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Journal of Mathematical Physics

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Cite as: Journal of Mathematical Physics **40**, 3643 (1999); https://doi.org/10.1063/1.532914
Submitted: 25 June 1998 . Accepted: 12 March 1999 . Published Online: 22 June 1999

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JOURNAL OF MATHEMATICAL PHYSICS VOLUME 40, NUMBER 7 JULY 1999

Invariant Painlevé analysis and coherent structures of two families of reaction-diffusion equations

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(Received 25 June 1998; accepted for publication 12 March 1999)

Exact closed-form coherent structures (pulses/fronts/domain walls) having the form of complicated traveling waves are constructed for two families of reaction–diffusion equations by the use of invariant Painlevé analysis. These analytical solutions, which are derived directly from the underlying PDE's, are investigated in the light of restrictions imposed by the ODE that any traveling wave reduction of the corresponding PDE must satisfy. In particular, it is shown that the coherent structures (a) asymptotically satisfy the ODE governing traveling wave reductions, and (b) are accessible to the PDE from compact support initial conditions. The solutions are compared with each other, and with previously known solutions of the equations. © *1999 American Institute of Physics.* [S0022-2488(99)01907-6]

I. INTRODUCTION

There has been considerable interest in coherent structure solutions of nonintegrable nonlinear partial differential equations (NLPDEs)[1–10] since these provide an organizing structure to the space of solutions. In a very rough sense, this is somewhat analogous to the way in which families of soliton solution act as basic building blocks for the solution space of integrable equations. Recent work, primarily in the context of generalized Ginzburg–Landau amplitude equations in pattern-forming systems, has included the existence of pulse (solitary wave), front (shock) and domain wall coherent structures using center manifold techniques,[11,12] as well as investigations of periodic and quasi-periodic solutions.[13–17] Another, more physics-oriented, approach was developed by van Saarloos[18,19] to investigate linear and nonlinear marginal stability of fronts. This approach has been comprehensively reviewed by van Saarloos and Hohenberg[20] in the context of generalized Ginzburg–Landau equations. Using the idea that spatio-temporal coherent structure solutions of NLPDEs, whether periodic, quasi-periodic, or chaotic, must obey the underlying singularity structure, Conte and co-workers[21,22] have used ideas related to the Painlevé test for integrability[23,24] and its modifications[25] to derive families of solutions of the complex cubic and quintic Ginzburg–Landau equation. Also, using phase-plane techniques on the ordinary differential equation (ODE) which must be satisfied by any traveling wave solution to the real Ginzburg–Landau equation, Powell *et al.*[26] have rederived and significantly elucidated several of van Saarloos' results[18,19] in a completely different manner. In addition, they use simple analytic solutions of the PDE obtained using truncated Painlevé expansions,[27] together with ideas from phase-plane analysis, as well as absolute versus convective instability of waves.[28] As a result, they show that front/pulse solution of the PDE must satisfy the traveling wave reduced ODE asymptotically. They also derive conditions for the accessibility of the solutions from compact support initial conditions.

In this paper, we consider coherent structures of the reaction-diffusion equation

$$u_t = u_{xx} + \frac{u}{b}(b+u)(1-u), \tag{1}$$

which has been considered in Refs. 4, 18 and 19. Note that other work on coherent structures of

[a)]Author for correspondence. Electronic mail: choudhur@longwood.cs.ucf.edu various reaction-diffusion equations is summarized in those papers, as well as in Section 1 of Ref. 26. For the purposes of comparison, we shall also consider coherent structures of the two families of reaction-diffusion equations

$$u_t = \beta u^2(1-u) + Du_{xx} \tag{2}$$

and

$$u_t = \beta u(1-u) + Du_{xx}. \tag{3}$$

Of these, (3) is the famous Fisher–Kolmogorov equation,[29,30] while (2) has second- and third-order nonlinearities, which is also true of (1). The primary difference between (1) and (2)/(3) is that the parameter b in the former adjusts the relative strength of the second- and third-order nonlinearities, while these strengths are fixed in (2) and (3).

