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Lattice-valued Convergence: Quotient Maps

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LATTICE-VALUED CONVERGENCE: QUOTIENT MAPS

by

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ABSTRACT

The introduction of fuzzy sets by Zadeh has created new research directions in many fields of mathematics. Fuzzy set theory was originally restricted to the lattice $[0, 1]$, but the thrust of more recent research has pertained to general lattices.

The present work is primarily focused on the theory of lattice-valued convergence spaces; the category of lattice-valued convergence spaces has been shown to possess the following desirable categorical properties: topological, cartesian-closed, and extensional. Properties of quotient maps between objects in this category are investigated in this work; in particular, one of our principal results shows that quotient maps are productive under arbitrary products.

A category of lattice-valued interior operators is defined and studied as well. Axioms are given in order for this category to be isomorphic to the category whose objects consist of all the stratified, lattice-valued, pretopological convergence spaces.

Adding a lattice-valued convergence structure to a group leads to the creation of a new category whose objects are called lattice-valued convergence groups, and whose morphisms are all the continuous homomorphisms between objects. The later category is studied and results related to separation properties are obtained.

For the special lattice $\{0, 1\}$, continuous actions of a convergence semigroup on convergence spaces are investigated; in particular, invariance properties of actions as well as properties of a generalized quotient space are presented.

To my family and in loving memory of my dear uncle Miloudi.

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CHAPTER 1: INTRODUCTION AND PRELIMINARIES

A general introduction is given in this chapter as well as some background information needed throughout this dissertation.

1.1 Introduction

The notion of a filter of subsets, introduced by Cartan [2], has been used as a valuable tool in the development of topology and its applications. Lowen [16] defined the concept of a prefilter as a subset of $[0,1]^X$ in order to study the theory of fuzzy topological spaces. Later, Lowen et al. [15] used prefilters to define the notion of an L-fuzzy convergence space, when $L = [0,1]$, and showed that the category of all such objects has several desirable properties, such as being cartesian closed, not possessed by the category of all fuzzy topological spaces. Höhle [6] introduced the idea of a (stratified) L-filter as a descriptive map from L^X into L rather than a subset of L^X in the investigation of MV-algebras. Stratified L-filters are shown by Höhle and Sostak [7] to be a fruitful tool employed in the development of general lattice-valued topological spaces.

In chapter 2, quotient maps in the category of stratified L-convergence spaces (SL-CS) are shown to be productive; that is : Given $(X_j, \bar{q_j})$, $(Y_j, \bar{p_j}) \in |$ SL-CS$|$, $j \in J$ and denote the product space by $(X, \bar{q}) = \underset{j \in J}{\times} (X_j, \bar{q_j})$ and $(Y, \bar{p}) = \underset{j \in J}{\times} (Y_j, \bar{p_j})$. Assume that $f_j : (X_j, \bar{q_j}) \to (Y_j, \bar{p_j})$ is a quotient map , for each $j \in J$; then $f : (X, \bar{q}) \to (Y, \bar{p})$ is also a quotient map, where $f((x_j)_{j \in J}) := (f_j(x_j))_{j \in J}$.

Jäger [10] defined and studied the category whose objects consist of all the stratified L-fuzzy pretopological convergence spaces, denoted by SL-PCS. Moreover, Jäger [10] investigated the notion of a stratified L-interior operator of the form int: $L^X \to L^X$ and showed that the category SL-INT, whose objects consist of all the interior operator spaces, is isomorphic to the category SL-PCS. Flores et al. [3] introduced a category, denoted by SL-P-CS, and discussed its relationship to SL-PCS. Jäger [10] showed that SL-INT is not isomorphic to SL-P-CS, and asked what the appropriate interior operators for SL-P-CS might be. The axioms needed for suitable interior operators which characterize the objects in SL-P-CS are presented in Chapter 3, and are of the form $INT : L^X \times L \to L^X$.

Denote by GRP the category whose objects consists of all groups and whose morphisms are all the homomorphism between groups. If $(X, .) \in |\mathrm{GRP}|$ and $(X, \bar{q}) \in |\mathrm{SL\text{-}CS}|$, then $(X, ., \bar{q})$ is called a stratified L-convergence group provided that the group operations, product and inverse, are continuous with respect to $\bar{q}$. In chapter 4, the notion of a lattice-convergence group is investigated along with some closeness and separation properties. In Particular, it is shown that SL-CG is a topological category over GRP.

In the case of the lattice $L = \{0, 1\}$, the notion of a topological group acting continuously on a topological space has been the subject of numerous research articles. Park [21, 22] and Rath [24] studied these concepts in the larger category of convergence spaces. This is a more natural category to work in since the homeomorphism group on a space can be equipped with a coarsest convergence structure making the group operations continuous. Moreover, unlike in the topological context, quotient maps are productive in the category of all convergence spaces with continuous maps as morphisms. This property plays a key role in the proof of several results contained in Chapter 5. Given a topological semigroup acting on a topological space, Burzyk et al. [1] introduced a "generalized quotient space." Elements of this space are equivalence classes determined by an abstraction of the method used to construct the rationals from the integers. General quotient spaces are used in the study of generalized functions [14, 18, 19]. Generalized quotients in the category of convergence spaces and invariance properties of continuous actions of convergence semigroups on convergence spaces are investigated in Chapter 5.

In summary, the author's principal contributions to this dissertation are included in Theorem

2.3, 2.4, 3.1, 4.6, 5.5, and Example 5.1. Preliminary results were needed prior to proving these theorems.

1.2 Notions in topology

1.2.1 Filters in topology

The notion of filters in topology was introduced by Cartan [2]. Filters are used as a tool in defining concepts such as point of closure and compactness.

Definition 1.1 *Let X and 2^X be a nonempty set and its power set. A subset $\mathcal{F}$ of 2^X is said to be a filter on X if :*

1. $\emptyset \notin \mathcal{F}$ *and* $\mathcal{F} \neq \emptyset$
2. *if* $A \in \mathcal{F}$ *and* $A \subseteq B$, *then* $B \in \mathcal{F}$
3. *if* $A, B \in \mathcal{F}$, *then* $A \cap B \in \mathcal{F}$

Example 1.1 *The following are examples of filters*

1. $\nu_\tau(x)$, *the set of all neighborhoods of* x, *is a filter provided* (X, τ) *is a topological space.*

2. $\{A \subseteq X : x \in A\}$ *is a filter on* X *denoted by* $\dot{x}$.

Let $\mathfrak{F}(X)$ denote the set of all filters on X. Define the following order in $\mathfrak{F}(X)$: $\mathcal{F} \leq \mathcal{G}$ means $\mathcal{F} \subseteq \mathcal{G}$ provided $\mathcal{F}, \mathcal{G} \in \mathfrak{F}(X)$. In such case, we say that $\mathcal{F}$ is coarser that $\mathcal{G}$ (or $\mathcal{G}$ is finer that $\mathcal{F}$). $(\mathfrak{F}(X), \leq)$ is a poset.

A filter $\mathcal{F}$ is said to be an ultrafilter of X if for any filter $\mathcal{G}$ on X, $\mathcal{F} \leq \mathcal{G}$ implies $\mathcal{F} = \mathcal{G}$. As an example, the filter $\dot{x}$ is an ultrafilter of X, provided $x \in X$. The set of all ultrafilters on X is denoted by $\mathfrak{U}(X)$.

Proposition 1.1 *For each filter* $\mathcal{G} \in \mathfrak{F}(X)$, *there exists an ultrafilter* $\mathcal{F} \in \mathfrak{F}(X)$ *such that* $\mathcal{G} \leq \mathcal{F}$.

The proof of this proposition is based on the Zorn's lemma. Details of the proof can be found in [20].

Let $\{\mathcal{F}_i, \; i \in I\}$ be a family of filters on X, then $\bigcap\limits_{i\in I} \mathcal{F}_i$ is also a filter and it is straightforward to verify that $\bigcap\limits_{i\in I} \mathcal{F}_i$ is coarser than each $\mathcal{F}_i$. Actually, $\bigcap\limits_{i\in I} \mathcal{F}_i$ is the finest filter on X that is coarser than each $\mathcal{F}_i$. We call $\bigcap\limits_{i\in I} \mathcal{F}_i$ the infimum of the set $\{\mathcal{F}_i, i \in I\}$ and we denote it by $\underset{i\in I}{\wedge} \mathcal{F}_i$. Similarly, we introduce the supremum of $\{\mathcal{F}_i, i \in I\}$ to be, when it exits, the coarsest filter that is finer than each $\mathcal{F}_i$ and we denote it by $\underset{i\in I}{\vee} \mathcal{F}_i$.

Definition 1.2 *A collection $\mathcal{B}$ of subsets of X is called a* **filter base** *if*

(B1) $\mathcal{B} \neq \emptyset$ and $\emptyset \notin \mathcal{B}$;

(B2) $A, B \in \mathcal{B}$ implies that there exists $C \in \mathcal{B}$ such that $C \subseteq A \cap B$.

Proposition 1.2 *We have:*

(a) $\underset{i\in I}{\wedge} \mathcal{F}_i = \{\underset{i\in I}{\cup} A_i, \;\; A_i \in \mathcal{F}_i\}$

(b) Let $\mathcal{F}$ and $\mathcal{G}$ be two filters on X. The supremum of $\mathcal{F}$ and $\mathcal{G}$, denoted by $\mathcal{F} \vee \mathcal{G}$, exists iff $A \cap B \neq \emptyset$ for every $(A, B) \in (\mathcal{F}, \mathcal{G})$. Under such condition : $\mathcal{F} \vee \mathcal{G}$ has $\{A \cap B : (A, B) \in (\mathcal{F}, \mathcal{G})\}$ as its base.

If $\mathcal{B}$ is a filter base, then the collection of subsets $\mathcal{F}_{\mathcal{B}} = \{A \subseteq X : \;\; A \supseteq B \text{ for some } B \in \mathcal{B}\}$ forms a filter. $\mathcal{F}_{\mathcal{B}}$ is said to be generated by $\mathcal{B}$. Conversely, A subcollection $\mathcal{B}$ of a filter $\mathcal{F}$ is a base filter for $\mathcal{F}$ if every set of $\mathcal{F}$ contains a set of $\mathcal{B}$.

Let $\mathcal{S}$ be a collection of subsets of X and let $\mathcal{B}$ contain all finite intersections of elements of $\mathcal{S}$. Then $\mathcal{B}$ forms a filter base iff no finite subset of $\mathcal{S}$ has an empty intersection. If $\mathcal{B}$ forms a filter base, then the filter $\mathcal{F}_{\mathcal{B}}$ is the coarsest filter which contains $\mathcal{S}$ and $\mathcal{S}$ is called a subbase of $\mathcal{F}_{\mathcal{B}}$.

Let $f : X \to Y$ be a map and $\mathcal{F} \in \mathfrak{F}(X)$. The **image of $\mathcal{F}$ under** f is defined to be the filter on Y whose base is $\{f(F) : F \in \mathcal{F}\}$, denoted by $f^{\to}\mathcal{F}$

Let $\{X_i : i \in I\}$ be a family of sets and let $\mathcal{F}_i$ be a filter on X_i for each $i \in I$. Let $\pi_i : \underset{j\in I}{\times} X_j \to X_i$ be the i$^{\text{th}}$ projection map, $i \in I$. Then $\{\pi_i^{-1}(F_i) : F_i \in \mathcal{F}_i\} = \underset{j\in I}{\times} F_j$ where $F_j = X_j$, whenever $j \neq i$, forms a subbase for filter on $X = \underset{i\in I}{\times} X_i$. The filter containing this subbase is called the **product of the filters** $\mathcal{F}_i$ and is denoted by $\underset{i\in I}{\times} \mathcal{F}_i$. It is straightforward to verify that the product filter is the coarsest filter $\mathcal{F}$ on X such that $\pi_i^{\rightarrow}(\mathcal{F}) = \mathcal{F}_i$ for each $i \in I$.

1.2.2 Convergence Spaces

Filters are used as a tool in defining convergence. Let 2^X denote the set of all subsets of X. Assume $q : \mathfrak{F}(X) \to 2^X$, and consider the following conditions:

(CS1) $x \in q(\dot{x})\ \forall x \in X$

(CS2) if $\mathcal{F} \leq \mathcal{G}$ then $q(\mathcal{F}) \subseteq q(\mathcal{G})$

(CS3) if $x \in q(\mathcal{F})$ then $x \in q(\mathcal{F} \cap \dot{x})$

(CS4) $q(\mathcal{F}) \cap q(\mathcal{G}) \subseteq q(\mathcal{F} \cap \mathcal{G})$

(CS5) $\forall \mathcal{F} \in \mathfrak{F}(X)$, $x \in q(\mathcal{F})$ iff $x \in q(\mathcal{G})$ for every ultrafilter $\mathcal{G}$ such that $\mathcal{F} \leq \mathcal{G}$

(CS6) Given $\nu_q(x) := \cap\{\mathcal{F}, x \in q(\mathcal{F})\}$, then $x \in q(\nu_q(x))$ for every $x \in X$;

$\nu_q(x) = \cap\{\mathcal{F}, \mathcal{F}$ is an ultrafilter and $x \in q(\mathcal{F})\}$.

Note that (CS6) $\Rightarrow$ (CS5) $\Rightarrow$ (CS4) $\Rightarrow$ (CS3) and the pair (X, q) is called

a **convergence space** if q satisfies (CS1) and (CS2);

a **K-convergence space** if q satisfies (CS1), (CS2) and (CS3);

a **limit space** if q satisfies (CS1), (CS2) and (CS4);

a **pseudotopological space** (or **Choquet space**) if q satisfies (CS1), (CS2) and (CS5).

a **pretopological space** if q satisfies (CS1), (CS2) and (CS6).

Definition 1.3 *A filter $\mathcal{F}$ is said to* **q-converge to** x *when $x \in q(\mathcal{F})$, and $x \in q(\mathcal{F})$ is denoted by $\mathcal{F} \xrightarrow{q} x$.*

A function $f:(X,q)\to(Y,p)$ between two convergence spaces is said to be **continuous** provided $\mathcal{F}\xrightarrow{q}x$ implies $f^{\rightarrow}\mathcal{F}\xrightarrow{p}f(x)$.

Definition 1.4 *(Preuss [23]) A category $\mathcal{C}$ consists of*

(1) a class $|\mathcal{C}|$ of **objects** *(which are denoted by A,B,C, ...),*

(2) a class of pairwise disjoint sets $[A,B]_{\mathcal{C}}$ for each pair (A,B) of objects (The members of $[A,B]_{\mathcal{C}}$ are called **morphisms** *from A to B), and*

(3) a **composition** *of morphisms, i.e for each triple (A,B,C) of objects there is a map*

$$[A,B]_{\mathcal{C}}\times[B,C]_{\mathcal{C}}\to[A,C]_{\mathcal{C}}$$

$$(f,g)\to g\circ f$$

(where $\times$ denotes the cartesian product) such that the following axioms are satisfied:

(Cat 1) (Associativity). If $f\in[A,B]_{\mathcal{C}}$, $g\in[B,C]_{\mathcal{C}}$ and $h\in[C,D]_{\mathcal{C}}$, then $h\circ(g\circ f)=(h\circ g)\circ f$

(Cat 2) (Existence of identities). For each $A\in|\mathcal{C}|$, there is an identity (morphism) $id_A\in[A,A]_{\mathcal{C}}$ such that for all $B,C\in|\mathcal{C}|$, all $f\in[A,B]_{\mathcal{C}}$ and all $g\in[A,B]_{\mathcal{C}}$, $f\circ id_A=f$ and $id_B\circ g=g$.

$f\in[A,B]_{\mathcal{C}}$ is denoted by $f:A\to B$ throughout this manuscript.

Denote the category of all convergence (K-convergence, pseudotopological, pretopological) spaces by **CONV** (**K-CONV,PSTOP,PTOP**) where the morphisms are all the continuous functions between the objects. For more details on category theory, refer to [23]. Another type of convergence spaces called probabilistic convergence spaces was introduced by Florescu [4] as an extension of the notion of a probabilistic metric space which arose from the work of Menger [17].

Definition 1.5 *Let $L=[0,1]$, $\mathcal{F}$, $\mathcal{G}\in\mathfrak{F}(X)$ and $\alpha,\beta\in L$. The pair $(X,\bar{Q})$, where $\bar{Q}=(Q_\alpha)_{\alpha\in L}$ and $Q_\alpha:X\to 2^{(\mathfrak{F}(X))}$, is called a probabilistic convergence space provided:*

(a) $\dot{x}\xrightarrow{Q_\alpha}x$ and $\dot{X}\xrightarrow{Q_0}x$ for each $x\in X$

(b) $\mathcal{G} \supseteq \mathcal{F} \xrightarrow{Q_\alpha} x$ *implies* $\mathcal{G} \xrightarrow{Q_\alpha} x$

(c) $\mathcal{F} \xrightarrow{Q_\alpha} x$ *implies* $\mathcal{F} \xrightarrow{Q_\beta} x$ *whenever* $\beta \leq \alpha$.

The probability of $\mathcal{F}$ *converging to* x *being at least* α *is the interpretation given that* $\mathcal{F} \xrightarrow{Q_\alpha} x$.

A map $f : (X, \bar{Q}) \to (Y, \bar{P})$ is said to be **continuous** whenever $\mathcal{F} \xrightarrow{Q_\alpha} x$ implies that $f^{\to}\mathcal{F} \xrightarrow{P_\alpha} f(x)$ for each $\mathcal{F} \in \mathfrak{F}(X), \quad x \in X$ and $\alpha \in L$.

Let **PCS** denote the category whose objects consist of all the probabilistic convergence spaces and whose morphisms are all the continuous maps between objects. Properties of PCS can be found in Kent and Richardson [25]

1.2.3 Initial and Final Structures

A category $\mathcal{C}$ has **initial structures** provided that for any set X and family $f_i : X \to (Y_i, \sigma_i)$, where $i \in I$ and $(Y_i, \sigma_i) \in |\mathcal{C}|$, there exists a unique structure τ such that a map $g : (Y, \delta) \to (X, \tau)$ is a $\mathcal{C}$-morphism iff $f_i \circ g : (Y, \delta) \to (X, \tau)$ is a $\mathcal{C}$-morphism for each $i \in I$.

Given that (X, τ) and (X, δ) in $|\mathcal{C}|$, τ **is finer than** δ, denoted by $\delta \leq \tau$, provided $id_X : (X, \tau) \to (X, \delta)$ is a $\mathcal{C}$-morphism.

Theorem 1.1 *Let* $\mathcal{C}$ *be a category possessing initial structures. Given* $f_i : X \to (Y_i, \sigma_i)$, $(Y_i, \sigma_i) \in |\mathcal{C}|$, *the initial structure* τ *is the coarsest structure such that* $f_i : (X, \tau) \to (Y_i, \sigma_i)$ *is a* $\mathcal{C}$*-morphism for each* $i \in I$.

Proof: Assume that $\mathcal{C}$ has initial structures and let τ be the initial structure for $f_i : X \to (Y_i, \sigma_i)$ is a $\mathcal{C}$-morphism, $i \in I$. Suppose that δ is another structure such that $f_i : (X, \delta) \to (Y_i, \sigma_i)$ is a $\mathcal{C}$-morphism. Then $id_X : (X, \delta) \to (X, \tau)$ obeys $f_i = f_i \circ id_X : (X, \delta) \to (Y_i, \sigma_i)$ is a $\mathcal{C}$-morphism and thus $id_X : (X, \delta) \to (X, \tau)$ is a $\mathcal{C}$-morphism. Hence $\tau \leq \delta$ and thus τ

is the coarsest structure such that each $f_i : (X, \tau) \to (Y_i, \sigma_i)$ is a $\mathcal{C}-$morphism. ■

The category $\mathcal{C}$ has **final structures** provided that for each family $f_i : (X_i, \tau_i) \to Y$, where $(X_i, \tau_i) \in |\mathcal{C}|$ and $i \in I$, there exists a unique structure σ such that $g : (Y, \sigma) \to (Z, \delta)$ is a $\mathcal{C}$-morphism iff $g \circ f_i : (X_i, \tau_i) \to (Z, \delta)$ is a $\mathcal{C}$-morphism for each $i \in I$. proof of the following result can be found in Preuss ([23], pages 34-35).

Theorem 1.2 *([23])Let $\mathcal{C}$ be a category.*

(a) The final structure σ where $f_i : (X_i, \tau_i) \to Y, \quad i \in I$, is the finest structure on Y such that each $f_i : (X_i, \tau_i) \to (Y, \sigma)$ is a $\mathcal{C}$-morphism.

(b) $\mathcal{C}$ has initial structures iff it has final structures.

Let $(X_i, \tau_i) \in |\mathcal{C}|, \quad i \in I$, and $X = \underset{i \in I}{\times} X_i$. Recall that $\pi_i : X \to X_i, \quad i \in I$, denotes the i^{th} projection map. The initial structure τ, related to the family π_i, is called the **product structure**. Likewise, if $f : (Y, \sigma) \to Z$ is a surjection map, the final structure δ on Z is called the **quotient structure**. Hence a category $\mathcal{C}$ possessing initial structures has product and quotient structures.

Definition 1.6 *A category $\mathcal{C}$ is called topological provided*

(a) $\mathcal{C}$ has initial structures

(b) Given set X, the class $\{(Y, \tau) \in |\mathcal{C}| : Y = X\}$ is a set

(c) if X has exactly one element, then there exists exactly one $\mathcal{C}$-structure on X

1.3 Notions in fuzzy topology

Let $I = [0, 1]$, A **fuzzy set** of X is a function from $X \to I$, that is, an element of I^X. The subset $\iota a := \{x : a(x) > 0\}$ where $a \in I^X$ is called the **support** of a. For every $x \in X$, $a(x)$ is called the **grade of membership** of x in X. As an extension, I can be any lattice instead of $[0, 1]$. More details about fuzzy sets can found in Zadeh [26].

Example 1.2 *Let $A \subseteq X$. $\alpha 1_A$, as defined below, is an example of a fuzzy set.*

$$\alpha 1_A(x) := \begin{cases} \alpha, & \text{if } x \in A \\ 0, & \text{if } x \notin A \end{cases}, x \in X.$$

The introduction of fuzzy sets made researchers interested in revising classic topology by trying to see what can be done to adopt fuzziness. Namely, new type of filters, new convergence notions, new type of categories...etc were created.

A poset L is a lattice when every finite subset of L has an infimum and has a supremum. Further, a complete lattice is a lattice that possesses arbitrary infima and suprema. Throughout the rest of this manuscript, L is assumed to be a complete lattice unless mentioned otherwise. Moreover, L is called **regular** if $\alpha \wedge \beta = 0$ iff either $\alpha = 0$ or $\beta = 0$ in L. For example, let X be any set having at least two elements. Define L to be the power set of X and define $\vee$ as $\cup$ and $\wedge$ as $\cap$. Then $0 = \emptyset$ and $1 = X$. Now, let A, B denote two disjoint, non empty subsets of X. Then $A \wedge B = 0$, yet neither $A = 0$ or $B = 0$. Hence, in this example, L is not regular. A lattice that is totally ordered is of course regular.

Let $f : X \to Y$ be a map, $a \in L^X$ and $b \in L^Y$. **The image of a under f** is defined by $f^{\to}(a)(y) := \vee\{a(x) : f(x) = y\}$ provided y belongs to the range of f; otherwise, $f^{\to}(a)(y) = 0$. Dually, $f^{\leftarrow}(b) := b \circ f$ is called **the inverse image of b under f**.

An implication operator, denoted by $\rightarrow$ and called **residual implication**, was defined in [8] by $\alpha \rightarrow \beta = \vee\{\lambda \in L | \alpha \wedge \lambda \leq \beta\}$. It is characterized by $\delta \leq \alpha \rightarrow \beta$ iff $\delta \wedge \alpha \leq \beta$.

1.3.1 Stratified L-filters and Stratified L-convergence spaces

Working in the fuzzy context, it is natural to replace a subset of a set X by a fuzzy subset of X. The question is what would be the new form of filters; by analogy, instead of a set of subsets, a fuzzy filter could be introduced as a fuzzy subset of L^X. This is exactly what Höhle and Sostak [7] have introduced as a new type of filters and called a stratified L-filter.

Definition 1.7 *([7]) Given a nonempty set X, a map $\mathcal{F} : L^X \rightarrow L$ is called a stratified L-filter provided that for each $\alpha \in L$ and $a, b \in L^X$:*

(F1) $\mathcal{F}(1_\phi) = 0, \quad \mathcal{F}(\alpha 1_X) \geq \alpha$

(F2) $\mathcal{F}(a) \leq \mathcal{F}(b)$ *whenever* $a \leq b$

(F3) $\mathcal{F}(a) \wedge \mathcal{F}(b) \leq \mathcal{F}(a \wedge b)$.

