i-1}) + d_{i-1} \tag{9}$$

for $i = 2, 3, \ldots, n-1$. If we let $h = x_i - x_{i-1}$ in equation (9), we have

$$d_i = a_{i-1}h^3 + b_{i-1}h^2 + c_{i-1}h + d_{i-1} \tag{10}$$

for $i = 2, 3, \ldots, n-1$.

3. The function $S'(x)$ will be continuous on the interval $[x_1, x_n]$. In order to ensure that the curve is smooth across the interval, the derivatives must be equal at the data points; that is,

$$s'_{i-1}(x_i) = s'_i(x_i) \tag{11}$$

for $i = 2, 3, \ldots, n-1$

Then using equation (3), we have

$$s'_i(x_i) = c_i$$

and

$$s'_{i-1}(x_i) = 3a_{i-1}(x_i - x_{i-1})^2 + 2b_{i-1}(x_i - x_{i-1}) + c_{i-1}$$

so

$$c_i = 3a_{i-1}(x_i - x_{i-1})^2 + 2b_{i-1}(x_i - x_{i-1}) + c_{i-1} \tag{12}$$

for $i = 2, 3, \ldots, n-1$. If we let $h = x_i - x_{i-1}$ in equation (12), we have

$$c_i = 3a_{i-1}h^2 + 2b_{i-1}h + c_{i-1} \tag{13}$$

for $i = 2, 3, \ldots, n-1$.

4. The function $S''(x)$ will be continuous on the interval $[x_1, x_n]$. In order to ensure that the curve is continuous across the interval, the second derivatives must be equal at the interior data points; that is, $s_i''(x_i) = s_{i+1}''(x_i)$ for $i = 1, 2, \ldots, n-1$.

From equation (4) we have $s_i''(x) = 6a_i(x - x_i) + 2b_i$, so

$$s_i''(x) = 6a_i(x - x_i) + 2b_i, \tag{14}$$

$$s_i''(x_i) = 6a_i(x_i - x_i) + 2b_i$$

$$s_i''(x_i) = 2b_i$$

for $i = 2, 3, \ldots, n-2$.

Since we must have $s_i''(x)$ continuous across the interval, $s_i''(x_i) = s_{i+1}''(x_i)$ for $i = 1, 2, \ldots, n-1$.

Then combining this and equation (14), we have that

$$s_i''(x_{i+1}) = 6a_i(x_{i+1} - x_i) + 2b_i \tag{15}$$

$$s_{i+1}''(x_{i+1}) = 6a_i(x_{i+1} - x_i) + 2b_i \tag{16}$$

If we let $h = x_{i+1} - x_i$ in equation (14) and (16), we have

$$s_{i+1}''(x_{i+1}) = 2b_{i+1} \tag{17}$$

$$2b_{i+1} = 6a_i h + 2b_i \tag{18}$$

At this point, we could substitute M for $s''(x)$ and express the equations above in terms of M_i and y_i to determine the weights of a_i , b_i , c_i , and d_i.[7] We will instead simplify these equations by making a substitution using equation (3):

$$\begin{aligned} s_i'(x_i) &= c_i \\ m_i &= c_i \end{aligned} \tag{19}$$

and expressing the equations above in terms of m_i and y_i. We can then determine the remaining coefficient a_i, b_i, and d_i. We want to relate the cubic spline function to boundary value problems and initial value problems with boundary conditions given in terms of function values and first derivatives. We note that d_i has already been determined to be

$$d_i = y_i . \tag{20}$$

We can similarly use equation (3) as

$$\begin{aligned} c_{i+1} &= 3a_i(x_{i+1} - x_i)^2 + 2b_i(x_{i+1} - x_i) + c_i \\ m_{i+1} &= 3a_i(x_{i+1} - x_i)^2 + 2b_i(x_{i+1} - x_i) + m_i \end{aligned} \tag{21}$$

Using equation (2) we can write

$$\begin{aligned} d_{i+1} &= a_i(x_{i+1} - x_i)^3 + b_i(x_{i+1} - x_i)^2 + c_i(x_{i+1} - x_i) + d_i \\ y_{i+1} &= a_i(x_{i+1} - x_i)^3 + b_i(x_{i+1} - x_i)^2 + m_i(x_{i+1} - x_i) + y_i \end{aligned} \tag{22}$$

We can now use equations (21) and (22) to determine expressions for a_i and b_i. Let $h = x_{i+1}$ x_i

and solve the system: $\begin{cases} m_{i+1} - m_i = 3a_i h + 2b_i h \\ (y_{i+1} - y_i) - m_i h = a_i h^3 + b_i h^2 \end{cases}$

Multiplying (21) by h , (22) by 2, and combining equations results in the following equation:

$$\begin{aligned} h(m_{i+1} - m_i) - 2(y_{i+1} - y_i) &= a_i h^3 \\ a_i &= \frac{(m_{i+1} - m_i)}{h^2} - \frac{2(y_{i+1} - y_i)}{h^3}. \end{aligned} \tag{23}$$

Multiplying (21) by h , (22) by 3, and combining equations results in the following equation:

$$\begin{aligned} h(m_{i+1} + 2m_i) - 3(y_{i+1} - y_i) &= -b_i h^2 \\ b_i &= \frac{3(y_{i+1} - y_i)}{h^2} - \frac{(m_{i+1} + 2m_i)}{h}. \end{aligned} \tag{24}$$

We can now write the function $S(x)$ strictly in terms of m_i and y_i , and we have

$$\begin{aligned} S(x) = &\left[\frac{(m_{i+1} - m_i)}{h^2} - \frac{2(y_{i+1} - y_i)}{h^3} \right] (x - x_i)^3 \\ &+ \left[\frac{3(y_{i+1} - y_i)}{h^2} - \frac{(m_{i+1} + 2m_i)}{h} \right] (x - x_i)^2 \\ &+ m_i (x - x_i) + y_i \end{aligned} \tag{25}$$

We will finally rewrite the cubic spline function $S(x)$ in terms of its first derivatives by collecting terms for m_i and y_i. We had let $h = x_{i+1} - x_i$ for the previous equations. Collecting terms for m_i , we have

$$\frac{m_i(x-x_i)^3}{h^2}-\frac{2m_i(x-x_i)^2}{h}+m_i(x-x_i)$$

$$\frac{m_i(x-x_i)}{h^2}\left[(x-x_i)^2-2(x-x_i)h+(x-x_i)h^2\right]$$

and expanding terms in the brackets yields

$$\begin{gathered}\frac{m_i(x-x_i)}{h^2}\left[x^2+(x_{i+1})^2-2x\cdot x_{i+1}\right]\\ \frac{m_i(x-x_i)(x_{i+1}-x)^2}{h^2}.\end{gathered} \tag{26}$$

Collecting terms for m_{i+1} , we have

$$\begin{gathered}\frac{m_{i+1}(x-x_i)^3}{h^2}-\frac{m_{i+1}(x-x_i)^2}{h}\\ \frac{m_{i+1}(x-x_i)^2}{h^2}[x-x_i-h]\end{gathered}$$

and rewriting h in the brackets, yields

$$-\frac{m_{i+1}(x-x_i)^2(x_{i+1}-x)}{h^2}\ . \tag{27}$$

Collecting terms for y_i , we have

$$\frac{2y_i(x-x_i)^3}{h^3}-\frac{3y_i(x-x_i)^2}{h^2}+y_i$$

$$\frac{y_i}{h^3}\left[2(x-x_i)^3-3(x-x_i)^2h+h^3\right]$$

and expanding terms in the brackets yields

$$\frac{y_i(x_{i+1}-x)^2}{h^3}[2x-2x_i-x_i+x_{i+1}]$$

$$\frac{y_i(x_{i+1}-x)^2[2(x-x_i)+h]}{h^3}. \tag{28}$$

Finally, collecting terms for y_{i+1} , we have

$$\frac{-2y_{i+1}(x-x_i)^3}{h^3}+\frac{3y_{i+1}(x-x_i)^2}{h^2}$$

$$\frac{y_i(x-x_i)^2}{h^3}[-2(x-x_i)+3h]$$

and since $h = x_{i+1} - x_i$, we have

$$\frac{y_{i+1}(x-x_i)^2[2(x_{i+1}-x)+h]}{h^3}. \tag{29}$$

The cubic spline function $S(x)$ in terms of its first derivatives $S'(x)$ is now given by

$$\begin{aligned} S(x) = \frac{m_i(x-x_i)(x_{i+1}-x)^2}{h^2} - \frac{m_{i+1}(x-x_i)^2(x_{i+1}-x)}{h^2} \\ + \frac{y_i(x_{i+1}-x)^2[2(x-x_i)+h]}{h^3} \\ + \frac{y_{i+1}(x-x_i)^2[2(x_{i+1}-x)+h]}{h^3} \end{aligned} \tag{30}$$

where $h = x_{i+1} - x_i$ will be used as the mesh size for calculating the spline function.

We will now differentiate with respect to x and simplify the equation.