To date, the approaches to the treatment of coherent structures may broadly be classified into three groups. First, there is the qualitative phase-plane/center manifold analysis of the traveling wave reduced ODEs to prove the existence of coherent structures. The second approach consists of actual construction of coherent structures via numerical simulation of the traveling wave reduced ODEs. The third approach comprises containment arguments wherein, starting from the correct boundary condition at one end of the interval, one shows that at the other end the solution asymptotes to a constant value. It thus corresponds to a coherent structure, rather than shooting off to infinity. Such containment arguments may often involve delicate analysis. The coherent structures derived in this paper are, in a sense, an attempt to connect the first two approaches by providing quantitative analytical expressions for nontrivial coherent structures. Clearly, these coherent structures are also of relevance in modeling the physics of the problems under consideration, although that is not the purpose of this paper. In fact, the next natural question to consider is their actual use in modeling applications. Some discussion regarding this follows the derivation of the coherent structures in Sec. IV.

In Sec. II we use invariant Painlevé expansions truncated at different orders to obtain nontrivial families of analytic solutions of the reaction-diffusion equations. Two points are worth noting in this context. First, the invariant Painlevé analysis[24] builds in invariance to the Möbius or homographic group ''*a priori*.'' In turn, this leads to simpler compatibility equations for the coefficients (the so-called Painlevé–Bäcklund equations) yielding more general solutions than obtained for (1) from the use of truncated noninvariant Painlevé expansions.[31] Second, although truncated invariant Painlevé expansions have been used fairly widely in recent years to derive analytic solutions (see Refs. 32–34, for instance), the Painlevé–Bäcklund equations which result from (1) and which are solved to obtain analytic solutions are quite complicated. Having obtained analytic solutions of the PDE in Sec. II, we next consider the properties of the ordinary differential equation governing traveling wave solutions in Sec. III. In Sec. IV, we discuss the compatibility of the PDE solutions and solutions derived earlier with those of the ODE, as well as accessibility from initial conditions. We also give numerical examples of various coherent structure solutions.

II. INVARIANT TRUNCATION PROCEDURE AND SPECIAL SOLUTIONS

A. Truncation procedure

For a NLPDE that is algebraic in u and its derivatives

$$E(u,x,t) = 0 \tag{4a}$$

around a movable singular manifold

$$\Phi - \Phi_0 = 0, \tag{4b}$$

one looks, in the invariant Painlevé formulation,[24] for a solution as an expansion of the form

$$u=\chi^{-\alpha}\sum_{j=0}^{\infty} u_j\chi^j, \tag{5}$$

where the coefficients u_j are invariant under a group of homographic (Möbius) transformations on Φ. The expansion variable χ, which must vanish as $(\Phi-\Phi_0)$, is chosen to be

$$\chi\equiv\frac{\psi}{\psi_x}=\left(\frac{\Phi_x}{\Phi-\Phi_0}-\frac{\Phi_{xx}}{2\Phi_x}\right)^{-1}, \quad \psi=(\Phi-\Phi_0)\Phi_x^{-1/2}.$$

The variable χ satisfies the Riccati equations

$$\chi_x=1+\tfrac{1}{2}S\chi^2, \tag{6a}$$

$$\chi_t=-C+C_x\chi-\tfrac{1}{2}(CS+C_{xx})\chi^2, \tag{6b}$$

and the variable ψ satisfies the linear equations

$$\psi_{xx}=-\tfrac{1}{2}S\psi, \tag{7a}$$

$$\psi_t=\tfrac{1}{2}C_x\psi-C\psi_x. \tag{7b}$$

Note that the systems of equations (6) and (7) are equivalent to each other. In (6) and (7), the quantities S (Schwarzian derivative) and C (the ''dimension of velocity'' or celerity) are defined by

$$S=\frac{\Phi_{xxx}}{\Phi_x}-\frac{3}{2}\left(\frac{\Phi_{xx}}{\Phi_x}\right)^2, \tag{8a}$$

$$C=-\frac{\Phi_t}{\Phi_x}, \tag{8b}$$

and are invariant under the group of homographic (Möbius) or fractional linear transformations[18]

$$\Phi\rightarrow\frac{a\Phi+b}{c\Phi+d}, \quad ad-bc=1. \tag{9}$$

These homographic invariants are linked by the cross-derivative condition ($\Phi_{xxxt}=\Phi_{txxx}$)

$$S_t+C_{xxx}+2C_xS+CS_x=0. \tag{10}$$

B. Solutions via invariant Painlevé analysis

We apply the above formalism to (1). The leading-order dominant balance yields $\alpha=2$. Using (5), truncated at the constant term

