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# Ising-Link Regge Gravity

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We define a simplified version of Regge quantum gravity where the link lengths can take on only two possible values, both always compatible with the triangle inequalities. This is therefore equivalent to a model of Ising spins living on the links of a regular lattice with somewhat complicated, yet local interactions. The measure corresponds to the natural sum over all $2^{\# \text{ links}}$ configurations, and numerical simulations can be efficiently implemented by means of look-up tables. In three dimensions we find a peak in the "curvature susceptibility" which grows with increasing system size. The value of the corresponding critical exponent appears to vary with the cosmological constant $\lambda$, agreeing with Regge gravity for at least one value of $\lambda$. However, the curvature does not go to zero at the transition.

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## I. INTRODUCTION

To date, two main formulations of lattice quantum gravity have been studied: the so-called "Regge gravity" [1, 2] and "dynamical triangulation" approaches [3–5]. The distinguishing feature is that the former has a fixed incidence matrix and varying link lengths while the latter has a varying incidence matrix and fixed link lengths.

Both formulations are technically and computationally demanding. For example, the Regge approach involves calculating areas and deficit angles involving general $d$ simplexes. In the dynamical triangulation approach these take on only a limited number of possible values, but the updating moves involve complicated interchanges of several simplexes at once. We introduce here a third lattice gravity approach which is structurally and computationally much simpler than either Regge gravity or dynamical triangulation, and as a result is amenable to analytic attack in more than two dimensions.

We call our formulation "Ising-link Regge gravity." It is easy to define. The incidence matrix is fixed exactly as in the conventional Regge approach. But the link lengths can only take on two values:

$$l_i = 1 + bs_i \ , \tag{1}$$

with $s_i = \pm 1$ and $b$ a positive constant.[1] $i$ is a link label. In order that the triangle inequality (or its higher dimensional generalization, that the simplex volume is real and positive) is always satisfied, it is straightforward to show that we must take $b < \frac{1}{3}$ in two dimensions, $b < 3 - \sqrt{8} \approx 0.17$ in three dimensions, etc. (See Sec. II.) We restrict $b$ to satisfy this inequality so that *all* $2^{N_1}$ configurations are allowed. ($N_1$ is the number of links.) This is quite different from either Regge gravity or dynamical triangulation, where most potential updates either violate the triangle inequalities or violate the manifold property. Furthermore, it provides us with a natural measure which gives all $2^{N_1}$ configurations equal weight. It is clear that our model is completely equivalent to a (regular lattice) Ising model with spins ($s_i$) living on the links. We will see that the spin interactions are local, albeit somewhat complicated.

The Ising-link model is analytically and computationally much simpler than either the Regge gravity or the dynamical triangulation approach. But is it too simple? In Sec. III we present mean field theory results on the model in three dimensions and in Sec. IV we give corresponding Monte Carlo results. We compare to results obtained by Hamber and Williams for the Regge theory in three dimensions (3D).

## II. ISING MODEL FORMULATION

In this section we discuss how to compute the Ising action corresponding to the discrete form of

$$S = \lambda V - \frac{k}{2}\int d^d x\sqrt{g}R \ , \tag{2}$$

where $V$ is the $d$-dimensional volume, $\int d^d x\sqrt{g}$, and $R$ is the scalar curvature. The lattice is formed out of hypercubes plus face, cubic ($d \geq 3$) and hypercubic diagonals ($d \geq 4$), with periodic boundary conditions [2]. First we will treat two dimensions, then three. Four dimensions are just like three, only harder.

*Two dimensions.* Consider a triangle with link lengths

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[1] $l_i = c(1 + bs_i)$ is no more general, as $c$ can be absorbed into the definitions of $\lambda$ and $k$ in (2).

$l_1$, $l_2$, and $l_3$. Define $l_i = 1 + bs_i$ as in (1). Since $s_i^2 = 1$ and the formula for the area of the triangle must be symmetric in the three spins, the most general form for the area is

$$A_{123} = C_0 + C_1(s_1 + s_2 + s_3) + C_2(s_1 s_2 + s_2 s_3 + s_3 s_1) + C_3 s_1 s_2 s_3 \,. \tag{3}$$

There are only four possible values for the area of the triangle, corresponding to 0, 1, 2, or all 3 of its spins being equal to +1. Computing these four areas and comparing to (3) gives four linear equations for the $C_\alpha$ in terms of the parameter $b$. Their solution is

$$\begin{aligned} 32C_0 &= 2\sqrt{3}(1 + b^2) + 3f(b) + 3g(b) \,, \\ 32C_1 &= 4b\sqrt{3} + f(b) - g(b) \,, \\ 32C_2 &= 2\sqrt{3}(1 + b^2) - f(b) - g(b) \,, \\ 32C_3 &= 4b\sqrt{3} - 3f(b) + 3g(b) \,, \end{aligned} \tag{4}$$

where $f(b) \equiv |1 - b|\sqrt{(1 + 3b)(3 + b)}$ and $g(b) \equiv (1 + b)\sqrt{(1 - 3b)(3 - b)}$. For example, $b = 0.1$ gives $C_1 \approx 0.0291$, $C_2 \approx 0.0039$ and $C_3 \approx -0.0008$. As stated in the Introduction, it is seen that we must have $b < 1/3$ for the triangle areas to be real and positive.

Since the Einstein term in (2) is a topological invariant, it is not relevant to the case of fixed topology being considered here. Thus, summing over all triangles and dropping the irrelevant constant term, the two-dimensional action is

$$S = \lambda \left( 2C_1 \sum_i s_i + C_2 \sum_{\langle ij \rangle} s_i s_j + C_3 \sum_{\langle ijk \rangle} s_i s_j s_k \right) \,, \tag{5}$$

where $i$, $j$, and $k$ are link labels. $\langle ij \rangle$ indicates $i$ and $j$ are two of three links forming a triangle. In this case, $s_i$ and $s_j$ may be termed nearest-neighbor links. $\langle ijk \rangle$ means that $i$, $j$, and $k$ are three links which form a triangle. The $C_2$ term is a "nearest-neighbor" ferromagnetic interaction. The $C_1$ term is a magnetic field term and the $C_3$ term is an additional symmetry-breaking term.