Example 1.3 *The following are examples of stratified L-filters*

1. $[x] : L^X \rightarrow L$, *where* $[x](a) = a(x)$, $a \in L^X$

2. $\mathcal{F}_0 : L^X \rightarrow L$ *where* $\mathcal{F}_0(a) = \bigwedge_{x \in X} a(x)$

3. Assume L is regular and let $\psi \in \mathfrak{F}(X)$. Then $\mathcal{F}_\psi$ is a stratified L-filter where

$$\mathcal{F}_\psi(a) := \begin{cases} 1, & \textit{if } \iota a \in \psi \\ 0, & \textit{otherwise} \end{cases}$$

Let $\mathfrak{F}_{SL}(X)$ denote the set of all stratified L-filters on X. When $L = \{0, 1\}$, $\Delta : \mathfrak{F}_{SL}(X) \rightarrow \mathfrak{F}(X)$ defined by $\Delta(\mathcal{F}) := \{A \subseteq X : \mathcal{F}(1_A) = 1\}$ is a bijection. For a general L, define $\mathcal{F} \leq \mathcal{G}$ by $\mathcal{F}(a) \leq \mathcal{G}(a)$, $(\bigwedge_{j \in J} \mathcal{F}_j)(a) := \bigwedge_{j \in J} \mathcal{F}_j(a)$ and recall that $\mathcal{F}_0(a) := \wedge\{a(x) : x \in X\}$, for each $x \in X$ and $a \in L^X$. Then $(\mathfrak{F}_{SL}(X), \leq)$ is a poset having least element $\mathcal{F}_0$. Note that when $L = \{0, 1\}$, $\Delta([x]) = \dot{x}$ and $\Delta(\mathcal{F}_0) = \dot{X}$, where $\dot{A}$ denotes the filter of all oversets of A. Furthermore, It is shown in [7] that the set $(\mathfrak{F}_{SL}(X), \leq)$ has maximal elements called **stratified L-ultrafilters**. The set of all stratified L-ultrafilters is denoted by $\mathfrak{U}_{SL}(X)$.

Lemma 1.1 *([8]) Let $\mathcal{F}_j \in \mathfrak{F}_{SL}(X)$, $j \in J$. Then $\bigvee_{j\in J} \mathcal{F}_j$ exists in $\mathfrak{F}_{SL}(X)$ iff for each $n \geq 1$, $\bigwedge_{k=1}^{n} \mathcal{F}_{j_k}(a_k) = 0$ whenever $a_k \in L^X$ and $\bigwedge_{k=1}^{n} a_k = 1_\phi$. Furthermore, if $\bigvee_{j\in J} \mathcal{F}_j$ exists, then $\bigvee_{j\in J} \mathcal{F}_j(a) = \vee\{\bigwedge_{k=1}^{n} \mathcal{F}_{j_k}(a_k) : a_k \in L^X, \bigwedge_{k=1}^{n} a_k \leq a, n \geq 1\}$ for each $a \in L^X$.*

Lemma 1.2 *([7]) Let $\mathcal{F} \in \mathfrak{F}_{SL}(X)$. The following are equivalent:*

(a) $\mathcal{F}$ is a stratified L-ultrafilter

(b) $\mathcal{F}(a) = \mathcal{F}(a \to 1_\emptyset) \to 0$, for each $a \in L^X$.

Lemma 1.3 *([9]) Assume that L is regular, and define $\delta : \mathfrak{U}(X) \to \mathfrak{U}_{SL}(X)$ by $\delta(\psi) = \mathcal{F}_\psi$. Then $\mathcal{F}_\psi \in \mathfrak{U}_{SL}(X)$ and δ is a bijection.*

Given $\mathcal{F} \in \mathfrak{F}_{SL}(X)$ and $\mathcal{G} \in \mathfrak{F}_{SL}(Y)$, the **image of $\mathcal{F}$ under f** is defined as $f^{\to}\mathcal{F}(b) := \mathcal{F}(f^{\leftarrow}(b))$ and the **inverse image of $\mathcal{G}$ under f** is given by $f^{\leftarrow}\mathcal{G}(a) := \vee\{\mathcal{G}(b) : b \in L^Y, f^{\leftarrow}(b) \leq a\}$ whenever the latter is a stratified L-filter, where $a \in L^X$. Furthermore, the image of an stratified L-ultrafilter under any map is again a stratified L-ultrafilter([7]).

Lemma 1.4 *([8]) Assume that $f : X \to Y$ and $\mathcal{G} \in \mathfrak{F}_{SL}(Y)$. Then $f^{\leftarrow}\mathcal{G}$ exists in $\mathfrak{F}_{SL}(X)$ iff for each $b \in L^Y$, $\mathcal{G}(b) = 0$ whenever $f^{\leftarrow}(b) = 1_\phi$.*

Lemma 1.5 *Suppose that $f : X \to Y$, $\psi \in \mathfrak{U}(X)$, and $\mathcal{F} \in \mathfrak{U}_{SL}(X)$. Then $f^{\to}\mathcal{F}_\psi = \mathcal{F}_{f^{\to}\psi}$.*

PROOF: Since $f^{\to}\mathcal{F}_\psi$ and $\mathcal{F}_{f^{\to}\psi}$ are L-utrafilters, it suffices to show that $f^{\to}\mathcal{F}_\psi \geq \mathcal{F}_{f^{\to}\psi}$. Let $b \in L^Y$; then $f^{\to}\mathcal{F}_\psi(b) = \mathcal{F}_\psi(f^{\leftarrow}(b))$. Suppose that $\mathcal{F}_{f^{\to}\psi}(b) = 1$; then $\iota b \in f^{\to}\psi$ and thus $f^{-1}(\iota b) \in \psi$. Note that $f^{-1}(\iota b) \subseteq \iota(f^{\leftarrow}(b))$. Indeed, if $x \in f^{-1}(\iota b)$, then $f(x) \in \iota b$. Then $f^{\leftarrow}(b)(x) = b(f(x)) > 0$ and thus $x \in \iota(f^{\leftarrow}(b))$. Hence $f^{-1}(\iota b) \subseteq \iota(f^{\leftarrow}(b)) \in \psi$, $\mathcal{F}_\psi(f^{\leftarrow}(b)) = 1$, and therefore $f^{\to}\mathcal{F}_\psi = \mathcal{F}_{f^{\to}\psi}$. ■

Definition 1.8 *Assume that* $\mathcal{F}, \mathcal{G} \in \mathfrak{F}_{SL}(X)$ *and* $\alpha, \beta \in L$. *The pair* $(X, \bar{q})$, *where* $\bar{q} = (q_\alpha)_{\alpha \in L}$, *is called a stratified L-convergence space whenever the following conditions are satisfied:*

(a) $[x] \xrightarrow{q_\alpha} x$ *and* $\mathcal{F}_0 \xrightarrow{q_0} x$ *for each* $x \in X$

(b) $\mathcal{G} \supseteq \mathcal{F} \xrightarrow{q_\alpha} x$ *implies* $\mathcal{G} \xrightarrow{q_\alpha} x$

(c) $\mathcal{F} \xrightarrow{q_\alpha} x$ *implies* $\mathcal{F} \xrightarrow{q_\beta} x$ *whenever* $\beta \leq \alpha$.

A map $f : (X, \bar{q}) \rightarrow (Y, \bar{p})$ is said to be **continuous** provided $\mathcal{F} \xrightarrow{q_\alpha} x$ implies that $f^{\rightarrow}\mathcal{F} \xrightarrow{p_\alpha} f(x)$, for each $\mathcal{F} \in \mathcal{F}_{SL}(X), x \in X$ and $\alpha \in L$.

Denote by **SL-CS** the category whose objects consist of all the stratified L-convergence spaces and whose morphisms are all the continuous maps between objects. Whenever $L = [0, 1]$, it is shown in Theorem 3.1 [3] that PCS, defined in 1.2.2, is embedded as a full subcategory of SL-CS. Denote the full subcategory of SL-CS consisting of all the objects $(X, \bar{q}) \in |\text{SL-CS}|$ for which $\mathcal{F} \xrightarrow{q_\alpha} x$ $\left(\nu_{q_\alpha}(x) := \wedge\{\mathcal{F} : \mathcal{F} \xrightarrow{q_\alpha} x\}\right)$ implies that $\mathcal{F} \wedge [x] \xrightarrow{q_\alpha} x$ $\left(\nu_{q_\alpha}(x) \xrightarrow{q_\alpha} x\right)$ <u>**(1.1)**</u> by **SL-K-CS (SL-P-CS)**, respectively.

CHAPTER 2: QUOTIENT MAPS IN SL-CS

2.1 Quotient Maps

It is shown in Theorem 5.1 [3] that SL-CS is a topological category and consequently quotient objects exists; that is, a continuous surjection $f : (X, \bar{q}) \to (Y, \bar{p})$ in SL-CS is a quotient map iff $\bar{p}$ is the unique structure on Y for which $g : (Y, \bar{p}) \to (Z, \bar{r})$ is continuous iff $g \circ f : (X, \bar{q}) \to (Z, \bar{r})$ is continuous. A characterization of $\bar{p}$ is given in Theorem 2.1, and quotient maps are shown to be productive in Theorem 2.3.

Lemma 2.1 *Suppose that $f : X \to Y$ is a surjection, $a_i \in L^X$ and $b_i \in L^Y, i = 1, 2$. Then*
(a) $b_1 \wedge f^{\to}(a_2) = 1_\phi$ *provided* $a_1 \wedge a_2 = 1_\phi$ *and* $f^{\leftarrow}(b_1) \leq a_1$
(b) $b_1 \wedge f^{\to}(a_2) \leq b_2$ *whenever* $a_1 \wedge a_2 \leq f^{\leftarrow}(b_2)$ *and* $f^{\leftarrow}(b_1) \leq a_1$.

Proof: Clearly (a) follows from (b) whenever $b_2 = 1_\phi$.
(b): Fix $y \in Y$. Since L is a complete Heyting algebra, the assumptions imply that $b_1(y) \wedge f^{\to}(a_2)(y) = b_1(y) \wedge \vee\{a_2(x) : f(x) = y\} = \vee\{f^{\leftarrow}(b_1)(x) \wedge a_2(x) : f(x) = y\} \leq \vee\{a_1(x) \wedge a_2(x) : f(x) = y\} \leq \vee\{f^{\leftarrow}(b_2)(x) : f(x) = y\} = b_2(y)$ and thus $b_1 \wedge f^{\to}(a_2) \leq b_2$. ■

Lemma 2.2 *Assume that $f : X \to Y$ is a surjection and $\mathfrak{F} \in \mathfrak{F}_{SL}(X), \mathcal{H} \in \mathfrak{F}_{SL}(Y)$ are such that $f^{\to}\mathfrak{F} \leq \mathcal{H}$. Then there exists $\mathcal{G} \in \mathfrak{F}_{SL}(X)$ for which $\mathcal{G} \geq \mathcal{F}$ and $f^{\to}\mathcal{G} = \mathcal{H}$.*

Proof: Since f is a surjection, $f^{\leftarrow}\mathcal{H}$ exists. Lemma 1.1 is used to verify the existence of $f^{\leftarrow}\mathcal{H} \vee \mathcal{F}$. Indeed, suppose that $a_i \in L^X$ such that $a_1 \wedge a_2 = 1_\phi$; it must be shown that $f^{\leftarrow}\mathcal{H}(a_1) \wedge \mathcal{F}(a_2) = 0, i = 1, 2$. Since L is a complete Heyting algebra, $f^{\leftarrow}\mathcal{H}(a_1) \wedge \mathcal{F}(a_2) = \vee\{\mathcal{H}(b_1) : f^{\leftarrow}(b_1) \leq a_1\} \wedge \mathcal{F}(a_2) = \vee\{\mathcal{H}(b_1) \wedge \mathcal{F}(a_2) : f^{\leftarrow}(b_1) \leq a_1\}$. It follows from $f^{\leftarrow}(f^{\to}(a_2)) \geq a_2$ and $f^{\to}\mathcal{F} \leq \mathcal{H}$ that $\mathcal{H}(b_1) \wedge \mathcal{F}(a_2) \leq \mathcal{H}(b_1) \wedge f^{\to}\mathcal{F}(f^{\to}(a_2)) \leq \mathcal{H}(b_1) \wedge \mathcal{H}(f^{\to}(a_2)) = \mathcal{H}(b_1 \wedge f^{\to}(a_2))$. According to Lemma 2.1 (a), $b_1 \wedge f^{\to}(a_2) = 1_\phi$ and thus $f^{\leftarrow}\mathcal{H}(a_1) \wedge \mathcal{F}(a_2) = 0$. Hence $\mathcal{G} := f^{\leftarrow}\mathcal{H} \vee \mathcal{F}$ exists.

Next, it is shown that $f^{\rightarrow}\mathcal{G} = \mathcal{H}$. Since f is a surjection, $f^{\rightarrow}(f^{\leftarrow}\mathcal{H}) = \mathcal{H}$ and thus $f^{\rightarrow}(f^{\leftarrow}\mathcal{H} \vee \mathcal{F}) \geq \mathcal{H}$. Fix $b_2 \in L^Y$; it remains to prove that $f^{\rightarrow}(f^{\leftarrow}\mathcal{H} \vee \mathcal{F})(b_2) \leq \mathcal{H}(b_2)$. Employing Lemmas 1.1,1.4,

$f^{\rightarrow}(f^{\leftarrow}\mathcal{H} \vee \mathcal{F})(b_2) = (f^{\leftarrow}\mathcal{H} \vee \mathcal{F})(f^{\leftarrow}(b_2))$

$= \vee\{f^{\leftarrow}\mathcal{H}(a_1) \wedge \mathcal{F}(a_2) : a_1 \wedge a_2 \leq f^{\leftarrow}(b_2)\}$

$= \vee\Big\{ \vee \{\mathcal{H}(b_1) : f^{\leftarrow}(b_1) \leq a_1\} \wedge \mathcal{F}(a_2) : a_1 \wedge a_2 \leq f^{\leftarrow}(b_2)\Big\}$

$= \vee\Big\{ \vee \{\mathcal{H}(b_1) \wedge \mathcal{F}(a_2) : f^{\leftarrow}(b_1) \leq a_1\} : a_1 \wedge a_2 \leq f^{\leftarrow}(b_2)\Big\}$

$\leq \vee\Big\{ \vee \{\mathcal{H}(b_1) \wedge f^{\rightarrow}\mathcal{F}\big(f^{\rightarrow}(a_2)\big) : f^{\leftarrow}(b_1) \leq a_1\} : a_1 \wedge a_2 \leq f^{\leftarrow}(b_2)\Big\}$

$\leq \vee\Big\{ \vee \{\mathcal{H}\big(b_1 \wedge f^{\rightarrow}(a_2)\big) : f^{\leftarrow}(b_1) \leq a_1\} : a_1 \wedge a_2 \leq f^{\leftarrow}(b_2)\Big\}$

$\leq \mathcal{H}(b_2)$ according to Lemma 2.1 (b). Hence $f^{\rightarrow}(f^{\leftarrow}\mathcal{H} \vee \mathcal{F}) \leq \mathcal{H}$ and thus $f^{\rightarrow}(\mathcal{G}) = \mathcal{H}$. ■

Theorem 2.1 *Assume that $(X, \bar{q}) \in |SL - CS|$ and $f : (X, \bar{q}) \to Y$ is a surjection. Then the quotient structure $\bar{p} = (p_\alpha)_{\alpha \in L}$ is given by: $\mathcal{G} \xrightarrow{p_\alpha} y$ iff these exists $\mathcal{F} \xrightarrow{q_\alpha} x$, for some $x \in f^{-1}(y)$, such that $f^{\rightarrow}\mathcal{F} = \mathcal{G}$.*

Proof: Note that $[y] \xrightarrow{p_\alpha} y$ since $f^{\rightarrow}([x]) = [y]$ whenever $x \in f^{-1}(y)$. Denote the coarsest member of $\mathfrak{F}_{SL}(Y)\big(\mathfrak{F}_{SL}(X)\big)$ by $\mathcal{G}_0(\mathcal{F}_0)$, respectively. Let $b \in L^Y$ and observe that $f^{\rightarrow}\mathcal{F}_0(b) = \mathcal{F}_0\big(f^{\leftarrow}(b)\big) = \wedge\{f^{\leftarrow}(b)(x) : x \in X\} = \wedge\{b(y) : y \in Y\} = \mathcal{G}_0(b)$. Then $f^{\rightarrow}\mathcal{F}_0 = \mathcal{G}_0 \xrightarrow{p_0} y$ since $\mathcal{F}_0 \xrightarrow{q_0} x$ for each $x \in X$. Next, if $\mathcal{H} \geq \mathcal{K} \xrightarrow{p_\alpha} y$, then there exists $\mathcal{F} \xrightarrow{q_\alpha} x \in f^{-1}(y)$ such that $f^{\rightarrow}\mathcal{F} = \mathcal{K}$. According to Lemma 2.2, there exists $\mathcal{G} \geq \mathfrak{F}$ such that $f^{\rightarrow}\mathcal{G} = \mathcal{H}$, and since $\mathcal{G} \xrightarrow{q_\alpha} x, \mathcal{H} \xrightarrow{p_\alpha} y$. Moreover, if $\beta \leq \alpha$ and $\mathcal{G} \xrightarrow{p_\alpha} y$, then $\mathcal{G} \xrightarrow{p_\beta} y$ and thus $(Y, \bar{p}) \in |SL - CS|$. It is straightforward to show that $f : (X, \bar{q}) \longrightarrow (Y, \bar{p})$ is a quotient map in SL-CS. ■

Suppose that $f_j : X_j \longrightarrow Y_j$ is a surjection and $\mathcal{F}_j \in \mathfrak{F}_{SL}(X_j)$ for each $j \in J$; denote

$X = \underset{j \in J}{\times} X_j \quad , \quad Y = \underset{j \in J}{\times} Y_j \quad , \quad \mathcal{F} = \underset{j \in J}{\times} \mathcal{F}_j := \underset{j \in J}{\vee} \pi_j^{\leftarrow}\mathcal{F}_j$ **(2.1)**, where $\pi_j : X \to X_j$ is the j^{th} projection map.

Lemma 2.3 *Suppose that $f_j : X_j \to Y_j$ is a surjection, $j \in J$. Let $a \in L^X$ and $b \in L^Y$. Then, using the notations given in (2.1),*

(a) $\mathcal{F}(a) = \vee\big\{ \vee \{ \overset{n}{\underset{k=1}{\wedge}} \mathcal{F}_{j_k}(c_k) : c_k \in L^{X_{j_k}}, \overset{n}{\underset{k=1}{\wedge}} \pi^{\leftarrow}_{X_{j_k}}(c_k) \leq a\} : n \geq 1\big\}$

(b) $\underset{j\in J}{\times}(f_j^{\rightarrow}\mathcal{F}_j)(b) = \vee\big\{ \vee \{ \overset{n}{\underset{k=1}{\wedge}} \mathcal{F}_{j_k}(f^{\leftarrow}_{j_k}(d_k)) : d_k \in L^{Y_{j_k}}, \overset{n}{\underset{k=1}{\wedge}} \pi^{\leftarrow}_{Y_{j_k}}(d_k) \leq b\} : n \geq 1\big\}$

(c) $f^{\rightarrow}\mathcal{F}(b) = \vee\big\{ \vee \{ \overset{n}{\underset{k=1}{\wedge}} \mathcal{F}_{j_k}(c_k) : c_k \in L^{X_{j_k}}, \overset{n}{\underset{k=1}{\wedge}} \pi^{\leftarrow}_{X_{j_k}}(c_k) \leq f^{\leftarrow}(b)\} : n \geq 1\big\}.$

Proof: (a): Using the fact that L is a complete Heyting algebra, and employing Lemmas 1.1,1.4, it follows that

$$\mathcal{F}(a) = \underset{j\in J}{\times} \mathcal{F}_j(a) := \big(\underset{j\in J}{\vee} \pi^{\leftarrow}_{X_j}\mathcal{F}_j\big)(a)$$
$$= \vee\{ \overset{n}{\underset{k=1}{\wedge}} (\pi^{\leftarrow}_{X_{j_k}}\mathcal{F}_{j_k})(a_k) : a_k \in L^X, \overset{n}{\underset{k=1}{\wedge}} a_k \leq a, n \geq 1\}$$
$$= \vee\big\{ \overset{n}{\underset{k=1}{\wedge}} \vee\{\mathcal{F}_{j_k}(c) : c \in L^{X_{j_k}}, \pi^{\leftarrow}_{X_{j_k}}(c) \leq a_k\} : \overset{n}{\underset{k=1}{\wedge}} a_k \leq a, n \geq 1\big\}$$
$$= \vee\big\{ \vee \{ \overset{n}{\underset{k=1}{\wedge}} \mathcal{F}_{j_k}(c_k) : c_k \in L^{X_{j_k}}, \overset{n}{\underset{k=1}{\wedge}} \pi^{\leftarrow}_{X_{j_k}}(c_k) \leq a\} : n \geq 1\big\}.$$

(b)-(c): Verification follows from (a) since $f^{\rightarrow}_{j_k}\mathfrak{F}_{j_k}(d_k) = \mathcal{F}_{j_k}\big(f^{\leftarrow}_{j_k}(d_k)\big)$ $\Big(f^{\rightarrow}\mathcal{F}(b) = \mathcal{F}\big(f^{\leftarrow}(b)\big)\Big)$, respectively. ■

Lemma 2.4 *Assume that $f_j : X_j \to Y_j$ is a surjection, $j \in J$. Let $b \in L^Y$. Then, using the notations listed in (2.1),*

(a) $f^{\rightarrow}\big(\overset{n}{\underset{k=1}{\wedge}} \pi^{\leftarrow}_{X_{j_k}}(c_k)\big)(y) = \overset{n}{\underset{k=1}{\wedge}} f^{\rightarrow}_{j_k}(c_k)(y_{j_k})$, *where* $c_k \in L^{X_{j_k}}$ *and* $y = (y_j)_{j\in J} \in Y$

(b) $\overset{n}{\underset{k=1}{\wedge}} \pi^{\leftarrow}_{X_{j_k}}(c_k) \leq f^{\leftarrow}(b)$ *implies that* $\overset{n}{\underset{k=1}{\wedge}} \pi^{\leftarrow}_{Y_{j_k}}(d_k) \leq b$ *whenever* $c_k \in L^{X_{j_k}}$ *and* $d_k = f^{\rightarrow}_{j_k}(c_k)$

(c) $\overset{n}{\underset{k=1}{\wedge}} \pi^{\leftarrow}_{Y_{j_k}}(d_k) \leq b$ *implies that* $\overset{n}{\underset{k=1}{\wedge}} \pi^{\leftarrow}_{X_{j_k}}(c_k) \leq f^{\leftarrow}(b)$ *provided* $d_k \in L^{Y_{j_k}}$ *and* $c_k = f^{\leftarrow}_{j_k}(d_k)$.

Proof: (a): Let $y = (y_j)_{j\in J} \in Y$. Since L is a complete Heyting algebra,

$$f^{\rightarrow}\big(\overset{n}{\underset{k=1}{\wedge}} \pi^{\leftarrow}_{X_{j_k}}(c_k)\big)(y) = \vee\big\{\big(\overset{n}{\underset{k=1}{\wedge}} \pi^{\leftarrow}_{X_{j_k}}(c_k)\big)(x) : f(x) = y\big\}$$
$$= \vee\{ \overset{n}{\underset{k=1}{\wedge}} (c_k \circ \pi_{X_{j_k}})(x) : f(x) = y\}$$
$$= \vee\{ \overset{n}{\underset{k=1}{\wedge}} c_k(x_{j_k}) : f_{j_k}(x_{j_k}) = y_{j_k}, \quad 1 \leq k \leq n\}$$
$$= \overset{n}{\underset{k=1}{\wedge}} \vee\{c_k(x_{j_k}) : f_{j_k}(x_{j_k}) = y_{j_k}\}$$
$$= \overset{n}{\underset{k=1}{\wedge}} f^{\rightarrow}_{j_k}(c_{j_k})(y_{j_k}).$$

(b): Note that $\overset{n}{\underset{k=1}{\wedge}} \pi^{\leftarrow}_{X_{j_k}}(c_k) \leq f^{\leftarrow}(b)$ is equivalent to $b = f^{\rightarrow}\big(f^{\leftarrow}(b)\big) \geq f^{\rightarrow}\big(\overset{n}{\underset{k=1}{\wedge}} \pi^{\leftarrow}_{X_{j_k}}(c_k)\big)$. Employing part (a), $f^{\rightarrow}\big(\overset{n}{\underset{k=1}{\wedge}} \pi^{\leftarrow}_{X_{j_k}}(c_k)\big)(y) = \overset{n}{\underset{k=1}{\wedge}} f^{\rightarrow}_{j_k}(c_k)(y_{j_k}) \leq b(y)$ for each $y \in L^Y$. Hence $\overset{n}{\underset{k=1}{\wedge}} \pi^{\leftarrow}_{Y_{j_k}}(d_k)(y) = \overset{n}{\underset{k=1}{\wedge}} d_k(y_{j_k}) = \overset{n}{\underset{k=1}{\wedge}} f^{\rightarrow}_{j_k}(c_k)(y_{j_k}) \leq b(y)$ for each $y \in Y$ and thus $\overset{n}{\underset{k=1}{\wedge}} \pi^{\leftarrow}_{Y_{j_k}}(d_k) \leq b$.

