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TILING WITH POLYOMINOES, POLYCUBES, AND RECTANGLES

by

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B.S. University of Central Florida, 2013

A thesis submitted in partial fulfillment of the requirements
for the degree of Master of Science
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Fall Term
2015

Major Professor: Michael Reid

ABSTRACT

In this paper we study the hierarchical structure of the 2-d polyominoes. We introduce a new infinite family of polyominoes which we prove tiles a strip. We discuss applications of algebra to tiling. We discuss the algorithmic decidability of tiling the infinite plane $\mathbb{Z} \times \mathbb{Z}$ given a finite set of polyominoes. We will then discuss tiling with rectangles. We will then get some new, and some analogous results concerning the possible hierarchical structure for the 3-d polycubes.

ACKNOWLEDGMENTS

I would like to express my deepest gratitude to my advisor, Professor Michael Reid, who spent countless hours mentoring me over my time at the University of Central Florida. His passion and genius in Mathematics and its educational pedagogy has pushed me to become, not only a better person, but a better mathematician and educator. Without him, this thesis would not have been possible.

I would also like to thank all of the wonderful friends that I have met during my time in graduate school, especially Arielle Gaudiello and Talon Ward. Without you pushing and believing in me this would not have been possible.

Lastly, I would like to thank my family for all of their love and support through this process.

TABLE OF CONTENTS

LIST OF FIGURES

1 INTRODUCTION

Solomon Golomb [3] generalized the concept of the domino to that of a polyomino. Like so many before me I fell in love with these geometric objects and the puzzles and problems that arise from them. These objects embody the spirit of mathematical exploration with their puzzles, problems, and structures. Consider the mutilated checker board and the domino.
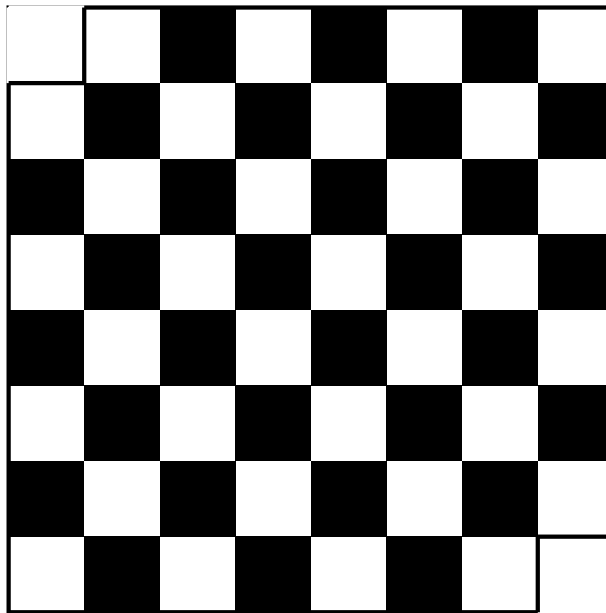
Figure 1: Mutilated Checkerboard.

Is it possible to cover the whole board with only ⊐ (dominos)? The answer is "no" and we will see why in later sections of the paper.

In this thesis we will cover many topics that I have researched during my Masters degree at the University of Central Florida. These topics include discrete tiling with polyominoes, polycubes, and rectangles. We will discuss 2-d polyominoes, algebraic applications to tiling, tiling the infinite plane with a finite set of polyominoes, tiling with rectangles, and then possible extensions to the 3-d polycubes.

2 $2 - D$ **POLYOMINOES**

Consider $\mathbb{Z} \times \mathbb{Z}$ and the unit square lattice. We will refer to each unit square in the square lattice as a **cell**.

Definition 2.1. *A **Polyomino** is a finite collection of lattice edge connected cells. In other words, it is a finite collection of cells whose interior is connected.*

We will refer to a polyomino that is made up of n cells a n-omino. Examples of n-ominos for $1 \le n \le 8$ are given next.

Monomino   Domino   L-tromino   Z-tetromino   W-pentomino   Hexomino   Heptomino   Octomino
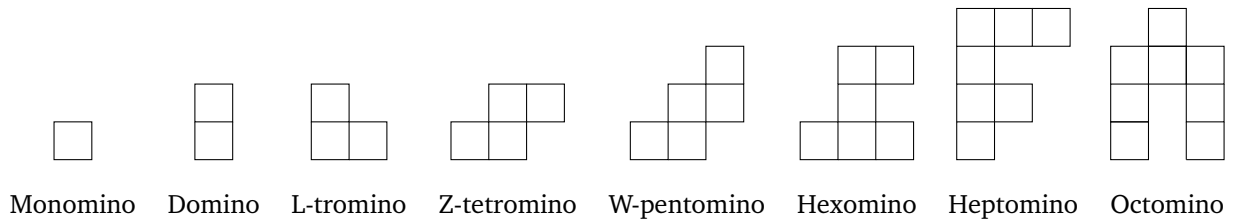
Figure 2: Examples of 2-d Polyominoes.

Definition 2.2. *A polyomino P is called **rectifiable** if a finite number of copies of P can be used to tile a rectangle R. The **order** of P is the smallest integer k so that k copies of P can be used to tile a rectangle.*

Definition 2.3. *Let X be the set of all integers k so that k copies of P can be used to tile a rectangle. If X contains an odd integer, then we will call P an **odd** polyomino. If X contains only even integers, then P is called **even**.*

An example of a rectifiable polyomino is given next.
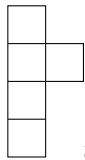
Example 2.1. *The Y-pentomino   tiles a 5×10 rectangle with 10 copies. This is the smallest rectangle it can tile. Hence, its order is 10 [11].*
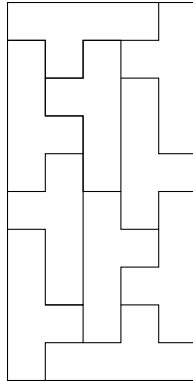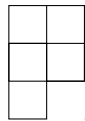
Figure 3: Smallest Rectangle that the Y-Pentomino Tiles. [11]

□

Definition 2.4. *A polyomino P is called a **reptile** if a finite number of copies of P can tile a scaled up version of P. The scaling factor of the reptile property is necessarily an integer $k > 1$.*

An example of a reptile is given next.
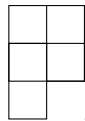
Example 2.2. *The P-pentomino,* *, is a reptile with a scaling factor $k = 2$.*
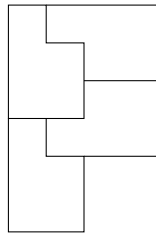
Figure 4: The First Reptile of the P-Pentomino.

□

Notice that if P is a reptile, then its reptile R is also a reptile. Hence, this process of scaling up and tiling can be repeated indefinitely.

Definition 2.5. *A set of polyominoes τ is called a **protoset**.*

With these few definitions we can now start asking some interesting questions. Some of them being:

- Which polyominoes tile a rectangle?

3

- If a polyomino does not tile a rectangle, then what regions can it tile?

- If a polyomino is a reptile, does it also tile a rectangle?

- Does there exist a polyomino that is odd?

- Does there exist a polyomino that is of odd order?

2.1 Polyomino Hierarchy for a Finite Protoset τ.

Golomb [5][6] introduced a hierarchy of fundamental regions that give us a way of characterizing a polyomino by which type of region that it can tile.
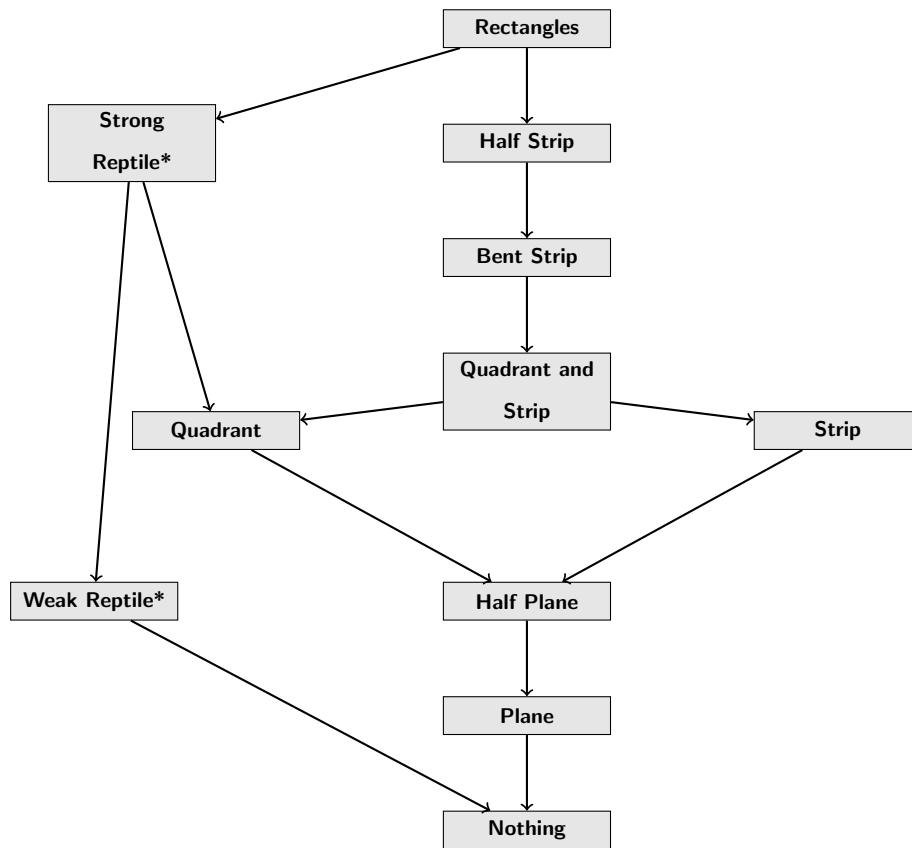
Figure 5: The 2-d Polyomino Hierarchy.

Definition 2.6. *Let a, b be positive integers. A __Half Strip__ is a region congruent to $\{(x, y) : 0 \leq y \leq a$ and $0 \leq x < \infty\}$.*

Definition 2.7. *A __Bent Strip__ is a region congruent to $\{(x, y) : 0 \leq y \leq a$ and $0 \leq x < \infty\} \bigcup \{(x, y) : 0 \leq y < \infty$ and $0 \leq x \leq b\}$.*

4

Definition 2.8. *A __Strip__ is a region congruent to $\{(x, y) : 0 \leq y \leq a \text{ and } -\infty < x < \infty\}$.*

The two categories that have an asterisk are exclusive to tiling with a finite protoset τ with two or more polyominoes.

Definition 2.9. *Let τ be a protoset of finitely many polyominoes. τ is said to have the __weak reptile property__ if every $P \in \tau$ can be scaled up and tiled by polyominoes in τ. τ has the __strong reptile property__ if it has the weak reptile property plus a common scaling factor for every $P \in \tau$.*

Remark 2.1. *When the protoset τ only has one polyomino the categories strong and weak reptile are logically equivalent and we call this the reptile category. It should also be mentioned that there are no known examples of protosets with the weak reptile property and not the strong reptile property.* △

In this hierarchy every arrow that we see is a logical implication. Golomb [5] proves all of these implications. Most of them are immediate and need almost no proof whatsoever (i.e. rectangle implies half strip, half strip implies bent strip, bent strip implies quadrant, quadrant implies half plane, half plane implies plane, and plane implies nothing). Two implications are not immediately obvious: bent strip implies strip and reptile implies quadrant.

Theorem 2.1. *(Golomb, 1966), [5] If P tiles a bent strip, then P tiles a strip.*
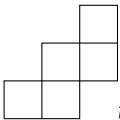
Proof.
Suppose the polyomino P tiles a bent strip. Without loss of generality assume this bent strip is in the corner of the first quadrant and is of the form $\{(x, y) : 0 \leq y \leq a \text{ and } 0 \leq x < \infty\} \bigcup \{(x, y) : 0 \leq x \leq b \text{ and } 0 \leq x < \infty\}$, where a and b are positive integers. Let w be the maximum width of P. Look at the leg of the bent strip which is on the x-axis. Now for every integer $i \geq 0$, look at the top to bottom patterns that P makes in each of the rectangles $R_i = \{(x, y) : 0 \leq y \leq a \text{ and } i \leq x \leq w + i\}$. For every i, R_i is a finite lattice. Thus, there are only finite number of patterns that this polyomino can make in each R_i. Since there are countably infinite R_i, we are guaranteed that within a finite number of steps a reemerging pattern will occur. We can now use this repeating pattern to obtain a tiling a strip.

□

Remark 2.2. *This proof still works if we have a finite protoset of polyominoes. However, if we have an infinite protoset of polyominoes, then the proof immediately breaks down and the result is not necessarily true.*

△

An example to illustrate the proof of the previous theorem is given next.

Example 2.3. *The W-Pentomino* *tiles a bent strip. One such tiling is given next.*
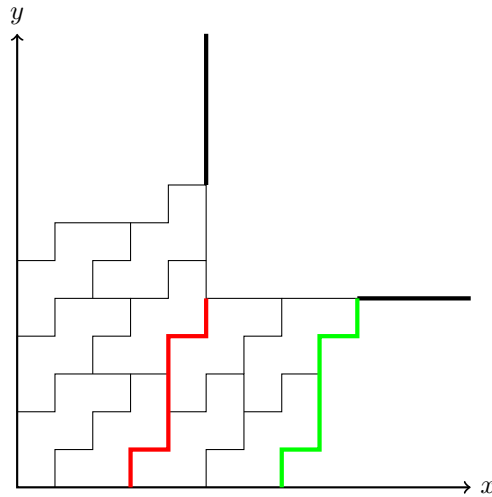
Here $w = 3$. The red path comes from the third examination of the 3×5 box and the green path comes from the seventh examination of the 3×5 box. This tiling of the W-pentomino leads to an obvious tiling of a strip. □

We need one more definition to prove the next theorem.

Definition 2.10. *The __rectangular box__ of a polyomino P is the minimum rectangle that contains P.*

Remark 2.3. *Notice that if P is a reptile, then it must be in at least one corner of its rectangular box. Assume for the sake of contradiction that it not in any corner of its rectangular box. Let P' be the reptile of P. Then since P' is a reptiling, we need to be able to tile every corner of P' using only P. However, every P that we have at our disposal can not occupy any corners. Even if P could some how reach around and tile some corner of P', then P would necessarily occupy at least one corner of its rectangular box. In the first case, P' cannot be tiled by P. In the second case, P would occupy one corner of its rectangular box. Both are contradictions.* △

Theorem 2.2. *(Golomb, 1966, [5]) If P is a reptile, then P tiles a Quadrant.*

Proof.

Suppose P is a reptile and fix a reptiling R. Let S be the set of all possible orientations of P at the corner of the quadrant. The set S is nonempty and finite, say $S = \{s_1, ..., s_n\}$, for some natural number n. Let

$f : S \to S$ be a function that takes in an $s_i \in S$ and replaces it with its scaled reptile in the same orientation. Each iteration of this function induces a tiling of a larger reptile. Moreover, since f is a function from a finite set to a finite set, it necessarily induces at least one cycle. If an element of S is not in a cycle it must then fall into a cycle after a finite number of compositions. Hence, we can always choose one orientation at the origin which is in a cycle to begin tiling with. Let $s_i \to s_k \to \cdots \to s_i$ be one such cycle of length l and choose s_i as our original orientation. Then after l compositions we obtain a reptiling that has the same orientation as the orientation we started with. Hence this tiling is consistent with our original choice of orientation at the origin. This process can be can be continued indefinitely, so if we took the number of compositions of f to infinity we will have a tiling of the quadrant consistent with our original choice of orientation.

$\square$

An example illustrating the proof of this theorem is given next.