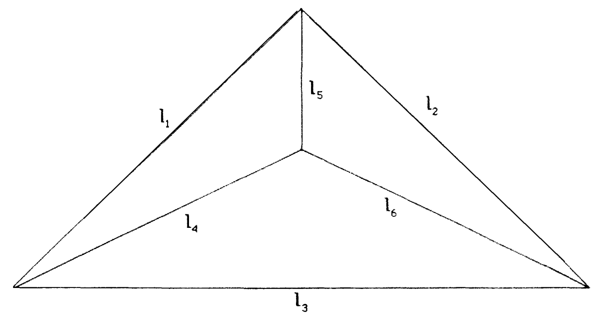


FIG. 1. A labeled tetrahedron.

One might hope that the continuum limit of Ising-link quantum gravity would correspond to a second-order magnetization phase transition. But with the explicit symmetry-breaking terms in (5), it is clear that this transition cannot be from order ($\langle s \rangle \neq 0$) to disorder ($\langle s \rangle = 0$). It would have to be an $\langle s \rangle \neq 0$ to $\langle s \rangle \neq 0$ transition. In two dimensions, as expected, we found no evidence of such a transition, at least in the mean field theory approximation.

The analytic results of Knizhnik *et al.* [6] and others [7] are at fixed area. These results are technically difficult to check in the case of the Ising-link model because changing any link length to its other value always changes the total area.

*Three dimensions.* We will now go on to discuss the form of the theory in three dimensions. In the next section we will compare numerical results in 3D to results for the unconstrained Regge theory.

Consider the labeled tetrahedron of Fig. 1. The volume $V_{\rm tet}$ is given by [2]

$$144V_{\rm tet}^2 = 4l_1^2 l_3^2 l_4^2 - l_1^2(l_3^2 - l_6^2 + l_4^2)^2 - l_3^2(l_1^2 + l_4^2 - l_5^2)^2 - l_4^2(l_1^2 - l_2^2 + l_3^2)^2 + (l_3^2 - l_6^2 + l_4^2)(l_1^2 + l_4^2 - l_5^2)(l_1^2 - l_2^2 + l_3^2) \,, \tag{6}$$

where the $l_i$ may be written in terms of spins $s_i$ using Eq. (1). There are only 11 distinct possible values for the volume of the tetrahedron, corresponding to the 11 *a priori* unknown constants in the most general equation for $V_{\rm tet}$ compatible with the symmetries of the labeled tetrahedron:

$$\begin{aligned} V_{\rm tet} = {} & C_0 + C_1 \sum_i s_i + C_2 \sum_{\langle ij \rangle} s_i s_j + C_3 \sum_{[ij]} s_i s_j + C_4 \sum_{\langle ijk \rangle} s_i s_j s_k + C_5 \sum_{(ijk)} s_i s_j s_k + C_6 \sum_{[ijk]} s_i s_j s_k \\ & + \left( \prod_{l=1}^{6} s_l \right) \left( C_0' + C_1' \sum_i s_i + C_2' \sum_{\langle ij \rangle} s_i s_j + C_3' \sum_{[ij]} s_i s_j \right) \,. \end{aligned} \tag{7}$$

Here $\langle ij \rangle$ are again two of three links that form a triangle in Fig. 1. $[ij]$ are the remaining pairs of links. $\langle ijk \rangle$ form a triangle, $(ijk)$ share a common site, and $[ijk]$ are the remaining triplets of links. Because $s_i^2 = 1$, the last four terms involve 4-, 5- and 6-link interactions. Evaluating (7) for each of the 11 distinct volumes results in 11 linear equations for the $C_i$ and $C_i'$. As in (4) for the 2D case, these can easily be solved to determine the $C_i$ and $C_i'$ as functions of $b$. The result is not particularly illuminating and we will not reproduce it here. It is worth noting, however, that the volumes are always real and positive if we choose $b < 3 - \sqrt{8} \approx 0.17$. In the mean field and numerical results described in the next two sections, $b$ is held equal to 0.1.

We see that after summing up the volumes of all the tetrahedrons, the volume term in (2) will consist of only local interactions of the spins, involving up to 6-spin interactions.

The second (Einstein) term in (2), $-\frac{k}{2}\int d^3x\sqrt{g}\,R$, takes the form [2]

$$S_E = k\sum_i l_i\left(\sum_{t/i}\theta_{t/i} - 2\pi\right) , \tag{8}$$

where $t/i$ denotes a tetrahedron containing the link $i$ and $\theta_{t/i}$ is the corresponding dihedral angle at link $i$. For the tetrahedron shown in Fig. 1, $\theta_{t/5}$ is given by

$$\cos(\theta_{t/5}) = \frac{1}{16A_{145}A_{256}}[2(l_4^2+l_6^2-l_3^2)l_5^2 \\ -(l_4^2+l_5^2-l_1^2)(l_5^2+l_6^2-l_2^2)] , \tag{9}$$

where $A_{ijk}$ is the triangle formed by links $i$, $j$, and $k$. The term $S_E$ can also be written in terms of local spin interactions, but we shall omit the details here.

## III. MEAN FIELD THEORY IN THREE DIMENSIONS

The Ising-link model is quite accessible to mean field theory (MFT) techniques. We write

$$Z = \sum_{\{s\}}\exp(-\beta H[s]) , \tag{10}$$

where $\beta \equiv 1$ and

$$H \equiv S = \lambda V - \frac{k}{2}\int d^3x\sqrt{g}R . \tag{11}$$

This is a functional of the spins since $l_i = 1 + bs_i$, by Eq. (1). We wish to minimize the free energy,

$$F = \langle H\rangle - \mathcal{S}/\beta , \tag{12}$$

for the spin probability distribution function $P[s]$, where $\mathcal{S}$ is the entropy.