$$u=u_0\chi^{-2}+u_1\chi^{-1}+u_2, \tag{11}$$

in (1), and eliminating the derivatives of χ using (6) yields a set of coupled nonlinear partial differential equations (the Painlevé–Bäcklund equations) order by order in powers of χ. These are given in Appendix A. The first four equations yield

$$u_0=0,$$

$$u_1=0 \text{ or } \pm\sqrt{2b}. \tag{12}$$

Inspection of (A5)–(A7) in Appendix A shows the need for further assumptions to allow their solutions. Making the further assumption[31–34] that C is a constant yields

$$\text{(a)}\quad 2+3b-3b^2-2b^3+(3\sqrt{2b}+3\sqrt{2}b^{5/2}+3\sqrt{2}b^{3/2})C-2\sqrt{2}b^{3/2}C^3=0,$$

$$\text{or}\ \ C=C_1\equiv\frac{-\sqrt{2}-2\sqrt{2}b}{2\sqrt{b}},\quad C=C_2\equiv\frac{-\sqrt{2}+\sqrt{2}b}{2\sqrt{b}} \tag{13a}$$

$$\text{or}\ \ C=C_3\equiv\frac{2\sqrt{2}+\sqrt{2}b}{2\sqrt{b}}\quad \text{for}\ \ u_1=\sqrt{2b},$$

with

$$S=\frac{C^2}{6}-\frac{b}{3}-\frac{1}{3b}-\frac{1}{3}=2Q^2, \tag{13b}$$

or

$$\text{(b)}\quad u_2=0,\ 1,\quad \text{or}\quad -b\quad \text{for}\ \ u_1=0 \tag{14}$$

[this yields only trivial constant solutions using (11)]. The cross-derivative condition (10) is now satisfied identically. The Schrödinger equation (6a) yields

$$\psi(x,t)=A(t)\cos Qx+B(t)\sin Qx$$

and hence, using (6b),

$$\chi\equiv\frac{\psi}{\psi_x}=\frac{c_1\cos(Q\xi)-c_2\sin(Q\xi)}{-Q[c_2\cos(Q\xi)+c_1\sin(Q\xi)]} \tag{15a}$$

with

$$\xi=x-Ct. \tag{15b}$$

Hence, using (12) and (13) in (11), traveling wave special solutions of (1) (for $u_1=\sqrt{2b}$) are

$$u^{(1)}=\pm\frac{\sqrt{2b}}{\chi}+\frac{1}{6}(2-2b-C\sqrt{2b}), \tag{16}$$

where χ is given by (15a), and C has one of the values in (13a). A solution may be derived analogously for $u_1=-\sqrt{2b}$, and a similar, less interesting, solution may be obtained using (11), (12), and (14) (for $u_2=0$, 1 or $-b$). Note that C and Q are connected via (13b), and hence Q is (implicitly) a function of b.

A similar process applied to (2) yields a solution

$$u_1^{(2)}=\pm\frac{\sqrt{2D/\beta}}{\chi}+\left(\frac{2D\mp C\sqrt{2D/\beta}}{6D}\right), \tag{17a}$$

where χ is given by (15), with

$$S=\frac{C^2-2D\beta}{6D^2}\equiv 2Q^2, \tag{17b}$$

and C is a solution of the cubic

$$2\beta\left(\frac{1}{3}\mp\frac{C}{3\sqrt{2\beta D}}\right)^3-2\beta\left(\frac{1}{3}\mp\frac{C}{3\sqrt{2\beta D}}\right)^2\pm\sqrt{\frac{2D}{\beta}}C\left(\frac{C^2-2D\beta}{6D^2}\right)=0. \tag{17c}$$

For instance, with $D=1$, $\beta=2$, we obtain $C=-1$, or $C=2$.

By contrast, the same procedure applied to (3) yields simpler, relatively trivial, traveling wave solutions. For completeness, we include a solution of (2) obtained earlier[31] using noninvariant Painlevé analysis:

$$u_{\mathrm{II}}^{(2)}=\gamma\sqrt{\frac{2D}{\beta}}\left[\frac{\alpha+\sqrt{2D/\beta}e^{\sqrt{\beta/2D}\,\xi_1}}{\alpha\gamma\xi_1+(2D\gamma/\beta)e^{\sqrt{\beta/2D}\,\xi_1}+\delta}\right], \tag{18a}$$

where

$$\xi_1\equiv x+\sqrt{\frac{\beta D}{2}}t, \tag{18b}$$

and α, γ, and δ are constants. Note that the discussion of this solution later in this paper is new, as is the framework of that discussion.