(c): Let $x \in X$. Since $\bigwedge\limits_{k=1}^{n} \pi_{Y_{j_k}}^{\leftarrow}(d_k) \leq b$,

$$\bigwedge\limits_{k=1}^{n} \pi_{X_{j_k}}^{\leftarrow}(c_k)(x) = \bigwedge\limits_{k=1}^{n} \pi_{X_{j_k}}^{\leftarrow}\big(f_{j_k}^{\leftarrow}(d_k)\big)(x)$$
$$= \bigwedge\limits_{k=1}^{n} f_{j_k}^{\leftarrow}(d_k)(x_{j_k}) = \bigwedge\limits_{k=1}^{n} d_k\big(f_{j_k}(x_{j_k})\big)$$
$$= \bigwedge\limits_{k=1}^{n} \pi_{j_k}^{\leftarrow}(d_k)\big(f(x)\big) \leq b\big(f(x)\big) = f^{\leftarrow}(b)(x) \text{ for each } x \in X.$$

Hence $\bigwedge\limits_{k=1}^{n} \pi_{X_{j_k}}^{\leftarrow}(c_k) \leq f^{\leftarrow}(b)$. ■

Theorem 2.2 *Let $f_j : X_j \to Y_j$ be a surjection for each $j \in J$. Then, using the notations given in (2.1), $f^{\rightarrow}\mathcal{F} = \underset{j\in J}{\times}(f_j^{\rightarrow}\mathcal{F}_j)$.*

Proof: Lemmas 2.3-2.4 are used to verify the result. Fix $b \in L^Y$ and suppose that $d_k \in L^{Y_{j_k}}, 1 \leq k \leq n$, satisfies $\bigwedge\limits_{k=1}^{n} \pi_{Y_{j_k}}^{\leftarrow}(d_k) \leq b$. According to Lemma 2.4 (c), $\bigwedge\limits_{k=1}^{n} \pi_{X_{j_k}}^{\leftarrow}(c_k) \leq f^{\leftarrow}(b)$ whenever $c_k = f_{j_k}^{\leftarrow}(d_k)$. Since $\mathcal{F}_{j_k}(c_k) = \mathcal{F}_{j_k}\big(f_{j_k}^{\leftarrow}(d_k)\big)$, it follows from Lemma 2.3 (b,c) that $f^{\rightarrow}\mathcal{F}(b) \geq \times_{j\in J}(f_j^{\rightarrow}\mathcal{F}_j)(b)$. Conversely, assume that $c_k \in L^{X_{j_k}}$ for $1 \leq k \leq n$, obeys $\bigwedge\limits_{k=1}^{n} \pi_{X_{j_k}}^{\leftarrow}(c_k) \leq f^{\leftarrow}(b)$ and let $d_k = f_{j_k}^{\rightarrow}(c_k)$. It follows from Lemma 2.4 (b) that $\bigwedge\limits_{k=1}^{n} \pi_{Y_{j_k}}^{\leftarrow}(d_k) \leq b$. Since $\mathcal{F}_{j_k}\big(f_{j_k}^{\leftarrow}(d_k)\big) = \mathcal{F}_{j_k}\big(f_{j_k}^{\leftarrow}f_{j_k}^{\rightarrow}(c_k)\big) \geq \mathcal{F}_{j_k}(c_k)$, Lemma 2.3 (b,c) implies that $\underset{j\in J}{\times}(f_j^{\rightarrow}\mathcal{F}_j)(b) \geq f^{\rightarrow}\mathcal{F}(b)$. Therefore $f^{\rightarrow}\mathcal{F} = \underset{j\in J}{\times}(f_j^{\rightarrow}\mathcal{F}_j)$. ■

Theorem 2.3 *Assume that $f_j : (X_j, \bar{q}_j) \to (Y_j, \bar{p}_j)$ is a quotient map in the SL-CS category, $(X, \bar{q}) = \underset{j\in J}{\times}(X_j, \bar{q}_j)$, and $(Y, \bar{p}) = \underset{j\in J}{\times}(Y_j, \bar{p}_j), j \in J$. Then $f = \underset{j\in J}{\times} f_j : (X, \bar{q}) \to (Y, \bar{p})$ is also a quotient map.*

Proof: Suppose that $\mathcal{H} \xrightarrow{p_\alpha} y$, $\alpha \in L$. Then $\pi_{Y_j}^{\rightarrow}\mathcal{H} \xrightarrow{p_{j\alpha}} y_j$ for each $j \in J$, and since f_j is a quotient map, Theorem 2.1 implies that there exists $\mathcal{F}_j \xrightarrow{q_{j\alpha}} x_j \in f_j^{-1}(y_j)$ such that $f_j^{\rightarrow}\mathcal{F}_j = \pi_{Y_j}^{\rightarrow}\mathcal{H}$. Denote $\mathcal{F} = \underset{j\in J}{\times}\mathcal{F}_j$; then $\mathcal{F} \xrightarrow{q_\alpha} x = (x_j)_{j\in J}$, and it follows from Theorem 2.2 that $f^{\rightarrow}\mathcal{F} = \underset{j\in J}{\times}(f_j^{\rightarrow}\mathcal{F}_j) = \underset{j\in J}{\times}(\pi_{Y_j}^{\rightarrow}\mathcal{H}) \leq \mathcal{H}$. According to Lemma 2.2, there exists $\mathcal{G} \geq \mathcal{F}$, $\mathcal{G} \xrightarrow{q_\alpha} x$, such that $f^{\rightarrow}\mathcal{G} = \mathcal{H}$. It follows from Theorem 2.1 that $f : (X, \bar{q}) \to (Y, \bar{p})$ is a quotient map in SL-CS. ■

It was shown in [3] that SL-CS is topological, cartesian-closed, and extensional. Then, in view of Theorem 2.3, SL-CS is a strong topological universe (Preuss,[23]).

2.2 Topological Objects

Gähler [5] proved that the "topological objects" in SL-CS can be characterized by a "diagonal condition." These definitions are listed below, and it is shown in Theorem 2.5 that, under a mild assumption, each object in SL-CS is the image of a topological object under a quotient map.

Given a set $X, \tau \subseteq L^X$ called a **stratified L-topology** [7] if it obeys the following conditions:

(a) $\alpha 1_X \in \tau$ for each $\alpha \in L$

(b) $a, b \in \tau$ implies that $a \wedge b \in \tau$

(c) $a_j \in \tau, j \in J$, implies that $\underset{j \in J}{\vee} a_j \in \tau$.

The pair (X, τ) is said to be a **stratified L-topological space**, and define $\nu_\tau(x) : L^X \to L$ by $\nu_\tau(x)(a) = \vee\{b(x) : b \in \tau, \quad b \leq a\}$. It is straightforward to show that $\nu_\tau(x) \in \mathcal{F}_{SL}(X)$. A map $f : (X, \tau) \to (Y, \sigma)$ between two stratified L-topological spaces is called **continuous** provided $f^{\leftarrow}(b) \in \tau$ wherever $b \in \sigma$. Moreover, it easily follows that $f : (X, \tau) \to (Y, \sigma)$ is continuous iff $f^{\rightarrow}\big(\nu_\tau(x)\big) \geq \nu_\sigma\big(f(x)\big)$ for each $x \in X$. An object $(X, \bar{q}) \in |\text{SL-CS}|$, where $\bar{q} = (q_\alpha)_{\alpha \in L}$, is said to be **topological** wherever there exists a stratified L-topological space (X, τ_α) such that $\nu_{\tau_\alpha}(x) = \nu_{q_\alpha}(x) \xrightarrow{q_\alpha} x$ for each $x \in X, \quad \alpha \in L$.

Objects $(X, \bar{q}) \in |\text{SL-P-CS}|$ are called **pretopological** (1.1). Gähler [4] characterized the objects in |SL-P-CS| that are topological. Indeed, define the **compression operator** $G : \mathfrak{F}_{SL}(\mathfrak{F}_{SL}(X)\big) \to \mathfrak{F}_{SL}(X)$ by $G(\Phi)(a) := \Phi(e_a)$, where $\Phi \in \mathfrak{F}_{SL}\big(\mathfrak{F}_{SL}(X)\big)$, and $e_a : \mathfrak{F}_{SL}(X) \to L$ is given by $e_a(\mathcal{G}) = \mathcal{G}(a)$, for each $a \in L^X$. It is easily verified that $G(\Phi) \in \mathfrak{F}_{SL}(X)$. An object $(X, \bar{q}) \in |\text{SL-P-CS}|$ obeys **diagonal axiom D** provided:

$$\sigma : X \to \mathfrak{F}_{SL}(X) \quad \text{such that}$$

(D) $\qquad \sigma(y) \xrightarrow{q_\alpha} y$ for each $y \in X$, $\mathcal{F} \in \mathfrak{F}_{SL}(X)$

and if $\mathcal{F} \xrightarrow{q_\alpha} x$, then $G(\sigma^{\rightarrow}\mathcal{F}) \xrightarrow{q_\alpha} x$, $\alpha \in L$.

The following result is proved by Gähler ([5], Proposition 20).

Theorem 2.4 *([5]) An object $(X, \bar{q}) \in |SL\text{-}P\text{-}CS|$ is topological iff it obeys axiom D.*

Recall that SL-K-CS denotes the full subcategory of SL-CS consisting of all objects $(X, \bar{q})$ for which $\mathcal{F} \xrightarrow{q_\alpha} x$ implies that $\mathcal{F} \wedge [x] \xrightarrow{q_\alpha} x$, $\alpha \in L$. Given $(X, \bar{q}) \in$ |SL-K-CS|, select $\mathcal{F} \xrightarrow{q_{\alpha_0}} x, \alpha_0 \in L$, and denote $j = (\mathcal{F}, x, \alpha_0)$. Define $\bar{q}_j = (q_{j_\alpha})_{\alpha \in L}$ as follows:

(i) q_{j_0} is the indiscrete structure on X

(ii) if $0 < \alpha \leq \alpha_0$,

$\mathcal{G} \xrightarrow{q_{j\alpha}} x$ iff $\mathcal{G} \geq \mathcal{F} \wedge [x]$ **(2.2)**

$\mathcal{G} \xrightarrow{q_{j\alpha}} y$, $y \neq x$, iff $\mathcal{G} \geq [y]$

(iii) if $\alpha \leq \alpha_0$ fails, $\mathcal{G} \xrightarrow{q_{j\alpha}} y$ iff $\mathcal{G} \geq [y]$.

Then $(X, \bar{q}_j) \in$ |SL-P-CS| wherever $j = (\mathcal{F}, x, \alpha_0)$.

Lemma 2.5 *Assume that $(X, \bar{q}) \in |SL\text{-}K\text{-}CS|$; then $(X, \bar{q}_j), j = (\mathcal{F}, x, \alpha_0)$, as defined in (2.2) is topological.*

Proof: According to Theorem 2.4, it suffices to show that condition D is satisfied. Clearly this condition is valid wherever $\alpha = 0$. Suppose that $0 < \alpha \leq \alpha_0$ and $\sigma : X \to \mathfrak{F}_{SL}(X)$ satisfies $\sigma(y) \xrightarrow{q_{j\alpha}} y$ for each $y \in X$. Note that if $a \in L^X$ and $y \in X$, $(e_a \circ \sigma)(y) = \sigma(y)(a) \geq \mathcal{F}(a) \wedge [y](a) = (\mathcal{F}(a) \cdot 1_X \wedge a)(y)$. Hence $e_a \circ \sigma \geq \mathcal{F}(a) \cdot 1_X \wedge a$ and thus $G(\sigma^{\rightarrow}\mathcal{F})(a) = \mathcal{F}(e_a \circ \sigma) \geq \mathcal{F}\big(\mathcal{F}(a) \cdot 1_X \wedge a\big) = \mathcal{F}\big(\mathcal{F}(a) \cdot 1_X\big) \wedge \mathcal{F}(a) \geq \mathcal{F}(a)$. Moreover, $G\big(\sigma^{\rightarrow}([y])\big)(a) = [y](e_a \circ \sigma) = \sigma(y)(a)$ and thus $G\big(\sigma^{\rightarrow}([y])\big) = \sigma(y)$. Likewise, if $\alpha \leq \alpha_0$ fails, then $G\big(\sigma^{\rightarrow}([y])\big) = \sigma(y)$ and thus it follows that $(X, \bar{q}_j)$ obeys condition D, and therefore is topological by Theorem 2.4. ■

Given a set X, assume that $(X_j, \bar{q}_j), j \in J$, is any collection of objects in SL-CS for which the X_j's are disjoint copies of X. Let $Y = \bigcup_{j \in J} X_j$, and define the **disjoint union (coproduct)** as follows:

$$
\begin{aligned}
&(Y, \bar{p}) = \bigcup_{j \in J} (X_j, \bar{q}_j), \quad p = (p_\alpha)_{\alpha \in L}, \\
&\mathcal{H} \xrightarrow{p_\alpha} y, y \in X_j, \text{ iff } \mathcal{H} \geq [\mathcal{G}] \qquad \underline{\mathbf{(2.3)}} \\
&\text{for some } \mathcal{G} \in \mathfrak{F}_{SL}(X_j), \quad \mathcal{G} \xrightarrow{q_{j\alpha}} y, \\
&\text{where } [\mathcal{G}](a) := \mathcal{G}(a 1_{X_j}) \text{ wherever } a \in L^Y.
\end{aligned}
$$

Lemma 2.6 *Each object in SL-K-CS is the quotient of a disjoint union of topological objects in SL-CS.*

Proof: Given $(X, \bar{q}) \in |\text{SL-K-CS}|$, $\mathcal{F} \xrightarrow{q_\alpha} x$, denote $J = \{j : j = (\mathcal{F}, x, \alpha), \mathcal{F} \xrightarrow{q_\alpha} x\}$ and define $(X_j, \bar{q}_j)$ as in (*), where the X_j's are disjoint copies of X. Let $(Y, \bar{p}) = \bigcup_{j \in J} (X_j, \bar{q}_j)$ denote the disjoint union defined in (**). Define $h : Y \to X$ to be the natural map, and observe that $h : (Y, \bar{p}) \to (X, \bar{q})$ is continuous. Indeed, if $\mathcal{H} \xrightarrow{p_\alpha} y, y \in X_j$, then there exists $\mathcal{G} \in \mathfrak{F}_{SL}(X_j)$ such that $\mathcal{G} \xrightarrow{q_{j\alpha}} y$ and $\mathcal{H} \geq [\mathcal{G}]$. Let $a \in L^X$; then $h^{\rightarrow}\mathcal{H}(a) = \mathcal{H}(a \circ h) \geq [\mathcal{G}](a \circ h) = \mathcal{G}(a)$, and thus $h^{\rightarrow}\mathcal{H} \geq \mathcal{G} \xrightarrow{q_\alpha} y$. Therefore, $h : (Y, \bar{p}) \to (X, \bar{q})$ is continous. Moreover, if $\mathcal{F} \xrightarrow{q_\alpha} x$, then $[\mathcal{F}] \xrightarrow{p_\alpha} x$ and $h^{\rightarrow}\big([\mathcal{F}]\big) = \mathcal{F}$. Hence h is a quotient map and each $(X_j, \bar{q}_j)$ is topological by Lemma 2.5. ■

Lemma 2.7 *Suppose that $(Y, \bar{p}) = \bigcup_{j \in J} (X_j, \bar{q}_j)$ is the disjoint union of objects as defined in (2.3). If each $(X_j, \bar{q}_j)$ obeys axiom D, then $(Y, \bar{p})$ also obeys axiom D.*

Proof: Assume that $\sigma : Y \to \mathfrak{F}_{SL}(Y)$ satisfies $\sigma(y) \xrightarrow{p_\alpha} y$ for each $y \in Y$ and $\mathcal{H} \xrightarrow{p_\alpha} x, x \in X_j$. Then there exists $\mathcal{F} \xrightarrow{q_{j\alpha}} x$ such that $\mathcal{H} \geq [\mathcal{F}]$ and it remains to show that $G(\sigma^{\rightarrow}\mathcal{H}) \xrightarrow{p_\alpha} x$. Since $\sigma(y) \xrightarrow{p_\alpha} y$ for each $y \in Y$, there exists $\mathcal{G}_y \in \mathfrak{F}_{SL}(X_j)$ such that $\mathcal{G}_y \xrightarrow{q_{j\alpha}} y$ and $\sigma(y) \geq [\mathcal{G}_y]$ whenever $y \in X_j$. Define $\Sigma_j : X_j \to \mathcal{F}_{SL}(X_j)$ by $\Sigma_j(y) = \mathcal{G}_y$ for each $y \in X_j$. Since $(X_j, \bar{q}_j)$ obeys axiom D, it suffices to prove that $G\big(\sigma^{\rightarrow}[\mathcal{F}]\big) \geq \big[G_j(\Sigma_j^{\rightarrow}\mathcal{F})\big]$, where G_j denotes the compression operator for X_j. Fix $a \in L^Y$, and let $b = a1_{X_j} \in L^{X_j}$. Then for $y \in X_j$, $(e_a \circ \sigma)(y) = \sigma(y)(a) \geq \big[\Sigma_j(y)\big](a) = \Sigma_j(y)(b) = (e_b \circ \Sigma_j)(y)$ and thus $(e_a \circ \sigma)1_{X_j} \geq e_b \circ \Sigma_j$. It follows that $G\big(\sigma^{\rightarrow}[\mathcal{F}]\big)(a) = [\mathcal{F}](e_a \circ \sigma) = \mathcal{F}(e_a \circ \sigma)1_{X_j} \geq \mathcal{F}(e_b \circ \Sigma_j) = G_j(\Sigma_j^{\rightarrow}\mathcal{F})(b) = \big[G_j(\Sigma_j^{\rightarrow}\mathcal{F})\big](a)$ for each $a \in L^Y$. Hence, $G(\sigma^{\rightarrow}\mathcal{H}) \geq G\big(\sigma^{\rightarrow}[\mathcal{F}]\big) \geq \big[G_j(\Sigma_j^{\rightarrow}\mathcal{F})\big] \xrightarrow{p_\alpha} x$ since $G_j(\Sigma_j^{\rightarrow}\mathcal{F}) \xrightarrow{q_{j\alpha}} x$. Therefore, $(Y, \bar{p})$ obeys axiom D. ■

Since an object $(X, \bar{q}) \in |\text{SL-P-CS}|$ is topological iff it obeys axiom D, Theorem 2.4, and Lemmas 2.5-2.6 imply the following result.

Theorem 2.5 *Assume that $(X, \bar{q}) \in |SL\text{-}K\text{-}CS|$. Then $(X, \bar{q})$ is the quotient of a topological object in SL-CS.*

Given a set X, let $\mathcal{C} = \big\{\bar{q} : (X, \bar{q}) \in |\text{SL-CS}|\big\}$. Define $\bar{q} \leq \bar{p}$ iff $\mathcal{F} \xrightarrow{p_\alpha} x$ implies that $\mathcal{F} \xrightarrow{q_\alpha} x, x \in X, \alpha \in L$. Then $(\mathcal{C}, \leq)$ is a complete lattice. Indeed, $\bigvee_{j \in J} \bar{p}_j = \bar{p}$ $(\bigwedge_{j \in J} \bar{q}_j = \bar{q})$, where $\mathcal{F} \xrightarrow{p_\alpha} x$ iff $\mathcal{F} \xrightarrow{p_{j\alpha}} x$ for each $j \in J$ ($\mathcal{F} \xrightarrow{q_\alpha} x$ iff there exists $j \in J$ such that $\mathcal{F} \xrightarrow{q_{j\alpha}} x$), respectively. Fix $(X, \bar{q}) \in |\text{SL-K-CS}|$ and denote $J = \big\{j : j = (\mathcal{F}, x, \alpha), \mathcal{F} \xrightarrow{q_\alpha} x\big\}$. According to Lemma 2.5, $(X, \bar{q}_j)$ as defined in (2.2) is topological and, by construction, $\bar{q} = \bigwedge_{j \in J} \bar{q}_j$, as stated below.

Corollary 2.1 *Suppose that $(X, \bar{q}) \in |SL\text{-}K\text{-}CS|$. Then $\bar{q}$ is the infimum of a collection of topological structures in SL-CS.*

Finally, let SL-TOP denote the full subcategory of SL-CS consisting of all the topological objects. It can be shown that SL-TOP is bireflective in SL-CS. Given $(X, \bar{q}) \in |\text{SL-CS}|$, let $(X, T\bar{q})$ denote the bireflection of $(X, \bar{q})$ in SL-TOP. Hence, the result below follows from categorical properties; for example, see Preuss (Theorem 2.2.12 [23]).

Theorem 2.6 *Assume that $f : (X, \bar{q}) \to (Y, \bar{p})$ is a quotient map in SL-CS. Then, using the notation above, $f : (X, T\bar{q}) \to (Y, T\bar{p})$ is also a quotient map in SL-TOP.*

CHAPTER 3: LATTICE-VALUED INTERIOR OPERATORS

3.1 Preliminaries

Definition 3.1 *(Jäger [10]). The pair* (X, J)*, where* $J : L^X \to L^X$*, is called a* **stratified L-interior space** *whenever it obeys:*

(J1) $\alpha 1_X \leq J(\alpha 1_X)$ *for each* $\alpha \in L$

(J2) $J(a) \leq a$ *for each* $a \in L^X$

(J3) $J(a) \leq J(b)$ *whenever* $a \leq b$

(J4) $J(a) \wedge J(b) \leq J(a \wedge b)$.

A map $f : (X, I) \to (Y, J)$ between two stratified L-interior spaces called **continuous** provided $J(b)(f(x)) \leq I(f^{\leftarrow}(b))(x)$ for each $x \in X$ and $b \in L^Y$. Denote by **SL-INT** the category whose objects consists of all the stratified L-interior spaces and whose morphisms are all the continuous maps between objects. Jäger [10] showed that SL-INT and SL-PCS are isomorphic but are not isomorphic to the category SL-P-CS introduced by Flores et al.[3]. Objects in SL-PCS are the pretopological objects in SL-FCS as defined by Jäger [10]. A suitable interior operator for objects in SL-P-CS is given in the next section.

3.2 Fuzzy Interior Operators : Pretopological

Höhle and Sostak ([7], p. 233) give the axioms needed for an "L-fuzzy interior operator" to characterize an "L-fuzzy topological space." A less restrictive interior operator is needed in this section.

Definition 3.2 *The pair* (X, I) *is called a* **stratified L-fuzzy interior space** *whenever* $I : L^X \times L \to L^X$ *obeys:*

(I1) $\beta 1_X \leq I(\beta 1_X, \alpha)$ *for each* $\alpha \in L$

(I2) $I(a,\alpha) \leq a$ *for each* $a \in L^X, \alpha \in L$

(I3) $I(a,\beta) \leq I(b,\alpha)$ *whenever* $a \leq b$ *and* $\alpha \leq \beta$

(I4) $I(a,\alpha) \wedge I(b,\alpha) \leq I(a \wedge b,\alpha)$.

Observe that (I3) implies that equality holds in (I4). A map $f : (X,I) \to (Y,J)$ between two stratified fuzzy L-interior spaces is said to be **continuous** whenever $J(b,\alpha)(f(x)) \leq I(f^{\leftarrow}(b),\alpha)(x)$ for each $x \in X$, $b \in L^Y$, and $\alpha \in L$. Note that if $f : (X,I) \to (Y,J)$ and $g : (Y,J) \to (Z,K)$ are each continuous, $x \in X$, $\alpha \in L$ and $b \in L^Z$, then $K(b,\alpha)(g(f(x))) \leq J(g^{\leftarrow}(b),\alpha)(f(x)) \leq I(f^{\leftarrow}(g^{\leftarrow}(b),\alpha))(x) = I((g \circ f)^{\leftarrow},\alpha)(x)$, and hence $g \circ f : (X,I) \to (Z,K)$ is continuous. Let **SL-FINT** denote the category whose objects are all the stratified L-fuzzy interior spaces and having all the continuous maps between objects as morphisms. Given set Y, the powerset is denoted by $P(Y)$.

Definition 3.3 *The pair* $(X,\overline{q})$, $\overline{q} = (q_\alpha)_{\alpha \in L}$, *is called a* **stratified L-pretopological space** *provided* $q_\alpha : X \to P(\mathcal{F}_{SL}(X))$ *obeys :*

(PS1) $[x] \xrightarrow{q_\alpha} x$ *(that is,* $[x] \in q_\alpha(x)$ *for each* $x \in X$, $\alpha \in L$*)*

(PS2) $\mathcal{G} \geq \mathcal{F} \xrightarrow{q_\alpha} x$ *implies* $\mathcal{G} \xrightarrow{q_\alpha} x$

(PS3) $\mathcal{U}_{q_\alpha}(x) := \wedge\{\mathcal{F} \in \mathfrak{F}_{SL}(X) : \mathcal{F} \xrightarrow{q_\alpha} x\} \xrightarrow{q_\alpha} x$

(PS4) $\mathcal{F} \xrightarrow{q_\alpha} x$ *and* $\alpha \leq \beta$ *implies* $\mathcal{F} \xrightarrow{q_\beta} x$.

The ordering in (PS4) is the reverse of that given in Flores et al. [3]. A map $f : (X,\overline{q}) \to (Y,\overline{p})$ between two stratified L-pretopological spaces is said to be **continuous** whenever $\mathcal{F} \xrightarrow{q_\alpha} x$ implies that $f^{\to}\mathcal{F} \xrightarrow{p_\alpha} f(x)$, for each $x \in X$ and $\alpha \in L$. Equivalently, $\mathcal{U}_{p_\alpha}(f(x)) \leq f^{\to}(\mathcal{U}_{q_\alpha}(x))$ for each $x \in X$ and $\alpha \in L$. The composition of two continuous functions is again continuous. Denote by **SL-P-CS** the category whose objects consist of all the stratified L-pretopological spaces and whose morphisms are all the continuous maps between objects. The next result answers a question posed by Jäger [10].