The mean field approximation [8] consists of replacing the true probability distribution for the spins by a factorized form:

$$P[s] \to p(s_1)p(s_2)p(s_3)\cdots p(s_{N_1}) , \tag{13}$$

where $N_1$ is the number of links. If all links were equivalent, we could write $p(s_i) = \frac{1+ms_i}{2} \Rightarrow \sum_{s_i} p(s_i) = 1$ and $\langle s_i\rangle = m$. But there are three different kinds of links in the lattice formed out of cubes with body and face diagonals: the body diagonals, the cube edges, and the face diagonals. Links of the same type have the same geometrical environment; links of different types do not. As a result we must use the more general distribution

$$p_j(s_i) \equiv \frac{1+m_js_i}{2} , \tag{14}$$

where $j = 1, 2$, and 3 for body diagonals, cube edges, and face diagonals, respectively.

A straightforward calculation allows us to determine $\langle H\rangle$ and $\mathcal{S}$ as functions of $m_1$, $m_2$, and $m_3$. Let

$$\begin{aligned} P_V &\equiv p_2(s_1)p_3(s_2)p_2(s_3)p_3(s_4)p_1(s_5)p_2(s_6) , \\ P_{R1} &\equiv p_1(s_1)p_3(s_2)p_2(s_3)p_3(s_4)p_2(s_5)p_2(s_6) , \\ P_{R2} &\equiv p_3(s_1)p_2(s_2)p_1(s_3)p_2(s_4)p_2(s_5)p_3(s_6) , \\ P_{R3} &\equiv p_2(s_1)p_1(s_2)p_3(s_3)p_2(s_4)p_3(s_5)p_2(s_6) . \end{aligned} \tag{15}$$

We find that

$$\langle V\rangle = 6N_0\sum_{s_1}\cdots\sum_{s_6}P_V(s)V_{\rm tet}(s) , \tag{16}$$

$$\left\langle\frac{1}{2}\int d^3x\sqrt{g}R\right\rangle = 2\pi N_0[7+b(m_1+3m_2+3m_3)] \\ -6N_0\sum_{s_1}\cdots\sum_{s_6}l_5(s)\theta_{t/5}(s)[P_V(s)+2P_{R1}(s)+P_{R2}(s)+2P_{R3}(s)] \tag{17}$$

and

$$\mathcal{S} \equiv -\langle \ln P[s]\rangle = -N_0[h(m_1)+3h(m_2)+3h(m_3)] , \tag{18}$$

where $V_{\rm tet}$ and $\theta_{t/5}$ are given by (6) and (9), respectively, $N_0$ is the number of lattice sites, and $h(x) \equiv \frac{1+x}{2}\ln(\frac{1+x}{2}) + \frac{1-x}{2}\ln(\frac{1-x}{2})$. $s$ is shorthand for $s_1, s_2, \ldots, s_6$. Now it is a simple matter to numerically minimize the free energy (12) as a function of $m_1$, $m_2$, and $m_3$. Then $\langle s\rangle$ is given by

$$\langle s\rangle = \frac{m_1+3m_2+3m_3}{7} , \tag{19}$$

and

$$\mathcal{R} \equiv \langle l^2\rangle\frac{\langle\int d^3x\sqrt{g}R\rangle}{\langle V\rangle} \\ = (1+b^2+2b\langle s\rangle)\frac{\langle\int d^3x\sqrt{g}R\rangle}{\langle V\rangle} . \tag{20}$$

Here we follow the notation of Hamber and Williams [9]. Also the "curvature susceptibility" is defined as

$$\chi_R = \frac{2}{\langle V\rangle}\frac{\partial}{\partial k}\left\langle\int d^3x\sqrt{g}\,R\right\rangle . \tag{21}$$

*Results.* Figure 2 shows typical MFT results for the case $\lambda = 1$. There is sharp crossover behavior seen in $\langle s \rangle$, $v$, and $\mathcal{R}$ at $k$ slightly negative. $\mathcal{R}$ and $\langle \int d^3x \sqrt{g} R \rangle$ (not shown in Fig. 2) are related by (20) and exhibit similar behavior in this region. $\chi_R$ was evaluated using (21), by taking a numerical derivative of $\langle \int d^3x \sqrt{g} R \rangle$ with increment $\Delta k = 1$. We see a peak in $\chi_R$ here, as expected from the rapid crossover behavior in $\mathcal{R}$. Hamber and Williams [9] found a second-order phase transition in the 3D Regge theory exhibiting

$$\chi_R \sim |k_c - k|^{\delta - 1} \tag{22}$$

for $k < k_c$, with $\delta = 0.80 \pm 0.06$, a very weak second-order phase transition. To investigate whether this kind of nonanalyticity is seen in the MFT approximation to the Ising-link model, we varied $\Delta k$ from 1 downward. ($\Delta k$ is the increment used to take the numerical derivative of $\langle \int d^3x \sqrt{g}\, R \rangle$.) The behavior (22) would result in the peak of $\chi_R$ growing with $\Delta k$ like $(\Delta k)^{\delta - 1}$. $\delta = 0.80$ would imply that the peak would grow by 58% in height for each factor of 10 decrease in $\Delta k$. However, for all values of $\lambda$ considered (up through $\lambda = 75$), there was *no* increase in the peak height as $\Delta k$ was decreased from 1 down to 0.01. As a result we have no evidence of (22) with $\delta - 1 < 0$ in the *mean field approximation* to the 3D Ising-link model. Nevertheless, in the next section we will present *Monte Carlo* evidence for (22) with $\delta - 1 < 0$ for $\lambda$ greater than about 15.