C. Preliminary discussion

We shall consider the behavior of the solutions (16)–(18) further in the next two sections. However, we first need to consider some results for the ODEs, derived from traveling wave reductions, of the underlying PDEs (1) and (2). This will be done in Sec. III. At this point, we make some preliminary observations regarding the solutions (16)–(18) which will be needed in Sec. III.

We note that we have aperiodic hyperbolic functions in (16) (corresponding to a coherent structure) for Q imaginary or, from (13a) and (13b),

$$\begin{gathered}0<b<\sqrt{\tfrac{3}{2}}\quad\text{or}\quad b<-\sqrt{\tfrac{3}{2}}\quad\text{for }C=C_1,\\ b>0\quad\text{or}\quad -3-\sqrt{6}<b<-3+\sqrt{6}\quad\text{for }C=C_2,\\ b>0\quad\text{for }C=C_3.\end{gathered} \tag{19}$$

For these cases, $\lim_{\xi\to\pm\infty}\chi=\pm 1/|Q|$ from (15), so that (16) yields

$$\lim_{\xi\to\pm\infty}u^{(1)}=\pm\sqrt{2b}|Q|+\tfrac{1}{6}(2-2b-C\sqrt{2b}). \tag{20}$$

For instance, for the first value of $C\equiv C_1$ given by (13a), this becomes

$$\lim_{\xi\to+\infty}u^{(1)}=1,\qquad \lim_{\xi\to-\infty}u^{(1)}=0. \tag{21}$$

Note that the solution $u^{(1)}$ tends to different values as $\xi\to\pm\infty$, and these values are independent of the constants c_1 and c_2 in (15a). Thus, the solutions (16) represent front solutions of (1).

Similarly, considering the solution $u_1^{(2)}$ of (2), if the roots of (17c) for C are C_4, C_5, and C_6 (say), then Q in (17b) may be imaginary for some ranges of β and D. For these cases, $Q\equiv i|Q|$, and $\lim_{\xi\to\pm\infty}\chi=\pm 1/|Q|$, so that (17) yields

$$\lim_{\xi\to\pm\infty}u_1^{(2)}=\pm\sqrt{\frac{2D}{\beta}}|Q|+\frac{1}{3}\pm\frac{C}{6D}\sqrt{\frac{2D}{\beta}}. \tag{22}$$

Note that $u_1^{(2)}$ tends to different values as $\xi \rightarrow \pm \infty$ and thus corresponds to a front solution of (2). For instance, with $D=1$, $\beta=2$, $C=\pm 1$ and the upper sign,

$$\lim_{\xi \rightarrow = -\infty} u_1^{(2)} = 0, \quad \lim_{\xi \rightarrow \infty} u_1^{(2)} = 1.$$

It may be shown in an analogous manner that the solutions $u_{\mathrm{II}}^{(2)}$ in (18) are also front solutions of (2). In the next section, we shall consider some properties of the ODE obtained by performing traveling wave reductions on the reaction-diffusion equations (1) and (2) before returning to discuss the solutions in this section further. We shall refer to (20)–(22) further during that discussion.

III. ANALYSIS OF TRAVELING WAVE REDUCED ODE

We shall look for traveling wave reductions of the PDE's (1) and (2). We present the results for (2) since the algebra is somewhat easier. The treatment for (1) is analogous.

Looking for traveling wave solutions of (2) of the form

$$u(x,t) = u(z) \equiv u(x - Ct) \tag{23a}$$

yields the ordinary differential equation

$$-Cu' = \beta u^2(1-u) + Du'', \tag{23b}$$

where the prime denotes d/dz. Note that we use z as an explicit traveling wave variable ''*a priori*,'' as distinct from the analogous variable ξ which emerged ''*a posteriori*'' in Sec. II from the Painlevé analysis.