Example 2.4. *We have already shown that the P-Pentomino is a reptile. Let*

$$S = \left\{ \;\; {}_{s_1 =} \qquad , {}_{s_2 =} \qquad , {}_{s_3 =} \qquad , {}_{s_4 =} \qquad , {}_{s_5 =} \qquad , {}_{s_6 =} \qquad \right\}$$

*and fix the reptiling we gave in **Example 2.2**. Then $f(s_2)$ gives the reptiling in **Example 2.2**. Notice that s_1 is now at the origin. Applying f again, we obtain the following even larger reptiling with s_2 at the origin.*
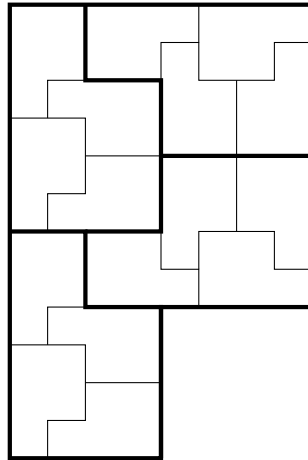
Figure 7: Illustration of Proof Technique of Thm 2.2.
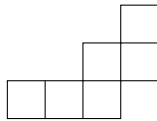
Thus, $s_2 \to s_1 \to s_2$ is a cycle of length two. So every two compositions of f starting with s_1 or s_2 will yield larger reptiles consistent with s_1 or s_2. It's also worth noting that $s_3 \to s_1 \to s_2 \to s_1$. Thus it would not be a good decision to begin our tiling with s_3. Likewise $s_4 \to s_5 \to s_6 \to s_5$, so s_4. Thus it would also not be a good choice to start with.

$\square$

So far we have seen some characteristic examples of the regions rectangle and bent strip. Golomb [5] gives a list of polyominoes and the regions they characteristically belong to. It's curious that he could not find definitive examples for reptile, half strip, quadrant and strip, or half plane. Not to say that examples don't exist, because clearly any polyomino that is rectifiable can tile any of the aforementioned regions. Perhaps some of these regions in the hierarchy are in fact logically equivalent regions. That is, maybe it's true that if P is a reptile, then it necessarily tiles a rectangle. After all, all known examples of reptiles in 2-d also tile rectangles. This is still an open problem in 2-d. Hochberg and Reid [9] showed that in all dimensions $d > 2$ there exist polycubes which are reptiles but do not tile any box.

It is also curious that we have yet to find a characteristic example of a polyomino which tiles a Half-Strip. Golomb [5] gives only one candidate, however, it has since been shown to tile a rectangle. Reid [14] gives examples of three infinite families of polyominoes which tile half strips, however, it is not known whether or not they are rectifiable.
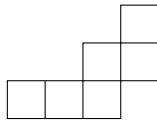
Golomb misclassified in his paper. Golomb claims that this hexomino characteristically tiles a strip. However, I can give an interesting tiling of the quadrant by the hexomino. Hence, this polyomino may be a characteristic example of quadrant and strip.
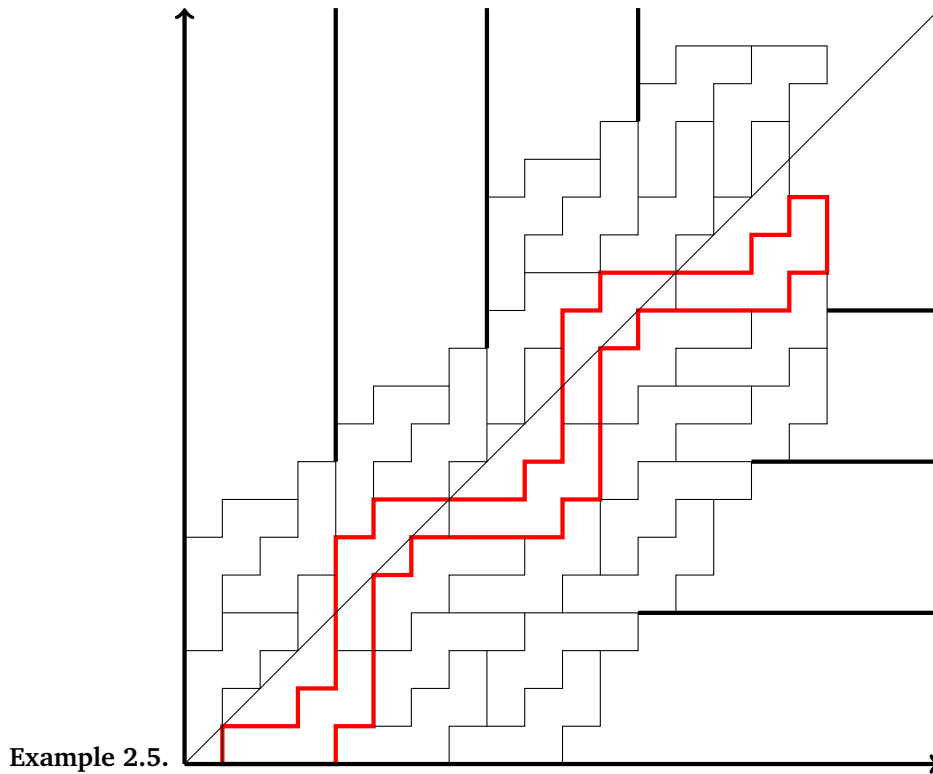
Example 2.5.

Figure 8: W-hexomino Tiling of the Quadrant.

□

This tiling is interesting because if you consider the path that is highlighted in red, the tiling has glide reflection. That is the 2nd column is same tiling as the 1st row, up to rotational symmetry. Likewise, the 3rd column is the same as the 2nd row, and so on. Reid has told me that the W-hexomino can tile a bent strip. I have not yet found a tiling.

2.2   A Infinite Family that Tiles a Strip

In this section I will present an infinite family of polyominoes which characteristically belong in the strip category of the hierarchy.

Figure 9: Infinite Family of Polyominoes.

For every $n \geq 5$ the dotted line represents $n - 5$ cells, I call this polyomino the V_n-omino. For example:

9

n=5, V_5 n=6, V_6 n=7, V_7
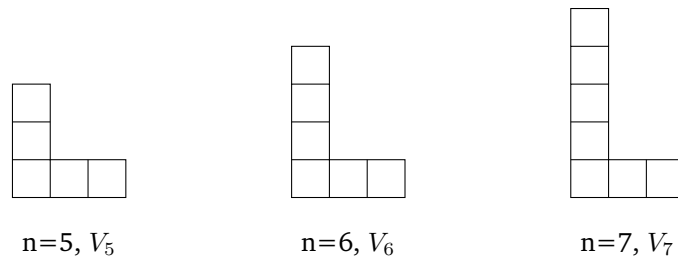
Figure 10: Example of n=5,6, and 7 for V_n.

I will now give a tiling of a strip of height of n.
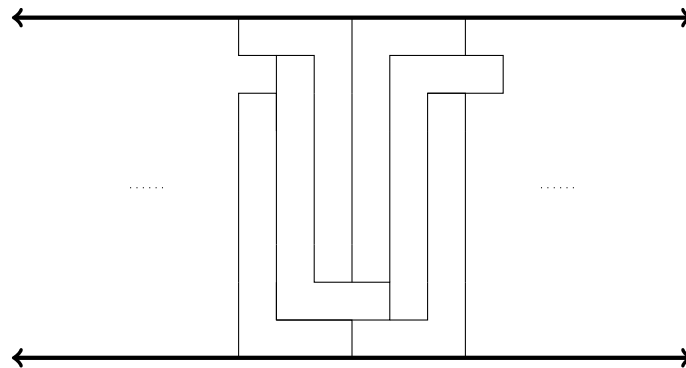
Figure 11: Strip Tiling for V_n.

To show that this family characteristically belongs to the strip category of the hierarchy we need to show that it can not tile a quadrant.

Theorem 2.3. *For all $n \geq 5$, V_n does not tile a quadrant.*

Proof.

For definiteness, we will be using the fact that $n = 9$ for all of the diagrams. However, the cases will hold for $n \geq 8$. For $n = 5$ and $n = 6$, see [5] and for $n = 7$ all but a few cases work. The cases which don't work for $n = 7$ we have a few extra placements to consider before we get the desired result.

Up to reflective symmetry, we have 3 cases at the corner of the first quadrant to consider.
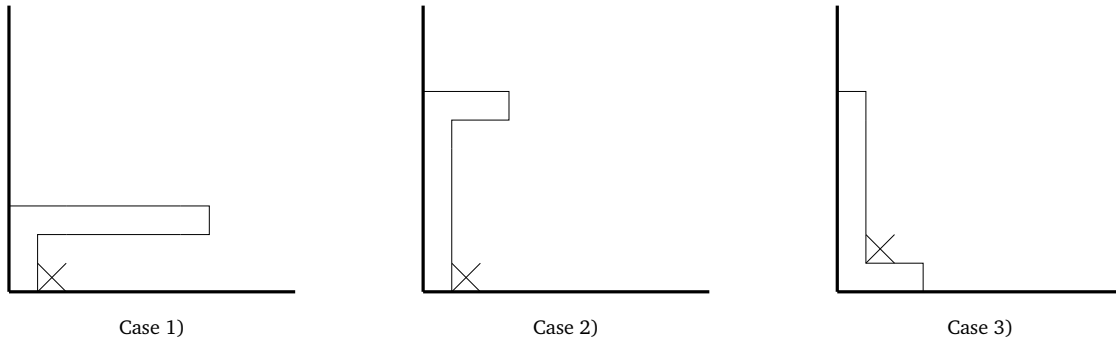
Figure 12: Three Cases, Up to Reflective Symmetry.

Case 1) The goal is to try and fill the marked cell. For the first case we have only one possible placement.
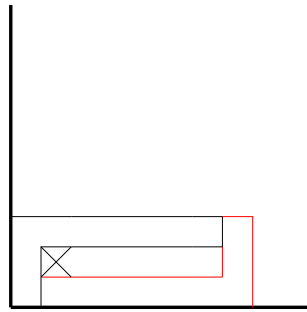
Figure 13: The Only Placement for Case 1.

We are immediately are blocked off. This completes Case 1.

Case 2) For case 2 we have four possible placements at the marked cell
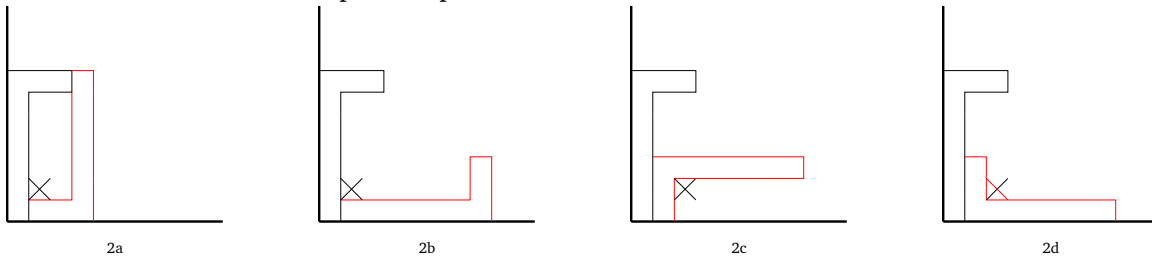
Figure 14: The Four Placements for 2.

For 2a we are again blocked from making any more placements. For 2b we have two options for covering the cell marked cell.

11

2b-i   2b-ii
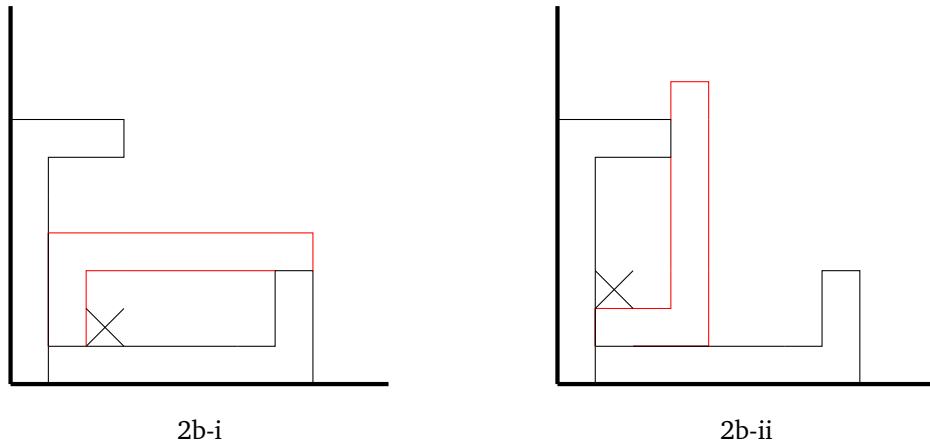
Figure 15: The Only Two Placements for 2b.

In both cases we are blocked from tiling the marked cell. For 2c there is only one possible placement.
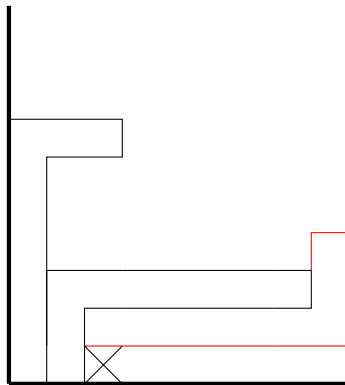
Figure 16: The Only Placement for 2c.

We are blocked off from tiling the marked cell. For 2d we have four possible placements.

2d-i   2d-ii   2d-iii   2d-iv
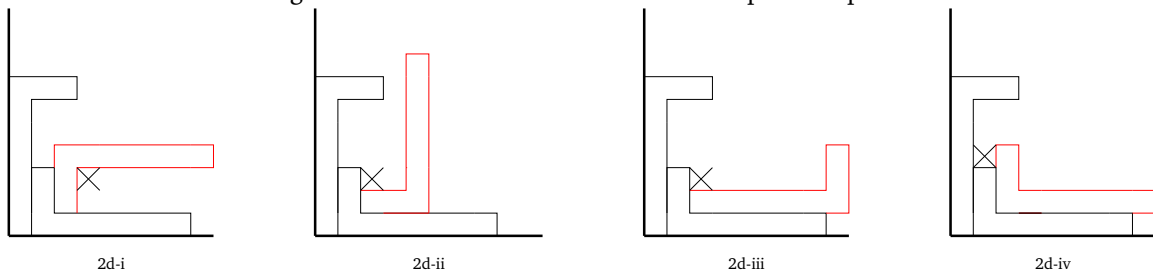
Figure 17: The Only Four Placements for 2d.

For 2d-i we only have two tile placements at the marked cell.
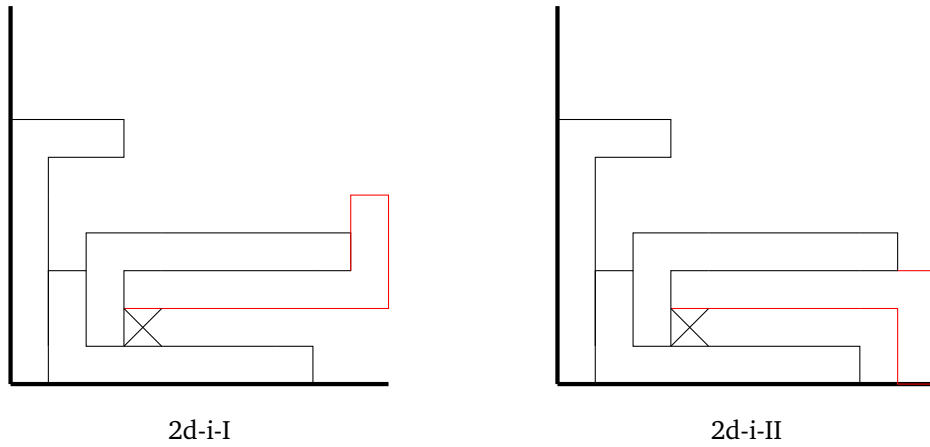
Figure 18: The Only Two Placements for 2d-i.