Differentiating (30) with respect to *x*, we obtain

$$S'(x) = \frac{m_i(x_{i+1}-x)(x_{i+1}+2x_i-3x)}{h^2} - \frac{m_{i+1}(x-x_i)(x_i+2x_{i+1}-3x)}{h^2} \\ - \frac{6y_i(x_{i+1}-x)(x-x_i)}{h^3} + \frac{6y_{i+1}(x-x_i)(x_{i+1}-x)}{h^3} \tag{31}$$

We differentiate (31) again with respect to *x*, and we have

$$S''(x) = -\frac{2m_i(2x_{i+1}+x_i-3x)}{h^2} - \frac{2m_{i+1}(x_{i+1}+2x_i-3x)}{h^2} \\ - \frac{6y_i(x_{i+1}+x_i-2x)}{h^3} + \frac{6y_{i+1}(x_{i+1}+x_i-2x)}{h^3} \tag{32}$$

which yields

$$S''(x_{i+1}) = -\frac{2m_i(2x_{i+1}+x_i-3x_{i+1})}{h^2} - \frac{2m_{i+1}(x_{i+1}+2x_i-3x_{i+1})}{h^2} \\ - \frac{6y_i(x_{i+1}+x_i-2x_{i+1})}{h^3} + \frac{6y_{i+1}(x_{i+1}+x_i-2x_{i+1})}{h^3}$$

Final simplification results in

$$S''(x_i) = \frac{2m_i}{h} + \frac{4m_{i+1}}{h} - \frac{6y_i}{h^2} + \frac{6y_{i+1}}{h^2} \\ S''(x_i) = \frac{2m_i}{h} + \frac{4m_{i+1}}{h} - \frac{6S_i}{h^2} + \frac{6S_{i+1}}{h^2} \tag{33}$$

If we consider an initial-value problem

$$\frac{dy}{dx} = f(x, y) \tag{34}$$

$$y(x_0) = y_0 \tag{35}$$

then from the chain rule, we have $\frac{d^2y}{dx^2} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\frac{dy}{dx}$ and

$$\begin{aligned} y''(x_{i+1}) &= \frac{\partial f}{\partial x}(x_{i+1}, y_{i+1}) + \frac{\partial f}{\partial y}(x_{i+1}, y_{i+1})\, f(x_{i+1}, y_{i+1}) \\ y''(x_{i+1}) &= \frac{\partial f}{\partial x}(x_{i+1}, S_{i+1}) + \frac{\partial f}{\partial y}(x_{i+1}, S_{i+1})\, f(x_{i+1}, S_{i+1}) \end{aligned} \tag{36}$$

When we equate the two expressions (33) and (36), we obtain

$$\frac{2m_i}{h} + \frac{4m_{i+1}}{h} - \frac{6S_i}{h^2} + \frac{6S_{i+1}}{h^2} = \frac{\partial f}{\partial x}(x_{i+1}, S_{i+1}) + \frac{\partial f}{\partial y}(x_{i+1}, y_{i+1})\, f(x_{i+1}, S_{i+1}) \tag{37}$$

and we will use this equation to compute the values of S_i and in turn use (30) to get S(x).

Let us consider a brief numerical example to illustrate the cubic spline method of solving a differential equation. Consider the initial value problem

$$\frac{dy}{dx} = y \tag{38}$$

$$y(0) = 1 \tag{39}$$

Equating (33) and (36) we have

$$\frac{2m_i}{h} + \frac{4m_{i+1}}{h} - \frac{6S_i}{h^2} + \frac{6S_{i+1}}{h^2} = S_{i+1} \tag{40}$$

From (38), we can replace the m_i in (40) and we have

$$\frac{2S_i}{h}+\frac{4S_{i+1}}{h}-\frac{6S_i}{h^2}+\frac{6S_{i+1}}{h^2}=S_{i+1} \tag{41}$$

and solving for S_{i+1} we have

$$S_{i+1}=\frac{(2h+6)}{(h^2-4h+6)}S_i \tag{42}$$

and we can now use the initial value to calculate next value in the sequence. The numerical results are given in Table 1 for h = 0.01. The graph of the function $y(x)$ is given in Figure 1.

Table 1: Numerical results with h=0.01

x	$y(x)$	exact solution
0	1	1
0.01	1.010050167	1.010050167
0.1	1.105170917	1.105170918
0.2	1.221402755	1.221402758
0.3	1.349858802	1.349858808
0.4	1.491824689	1.491824698
0.5	1.648721259	1.648721271
0.6	1.822118785	1.8221188
0.7	2.013752688	2.013752707
0.8	2.225540904	2.225540928
0.9	2.45960308	2.459603111
0.99	2.691234435	2.691234472
1	2.718281791	2.718281828

We see that the data points are remarkably close the exact solution for a reasonably small step size, and the data points can be used now to provide a solution for the differential equation.

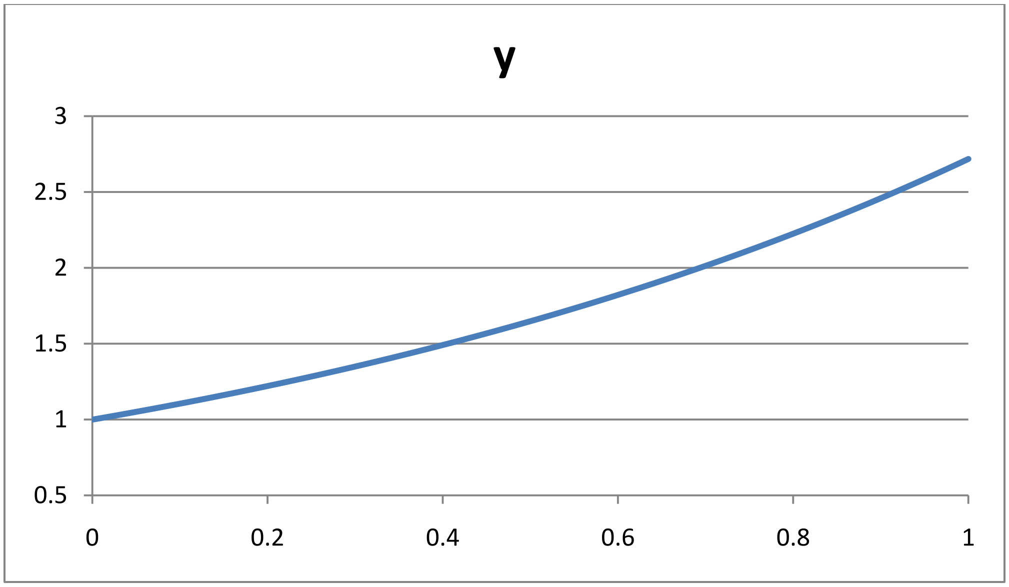

Figure 1: Graph of the solution $y(x)$ for the differential equation (30) and (31)

The Initial-Value Technique

In their paper, Kumar, Singh, and Mishra [12] introduce an initial-value technique for singularly perturbed two-point boundary value problems with a layer on the left (or right) end of the underlying interval in which the original second order problem is replaced by an asymptotically equivalent three first-order initial-value problems, which are then solved via cubic spline. For convenience, we will call this method the initial-value technique. We first consider a linear singularly perturbed two-point boundary problem of the form:

$$\varepsilon u''(x) + p(x)u'(x) + q(x)u(x) = r(x),\ x \in [a,b] \tag{43}$$

$$u(a) = \alpha \tag{44}$$

$$u(b) = \beta \tag{45}$$

where ε is a small positive parameter $(0 < \varepsilon << 1)$ and α , β are known constants. We assume that $p(x)$, $q(x)$, and $r(x)$ are sufficiently continuously differentiable functions in $[a, b]$, and furthermore, we assume that $p(x) \geq M > 0$ throughout the interval $[a, b]$, where M is some positive constant. This assumption implies that the solution of (43), (44), and (45) will be in the neighborhood of $x = a$.

Since singular perturbation problems exhibit boundary layer behavior of the solution, the solution of (43), (44), and (45) is given by

$$u(x,\varepsilon) = v(x) + w(x)e^{-t(x)/\varepsilon} \tag{46}$$

with

$$t(x)=\int_a^x p(\tau)d\tau$$

where $v(x,\varepsilon)=\sum_{n=0}^{\infty} v_n(x)\varepsilon^n$ and $w(x,\varepsilon)=\sum_{n=0}^{\infty} w_n(x)\varepsilon^n$ (cf.[17, p.292]), so we have

$$u(x,\varepsilon)=\sum_{n=0}^{\infty} v_n(x)\varepsilon^n+\left(\sum_{n=0}^{\infty} v_n(x)\varepsilon^n\right)e^{-t(x)/\varepsilon} \tag{47}$$

with

$$t(x)=\int_a^x p(\tau)d\tau \ . \tag{48}$$

Differentiating (47) with respect to *x* yields

$$u'(x,\varepsilon)=\sum_{n=0}^{\infty} v_n'(x)\varepsilon^n+\left(\sum_{n=0}^{\infty} w_n'(x)\varepsilon^n\right)e^{-t(x)/\varepsilon}-\left(\sum_{n=0}^{\infty} w_n(x)\varepsilon^n\right)e^{-t(x)/\varepsilon}\left(\frac{p(x)}{\varepsilon}\right) \tag{49}$$

$$\begin{aligned} u''(x,\varepsilon)=\sum_{n=0}^{\infty} v_n''(x)\varepsilon^n+\left(\sum_{n=0}^{\infty} w_n''(x)\varepsilon^n\right)e^{-t(x)/\varepsilon}-2\left(\sum_{n=0}^{\infty} w_n'(x)\varepsilon^n\right)e^{-t(x)/\varepsilon}\left(\frac{p(x)}{\varepsilon}\right) \\ +\left(\sum_{n=0}^{\infty} w_n(x)\varepsilon^n\right)e^{-t(x)/\varepsilon}\left(\frac{p(x)}{\varepsilon}\right)^2-\left(\sum_{n=0}^{\infty} w_n(x)\varepsilon^n\right)e^{-t(x)/\varepsilon}\left(\frac{p'(x)}{\varepsilon}\right) \end{aligned} \tag{50}$$