Theorem 3.1 *The categories SL-P-CS and SL-FINT are isomorphic.*

PROOF: Denote θ : SL-P-CS $\to$ SL-FINT by $\theta(X, \overline{q}) = (X, I_{\overline{q}})$, $\overline{q} = (q_\alpha)_{\alpha \in L}$, where $I_{\overline{q}} : L^X \times L \to L^X$ is defined by $I_{\overline{q}}(a, \alpha)(x) := \mathcal{U}_{q_\alpha}(x)(a)$, for each $x \in X$, $\alpha \in L$, and $a \in L^X$. For sake of brevity, the above is written as : $I_{\overline{q}}(a, \alpha) = \mathcal{U}_{q_\alpha}(.)(a)$. First, it is shown that $(X, I_{\overline{q}}) \in$ |SL-FINT|. It follows from (F1) of Definition 1.7 that $\beta 1_X \leq \mathcal{U}_{q_\alpha}(.)(\beta 1_X) = I_{\overline{q}}(\beta 1_X, \alpha)$ and thus (I1) is satisfied. Since $[x] \overset{q_\alpha}{\to} x$, $\mathcal{U}_{q_\alpha}(x) \leq [x]$, and hence $I_{\overline{q}}(a, \alpha)(x) = \mathcal{U}_{q_\alpha}(x)(a) \leq [x](a) = a(x)$ for each $x \in X$. Therefore $I_{\overline{q}}(a, \alpha) \leq a$ and (I2) is valid. If $a \leq b$ and $\alpha \leq \beta$, then using (F2) of Definition 1.7 and (PS4), $I_{\overline{q}}(a, \beta) = \mathcal{U}_{q_\beta}(.)(a) \leq \mathcal{U}_{q_\alpha}(.)(b) = I_{\overline{q}}(b, \alpha)$ and thus (I3) holds. Finally, if a, $b \in L^X$ and $\alpha \in L$, then employing (F3) of Definition 1.7, $I_{\overline{q}}(a, \alpha) \wedge I_{\overline{q}}(b, \alpha) = \mathcal{U}q_\alpha(.)(a) \wedge \mathcal{U}q_\alpha(.)(b) \leq \mathcal{U}q_\alpha(.)(a \wedge b) = I_{\overline{q}}(a \wedge b, \alpha)$ and (I4) is satisfied. Hence $(X, I_{\overline{q}}) \in$ |SL-FINT|. Observe that θ : SL-P-CS $\to$ SL-FINT is a functor. Indeed, assume that $f : (X, \overline{q}) \to (Y, \overline{p})$ is continuous, $\theta(X, \overline{q}) = (X, I_{\overline{q}})$ and $\theta(Y, \overline{p}) = (Y, I_{\overline{p}})$. If $b \in L^Y$, $\alpha \in L$ and $x \in X$, then $I_{\overline{p}}(b, \alpha)(f(x)) = \mathcal{U}_{p_\alpha}(f(x))(b) \leq f^{\to}(\mathcal{U}_{q_\alpha}(x))(b) = \mathcal{U}_{q_\alpha}(x)(f^{\leftarrow}(b)) = I_{\overline{q}}(f^{\leftarrow}(b), \alpha)(x)$. Therefore $f : (X, I_{\overline{q}}) \to (Y, I_{\overline{p}})$ is continuous and thus θ is a functor.

Conversely, denote ψ : SL-FINT $\to$ SL-P-CS by $\psi(X, I) = (X, \overline{q}_I)$, $\overline{q}_I = (q_{\alpha,I})_{\alpha \in L}$, where $\mathcal{F} \overset{q_{\alpha,I}}{\to} x$ iff $I(a, \alpha)(x) \leq \mathcal{F}(a)$, for each $a \in L^X$. Employing (I1)-(I4), it is straightforward to show that for each fixed $x \in X$ and $\alpha \in L$, $I(., \alpha)(x) \in \mathfrak{F}_{SL}(X)$. Next, it is shown that $(X, \overline{q}_I) \in$ |SL-P-CS|. Using (I2), $I(a, \alpha)(x) \leq a(x) = [x](a)$ for each $a \in L^X$, and thus $[x] \overset{q_{\alpha,I}}{\to} x$ and (PS1) is satisfied. Verification of (PS2) and (PS3) follows from the definition of $q_{\alpha,I}$. Assume that $\alpha \leq \beta$ and $\mathcal{F} \overset{q_{\alpha,I}}{\to} x$. According to (I3), $I(a, \beta)(x) \leq I(a, \alpha)(x) \leq \mathcal{F}(a)$ for each $a \in L^X$, and thus $\mathcal{F} \overset{q_{\beta,I}}{\to} x$. Hence (PS4) is valid and $(X, \overline{q}_I) \in$ |SL-P-CS|. Moreover, ψ : SL-FINT $\to$ SL-P-CS is a functor. Indeed, suppose that $f : (X, I) \to (Y, J)$ is continuous, $\psi(X, I) = (X, \overline{q}_I)$ and $\psi(Y, J) = (Y, \overline{p}_J)$. Assume that $\mathcal{F} \overset{q_{\alpha,I}}{\to} x$; then $I(a, \alpha)(x) \leq \mathcal{F}(a)$ for each $a \in L^X$. Hence for each $b \in L^Y$, $J(b, \alpha)(f(x)) \leq I(f^{\leftarrow}(b), \alpha)(x) \leq \mathcal{F}(f^{\leftarrow}(b)) = f^{\to}\mathcal{F}(b)$, and it follows that $f^{\to}\mathcal{F} \overset{p_{\alpha,J}}{\to} f(x)$. Then $f : (X, \overline{q}_I) \to (Y, \overline{p}_J)$ is continuous, and thus ψ is a functor.

It follows from the definitions of θ and ψ that $\psi \circ \theta = \mathrm{id}_{\text{SL-P-CS}}$ and $\theta \circ \psi = \mathrm{id}_{\text{SL-FINT}}$. Hence θ : SL-P-CS $\to$ SL-FINT is an isomorphism. ■

3.3 Fuzzy Interior Operators : Topological

Objects in SL-FINT satisfying additional axioms are investigated in this section.

Definition 3.4 *The pair (X, τ) is called a* **stratified L-fuzzy topological space** *provided $\tau : L^X \to L$ obeys:*

(FT1) $\tau(1_\emptyset) = 1$ and $\beta \leq \tau(\beta 1_X)$, for each $\beta \in L$

(FT2) $\tau(a) \wedge \tau(b) \leq \tau(a \wedge b)$, $a, b \in L^X$

(FT3) $\bigwedge\limits_{j \in J} \tau(a_j) \leq \tau(\bigvee\limits_{j \in J} a_j)$, $a_j \in L^X$, $i \in J$.

A map $f : (X, \tau) \to (Y, \sigma)$ between two stratified L-fuzzy topological spaces is called **continuous** whenever $\sigma(b, \alpha) \leq \tau(f^{\leftarrow}(b), \alpha)$ is satisfied for each $\alpha \in L$ and $b \in L^Y$. Let **SL-FTOP** denote the category whose objects consist of all the stratified L-fuzzy topological spaces and whose morphisms are all the continuous maps between objects.

Definition 3.5 *Let* **SL-R-FINT (SL-S-FINT, SL-T-FINT, SL-U-FINT, SL-V-FINT)** *denote the full subcategory of SL-FINT whose objects (X, I) fulfill the additional property I5 (I6-I7,I5-I7,I7,I5 and I7), respectively, where*

(I5) $I(a, \alpha) = b$ for each $\alpha \in A \subseteq L$ implies that $I(a, \vee A) = b$, where $a, b \in L^X$

(I6) $I(a, \alpha) \leq I(I(a, \alpha), \alpha)$ for each $\alpha \in L$, $a \in L^X$

(I7) $I(a, 0) = a$ for each $a \in L^X$.

The reader is asked to refer to Höhle and Sostak ([7], Theorem 8.1.2) for the proof of the following result.

Theorem 3.2 *The categories SL-TOP and SL-T-FINT are isomorphic.*

Lemma 3.1 *Given $(X, I) \in |SL\text{-}FINT|$, $\alpha \in L$, and $a \in L^X$. There exists a largest $a_\alpha \in L^X$ satisfying $a_\alpha \leq a$ and $I(a_\alpha, \alpha) = a_\alpha$.*

Proof: Denote $C = \{c \in L^X : c \leq a, I(c, \alpha) = c\}$. Note that C is nonempty since $1_\emptyset \in C$. Let $b = \vee C$. If $c \in C$, then $c = I(c, \alpha) \leq I(b, \alpha)$ and thus $b \leq I(b, \alpha)$. According to (I2), $I(b, \alpha) \leq b$; hence $I(b, \alpha) = b$ and $b \leq a$ ■

Definition 3.6 *Let $(X, I) \in |SL\text{-}FINT|$. Define $I_* : L^X \times L \to L^X$ by*

$$I_*(a, \alpha) = \begin{cases} a_\alpha, & \alpha > 0 \\ a, & \alpha = 0 \end{cases}$$ *, where a_α is determined as in Lemma 3.1.*

The ordering of objects in SL-FINT is needed below. Given $(X, I) \in |\text{SL-FINT}|$, $I_*(a, \alpha) \leq I(a, \alpha)$ for each $a \in L^X$ and $\alpha \in L$ with $\alpha > 0$. This leads to the following definition.

Definition 3.7 *Assume $(X, I), (X, J) \in |SL\text{-}FINT|$. Then (X, I) is called* **coarser (almost coarser)** *than (X, J), denoted by $I \leq J$ ($I \lesssim J$), provided $I(a, \alpha) \leq J(a, \alpha)$ for each $a \in L^X$ and $\alpha \in L$ ($\alpha > 0$), respectively.*

Theorem 3.3 *Assume that $(X, I) \in |SL\text{-}FINT|$. Then*

(a) (X, I_*) is the finest object in SL-S-FINT which is almost coarser than (X, I)

(b) $(X, I_*) \in |SL\text{-}T\text{-}FINT|$ whenever $(X, I) \in |SL\text{-}R\text{-}FINT|$.

Proof: (a): First, it is shown that $(X, I_*) \in |\text{SL-S-FINT}|$. Since $(X, I) \in |\text{SL-FINT}|$, it follows from (I1) and (I2) that $I(\beta 1_X, \alpha) = \beta 1_X$ and thus $I_*(\beta 1_X, \alpha) = \beta 1_X$. Hence (X, I_*) obeys (I1). By definition $I_*(a, \alpha) \leq a$ and thus (X, I_*) satisfies (I2). Next, assume that $a \leq b$ and $\alpha \leq \beta$. Denote $I_*(a, \beta) = a_\beta$ and $I_*(b, \alpha) = b_\alpha$. Since (X, I) satisfies (I2) and (I3), $a_\beta = I_*(a, \beta) = I(a_\beta, \beta) \leq I(a_\beta, \alpha) \leq a_\beta$ and thus $I(a_\beta, \alpha) = a_\beta \leq a \leq b$. It follows from the definition of b_α that $a_\beta \leq b_\alpha$, and thus $I_*(a, \beta) = a_\beta \leq b_\alpha = I_*(b, \alpha)$. Hence (X, I_*) obeys (I3). Next, (X, I_*) satisfies (I4). Indeed , let $I_*(a, \alpha) = a_\alpha$, $I_*(b, \alpha) = b_\alpha$, and since (X, I) obeys (I3) and (I4), $a_\alpha \wedge b_\alpha = I(a_\alpha, \alpha) \wedge I(b_\alpha, \alpha) = I(a_\alpha \wedge b_\alpha, \alpha)$. Moreover, $a_\alpha \wedge b_\alpha \leq a \wedge b$, and thus it follows that $I_*(a,\alpha) \wedge I_*(b,\alpha) = a_\alpha \wedge b_\alpha \leq I_*(a \wedge b, \alpha)$. Hence (X, I_*) satisfies (I4). It is shown that (X, I_*) satisfies (I6). Let $I_*(a,\alpha) = a_\alpha$; then $I(a_\alpha, \alpha) = a_\alpha$ and thus $I_*(a_\alpha, \alpha) = a_\alpha$. Hence $I_*(I_*(a,\alpha),\alpha) = I_*(a_\alpha,\alpha) = a_\alpha = I_*(a,\alpha)$ and thus (X, I_*) obeys (I6). Finally, by definition, $I_*(a, 0) = a$ and therefore $(X, I_*) \in |\text{SL-S-FINT}|$.

Assume that $(X, J) \in |\text{SL-S-FINT}|$ and $J \lesssim I$. It is shown that $J \leq I_*$. Since (X, I) obeys (I2) and (X, J) obeys (I2) and (I6), $I(J(a,\alpha),\alpha) \leq J(a,\alpha) = J(J(a,\alpha),\alpha) \leq I(J(a,\alpha),\alpha)$ implies that $I(J(a,\alpha),\alpha) = J(a,\alpha) \leq a$, for each $\alpha > 0$. Hence by definition of I_*, $J(a,\alpha) \leq I_*(a,\alpha)$ for each $\alpha \in L$ and $a \in L^X$. Hence $J \leq I_*$ and thus (X, I_*) is the finest object in SL-S-FINT which is almost coarser than (X, I).

(b): Suppose that $(X, I) \in |\text{SL-R-FINT}|$. It is shown that (X, I_*) also satisfies (I5). Assume that $A \subseteq L$ such that $I_*(a,\alpha) = b$ for each $\alpha \in A$. Denote $I_*(a,\alpha) = a_\alpha = b$ for each $\alpha \in A$. Then $b = I_*(a,\alpha) = I(a_\alpha,\alpha) = I(b,\alpha)$ for each $\alpha \in A$. Since (X, I) obeys (I5), $b = I(b, \vee A) = I_*(b, \vee A)$. Using the fact that $b \leq a$ and that (X, I_*) obeys (I3), $b = I_*(b, \vee A) \leq I_*(a, \vee A) \leq I_*(a,\alpha) = b$ when $\alpha \in A$. Hence $I_*(a, \vee A) = b$ and thus (X, I_*) satisfies (I5). Employing part (a), it follows that $(X, I_*) \in |\text{SL-T-FINT}|$. ■

Theorem 3.4 *Assume that $(X, I), (Y, J) \in |SL\text{-}FINT|$ and $f : (X, I) \to (Y, J)$ is continuous. Then*

(a) $f : (X, I_*) \to (Y, J_*)$ is continuous

(b) $id_X : (X, I) \to (X, I_*)$ is continuous iff $(X, I) \in |SL\text{-}U\text{-}FINT|$

(c) SL-S-FINT (SL-T-FINT) is a bireflective subcategory of SL-U-FINT (SL-V-FINT), respectively.

PROOF: (a): It is shown that if $x \in X$, $\alpha \in L$, and $b \in L^Y$, then $J_*(b,\alpha)(f(x)) \leq I_*(f^{\leftarrow}(b),\alpha)(x)$. Indeed, denote $J_*(b,\alpha) = b_\alpha$, and by Lemma 3.1, b_α is the largest member of L^Y such that $b_\alpha \leq b$ and $J(b_\alpha,\alpha) = b_\alpha$, $\alpha > 0$. Employing the continuity of $f : (X, I) \to (Y, J)$ and the fact that (X, I) satisfies (I2), $f^{\leftarrow}(b_\alpha)(x) = b_\alpha(f(x)) = J(b_\alpha,\alpha)(f(x)) \leq I(f^{\leftarrow}(b_\alpha),\alpha)(x) \leq f^{\leftarrow}(b_\alpha)(x)$, for each $x \in X$. Hence $I(f^{\leftarrow}(b_\alpha),\alpha) = f^{\leftarrow}(b_\alpha)$, and since

$b_\alpha \leq b$ implies that $f^{\leftarrow}(b_\alpha) \leq f^{\leftarrow}(b)$, $f^{\leftarrow}(b_\alpha) \leq I_*(f^{\leftarrow}(b), \alpha)$. It follows that $J_*(b, \alpha)(f(x)) = b_\alpha(f(x)) = f^{\leftarrow}(b_\alpha)(x) \leq I_*(f^{\leftarrow}(b), \alpha)(x)$, and thus $f : (X, I_*) \to (Y, J_*)$ is continuous.

(b): Since $I_*(a, \alpha) \leq I(a, \alpha)$ for each $\alpha > 0$, $id_X : (X, I) \to (X, I_*)$ is continuous iff $I(a, 0) = a$ for each $a \in L^X$.

(c): Verification here follows from parts (a)-(b) above and Theorem 3.3. ■

The next result is a refinement of Theorem 3.4. The proof employs transfinite induction but the details are not provided here.

Definition 3.8 *Given that $(X, I) \in |SL\text{-}FINT|$. Define inductively , $I^\sigma : L^X \times L \to L^X$, for each ordinal $\sigma \geq 1$ as follows:*

$$I^\sigma(a, \alpha) = \begin{cases} I(I^{\sigma-1}(a, \alpha), \alpha), & \sigma - 1 \text{ exists, } \alpha > 0 \\ \bigwedge_{\delta < \sigma} I^\delta(a, \alpha), & \sigma \text{ a limit ordinal, } \alpha > 0 \\ a, & \alpha = 0 \end{cases} .$$

Theorem 3.5 *Suppose that $(X, I), (Y, J) \in |SL\text{-}FINT|$ and assume that $f : (X, I) \to (Y, J)$ is continuous. Then for $\sigma \geq 1$,*

(a) $(X, I^\sigma) \in |SL\text{-}FINT|$

(b) $I^\sigma = I_*$ and $J^\sigma = J_*$ whenever σ is sufficiently large

(c) $f : (X, I^\sigma) \to (Y, J^\sigma)$ is continuous.

It follows from Theorem 5.1-5.2 [3] that SL-P-CS is a topological category. It is shown in Theorem 3.1 above that SL-P-CS and SL-FINT are isomorphic; hence, quotient objects in SL-FINT exists. The exact form of quotient structures in SL-FINT is given below.

Theorem 3.6 *Assume that $(X, I) \in |SL\text{-}FINT|$ and $f : (X, I) \to Y$ is a surjection. Define $J : L^Y \times L \to L^Y$ as follows:*

$J(b, \alpha)(y) := \bigwedge_{x \in f^{-1}(y)} I(f^{\leftarrow}(b), \alpha)(x)$, where $b \in L^Y$ and $y \in Y$. Then

(a) $(Y, J) \in |SL\text{-}FINT|$

(b) $f : (X, I) \to (Y, J)$ is a quotient map in SL-FINT

(c) $f : (X, I_*) \to (Y, J_*)$ is a quotient map in SL-S-FINT.

PROOF: (a): It is straightforward to verify that axioms (I1)-(I4) are satisfied and thus $(Y, J) \in |\text{SL-FINT}|$.

(b): Since SL-FINT is a topological construct, according to Preuss ([23], Proposition 1.2.1.2) it suffices to show that J is the finest structure for Y such that $f : (X, I) \to (Y, J)$ is continuous. Suppose that $f : (X, I) \to (Y, K)$ is continuous in SL-FINT. Let $y \in Y$, $b \in L^Y$, and $\alpha \in L$. The continuity of f implies that $K(b, \alpha)(y) \leq I(f^{\leftarrow}(b), \alpha)(x)$ for each $x \in f^{-1}(y)$. Hence $K(b, \alpha)(y) \leq \bigwedge_{x \in f^{-1}(y)} I(f^{\leftarrow}(b), \alpha)(x) = J(b, \alpha)(y)$ for each $y \in Y$. Therefore, $K \leq J$ and thus $f : (X, I) \to (Y, J)$ is a quotient map in SL-FINT.

(c): Since $(X, I) \in |\text{SL-FINT}|$, define $I^* : L^X \times L \to L^X$ as follows: $I^*(a, \alpha) = \begin{cases} I(a, \alpha) & , \alpha > 0 \\ a & , \alpha = 0 \end{cases}$.
It is straightforward to check that $(X, I^*) \in |\text{SL-U-FINT}|$, $(I^*)_* = I_*$, and SL-U-FINT is a bicoreflective subcategory of SL-FINT. Since SL-FINT is topological, SL-U-FINT is also topological. According to the hypothesis, $f : (X, I) \to (Y, J)$ is a quotient map in SL-FINT, and it easily follows that $f : (X, I^*) \to (Y, J^*)$ is a quotient map in SL-U-FINT. By Theorem 3.4(c), SL-S-FINT is bireflective in SL-U-FINT, and thus it follows from Preuss ([23], Theorem 2.2.12) that $f : (X, (I^*)_*) \to (Y, (J^*)_*)$ is a quotient map in SL-S-FINT. Since $(I^*)_* = I_*$, $f : (X, I_*) \to (Y, J_*)$ is a quotient map in SL-S-FINT. ■

3.4 Examples

Let's conclude with an elementary example illustrating objects in SL-FINT that fail to satisfy various axioms (I5)-(I7).

Example 3.1 *Let $L = \{0, 1, \alpha, \beta\}$ denote a lattice of distinct elements with ordering : $0 \leq \alpha$, $\beta \leq 1$. Suppose that X is any set having at least two members. Define the following four structures for X, where $\delta \in L$ and $a \in L^X$:*

(a) $I(a,\delta) = \begin{cases} \bigwedge\limits_{x\in X} a(x).1_X & , \delta = 1 \\ a & , \delta = 0, \alpha, \beta \end{cases}$

(b) $K(a,\delta) = \begin{cases} \bigwedge\limits_{x\in X} a(x).1_X & , \delta = \alpha, 1 \\ a & , \delta = 0, \beta \end{cases}$

(c) $M(a,\delta) = \begin{cases} \bigwedge\limits_{x\in X} a(x).1_X & , \delta = \beta, 1 \\ a & , \delta = 0, \alpha \end{cases}$

(d) $N(a,\delta) = \bigwedge\limits_{x\in X} a(x).1_X$.

Observe that (X, I) obeys axioms (I1)-(I4) and axioms (I6)-(I7). Let $A = \{0, \alpha, \beta\} \subseteq L$. Choose a non-constant $a \in L^X$ and note that $I(a, \delta) = a$ for each $\delta \in A$. Since $\vee A = 1$, $I(a, \vee A) = \bigwedge\limits_{x\in X} a(x).1_X \neq a$, and thus $(X, I) \in |$SL-S-FINT$|$ but it fails to satisfy (I5). Moreover, note that (X, K), $(X, M) \in |$SL-T-FINT$|$ and each is coarser that (X, I). However, for each $\delta \in L$ and $a \in L^X$, $K(a, \delta) \vee M(a, \delta) = I(a, \delta)$. This implies that there fails to exist a finest $(X, H) \in |$SL-T-FINT$|$ which is coarser than (X, I), and hence SL-T-FINT is not a bireflective subcategory of SL-S-FINT. Finally, observe that (X, N) satisfies axioms (I1)-(I6) but fails to obey (I7) since $N(a, 0) = \bigwedge\limits_{x\in X} a(x).1_X \neq a$, whenever $a \in L^X$ is non-constant. ■

CHAPTER 4: LATTICE-VALUED CONVERGENCE GROUPS

4.1 Introduction

The notion of a lattice-valued convergence group and some separation properties are investigated in this chapter. The group operations of product and inversion are required to be continuous. Lemma 4.1 contains some elementary properties that will be utilized throughout this chapter. Let **GRP** denote the category whose objects consist of all groups and whose morphisms are all the homomorphisms between groups. In order to simplify the exposition, the same symbol "." will be used to denote all group multiplications. Suppose that $(X, .) \in |\mathrm{GRP}|$; define $\gamma : X \times X \to X$ by $\gamma(x, y) = x.y$ and $\psi : X \to X$ by $\psi(x) = x^{-1}$. Let $\mathcal{F}, \mathcal{G} \in \mathfrak{F}_{SL}(X)$, and denote $\mathcal{F}.\mathcal{G} := \gamma^{\rightarrow}(\mathcal{F} \times \mathcal{G})$ and $\mathcal{F}^{-1} := \psi^{\rightarrow}\mathcal{F}$. Recall that $\mathcal{F} \times \mathcal{G} := \pi_1^{\leftarrow}\mathcal{F} \vee \pi_2^{\leftarrow}\mathcal{G}$, and it is shown by Jäger ([8], p. 505) that $(\mathcal{F} \times \mathcal{G})(b) = \vee\{\mathcal{F}(a_1) \wedge \mathcal{G}(a_2) : a_1 \times a_2 \leq b\}$, where $(a_1 \times a_2)(s, t) := a_1(s) \wedge a_2(t)$. Moreover, if $b \in L^X$, then define $b^{-1} \in L^X$ by $b^{-1}(x) = b(x^{-1})$ for each $x \in X$.

Lemma 4.1 *Assume that $(X, .), (Y, .) \in |GRP|$ and let $f : (X, .) \to (Y, .)$ be a homomorphism. Let $\mathcal{F}, \mathcal{G} \in \mathfrak{F}_{SL}(X)$ and $b, c_1, c_2 \in L^X$. Then*

(a) $c_1 \times c_2 \leq b \circ \gamma$ *iff* $c_2^{-1} \times c_1^{-1} \leq b^{-1} \circ \gamma$

(b) $f \circ \gamma = \gamma \circ (f \times f)$

(c) $[x].[y] = [x.y]$

(d) $\mathcal{F}.[e] \leq \mathcal{F}$, $[e].\mathcal{F} \leq \mathcal{F}$

(e) $(\mathcal{F}.\mathcal{G})^{-1} = \mathcal{G}^{-1}.\mathcal{F}^{-1}$

(f) $f^{\rightarrow}(\mathcal{F}.\mathcal{G}) = f^{\rightarrow}\mathcal{F}.f^{\rightarrow}\mathcal{G}$

(g) $f^{\rightarrow}(\mathcal{F}^{-1}) = (f^{\rightarrow}\mathcal{F})^{-1}$.