For 2d-i-I we can not make a tile placement at the marked cell without either overlapping a previous placement or going out of bounds. As for 2d-i-II we are immediately blocked off and can not get to the marked cell without overlapping a previous tile or going out of bounds. For 2d-ii, there are no possible placements without overlapping any previous placements. For 2d-iii we again have two possible tile placements.
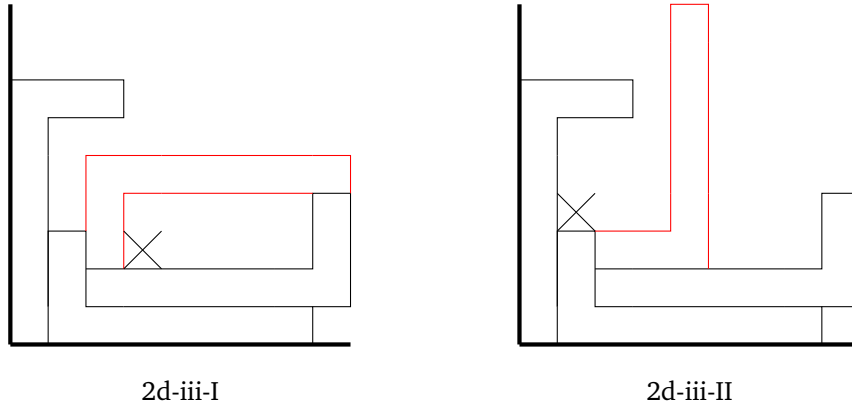
Figure 19: The Only Two Placements for 2d-iii.

In 2d-iii-I we are immediately blocked off. As for 2d-iii-II we have only one possible placement at the marked cell.
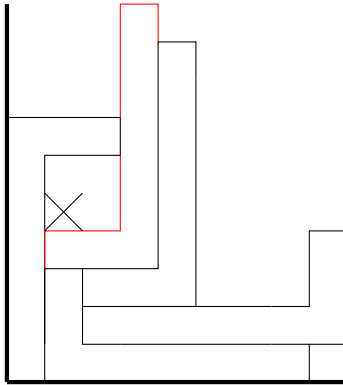
Figure 20: The Only Placement for 2d-iii-II.

We immediately see that we are blocked from tiling the marked cell. There is only one possible placement for 2d-iv.
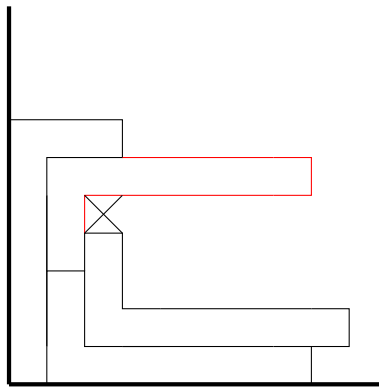
Figure 21: The Only Placement for 2d-iv.

There are only two possible placements for the marked cell.

2d-iv-I   2d-iv-II
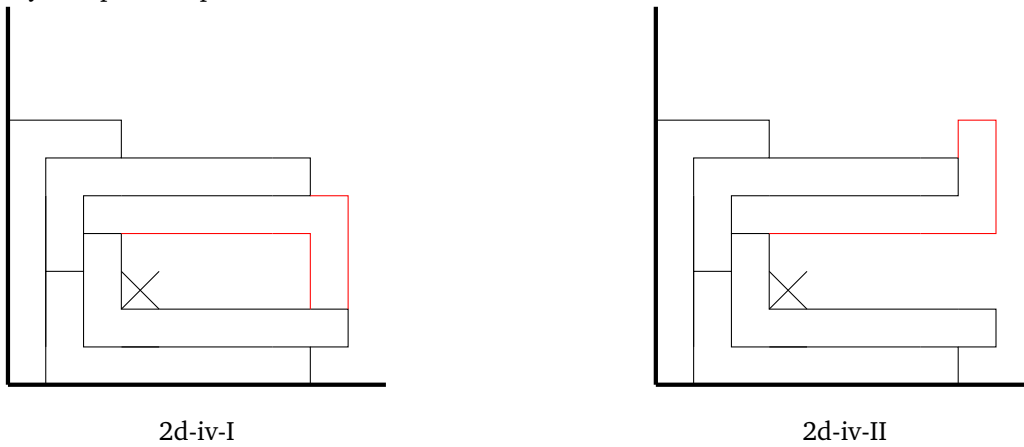
Figure 22: The Only Two Placements for 2d-iv.

For 2d-iv-I we can immediately see that we are blocked off from the marked cell. For 2d-iv-II we have only two possible tile placements.

2d-iv-II-A   2d-iv-II-B

Figure 23: The Only Two Placements for 2d-iv-II.

For both of these sub cases we are immediately blocked off from tiling the marked cells. This completes Case 2.

Case 3) For this case we have six tile placements to tile the marked cell.

3a   3b   3c

3d   3e   3f

Figure 24: The Six Possible Placements for Case 3.

For 3a we have only one possible placement.

Figure 25: The Only Tile Placement for 3a.

We immediately see that we are blocked from tiling the marked cell. For 3b we have four possible placements at the marked spot.

3b-i   3b-ii   3b-iii   3b-iv
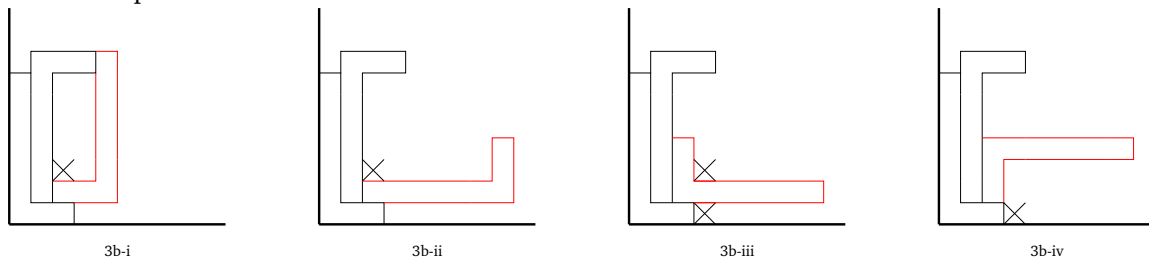
Figure 26: The Four Placements for 3b.

We are already blocked off in 3b-i. For 3b-ii there are only two possible placements at the marked cell.

3b-ii-I   3b-ii-II
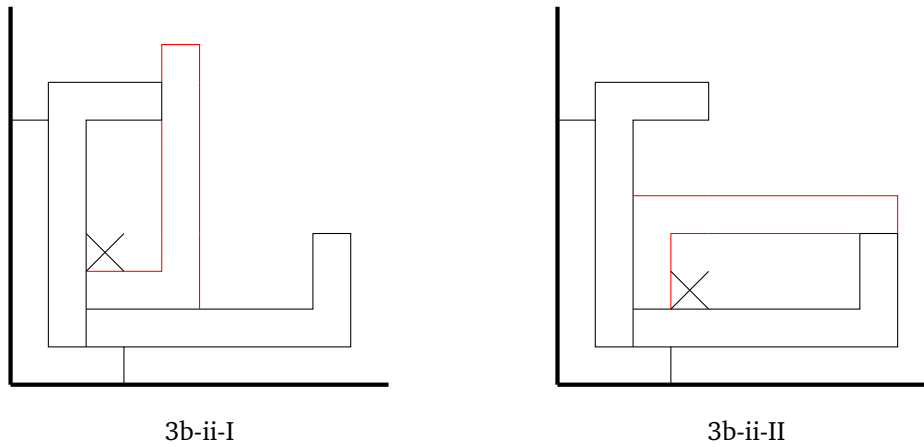
Figure 27: The Two Placements for 3b-ii.

As we can see in both cases we are blocked from making any further placements on the marked cell. For 3b-iii we have only one way of tiling the bottom marked cell and then two ways of tiling the upper marked

cell. Hence we have two possible cases to deal with.
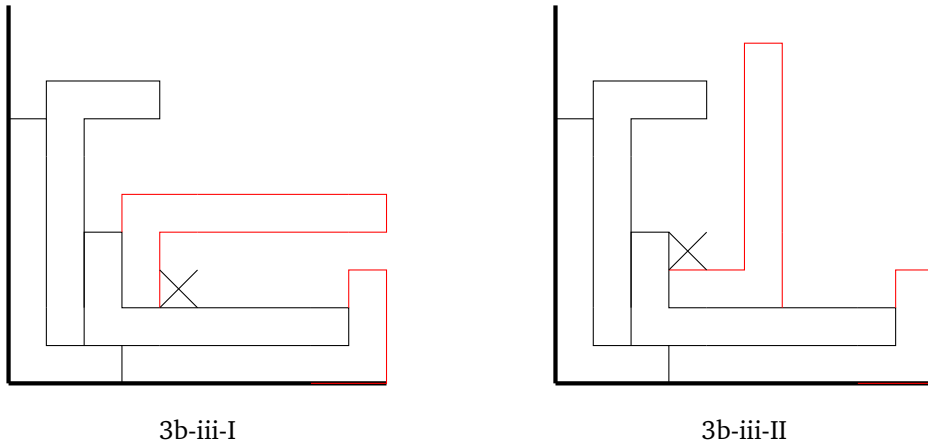
3b-iii-I   3b-iii-II

Figure 28: The Two Placements for 3b-iii.

In both cases we have no way of tiling the marked cell without overlapping a previous tile. For b-iv we have three possible placements at the marked cell.

3b-iv-I   3b-iv-II   3b-iv-III
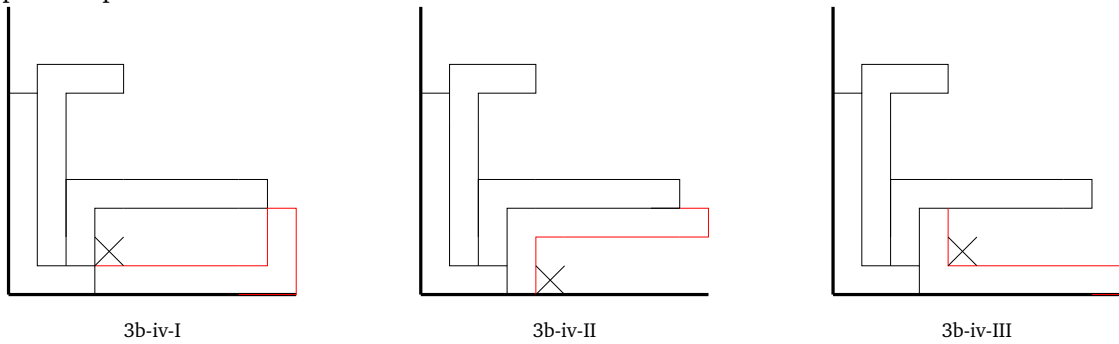
Figure 29: The Only Three Placements of b-iv.

In 3b-iv-I we are immediately blocked off from tiling the marked cell. For 3b-iv-II we have only one possible placement at the marked cell.
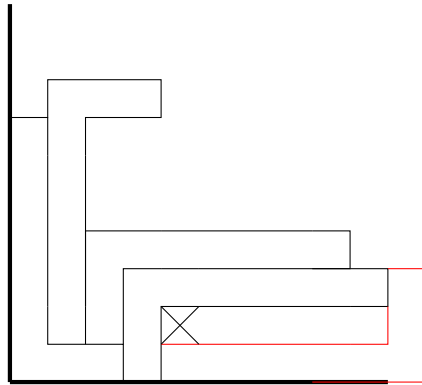
Figure 30: The Only Placement for 3b-iv-II.

We are immediately blocked off from tiling the marked cell. For 3b-iv-III we also have only one possible placement.
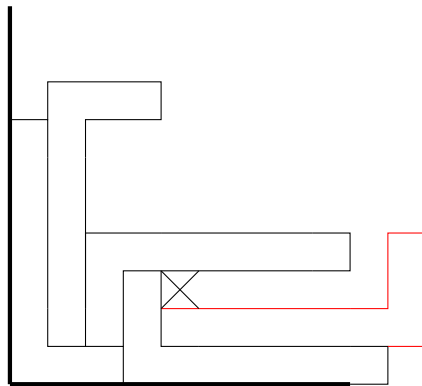
Figure 31: The Only Placement for 3b-iv-III.

There is no possible way to tile the marked cell without overlapping any other tile. For 3c there are only two possibilities to fill in the marked cell.
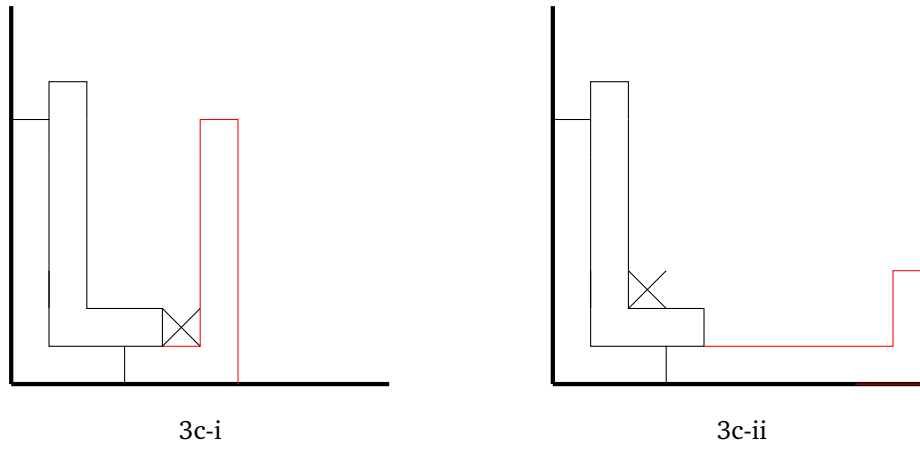
Figure 32: Two Placements for 3c.

For c-i there are only two ways to tile the marked cell.
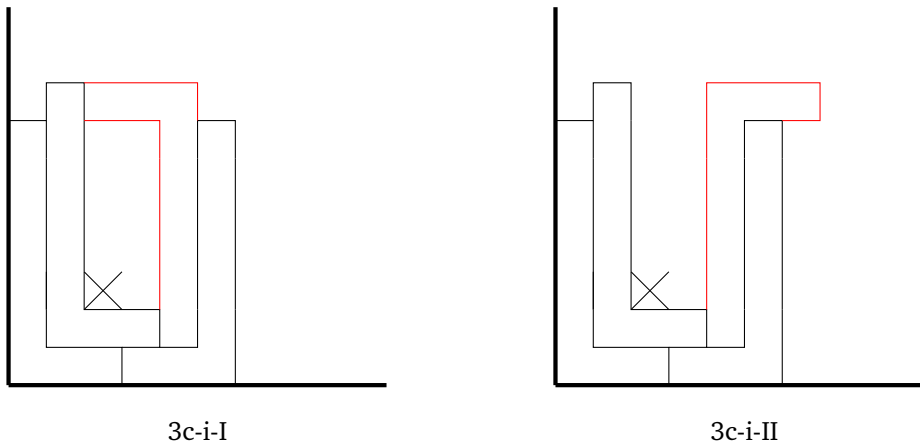
Figure 33: The Only Two Placements for 3c-i.

For 3c-i-I we are immediately blocked off. For 3c-i-II there are two possible placements.

3c-i-II-a   3c-i-II-b

Figure 34: The Only Two Placements for 3c-i-II.

In both of the cases given above we are blocked off from tiling the marked cell. For c-ii we have six placements at the marked cell.

3c-ii-I   3c-ii-II   3c-ii-III

3c-ii-IV   3c-ii-V   3c-ii-VI

Figure 35: Six Placements After c-ii.

For the 3c-ii-I and 3c-ii-II we are immediately blocked off from tiling the marked cell. For 3c-ii-III we can not tile the marked cell without overlapping another tile. For 3c-ii-IV we only have two possible placements.

3c-ii-IV-A   3c-ii-IV-B

Figure 36: The Only Two Placements for 3c-ii-IV.

We immediately see that we are blocked from tiling the marked cell in 3c-ii-III-A. There is only one way to tile the marked cell in 3c-ii-IV-B.
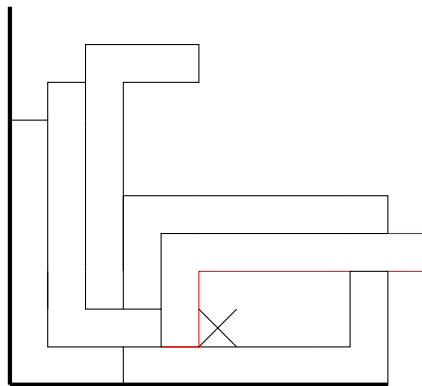
Figure 37: The Only Placements for 3c-ii-IV-B.