We can now substitute (47), (48), and (49) in (43), and we have

$$
\begin{aligned}
&\sum_{n=0}^{\infty} v_n''(x)\varepsilon^{n+1} + \left(\sum_{n=0}^{\infty} w_n''(x)\varepsilon^{n+1}\right)e^{-t(x)/\varepsilon} - 2\left(\sum_{n=0}^{\infty} w_n'(x)\varepsilon^{n}\right)e^{-t(x)/\varepsilon} p(x) \\
&\quad + \left(\sum_{n=0}^{\infty} w_n(x)\varepsilon^{n-1}\right)e^{-t(x)/\varepsilon}[p(x)]^2 - \left(\sum_{n=0}^{\infty} w_n(x)\varepsilon^{n}\right)e^{-t(x)/\varepsilon} p'(x) \\
&\quad - [p(x)]^2\left(\sum_{n=0}^{\infty} w_n(x)\varepsilon^{n-1}\right)e^{-t(x)/\varepsilon} + q(x)\left(\sum_{n=0}^{\infty} v_n(x)\varepsilon^{n}\right) \\
&\quad + q(x)\left(\sum_{n=0}^{\infty} w_n(x)\varepsilon^{n}\right)e^{-t(x)/\varepsilon} = r(x)
\end{aligned} \tag{51}
$$

By restricting these series to their first terms, we the get

$$
\begin{aligned}
&p(x)v_0'(x) + q(x)v_0(x) + [-2p(x)w_0'(x) - p'(x)w_0(x) \\
&\quad + p(x)w_0'(x) + q(x)w_0(x)]e^{-t(x)/\varepsilon} = r(x)
\end{aligned}
$$

We therefore have the following:

$$p(x)v_0'(x) + q(x)v_0(x) = r(x) \tag{52}$$

and

$$\frac{d}{dx}[p(x)w_0(x)] = q(x)w_0(x) \tag{53}$$

The representations (47) and (48) can be inserted to the boundary conditions (44) and (45), and the boundary conditions become

$$v_0(a) + w_0(a) = \alpha \tag{54}$$

and

$$v_0(b) = \beta \tag{55}$$

where the exponentially small term $e^{-t(b)/\varepsilon}$ is neglected in order to obtain the boundary condition (55) at $x = b$. The differential equation (52) can be solved with the boundary condition (55) to determine $v_0(x)$, then $w_0(x)$ is determined by solving the differential equation (53) with the boundary condition $w_0(a) = \alpha - v_0(a)$.

From (48), we have $t(x) = \int_a^x p(\tau)d\tau$, so $t'(x) = p(x)$ with $t(a) = 0$

The three initial-value problems corresponding to (43), (44), and (45) are given by [12]

(IVP.I) $p(x)v_0'(x) + q(x)v_0(x) = r(x)$ with $v_0(b) = \beta$ (56)

(IVP.II) $\dfrac{d}{dw}\left[p(x)w_0(x)\right] = q(x)w_0(x)$ with $w_0(a) = \alpha - v_0(a)$ (57)

(IVP.III) $t'(x) = p(x)$ with $t(a) = 0$ (58)

These initial-value problems are independent of the perturbation parameter and will be solved by the cubic spline method. For the problems exhibiting right-end behavior, we use the three initial-value problems as well.

We consider a linear singularly perturbed two-point boundary problem of the form:

$$\varepsilon u''(x) + p(x)u'(x) + q(x)u(x) = r(x), \; x \in [a,b] \tag{59}$$

$$u(a) = \alpha \tag{60}$$

$$u(b) = \beta \tag{61}$$

where ε is a small positive parameter $(0 < \varepsilon << 1)$ and α , β are known constants. We assume that $p(x)$, $q(x)$, and $r(x)$ are sufficiently continuously differentiable functions in $[a, b]$, and furthermore, we assume that $p(x) \geq M > 0$ throughout the interval $[a, b]$, where M is some negative constant. This assumption implies that the solution of (43), (44), and (45) will be in the neighborhood of $x = b$.

Therefore the three initial-value problems corresponding to (43), (44), and (45) are given by[12]

$$\text{(IVP.I)} \quad p(x)v_0'(x) + q(x)v_0(x) = r(x) \text{ with } v_0(a) = \alpha \tag{62}$$

$$\text{(IVP.II)} \quad \frac{d}{dw}\left[p(x)w_0(x)\right] = q(x)w_0(x) \text{ with } w_0(a) = \beta - v_0(b) \tag{63}$$

$$\text{(IVP.III)} \quad t'(x) = p(x) \text{ with } t(b) = 0 \tag{64}$$

These initial-value problems are independent of the perturbation parameter and will be solved by the cubic spline method.

CHAPTER 3: NUMERICAL EXAMPLES

Linear Singular Perturbation Problems with Left-end Boundary Layer

Example 1

First, consider the following homogeneous singular perturbation problem from Bender and Orzag [3, p.480, Problem 9.17 with α =0]

$$\begin{aligned} &\varepsilon y''(x) + y'(x) - y(x) = 0 \\ &x \in [0,1]\ ,\ y(0) = 1\ ,\ y(1) = 1. \end{aligned} \tag{65}$$

Here we note that $p(x) = 1,\ q(x) = -1,\ r(x) = 0$.

From (56), we have $v_0'(x) - v_0(x) = 0$ with $v_0(1) = 1$, so $v_0''(x) = v_0'(x) = v_0(x)$ we can set $v_i = m_i$ and we have $v_n = 1$, so the resulting equation from (37) is

$$6v_i - 6v_{i+1} + 2hm_i + 4hm_{i+1} = h^2 v_{i+1}.$$

$$6v_i - (h^2 + 6)v_{i+1} + 2hm_i + 4hm_{i+1} = 0$$

$$(6 + 2h)v_i + [-4h + h^2 + 6]v_{i+1} = 0$$

$$v_i = \frac{[-4h + h^2 + 6]}{(6 + 2h)} v_{i+1}$$

In an effort to make the method readily accessible, we will be using an industry-standard software package for spreadsheets (Microsoft EXCEL) to solve for $v_0(x)$ and then, $w_0(x)$.

From (57), we have $w_0'(x) + w_0(x) = 0$, $w_0(0) = 1 - v_0(0)$, so $w_0'(x) = -w_0(x)$ we can set $w_i = -m_i$ and we have $w_1 = 1 - v_0(0)$, so the resulting equation from (37) is

$$6w_i - 6w_{i+1} + 2hm_i + 4hm_{i+1} = -h^2 w_{i+1}.$$

$$6w_i - (6 - h^2)w_{i+1} + 2hm_i + 4hm_{i+1} = 0$$

$$(6 - 2h)w_i - [4h - h^2 + 6]w_{i+1} = 0$$

$$w_{i+1} = \frac{(6 - 2h)}{[4h - h^2 + 6]} w_i$$

From (58) we have $t(x) = x$ after integrating and applying the initial condition. The numerical results are given in Table 2 for $\varepsilon = 10^{-3}$.

Table 2: Numerical results of Example 1 with $\varepsilon = 10^{-3}$, $h = 10^{-3}$

x	$v(x)$	$w(x)$
0	0.367879441	0.632120559
0.001	0.368247505	0.631488965
0.01	0.371576691	0.625832939
0.02	0.375311099	0.619607861
0.03	0.379083038	0.613444703
0.04	0.382892886	0.60734285
0.05	0.386741023	0.60130169
0.1	0.40656966	0.571985387
0.3	0.496585304	0.468333227
0.5	0.60653066	0.383464362
0.7	0.740818221	0.313974983
0.9	0.904837418	0.257078101
1	1	0.232621634

The solution to (65) using (46) has the form $u(x)=v(x)+w(x)e^{-t(x)/\varepsilon}$ with $t(x)=x$. The results are given in Table 3 and Table 4 for $\varepsilon=10^{-3}$ and $\varepsilon=10^{-4}$ respectively, and compared to the exact solution. The graph of the cubic spline function is given in Figure 2. The exact solution to (65) is given by

$$y(x)=\frac{(e^{m_2}-1)e^{m_1x}+(1-e^{m_1})e^{m_2x}}{e^{m_2}-e^{m_1}},$$

where $m_1=\dfrac{-1+\sqrt{1+4\varepsilon}}{2\varepsilon}$ and $m_1=\dfrac{-1+\sqrt{1+4\varepsilon}}{2\varepsilon}$.