Treating (23) as a dynamical system in the $(u,u') \equiv (u,v)$ plane in the standard way, we find the fixed (critical) points in the (u,v) plane:

$$(u_0,0) \equiv (0,0), \tag{24a}$$

$$(u_+,0) \equiv (1,0), \tag{24b}$$

whose linear stability is governed by the eigenvalues (which are also the spatial wave numbers in z space)

$$\lambda_0^{1,2} = 0, \ \frac{-C}{D}, \tag{25a}$$

$$\lambda_+^{1,2} = \frac{-C/D \pm \sqrt{C^2/D^2 + 4\beta/D}}{2}. \tag{25b}$$

Since β and D are positive, the fixed point $(u_+,0)$ is thus a saddle-point, while $(u_0,0)$ is a nonhyperbolic fixed point. The system (23) may thus have a heteroclinic orbit connecting $(u_0,0)$ and $(u_+,0)$. In the context of the underlying PDE (2), this corresponds to a front solution connecting $u_0(u_+)$ and $u_+(u_0)$ as z goes from $-\infty$ to $+\infty$. From (22), we see that the solutions (17) of the PDE (directly obtained from the PDE) are indeed heteroclinic orbits of (23). For instance, with $D=1$, $\beta=1$, $C=\pm 1$, and the lower sign in (22), $\lim_{\xi \rightarrow +\infty} u_1^{(2)} = 0$ and $\lim_{\xi \rightarrow -\infty} u_1^{(2)} = 1$, so that this front solution $u_{\mathrm{II}}^{(2)}$ is a heteroclinic orbit of (23) joining $(u_0,0)$ to $(u_+,0)$ as $z \rightarrow \pm \infty$. Note that for some PDEs (such as the long-wave equations[35]) this may not happen automatically—the integrated version of the traveling wave reduced ODE may contain unknown constants of integration which must be chosen to ensure this. An analogous treatment of (1) yields the traveling wave reduced ODE

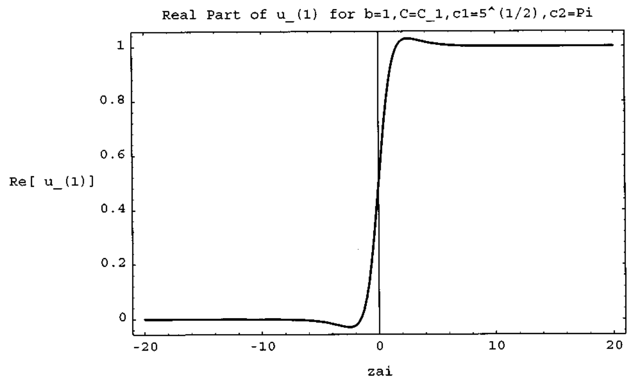

FIG. 1. Real part of $u\_(1)$ for $b=1$, $C=C\_1$, $c1=5\^(1/2)$, $c2=Pi$.

$$-Cu'=u''+\frac{u}{b}(b+u)(1-u), \tag{26}$$

which, treated as a dynamical system in $(u,u')\equiv(u,v)$, has fixed points $(u_0,0)=(0,0)$, $(u_+,0)=(1,0)$ and $(u_-,0)=(-b,0)$. As for (2), it may be shown using (20) and (21) that the front solutions $u^{(1)}$ of (1) correspond to heteroclinic orbits of (26) joining two of the above fixed points. For instance, for $C\equiv C_1$, $\lim_{\xi\to-\infty}u^{(1)}=0$, $\lim_{\xi\to\infty}u^{(1)}=1$, so that the front solution $u^{(1)}$ of (1) is a heteroclinic orbit of (26) joining $(u_0,0)$ to $(u_+,0)$ as $z\to\pm\infty$. Once again, this need not happen automatically, as it does not for the long-wave equations,[35] for instance.

As extensively investigated and stressed by Powell *et al.*[26] the front solutions represented by the heteroclinic orbits of the traveling wave reduced ODEs (23) and (26) need not correspond to fronts obtained directly from the PDE (1). We shall now consider this further.

IV. DISCUSSION

In this section, we consider further features of the solutions $u^{(1)}$ and $u_1^{(2)}$ [in (16) and (17)] of (1) and (2) obtained by use of invariant Painlevé expansions. Powell *et al.*[26] have, among numerous other things, made the points that coherent structure solutions such as (16) and (17), which are directly obtained from a PDE, (a) must asymptotically satisfy the ODE governing traveling wave reductions, and (b) be accessible to the PDE from compact support initial conditions. Considering the traveling wave reduced ODE (23) obtained from (2), we have $z\equiv x-Ct\to-\infty$ as $t\to\infty$ (for $C>0$), and so u tends to the saddle-point $(u_+,0)$ in (24b) along its unstable manifold. From (25b), the eigenvalue along this direction is