Again we are immediately blocked from tiling the marked cell. For 3c-ii-V we only have two possible tile placements.
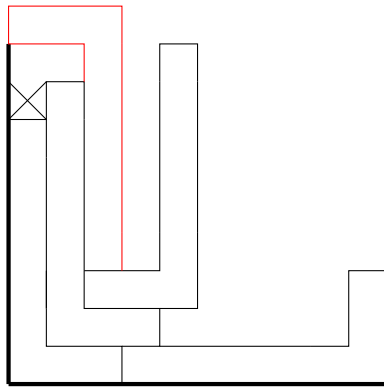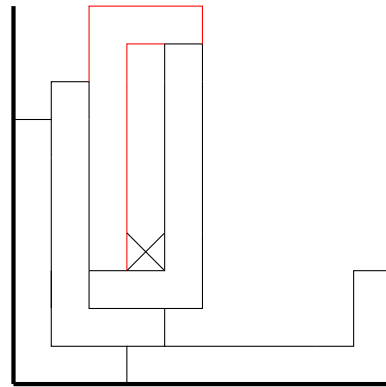
3c-ii-V-A   3c-ii-V-B

Figure 38: The Only Two Placements for 3-ii-V.

In both of the placements above we are blocked from tiling the marked cell. For 3c-ii-VI, there is only one possible placement.
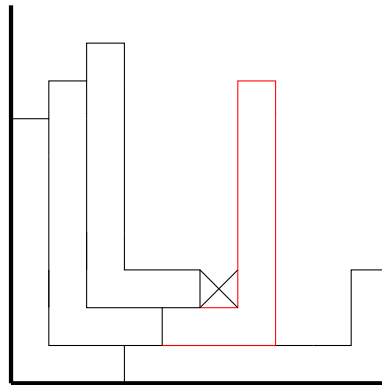
Figure 39: The Only Placement of 3c-ii-VI.

There are then only two possible placements at the marked cell for 3c-ii-VI.
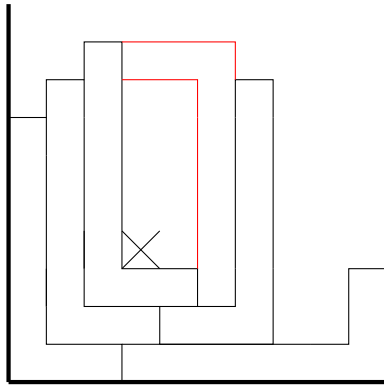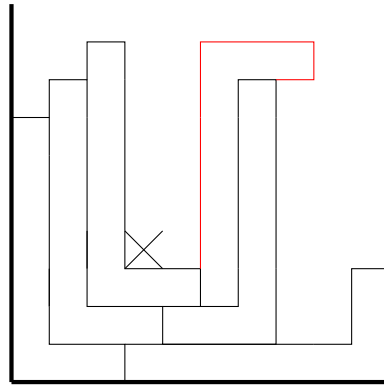
Figure 40: The Only Two Placements for 3c-ii-VI.

We are immediately blocked from tiling the marked cell in 3c-ii-VI-A. However, for 3c-ii-VI-B we have two possible placements.
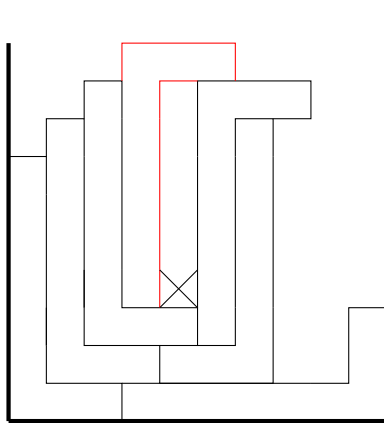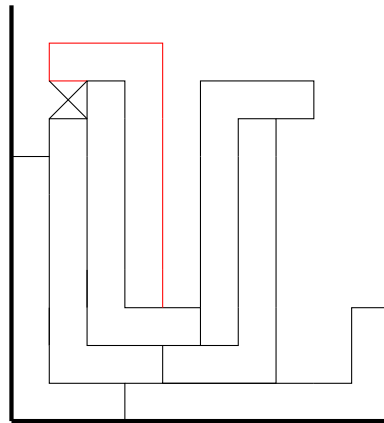
Figure 41: The Only Two Placements for 3c-ii-VI-B.

For 3c-ii-VI-B-1 we see that we are immediately blocked off from tiling the marked cell. For 3c-ii-VI-B-1we can not tile the marked cell without going out of bound.

For case 3d there is only one placement for the bottom marked cell and then there are seven possible placements for the upper marked cell.
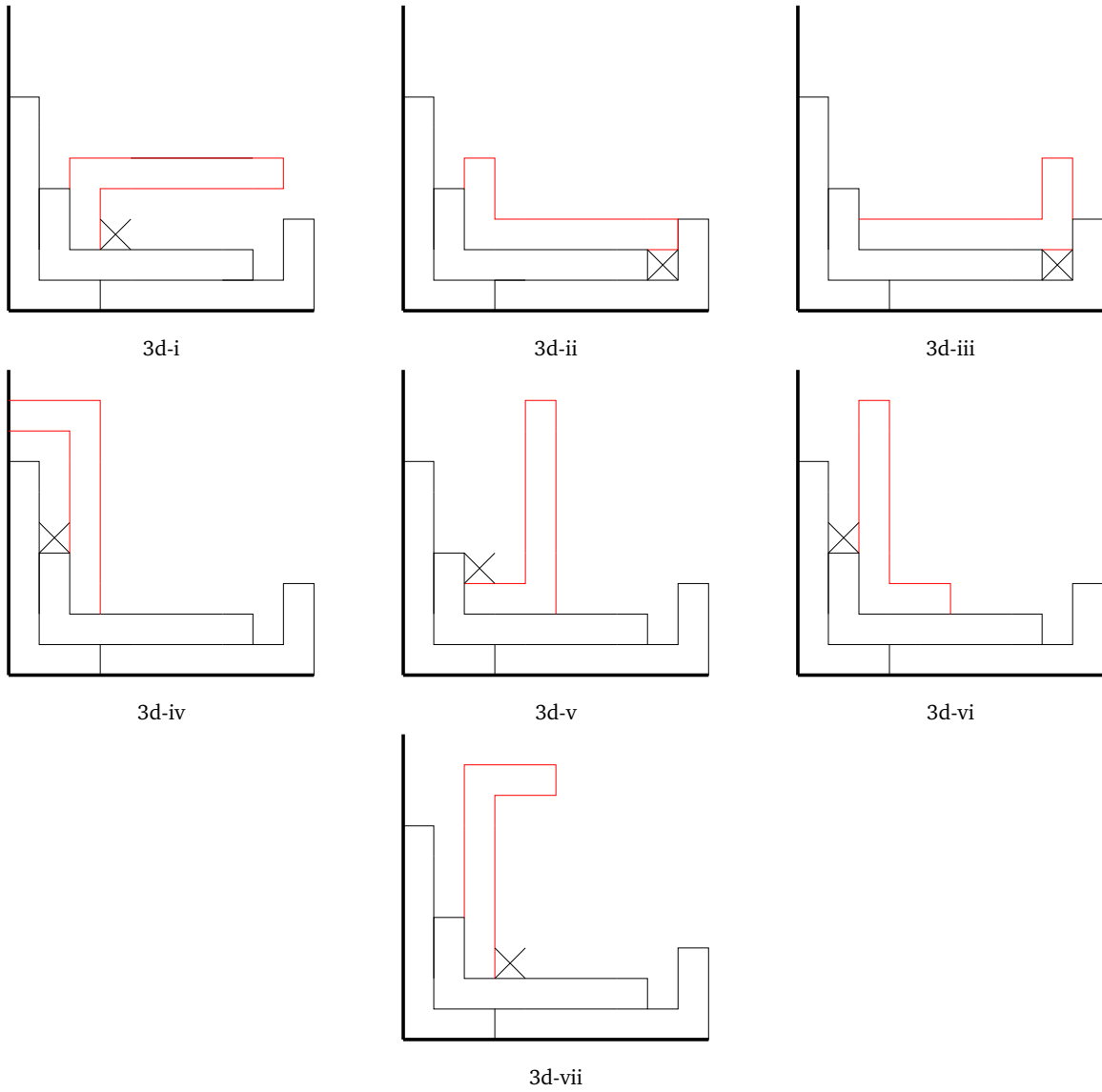
Figure 42: The Seven Placements for 3d.

For 3d-i there is no possible tile placement that can tile the marked cell without overlapping an already placed tile. For 3d-ii, 3d-iii, and 3d-iv we are immediately blocked off. For 3d-v there are two possible placements.
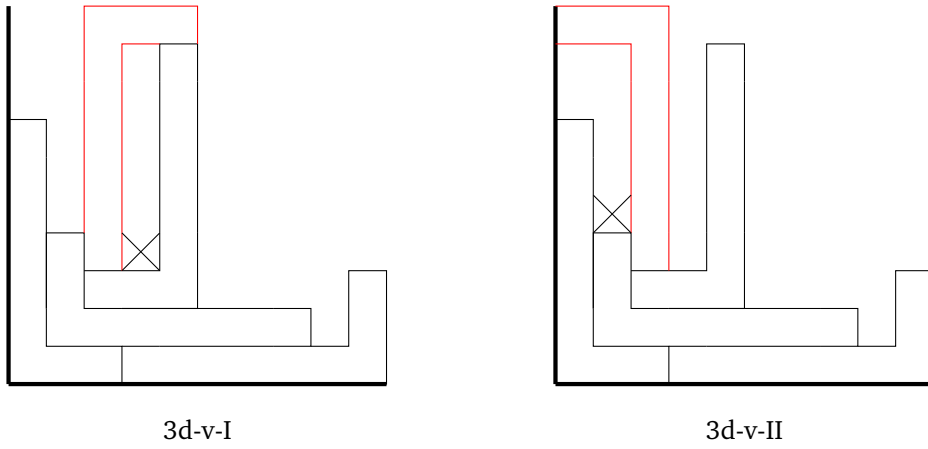
Figure 43: The Only Two Placements for 3d-v.

Again we see that we are immediately blocked off from tiling the marked cell. There is only one placement for 3d-vi.
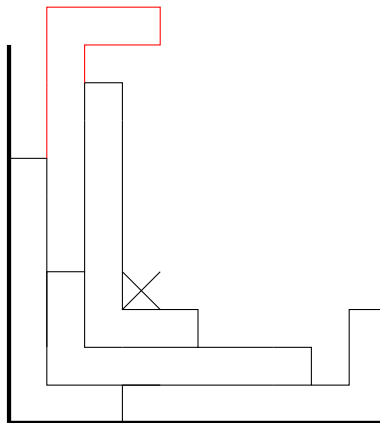
Figure 44: The Only Placement for 3d-vi.

After this placement we have six possible tile placements at the marked cell.

3d-vi-I

3d-vi-II

3d-vi-III

3d-vi-IV
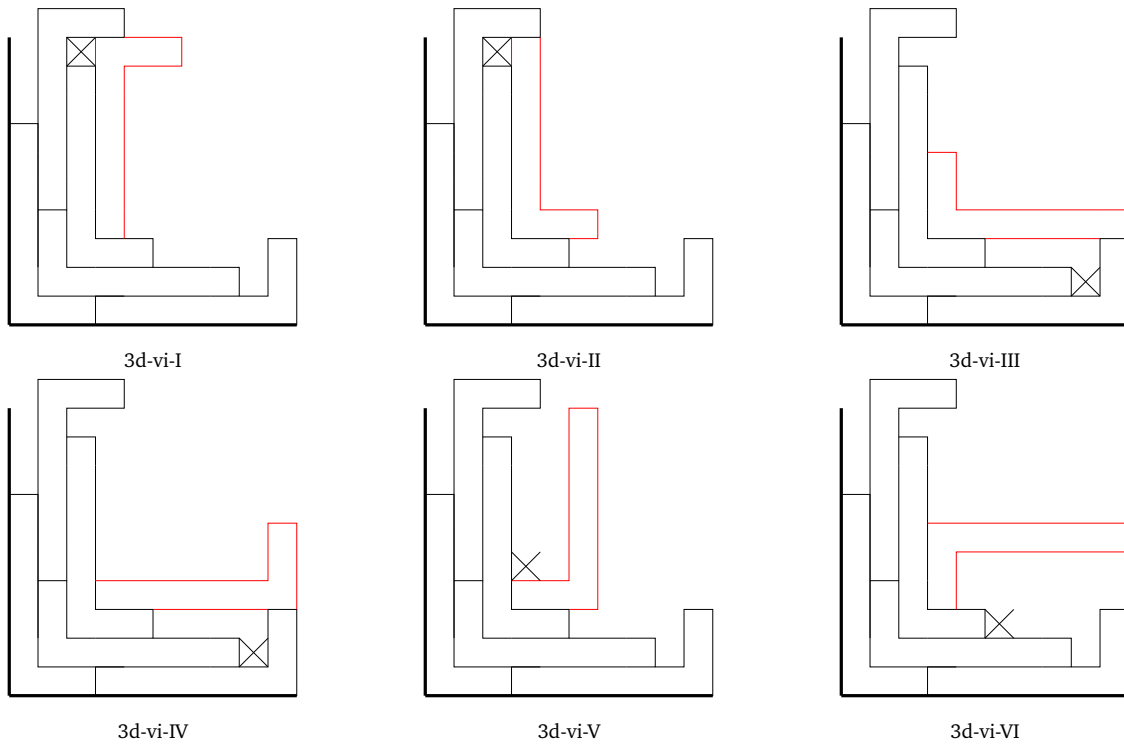
3d-vi-V

3d-vi-VI

Figure 45: The Six Placements for 3d-vi.

For 3d-vi-I, 3d-vi-II, 3d-vi-III, and 3d-vi-IV we are immediately blocked off. For 3d-vi-V we have no possible placement that tiles the marked cell without overlapping a previous placement. For 3d-vi-VI there is only possible placement at the marked cell.
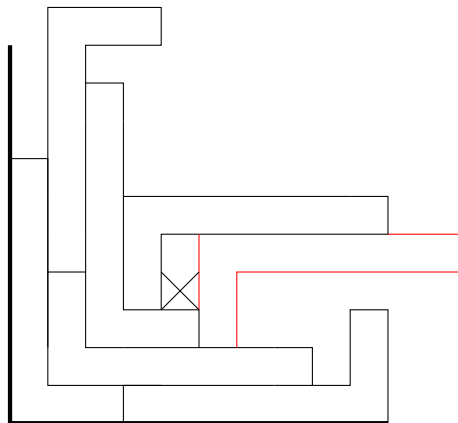

Figure 46: The Only Possible Placement for 3d-vi-VI.

We are immediately blocked off from tiling the marked cell. For 3d-vii we have two possible placements.
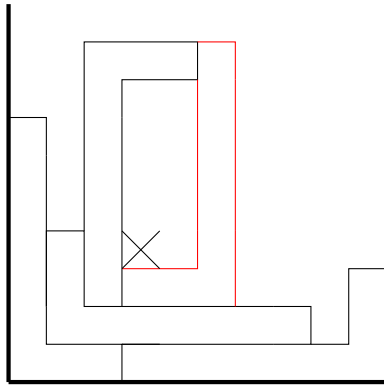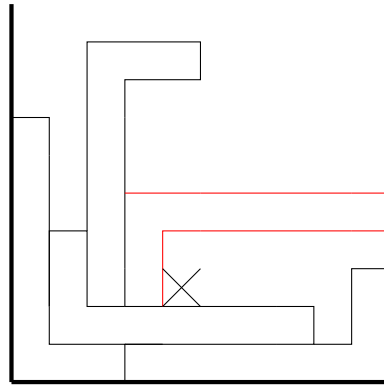
3d-vii-I   3d-vii-II

Figure 47: The Only Two Placements for 3d-vii.

For 3d-vii-I we are immediately blocked off. For 3d-vii-II we again have no way of tiling the marked cell without overlapping another tile.