Proof: Suppose that $c_1 \times c_2 \leq b \circ \gamma$. Then $(c_2^{-1} \times c_1^{-1})(x, y) = c_2^{-1}(x) \wedge c_1^{-1}(y) = c_1(y^{-1}) \times c_2(x^{-1}) = (c_1 \times c_2)(y^{-1}, x^{-1}) \leq b(y^{-1}.x^{-1}) = b^{-1}(x.y) = (b^{-1} \circ \gamma)(x, y)$. Hence $c_2^{-1} \times$

$c_1^{-1} \leq b^{-1} \circ \gamma$. Conversely, if $c_2^{-1} \times c_1^{-1} \leq b^{-1} \circ \gamma$, then $(c_1 \times c_2)(x,y) = c_1(x) \wedge c_2(y) = c_2^{-1}(y^{-1}) \wedge c_1^{-1}(x^{-1}) = (c_2^{-1} \times c_1^{-1})(y^{-1}, x^{-1}) \leq b^{-1}(y^{-1}.x^{-1}) = b(x.y) = (b \circ \gamma)(x,y)$. Hence $c_1 \times c_2 \leq b \circ \gamma$.

(b): Note that $[\gamma \circ (f \times f)](x,y) = \gamma(f(x), f(y)) = f(x).f(y) = f(x.y) = f(\gamma(x,y)) = (f \circ \gamma)(x,y)$, and thus $f \circ \gamma = \gamma \circ (f \times f)$.

(c): It follows that $([x].[y])(b) = (\gamma^{-1})^{\rightarrow}([x] \times [y])(b) = ([x] \times [y])(b \circ \gamma) = \vee\{[x](a_1) \wedge [y](a_2) : a_1 \times a_2 \leq b \circ \gamma\} = \vee\{a_1(x) \wedge a_2(y) : a_1 \times a_2 \leq b \circ \gamma\}$. Since $a_1(x) \wedge a_2(y) = (a_1 \times a_2)(x,y) \leq b(\gamma(x,y)) = b(x.y) = [x.y](b)$, $[x].[y] \leq [x.y]$. Conversely, choose $a_1 = b(x.y)1_x$ and $a_2 = 1_y$. Then $(a_1 \times a_2)(x,y) = a_1(x) \wedge a_2(y) = b(x.y) = (b \circ \gamma)(x,y)$, and thus $a_1 \times a_2 \leq b \circ \gamma$. Hence $([x].[y])(b) \geq a_1(x) \wedge a_2(y) = b(x.y) = [x.y](b)$. Therefore $[x].[y] \geq [x.y]$ and thus $[x].[y] = [x.y]$.

(d): Assume that $b \in L^X$. Then $(\mathcal{F}.[e])(b) = \gamma^{\rightarrow}(\mathcal{F} \times [e])(b) = (\mathcal{F} \times [e])(b \circ \gamma) = \vee\{\mathcal{F}(a_1) \wedge [e](a_2) : a_1 \times a_2 \leq b \circ \gamma\} = \vee\{\mathcal{F}(a_1) \wedge a_2(e) : a_1 \times a_2 \leq b \circ \gamma\}$. If $a_1 \times a_2 \leq b \circ \gamma$, then $(a_1 \times a_2)(x,y) \leq b(x.y)$ and thus $(a_1 \wedge a_2(e)1_X)(x) = a_1(x) \wedge a_2(e) = (a_1 \times a_2)(x,e) \leq b(x.e) = b(x)$, for each $x \in X$. Hence $a_1 \wedge a_2(e)1_X \leq b$ and $\mathcal{F}(a_1) \wedge a_2(e) \leq \mathcal{F}(a_1) \wedge \mathcal{F}(a_2(e)1_X) \leq \mathcal{F}(a_1 \wedge a_2(e)1_X) \leq \mathcal{F}(b)$. It follows from the above that $(\mathcal{F}.[e])(b) \leq \mathcal{F}(b)$, for each $b \in L^X$, and thus $\mathcal{F}.[e] \leq \mathcal{F}$. Likewise, $[e].\mathcal{F} \leq \mathcal{F}$.

(e): Observe that $(\mathcal{G}^{-1}.\mathcal{F}^{-1})(b) = \gamma^{\rightarrow}(\mathcal{G}^{-1} \times \mathcal{F}^{-1})(b) = (\mathcal{G}^{-1} \times \mathcal{F}^{-1})(b \circ \gamma) = \vee\{\mathcal{G}^{-1}(c_1) \wedge \mathcal{F}^{-1}(c_2) : c_1 \times c_2 \leq b \circ \gamma\} = \vee\{\mathcal{F}^{-1}(c_2) \wedge \mathcal{G}^{-1}(c_1) : c_1 \times c_2 \leq \gamma\} = \vee\{\mathcal{F}(c_2^{-1}) \wedge \mathcal{G}(c_1^{-1}) : c_1 \times c_2 \leq b \circ \gamma\}$. Using (a) above, $c_1 \times c_2 \leq b \circ \gamma$ iff $c_2^{-1} \times c_1^{-1} \leq b^{-1} \circ \gamma$, and thus $\vee\{\mathcal{F}(c_2^{-1}) \wedge \mathcal{G}(c_1^{-1}) : c_1 \times c_2 \leq b \circ \gamma\} = \vee\{\mathcal{F}(a_1) \wedge \mathcal{G}(a_2) : a_1 \times a_2 \leq b^{-1} \circ \gamma\}$. Hence $(\mathcal{G}^{-1}.\mathcal{F}^{-1})(b) = \vee\{\mathcal{F}(a_1) \wedge \mathcal{G}(a_2) : a_1 \times a_2 \leq b^{-1} \circ \gamma\} = (\mathcal{F} \times \mathcal{G})(b^{-1} \circ \gamma) = \gamma^{\rightarrow}(\mathcal{F} \times \mathcal{G})(b^{-1}) = (\mathcal{F}.\mathcal{G})(b^{-1}) = (\mathcal{F}.\mathcal{G})^{-1}(b)$, for each $b \in L^X$. Hence $(\mathcal{F}.\mathcal{G})^{-1} = \mathcal{G}^{-1}.\mathcal{F}^{-1}$.

(f): Employing (b) above, $f^{\rightarrow}(\mathcal{F}.\mathcal{G})(b) = f^{\rightarrow}[(\gamma^{\rightarrow}(\mathcal{F} \times \mathcal{G})](b) = (f \circ \gamma)^{\rightarrow}(\mathcal{F} \times \mathcal{G})(b) = [\gamma \circ (f \times f)]^{\rightarrow}(\mathcal{F} \times \mathcal{G})(b) = \gamma^{\rightarrow}[f^{\rightarrow}\mathcal{F} \times f^{\rightarrow}\mathcal{F}](b) = (f^{\rightarrow}\mathcal{F}.f^{\rightarrow}\mathcal{G})(b)$, for each $b \in L^Y$. Hence $f^{\rightarrow}(\mathcal{F}.\mathcal{G}) = f^{\rightarrow}\mathcal{F}.f^{\rightarrow}\mathcal{G}$.

(g): Let $b \in L^Y$ and note that $(b \circ f \circ \psi)(x) = b(f(x^{-1})) = b((f(x))^{-1}) = b^{-1}(f(x)) = (b^{-1} \circ f)(x)$, for each $x \in X$. Hence $b \circ f \circ \psi = b^{-1} \circ f$. Then $f^{\rightarrow}(\mathcal{F}^{-1})(b) = f^{\rightarrow}(\psi^{\rightarrow}\mathcal{F})(b) = (f \circ \psi)^{\rightarrow}\mathcal{F}(b) = \mathcal{F}(b \circ f \circ \psi) = \mathcal{F}(b^{-1} \circ f) = f^{\rightarrow}\mathcal{F}(b^{-1}) = (f^{\rightarrow}\mathcal{F})^{-1}(b)$, for each $b \in L^Y$. Therefore $f^{\rightarrow}(\mathcal{F}^{-1}) = (f^{\rightarrow}\mathcal{F})^{-1}$. ■

4.2 Stratified L-convergence groups

Let $(X, .) \in |\text{GRP}|$ and $(X, \bar{q}) \in |\text{SL-CS}|$. Then $(X, ., \bar{q})$ is called a **stratified L-convergence group** provided $\gamma : (X, ., \bar{q}) \times (X, ., \bar{q}) \to (X, ., \bar{q})$ and $\psi : (X, ., \bar{q}) \to (X, ., \bar{q})$ are each continuous. Moreover, **SL-CG** denotes the category whose objects consist of all the stratified L-convergence groups and whose morphisms are all the continuous homomorphisms between objects.

Theorem 4.1 *SL-CG is a topological category over GRP.*

Proof: First, fix $(X, .) \in |\text{GRP}|$. Then $\{(Y, ., \bar{q}) \in |\text{SL-CG}| : X = Y\}$ is a set. Next, assume that $X = \{x_0\}$. Suppose that $\mathcal{F} \in \mathfrak{F}_{SL}(X)$ and $a \in L^X$; then $a = \beta 1_X$, where $\beta = a(x_0)$, and thus $\mathcal{F}(a) = \mathcal{F}(\beta 1_X) \geq \beta = a(x_0) = [x_0](a)$. Hence $\mathcal{F} \geq [x_0]$ and $\mathcal{F} \xrightarrow{q_\alpha} x_0$, for each $\alpha \in L$. It follows that there is exactly one $(X, \bar{q}) \in |\text{SL-CS}|$ whenever $X = \{x_0\}$. Also, since $x_0 = e$, $\mathcal{F}.\mathcal{G} \geq [x_0].[x_0] = [x_0]$ and $\mathcal{F}^{-1} \geq [x_0]$. Hence $(X, ., \bar{q})$ is the only object in SL-CG provided $(X, .) \in |\text{GRP}|$ with X consisting of only the identity element.

Finally, it remains to show that SL-CG possesses initial structures. Consider the source $f_j : (X, .) \to (Y_j, ., \bar{q}_j)$, $j \in J$, where each f_j is a homomorphism and $(Y_j, ., \bar{q}_j) \in |\text{SL-CG}|$. Define $\mathcal{F} \xrightarrow{q_\alpha} x$ iff $f_j^{\rightarrow}\mathcal{F} \xrightarrow{q_{\alpha j}} f_j(x)$, for each $j \in J$, $\alpha \in L$, and denote $\bar{q} = (q_\alpha)_{\alpha \in L}$. Then $(X, \bar{q}) \in |\text{SL-SC}|$. It is shown that $(X, ., \bar{q}) \in |\text{SL-CG}|$. Suppose that $\mathcal{F} \xrightarrow{q_\alpha} x$ and $\mathcal{G} \xrightarrow{q_\alpha} z$. Then by Lemma 4.1 (f), $f_j^{\rightarrow}(\mathcal{F}.\mathcal{G}) = f_j^{\rightarrow}\mathcal{F}.f_j^{\rightarrow}\mathcal{G} \xrightarrow{q_{\alpha j}} f_j(x).f_j(z) = f_j(x.z)$, for each $j \in J$. Hence $\mathcal{F}.\mathcal{G} \xrightarrow{q_\alpha} x.z$. Similarly, by Lemma 4.1 (g), $f_j^{\rightarrow}(\mathcal{F}^{-1}) = (f_j^{\rightarrow}\mathcal{F})^{-1} \xrightarrow{q_{\alpha j}} (f_j(x))^{-1} = f_j(x^{-1})$, for each $j \in J$, and thus $\mathcal{F}^{-1} \xrightarrow{q_\alpha} x^{-1}$. Therefore $(X, ., \bar{q}) \in |\text{SL-CG}|$.

Since $\bar{q}$ is the initial structure for $f_j : X \to (Y_j, \bar{q}_j)$, $j \in J$, in SL-CS and $(X, ., \bar{q}) \in |\text{SL-CG}|$, it follows that SL-CG possesses initial structures and is topological over GRP. ■

Recall that $(X, \bar{q}) \in |\text{SL-CS}|$ is called α-**Hausdorff** provided that each $\mathcal{F} \in \mathfrak{F}_{SL}(X)$ q_α-converges to at most one element in X. Also, $x \in \mathrm{cl}_{q_\alpha} A$ means that there exists $\mathcal{F} \in \mathfrak{F}_{SL}(A)$ such that $[\mathcal{F}] := i^{\rightarrow}\mathcal{F} \xrightarrow{q_\alpha} x$, where $i : A \to X$ is the natural injection.

Theorem 4.2 *Assume that $(X, ., \bar{q}) \in |SL\text{-}CG|$. Then $(X, ., \bar{q})$ is α-Hausdorff iff $\{e\}$ is α-closed.*

PROOF: Suppose that $(X, ., \bar{q})$ is α-Hausdorff and $x \in \mathrm{cl}_{q_\alpha}\{e\}$. Then there exists $\mathcal{F} \in \mathfrak{F}_{SL}(\{e\})$ such that $[\mathcal{F}] = i^{\rightarrow}\mathcal{F} \xrightarrow{q_\alpha} x$, where $i : \{e\} \to X$, $i(e) = e$. Observe that $[\mathcal{F}] \geq [e]$. Indeed, if $b \in L^X$, then $[\mathcal{F}](b) = i^{\rightarrow}\mathcal{F}(b) = \mathcal{F}(b \circ i) = \mathcal{F}(b(e).1_{\{e\}}) \geq b(e) = [e](b)$. Hence $[\mathcal{F}] \geq [e]$ and thus $[\mathcal{F}] \xrightarrow{q_\alpha} e$. Since $(X, ., \bar{q})$ is α-Hausdorff, $x = e$ and thus $\{e\}$ is α-closed. Conversely, assume that $\{e\}$ is α-closed and $\mathcal{F} \xrightarrow{q_\alpha} x, z$. Then $\mathcal{H} = \mathcal{F}.\mathcal{F}^{-1} \xrightarrow{q_\alpha} x.z^{-1}$. It is shown that $i^{\leftarrow}\mathcal{H}$ exists, where $i : \{e\} \to X$, $i(e) = e$. Note that $\mathcal{H}(1_{\{e\}^c}) = (\mathcal{F}.\mathcal{F}^{-1})(1_{\{e\}^c}) = \gamma^{\rightarrow}(\mathcal{F} \times \mathcal{F}^{-1})(1_{\{e\}^c}) = (\mathcal{F} \times \mathcal{F}^{-1})(1_{\{e\}^c} \circ \gamma)$. Observe that $(1_{\{e\}^c} \circ \gamma)(s, t) = 1_{\{e\}^c}(s.t) = 1_B(s, t)$, where $B = \{(s, t) : t \neq s^{-1}\}$. Hence $1_{\{c\}^c} \circ \gamma = 1_B$, $(\mathcal{F} \times \mathcal{F}^{-1})(1_B) = \vee\{\mathcal{F}(a_1) \wedge \mathcal{F}^{-1}(a_2) : a_1 \times a_2 \leq 1_B\} = \vee\{\mathcal{F}(a_1 \wedge a_2^{-1}) : a_1 \times a_2 \leq 1_B\}$. Moreover $(a_1 \times a_2^{-1})(s, s) = a_1(s) \wedge a_2(s^{-1}) = (a_1 \times a_2)(s, s^{-1}) \leq 1_B(s, s^{-1}) = 0$, and thus $\mathcal{F}(a_1 \wedge a_2^{-1}) = 0$. Therefore $(\mathcal{F} \times \mathcal{F}^{-1})(1_B) = 0$ and $i^{\leftarrow}\mathcal{H}$ exists. It follows that $\mathcal{K} := i^{\rightarrow}(i^{\leftarrow}\mathcal{H}) \geq \mathcal{H}$, thus $\mathcal{K} \xrightarrow{q_\alpha} x.z^{-1}$. Then $x.z^{-1} \in \mathrm{cl}_{q_\alpha}\{e\}$ and hence $x = z$. Therefore $(X, ., \bar{q})$ is α-Hausdorff. ■

Theorem 4.3 *Suppose that $(X, ., \bar{q}) \in |SL\text{-}CG|$. Let $f : (X, ., \bar{q}) \to (Y, .)$ be an onto homomorphism, and choose $\bar{p}$ such that $f : (X, \bar{q}) \to (Y, \bar{p})$ is a quotient map in SL-CS. Then $f : (X, ., \bar{q}) \to (Y, ., \bar{p})$ is a quotient map in SL-CG.*

PROOF: It must be shown that γ_Y and ψ_Y are continuous. Assume that $\mathcal{G}_i \xrightarrow{p_\alpha} y_i$; then there exists $\mathcal{F}_i \xrightarrow{q_\alpha} x_i$ such that $f^{\rightarrow}\mathcal{F}_i = \mathcal{G}_i$, for $i = 1, 2$. Then $\mathcal{F}_1.\mathcal{F}_2 \xrightarrow{q_\alpha} x_1.x_2$, and it follows from Lemma 4.1(f) that $\gamma_Y^{\rightarrow}(\mathcal{G}_1 \times \mathcal{G}_2) = \mathcal{G}_1.\mathcal{G}_2 = f^{\rightarrow}\mathcal{F}_1.f^{\rightarrow}\mathcal{F}_2 = f^{\rightarrow}(\mathcal{F}_1.\mathcal{F}_2) \xrightarrow{p_\alpha}$

$f(x_1.x_2) = f(x_1).f(x_2) = y_1.y_2$. Hence γ_Y is continuous. Next, suppose that $\mathcal{G} \xrightarrow{p_\alpha} y$ and thus there exists $\mathcal{F} \xrightarrow{q_\alpha} x$ such that $f^{\rightarrow}\mathcal{F} = \mathcal{G}$. Then $\mathcal{F}^{-1} \xrightarrow{q_\alpha} x^{-1}$, and by Lemma 4.1(g), $\psi_Y^{\rightarrow}\mathcal{G} = \psi_Y(f^{\rightarrow}\mathcal{F}) = (f^{\rightarrow}\mathcal{F})^{-1} = f^{\rightarrow}(\mathcal{F}^{-1}) \xrightarrow{p_\alpha} f(x^{-1}) = (f(x))^{-1} = y^{-1}$. Therefore ψ is continuous, and $(Y, ., \bar{p}) \in |\text{SL-CG}|$. Since $f : (X, \bar{q}) \to (Y, \bar{p})$ is a quotient map in SL-CS, it follows that $f : (X, ., \bar{q}) \to (Y, ., \bar{p})$ is a quotient map in SL-CG. ■

Assume that $f : (X, \bar{q}) \to (Y, \bar{p})$ is a continuous surjection in SL-CS. Define $R = \{(x, z) \in X \times X : f(x) = f(z)\}$; then R is an equivalence relation on X, and let $\langle x \rangle := \{z \in X : (x, z) \in R\}$. Let $\theta : X \to X/R$ denote the canonical map $\theta(x) = \langle x \rangle$, $x \in X$. Moreover, $\bar{r} = (r_\alpha)_{\alpha \in L}$ denotes the quotient structure determined by $\theta : (X, \bar{q}) \to X/R$ in SL-CS. Define $f_R : X/R \to Y$ by $f_R(\langle x \rangle) = f(x)$; then f_R is well-defined, and f_R is a bijection.

Theorem 4.4 *Given the notations described above, let $f : (X, \bar{q}) \to (Y, \bar{p})$ be a continuous surjection. Then $f_R : (X/R, \bar{r}) \to (Y, \bar{p})$ is a homeomorphism iff f is a quotient map in SL-CS.*

Proof: Assume that $f : (X, \bar{q}) \to (Y, \bar{p})$ is a quotient map in SL-CS. It is shown that f_R is a homeomorphism . Suppose that $\mathcal{G} \xrightarrow{r_\alpha} \langle x \rangle$. Since θ is a quotient map, it follows from Theorem 2.1 that there exists $x_1 \in \langle x \rangle$ and $\mathcal{F} \xrightarrow{q_\alpha} x_1$, such that $\theta^{\rightarrow}\mathcal{F} = \mathcal{G}$. Hence $f_R^{\rightarrow}\mathcal{G} = f_R^{\rightarrow}(\theta^{\rightarrow}\mathcal{F}) = (f_R \circ \theta)^{\rightarrow}\mathcal{F} = f^{\rightarrow}\mathcal{F} \xrightarrow{p_\alpha} f(x_1) = f_R(\langle x \rangle)$, and thus f_R is continuous.
Next, it is shown that $f_R^{-1} : (Y, \bar{p}) \to (X/R, \bar{r})$ is continuous. Assume that $\mathcal{H} \xrightarrow{p_\alpha} y$. Since f is a quotient map, there exist $x \in f^{-1}(y)$ and $\mathcal{F} \xrightarrow{q_\alpha} x$ such that $f^{\rightarrow}\mathcal{F} = \mathcal{H}$. Denote $\mathcal{G} = \theta^{\rightarrow}\mathcal{F} \xrightarrow{r_\alpha} \langle x \rangle$ and thus $f_R^{\rightarrow}\mathcal{G} = f_R^{\rightarrow}(\theta^{\rightarrow}\mathcal{F}) = f^{\rightarrow}\mathcal{F} = \mathcal{H}$. Hence $(f_R^{-1})^{\rightarrow}\mathcal{H} = (f_R^{-1})^{\rightarrow}(f_R^{\rightarrow}\mathcal{G}) = (f_R^{-1} \circ f_R)^{\rightarrow}\mathcal{G} = \mathrm{id}_{X/R}^{\rightarrow}\mathcal{G} = \mathcal{G} \xrightarrow{r_\alpha} \langle x \rangle = f_R^{-1}(y)$. Therefore f_R is a homeomorphism.
Conversely, assume that f_R is a homeomorphism, and it is shown that f is a quotient map. Suppose that $\mathcal{G} \xrightarrow{p_\alpha} y$, and thus $\mathcal{H} = (f_R^{-1})^{\rightarrow}\mathcal{G} \xrightarrow{r_\alpha} f_R^{-1}(y) = \langle x \rangle$. Since θ is a quotient map, there exist $x_1 \in \langle x \rangle$ and $\mathcal{F} \xrightarrow{q_\alpha} x_1$, such that $\theta^{\rightarrow}\mathcal{F} = \mathcal{H}$. Hence $f^{\rightarrow}\mathcal{F} = (f_R \circ \theta)^{\rightarrow}\mathcal{F} = f_R^{\rightarrow}\mathcal{H} = f_R^{\rightarrow}[(f_R^{-1})^{\rightarrow}\mathcal{G}] = \mathrm{id}_Y^{\rightarrow}\mathcal{G} = \mathcal{G}$. Therefore f is a quotient map in SL-CS. ■

Lemma 4.2 *Assume that $f : (X, \bar{q}) \to (Y, \bar{p})$ is continuous in SL-CS. Then $f^{-1}(B)$ is α-closed in $(X, \bar{q})$ whenever B is α-closed in $(Y, \bar{p})$.*

PROOF: Denote $A = f^{-1}(B)$ and let $x \in \mathrm{cl}_{q_\alpha} A$. Then there exists $\mathcal{F} \in \mathfrak{F}_{SL}(A)$ such that $[\mathcal{F}] := i_A^{\to}\mathcal{F} \xrightarrow{q_\alpha} x$, where $i_A : A \to X$ is the natural injection. Define $f_A = f|A : A \to B$ and thus $f_A^{\to}\mathcal{F} \in \mathfrak{F}_{SL}(B)$. Observe that $[f_A^{\to}\mathcal{F}] = f^{\to}([\mathcal{F}])$. Indeed, if $b \in L^Y$, then $[f_A^{\to}\mathcal{F}](b) = f_A^{\to}\mathcal{F}(b|_B) = \mathcal{F}(b|_B \circ f_A) = \mathcal{F}((b \circ f)|_A) = [\mathcal{F}](b \circ f) = f^{\to}([\mathcal{F}])(b)$, and thus $[f_A^{\to}\mathcal{F}] = f^{\to}([\mathcal{F}]) \overset{p_\alpha}{=\!\!\longrightarrow} f(x)$. It follows that $f(x) \in \mathrm{cl}_{p_\alpha} B = B$, and hence $x \in f^{-1}(B) = A$. Therefore A is also α-closed. ■

Remark 4.1 *Suppose that $x_0 \in X$; define $\mathcal{H}(a) := a(x_0, x_0)$ for each $a \in L^{X \times X}$. Then $\mathcal{H} = [x_0] \times [x_0]$. Indeed, Jäger ([8], p. 505) shows that $(\mathcal{F} \times \mathcal{G})(a) = \vee\{\mathcal{F}(a_1) \wedge \mathcal{G}(a_2) : a_1 \times a_2 \leq a\}$, where $(a_1 \times a_2)(s, t) := a_1(s) \wedge a_2(t)$. Note that if $a_1 \times a_2 \leq a$, then $[x_0](a_1) \wedge [x_0](a_2) = a_1(x_0) \wedge a_2(x_0) = (a_1 \times a_2)(x_0, x_0) \leq a(x_0, x_0) = \mathcal{H}(a)$. Hence $[x_0] \times [x_0] \leq \mathcal{H}$. Conversely, define $a_1 = a_2 := a(x_0, x_0)1_{\{x_0\}}$; then $(a_1 \times a_2)(s, t) \leq a(s, t)$. Hence $a_1 \times a_2 \leq a$, $[x_0](a_1) \wedge [x_0](a_2) = a(x_0, x_0)$, and thus $\mathcal{H} = [x_0] \times [x_0]$.*

Lemma 4.3 *Assume that $f : (X, \bar{q}) \to (Y, \bar{p})$ is a quotient map in SL-CS and let $y \in cl_{p_\alpha} B$. Then there exists $x \in f^{-1}(y)$ such that $x \in cl_{q_\alpha} f^{-1}(B)$.*

PROOF: Let $A = f^{-1}(B)$ and $y \in \mathrm{cl}_{p_\alpha} B$. There exists $\mathcal{G} \in \mathfrak{F}_{SL}(B)$ such that $i_B^{\to}\mathcal{G} = [\mathcal{G}] \xrightarrow{p_\alpha} y$. Since f is a quotient map , there exist $x \in f^{-1}(y)$ and $\mathcal{F} \xrightarrow{q_\alpha} x$ for which $f^{\to}\mathcal{F} = [\mathcal{G}]$. Since $i_B^{\leftarrow}([\mathcal{G}])$ exists, $[\mathcal{G}](1_{B^c}) = 0$. Hence $0 = [\mathcal{G}](1_{B^c}) = f^{\to}\mathcal{F}(1_{B^c}) = \mathcal{F}(f^{\leftarrow}(1_{B^c})) = \mathcal{F}(1_{f^{-1}(B^c)}) = \mathcal{F}(1_{A^c})$, and thus $i_A^{\leftarrow}\mathcal{F}$ exists. Hence $i_A^{\to}(i_A^{\leftarrow}\mathcal{F}) \geq \mathcal{F} \xrightarrow{q_\alpha} x$ and thus $x \in \mathrm{cl}_{q_\alpha} A$. ■

An object $(X, \bar{q}) \in |\text{SL-CS}|$ is called $\alpha - \mathbf{T}_1$ provided $[x] \xrightarrow{q_\alpha} z$ implies $x = z$.