Table 3: Numerical results of Example 1 with $\varepsilon=10^{-3}$, $h=10^{-3}$

x	*y*(*x*)	exact solution	**error**
0	1.0000000000	1.0000000000	0.0000E+00
0.001	0.6005593120	0.6007917971	2.3249E-04
0.01	0.3716051038	0.3719723959	3.6729E-04
0.02	0.3753111001	0.3756783508	3.6725E-04
0.03	0.3790830381	0.3794501927	3.6715E-04
0.04	0.3828928860	0.3832599056	3.6702E-04
0.05	0.3867410235	0.3871078683	3.6684E-04
0.1	0.4065696597	0.4069350065	3.6535E-04
0.3	0.4965853038	0.4969323412	3.4704E-04
0.5	0.6065306597	0.6068333956	3.0274E-04
0.7	0.7408182207	0.7410400560	2.2184E-04
0.9	0.9048374180	0.9049277258	9.0308E-05
1	1.0000000000	1.0000000000	0.0000E+00

Table 4: Numerical results of Example 1 with $\varepsilon = 10^{-4}$, $h = 10^{-4}$

x	$y(x)$	exact solution	**error**
0	1.0000000000	1.0000000000	0.0000E+00
0.0001	0.6004371364	0.6004603902	2.3254E-05
0.001	0.3682761742	0.3683129549	3.6781E-05
0.002	0.3686159376	0.3686527200	3.6782E-05
0.003	0.3689847366	0.3690215189	3.6782E-05
0.004	0.3693539059	0.3693906880	3.6782E-05
0.005	0.3697234445	0.3697602265	3.6782E-05
0.1	0.4065696597	0.4066062453	3.6586E-05
0.3	0.4965853038	0.4966200590	3.4755E-05
0.5	0.6065306597	0.6065609809	3.0321E-05
0.7	0.7408182207	0.7408404411	2.2220E-05
0.9	0.9048374180	0.9048464646	9.0466E-06
1	1.0000000000	1.0000000000	0.0000E+00

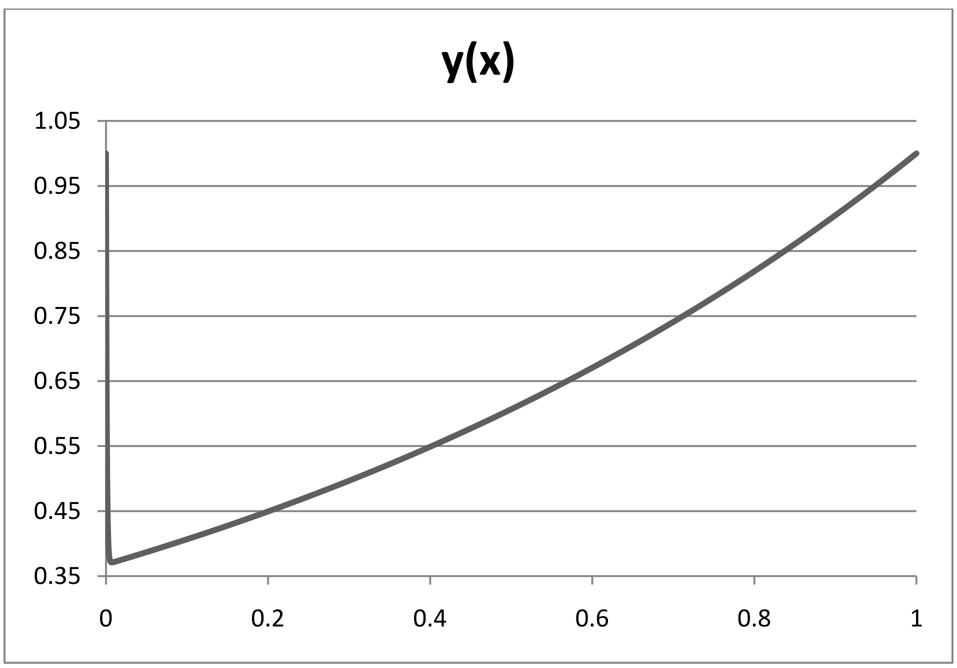

Figure 2: Graph of the spline solution of Example 1

Example 2

Next, consider the following non-homogeneous singular perturbation problem from fluid dynamics for fluid of small viscosity [15, Example 2]

$$\begin{aligned} &\varepsilon y''(x) + y'(x) = 1 + 2x \\ &x \in [0,1] \text{ , } y(0) = 0 \text{ , } y(1) = 1. \end{aligned} \tag{66}$$

Here we note that $p(x) = 1,\ q(x) = 0,\ r(x) = 1 + 2x$.

From (56), we have $v_0'(x) = 1 + 2x$ with $v_0(1) = 1$, so we can set $m_{i+1} = 1 + 2x_i$, so the resulting equation from (37) is

$$6v_i - 6v_{i+1} + 2hm_i + 4hm_{i+1} = 2h^2 .$$

$$6v_i - 6v_{i+1} + 2h(1 + 2x_i) + 4h(1 + 2x_{i+1}) = 2h^2$$

$$v_i = \frac{6v_{i+1} - 2h(1 + 2x_i) - 4h(1 + 2x_{i+1}) + 2h^2}{6}$$

From (57), we have $w_0'(x) = 0$, $w_0(0) = 1 - v_0(0)$, so $w_0(x) = 1$ and the resulting equation from (37) is simply

$$w_{i+1} = w_i$$

From (58) we have $t(x) = x$ after integrating and applying the initial condition. The numerical results are given in Table 5 for $\varepsilon = 10^{-3}$. The spline function is graphed in Figure 3.

Table 5: Numerical results of Example 2 with $\varepsilon = 10^{-3}$, $h = 10^{-3}$

x	$v(x)$	$w(x)$
0	-0.999999833	1
0.001	-0.998998667	1
0.01	-0.989898185	1
0.02	-0.97959657	1
0.03	-0.969094988	1
0.04	-0.95839344	1
0.05	-0.947491925	1
0.1	-0.88998485	1
0.3	-0.609964883	1
0.5	-0.24995825	1
0.7	0.19003505	1
0.9	0.710015017	1
1	1	1

The solution to (66) using (46) has the form $u(x) = v(x) + w(x)e^{-t(x)/\varepsilon}$ with $t(x) = x$.

The results are given in Table 6 for $\varepsilon = 10^{-3}$ and Table 7 for $\varepsilon = 10^{-4}$ and compared to the exact solution [12]. The exact solution is given by

$$y(x) = x(x+1-2\varepsilon) + \frac{(2\varepsilon-1)(1-e^{-x/\varepsilon})}{(1-e^{-1/\varepsilon})}.$$

Table 6: Numerical results of Example 2 with $\varepsilon = 10^{-3}$, $h = 10^{-3}$

x	$y(x)$	exact solution	error
0.0000	0.0000000000	0.0000000000	0.0000E+00
0.0010	-0.6311195588	-0.6298573177	1.2622E-03
0.0100	-0.9898546001	-0.9878746909	1.9799E-03
0.0200	-0.9795999979	-0.9776399979	1.9600E-03
0.0300	-0.9691000000	-0.9671600000	1.9400E-03
0.0400	-0.9584000000	-0.9564800000	1.9200E-03
0.0500	-0.9475000000	-0.9456000000	1.9000E-03
0.1000	-0.8900000000	-0.8882000000	1.8000E-03
0.3000	-0.6100000000	-0.6086000000	1.4000E-03
0.5000	-0.2500000000	-0.2490000000	1.0000E-03
0.7000	0.1900000000	0.1906000000	6.0000E-04
0.9000	0.7100000000	0.7102000000	2.0000E-04
1.0000	1.0000000000	1.0000000000	0.0000E+00

Table 7: Numerical results of Example 2 with $\varepsilon = 10^{-4}$, $h = 10^{-4}$

x	$y(x)$	exact solution	error
0.0000	0.0000000000	0.0000000000	0.0000E+00
0.0010	-0.6320205488	-0.6318941447	1.2640E-04
0.0100	-0.9989536001	-0.9987538092	1.9979E-04
0.0200	-0.9979959979	-0.9977963979	1.9960E-04
0.0300	-0.9969910000	-0.9967916000	1.9940E-04
0.0400	-0.9959840000	-0.9957848000	1.9920E-04
0.0500	-0.9949750000	-0.9947760000	1.9900E-04
0.1000	-0.8900000000	-0.8898200000	1.8000E-04
0.3000	-0.6100000000	-0.6098600000	1.4000E-04
0.5000	-0.2500000000	-0.2499000000	1.0000E-04
0.7000	0.1900000000	0.1900600000	6.0000E-05
0.9000	0.7100000000	0.7100200000	2.0000E-05
1.0000	1.0000000000	1.0000000000	0.0000E+00