For 3e we have only four possible tile placements.
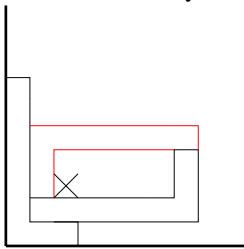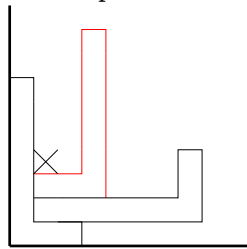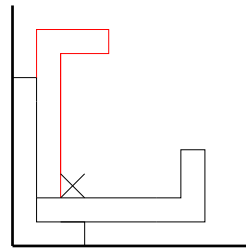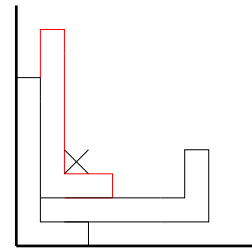
3e-i   3e-ii   3e-iii   3e-iv

Figure 48: The Only Four Placements for 3e.

For 3e-i we are immediately blocked off. For 3e-ii we have only one tile placement.
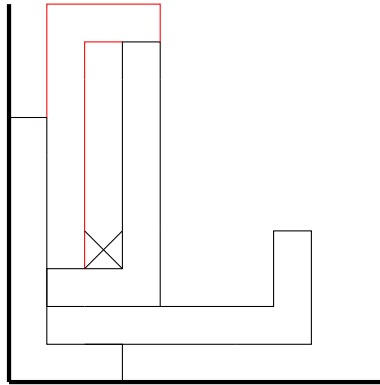
Figure 49: The Only Placement of 3e-ii.

We can immediately see that we are blocked from tiling the marked cell. For 3e-iii we have two possible tile placements.
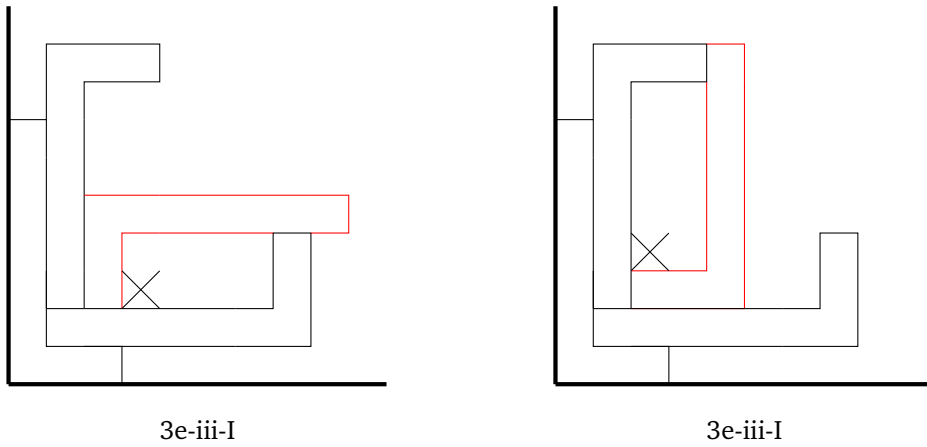
3e-iii-I   3e-iii-I

Figure 50: The Two Placements of 3e-iii.

In both cases we are immediately blocked off. For 3e-iv there are five possible tile placements at the marked cell.

3e-iv-I   3e-iv-II   3e-iv-III   3e-iv-IV   3e-iv-V
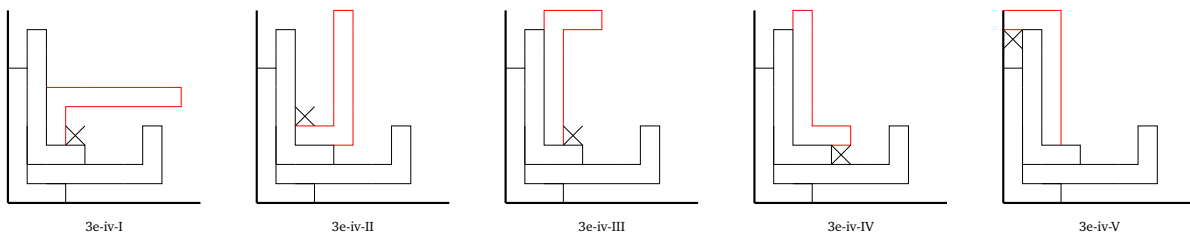
Figure 51: The Five Sub-Cases for 3-e-iv.

For 3e-iv-I and 3e-iv-V we are blocked off from tiling the marked cell without overlapping any other tile placement. For 3e-iv-II we have two possible tile placements.

3e-iv-II-A   3e-iv-II-B
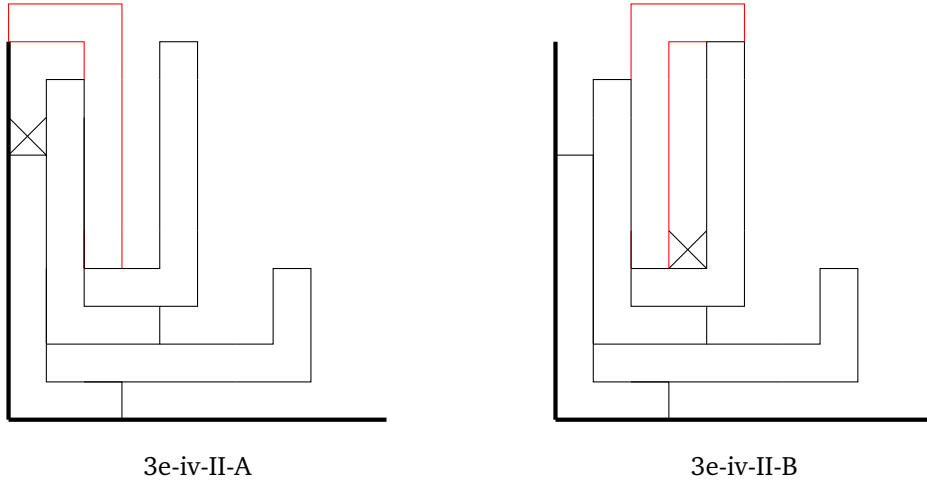
Figure 52: The Only Two Placements of 3e-iv-II.

In both cases we are immediately blocked off from tiling the marked cell. In 3e-iv-III we also have two possible placements.
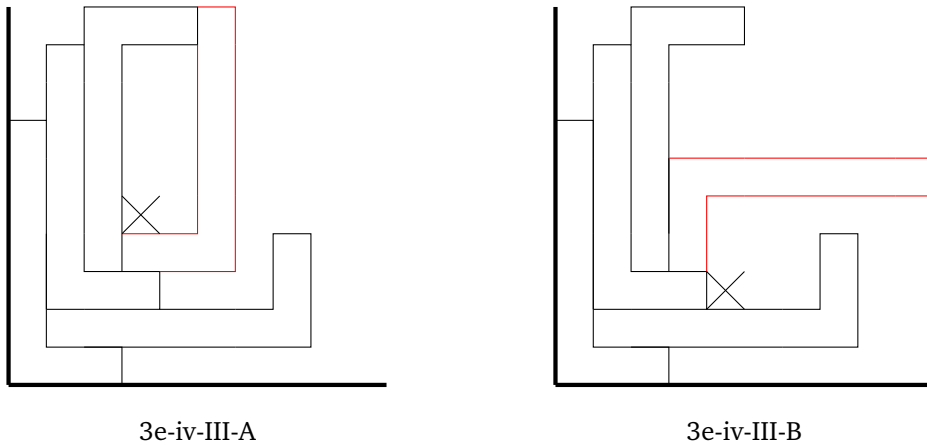
3e-iv-III-A   3e-iv-III-B

Figure 53: The Only Two Placements of 3e-iv-III.

In 3e-iv-III-A we are immediately blocked off from tiling the marked cell. In 3e-iv-III-B we only have one possible placement.
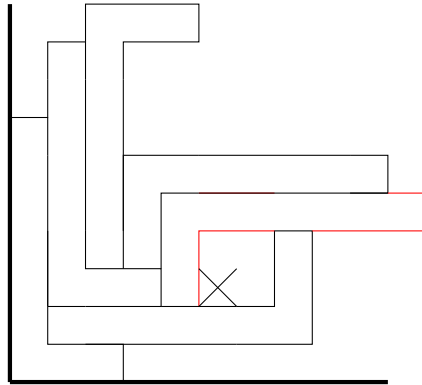
Figure 54: The Only Placement of 3e-iv-III-B.

We immediately see that we are blocked from tiling the marked cell. It should be noted that the blocked off region would not exist if $n = 7$. For 3e-iv-IV we only have one possible placement at the marked cell.
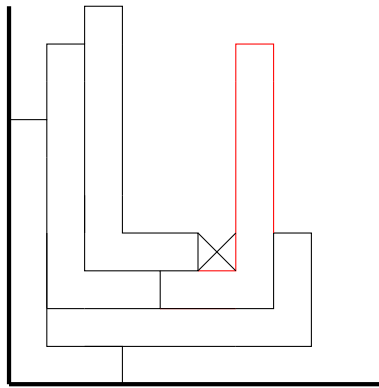
Figure 55: The Only Placement for 3e-iv-IV.

Now there are two possible placements at the marked cell.
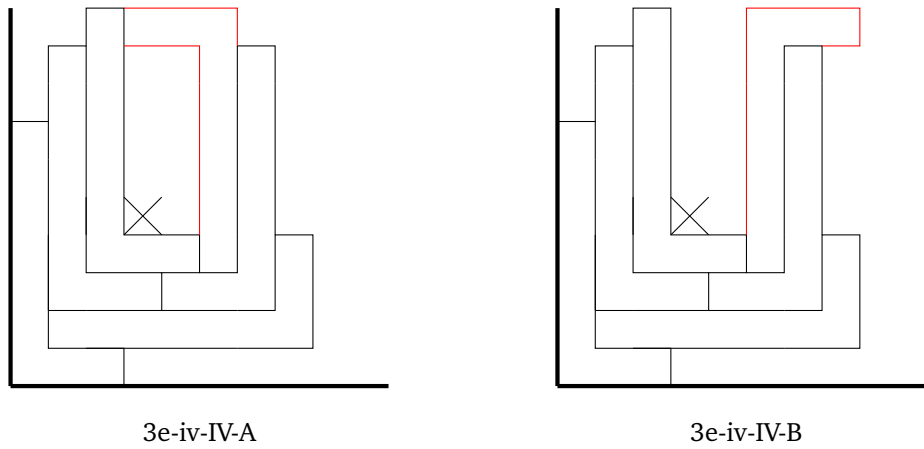
Figure 56: The Only Two Placements for 3e-iv-IV.

In 3e-iv-IV-A we see that we are immediately blocked from tiling the marked cell. For 3e-iv-IV-B we only have two possible tile placements at the marked cell.
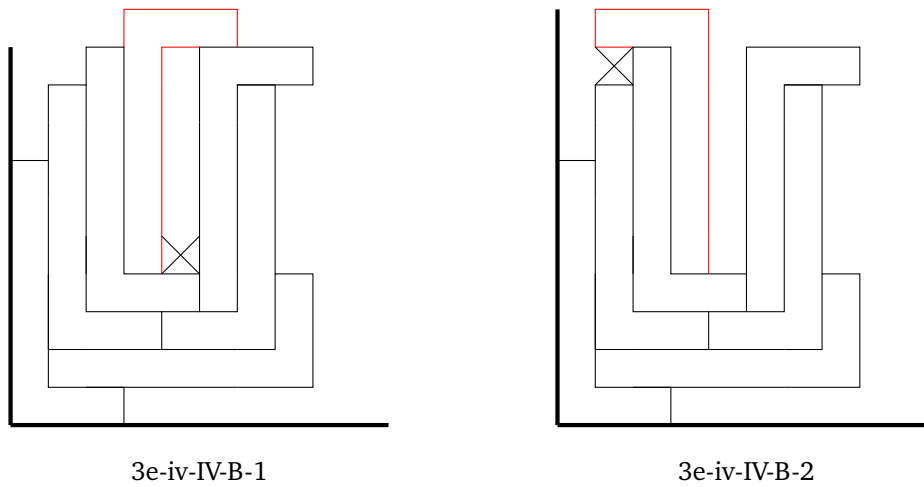
Figure 57: The Only Two Possible Placements for 3e-iv-IV-B.

For 3e-iv-IV-B-1 we are blocked off from placing any further tiles. In 3e-iv-IV-B-2 we can not tile the marked cell without going out of bounds.

For 3f there is only one way to tile the marked cell.
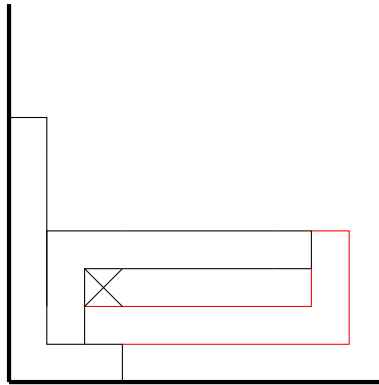
Figure 58: The Only Possible Tile Placement for 3f.

We are again blocked off from tiling the marked cell. This completes case 3 and the proof. □

For the cases 2d-iii-II and 3e-iv-III-B we would need to consider more tile placements for $n = 7$ to get the desired result.

2.3   Even and Odd Polyominoes

In example 2.1 we have already seen an example of a polyomino which is rectifiable and of even order (the smallest rectangle that it tiles requires an even number of polyominoes). It is an interesting question to ask whether or not there are any polyominoes that are odd. What's more, are there polyominoes that are of odd order (the smallest rectangle that it tiles requires an odd number of polyominoes)? The answer to the former is, "yes". Klarner [12] shows that there are an infinite number of distinct odd polyominoes. The general family of these polyominoes is given next.
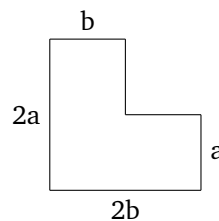
Figure 59: Infinite Family of Odd Polyominoes.

One such example of an odd tiling coming from this family is given next.

Example 2.6. *Let $a = 1$ and $b = 2$. Then* *uses 11 copies to tile a 11×6 rectangle, [12].*

32
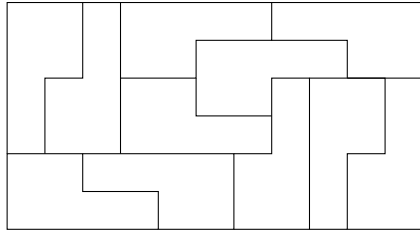
Figure 60: Example of Odd Tiling, [12].

□

This is an interesting example because it is the smallest odd number that permits a non-rectangular polyomino to be rectifiable. There are many other polyominoes that are odd. Reid [18] presents another infinite family of polyominoes that are odd under some special conditions. Reid proved that $L(a, b, c)$ is odd when its basic rectangle has sides that are relatively prime. This family is given next.
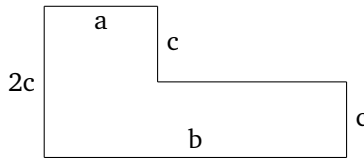
Figure 61: The $L(a, b, c)$ Polyomino.

Where the basic rectangle is $(a + b) \times 2c$.
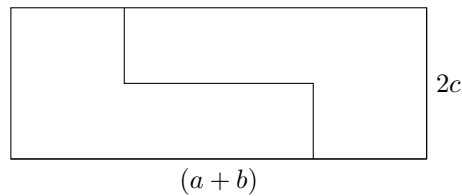
Figure 62: $(a + b) \times 2c$ Basic Rectangle.

Reid [18] shows that if **GCD**$(a + b, 2c) = 1$, then $L(a, b, c)$ is odd. We give an example of an application of his theorem next.

Example 2.7. *The polyomino P_{2n+1} is odd for all natural numbers n.*
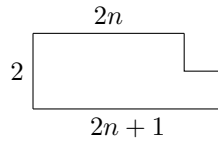
Figure 63: P_{2n+1}.

We first notice that $P_{2n+1} \cong L(2n, 2n+1, 1)$ and that P_{2n+1} has basic rectangle $(4n+1) \times 2$.
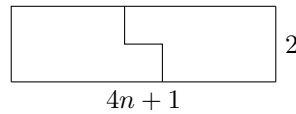
Figure 64: $(4n+1) \times 2$ *Basic Rectangle.*

Then, $4n+1$ is an odd integer for every natural number n and it follows that **GCD**$(4n+1, 2) = 1$. Thus by **Theorem 2.2** [18], P_{2n+1} is odd for every n.