For $\lambda \neq 1$ the situation looks similar to that shown in Fig. 2. There is a finite peak in $\chi_R$ at $k$ slightly negative and a first-order phase transition at large positive $k$. The height of the peak in $\chi_R$ and the size of the discontinuity at the first-order phase transition vary appreciably with $\lambda$. Figure 3 shows a dashed curve in $\lambda$-$k$ space where there is a peak in $\chi_R$ and a solid curve of first-order phase transitions. For $k$ to the right of the first-order phase transition, $\mathcal{R}$ rapidly approaches zero from below as the system approaches a state with $m_1 = m_3 = -m_2 = 1$. As $k \to -\infty$, the system approaches a state with $m_1 = m_2 = -m_3 = 1$.

## IV. NUMERICAL RESULTS IN THREE DIMENSIONS

The 3D Ising-link model was also analyzed by the Monte Carlo method. The discrete form of (2) was used, with toroidal topology and without higher derivative terms, as discussed in the previous section. As in

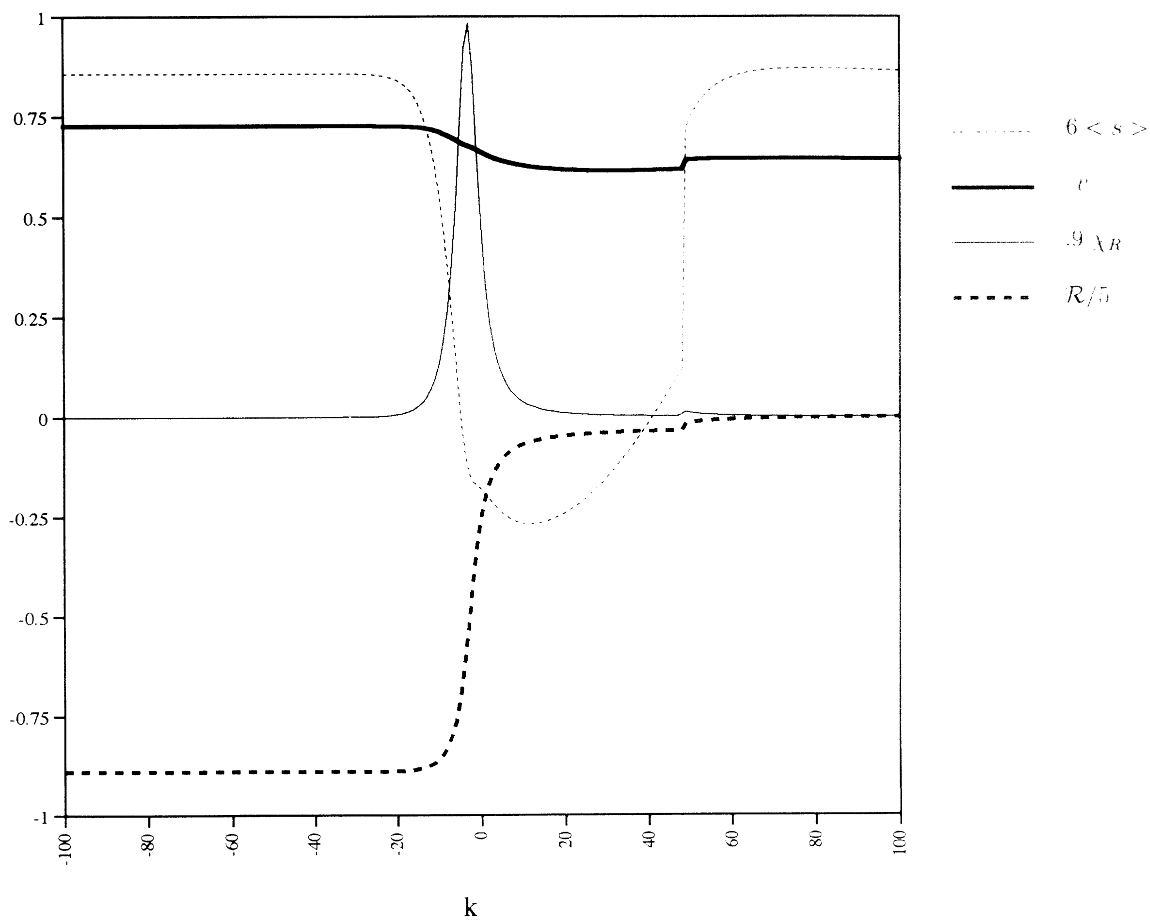


FIG. 2. A plot of the indicated quantities versus $k$ for $\lambda = 1$ as calculated in MFT. $v$ is the average volume per site, $\langle V \rangle / N_0$.

that section, $b$ was chosen to be 0.1 throughout. [See Eq. (1).]

Since there are only two possible lengths per link, there are at most $2^6 = 64$ possible configurations for a particular tetrahedron. (Actually there are significantly fewer due to symmetry.) Because all terms in the action are determined solely by the link lengths in the lattice, the limited number of distinct tetrahedrons allows many of the calculations to be performed only once at program entry and stored for later use in the form of "look-up tables." These tables are accessed during the Monte Carlo updating. As a result, the Ising-link model proved to be quite computationally efficient; run times were reduced by as much as a factor of 10 over continuous-link (Regge gravity) simulations.

Except for the reliance on look-up tables, the simulations were carried out in the usual way. An initial random configuration of link lengths is chosen, generating a particular initial geometry. A link update consists of choosing a particular link in the lattice, calculating the change in the action if the link takes on its other possible value, and accepting the new link value with probability proportional to the exponential of the negative change in the action (heat bath). Link updates are performed for each link in the lattice; this constitutes one sweep. The quantities of interest are calculated after each sweep of the lattice, and the values for each new geometry are binned for statistical analysis. Runs of up to 128k sweeps on the $4^3$ and $8^3$ lattices, and up to 500k sweeps on the $16^3$ lattice were performed for various values of $\lambda$ and $k$.