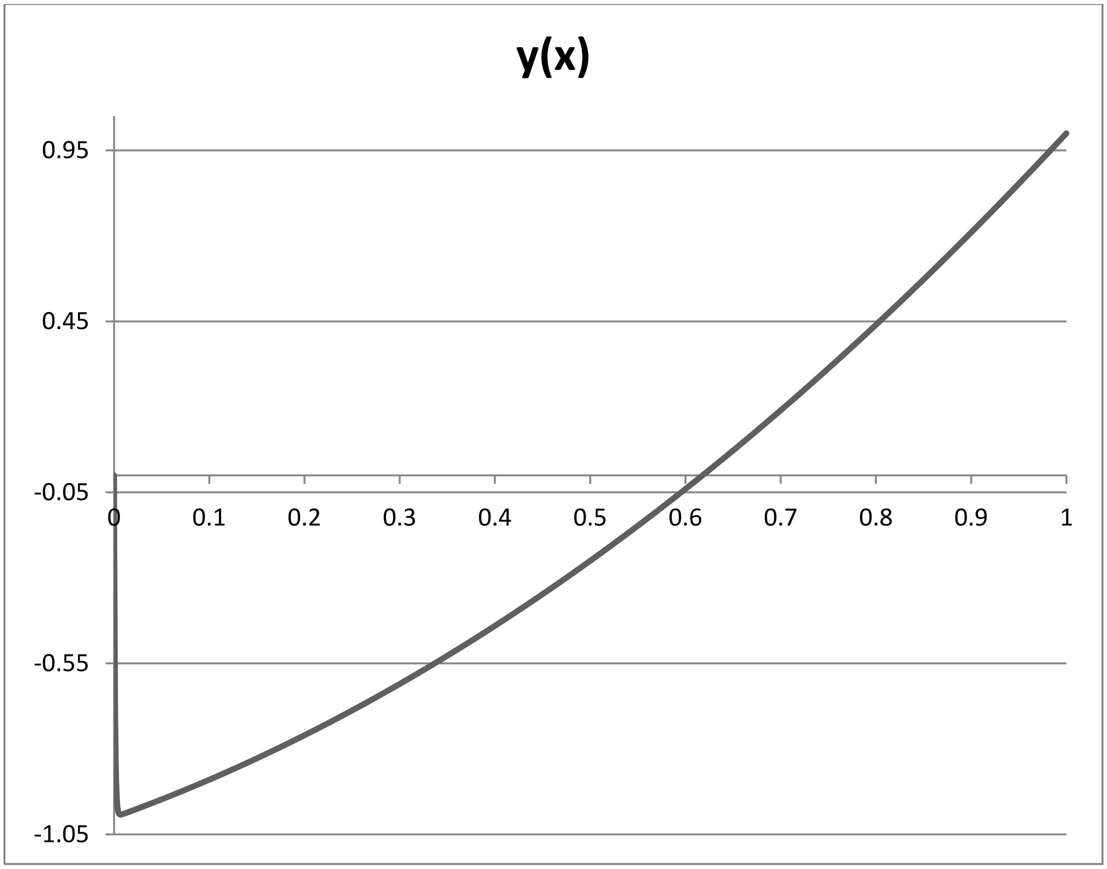

Figure 3: Graph of the spline solution of Example 2

Example 3

Now, consider the following variable coefficient singular perturbation problem from Kevorkian and Cole [8, Eqs. (2.3.26) and (2.3.27) with $\alpha = -1/2$]

$$\varepsilon y''(x) + \left(1 - \frac{x}{2}\right) y'(x) - \frac{1}{2} y(x) = 0$$
$$x \in [0,1] \text{ , } y(0) = 0 \text{ , } y(1) = 1. \tag{67}$$

Here we note that $p(x) = 1 - \dfrac{x}{2}$, $q(x) = -\dfrac{1}{2}$, $r(x) = 0$.

The solution to (67) using (46) has the form $u(x) = v(x) + w(x)e^{-t(x)/\varepsilon}$ with $t(x) = x - x^2/4$. The results are given in Table 8 and Table 9 for $\varepsilon = 10^{-3}$ and in Table 10 for $\varepsilon = 10^{-4}$ and compared to the exact solution [12]. The spline function is graphed in Figure 4. The exact solution is given by

$$y(x) = \frac{1}{2-x} + \frac{1}{2}\exp\left(-(x - x^2/4)/\varepsilon\right).$$

Table 8: Numerical results of Example 3 with $\varepsilon = 10^{-3}$, $h = 10^{-3}$

x	$v(x)$	$w(x)$
0	0.5	-0.5
0.001	0.500250125	-0.5
0.01	0.502512563	-0.5
0.02	0.505050505	-0.5
0.03	0.507614213	-0.5
0.04	0.510204082	-0.5
0.05	0.512820513	-0.5
0.1	0.52631579	-0.5
0.3	0.588235294	-0.5
0.5	0.666666667	-0.5
0.7	0.769230769	-0.5
0.9	0.909090909	-0.5
1	1	-0.5

Table 9: Numerical results of Example 3 with $\varepsilon = 10^{-3}$, $h = 10^{-3}$

x	$y(x)$	exact solution	error
0	0.0000000000	0.0000000000	0.0000E+00
0.001	0.3162644138	0.3162644138	-3.0755E-11
0.01	0.5024892882	0.5024892882	-4.8775E-11
0.02	0.5050505040	0.5050505039	-4.8915E-11
0.03	0.5076142132	0.5076142132	-4.9052E-11
0.04	0.5102040817	0.5102040816	-4.9188E-11
0.05	0.5128205129	0.5128205128	-4.9323E-11
0.1	0.5263157895	0.5263157895	-4.9981E-11
0.3	0.5882352942	0.5882352941	-5.2084E-11
0.5	0.6666666667	0.6666666667	-5.2156E-11
0.7	0.7692307693	0.7692307692	-4.6595E-11
0.9	0.9090909091	0.9090909091	-2.5136E-11
1	1.0000000000	1.0000000000	0.0000E+00

Table 10: Numerical results of Example 3 with $\varepsilon = 10^{-4}$, $h = 10^{-4}$

x	$y(x)$	exact solution	error
0	0.0000000000	0.0000000000	0.0000E+00
0.001	0.3160806821	0.3160806821	-2.9532E-14
0.01	0.5002273683	0.5002273683	-4.6962E-14
0.02	0.5005004995	0.5005004995	-4.6851E-14
0.03	0.5007511267	0.5007511267	-4.6851E-14
0.04	0.5010020040	0.5010020040	-4.6851E-14
0.05	0.5012531328	0.5012531328	-4.6740E-14
0.1	0.5263157895	0.5263157895	-4.8184E-14
0.3	0.5882352941	0.5882352941	-5.2180E-14
0.5	0.6666666667	0.6666666667	-5.3735E-14
0.7	0.7692307692	0.7692307692	-4.7184E-14
0.9	0.9090909091	0.9090909091	-2.6867E-14
1	1.0000000000	1.0000000000	0.0000E+00

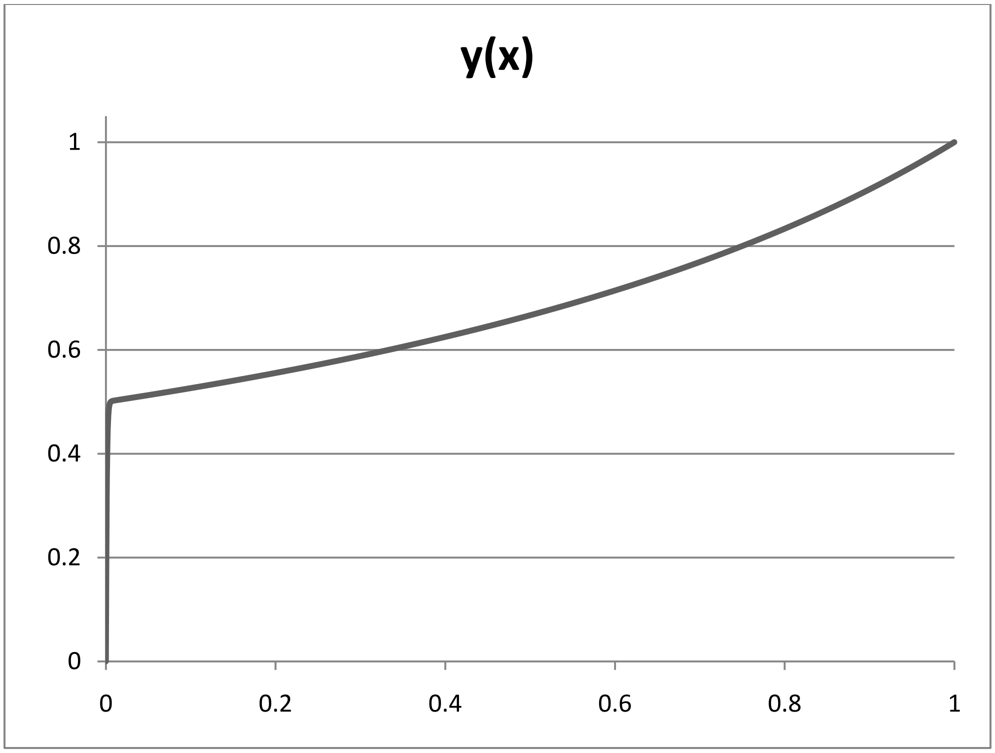

Figure 4: Graph of the spline solution of Example 3

Linear Singular Perturbation Problems with Right-end Boundary Layer

Example 4

First, consider the following homogeneous singular perturbation problem

$$\begin{aligned} &\varepsilon y''(x) - y'(x) = 0 \\ &x \in [0,1]\ ,\ y(0) = 1\ ,\ y(1) = 0. \end{aligned} \tag{68}$$

Here we see that $p(x) = -1,\ q(x) = 0,\ r(x) = 0$.

The solution to (68) using (46) has the form $u(x) = v(x) + w(x)e^{-t(x)/\varepsilon}$ with $t(x) = -x - 1.$

The results are given in Table 11 and Table 12 for $\varepsilon = 10^{-3}$ and Table 13 for $\varepsilon = 10^{-4}$ and compared to the exact solution [11]. The spline function is graphed in Figure 5. The exact solution is given by: $y(x) = \dfrac{e^{(x-1)/\varepsilon} - 1}{e^{-1/\varepsilon} - 1}$.