$\square$

It is still an open question as to whether or not there exists a polyomino P so that $5, 7$, or 9 copies of P permits a tiling of a rectangle. Moreover, it is still not known whether or not there exist a polyomino P whose order is odd. Stewart and Wormstien [19] show that there exists no nontrivial polyomino (a polyomino which is not already a rectangle) that is of order 3, however, they are skeptical that their argument can be extended to show the result for $5, 7, 9, \dots$. All evidence thus far has suggested that every nontrivial polyomino which tiles a rectangle has even order.

3 ALGEBRAIC APPLICATIONS TO TILING

Up until this point we have only seen proofs, or rather disproofs, of a polyomino tiling a region using a method that Golomb calls backtracking [8]. Is there a more "elegant" way of proving such results? If the region is finite, then sometimes there is. If the region is infinite, however, this becomes a significantly harder problem. The arguments almost always depend on the geometric properties of the polyomino. In this chapter, we will explore some techniques for proving or disproving that a finite protoset of polyominoes tiles a finite region R.

3.1 Coloring Arguments

Let us look at the problem that we posed in the introduction.
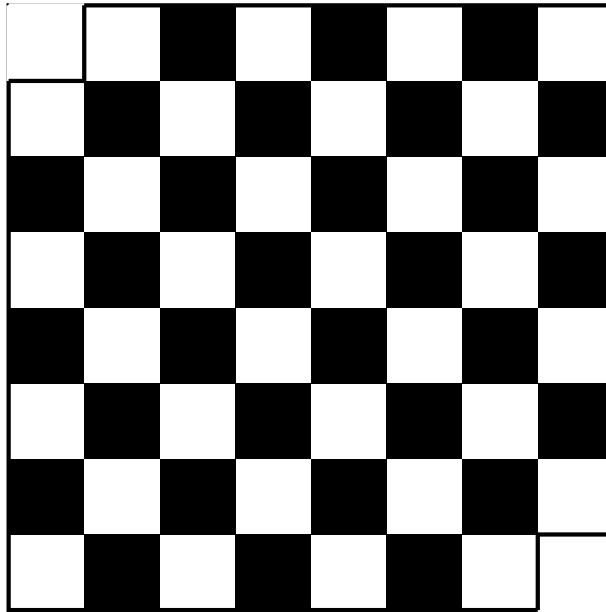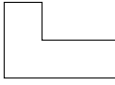
Figure 65: Mutilated Checkerboard.

Is it possible to cover the whole board with only ⊔ (dominos)? We can solve this problem by the backtracking method, however, this approach is not very efficient. After careful examination of the problem one can easily see that any placement of the domino necessarily covers one white cell and one black cell, i.e. ▪ and ▪⊔. The mutilated 8×8 checkerboard has 32 white cells and 30 black cells. Hence, there is no possible combination of domino placements that will cover the board since each placement covers an equal amount of white and black cells.

The next result was proposed by Golomb [4] and solved by Klarner [10] using a coloring argument. The result is as follows:

Theorem 3.1. *The L-tetromino* *tiles an $a \times b$ rectangle if and only if a and b are greater than 1 and $ab \equiv 0 \mod (8)$.*

Proof.

Since we are tiling with the tetromino it is necessary that the area of the rectangle is divisible by 4, that is $ab \equiv 0 \mod (4)$. Therefore, without loss of generality, we may choose a to be even. Color the first row all black and the second row all white. Then in an alternating fashion, color each row thereafter white or black. i.e.
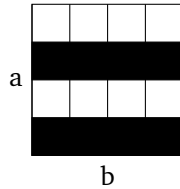
Figure 66: Example of the Coloring.

There are $\dfrac{a}{2}$ black rows. Notice that every placement of the L-tetromino covers either 3 white squares and 1 black square or, vice versa, 3 black squares and 1 white square. If the $a \times b$ rectangle is tiled by the L-tetromino then it is tiled by n which cover 3 white squares and 1 black square and m which cover 3 black squares and 1 white square. Then we have the following equalities:
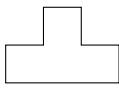
$$\frac{ab}{2} = 3m + n$$
$$\frac{ab}{2} = 3n + m$$

This implies that $m = n$. So the total number, T, of L-tetrominos used is $T = m + n = m + m = 2m$ which is even. Since we must use an even number of L-tetrominos, the area must be divisible by 8.

Lastly, without loss of generality, suppose $a = 1$ and $b = 8k$ for some natural number k. The L-tetromino obviously can not tile any $1 \times 8k$ rectangle. Thus, we also have the restriction that a and b must be greater than 1.

$\square$

From these examples we can immediately see the power and beauty of coloring arguments. However, it should be noted that sometimes coloring arguments can not be formed. It seems that in these cases one must rely on the geometric properties of the polyomino to prove the desired result. One example of this is the classic result from Walkup [21].

Theorem 3.2. *The T-tetromino* tiles an a×b rectangle if and only if a and b are both divisible by 4.

The proof of this theorem is rather involved and relies heavily on how these T-tetrominos fit together in the infinite plane [21], [13]. It is a purely geometric argument and can be shown that one can not form a coloring argument to prove this result.

3.2   Linear Systems and Signed Tilings

A natural way to think about tiling a finite region is by use of linear systems. Consider the domino and the following region R.
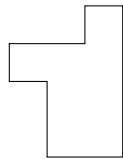
Figure 67: Region for Tiling, R.

Now consider all possible placements at each cell in the region.
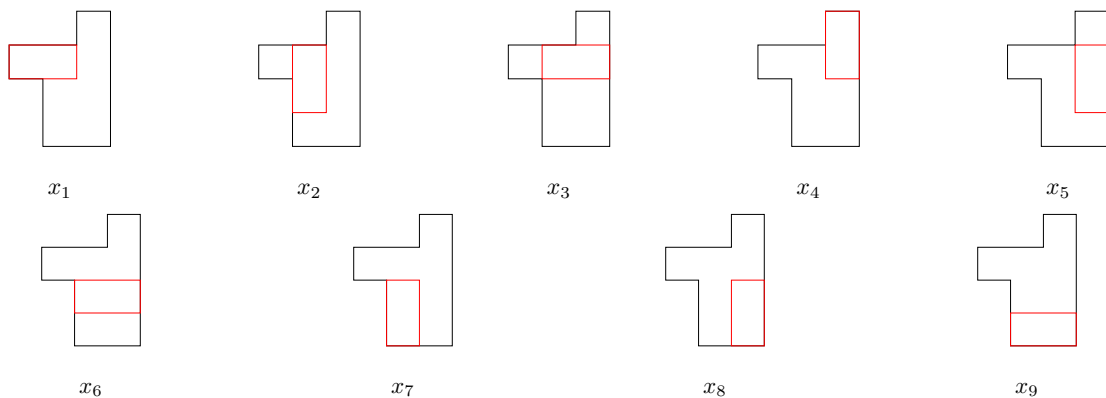
Figure 68: All Possible Placements in the Region.

Let the columns of our linear system be the placements $x_1, ..., x_9$ and the rows be each cell in the region. Then we will have the following linear system to solve.

37

$$\begin{aligned}
x_1 &&&&&&&&&&&&&&&&&= 1\\
x_1 &+& x_2 &+& x_3 &&&&&&&&&&&&&= 1\\
&& && x_3 &+& x_4 &+& x_5 &&&&&&&&&= 1\\
&&&& && x_4 &&&&&&&&&&&= 1\\
&& x_2 &+& &&&&&& x_6 &+& x_7 &&&&&= 1\\
&&&&&& && x_5 &+& x_6 &+& && x_8 &&&= 1\\
&&&&&&&&&&&& x_7 &+& && x_9 &= 1\\
&&&&&&&&&&&&&& x_8 &+& x_9 &= 1
\end{aligned}$$

Figure 69: System of Equations for Tiling Region.

If we only consider the variables taking on values 0 or 1 (a 0 corresponding to not placing a tile and a 1 corresponding to placing a tile), then a solution to the above system indeed corresponds to a tiling and a tiling corresponds to a solution. There are two solutions to this system in $\{0,1\}$ and they are given next.

	1	2
x_1	1	1
x_2	0	0
x_3	0	0
x_4	1	1
x_5	0	0
x_6	1	0
x_7	0	1
x_8	0	1
x_9	1	0

Figure 70: The Two Solutions for the System in $\{0,1\}$.

Suppose that $x_1, ..., x_9$ are all integers. An "anti-tile" [15] is one which you subtract from a region(i.e. x_i is a negative integer).

Definition 3.1. *A **_Signed Tiling_** of R is a combination of tiles and anti-tiles that leads to a weighted tiling of 1 for every cell in the region R and 0 for every cell outside the region.*

Remark 3.1. *Signed tilings can utilize cells outside of the region as long as its "weighted" tiling is 1 for every cell inside and 0 for every cell outside after all the tiles and anti-tiles have been placed.*

One example of a signed tiling was given by Conway and Lagarias [2].

Example 3.1. *The L-tromino* permits *a signed tiling of the region R that follows.*
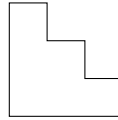
Figure 71: Region that has Signed Tiling [2].

The signed tiling is given next.
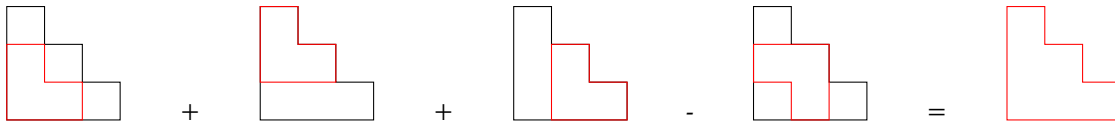
Figure 72: Sequence of Tile and Anti-Tile Placements for R [2].

□

If you have a tiling, then all cells inside the regions have a weight of 1 and every cell outside has a weight 0. Thus tilings are also signed tilings.

Remark 3.2. *If a set of tiles τ tiles a region R, then it also has a signed tiling of the region R. Equivalently, if a set does not have a signed tiling of the region R, then it does not have a tiling of R. In other words, the space of signed tilings contains the space of tilings.* △

Hence, if we can find some way of showing that a region R does not have a signed tiling from a finite protoset τ, then we can show that the region can also not be tiled by τ.
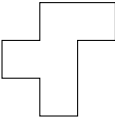
3.3   The Tile Homology Group

We would like some way of measuring whether or not a given set τ permits a signed tiling of a region R. If we can somehow show a region is completely obstructed from having a signed tiling, then we can also show that it does not permit a tiling. In the spirit of Conway, Lagarias [2], and later Reid [15], this section will be dedicated to the tile homology group of a finite protoset of polyominoes τ.

First let A be the free abelian group on the square lattice whose elements correspond to the coordinates that have weight 1 from a tile placement in τ and 0 in every other coordinate. In a similar manner we can associate a region R with an element in A. Let $B(\tau) \subseteq A$ be the subgroup generated by all possible placements of tiles in τ [15].

Definition 3.2. *The **tile homology group** is the quotient $H(\tau) = A/B(\tau)$.*

Let $a_{i,j}$ denote the elements of A and let their image in $H(\tau)$ be denoted by $\bar{a}_{i,j}$. Now we will do an example.

Example 3.2. *Consider the F-pentomino,* *, with all orientations permitted.*
The generators for the tile homology group are $\bar{a}_{i,j}$, where $i, j \in \mathbb{Z}$. We have an infinite number of relations. Let us consider the following.
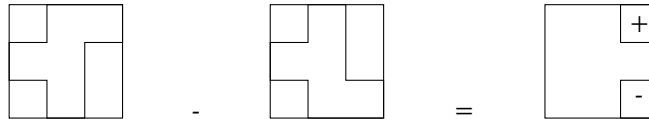
Figure 73: Shifting a Cell by Negative Two in the y-direction.

This shows that $\bar{a}_{i,j} - \bar{a}_{i,j-2} = 0$ and also that $\bar{a}_{i,j} - \bar{a}_{i,j+2} = 0$. In other words, we can shift cells up or down two units with signed tilings. Likewise it can be shown that $\bar{a}_{i,j} - \bar{a}_{i-2,j} = 0$ and also that $\bar{a}_{i,j} - \bar{a}_{i+2,j} = 0$. In other words, we can shift cells right or left two units with signed tiling. This shows that $H(\tau)$ is generated by the four elements $\bar{a}_{0,0}, \bar{a}_{0,1}, \bar{a}_{1,0}$ and $\bar{a}_{1,1}$. Let type A, B, C and D cells be the ones generated by $\bar{a}_{0,0}, \bar{a}_{0,1}, \bar{a}_{1,0}$ and $\bar{a}_{1,1}$, respectively.
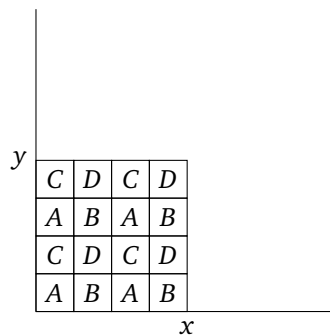
Figure 74: Example of Cell Types in Quadrant I.

Note that because we can shift cells left, right, up, and down these cells generate the whole square lattice. The infinitely many relations that we started with now condense down into four and they are as follows.

$$2A + B + C + D \;=\; 0 \tag{1}$$

$$A + 2B + C + D \;=\; 0 \tag{2}$$

$$A + B + 2C + D \;=\; 0 \tag{3}$$

$$A + B + C + 2D \;=\; 0 \tag{4}$$

Figure 75: *System of Relations.*

First we need to do row reduction over the integers and get the matrix into its reduced form. The original system A is given next.

$$A = \begin{pmatrix} 2 & 1 & 1 & 1 \\ 1 & 2 & 1 & 1 \\ 1 & 1 & 2 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix}$$

Figure 76: *System of the Four Relations.*

Its reduced form is given next.

$$H = \begin{pmatrix} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 5 \end{pmatrix}$$

Figure 77: *Reduced Form of Matrix A.*

Let a, b, c and d be the number of cells that are of the type A, B, C and D, respectively. The vector (a, b, c, d) is the element associated with R. R has a signed tiling if and only if $(a, b, c, d) \in B(\tau)$. That is, we wish to write the row vector (a, b, c, d) as an integer linear combination of the rows in H.

$$(a, b, c, d) = x(1, 0, 0, 4) + y(0, 1, 0, -1) + z(0, 0, 1, -1) + w(0, 0, 0, 5)$$

We immediately see that $x = a$, $y = b$, $z = c$. We can then deduce that $d = 4x - y - z - 5w$ or, equivalently, $w = (d + b + c - 4a) \mod (5)$, whence w is an integer. Hence, $H(\tau) \simeq \mathbb{Z}/5\mathbb{Z}$ by the following map $\phi : [R] \rightarrow \mathbb{Z}/5\mathbb{Z}$, where $(a, b, c, d) \rightarrow (d + b + c - 4a) \mod (5)$. Since $-4 \equiv 1 \mod (5)$, this map just counts the area of the region R. Therefore, there can be a signed tiling of a region R if and only if its area is a multiple of 5.

$\square$
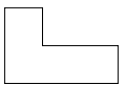
Reid [15] presents a proposition that shows that the colorings that we gave in the first section of this chapter can always be given a suitable numbering of the cells in the square lattice to achieve the same result.

Proposition 3.3 (Reid, [15])**.** *Let R be a region that does not have a signed tiling by the protoset τ. Then there is an integer numbering of the cells of the square lattice such that for some integer N we have:*

1. *Any placement of a tile covers a total divisible by N.*

2. *The region covers a total that is not divisible by N.*

Now, we will prove a result similar to that of **Theorem 3.1** using the tile homology numbering method.