The two physical quantities of greatest interest are $\mathcal{R}$ and $\chi_R$, defined in Eqs. (20) and (21). Hamber and Williams found that in the Regge theory, the curvature susceptibility diverges at points where $\mathcal{R}$ vanishes [9]. Thus, a portion of the curve $\mathcal{R} = 0$ was first identified (Fig. 4), and the behavior of the model was studied along that curve. But peaks in $\chi_V$ (defined in [9]) and not in $\chi_R$ appeared along the $\mathcal{R} = 0$ curve. The peaks in $\chi_R$ turned out to be located at values of $k$ that correspond to local inflection points in $\mathcal{R}$. This behavior is consistent with Eq. (21), which relates $\chi_R$ to the first derivative of $\mathcal{R}$ with respect to $k$. Also plotted in Fig. 4 is a dashed curve of peaks in $\chi_R$. The cases $\lambda = 1$ and $\lambda = 10$ were studied for values of $k$ close to that curve, but no growth of the peak with increasing system size was observed, indicating the absence of a second-order transition in this

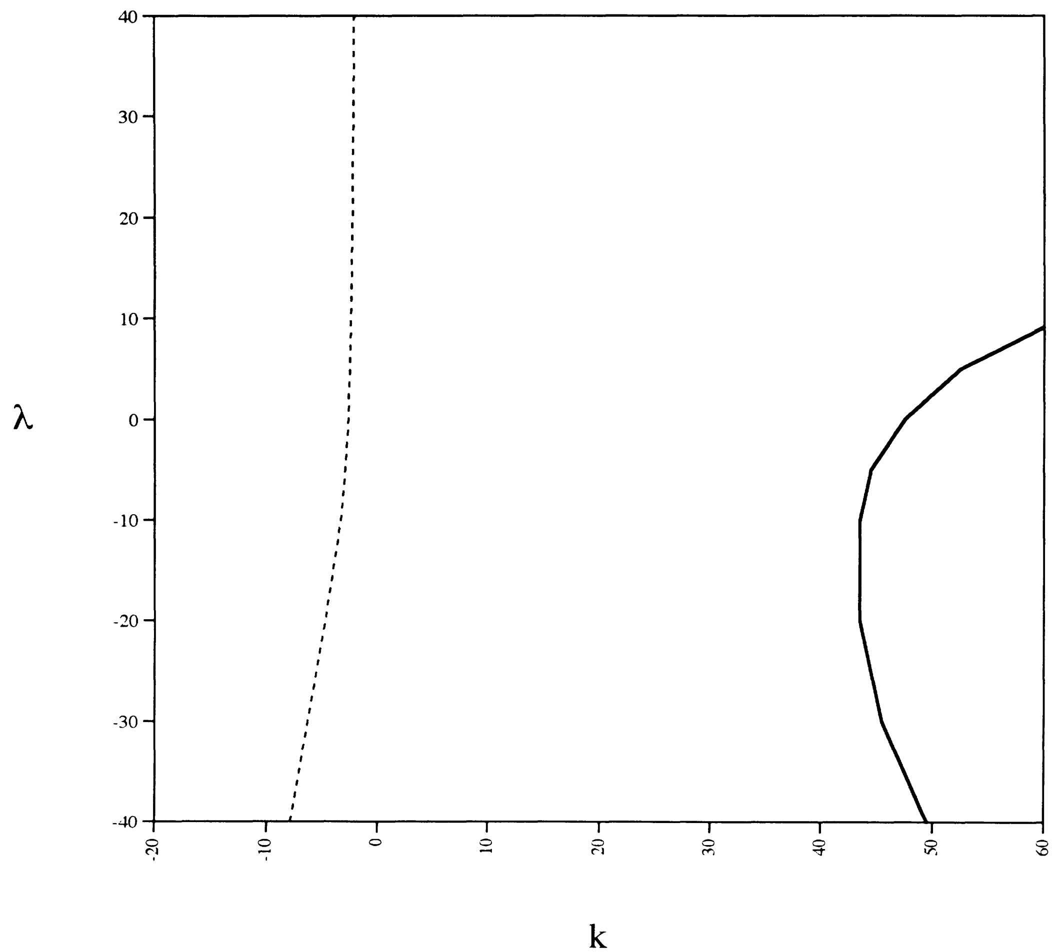


FIG. 3. The "phase diagram" of the 3D Ising-link model in the MFT approximation. The dashed curve shows the location of the peak in $\chi_R$ and the solid curve is a first-order phase transition.

region of parameter space. For $\lambda = 20$ and above, however, we did find growth in the $\chi_R$ peak with increasing system size indicative of a second-order phase transition (see below). Thus the dashed curve appears to become a curve of second-order phase transitions somewhere between $\lambda = 10$ and $\lambda = 20$.

Figure 4 may be compared with Fig. 3 determined by MFT. The location of the peak in $\chi_R$ is the same in both plots to within Monte Carlo statistical errors. However, no second-order phase transition occurred in the MFT approximation for any value of $\lambda$. The agreement at large $k$ between MFT and Monte Carlo is not very good. As discussed in the preceding section, MFT exhibited a first-order phase transition along the solid curve of Fig. 3, and for $k$ to the right of that curve, $\mathcal{R}$ asymptotically approached 0 from below. This first-order phase transition was not found in Monte Carlo, and $\mathcal{R}$ went from negative to positive values at the location of the solid curve in Fig. 4.

The comparison with MFT is also seen by comparing Monte Carlo $\lambda = 1$ data displayed in Fig. 5 with corresponding MFT data shown in Fig. 2. Note the difference in the scales for $\langle s \rangle$ and $\chi_R$. For $k$ negative the agreement is quite good; in fact the Monte Carlo results agree completely with MFT at $k \to -\infty$; both indicate that the system freezes into a state with the body diagonals and cube edges long ($s = 1$) and the face diagonals short ($s = -1$). The main negative $k$ disagreement occurs at the peak in $\chi_R$ which is much lower in the MFT approximation than even on a $4^3$ lattice. For $k$ positive the agreement is much poorer. For large $k$ in MFT, $\mathcal{R}$ never goes positive, a spurious jump in $\chi_R$ is predicted ($k \approx 48$) and $\langle s \rangle$ is off by about a factor of 3.