Table 11: Numerical results of Example 4 with $\varepsilon = 10^{-3}$, $h = 10^{-3}$

x	$v(x)$	$w(x)$
0	1	-1
0.2	1	-1
0.4	1	-1
0.6	1	-1
0.8	1	-1
0.9	1	-1
0.92	1	-1
0.94	1	-1
0.96	1	-1
0.98	1	-1
0.99	1	-1
0.999	1	-1
1	1	-1

Table 12: Numerical results of Example 4 with $\varepsilon = 10^{-3}$, $h = 10^{-3}$

x	$y(x)$	exact solution	error
0	1.0000000000	1.0000000000	0.0000E+00
0.2	1.0000000000	1.0000000000	0.0000E+00
0.4	1.0000000000	1.0000000000	0.0000E+00
0.6	1.0000000000	1.0000000000	0.0000E+00
0.8	1.0000000000	1.0000000000	0.0000E+00
0.9	1.0000000000	1.0000000000	0.0000E+00
0.92	1.0000000000	1.0000000000	0.0000E+00
0.94	1.0000000000	1.0000000000	0.0000E+00
0.96	1.0000000000	1.0000000000	0.0000E+00
0.98	0.9999999979	0.9999999979	0.0000E+00
0.99	0.9999546001	0.9999546001	0.0000E+00
0.999	0.6321205588	0.6321205588	0.0000E+00
1	0.0000000000	0.0000000000	0.0000E+00

Table 13: Numerical results of Example 4 with $\varepsilon = 10^{-4}$, $h = 10^{-4}$

x	$y(x)$	exact solution	error
0	1.0000000000	1.0000000000	0.0000E+00
0.2	1.0000000000	1.0000000000	0.0000E+00
0.4	1.0000000000	1.0000000000	0.0000E+00
0.6	1.0000000000	1.0000000000	0.0000E+00
0.8	1.0000000000	1.0000000000	0.0000E+00
0.9	1.0000000000	1.0000000000	0.0000E+00
0.92	1.0000000000	1.0000000000	0.0000E+00
0.94	1.0000000000	1.0000000000	0.0000E+00
0.96	1.0000000000	1.0000000000	0.0000E+00
0.998	0.9999999979	0.9999999979	0.0000E+00
0.999	0.9999546001	0.9999546001	0.0000E+00
0.9999	0.8646647168	0.8646647168	0.0000E+00
1	0.6321205588	0.6321205588	0.0000E+00

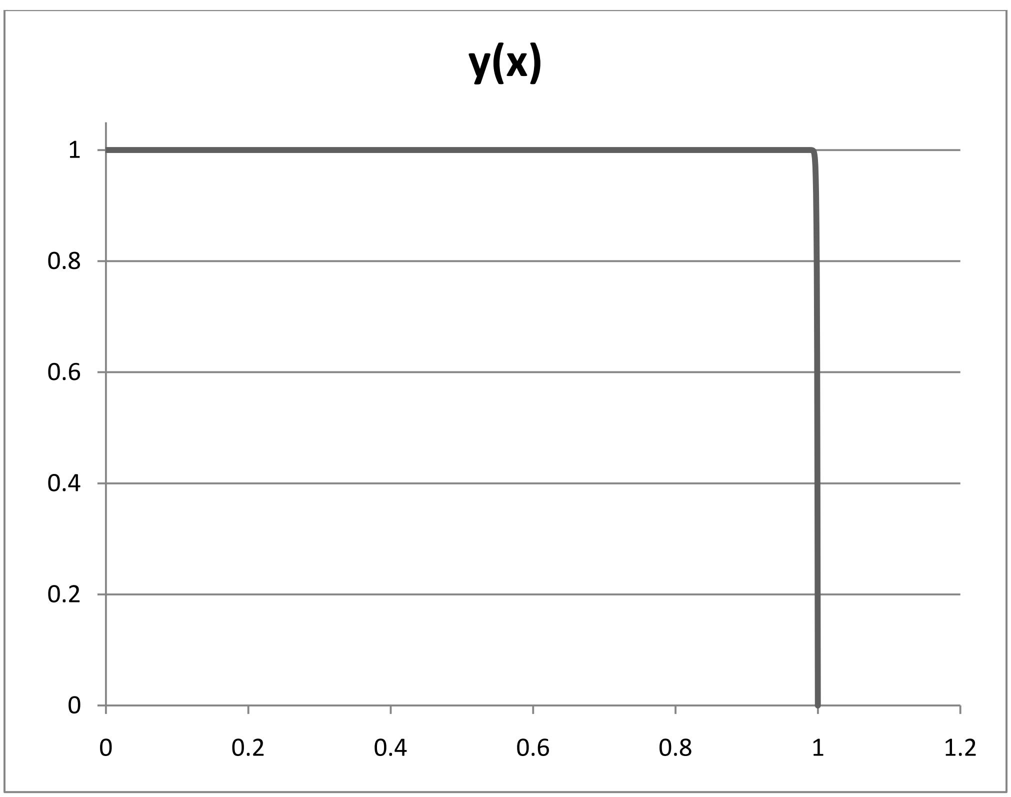

Figure 5: Graph of the spline solution of Example 4

Example 5

First, consider the following homogeneous singular perturbation problem

$$\varepsilon y''(x) - y'(x) - (1+\varepsilon) y(x) = 0$$
$$x \in [0,1] \,,\, y(0) = 1 + e^{-(1+\varepsilon)/\varepsilon} \,,\, y(1) = 1 + 1/\varepsilon. \tag{69}$$

Here we see that $p(x) = -1, \; q(x) = -(1+\varepsilon), \; r(x) = 0$.

The solution to (69) using (46) has the form $u(x) = v(x) + w(x)e^{-t(x)/\varepsilon}$ with $t(x) = -x + 1$. The results are given in Table 14 and Table 15 for $\varepsilon = 10^{-3}$ and in Table 16 for $\varepsilon = 10^{-4}$, and compared to the exact solution [11]. The spline function is graphed in Figure 6. The exact solution is given by

$$y(x) = e^{(1+\varepsilon)(x-1)/\varepsilon}.$$

Table 14: Numerical results of Example 5 with $\varepsilon = 10^{-3}, \; h = 10^{-3}$

x	$v(x)$	$w(x)$
0	1	0.367646878
0.2	0.818567023	0.449134729
0.4	0.670051972	0.548684122
0.6	0.548482448	0.670298347
0.8	0.448969645	0.818868007
0.9	0.406203912	0.90507961
0.92	0.398152572	0.923381899
0.94	0.390260818	0.942054291
0.96	0.382525485	0.961104272
0.98	0.374943474	0.980539477
0.99	0.371209012	0.990403967
0.999	0.367879809	0.999366829
1	0.367511746	1.000367696

Table 15: Numerical results of Example 5 with $\varepsilon = 10^{-3},\ h = 10^{-3}$

x	$y(x)$	exact solution	error
0	1.0000000000	1.0000000000	0.0000E+00
0.2	0.8185670233	0.8187307531	1.6373E-04
0.4	0.6700519716	0.6703200460	2.6807E-04
0.6	0.5484824479	0.5488116361	3.2919E-04
0.8	0.4489696447	0.4493289641	3.5932E-04
0.9	0.4062039117	0.4065696597	3.6575E-04
0.92	0.3981525722	0.3985190411	3.6647E-04
0.94	0.3902608177	0.3906278354	3.6702E-04
0.96	0.3825254852	0.3828928860	3.6740E-04
0.98	0.3749434762	0.3753111009	3.6762E-04
0.99	0.3712539764	0.3716216392	3.6766E-04
0.999	0.7355263194	0.7357592502	2.3293E-04
1	1.3678794412	1.3678794412	0.0000E+00

Table 16: Numerical results of Example 5 with $\varepsilon = 10^{-4},\ h = 10^{-4}$

x	$y(x)$	exact solution	error
0	1.0000000000	1.0000000000	0.0000E+00
0.2	0.818717108	0.818730753	1.3645E-05
0.4	0.670297703	0.670320046	2.2343E-05
0.6	0.548784197	0.548811636	2.7439E-05
0.8	0.449299011	0.449328964	2.9953E-05
0.9	0.4065391690	0.4065696597	3.0491E-05
0.92	0.3984884899	0.3985190411	3.0551E-05
0.94	0.3905972382	0.3906278354	3.0597E-05
0.96	0.3828622566	0.3828928860	3.0629E-05
0.98	0.3752804505	0.3753110989	3.0648E-05
0.999	0.3682622060	0.3682928592	3.0653E-05
0.9999	0.7357395039	0.7357588860	1.9382E-05
1	1.3678794412	1.3678794412	0.0000E+00