Theorem 3.4. *Let $\tau = \left\{ \vcenter{\hbox{▢}} \right\}$ all orientations allowed. If τ tiles an $a \times b$ rectangle then its area is divisible by 8.*

Proof. It suffices to prove that ▢ can not tile any of the following rectangles.

$$R_1 = (2m + 1) \times (8n + 4)$$

$$R_2 = (4m + 2) \times (4n + 2)$$

$$R_3 = (8m + 4) \times (2n + 1)$$

It's not hard to show that we can shift cells up, down, left, and right by two units in the tile homology group. We then get the following isomorphism $\phi[R] \rightarrow (A - B - C + D, (2A + B - C) \mod (4))$. This map leads to two separate numberings.
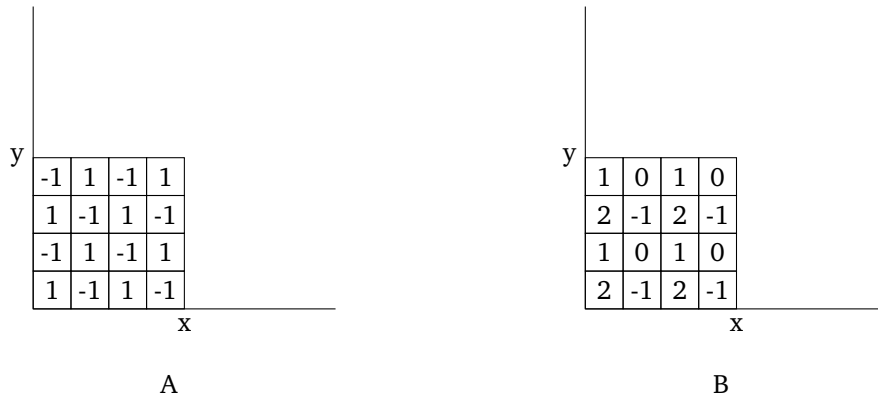
Figure 78: The Two Numberings of Square Lattice for L-Tetromino.

Now, if we take the linear combination $A+2B$ we get another numbering of the cells that is more convenient to work with.
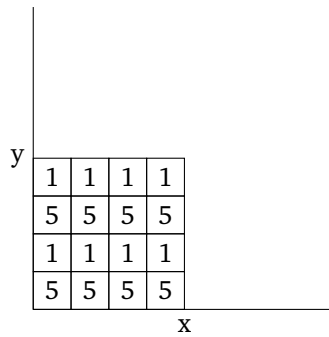
Figure 79: Numbering from Linear Combination from the Two Above.

Every placement of the L-tetromino covers a total of 8 or 16 which is congruent to $0 \mod (8)$. Place the three types of rectangles listed above at the corner of the first quadrant. Then these rectangles all cover a total that is congruent to $4 \mod (8)$. The totals are:

$$R_1 : 48mn + 24m + 40n + 20 \equiv 4 \mod (8)$$

$$R_2 : 48mn + 24m + 24n + 12 \equiv 4 \mod (8)$$

$$R_3 : 48mn + 24m + 24n + 12 \equiv 4 \mod (8)$$

Therefore, there is no possible way of having a signed tiling of any of the three rectangles. □

Remark 3.3. *The totals of the three rectangles depend on where they are in the lattice. It suffices to show the total for the rectangles at the corner of the first quadrant is not divisible by 8.* △

4 TWO INTERESTING RESULTS AND POSSIBLE EXTENSIONS TO 3D

For this section of the paper we will explore two interesting results that I studied during my masters and possible extensions to the 3rd dimension. One of the results is concerned with the possibility of finding an efficient algorithm to decide whether or not a given finite protoset τ of polyominoes can tile the infinite plane. The other result is concerned with tiling rectangles. We will also develop a possible hierarchy for polycubes in 3-d.

4.1   Tiling the Infinite Plane

Let τ be a protoset of a finite number of polyominoes. Given this set of polyominoes, can we tile the infinite plane? One way of trying to solve this problem is by the backtracking method described earlier. That is, show that there exists no way of tiling the plane by applying exhaustive case by case analysis similar to the proof of **Theorem 2.3**. However, if we want to find a tiling all we can hope for is to fortuitously stumble upon a periodic tiling by the set. It would be nice to know if there exists an efficient procedure for determining this "yes" or "no" question.

Definition 4.1. *A **decision problem** is a problem that definitively has a "yes" or "no" response. A **decision algorithm** is a procedure for determining "yes" or "no" given any set of inputs. If there does not exist such a algorithm for the decision problem, then the problem is called **undecidable**.*

The decision problem that we are concerned with is as such: Can τ tile the infinite plane? Golomb [6] shows that this decision problem is undecidable. Before we get into his proof we must first introduce some more terminology and background information.

Definition 4.2. *A **Wang tile** is a unit square with colored edges. The only way to tile with Wang tiles is to match colored edges.*

An example of a tiling with Wang tiles follows.

Example 4.1. *A tiling of a 2×4 rectangle using five distinct Wang tiles, $W = \left\{ \boxed{\ }, \boxed{\ }, \boxed{\ }, \boxed{\ }, \boxed{\ } \right\}$.*
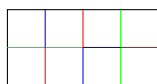
Figure 80: *Tiling of 2×4 Rectangle Using Five Distinct Wang Tiles.*

□

45

We introduce these objects because Berger [1] proved that tiling the infinite plane with any finite number of Wang tiles is algorithmically undecidable. It turns out that given any finite set of polyominoes, we can make an isomorphism to a finite set of Wang tiles. Likewise, given any finite set of Wang tiles we can make a isomorphism to a finite set of polyominoes. Hence, it can be shown that tiling the plane with any finite set of polyominoes is also algorithmically undecidable [6].

Remark 4.1. *τ must be a finite set with more that one polyomino.* △

Theorem 4.1 (Golomb, [6])**.** *Tiling the infinite plane with a finite set τ of polyominoes is algorithmically undecidable.*

Proof (sketch).

1. Mapping τ to a finite set of Wang tiles.

   First generate all of the possible orientations of each of the polyominoes in τ and put them in a new set τ'. Color each outer lattice edge of the polyominoes in τ' '0' and each interior lattice of every polyomino in τ' a unique color. Dissect every colored polyomino into unit squares and put them all into a set called W. W is a finite set of Wang tiles. By the unique coloring, this process is isomorphic because tiling with W we can only generate the original polyominoes we started with.

2. Mapping a set of Wang tiles to a finite set of polyominoes.

   Let W be a finite set of Wang tiles. Then there are a finite number of colors, so let us enumerate the colors. Write each of these numbered colors in its binary representation. Now take a square that is sufficiently large and attach corners with a distinct shape so that the left side of the new object fits only with the right side (likewise top to bottom). Use the binary to make the following modification always going top to bottom or left to right on the modified square edge. On the top and right edge of the large square if there is a 1 in the ith slot of the binary expansion, modify the ith lattice edge by extruding it by one unit square, if there is a 0, leave it unchanged. On the bottom and left edge of the large square if there is a 1 in the ith slot in the binary expansion, modify the ith lattice edge by intruding it by one unit square, if there is a 0, leave it unchanged. After this modification we have made a finite set of polyominoes that will fit together in a unique way. This process is isomorphic because given this set of polyominoes, we can tell exactly which Wang tile they were derived from by the unique code. Moreover, these polyominoes tile the same way as the original Wang tiles.

$\square$

In the sketch of the proof we used the phrase "sufficiently large square". What constitutes sufficiently large? Let N be the number of distinct colors on the Wang tiles. We will modify the $s \times s$ square where $s = \lfloor \log_2(N) \rfloor + 1$.

Example 4.2. *First we will illustrate the process of turning polyominoes into Wang tiles. Let*

$$\tau = \left\{ \; \begin{array}{c}\square\end{array} \; , \; \begin{array}{c}\square\end{array} \; \right\}$$

Generate all possible orientations.

$$\tau' = \left\{ \; \cdots \; \right\}$$

Color all of the outer edges 0.

$$\tau' = \left\{ \; \cdots \; \right\}$$

Dissect these polyominoes into unit squares and we achieve a set of 28 distinct Wang tiles. Notice that when we tile with these Wang tiles it is inevitable that we re-obtain our original polyominoes.

Now, we illustrate turning a finite set of Wang tiles into a finite protoset of polyominoes. Let

$$W = \left\{ \; {}_1\begin{array}{c}{}^2\square\end{array} \; , \; \begin{array}{c}\boxed{2}\end{array}{}_1 \; \right\}$$

and do not allow rotations. There are three colors so $s = \lfloor \log_2(3) \rfloor + 1 = 2$. Hence, will use the following 2×2 square with special corners attached.
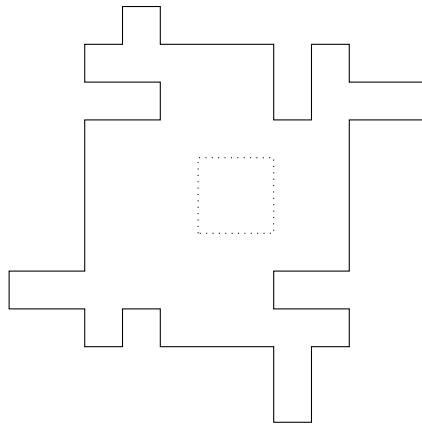
Figure 81: *The Modified 2×2 Square.*

Here the dotted square in the middle represents the 2×2 square. Notice that by construction of this geometric object, only the top will fit with the bottom as well as the right only to the left. Blank has binary expansion 00, 1 has binary expansion 01, and 2 has binary expansion 10. Now, using the coding as described in the pseudo-proof above we can construct the following two polyominoes.
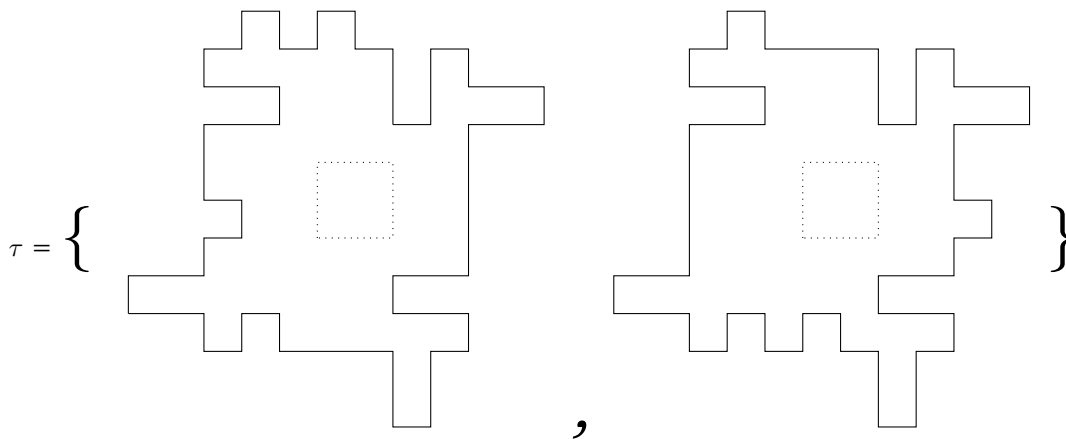
Notice that these polyominoes only fit together in the same way that their corresponding Wang tiles fit together. Moreover, given this set of polyominoes we can directly find which Wang tile they came from by the unique binary code that we have encoded. □

Remark 4.2. *We fixed the orientations of the Wang tiles in the previous example for illustrative purposes. If we allowed rotations, then we would just generate all rotations and put them in a new set W' and do the same process.* △

4.2   Tiling with Rectangles

In this section of the paper we will explore an interesting result concerned with tiling rectangles with smaller rectangles. It's not hard to see that if the domino can tile an $a \times b$ rectangle then a, b, or possibly both are multiples of 2. Is there an analogous result for the $1 \times n$ straight polyomino? That is, if a $a \times b$ rectangle is tiled by the $1 \times n$ polyomino, does a, b, or possibly both have to be a multiple of n? For example, can the 1×4 straight tetromino tile a 30×30 rectangle?

Let us first discuss the more general theorem on tiling rectangles with rectangles before we discuss the discrete version. Stan Wagon [20] gives 14 proofs of the following theorem. The proof method that we will use is similar to one that Wagon gives.

Theorem 4.2 (N. G. De Bruijn, [20]). *If an $a \times b$ rectangle R is tiled by rectangles, each of which has one integral side, then either a, b, or both are integral.*

Proof.

Consider an $a \times b$ rectangle R, which is tiled by smaller rectangles with at least one integral side. Place one corner of R at the corner of the first quadrant. Make vertices at each of the corners of the smaller rectangles. Color all of the integral edges of the smaller rectangles all the same color. Multiple edges can occur in this graph. Notice that the only vertices with degree equal to 1 are the outer vertices of the big rectangle, R, while all other vertices are of degree 2 or 4. Starting from the vertex at the corner of the first quadrant make a walk along the colored edges such that no one edges is traversed more than once. Since there are only a finite number of colored edges we will eventually end at one of the other corners of, R. Moreover, we can not end at the same corner we started with since it has degree 1 and that would then require we traverse at least one colored edge twice.

Lastly, consider the distance of each vertex to the origin in the usual Euclidean sense. Consider L, the total directed distance walked. If we walk from one vertex to another vertex that is further away from the origin, then we will add the length of that edge to L. If we walk from one vertex to a vertex that is closer to the origin, then we will subtract that length from L. Since the integers are closed under addition, L will also be an integer. Moreover, since we started at the vertex at the origin and ended at another vertex of R at least one of its sides, either a or b, must be an integer.

□

Let us now take an example to illustrate the proof method. For the example we let a W rectangle be one whose width is an integer and a H rectangle be one whose height is an integer.

Example 4.3. *Consider a rectangle R that is tiled as in **Theorem 5.2**.*
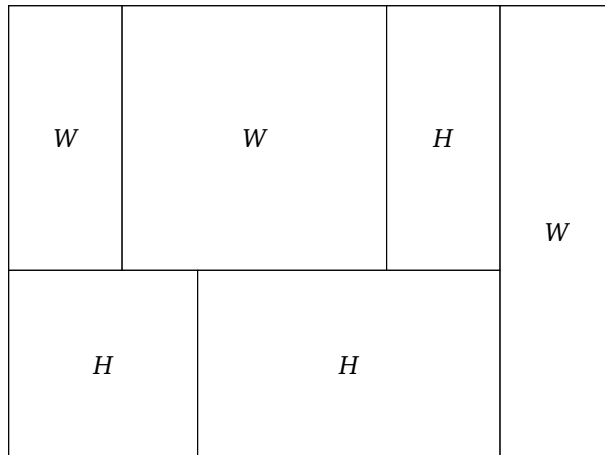
Figure 82: Rectangle R, Tiled by Smaller Rectangles of Integer Width or Height.

Now we color the integral edges.
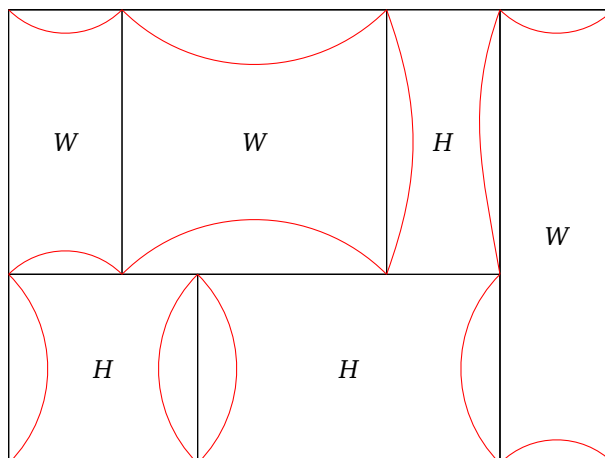
Figure 83: Coloring of the Integer Sides.

Then the colored walk starting at the origin is as follows in green.
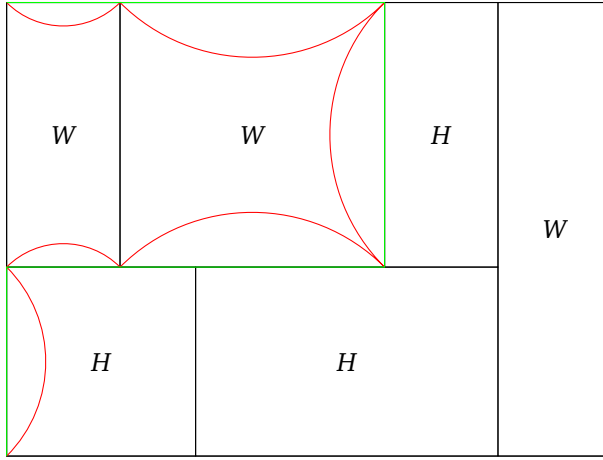
Figure 84: The Final Coloring of the Rectangle, Showing it has Integer Side.