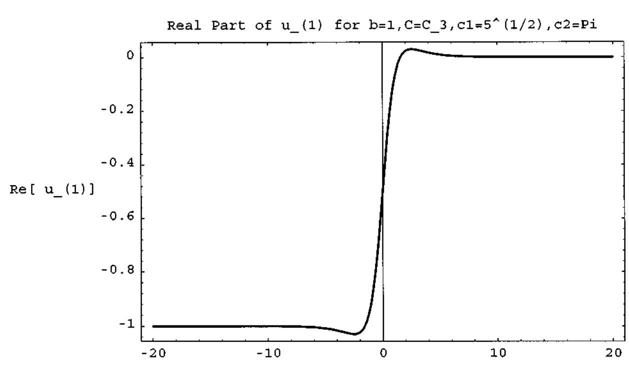

FIG. 2. Real part of $u\_(1)$ for $b=1$, $C=C\_3$, $c1=5\^(1/2)$, $c2=Pi$.

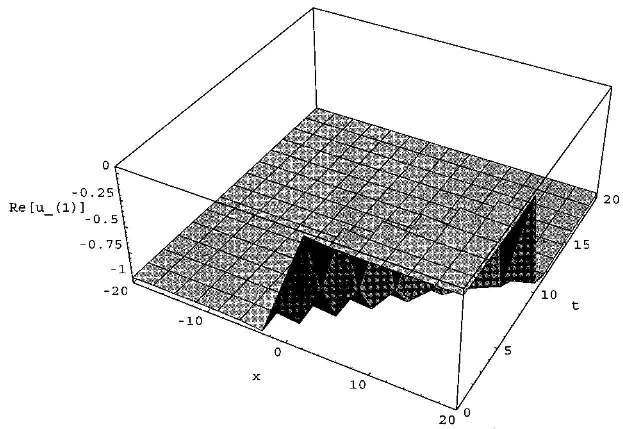

FIG. 3. Real part of Front $u\_(1)$ for $b=1$, $C=C\_3$, $c1=5\hat{}(1/2)$, $c2=Pi$.

$$\lambda_{\mathrm{ODE}}=\lambda_{+}^{1}=\frac{-C/D+\sqrt{C^2/D^2+4\beta/D}}{2}. \tag{27}$$

By the Unstable Manifold Theorem,[36] (27) gives the time asymptotic spatial wave number of the front solutions (2) [along the global unstable manifold of $(u_+,0)$] satisfying the ODE (23) and with solution values u_0 and u_+ for $t\rightarrow\mp\infty$. Inspection of the solutions $u_1^{(2)}$ in (17) reveals that the wave number ($\lambda_{\mathrm{PDE}}\equiv|Q|$) of these solutions obtained directly from the PDE are exactly the same as λ_{ODE} [this may be seen from (17b) and (27), using the fact that C satisfies (17c)]. In Sec. III, we verified that the values of (a) u_{PDE} for $\xi\rightarrow\pm\infty$, and (b) u_{ODE} for $z\rightarrow\pm\infty$ are matched; here we see that the resulting time asymptotic wavenumbers in the ODE and PDE solutions are also the same. Thus, as conjectured in Powell *et al.*, the solutions obtained via Painlevé analysis are indeed the so-called nonlinear solutions;[19] note that Powell *et al.* equivalently think of C as a function of λ, instead of λ as a function of C as done here. As pointed out by both Powell *et al.*[26] and Marcq *et al.*[22] for the GL equation, this is because the Painlevé analysis builds in ''*a priori*'' the singularity structure which must be satisfied by any coherent structure solution of the PDE.

Although the front solutions (16) and (17) satisfy the traveling wave reduced ODEs (26) and (23), we must also check the accessibility of these solutions to the PDEs (1) and (2) from compact support initial conditions as stressed in Ref. 26. Following the treatment of absolute versus convective instability in Ref. 28, we find the temporal growth rate at any fixed x spatial position

$$\sigma=|-C\lambda_{\mathrm{PDE}}|\equiv\left|-C|Q|=-C\left[\frac{2D\beta-C^2}{12D^2}\right]^{1/2}\right|. \tag{28}$$

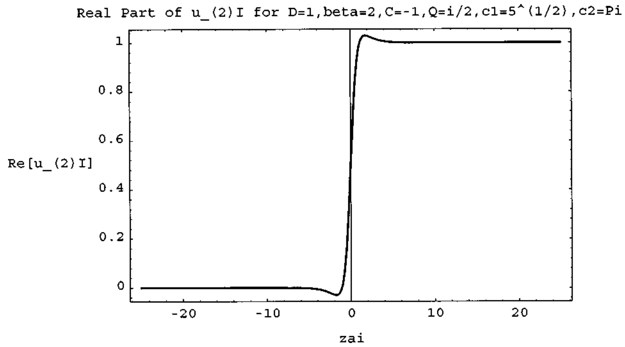

FIG. 4. Real part of $u\_(2)I$ for $D=1$, beta$=2$, $C=-1$, $Q=i/2$, $c1=5\hat{}(1/2)$, $c2=Pi$.