We now return to the evidence for a second-order phase transition at large $\lambda$. The desired signature for critical behavior would be

$$\mathcal{R} \approx \mathcal{R}_0 + A|k_c - k|^\delta \quad (23)$$

and

$$\chi_R \approx B|k_c - k|^{\delta-1} \quad (24)$$

for $k \approx k_c$, where $k_c$ is the critical point for fixed $\lambda$ and

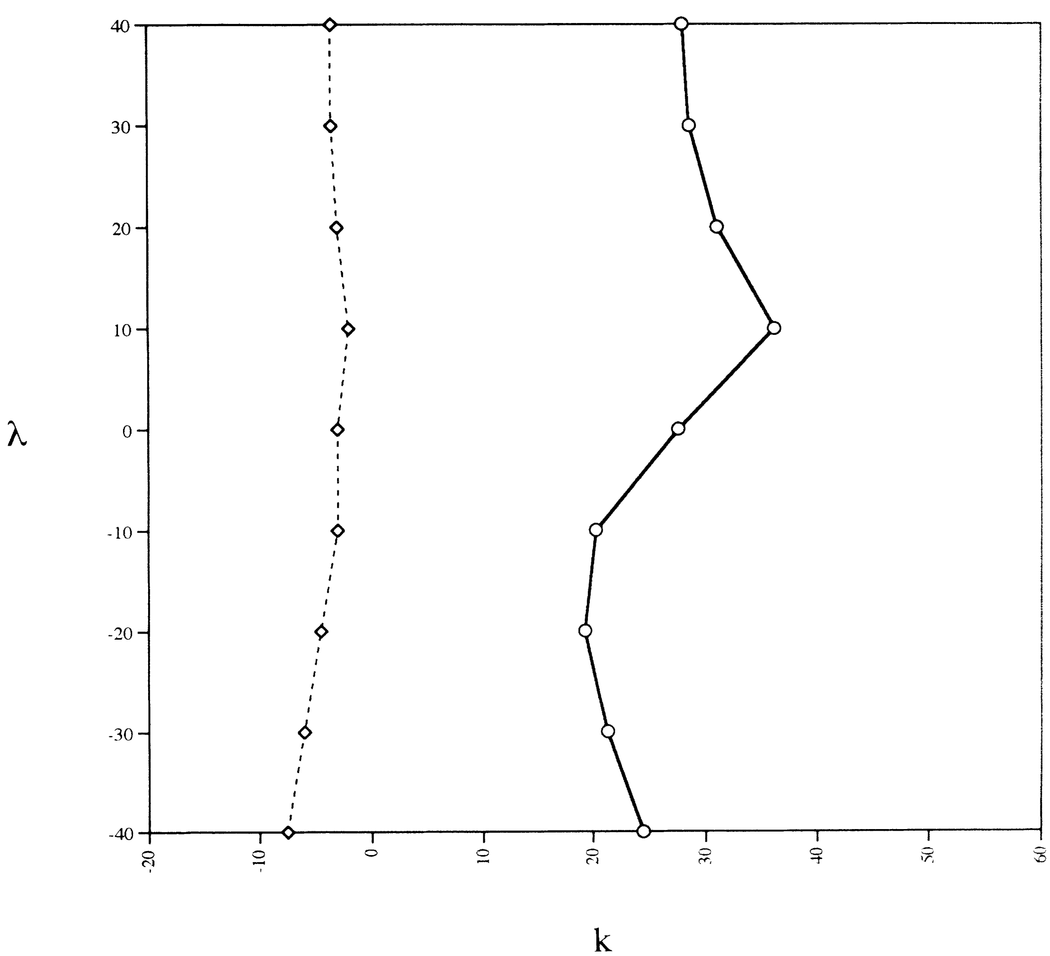


FIG. 4. The "phase diagram" of the 3D Ising-link model as determined by Monte Carlo simulations. The dashed curve shows the location of the peak in $\chi_R$ which is found to scale with system size at large $\lambda$. The solid curve is $\mathcal{R} = 0$. Lattice sizes up to $16^3$ were used, and error bars are of order the size of the data points.

$\delta$ is the critical exponent characteristic of the transition [9].

Previous work on the full Regge theory in three dimensions at $\lambda = 1$ determined that $\mathcal{R}_0 \approx 0$ and $\delta = 0.80 \pm 0.06$ [9]. In that theory $k$ had to approach $k_c$ from below; the theory was sick for $k > k_c$. Here we found that the Ising-link model has $\mathcal{R}_0 \neq 0$ at the peaks in $\chi_R$. The curvature susceptibility, though, does show behavior expected of a second-order phase transition. Near criticality there is an observed narrowing in the curvature susceptibility and an increase in peak height with increasing system size. This is readily seen in Fig. 6. Following Hamber and Williams [9], the finite-size scaling relation for the peak of the curvature susceptibility is

$$\ln(\chi_R) \sim c + \frac{\alpha}{\nu} \ln L \,, \tag{25}$$

where $L$ is the system length and $\alpha/\nu = d(1-\delta)/(1+\delta)$, with $d = 3$. Using this relation, the critical exponent was determined from the curvature susceptibility data for various values of $\lambda$ as shown in the following table:

| $\lambda$ | $\delta$ |
|---|---|
| 0 | 1.00(1) |
| 10 | 1.00(1) |
| 20 | 0.92(2) |
| 30 | 0.82(2) |
| 40 | 0.82(1) |
| 50 | 0.73(4) |
| 60 | 0.64(4) |
| 75 | 0.55(2) |