Thus, the height of this particular rectangle is an integer. □

Remark 4.3. *This proof can be generalized to multiple dimensions. That is, if we have a d-dimensional box D that is packed with smaller d-dimensional boxes all with at least one integral side, then D also must have at least one integral side. Moreover, the only fact that we used about the integers is that they are closed under addition. There is nothing special about the integers in this regard so we can also generalize this result to a general class of groups that are closed under the binary operation "addition" (i.e. rationals, reals, algebraic, etc.) That is, if any box is tiled by smaller boxes all having at least one side length from a group G, which is closed under addition, then at least one side of the box has a side length which is also in the group G.* △

We now introduce the discrete version of this theorem, which can easily be derived from the more general one.

Theorem 4.3 (Discrete Version, [8])**.** *Suppose that an $a \times b$ rectangle R, is tiled by $1 \times n$ rectangles, then at least one a or b is a multiple of n.*

Proof.

Suppose R is a rectangle tiled as described in the theorem. Let us then scale the entire tiling by a factor of $\frac{1}{n}$. Then, R' is a $\frac{a}{n} \times \frac{b}{n}$ rectangle tiled by $\frac{1}{n} \times 1$ rectangles. Since each of the smaller rectangles has one side that is an integer, 1, then by **Theorem 4.2** R' must have at least one side that is an integer. Without loss of generality, assume $\frac{a}{n}$ is an integer. This means that $n|a$. Hence, a must be a multiple of n. □

Hence, the 1×4 straight tetromino can not tile a 30×30 rectangle because 30 is not a multiple of 4.

51

4.3   D=3 Polycubes

In this section we will explore 3-d polycubes. We develop an analagous hierarchy for 3-d polycubes, in spirit of Golomb in 2-d [5]. First, consider $\mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}$ and the unit cube lattice. As before, we will refer to each unit cube in the 3-d space again as a **cell**.

Definition 4.3. *A 3-d __Polycube__ is a finite union of unit cells, aligned on the lattice, whose interior is connected.*

We will refer to a polycube made of n cells as an n-cube. We will now give some examples of 3-D polycubes.

| Z-Tetracube | TetraCube | Tetracube | Pentacube | Notched Cube |
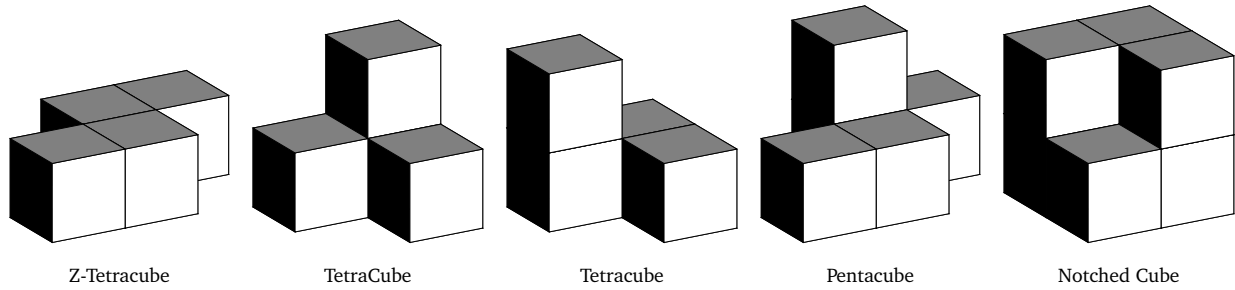
Figure 85: Examples of 3-d Polycubes.

We also have analogous definitions for a polycube to be **Boxifiable** and a **Reptile**.

Definition 4.4. *A 3-d __Box__ is a region B which is congruent to*

$$B \cong \{(x, y, z) : 0 \leq x \leq a, 0 \leq y \leq b, 0 \leq z \leq c \text{ for some natural numbers } a, b, c\}$$

Definition 4.5. *A 3-d __Half-Beam__ is a region H which is congruent to*

$$H \cong \{(x, y, z) : 0 \leq x \leq a, 0 \leq y \leq b, 0 \leq z < \infty \text{ for some natural numbers } a, b\}$$

This is just a 2-d half-strip extruded in the 3rd dimension by a finite quantity.

Definition 4.6. *A 3-d __Beam__ is a region Be which is congruent to*

$$Be \cong \{(x, y, z) : 0 \leq x \leq a, 0 \leq y \leq b, -\infty < z < \infty \text{ for some natural numbers } a, b\}$$

This is just a 2-d strip extruded in the 3rd dimension by a finite quantity.

Definition 4.7. *A 3-d __Bent-Beam__ is a region BB which is congruent to the following union of two half beams*

$$BB \cong \{(x, y, z) : 0 \leq x \leq a, 0 \leq y \leq b, 0 \leq z < \infty \text{ for some natural numbers } a, b\}$$

$$\bigcup$$

$$\{(x, y, z) : a \leq x < \infty, 0 \leq y \leq b, 0 \leq z \leq a \text{ for the same natural numbers } a, b\}$$

52

This is just a 2-d bent-strip extruded in the 3rd dimension by a finite quantity.

Definition 4.8. *A **Quarter Plank** is a region HC that is congruent to*

$$HC \cong \{(x, y, z) : 0 \leq x < a, 0 \leq y < \infty, 0 \leq z < \infty \text{ from some natural number } a\}$$

This is just a 2-d quadrant extruded in the 3rd dimension by a finite quantity.

Definition 4.9. *A **Plank** is a region C that is congruent to*

$$C \cong \{(x, y, z) : 0 \leq x \leq a, -\infty < y < \infty, -\infty < z < \infty \text{ for some natural number } a\}$$

This is just a 2-d plane extruded in the 3rd dimension by a finite quantity.

Definition 4.10. *A **Octant** is a region O that is congruent to*

$$O \cong \{(x, y, z) : 0 \leq x < \infty, 0 \leq y < \infty, 0 \leq z < \infty\}$$

Definition 4.11. *A **Shell** of an octant is a region S that is congruent to the union*

$$S \cong \{(x, y, z) : 0 \leq x \leq a, 0 \leq y < \infty, 0 \leq z < \infty \text{ for some natural number } a\}$$
$$\bigcup$$
$$\{(x, y, z) : 0 \leq x < \infty, 0 \leq y \leq b, 0 \leq z < \infty \text{ for some natural number } b\}$$
$$\bigcup$$
$$\{(x, y, z) : 0 \leq x < \infty, 0 \leq y < \infty, 0 \leq z \leq c \text{ for some natural number } c\}$$

A shell is an octant cut out of another octant.

Definition 4.12. *A **Half-Space** is a region HS which is congruent to the union of four octants whose interior is connected.*

Now that we have defined the regions that we will be working with we can establish a hierarchy for the 3-d polycubes. One possible representation of the hierarchy is given in the next figure.
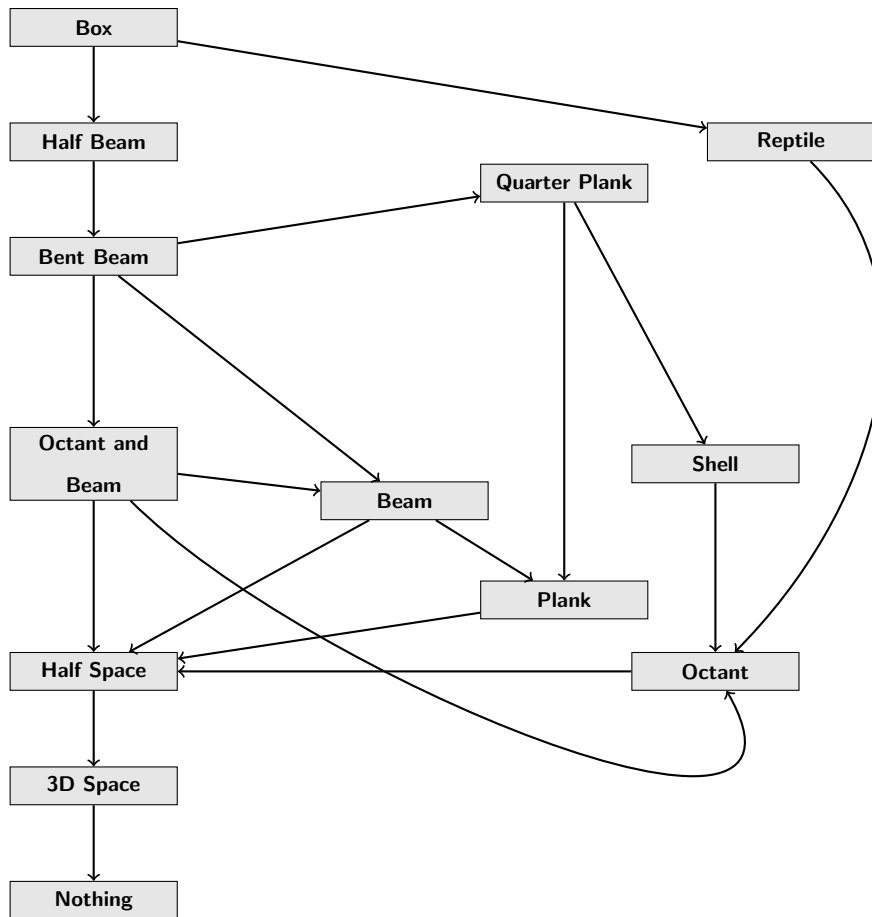
Figure 86: The 3-D Polycube Hierachy.

Again each arrow represents a logical implication. Most of these implications are trivial (in fact, most of the proofs work exactly the same way as their 2-d analog). The next theorem will take care of most of the cases which we have already considered in 2-d.

Theorem 4.4. *We have the following list of implications from the hierarchy:*

1. *If a polycube P packs a box, then it is a reptile and it packs a half beam.*

2. *If a polycube P packs a half beam, then it also packs a bent beam.*

3. *If a polycube P packs a bent beam, then it also packs a beam.*

4. *If a polycube P packs a octant, then it also packs the half space.*

5. *If a polycube P packs half space, then it also packs all of 3-d space.*

6. *If a polycube P packs 3-d space, then it also packs nothing.*

7. *If a polycube P is a reptile, then it also packs an octant.*

Proof.

1. If a polycube packs a $a \times b \times c$ box, then it can pack a $abc \times abc \times abc$ box. If the polycube is made up of n cells, then we can use n of the $abc \times abc \times abc$ boxes to make a scaled up version of itself.

   If we place the first $a \times b \times c$ box at the origin in the first octant, then we can stack consecutive $a \times b \times c$ boxes resulting in a region $\{(x, y, z) : 0 \leq x < a, 0 \leq y \leq b, 0 \leq z < \infty\}$. Thus, P also packs a half beam.

2. Suppose P packs a half beam $H \cong \{(x, y, z) : 0 \leq x \leq a, 0 \leq y \leq b, 0 \leq z < \infty$ for some natural numbers $a, b\}$. Then if we rotate H by $\frac{\pi}{2}$ with respect to the xy axis and shift the entire tiling by a units along the x-axis we get another tiling of a half beam $H' \cong \{(x, y, z) : a \leq x < \infty, 0 \leq y \leq b, 0 \leq z \leq a\}$. If we union $H \bigcup H'$ then, by definition, we get a tiling of a bent beam.

3. This is essentially the same proof as the one given in the analogous 2-d bent strip implies strip version. We only need to make minor modifications to account for the extra dimension. Without loss of generality, let us orient the bent beam so that it has one arm along the x-axis. Let h be the height of this arm, d be the depth of this arm, and w be the maximum width of the polycube P. Then we examine all of the possible patterns in these consecutive boxes (as in 2-d). Since these are all finite lattices we are bound to find a repeating pattern. We can repeat this pattern indefinitely in a packing that results in a beam.

4. Without loss of generality, suppose that P packs the first octant $O_1 \cong \{(x, y, z) : 0 \leq x < \infty, 0 \leq y < \infty, 0 \leq z < \infty\}$. Then we can reflect our packing about the x-axis, y-axis, and $y = -x$, obtaining three more packings $O_2 \cong \{(x, y, z) : -\infty < x \leq 0, 0 \leq y < \infty, 0 \leq z < \infty\}$, $O_3 \cong \{(x, y, z) : 0 \leq x < \infty, -\infty < y \leq 0, 0 \leq z < \infty\}$, and $O_4 \cong \{(x, y, z) : -\infty < x \leq 0, -\infty < y \leq 0, 0 \leq z < \infty\}$. The union $O_1 \bigcup O_2 \bigcup O_3 \bigcup O_4$ results in a packing of a half space.

5. Suppose P packs a half space. Without loss of generality, suppose that it is the upper half space. Then if we reflect our packing through the xy plane, make a copy, and then take the union of the two packings and we achieve a packing for all of 3-d space.

6. The fact that P packs 3-d space implies nothing is vacuously true.

7. This argument is exactly the same as the one given in the 2-d case.

$\square$

Now we give a theorem concerning the new implications on the hierarchy.

Theorem 4.5. *We also have the following list of new implications from the hierarchy:*

1. *If a polycube P packs a bent beam, then it packs a quarter plank.*

2. *If a polycube P packs a quarter plank, then it packs a shell.*

3. *If a polycube P packs a shell, then it packs a octant.*

Proof.

1. Suppose P packs a bent beam. Then if we make a copy of this packing and indefinitely nest the copies inside of each other we will achieve a packing of a quarter plank.

2. Suppose P packs a quarter plank $Q_0 \cong \{(x,y,z) : 0 \leq x \leq a, 0 \leq y < \infty, 0 \leq z < \infty$ for some natural number $a\}$. Then we can make a copy of this packing, rotate it by $\frac{\pi}{2}$ with respect to the xz plane, and shift it in the x direction by a units we achieve another quarter plank tiling $Q_1 \cong \{(x,y,z) : a \leq x < \infty, 0 \leq y < \infty, 0 \leq z \leq a\}$. Similarly, if we make another copy of the original packing of Q_0, rotate it by $\frac{\pi}{2}$ with respect to the xy plane and shift it over by a in the x and z directions we achieve another quarter plank packing $Q_2 \cong \{(x,y,z) : a \leq x < \infty, 0 \leq y \leq a, a \leq z < \infty\}$. Then the union $S = Q_0 \bigcup Q_1 \bigcup Q_2$ is a packing of a shell.

3. Suppose P packs a shell. Then if we make copies of this packing and indefinitely nest the copies inside of each other we will achieve a packing of an octant.

$\square$

5   CONCLUSIONS AND FURTHER WORK

In this thesis we have explored some interesting topics in discrete tiling. Most of them regarding the enchanting geometric objects called polyominoes. We have explored the characterization of polyominoes and their hierarchical structures presented by Golomb [5]. We have given a tiling of a strip for a infinite family of polyominoes. We then showed that this family characteristically belongs to the strip category in the hierarchy. We discussed some interesting problems that are concerned with orders of polyominoes. We then explored the art of coloring arguments, signed tilings, and the structure of the tile homology group. We also explored two interesting results concerning decidability and tiling rectangles. Lastly, we presented a possible 3-d hierarchy for the polycubes.

In the future, I would like to further develop the 3-d polycube hierarchy and find some interesting characteristic examples for the regions that I have defined. I would like to continue my study in algebraic applications to tiling. I would like to understand and utilize the even more interesting tile homotopy group. I would like to explore whether or not the reptile category is logically distinguishable from the rectangle category in Golomb's 2-d hierarchy. I would also like to study the possible logical equivalence of the rectangle and half-strip category in the 2-d hierarchy. I would like to explore the possibility of there being a function $f(n)$, dependent on the size of a polyomino, that gives us a limit for checking whether or not that polyomino tiles a quadrant. In other words, if there exists an algorithm for deciding whether or not a finite protoset of polyominoes tiles a quadrant of the plane. I would like to study discrete tilings in different media such as the triangular and hexagonal lattice.

Mathematicans have been engrossed by polyominoes ever since Golomb generalized the domino to that of a polyomino in 1954 [3]. These objects completely embody the spirit of Mathematics and like so many before me, I fell in love with their interesting puzzles, patterns, and mathematical structure. Polyominoes started as a simple interest and curiosity and have since turned into a lifelong passion.

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