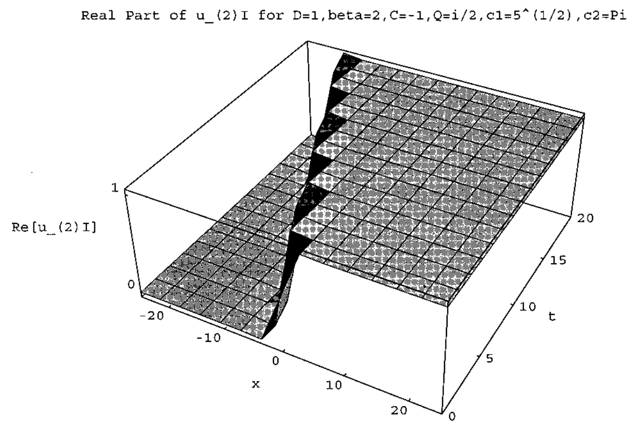

FIG. 5. Real part of $u\_(2)I$ for $D=1$, beta=2, $C=-1$, $Q=i/2$, $c1=5\char`^(1/2)$, $c2=Pi$.

Thus, we expect that the front which emerges asymptotically in time from compact support initial conditions corresponds to the root of (17c) for which we have the maximum temporal growth rate. For instance, with $D=1$, $\beta=2$, we obtain $C=-1$ or 2 from (17c). From (17b), $|Q|=\frac{1}{2}$ for $C=-1$, and $|Q|=0$ for $C=2$. Thus the maximum σ occurs for $C=-1$.

Analogous results apply to the wave numbers and temporal growth rates of the solution (16) obtained directly from the PDE (1) and its traveling wave reduced ODE (26). The algebra is harder, but is tractable using a computer algebra system.

One should note that some of the results obtained by phase-plane analysis of the traveling wave reduced ODE are equivalent to those obtained by van Saarloos' linear and nonlinear marginal stability analysis and steepest envelope technique.[29] For completeness, this treatment is summarized in Appendix B for Eq. (2). An analogous treatment holds for (1).

Finally, let us consider plots of the solutions of (1) and (2) given by (16) and (17), and compare them further to predictions from the traveling wave reduced ODEs. For both (16) and (17), we choose representative parameter values corresponding to the front solutions discussed earlier. Note that we may choose the constants c_1 and c_2 to make (16) and (17) correspond to physically relevant real solutions. We pick arbitrary c_1 and c_2 values instead, and plot the real parts of the solutions.

Figures 1 and 2 show the real part of $u^{(1)}$ [given by (16)] for $b=1$, $c_1=\sqrt{5}$, $c_2=\pi$, and (a) $C=C_1$ for Fig. 1 and (b) $C=C_3$ for Fig. 2. Note that the primary difference between these plots is that the former, with $C=C_1$, corresponds to a front connecting the states 0 and 1 or u_0 and u_+ as $\xi\rightarrow\pm\infty$, while the latter front with $C=C_3$ connects the state -1 and 0 [note that the third fixed point $(u_-,0)=(-b,0)$ of (26) is $(-1,0)$ for $b=1$]. Figure 3 shows the same front as Fig. 2, but in (x,t) coordinates. Note the rightward propagation of the front (towards larger x) as t increases due to the phase speed $C=C_3=3/\sqrt{2}$ being positive.

Figures 4 and 5 show the solution (17) of the PDE (2) for $c_1=\sqrt{5}$, $c_2=\pi$, $D=1$, $\beta=2$, $Q=i/2$, and $C=-1$ [note that $Q=\pm i/2$, and $C=-1$ or 2 by (17b) and (17c)—we pick $Q=i/2$ and $C=-1$]. The solution corresponds to a front joining the states 0 and 1 [or the fixed points $(u_0,0)$ and $(u_+,0)$ of Eq. (23) given in (24)]. In Fig. 5, note the leftward propagation of the front due to the negative phase speed $C=-1$.