*The stated errors are purely statistical.* Only $L = 4, 8$ data was used except for $\lambda = 40$ and 75, which included $L = 16$. $\lambda = 75$ showed evidence of scaling: $\delta$ determined from $L = 4$ and 8 was compatible with $\delta$ determined from $L = 8$ and 16. With $\lambda = 40$ we found $L = 4$ and 8 gave $\delta = 0.78(1)$, whereas $L = 8$ and 16 gave $\delta = 0.83(1)$. Thus finite size effects are probably significant and the systematic errors in the above table probably exceed the given statistical errors.[2]

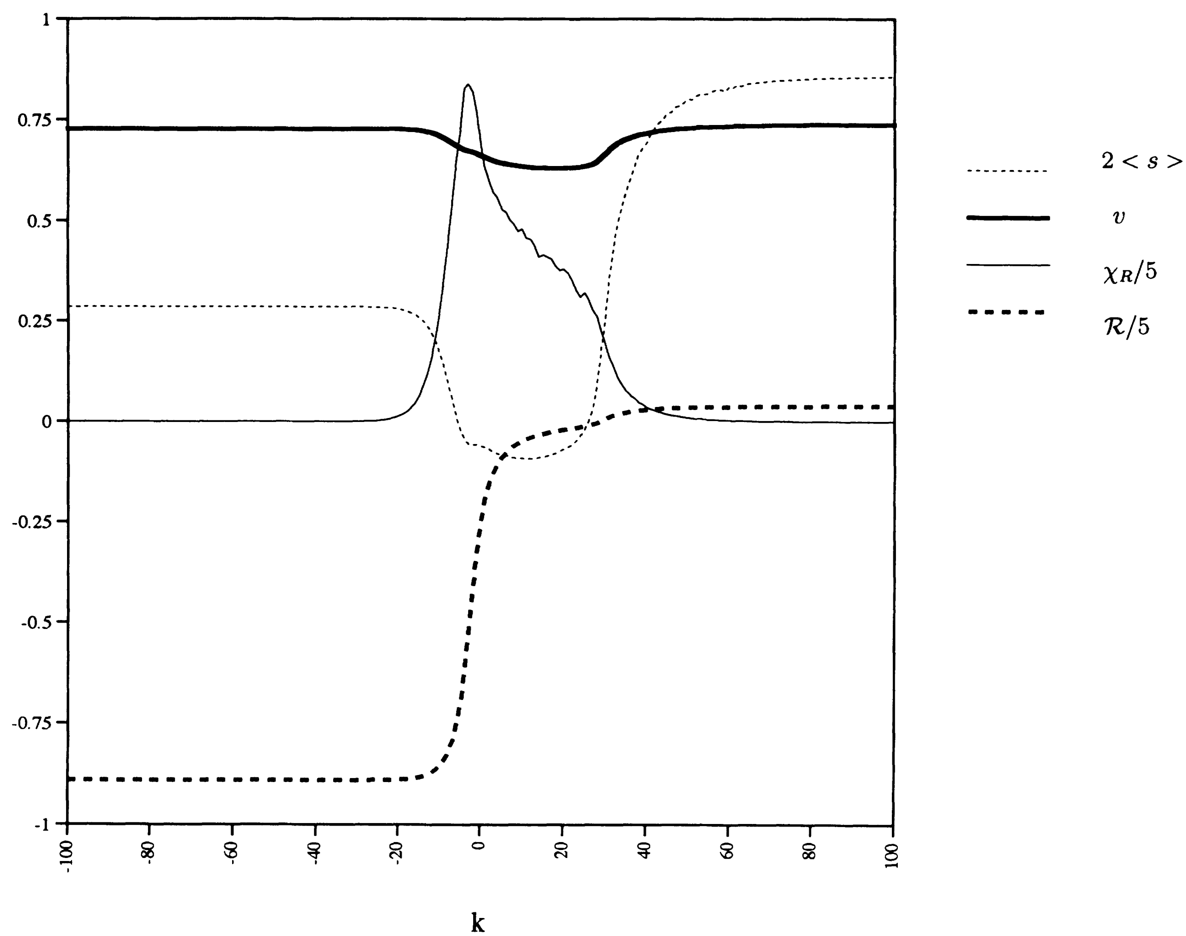


FIG. 5. Monte Carlo data for $\lambda = 1$ on a $4^3$ lattice. Data were taken every $\Delta k = 1$, and error bars are 0.013 or smaller.

[2]It should be mentioned that fairly severe critical slowing down was observed at $\lambda = 40$, $L = 16$. 500k sweeps were needed before different runs with the same coupling values gave results for $\chi_R$ which were compatible within the statistically determined error bars.

Nevertheless we have a preliminary indication that the critical exponent varies with coupling $\lambda$. Variable exponents are known to exist in the Ashkin-Teller model, the eight-vertex model, and the Ising model with next-nearest-neighbor antiferromagnetic interactions [10].

The possibility that $\delta$ varies with cosmological constant may indicate that the Ising-link model is sick or that it may exhibit a family of continuum limits. Further studies where $L$ is allowed to vary over a larger range for each value of $\lambda$ are needed to convincingly demonstrate scaling and critical behavior varying with cosmological constant.

Assuming that $\delta$ does not vary with $\lambda$ in the standard Regge theory, we note that the curvature susceptibilities of the two models appear to agree for $\lambda \sim 40$.

## V. DISCUSSION AND SUGGESTIONS FOR FURTHER WORK

In three dimensions we have uncovered critical behavior in the Ising-link model for $\lambda$ greater than about 15. The critical behavior takes the form (23) and (24) as found by Hamber and Williams for the full Regge theory in 3D. But $\mathcal{R}_0$ was 0 in their model and not in ours. Their model was sick for $k > k_c$; ours was not. Surprisingly, the Ising-link curvature susceptibility exponent appeared to vary with cosmological constant, agreeing with the (presumably fixed) Regge value only for $\lambda \sim 40$.