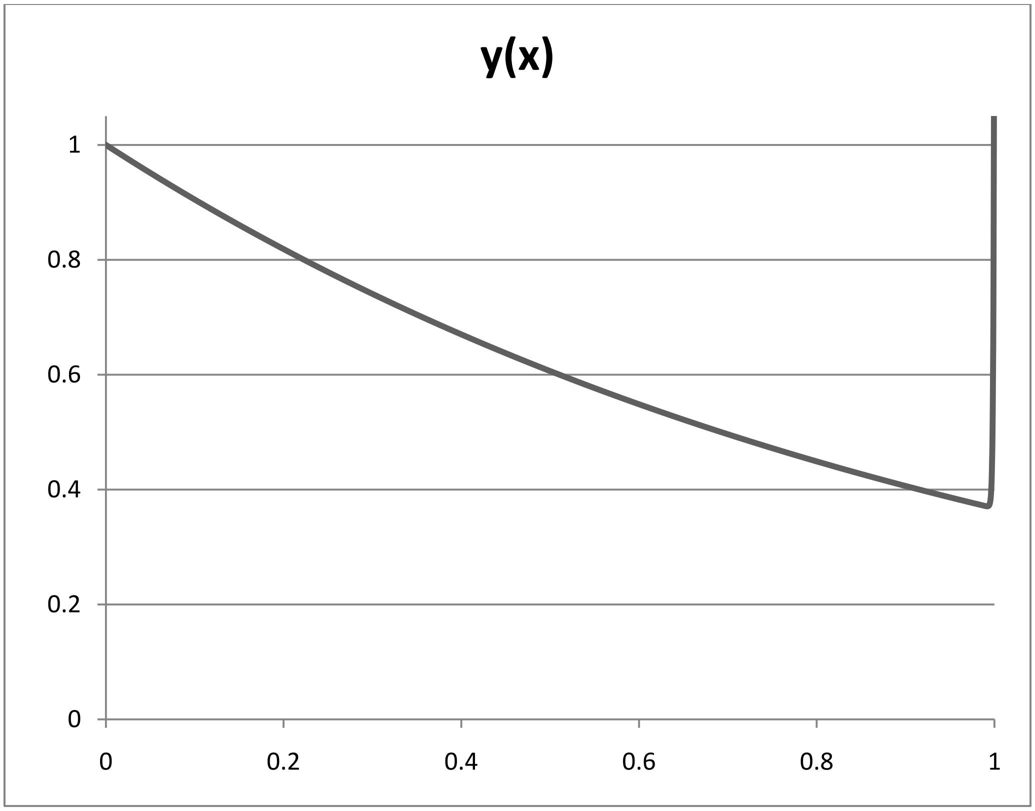

Figure 6: Graph of the spline solution of Example 5

Nonlinear Singular Perturbation Problems with Left-end Boundary Layer

Example 6

For the nonlinear boundary value problems, we convert the nonlinear singular perturbation problem is converted to a sequence of linear singular perturbation problems by using quasilinearization, and then the outer layer solution is taken to be the initial approximation. First, consider the following singular perturbation problem from Kevorkian and Cole [8 and Cole, p. 56, Eq. (2.5.1)]

$$\begin{aligned} &\varepsilon y''(x) - y(x)y'(x) - y(x) = 0 \\ &x \in [0,1] \text{ , } y(0) = -1 \text{ , } y(1) = 3.9995. \end{aligned} \tag{69}$$

The initial approximation can be taken from the problem

$$-y(x)y'(x)-y(x)=0\,. \tag{70}$$

Given the value at x = 0, we must suppose $y'(x)=0$, and $y(x)=C$ and further that $y(x)=x+2.9995$ in order to satisfy the condition at x = 1.We can the use the linear problem

$$\begin{aligned}&\varepsilon y''(x)-\left(x+2.9995\right)y'(x)-\left(x+2.9995\right)=0\\&x\in[0,1]\;,\;y(0)=-1\;,\;y(1)=3.9995\end{aligned} \tag{71}$$

as the linear problem concerned to (69). We can now solve the linear problem as the approximation to the nonlinear problem (69) and we have $p(x)=x+2.9995,\; q(x)=0,$ and $r(x)=x+2.9995$.

The solution to (69) using (46) has the form $u(x)=v(x)+w(x)e^{-t(x)/\varepsilon}$ with $t(x)=x^2/2+2.9995x$. The results are given in Table 17 and Table 18 for $\varepsilon=10^{-3}$ and in Table 19 for $\varepsilon=10^{-4}$, and compared to the exact solution [8, p. 57-58, Eqs. (2.5.5), (2.5.11), and (2.5.14)]. The spline function is graphed in Figure 7. The exact solution is given by

$$y(x)=x+c_1\tanh(c_1(x/\varepsilon+c_2)/2)$$

where $c_1=2.9995$ and $c_2=(1/c_1)\log_e[(c_1-1)/(c_1+1)]$. For this example, we have a boundary layer of width $O(\varepsilon)$ at x = 0 [12]

Table 17: Numerical results of Example 6 with $\varepsilon = 10^{-3}$, $h = 10^{-3}$

x	$v(x)$	$w(x)$
0	2.9995	-3.9995
0.001	3.0005	-3.998167055
0.01	3.0095	-3.986210417
0.02	3.0195	-3.973008859
0.03	3.0295	-3.959894455
0.04	3.0395	-3.946866343
0.05	3.0495	-3.933923676
0.1	3.0995	-3.870463059
0.3	3.2995	-3.635853993
0.5	3.4995	-3.428061223
0.7	3.6995	-3.242735572
0.9	3.8995	-3.076420118
1	3.9995	-2.019543562

Table 18: Numerical results of Example 6 with $\varepsilon = 10^{-3}$, $h = 10^{-3}$

x	$y(x)$	exact solution	error
0	-1.0000000000	-1.0000000000	0.0000E+00
0.001	2.8014429835	2.4569396723	-3.4450E-01
0.01	3.0095000000	3.0095000000	-8.8063E-13
0.02	3.0195000000	3.0195000000	-1.0836E-13
0.03	3.0295000000	3.0295000000	-1.0703E-13
0.04	3.0395000000	3.0395000000	-1.0569E-13
0.05	3.0495000000	3.0495000000	-1.0525E-13
0.1	3.0995000000	3.0995000000	-9.9032E-14
0.3	3.2995000000	3.2995000000	-7.7272E-14
0.5	3.4995000000	3.4995000000	-5.5067E-14
0.7	3.6995000000	3.6995000000	-3.3307E-14
0.9	3.8995000000	3.8995000000	-1.1546E-14
1	3.9995000000	3.9995000000	0.0000E+00

Table 19: Numerical results of Example 6 with $\varepsilon = 10^{-4},\ h = 10^{-4}$

x	$y(x)$	exact solution	error
0	-1.0000000000	-1.0000000000	0.0000E+00
0.0001	2.8003936357	2.4560396723	-3.4435E-01
0.001	3.0005000000	3.0005000000	1.3540E-12
0.002	3.0015000000	3.0015000000	2.1059E-12
0.003	3.0025000000	3.0025000000	2.1041E-12
0.004	3.0035000000	3.0035000000	2.1019E-12
0.005	3.0045000000	3.0045000000	2.0997E-12
0.1	3.0995000000	3.0995000000	1.8994E-12
0.3	3.2995000000	3.2995000000	1.4770E-12
0.5	3.4995000000	3.4995000000	1.0552E-12
0.7	3.6995000000	3.6995000000	6.3283E-13
0.9	3.8995000000	3.8995000000	2.1094E-13
1	3.9995000000	3.9995000000	0.0000E+00

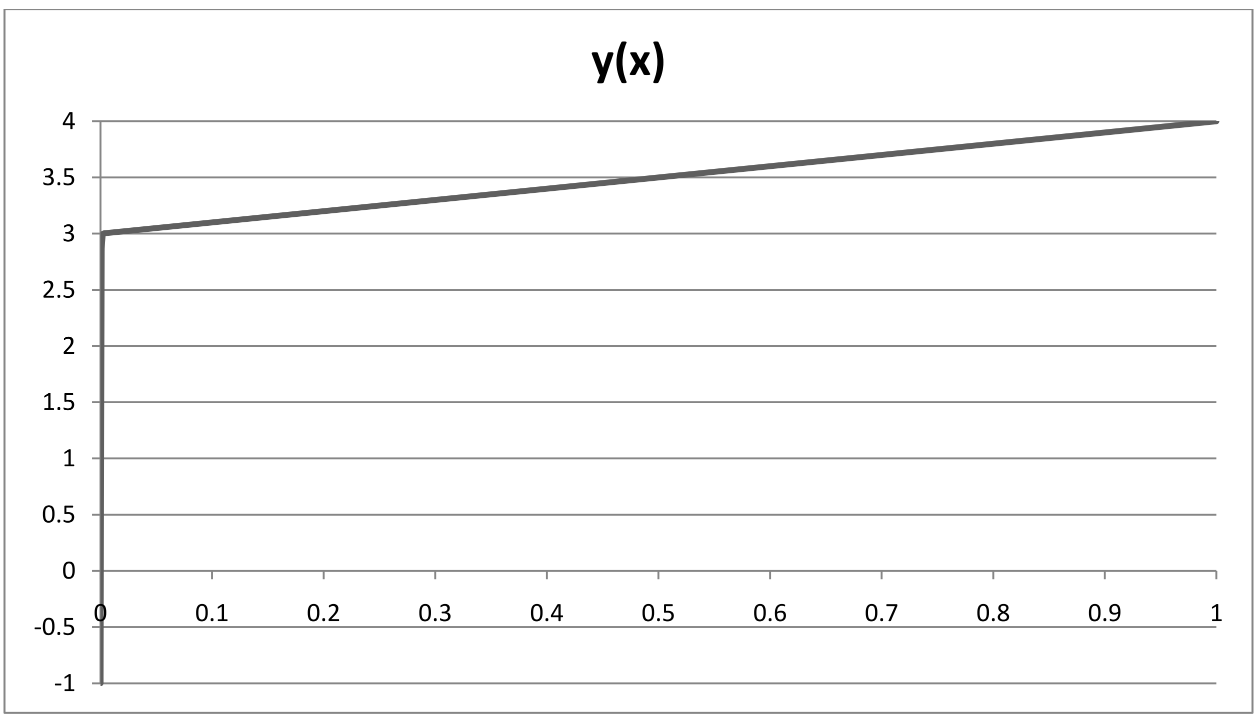

Figure 7: Graph of the spline solution of Example 6

Example 7

Finally, let us consider the following singular perturbation problem from Bender and Orszag [3, p. 463, Eq (9.7.1)]

$$\begin{aligned} &\varepsilon y''(x) - 2y'(x) + e^{y(x)} = 0 \\ &x \in [0,1]\ ,\ y(0) = 0\ ,\ y(1) = 0. \end{aligned} \tag{72}$$