In conclusion, we have derived two nontrivial families of analytical solutions of (1) and (2), which may sometimes be coherent structures, and analyzed several of their properties. As mentioned in Sec. I, these analytical solutions act as a sort of bridge between two of the common approaches to the analysis of coherent structures. These two approaches are, first, proofs of the existence of coherent structure solutions of the traveling wave reduced ODEs, and, second, construction of coherent structures by numerical simulations of these ODEs. The analytical solutions may also be of relevance in modeling the physics of the problem under consideration. Although it is not the purpose of this article to consider detailed modeling issues, some of the approaches which may be relevant to modeling of reaction-diffusion equations include those in Ref. 29, Chap. 3 of Ref. 37, Chap. 6 of Ref. 38, as well as numerous research papers. Related modeling issues for other nonlinear PDEs are discussed, for instance, in Chap. 10 of Ref. 39 (this also discusses nonintegrable equations, not just integrable equations as the chapter title might seem to indicate), and the recent review article by Balmforth.[40]

ACKNOWLEDGMENT

The authors acknowledge the helpful comments of the referee regarding stylistic matters, inclusion of Appendix B, as well as a discussion of the relevance of the results in this paper. These improved the clarity and completeness of the paper greatly.

APPENDIX A

The equations obtained at different powers of χ are

$$O(\chi^{-6}):\quad u_0=0, \tag{A1}$$

$$O(\chi^{-5}):\quad u_0^2u_1=0, \tag{A2}$$

$$O(\chi^{-4}):\quad 6u_0-u_0^2+\frac{u_0^2}{b}-\frac{3u_0u_1^2}{b}-\frac{3u_0^2u_2}{b}=0, \tag{A3}$$

$$O(\chi^{-3}):\quad -2Cu_0+2u_1-2u_0u_1+\frac{2u_0u_1}{b}-\frac{u_1^3}{b}-\frac{6u_0u_1u_2}{b}-4u_{0x}=0, \tag{A4}$$

$$O(\chi^{-2}):\quad u_0+4Su_0-Cu_1-u_1^2+\frac{u_1^2}{b}-2u_0u_2+\frac{2u_0u_2}{b}-\frac{3u_1^2u_2}{b}-\frac{3u_0u_2^2}{b}-u_{0t}+2u_0C_x-2u_{1x}+u_{0xx}=0, \tag{A5}$$

$$O(\chi^{-1}):\quad -CSu_0+u_1+Su_1-2u_1u_2+\frac{2u_1u_2}{b}-\frac{3u_1u_2^2}{b}-u_{1t}+u_1C_x-u_0S_x-2Su_{0x}-u_0C_{xx}+u_{1xx}=0, \tag{A6}$$

$$O(\chi^{0}):\quad \frac{1}{2}S^2u_0-\frac{1}{2}CSu_1+u_2-u_2^2+\frac{u_2^2}{b}-\frac{u_2^3}{b}-u_{2t}-\frac{1}{2}u_1S_x-Su_{1x}-\frac{1}{2}u_1C_{xx}+u_{2xx}=0. \tag{A7}$$

APPENDIX B: VAN SAARLOOS' TECHNIQUE

The tail of the coherent structure must obey the linear RD equation (2) as $x\to\infty$ (since $u\to 0$ or 1) so that

$$u_t=Du_{xx}\,. \tag{B1}$$

Consider the behavior of a linear mode

$$u=\exp[i(\omega-i\sigma)t+i(k-i\lambda)z],$$

where $z=x-Ct$ and C is the front speed. For this mode to be part of a persistent front, it must have zero temporal growth $\sigma=0$ in a frame moving at speed C. This gives the dispersion relation

$$i\omega-iC(k-i\lambda)=-D(k-i\lambda)^2. \tag{B2}$$

For a nonoscillatory mode with $\omega=0$, separating the real and imaginary parts of (B2) yields either (a) $k=0$, $\lambda=-C/D$, or (b) $\lambda=-C/(2D)$, $k^2=-(C/2D)^2$. Case (b) with imaginary k implies no coherent structures, so we have

$$\lambda=-C/D,$$

which is equal to λ_{PDE}. Various other features discussed in the text may also be derived by this approach (see Refs. 18, 19, and 26).

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