We should mention a quite different interpretation of our results suggested by a referee of this paper. In [9], the possibility of an additional $\chi_R$ phase transition at negative $k$ in the 3D Regge theory was discussed, although little supporting evidence was presented. This second transition would have $\mathcal{R}$ nonzero and could correspond to the negative $k$ Ising-link transition studied in this paper. In that case it would be natural to identify the positive $k$, $\mathcal{R} = 0$ transition in the Ising-link model with the positive $k$, $\mathcal{R} = 0$ transition of the Regge theory. Unfortunately, the positive $k$ Ising-link transition exhibits no peak in $\chi_R$ so the correspondence between the two models in this scenario would not be complete.

Many questions remain: What connections can be made with the Ponzano-Regge theory [11] (which also allows only two link lengths) and its variants? Does the critical behavior really vary with cosmological constant? If so does this imply a family of continuum limits associated with the Ising-link model? Is there any dependence

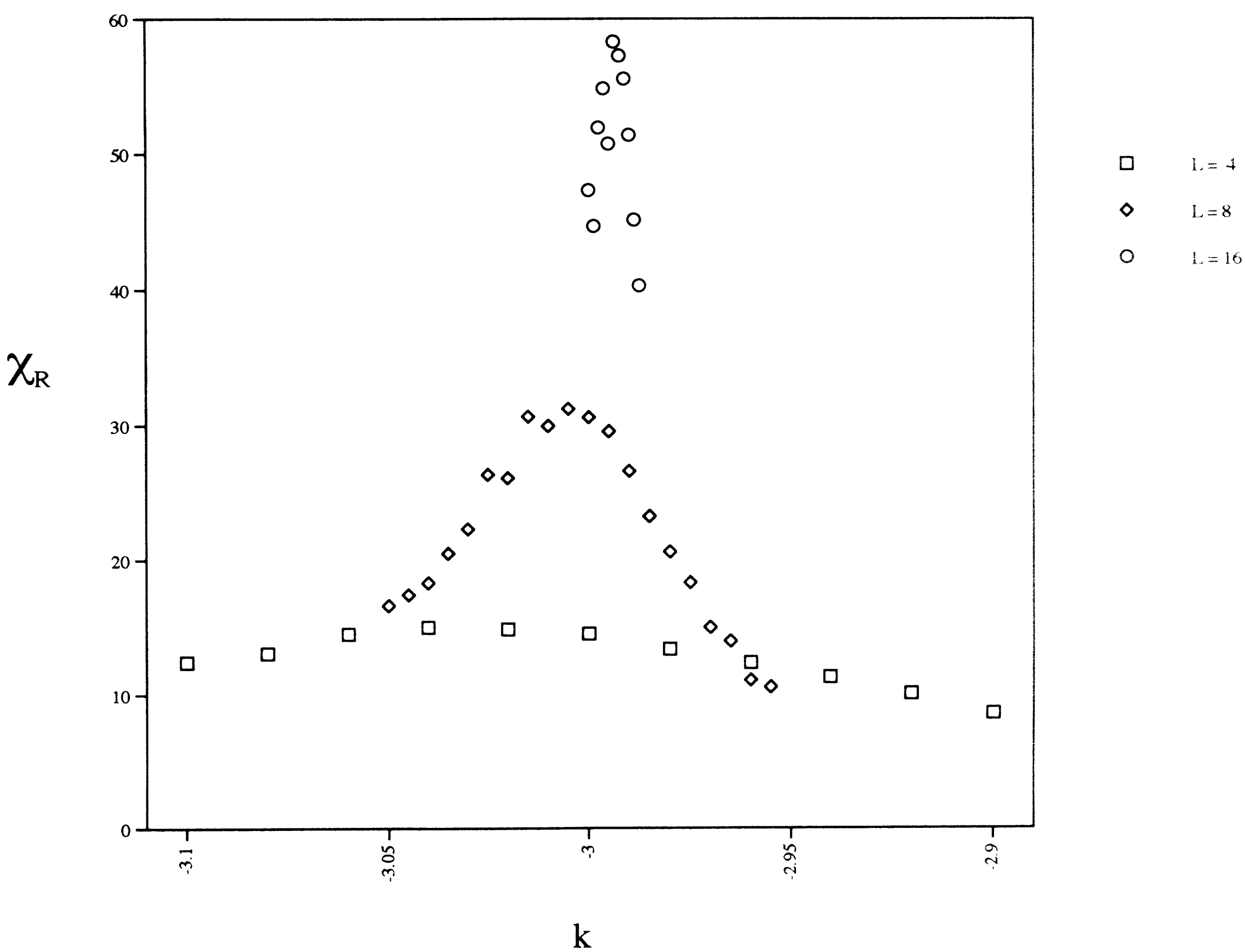


FIG. 6. The peak in $\chi_R$ at $\lambda = 75$ for lattices of lengths 4, 8, and 16. The largest statistical errors of the data points are respectively, 0.3, 1.3, and 3.3.

on the parameter $b$ of Eq. (1)? Can sense be made of a nonzero $\mathcal{R}_0$ at the ($k < 0$) critical point [12,13]? What happens if one allows the link lengths to vary among $3, 4, \ldots, \infty$ values?

The Ising-link model can and should be investigated in four dimensions as well. The mean field approximation is somewhat more difficult but still quite feasible, and it would be expected to be more accurate in four dimensions than in three.[3] Other analytic methods should also be considered. Four-dimensional Monte Carlo computations can still be performed using look-up tables and as a result should be many times faster than for the full Regge theory.

*Note added.* While writing up this work we received two related papers by W. Beirl, H. Markum, and J. Riedler [14]. They independently defined the Ising-link model and investigated its behavior in two dimensions. J. Riedler [15] has recently presented the first results for the Ising-link model in the physically relevant case of four dimensions.

[3]This is because as the dimension increases, the fluctuations in the environment of each link should become relatively smaller. These fluctuations are neglected in the mean field approximation.

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