The initial approximation can be taken from the problem and we can then use the linear problem

$$\begin{aligned} &\varepsilon y''(x) + 2y'(x) + \frac{2}{1+x} y(x) = \left(\frac{2}{1+x}\right)\left[\ln\left(\frac{2}{1+x}\right) - 1\right] \\ &x \in [0,1]\ ,\ y(0) = 0\ ,\ y(1) = 0 \end{aligned} \tag{73}$$

[11] as the linear problem concerned to (72). We can now solve the linear problem as the approximation to the nonlinear problem (72) and we have $p(x) = 2,\ q(x) = \dfrac{2}{1+x},$ and

$r(x) = \left(\dfrac{2}{1+x}\right)\left[\ln\left(\dfrac{2}{1+x}\right) - 1\right].$

The solution to (72) using (46) has the form $u(x) = v(x) + w(x)e^{-t(x)/\varepsilon}$ with $t(x) = 2x$. The results are given in Table 20 and Table 21 for $\varepsilon = 10^{-3}$ and in Table 22 for $\varepsilon = 10^{-4}$, and compared to the exact solution [3, p. 463, Eq. (9.7.6)]. The exact solution is given by

$$y(x) = \log_e(2/(1+x)) - \log_e(2)e^{-2x/\varepsilon}$$

For this example, we have a boundary layer of width $O(\varepsilon)$ at x = 0 [3]. The spline function is graphed in Figure 8.

Table 20: Numerical results of Example 7 with $\varepsilon = 10^{-3}$, $h = 10^{-3}$

x	$v(x)$	$w(x)$
0	0.693147181	-0.693147181
0.001	0.69214768	-0.69349361
0.01	0.68319685	-0.696603721
0.02	0.673344553	-0.700043194
0.03	0.663588378	-0.703465851
0.04	0.653926467	-0.706871936
0.05	0.644357016	-0.710261687
0.1	0.597837001	-0.726973392
0.3	0.430782916	-0.790294192
0.5	0.287682072	-0.848904887
0.7	0.16251893	-0.903722359
0.9	0.051293294	-0.955399758
1	0	-0.980217321

Table 21: Numerical results of Example 7 with $\varepsilon = 10^{-3}$, $h = 10^{-3}$

x	$y(x)$	exact solution	error
0	0.0000000000	0.0000000000	0.0000E+00
0.001	0.5982935261	0.5983404102	4.6884E-05
0.01	0.6831968483	0.6831968483	-2.2989E-11
0.02	0.6733445533	0.6733445533	-2.9037E-11
0.03	0.6635883783	0.6635883783	-2.8005E-11
0.04	0.6539264674	0.6539264674	-2.7013E-11
0.05	0.6443570164	0.6443570164	-2.6060E-11
0.1	0.5978370008	0.5978370008	-2.1825E-11
0.3	0.4307829161	0.4307829161	-1.0948E-11
0.5	0.2876820725	0.2876820725	-5.3993E-12
0.7	0.1625189295	0.1625189295	-2.3527E-12
0.9	0.0512932944	0.0512932944	-5.9214E-13
1	0.0000000000	0.0000000000	0.0000E+00

Table 22: Numerical results of Example 7 with $\varepsilon = 10^{-4},\ h = 10^{-4}$

x	$y(x)$	exact solution	error
0	0.0000000000	0.0000000000	0.0000E+00
0.0001	0.5992363968	0.5992399156	3.5188E-06
0.001	0.6921476788	0.6921476788	4.0590E-13
0.002	0.6911491779	0.6911491779	-2.8977E-14
0.003	0.6901516716	0.6901516716	-2.8755E-14
0.004	0.6891551593	0.6891551593	-2.8866E-14
0.005	0.6881596390	0.6881596390	-2.8866E-14
0.1	0.5978370008	0.5978370008	-2.2760E-14
0.3	0.4307829161	0.4307829161	-1.0658E-14
0.5	0.2876820725	0.2876820725	-5.2736E-15
0.7	0.1625189295	0.1625189295	-2.4147E-15
0.9	0.0512932944	0.0512932944	-8.3267E-16
1	0.0000000000	0.0000000000	0.0000E+00

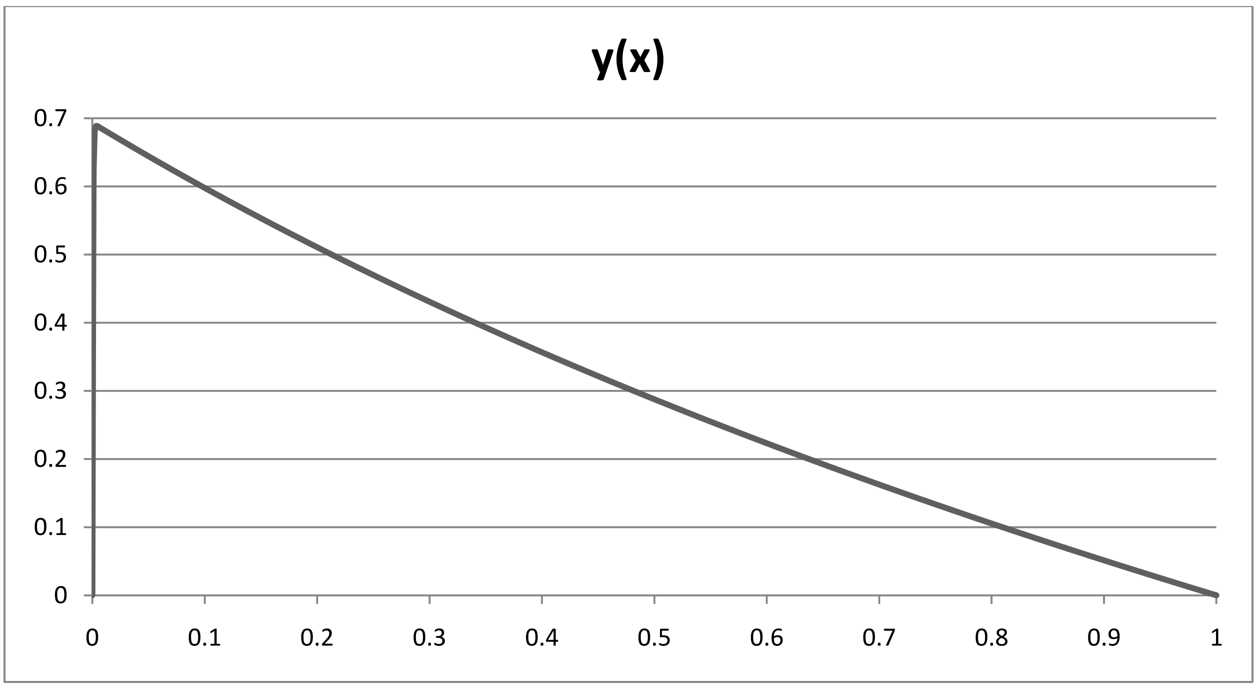

Figure 8: Graph of the spline solution of Example 7

CONCLUSION

The initial-value technique described by Kumar, et al. has been examined as a method for solving singularly perturbed two-point boundary value problems. For each problem examined, we computed the solution numerically by solving three initial value problems, which are deduced from the original problem. In general, the numerical solution of a boundary value problem will be more difficult to calculate than the numerical solutions of the initial-value problems. It is generally preferable to convert the second order problem into first order problems.

This technique was implemented using standard, readily available software spreadsheet program (Microsoft EXCEL), which makes it easy to implement on any computer and requires only modest preparation. No knowledge of differential equations is required to complete the process, and minimal knowledge of Calculus is required for the process. We implemented the method on linear boundary value problems with both right-end and left-end behavior. Using quasilinearization, we were able to approximate the solutions to nonlinear boundary value problems with left-end behavior.

We used two specific values of ε and used the same value for the mesh size, but the data suggests that increasing the mesh size provides a proportionally better approximation. The approximation became more accurate for smaller values of ε. Thus the initial-value technique provides a reasonable approximation for the solution of the problem, and the cubic spline method was easily implemented to solve the initial-value problems. In addition, the step size used for the calculations could easily be varied in order to more closely approximate the values near the boundary points.

By calculating the first derivatives along with the point values, we were also able to build a spline function to graph the solutions and to calculate any values other than the nodes. The numerical results indicate that the initial-value technique is accurate and suitable for solving linear and nonlinear problems with thin layers. The spline is also useful for providing the additional data points that can then be used to refine the spline near the boundary points. Once the spline is calculated using an equal step size, another calculation can be used to refine the spline near the boundary values using a smaller step size as needed.

The error estimate for the cubic spline method for the first-order problems is described in Kumar's papers. For a function $y \in C^4[a,b]$, the cubic spline method described provides a fourth-order approximation to the solution of the initial-value problems used for the initial-value technique. Also, the error bounds for both cubic splines and cubic Hermite splines have an error bound that is also $O\left(h^4\right)$. For the solution to the nonlinear problems, the convergence is quadratic.

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