Theorem 4.5 *Using the notation defined earlier, assume that $f : (X, \bar{q}) \to (Y, \bar{p})$ is a continuous surjection, and let $R = \{(x, z) \in X \times X : f(x) = f(z)\}$. Then* *(a) R is α-closed provided $(X/R, \bar{r})$ is α-Hausdorff*

(b) R is α-closed whenever $(Y, \bar{p})$ is α-Hausdorff

(c) $(X/R, \bar{r})$ is $\alpha - T_1$ whenever R is α-closed.

Proof: (a): Let $\langle x\rangle = \{z \in X : (x,z) \in R\}$, and define $\Delta = \{(\langle x\rangle, \langle x\rangle) : x \in X\}$. Note that $(\theta \times \theta)^{-1}(\Delta) = R$. Since $\theta \times \theta : (X \times X, \bar{q} \times \bar{q}) \to (X/R \times X/R, \bar{r} \times \bar{r})$ is continuous, it follows from Lemma 4.2 that $(\theta \times \theta)^{-1}(\Delta) = R$ is α-closed provided that Δ is α-closed in $(X/R \times X/R, \bar{r} \times \bar{r})$. It is shown that Δ is α-closed. Indeed, assume that $(\langle s\rangle, \langle t\rangle) \in \mathrm{cl}_{r_\alpha \times r_\alpha}\Delta$. Then there exists $\mathcal{H} \in \mathcal{F}_{SL}(\Delta)$ such that $[\mathcal{H}] = i_\Delta^{\rightarrow}\mathcal{H} \xrightarrow{r_\alpha \times r_\alpha} (\langle s\rangle, \langle t\rangle)$. Observe that $\pi_1 \circ i_\Delta = \pi_2 \circ i_\Delta : \Delta \to X/R$, and thus $(\pi_1 \circ i_\Delta)^{\rightarrow}\mathcal{H} = (\pi_2 \circ i_\Delta)^{\rightarrow}\mathcal{H} = \pi_1^{\rightarrow}([\mathcal{H}]) = \pi_2^{\rightarrow}([\mathcal{H}]) \xrightarrow{r_\alpha} \langle s\rangle, \langle t\rangle$. Since $(X/R, \bar{r})$ is α-Hausdorff, $\langle s\rangle = \langle t\rangle$, and thus Δ is α-closed. Therefore R is α-closed by Lemma 4.2.

(b): Assume that $(Y, \bar{p})$ is α-Hausdorff and $\mathcal{H} \xrightarrow{r_\alpha} \langle x\rangle, \langle z\rangle$. Then there exist $x_1 \in \langle x\rangle$, $z_1 \in \langle z\rangle$, $\mathcal{F} \xrightarrow{q_\alpha} x_1$, $\mathcal{G} \xrightarrow{q_\alpha} z_1$, such that $\theta^{\rightarrow}\mathcal{F} = \theta^{\rightarrow}\mathcal{G} = \mathcal{H}$. Since $f_R^{\rightarrow}\mathcal{H} = f_R^{\rightarrow}(\theta^{\rightarrow}\mathcal{F}) = f^{\rightarrow}\mathcal{F} \xrightarrow{p_\alpha} f(x_1)$ and $f_R^{\rightarrow}\mathcal{H} = f_R^{\rightarrow}(\theta^{\rightarrow}\mathcal{G}) = f^{\rightarrow}\mathcal{G} \xrightarrow{p_\alpha} f(z_1)$, it follows that $f(x_1) = f(z_1)$. Hence $\langle x\rangle = \langle z\rangle$ and thus $(X/R, \bar{r})$ is α-Hausdorff, and by (a), R is α-closed.

(c): Suppose that R is α-closed and $[\langle x\rangle] \xrightarrow{r_\alpha} \langle z\rangle$. Again, let $\Delta = \{(\langle x\rangle, \langle x\rangle) : x \in X\}$ and define $\mathcal{H} \in \mathfrak{F}_{SL}(\Delta)$ by $\mathcal{H}(a) = a(\langle x\rangle, \langle x\rangle)$ for each $a \in L^\Delta$. it follows from Remark 4.1 that $i_\Delta^{\rightarrow}\mathcal{H} = [(\langle x\rangle, \langle x\rangle))] = [\langle x\rangle] \times [\langle x\rangle]$, and thus $i_\Delta^{\rightarrow}\mathcal{H} \xrightarrow{r_\alpha \times r_\alpha} (\langle x\rangle, \langle z\rangle)$. Then $(\langle x\rangle, \langle z\rangle) \in \mathrm{cl}_{r_\alpha \times r_\alpha}\Delta$. It is shown in Theorem 2.3 that the product of quotient maps is again a quotient map, $\theta \times \theta : (X \times X, \bar{q} \times \bar{q}) \to (X/R \times X/R, \bar{r} \times \bar{r})$ is a quotient map. Since $(\langle x\rangle, \langle z\rangle) \in \mathrm{cl}_{r_\alpha \times r_\alpha}\Delta$ and $(\theta \times \theta)^{-1}(\Delta) = R$, it follows from Lemma 4.3 that there exist $x_1 \in \langle x\rangle$ and $z_1 \in \langle z\rangle$ such that $(x_1, z_1) \in \mathrm{cl}_{q_\alpha \times \alpha}R$. Since R is α-closed, $(x_1, z_1) \in R$ and thus $\langle x\rangle = \langle z\rangle$. It follows that $(X/R, \bar{r})$ is $\alpha - \mathrm{T}_1$. ■

Now, suppose that $f : (X, .) \to (Y, .)$ is an onto homomorphism, and recall that $R = \{(x,z) \in X \times X : f(x) = f(z)\}$ and $\langle x\rangle := \{z \in X : (x,z) \in R\}$, define $\langle x\rangle.\langle y\rangle = \langle x.y\rangle$, and note that $(X/R, .) \in |\mathrm{GRP}|$. Let $\theta : X \to X/R$ be the canonical map $\theta(x) = \langle x \rangle$, and denote $f_R(\langle x \rangle) = f(x)$, $x \in X$

Theorem 4.6 *Given the notations described above, assume that $f : (X, ., \bar{q}) \to (Y, ., \bar{p})$ is a continuous onto homomorphism in SL-CG. Let $\bar{r}$ denote the quotient structure in SL-CS determined by $\theta : (X, \bar{q}) \to X/R$. Then*

(a) $f_R : (X/R, ., \bar{r}) \to (Y, ., \bar{p})$ is a continuous onto homomorphism

(b) f_R is a homeomorphism iff f is a quotient map in SL-CG

(c) R is α-closed in $(X \times X, \bar{q} \times \bar{q})$ iff $(X/R, ., \bar{r})$ is α-Hausdorff.

PROOF: According to Theorem 4.3, $\theta : (X, ., \bar{q}) \to (X/R, ., \bar{r})$ is a quotient map in SL-CG. Verification of parts (a)-(b) follows as in Theorem 4.4. Employing parts (a) and (c) of Theorem 4.5 along with Theorem 4.2 establishes part (c). ■

Let $(X, .) \in |\mathrm{GRP}|$ and $\mathfrak{C}_\alpha \subseteq \mathfrak{F}_{SL}(X)$, for each $\alpha \in L$. Consider the following conditions:

(c1) $[e] \in \mathfrak{C}_\alpha$ and $\mathfrak{C}_\alpha \subseteq \mathfrak{C}_\beta$ whenever $\beta \leq \alpha$

(c2) $\mathcal{G} \in \mathfrak{C}_\alpha$ if $\mathcal{G} \geq \mathcal{F} \in \mathfrak{C}_\alpha$

(c3) $\mathcal{F}.\mathcal{G} \in \mathfrak{C}_\alpha$ provided $\mathcal{F}, \mathcal{G} \in \mathfrak{C}_\alpha$

(c4) $\mathcal{F}^{-1} \in \mathfrak{C}_\alpha$ whenever $\mathcal{F} \in \mathfrak{C}_\alpha$

(c5) $[x].\mathcal{F}.[x^{-1}] \in \mathfrak{C}_\alpha$ if $\mathcal{F} \in \mathfrak{C}_\alpha$, $x \in X$.

Theorem 4.7 *Let $(X, .) \in |GRP|$ and assume that $\mathfrak{C}_\alpha$ obeys (c1)-(c5) above, for each $\alpha \in L$. Define $\mathcal{H} \xrightarrow{q_\alpha} x$ iff there exists $\mathcal{F} \in \mathfrak{C}_\alpha$ such that $\mathcal{H} \geq [x].\mathcal{F}$, and denote $\bar{q} = (q_\alpha)_{\alpha \in L}$. Then $(X, ., \bar{q}) \in |SL\text{-}CG|$. Conversely, if $(X, ., \bar{q}) \in |SL\text{-}CG|$ and $\mathfrak{C}_\alpha = \{\mathcal{F} : \mathcal{F} \xrightarrow{q_\alpha} e\}$, $\alpha \in L$, then $\mathfrak{C}_\alpha$ satisfies (c1)-(c5). Moreover, $(X, ., \bar{q})$ is the only object in SL-CG for which $\{\mathcal{F} : \mathcal{F} \xrightarrow{q_\alpha} e\}$ coincides with $\mathfrak{C}_\alpha$ obeying (c1)-(c5), for each $\alpha \in L$*

PROOF: Suppose that $\mathfrak{C}_\alpha$ obeys (c1)-(c5). First, it is shown that $(X, \bar{q}) \in |\mathrm{SL\text{-}CS}|$. According to Lemma 4.1 (c), $[x] = [x].[e]$ and since $[e] \in \mathfrak{C}_\alpha$, $[x] \xrightarrow{q_\alpha} x$ for each $\alpha \in L$. If $\mathcal{K} \geq \mathcal{H} \xrightarrow{q_\alpha} x$, then $\mathcal{K} \geq \mathcal{H} \geq [x].\mathcal{F}$ for some $\mathcal{F} \in \mathfrak{C}_\alpha$, and thus $\mathcal{K} \xrightarrow{q_\alpha} x$. Next, assume that $\beta \leq \alpha$ and $\mathcal{H} \xrightarrow{q_\alpha} x$. Then $\mathcal{H} \geq [x].\mathcal{F}$ for some $\mathcal{F} \in \mathfrak{C}_\alpha$, and by (c1), $\mathcal{F} \in \mathfrak{C}_\beta$. Hence $\mathcal{H} \xrightarrow{q_\beta} x$ and $(X, \bar{q}) \in |\text{SL-CS}|$.

It is shown that γ and ψ are continuous. Suppose that $\mathcal{H} \xrightarrow{q_\alpha} x$ and $\mathcal{K} \xrightarrow{q_\alpha} y$; then $\mathcal{H} \geq [y].\mathcal{G}$, for some $\mathcal{F}, \mathcal{G} \in \mathfrak{C}_\alpha$. Employing (c3) and (c5) along with Lemma 4.1 (a), $\mathcal{H}.\mathcal{K} \geq [x].\mathcal{F}.[y].\mathcal{G} = [x].\mathcal{F}.([y].\mathcal{G}.[y^{-1}]).[y] = [x].\mathcal{F}.\mathcal{G}_1.[y] = [x].\mathcal{G}_2.[y] = [x].[y].([y^{-1}].\mathcal{G}_2.[y]) = [x.y].\mathcal{G}_3$, for some $\mathcal{G}_i \in \mathfrak{C}_\alpha$, $i = 1, 2, 3$. Hence $\mathcal{H}.\mathcal{K} \xrightarrow{q_\alpha} [x.y]$, and γ is continuous. Moreover, using Lemma 4.1(e), (c4) and (c5), $\mathcal{H}^{-1} \geq ([x].\mathcal{F})^{-1} = \mathcal{F}^{-1}.[x^{-1}] = [x^{-1}].([x].\mathcal{F}^{-1}.[x^{-1}]) = [x^{-1}].\mathcal{F}_1$, $\mathcal{F}_1 \in \mathfrak{C}_\alpha$. Hence $\mathcal{H}^{-1} \xrightarrow{q_\alpha} [x^{-1}]$ and thus ψ is continuous. Therefore $(X, ., , \bar{q}) \in |\text{SL-CG}|$. Moreover, if $\mathcal{F} \in \mathfrak{C}_\alpha$, then by Lemma 4.1(d), $\mathcal{F} \geq [e].\mathcal{F}$ and thus $\mathcal{F} \xrightarrow{q_\alpha} x$. Conversely, suppose that $\mathcal{H} \xrightarrow{q_\alpha} e$. Then $\mathcal{H} \geq [e].\mathcal{F}$ for some $\mathcal{F} \in \mathfrak{C}_\alpha$. Using Lemma 4.1 (d), (c2) and (c5), $\mathcal{H} \geq \mathcal{H}.[e] \geq [e].\mathcal{F}.[e] = \mathcal{F}_1$ and thus $\mathcal{H} \in \mathfrak{C}_\alpha$. Therefore $\mathcal{G} \xrightarrow{q_\alpha} e$ iff $\mathcal{G} \in \mathfrak{C}_\alpha$.

Assume that $(X, ., \bar{p}) \in |\text{SL-CG}|$ and $\{\mathcal{F} : \mathcal{F} \xrightarrow{p_\alpha} e\} = \mathfrak{C}_\alpha$ for each $\alpha \in L$. It is shown that $\bar{q} = \bar{p}$. If $\mathcal{H} \xrightarrow{q_\alpha} x$, then there exists $\mathcal{F} \in \mathfrak{C}_\alpha$ such that $\mathcal{H} \geq [x].\mathcal{F}$. Since $\mathcal{F} \xrightarrow{p_\alpha} e$ and $[x].\mathcal{F} \xrightarrow{p_\alpha} x$, $\mathcal{H} \xrightarrow{p_\alpha} x$. Hence $\bar{q} \geq \bar{p}$. Conversely, suppose that $\mathcal{H} \xrightarrow{p_\alpha} x$. Then by Lemma 4.1 (c,d), $\mathcal{H} \geq [e].\mathcal{H} = [x].([x^{-1}].\mathcal{H})$. Since $[x^{-1}].\mathcal{H} \xrightarrow{p_\alpha} e$, $[x^{-1}].\mathcal{H} \in \mathfrak{C}_\alpha$ and thus $\mathcal{H} \xrightarrow{q_\alpha} x$. Therefore $\bar{p} \geq \bar{q}$ and thus $(X, ., \bar{q})$ is the only object in SL-CG for which $\mathcal{F} \xrightarrow{q_\alpha} x$ iff $\mathcal{F} \in \mathfrak{C}_\alpha$ which obeys (c1)-(c5), $\alpha \in L$. ■

Remark 4.2 *Assume that $(X, ., \bar{q}), (Y, ., \bar{p}) \in |SL\text{-}CG|$ and $f : (X, ., \bar{q}) \to (Y, ., \bar{p})$ is homomorphism that is continuous at the identity e_X. Then f is continuous. Indeed, if $\mathcal{H} \xrightarrow{q_\alpha} x$, then by Theorem 4.7, there exists $\mathcal{F} \xrightarrow{q_\alpha} e_X$ such that $\mathcal{H} \geq [x].\mathcal{F}$. According to Lemma 4.1(f), $f^{\rightarrow}\mathcal{H} \geq f^{\rightarrow}([x]).f^{\rightarrow}\mathcal{F} = [f(x)].f^{\rightarrow}\mathcal{F} \xrightarrow{p_\alpha} f(x).e_Y = f(x)$, and thus f is continuous.* ■

CHAPTER 5: CONVERGENCE SEMIGROUP ACTIONS: GENERALIZED QUOTIENTS

5.1 Preliminaries

Recall that unlike the category of all topological spaces, CONV is cartesian closed and thus has suitable function spaces. In particular, let (X, q), $(Y, p) \in |\mathrm{CONV}|$ and let $C(X, Y)$ denote the set of all continuous functions from X to Y. Define $\omega : (X, q) \times C(X, Y) \to (Y, p)$ to be the evaluation map given by $\omega(x, f) = f(x)$. There exists a coarsest convergence structure **c** on $C(X, Y)$ such that w is jointly continuous. More precisely, c is defined by : $\Phi \xrightarrow{c} f$ iff $w^{\to}(\mathcal{F} \times \Phi) \xrightarrow{p} f(x)$ whenever $\mathcal{F} \xrightarrow{q} x$. This compatibility between (X, q) and $(C(X, Y), c)$ is an example of a continuous action in CONV. Continuous actions which are invariant with respect to a convergence space property P are studied. Choices for P include : locally compact, locally bounded, regular, Choquet(pseudotopological), and first-countable. An object $(X, q) \in |\mathrm{CONV}|$ is said to be **locally compact** (**locally bounded**) if $\mathcal{F} \xrightarrow{q} x$ implies that $\mathcal{F}$ contains a compact (bounded) subset of X, respectively. A subset B of X is **bounded** provided that each ultrafilter containing B q-converges in X. Further, (X, q) is called **regular** (**Choquet**) provided $\mathrm{cl}_q\mathcal{F} \xrightarrow{q} x$ ($\mathcal{F} \xrightarrow{q} x$) whenever $\mathcal{F} \xrightarrow{q} x$ (each ultrafilter containing $\mathcal{F}$ q-converges to x), respectively. Here $\mathrm{cl}_q\mathcal{F}$ denotes the filter on X whose base is $\{\mathrm{cl}_q F : F \in \mathcal{F}\}$. Some authors use the term "pseudotopological space" for a Choquet space. Finally, (X, q) is said to be **first-countable** whenever $\mathcal{F} \xrightarrow{q} x$ implies the existence of a coarser filter on X having a countable base and q-converging to x.

Let **SG** denote the category whose objects consist of all the semigroups (with an identity element), and whose morphisms are all the homomorphisms between objects. Further, $(S, ., p)$ is said to be a **convergence semigroup** provided : $(S, .) \in |\mathrm{SG}|$, $(S, p) \in |\mathrm{CONV}|$, and $\gamma : (S, p) \times (S, p) \to (S, p)$ is continuous, where $\gamma(x, y) = x.y$. Let **CSG** be the category whose objects consist of all the convergence semigroups, and whose morphisms are all the continuous homomorphisms between objects.

5.2 Continuous Actions

An action of a semigroup on a topological space is used to define "generalized quotients" in [1]. Below is Rath's [24] definition of an action in the convergence space context. Let $(X, q) \in |\mathrm{CONV}|$, $(S, ., p) \in |\mathrm{CSG}|$, $\lambda : X \times S \to X$, and consider the following conditions :

(a1) $\lambda(x, e) = x$ for each $x \in X$ (e is the identity element)

(a2) $\lambda(\lambda(x, g), h) = \lambda(x, g.h)$ for each $x \in X, \quad g, h \in S$

(a3) $\lambda : (X, q) \times (S, ., p) \to (X, q)$ is continuous.

Then $(S, .)((S, ., p))$ is said to **act**(**act continuously**) on (X, q) whenever a1-a2 (a1-a3) are satisfied and, in this case, λ is called the **action** (**continuous action**), respectively. For sake of brevity, $(X, S) \in \mathbf{A}(\mathbf{AC})$ denotes the fact that $(S, ., p) \in |\mathrm{CSG}|)$ acts (acts continuously) on $(X, q) \in |\mathrm{CONV}|$, respectively. Moreover, $\boldsymbol{(X, S, \lambda)} \in \mathrm{A}$ indicates that the action is λ.

The notion of "generalized quotients" determined by commutative semigroup acting on a topological space is investigated in [1]. Elements of the semigroup in [1] are assumed to be injections on the given topological space.

Lemma 5.1 *([1]) Suppose that $(S, X, \lambda) \in A$, $(S, .)$ is commutative and $\lambda(., g) : X \to X$ is an injection, for each $g \in S$. Define $(x, g) \sim (y, h)$ on $X \times S$ iff $\lambda(x, h) = \lambda(y, g)$. Then $\sim$ is an equivalence relation on $X \times S$.*

In the context of Lemma 5.1, let $\langle (x, g) \rangle$ be the equivalence class containing (x, g), $\boldsymbol{B(X, S)}$ denote the quotient set $(X \times S)/\sim$, and define $\varphi : (X \times S, r) \to B(X, S)$ to be the canonical map, where $r = q \times p$ is the product convergence structure. Equip $B(X, S)$ with the convergence quotient structure $\boldsymbol{\sigma}$. Then $\mathcal{K} \xrightarrow{\sigma} \langle (y, h) \rangle$ iff there exist $(x, g) \sim (y, h)$ and $\mathcal{H} \xrightarrow{r} (x, g)$ such that $\varphi^{\to}\mathcal{H} = \mathcal{K}$. The space $(B(X, S), \sigma)$ is investigated in section 5.3.

Remark 5.1 *Fix a set X. the set of all convergence structures on X with the ordering $p \leq q$ defined in section 1.2 is a complete lattice. Indeed, if $(X, q_j) \in |CONV|$, $j \in J$, then $\sup_{j \in J} q_j = q^1$ is given by $\mathcal{F} \xrightarrow{q^1} x$ iff $\mathcal{F} \xrightarrow{q_j} x$, for each $j \in J$. Dually, $\inf_{j \in J} q_j = q^0$ is defined by $\mathcal{F} \xrightarrow{q^0} x$ iff $\mathcal{F} \xrightarrow{q_j} x$, for some $j \in J$. It is easily verified that if $((X, q_j), (S, ., p), \lambda) \in AC$ for each $j \in J$, then both $((X, q^1), (S, ., p), \lambda)$ and $((X, q^0), (S, ., p), \lambda)$ belong to AC.* ■

Theorem 5.1 *Assume that $((X, q), (S, ., p), \lambda) \in AC$. Then*

(a) there exists a finest convergence structure q^F on X such that $((X, q^F), (S, ., p), \lambda) \in AC$

(b) there exists a coarsest convergence structure p^c on S for which $((X, q), (S, ., p^c), \lambda) \in AC$

(c) $((B(X, S), \sigma), (S, ., p)) \in AC$ provided $(S, .)$ is commutative and $\lambda(., g)$ is an injection, for each $g \in S$.

Proof: (a): Define q^F as follows: $\mathcal{F} \xrightarrow{q^F} x$ iff there exist $z \in X$, $\mathcal{G} \xrightarrow{p} g$ such that $x = \lambda(z, g)$ and $\mathcal{F} \geq \lambda^{\to}(\dot{z} \times \mathcal{G})$. Then $(X, q^F) \in$ |CONV|. Indeed, $\dot{x} \xrightarrow{q^F} x$ since $x = \lambda(x, e)$ and $\dot{x} = \lambda^{\to}(\dot{x} \times \dot{e})$. Hence (CS1) is satisfied. Clearly (CS2) is valid, and $(X, q^F) \in$ |CONV|. It is shown that $\lambda : (X, q^F) \times (S, p) \to (X, q^F)$ is continuous. Suppose that $\mathcal{F} \xrightarrow{q^F} x$ and $\mathcal{H} \xrightarrow{p} h$; then there exist $z \in X$, $\mathcal{G} \xrightarrow{p} g$ such that $x = \lambda(z, g)$ and $\mathcal{F} \geq \lambda^{\to}(\dot{z} \times \mathcal{G})$. Hence, $\mathcal{F} \times \mathcal{H} \geq \lambda^{\to}(\dot{z} \times \mathcal{G}) \times \mathcal{H}$, and employing (a2), $\lambda^{\to}(\mathcal{F} \times \mathcal{H}) \geq \lambda^{\to}(\lambda^{\to}(\dot{z} \times \mathcal{G}) \times \mathcal{H}) = [\{\lambda(\{z\} \times G.H) : G \in \mathcal{G}, H \in \mathcal{H}\}] = \lambda^{\to}(\dot{z} \times \mathcal{G}.\mathcal{H})$. Since $\mathcal{G}.\mathcal{H} \xrightarrow{p} g.h$ and $\lambda(z, g.h) = \lambda(\lambda(z, g), h) = \lambda(x, h)$, it follows from the definition of q^F that $\lambda^{\to}(\mathcal{F} \times \mathcal{H}) \xrightarrow{q^F} \lambda(x, h)$. Hence $((X, q^F), (S, ., p), \lambda) \in$ AC.

Assume that $((X, s), (S, ., p), \lambda) \in$ AC. It is shown that $s \leq q^F$. Suppose that $\mathcal{F} \xrightarrow{q^F} x$; then there exist $z \in X$, $\mathcal{G} \xrightarrow{p} g$ such that $x = \lambda(z, g)$ and $\mathcal{F} \geq \lambda^{\to}(\dot{z} \times \mathcal{G})$. Since $\lambda^{\to}(\dot{z} \times \mathcal{G}) \xrightarrow{s} \lambda(z, g)$, it follows that $\mathcal{F} \xrightarrow{s} x$ and thus $s \leq q^F$. Hence q^F is the finest convergence structure on X such that $((X, q^F), (S, ., p), \lambda) \in$ AC.

(b): Define p^c as follows: $\mathcal{G} \xrightarrow{p^c} g$ iff for each $\mathcal{F} \xrightarrow{q} x$, $\lambda^{\to}(\mathcal{F} \times \mathcal{G}) \xrightarrow{q} \lambda(x, g)$. Then $(S, p^c) \in$ |CONV|. First, it is shown that $(S, ., p^c) \in$ |CSG|; that is, if $\mathcal{G} \xrightarrow{p^c} g$ and $\mathcal{H} \xrightarrow{p^c} h$, then $\mathcal{G}.\mathcal{H} \xrightarrow{p^c} g.h$. Assume that $\mathcal{F} \xrightarrow{q} x$; then using (a2), $\lambda^{\to}(\mathcal{F} \times \mathcal{G}.\mathcal{H}) =$

$[\{\lambda(F \times G.H) : F \in \mathcal{F}, G \in \mathcal{G}, H \in \mathcal{H}\}] = [\{\lambda(\lambda(F \times G) \times H) : F \in \mathcal{F}, G \in \mathcal{G}, H \in \mathcal{H}\}] = \lambda^{\rightarrow}(\lambda^{\rightarrow}(\mathcal{F} \times \mathcal{G}) \times \mathcal{H})$. It follows from the definition of p^c that $\lambda^{\rightarrow}(\mathcal{F} \times \mathcal{G}) \xrightarrow{q} \lambda(x, g)$, and thus $\lambda^{\rightarrow}(\lambda^{\rightarrow}(\mathcal{F} \times \mathcal{G}) \times \mathcal{H}) \xrightarrow{q} \lambda(\lambda(x, g), h) = \lambda(x, g.h)$. Hence $\mathcal{G}.\mathcal{H} \xrightarrow{p^c} g.h$, and thus $(S, ., p^c) \in |\mathrm{CSG}|$. According to the construction, p^c is the coarsest convergence structure on S such that $\lambda : (X, q) \times (S, p^c) \to (X, q)$ is continuous.

(c): Define $\lambda_B : (B(X, S), \sigma) \times (S, ., p) \to (B(X, S), \sigma)$ by $\lambda_B(\langle(x, g)\rangle, h) = \langle(x, g.h)\rangle$. It is shown that λ_B is a continuous action. Indeed, $\lambda_B(\langle(x, g)\rangle, e) = \langle(x, g)\rangle$, and $\lambda_B(\lambda_B(\langle(x, g)\rangle, h), k) = \lambda_B(\langle(x, g.h)\rangle, k) = \langle(x, g.h.k)\rangle = \lambda_B(\langle(x, g)\rangle, h.k)$. Hence λ_B is an action. It remains to show that λ_B is continuous. Suppose that $\mathcal{K} \xrightarrow{\sigma} \langle(x, g)\rangle$ and $\mathcal{L} \xrightarrow{p} l$. Since φ is a quotient map in CONV, there exists $\mathcal{H} \xrightarrow{r} (x_1, g_1) \sim (x, g)$ such that $\varphi^{\rightarrow}\mathcal{H} = \mathcal{K}$. Then $\lambda_B^{\rightarrow}(\mathcal{K} \times \mathcal{L}) = \lambda_B^{\rightarrow}(\varphi^{\rightarrow}\mathcal{H} \times \mathcal{L})$. Let $K \in \mathcal{K}$ and $L \in \mathcal{L}$, and note that $\lambda_B(\varphi(H) \times L) \subseteq \lambda_B(\varphi(\pi_1(H) \times \pi_2(H)) \times L) = \varphi(\pi_1(H) \times \pi_2(H).L)$. Hence $\lambda_B^{\rightarrow}(\varphi^{\rightarrow}\mathcal{H} \times \mathcal{L}) \geq \varphi^{\rightarrow}(\pi_1^{\rightarrow}\mathcal{H} \times \pi_2^{\rightarrow}\mathcal{H}.\mathcal{L}) \xrightarrow{\sigma} \varphi(x_1, g_1.l) = \langle(x_1, g_1.l)\rangle = \lambda_B(\langle(x_1, g_1)\rangle, l) = \lambda_B(\langle(x, g)\rangle, l)$. Therefore $(B(X, S), S, \lambda_B) \in$ AC. ■

Remark 5.2 *Let $(X, q) \in |CONV|$ and let $(C(X, X), c)$ denote the space defined in section 5.1. Since c is the coarsest convergence structure for which the evaluation map $\omega : (X, q) \times (C(X, X), c) \to (X, q)$ is continuous, this is a particular case of Theorem 5.1(b), where $\lambda = \omega$, $(S, ., p^c) = (C(X, X), ., c)$, and the group operation is composition. Moreover, it is well-known that, in general, there fails to exist a coarsest topology on $C(X, X)$ for which $\omega : (X, q) \times C(X, X) \to (X, q)$ is jointly continuous (even when q is a topology).* ■

Assume that $(X, S, \lambda) \in$ A; then λ is said to **distinguish elements in S** whenever $\lambda(x, g) = \lambda(x, h)$ for all $x \in X$ implies that $g = h$. In this case, define $\theta : S \to C(X, X)$ by $\theta(g)(x) = \lambda(x, g)$, for each $x \in X$. Note that θ is an injection iff λ separates elements in S. Moreover, θ is a homomorphism whenever the operation in $C(X, X)$, $k.l = l \circ k$, is composition.

Theorem 5.2 *Suppose that* $((X,q),(S,.,p),\lambda) \in AC$*, and assume that* λ *distinguishes elements in* S*. Then the following are equivalent:*

(a) $\theta : (S,p) \to (C(X,X),c)$ *is an embedding*

(b) $p = p^c$

(c) *if* $\mathcal{G} \stackrel{p}{\nrightarrow} g$*, then there exists* $\mathcal{F} \xrightarrow{q} x$ *such that* $\lambda^{\rightarrow}(\mathcal{F} \times \mathcal{G}) \stackrel{q}{\nrightarrow} \lambda(x,g)$*.*

Proof: (a) $\Rightarrow$ (b): Assume that $\theta : (S,p) \to (C(X,X),c)$ is an embedding. According to Theorem 5.1(b), $p^c \le p$. Suppose that $\mathcal{G} \xrightarrow{p^c} g$; then if $\mathcal{F} \xrightarrow{q} x$, $\lambda^{\rightarrow}(\mathcal{F} \times \mathcal{G}) \xrightarrow{q} \lambda(x,g)$. It is shown that $\theta^{\rightarrow}\mathcal{G} \xrightarrow{c} \theta(g)$. Indeed, note that $\omega^{\rightarrow}(\mathcal{F} \times \theta^{\rightarrow}\mathcal{G}) = [\{\omega(F \times \theta(G)) : F \in \mathcal{F}, G \in \mathcal{G}\}] = [\{\lambda(F \times G) : F \in \mathcal{F}, G \in \mathcal{G}\}] = \lambda^{\rightarrow}(\mathcal{F} \times \mathcal{G}) \xrightarrow{q} \lambda(x,g) = \omega(x,\theta(g))$. Hence $\theta^{\rightarrow}\mathcal{G} \xrightarrow{c} \theta(g)$, and thus $\mathcal{G} \xrightarrow{p} g$. Therefore $p = p^c$.

(b) $\Rightarrow$ (c): Verification follows directly from the definition of p^c.

(c) $\Rightarrow$ (a): Suppose that $\mathcal{G} \xrightarrow{p} g$ and $\mathcal{F} \xrightarrow{q} x$. Since $\lambda : (X,q) \times (S,p) \to (X,q)$ is continuous, $\lambda^{\rightarrow}(\mathcal{F} \times \mathcal{G}) \xrightarrow{q} \lambda(x,g)$. Hence $\omega^{\rightarrow}(\mathcal{F} \times \theta^{\rightarrow}\mathcal{G}) = \lambda^{\rightarrow}(\mathcal{F} \times \mathcal{G}) \xrightarrow{q} \lambda(x,g) = \omega(x,\theta(g))$, and thus $\theta^{\rightarrow}\mathcal{G} \xrightarrow{c} \theta(g)$. Conversely, if $\mathcal{G} \in \mathfrak{F}(S)$ such that $\theta^{\rightarrow}\mathcal{G} \xrightarrow{c} \theta(g)$, then the hypothesis implies that $\mathcal{G} \xrightarrow{p} g$. Hence $\theta : (S,p) \to (C(X,X),c)$ is an embedding. ■

Remark 5.3 *The map* θ *given in Theorem 5.2 is called a* **continuous representation** *of* $(S,.,p)$ *on* (X,q)*. Rath [24] discusses this concept in the context of a group with* $(C(X,X),.,c)$ *replaced by* $(H(X),.,\gamma)$*, where* $(H(X),.)$ *is the group of all homeomorphisms on* X *with composition as the group operation, and* γ *is the coarsest convergence structure making the operations of composition and inversion continuous.* ■

Quite often it is desirable to consider modifications of convergence structures. For example, given $(X,q) \in |\mathrm{CONV}|$, there exists a finest regular convergence structure on X which is coarser than q [13]. The notation $\boldsymbol{Pq}$ denotes the P-modification of q. Generally, P represents a convergence space property; however, it is convenient to include the case whenever $Pq = q$. Let **PCONV** denote the full subcategory of CONV consisting of all the objects in CONV that satisfy condition P. Condition P is said to be **finitely productive**(**productive**) provided that for each collection $(X_j, q_j) \in |\text{CONV}|$, $j \in J$, $P(\underset{j\in J}{\times} q_j) = \underset{j\in J}{\times} Pq_j$ whenever J is a finite (arbitrary) set, respectively.

Theorem 5.3 *Assume that F_P : CONV $\rightarrow$ PCONV is a functor obeying $F_P(X, q) = (X, Pq)$, $F_P(f) = f$, and suppose that P is finitely productive. If $((X, q), (S, ., p), \lambda) \in AC$ and $h : (T, ., \xi) \rightarrow (S, ., p)$ is a continuous homomorphism in CSG, then $((X, Pq), (T, ., P\xi)) \in AC$; in particular, $((X, Pq), (S, ., Pp), \lambda) \in AC$.*

Proof: Given that $((X, q), (S, ., p), \lambda) \in$ AC, define $\Lambda : (X, q) \times (T, \xi) \rightarrow (X, q)$ by $\Lambda(x, t) = \lambda(x, h(t))$. Clearly Λ is an action; moreover, Λ is continuous. Indeed, suppose that $\mathcal{F} \xrightarrow{q} x$ and $\mathcal{G} \xrightarrow{\xi} t$; then $\Lambda^{\rightarrow}(\mathcal{F} \times \mathcal{G}) = [\{\Lambda(F \times G) : F \in \mathcal{F}, G \in \mathcal{G}\}] = [\{\lambda(F \times h(G)) : F \in \mathcal{F}, G \in \mathcal{G}\}] = \lambda^{\rightarrow}(\mathcal{F} \times h^{\rightarrow}\mathcal{G}) \xrightarrow{q} \lambda(x, h(t)) = \Lambda(x, t)$. Therefore Λ is continuous.
Since F_P is a functor and P is finitely productive, continuity of the operation $\gamma : (T, ., \xi) \times (T, ., \xi) \rightarrow (T, ., \xi)$, defined by $\gamma(t_1, t_2) = t_1.t_2$, implies continuity of $\gamma : (T, ., P\xi) \times (T, ., P\xi) \rightarrow (T, ., P\xi)$. Hence $(T, ., P\xi) \in |\text{CSG}|$. Likewise, $\Lambda : (X, Pq) \times (T, P\xi) \rightarrow (X, Pq)$ is continuous, and thus $((X, Pq), (T, ., P\xi), \Lambda) \in$ AC. ■

Let $(S_j, ., p_j) \in |\text{CSG}|$, $j \in J$, and denote the product by $(S, ., p) = \underset{j\in J}{\times}(S_j, ., p_j)$. The **direct sum** of $(S_j, .)$, $j \in J$, is the subsemigroup of $(S, .)$ defined by $\underset{j\in J}{\oplus} S_j = \{(g_j) \in S : g_j = e_j$ for all but finitely many $j \in J\}$. Denote $\theta_j : S_j \rightarrow \underset{j\in J}{\oplus} S_j$ to be the map $\theta_j(g) = (g_k)$, where $g_j = g$ and $g_k = e_k$ whenever $k \neq j$, and let $\theta : \underset{j\in J}{\oplus} S_j \rightarrow \underset{j\in J}{\times} S_j$ be the inclusion map. Define $\mathcal{H} \xrightarrow{\eta} (g_j)$ in $\underset{j\in J}{\oplus} S_j$ iff $\mathcal{H} \geq \theta_{k_1}^{\rightarrow}\mathcal{G}_1.\theta_{k_2}^{\rightarrow}\mathcal{G}_2...\theta_{k_n}^{\rightarrow}\mathcal{G}_n$, where $\mathcal{G}_j \xrightarrow{p_{k_j}} g_{k_j}$ in $(S_{k_j}, ., p_{k_j})$ and $n \geq 1$. Then $(\underset{j\in J}{\oplus} S_j, ., \eta) \in |\text{CSG}|$, and $\theta : (\underset{j\in J}{\oplus} S_j, ., \eta) \rightarrow (S, ., p)$ is a continuous homomorphism.

Theorem 5.4 *Suppose that F_P : CONV $\rightarrow$ PCONV is a functor satisfying $F_P(X, q) = (X, Pq)$, $F_P(f) = f$, and P is productive. Assume that $((X_j, q_j), (S_j, ., p_j), \lambda_j) \in AC$ for each $j \in J$. Then*

(a) $\left(\underset{j\in J}{\times}(X_j, Pq_j), \underset{j\in J}{\times}(S_j,.,Pp_j)\right) \in AC$

(b) $\left(\underset{j\in J}{\times}(X_j, Pq_j), (\underset{j\in J}{\oplus} S_j,.,P\eta)\right) \in AC.$

Proof: (a): Denote $(X,q) = \underset{j\in J}{\times}(X_j, q_j)$, $(S,.,p) = \underset{j\in J}{\times}(S_j,.,p_j)$, and define $\lambda : (X,q) \times (S,p) \to (X,q)$ by $\lambda((x_j),(g_j)) = (\lambda_j(x_j,g_j))$. Clearly λ is an action. Then, according to Theorem 5.3 and the assumption that P is productive, it suffices to show that $((X,q),(S,p),\lambda) \in$ AC. The latter follows from a routine argument, and thus $(\underset{j\in J}{\times}(X_j, Pq_j), \underset{j\in J}{\times}(S_j,.,Pp_j),\lambda) \in$ AC.
(b): Since $\theta : (\oplus S_j,.,\eta) \to (S,.,p)$ is a continuous homomorphism in CSG and P is productive, it follows from Theorem 5.3 that $(\underset{j\in J}{\times}(X_j, Pq_j), (\oplus S_j,.,P\eta)) \in$ AC. ■

Corollary 5.1 *Assume that F_P : CONV $\to$ PCONV is a functor satisfying $F_P(X,q) = (X,Pq)$, $F_P(f) = f$, and P is finitely productive. Suppose that $((X_j,q_j),(S_j,.,p_j)) \in AC$ for each $j \in J$. Denote $(X,q) = \underset{j\in J}{\times}(X_j,q_j)$ and $(S,.,p) = \underset{j\in J}{\times}(S_j,.,p_j)$. Then*
(a) $((X,Pq),(S,.,Pp)) \in AC$

(b) $((X,Pq),(\underset{j\in J}{\oplus} S_j,.,P\eta)) \in AC.$

Verification of Corollary 5.1 follows the proof of Theorem 5.4 with the exception that since P is only finitely productive, (X,Pq) and $\underset{j\in J}{\times}(X_j,Pq_j)$, as well as $(S,.,Pp)$ and $\underset{j\in J}{\times}(S_j,.,Pp_j)$, may differ. Of course equality holds whenever the index set is finite. Choices of P that are finitely productive, and preserve continuity when taking P-modifications include: locally compact, locally bounded, regular, and first-countable. The property of being Choquet is productive, and continuity is preserved under taking Choquet modifications.

5.3 Generalized Quotients

Recall that if $((X,q),(S,.,p),\lambda) \in$ AC, $(S,.)$ is commutative, $\lambda(.,g)$ is an injection, then by Lemma 5.1, $(x,g) \sim (y,h)$ iff $\lambda(x,h) = \lambda(y,g)$ is an equivalence relation. Denote $R = \{((x,g),(y,h)) : (x,g) \sim (y,h)\}$, $r = q \times p$, and $\varphi : (X \times S, r) \to ((X \times S)/\sim, \sigma)$ the convergence quotient map defined by $\varphi(x,g)=\langle(x,g)\rangle$. Then $\boldsymbol{(B(X,S),\sigma)}:=((X\times S)/\sim,\sigma)$ is called the **generalized quotient space**. Convergence space properties of $(B(X,S),\sigma)$ are investigated in this section.

For ease of exposition, $((X,q),(S,.,p),\lambda)\in$ **GQ** denotes that $((X,q),(S,.,p),\lambda)\in$ AC, $(S,.)$ is commutative, and $\lambda(.,g)$ is an injection, for each $g\in S$. The generalized quotient space $(B(X,S),\sigma)$ exists whenever $((X,q),(S,.,p),\lambda)\in$ GQ.

Theorem 5.5 *Assume that $((X,q),(S,.,p),\lambda)\in GQ$. Then the following are equivalent:*

(a) (X,q) *is Hausdorff*

(b) R *is closed in* $((X\times S)\times(X\times S),r\times r)$

(c) $(B(X,S),\sigma)$ *is Hausdorff.*

Proof: (a) $\Rightarrow$ (b): Let π_{ij} denote the projection map defined by : $\pi_{ij}:(X\times S)\times(X\times S)\rightarrow X\times S$ where $\pi_{ij}(((x,g),(y,h)))=(x,g)$ when $i,j=1,2$ and $\pi_{ij}(((x,g),(y,h)))=(y,h)$ when $i,j=3,4$. Suppose that $\mathcal{H}\xrightarrow{r\times r}((x,g),(y,h))$ and $R\in\mathcal{H}$. Let $H\in\mathcal{H}$; then $H\cap R\neq\emptyset$, and thus there exists $((x_1,g_1),(y_1,h_1))\in H\cap R$. Hence $\lambda(x_1,h_1)=\lambda(y_1,g_1)$, and consequently $\lambda((\pi_1\circ\pi_{12})(H)\times(\pi_2\circ\pi_{34})(H))\cap\lambda((\pi_1\circ\pi_{34})(H)\times(\pi_2\circ\pi_{12})(H))\neq\emptyset$, for each $H\in\mathcal{H}$. It follows that $\mathcal{K}:=\lambda^{\rightarrow}((\pi_1\circ\pi_{12})^{\rightarrow}\mathcal{H}\times(\pi_2\circ\pi_{34})^{\rightarrow}\mathcal{H})\vee\lambda^{\rightarrow}((\pi_1\circ\pi_{34})^{\rightarrow}\mathcal{H}\times(\pi_2\circ\pi_{12})^{\rightarrow}\mathcal{H})$ exists. However, $(\pi_1\circ\pi_{12})^{\rightarrow}\mathcal{H}\xrightarrow{q}x$, $(\pi_2\circ\pi_{34})^{\rightarrow}\mathcal{H}\xrightarrow{p}h$, $(\pi_1\circ\pi_{34})^{\rightarrow}\mathcal{H}\xrightarrow{q}y$, $(\pi_2\circ\pi_{12})^{\rightarrow}\mathcal{H}\xrightarrow{p}g$, and thus $\mathcal{K}\xrightarrow{q}\lambda(x,h),\lambda(y,g)$. Since (X,q) is Hausdorff, $\lambda(x,h)=\lambda(y,g)$ and thus $(x,g)\sim(y,h)$. Therefore, $((x,g),(y,h))\in R$, and thus R is closed.

(b) $\Rightarrow$ (c): Assume that $\mathcal{K}\xrightarrow{\sigma}\langle(y_i,h_i)\rangle$, $i=1,2$. Since $\varphi:(X\times S,r)\rightarrow(B(X,S),\sigma)$ is a quotient map in CONV, there exist $(x_i,g_i)\sim(y_i,h_i)$ and $\mathcal{H}_i\xrightarrow{r}(x_i,g_i)$ such that $\varphi^{\rightarrow}\mathcal{H}_i=\mathcal{K}$, $i=1,2$. Then for each $H_i\in\mathcal{H}_i$, $\varphi(H_1)\cap\varphi(H_2)\neq\emptyset$ and thus there exists $(s_i,t_i)\in H_i$ such that $(s_1,t_1)\sim(s_2,t_2)$, $i=1,2$. Hence the least upper bound filter $\mathcal{L}:=(\mathcal{H}_1\times\mathcal{H}_2)\vee\dot{R}$ exists, and $\mathcal{L}\xrightarrow{r\times r}((x_1,g_1),(x_2,g_2))$. Since R is closed, $(x_1,g_1)\sim(x_2,g_2)$ and thus $\langle(y_1,h_1)\rangle=\langle(y_2,h_2)\rangle$. Therefore $(B(X,S),\sigma)$ is Hausdorff.

(c) $\Rightarrow$ (a): Suppose that $(B(X,S),\sigma)$ is Hausdorff and $\mathcal{F}\xrightarrow{q}x,y$. Then $\varphi^{\rightarrow}(\mathcal{F}\times\dot{e})\xrightarrow{\sigma}$

$\langle(x,e)\rangle, \langle(y,e)\rangle$, and thus $(x,e) \sim (y,e)$. Therefore, $x = \lambda(x,e) = \lambda(y,e) = y$, and thus (X,q) is Hausdorff. ■

Conditions for which $(B(X,S),\sigma)$ is T_1 are given below. In the topological setting, sufficient conditions in order for the generalized quotient space to be T_2 are given in [1] whenever $(S,.)$ is equipped with the discrete topology.

Theorem 5.6 *Suppose that $((X,q),(S,.,p),\lambda) \in GQ$. Then $(B(X,S),\sigma)$ is T_1 iff $\varphi^{-1}(\langle(y,h)\rangle)$ is closed in $(X \times S, r)$, for each $(y,h) \in X \times S$.*

PROOF: The "only if" is clear since $\{\langle(y,h)\rangle\}$ is closed and φ is continuous. Conversely, assume that $\varphi^{-1}(\langle(y,h)\rangle)$ is closed, for each $(y,h) \in X \times S$, and suppose that $\langle(\dot{x,g})\rangle \xrightarrow{\sigma} \langle(y,h)\rangle$. Since φ is a quotient map in CONV, there exist $(s,t) \sim (y,h)$ and $\mathcal{H} \xrightarrow{r} (s,t)$ such that $\varphi^{\rightarrow}\mathcal{H} = \langle(\dot{x,g})\rangle$. Then $\varphi^{-1}(\langle(x,g)\rangle) \in \mathcal{H}$, and thus $(s,t) \in \mathrm{cl}_r\varphi^{-1}(\langle(x,g)\rangle) = \varphi^{-1}(\langle(x,g)\rangle)$. Hence $(x,g) \sim (s,t) \sim (y,h)$, and thus $\langle(x,g)\rangle = \langle(y,h)\rangle$. Therefore $(B(X,S),\sigma)$ is T_1. ■

Corollary 5.2 *Assume that $((X,q),(S,.,p),\lambda) \in GQ$, and let p denote the discrete topology. Then $(B(X,S),\sigma)$ is T_1 iff (X,q) is T_1.*

PROOF: Suppose that $(B(X,S),\sigma)$ is T_1 and $\dot{x} \xrightarrow{q} y$. Then $(\dot{x,e}) \xrightarrow{r} (y,e)$, and thus $\langle(\dot{x,e})\rangle = \varphi^{\rightarrow}((\dot{x,e})) \xrightarrow{\sigma} \langle(y,e)\rangle$. It follows that $\langle(x,e)\rangle = \langle(y,e)\rangle$ and hence $x = y$. Therefore (X,q) is T_1.

Conversely, assume that (X,q) is T_1 and $(y,h) \in \mathrm{cl}_r\varphi^{-1}(\langle(x,g)\rangle)$. Then there exists $\mathcal{H} \xrightarrow{r} (y,h)$ such that $\varphi^{-1}(\langle(x,g)\rangle) \in \mathcal{H}$, $\pi_1^{\rightarrow}\mathcal{H} \xrightarrow{q} y$, $\pi_2^{\rightarrow}\mathcal{H} \xrightarrow{p} h$, and since p is the discrete topology, choose $H \in \mathcal{H}$ for which $\pi_2(H) = \{h\}$ and $\varphi(H) = \{\langle(x,g)\rangle\}$. If $(s,t) \in H$, then $(s,t) \sim (x,g)$, $t = h$, and thus $\lambda(s,g) = \lambda(x,h)$. Hence $\lambda(\pi_1(H) \times \{g\}) = \{\lambda(x,h)\}$, and thus $\lambda(\dot{x,h}) = \lambda^{\rightarrow}(\pi_1^{\rightarrow}\mathcal{H} \times \dot{g}) \xrightarrow{q} \lambda(y,g)$. Then $\lambda(x,h) = \lambda(y,g)$, $(x,g) \sim (y,h)$, and thus $\varphi^{-1}(\langle(x,g)\rangle)$ is r-closed. Hence it follows from Theorem 5.6 that $(B(X,S),\sigma)$ is T_1. ■

Corollary 5.3 *([1]) Suppose that the hypotheses of Corollary 5.2 are satisfied with the exception that (X, q) is a topological space and $B(X, S)$ is equipped with the quotient topology τ. Then $(B(X, S), \tau)$ is T_1 iff (X, q) is T_1.*

PROOF: It follows from Theorem 2 [11] that since $\varphi : (X \times S, r) \to (B(X, S), \sigma)$ is a quotient map in CONV, $\varphi : (X \times S, r) \to (B(X, S), t\sigma)$ is a topological quotient map, where $t\sigma$ is the largest topology on $X \times S$ which is coarser than σ. Moreover, $\tau = t\sigma$, and $A \subseteq B(X, S)$ is σ-closed iff it is τ-closed. Hence the desired conclusion follows from Corollary 5.2. ■

An illustration is given to show that the generalized quotient space may fail to be T_1 even though (X, q) is a T_1 topological space.

Example 5.1 *Denote $X = (0, 1)$, q the cofinite topology on X, and define $f : X \to X$ by $f(x) = ax$, where $0 < a < 1$ is fixed. Let $S = \{f^n : n \geq 0\}$, where $f^0 = id_X$ and f^n denotes the n-fold composition of f with itself. Then $(S, .) \in |SG|$ is commutative with composition as the operation. Also equip $(S, .)$ with the cofinite topology p. It is shown that the operation $\gamma : (S, p) \times (S, p) \to (S, p)$ defined by $\gamma(g, h) = g.h := h \circ g$ is continuous at (f^m, f^n). Define $C = \{f^k : k \geq k_0\}$; then $\{f^{m+n}\} \cup C$ is a basic p-neighborhood of f^{m+n}, where $k_0 \geq 0$. Observe that if $A = \{f^m\} \cup C$ and $B = \{f^n\} \cup C$, then $\gamma(A \times B) \subseteq C \cup \{f^{m+n}\}$. Therefore γ is continuous, and $(S, ., p) \in |CSG|$.*

Define $\lambda : X \times S \to X$ by $\lambda(x, g) = g(x)$, for each $x \in X$, $g \in S$, and note that λ is an action. It is shown that $\lambda : (X, q) \times (S, p) \to (X, q)$ is continuous at (x_0, f^n) in $X \times S$. A basic q-neighborhood of $\lambda(x_0, f^n) = f^n(x_0)$ is of the form $W = X - F$, where $f^n(x_0) \notin F$ and F is a finite subset of X. Let y_0 be the smallest member of F, and choose k_0 to be a natural number such that $a^{k_0} < y_0$. Then for each $k \geq k_0$, $f^k(x) = a^k x < y_0$ for each $x \in X$. Since f^n is injective, $F_0 = (f^n)^{-1}(F)$ is a finite subset of X. Then $U = X - F_0$ is a q-neighborhood of x_0, $V = \{f^n\} \cup \{f^k : k \geq k_0\}$ is a p-neighborhood of f^n, and $\lambda(U \times V) \subseteq W$. Indeed, if $x \in U$ and $k \geq k_0$, then $\lambda(x, f^k) = f^k(x) < y_0$, and thus $f^k(x) \in W$. Further, if $x \in U$, then

$f^n(x) \notin F$, and hence $f^n(x) \in W$. It follows that $\lambda(U \times V) \subseteq W$, and thus λ is a continuous action.

It is shown that $\varphi^{-1}(\langle(x_0, id_X)\rangle)$ is not closed in $(X \times S, r)$. Note that $(x, f^n) \in \varphi^{-1}(\langle(x_0, id_X)\rangle)$ iff $id_X(x) = f^n(x_0)$. Hence $\varphi^{-1}(\langle(x_0, id_X)\rangle) = \{(f^n(x_0), f^n) : n \geq 0\}$. Since $id_X = f^0 > f^1 > f^2 > ...$, it easily follows that $cl_r\varphi^{-1}(\langle(x_0, id_X)\rangle) = X \times S$, and thus $\varphi^{-1}(\langle(x_0, id_X)\rangle)$ is not r-closed. It follows from Theorem 5.6 that $(B(X, S), \sigma)$ is not T_1 even though both (X, q) and (S, p) are T_1 topological spaces. ■

A continuous surjection $f : (X, q) \to (Y, p)$ in CONV is said to be **proper map** provided that for each ultrafilter $\mathcal{F}$ on X, $f^{\to}\mathcal{F} \xrightarrow{p} y$ implies that $\mathcal{F} \xrightarrow{q} x$, for some $x \in f^{-1}(y)$. Proper maps in CONV are discussed in [12]; in particular, proper maps preserve closures. A proper convergence quotient map is called a **perfect map** [13].

Remark 5.4 *Assume that $((X, q), (S, ., p), \lambda) \in GQ$, (X, q) and (S, p) are regular, and $\varphi : (X \times S, r) \to ((B(X, S), \sigma)$ is a perfect map. Then $(B(X, S), \sigma)$ is also regular. Indeed, suppose that $\mathcal{H} \in \mathfrak{F}(B(X, S))$ such that $\mathcal{H} \xrightarrow{\sigma} \langle(y, h)\rangle$. Since φ is a quotient map in CONV, there exists $(x, g) \sim (y, h)$ and $\mathcal{K} \xrightarrow{r} (x, g)$ such that $\varphi^{\to}\mathcal{K} = \mathcal{H}$. Moreover, the regularity of $(X \times S, r)$ implies that $cl_r\mathcal{K} \xrightarrow{r} (x, g)$. Since φ is a proper map and thus preserves closures, $\varphi^{\to}(cl_r\mathcal{K}) = cl_\sigma\varphi^{\to}\mathcal{K} = cl_\sigma\mathcal{H} \xrightarrow{\sigma} \langle(y, h)\rangle$. Hence $(B(X, S), \sigma)$ is regular.* ■

The proof of the following result is straightforward to verify.

Lemma 5.2 *Suppose that $(S, ., p) \in |CSG|$ and $(T, .) \in |SG|$. Assume that $f : (S, ., p) \to (T, ., \sigma)$ is both a homomorphism and a quotient map in CONV. Then $(T, ., \sigma) \in |CSG|$.*

Assume that $((X, q), (S, ., p), \lambda) \in$ AC. Recall that λ distinguishes elements in S whenever $\lambda(x, g) = \lambda(x, h)$ for each $x \in X$ implies $g = h$. This property was needed in the verification of Theorem 5.2. In the event that λ fails to distinguish elements in S, define $g \sim h$ iff $\lambda(x, g) = \lambda(x, h)$ for each $x \in X$. Then $\sim$ is an equivalence relation on S; denote $\boldsymbol{S_1} =$

$S/\sim = \{[g] : g \in S\}$, and define the operation $[g].[h] = [g.h]$, for each $g, h \in S$. The operation is well defined and $(S_1, .) \in |\mathrm{SG}|$. Let $\boldsymbol{p_1}$ denote the quotient convergence structure on S_1 determined by $\rho : (S, p) \to S_1$, where $\rho(g) = [g]$. Then $\rho : (S, .) \to (S_1, .)$ is a homomorphism, and it follows from Lemma 5.2 that $(S_1, ., p_1) \in |\mathrm{CSG}|$. Define $\lambda_1 : X \times S_1 \to X$ by $\lambda_1(x, [g]) = \lambda(x, g)$.

Theorem 5.7 *Assume $((X, q), (S, ., p), \lambda) \in GQ$, λ fails to distinguish elements in S, and let $(B(X \times S), \sigma), (B(X \times S_1), \sigma_1)$ denote the generalized quotient spaces corresponding to $(X \times S, r)$ and $(X \times S_1, r_1)$, where $r = q \times p$ and $r_1 = q \times p_1$. Then*

(a) $\lambda_1 : (X \times S_1, r_1) \to (X, q)$ *is a continuous action*

(b) λ_1 *separates elements in* S_1

(c) $(B(X, S), \sigma)$ *and* $(B(X, S_1), \sigma_1)$ *are homeomorphic.*

Proof: (a): It is routine to verify that λ_1 is an action. Let us show that λ_1 is continuous. Suppose that $\mathcal{F} \xrightarrow{q} x$ and $\mathcal{G} \xrightarrow{p_1} [g]$; then since p_1 is a quotient structure in CONV, there exists $\mathcal{G}_1 \xrightarrow{p} g_1 \sim g$ such that $\rho^{\to}\mathcal{G}_1 = \mathcal{G}$. Hence $\lambda_1^{\to}(\mathcal{F} \times \mathcal{G}) = \lambda_1^{\to}(\mathcal{F} \times \rho^{\to}\mathcal{G}_1) = [\{\lambda_1(F \times \rho(G_1)) : F \in \mathcal{F}, G_1 \in \mathcal{G}_1\}] = [\{\lambda(F \times G_1) : F \in \mathcal{F}, G_1 \in \mathcal{G}_1\}] = \lambda^{\to}(\mathcal{F} \times \mathcal{G}_1) \xrightarrow{q} \lambda(x, g_1) = \lambda_1(x, [g])$, and thus λ_1 is continuous.

(b): Suppose that $\lambda_1(x, [g]) = \lambda_1(x, [h])$ for each $x \in X$. Then $\lambda(x, g) = \lambda(x, h)$ for each $x \in X$, and thus $[g] = [h]$. Hence λ_1 distinguishes elements in S_1.

(c): It easily follows that the diagram below is commutative:

$$\begin{tikzcd} X \times S \arrow[r, "\varphi_1"] \arrow[d, "\psi_1"] & B(X, S) \arrow[d, "\psi_2"] \\ X \times S_1 \arrow[r, "\varphi_2"] & B(X, S_1) \end{tikzcd}$$

where φ_1, φ_2 are quotient maps, $\psi_1(x, g) = (x, [g])$, and $\psi_2(\langle x, g\rangle) = \langle (x, [g])\rangle$. Moreover, ψ_2 is an injection. Indeed, assume that $\langle (x, [g])\rangle = \psi_2(\langle (x, g)\rangle) = \psi_2(\langle (y, h)\rangle) = \langle (y, [h])\rangle$;

then $\lambda_1(x,[h]) = \lambda_1(y,[g])$ and thus $\lambda(x,h) = \lambda(y,g)$. Hence $\langle(x,g)\rangle = \langle(y,h)\rangle$ and ψ_2 is an injection. Clearly ψ_2 is a surjection.

It is shown that ψ_2 is continuous. Indeed, suppose that $\mathcal{H} \xrightarrow{\sigma} \langle(y,h)\rangle$; then there exist $(x,g) \sim (y,h)$ and $\mathcal{K} \xrightarrow{r} (x,g)$ such that $\varphi_1^{\rightarrow}\mathcal{K} = \mathcal{H}$. Since the diagram above commutes with ψ_1 and φ_2 continuous, it follows that $\psi_2^{\rightarrow}\mathcal{H} = (\psi_2 \circ \varphi_1)^{\rightarrow}\mathcal{K} = (\varphi_2 \circ \psi_1)^{\rightarrow}\mathcal{K} \xrightarrow{\sigma_1} (\varphi_2 \circ \psi_1)(x,g) = (\psi_2 \circ \varphi_1)(x,g) = \psi_2(\langle(x,g)\rangle) = \psi_2(\langle(y,h)\rangle)$. Hence ψ_2 is continuous.

Finally, let us show that ψ_2^{-1} is continuous. Assume that $\mathcal{H} \xrightarrow{\sigma_1} \langle(y,[h])\rangle$. Since φ_2 is a quotient map, there exist $(x,[g]) \sim (y,[h])$ and $\mathcal{K} \xrightarrow{r_1} (x,[g])$ such that $\varphi_2^{\rightarrow}\mathcal{K} = \mathcal{H}$. In particular, $\mathcal{F} = \pi_1^{\rightarrow}\mathcal{K} \xrightarrow{q} x$ and $\mathcal{G} = \pi_2^{\rightarrow}\mathcal{K} \xrightarrow{p_1} [g]$. Since $\rho : (S,p) \to (S_1,p_1)$ is a quotient map, there exist $g_1 \sim g$ and $\mathcal{G}_1 \xrightarrow{p} g_1$ such that $\rho^{\rightarrow}\mathcal{G}_1 = \mathcal{G}$. Then $\mathcal{F} \times \mathcal{G}_1 \xrightarrow{r} (x,g_1)$, and thus $\psi_1^{\rightarrow}(\mathcal{F} \times \mathcal{G}_1) = \mathcal{F} \times \rho^{\rightarrow}\mathcal{G}_1 = \mathcal{F} \times \mathcal{G} \leq \mathcal{K}$. Hence $(\varphi_2 \circ \psi_1)^{\rightarrow}(\mathcal{F} \times \mathcal{G}_1) \leq \varphi_2^{\rightarrow}\mathcal{K} = \mathcal{H}$, and since the diagram commutes, $\psi_2^{\leftarrow}\mathcal{H} \geq (\psi_2^{-1} \circ \varphi_2 \circ \psi_1)^{\rightarrow}(\mathcal{F} \times \mathcal{G}_1) = \varphi_1^{\rightarrow}(\mathcal{F} \times \mathcal{G}_1) \xrightarrow{\sigma} \langle(x,g)\rangle = \psi_2^{-1}(\langle(y,[h])\rangle)$. Therefore ψ_2 is a homeomorphism. ■

Sufficient conditions in order for (X,q) to be embedded in $(B(X,S),\sigma)$ are presented below.

Theorem 5.8 *Suppose that $((X,q),(S,.,p),\lambda) \in GQ$. Define $\beta : (X,q) \to (B(X,S),\sigma)$ by $\beta(x) = \langle(x,e)\rangle$, for each $x \in X$. Then*

(a) β is a continuous injection

(b) β is an embedding provided that (X,q) is a Choquet space, p is discrete, and λ is a proper map.

Proof: (a): Clearly β is an injection. Next, assume that $\mathcal{F} \xrightarrow{q} x$; then $\beta^{\rightarrow}\mathcal{F} = [\{\beta(F) : F \in \mathcal{F}\}] = [\{\varphi(F \times \{e\}) : F \in \mathcal{F}\}] = \varphi^{\rightarrow}(\mathcal{F} \times \dot{e}) \xrightarrow{\sigma} \varphi(x,e) = \beta(x)$. Therefore β is continuous.

(b): First, suppose that $\mathcal{F}$ is an ultrafilter on X such that $\beta^{\rightarrow}\mathcal{F} \xrightarrow{\sigma} \beta(x) = \langle(x,e)\rangle$. Since $\varphi : (X \times S, r) \to (B(X,S),\sigma)$ is a quotient map in CONV, there exist $(y,g) \sim (x,e)$ and $\mathcal{K} \xrightarrow{r} (y,g)$ such that $\varphi^{\rightarrow}\mathcal{K} = \beta^{\rightarrow}\mathcal{F}$. Denote $\mathcal{F}_1 = \pi_1^{\rightarrow}\mathcal{K} \xrightarrow{q} y$ and $\mathcal{G}_1 = \pi_2^{\rightarrow}\mathcal{K} \xrightarrow{p} g$.

Since p is the discrete topology, $\mathcal{G}_1 = \dot{g}$, and thus $\mathcal{K} \geq \pi_1^{\rightarrow}\mathcal{K} \times \pi_2^{\rightarrow}\mathcal{K} = \mathcal{F}_1 \times \dot{g}$. Let $F_1 \in \mathcal{F}_1$; then $\varphi^{\rightarrow}(\mathcal{F}_1 \times \dot{g}) \leq \varphi^{\rightarrow}\mathcal{K} = \beta^{\rightarrow}\mathcal{F}$ implies that there exists $F \in \mathcal{F}$ such that $\beta(F) \subseteq \varphi(F_1 \times \{g\})$. If $z \in F$, then $\beta(z) = \langle(z,e)\rangle = \langle(z_1,g)\rangle$, for some $z_1 \in F_1$, and thus $\lambda(z,g) = \lambda(z_1,e) = z_1 \in F_1$. It follows that $\lambda(F \times \{g\}) \subseteq F_1$, and thus $\lambda^{\rightarrow}(\mathcal{F} \times \dot{g}) \geq \mathcal{F}_1 \xrightarrow{q} y$. Since $\mathcal{F} \times \dot{g}$ is an ultrafilter on $X \times S$ and λ is a proper map, $\mathcal{F} \times \dot{g} \xrightarrow{r} (s,t)$, for some $(s,t) \in \lambda^{-1}(y)$. Then $\mathcal{F} \xrightarrow{q} s$ and $g = t$ since p is discrete. It follows that $\lambda(y,e) = y = \lambda(s,t) = \lambda(s,g)$, and thus $(s,e) \sim (y,g)$. As shown above, $(y,g) \sim (x,e)$, and thus $(x,e) \sim (s,e)$. Therefore $x = s$, and $\mathcal{F} \xrightarrow{q} x$.

Finally, let $\mathcal{F}$ be any filter on X such that $\beta^{\rightarrow}\mathcal{F} \xrightarrow{\sigma} \beta(x)$. If $\mathcal{H}$ is any ultrafilter on X containing $\mathcal{F}$, then $\beta^{\rightarrow}\mathcal{H} \xrightarrow{\sigma} \beta(x)$, and from the previous case, $\mathcal{H} \xrightarrow{q} x$. Since (X,q) is a Choquet space, $\mathcal{F} \xrightarrow{q} x$ and hence β is an embedding. ■

Assume that $((X,q),(S,.,p),\lambda) \in$ GQ, $(X,\bar{q})$ is the finest Choquet space such that $\bar{q} \leq q$, $\bar{r} = \bar{q} \times p$, and let $\bar{\sigma}$ denote the quotient convergence structure on $B(X,S)$ determined by $\varphi : (X \times S, \bar{r}) \to B(X,S)$.

Corollary 5.4 *Assume $((X,q),(S,.,p),\lambda) \in GQ$, p is discrete, and λ is a proper map. Then, using the above notations, $\beta : (X,\bar{q}) \to (B(X,S),\bar{\sigma})$ is an embedding.*

Proof: It follows from Theorem 5.3 that $((X,\bar{q}),(S,.,p),\lambda) \in$ AC. Since q and $\bar{q}$ agree on ultrafilter convergence, $\lambda : (X,\bar{q}) \times (S,p) \to (X,\bar{q})$ is also a proper map, and $(X,\bar{q})$ is a Choquet space. Then according to Theorem 5.8, $\beta : (X,\bar{q}) \to (B(X \times S),\bar{\sigma})$ is an embedding. ■

Let us conclude by showing that the generalized quotient of a product is homeomorphic to the product of the generalized quotients. Assume that $((X_j,q_j),(S_j,.,p_j),\lambda_j) \in$ GQ, for each $j \in J$. Let $(X,q) = \underset{j\in J}{\times}(X_j,q_j)$ and $(S,.,p) = \underset{j\in J}{\times}(S_j,.,p_j)$ denote the product spaces, and define $\lambda : X \times S \to X$ by $\lambda((x_j),(g_j)) = (\lambda_j(x_j,g_j))$. According to Corollary 5.1, $((X,q),(S,.,p),\lambda) \in$ AC. Moreover, since each $(S_j,.,p_j)$ is commutative and $\lambda_j(.,g)$

is an injection for each $j \in J$, $(S, ., p)$ is commutative and $\lambda(., g)$ is an injection. Hence $((X, q), (S, ., p), \lambda) \in$ GQ. Let $\varphi_j : (X_j, q_j) \times (S_j, ., p_j) \to (B(X_j, S_j), \sigma_j)$ denote the convergence quotient map, $r_j = q_j \times p_j$, $\varphi = \underset{j \in J}{\times} \varphi_j$, for each $j \in J$. Since the product of quotient maps in CONV is again a quotient map, $\varphi : \underset{j \in J}{\times}(X_i \times S_j, r_j) \to \underset{j \in J}{\times}(B(X_j, S_j), \sigma_j)$ is also a quotient map. Denote $\sigma = \underset{j \in J}{\times} \sigma_j$.

Define $((x_j), (g_j)) \sim ((y_j), (h_j))$ in $X \times S$ iff $\lambda((x_j), (h_j)) = \lambda((y_j), (g_j))$. This is an equivalence relation on $X \times S$, and it follows from the definition of λ that $((x_j), (g_j)) \sim ((y_j), (h_j))$ iff $(x_j, g_j) \sim (y_j, h_j)$, for each $j \in J$. Let $(B(X, S), \Sigma)$ denote the corresponding generalized quotient space, where $\Phi : (X \times S, r) \to (B(X, S), \Sigma)$ is the quotient map and $r = \underset{j \in J}{\times} r_j$.

Theorem 5.9 *Suppose that $((X_j, q_j), (S_j, ., p_j), \lambda_j) \in GQ$, for each $j \in J$. Then, employing the notations defined above, $\underset{j \in J}{\times}(B(X_j, S_j), \sigma_j)$ and $(B(X, S), \Sigma)$ are homeomorphic.*

Proof: Consider the following diagram:

$$\begin{tikzcd}
\underset{j \in J}{\times}(X_j \times S_j, r_j) \arrow[r, "\delta"] \arrow[d, "\varphi"] & (X \times S, r) \arrow[d, "\Phi"] \\
\underset{j \in J}{\times}(B(X_j, S_j), \sigma_j) \arrow[r, "\Delta"] & (B(X, S), \Sigma),
\end{tikzcd}$$

where $\delta(((x_j, g_j)_j)) = ((x_j), (g_j))$ and $\Delta((\langle(x_j, g_j)\rangle_j)) = \langle((x_j), (g_j))\rangle$. Then δ is a homeomorphism, and the diagram commutes. Note that Δ is a bijection. Indeed, if $\Delta((\langle(x_j, g_j)\rangle_j)) = \Delta((\langle(y_j, h_j)\rangle_j))$, then $((x_j), (g_j)) \sim ((y_j), (h_j))$ and thus $(x_j, g_j) \sim (y_j, h_j)$, for each $j \in J$. Hence $\langle(x_j, g_j)\rangle_j = \langle y_j, g_j\rangle_j$ for each $j \in J$, and thus Δ is an injection. Clearly Δ is a surjection.

It is shown that Δ is continuous. Assume that $\mathcal{H} \xrightarrow{\sigma} (\langle(y_j, h_j)\rangle_j)$; then since φ is a quotient map, there exist $((x_j), (g_j)) \sim ((y_j), (h_j))$ and $\mathcal{K} \xrightarrow{r} ((x_j, g_j)_j)$ such that $\varphi^{\to}\mathcal{K} = \mathcal{H}$. However, the diagram commutes, and thus $\Delta^{\to}\mathcal{H} = (\Delta \circ \varphi)^{\to}\mathcal{K} = (\Phi \circ \delta)^{\to}\mathcal{K} \xrightarrow{\Sigma} \Phi((x_j), (g_j)) = \Phi((y_j), (h_j)) = \langle((y_j), (h_j))\rangle$. Hence Δ is continuous.

Conversely, suppose that $\mathcal{H} \xrightarrow{\Sigma} \langle((y_j),(h_j))\rangle$; then since Φ is a quotient map, there exist $((x_j),(g_j)) \sim ((y_j),(h_j))$ and $\mathcal{K} \xrightarrow{r} ((x_j),(g_j))$ such that $\Phi^{\rightarrow}\mathcal{K} = \mathcal{H}$. Using the fact that δ is a homeomorphism and that the diagram commutes, $\Delta^{\leftarrow}\mathcal{H} = (\varphi \circ \delta^{-1})^{\rightarrow}\mathcal{K} \xrightarrow{\sigma} \varphi((x_j, g_j)_j) = \varphi((y_j, h_j)_j) = (\langle(y_j, h_j)\rangle_j)$, and thus Δ^{-1} is continuous. Therefore Δ is a homeomorphism. ∎

Remark 5.5 *In general, quotient maps are not productive in the category of all topological spaces with the continuous maps as morphisms. Whether or not Theorem 5.9 is valid in the topological context is still unknown.* ∎

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