2011

Computational Study Of The Near Field Spontaneous Creation Of Photonic States Coupled To Few Level Systems

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COMPUTATIONAL STUDY OF THE

NEAR FIELD SPONTANEOUS CREATION

OF PHOTONIC STATES

COUPLED TO FEW LEVEL SYSTEMS

by

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A dissertation submission in partial fulfillment of the requirements
for the degree of Doctor of Philosophy
in the Department of Physics
in the College of Sciences
at the University of Central Florida
Orlando, Florida

Summer Term
2011

Major Professor: Michael N. Leuenberger

ABSTRACT

Models of the spontaneous emission and absorption of photons coupled to the electronic states of quantum dots, molecules, N-V (single nitrogen vacancy) centers in diamond, that can be modeled as artificial few level atoms, are important to the development of quantum computers and quantum networks. A quantum source modeled after an effective few level system is strongly dependent on the type and coupling strength the allowed transitions. These selection rules are subject to the Wigner-Eckert theorem which specifies the possible transitions during the spontaneous creation of a photonic state and its subsequent emission. The model presented in this dissertation describes the spatio-temporal evolution of photonic states by means of a Dirac-like equation for the photonic wave function within the region of interaction of a quantum source. As part of this aim, we describe the possibility to shift from traditional electrodynamics and quantum electrodynamics, in terms of electric and magnetic fields, to one in terms of a photonic wave function and its operators. The mapping between these will also be presented herein. It is further shown that the results of this model can be experimentally verified. The suggested method of verification relies on the direct comparison of the calculated density matrix or Wigner function, associated with the quantum state of a photon, to ones that are experimentally reconstructed through optical homodyne tomography techniques. In this non-perturbative model we describe the spontaneous creation of photonic state in a non-Markovian limit which does not implement the Weisskopf-Wigner approximation. We further show that this limit is important for the description of how a single photonic mode is created from the possibly infinite set of photonic frequencies ν_k that can be excited in a dielectric-cavity from the vacuum state. We use discretized central-difference approximations to the space and time partial derivatives, similar to finite-difference time domain models, to compute these results. The results presented herein show that near field effects need considered when describing adjacent quantum sources that are separated by distances that are small with respect to the wavelength of

their spontaneously created photonic states. Additionally, within the future scope of this model, we seek results in the Purcell and Rabi regimes to describe enhanced spontaneous emission events from these few-level systems, as embedded in dielectric cavities. A final goal of this dissertation is to create novel computational and theoretical models that describe single and multiple photon states via single photon creation and annihilation operators.

This work is dedicated to Bonnie, Dianita, Mom, and Chuck

ACKNOWLEDGMENTS

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δ_{nm} Kronecker delta

$\eta_{\mu\nu}$ Minkowski Metric

$\hat{\epsilon}_{\vec{k},\lambda}$ Spin (Helicity) polarized vector.

$\hat{\vec{\Gamma}}$ Single photon operator

$\hbar$ Planck's constant

$\mathcal{H}$ Hamiltonian

$\mathcal{V}$ Hamiltonian in the interaction picture

ν_k Optical frequency corresponding to wave vectro $\vec{k}$

ω_γ Resonant frequency of an optical cavity

ω_{nm} Electronic transition frequecy between states with quantum numbers n and m

∂_t Partial derivative with respect to time

σ_+ Positive Spin (Helicity) optical transition

σ_- Negative Spin (Helicity) optical transition

$\Box$ D'Alembertian $c^{-2}\partial_t^2 - \nabla^2$

$\varphi\left(\vec{r}, t\right)$ Wave function as a function of $\vec{r}$ and t

$\vec{\mathcal{F}}_\gamma^{\pm}$ Riemann-Silberstein Vector

$\vec{\nabla}$ Nabla gradient/differential operator $\hat{x}\partial x + \hat{y}\partial y + \hat{z}\partial z$

$\vec{A}$ Electromagnetic vector potential

$\vec{B}$ Magnetic Field

$\vec{D}$ Displacement Field

$\vec{E}$ Electric Field

$\vec{k}$ Wave vector

$\vec{r}$ Position vector, sometimes also written as $\vec{x}$

$\vec{r}_0$ Position vector to the origin of a coordinate system, also represented as $\vec{x}_0$

$\wp_{nm}$ Transition dipole moments

$\zeta_{\vec{k},\pm}$ PWF unit normalized probability amplitudes

A_μ Four vector potential of an electromagnetic or Maxwell Field

$a_{\vec{k},\lambda}$ Second Quantization Annihilation Operator associated with a Fock number state of spin (helicity) λ and wave vector $\vec{k}$

$a_{\vec{k},\lambda}^\dagger$ Second Quantization Creation Operator (Adjoint of the Creation Operator) associated with a Fock number state of spin (helicity) λ and wave vector $\vec{k}$

c speed of light

e Electric charge

e Euler's or natural number which satisfies $\ln e = 1$

E_n Energy of a state enumerated by n

$g^{(2)}$ Electromagnetic Correlation Function

i Imaginary unit representing $\sqrt{-1}$ and $e^{i\frac{\pi}{2}}$

m	Mass of a particle
p_μ	Four momentum
t	Time
x_μ	Position four vector
$\hat{\vec{\Psi}}_{\gamma,\pm}$	Energy normalized photonic wave function operator
bits	Binary Digit
FDTD	Finite Difference Time Domain a la Yee grid and algorithm
FEM	Finite Element Method
MEEP	MIT Electromagnetic Equation Propagation software package
MPB	MIT Photonics Bands software package
MPI	Message Passing Interface
N-V	Nitrogen-Vacancy center in diamond
PWF	Photonic Wave Function
QC	Quantum Computing
QD	Quantum Dot
QEC	Quantum Error Correction
QED	Quantum Electro-Dynamics
QFT	Quantum Field Theory
QIT	Quantum Information Technology
QO	Quantum Optics

QSM   Quasi-Spectral Method

qubits Quantum Binary Digits

RWA   Rotating Wave Approximation

SM   Spectral Method

1 INTRODUCTION, MOTIVATION, AND WORK PLAN

1.1 Summary

Modeling the spontaneous emission of a single-photon emitted from an electrically or optically injected few-level system is important for several proposed solid-state quantum computers and quantum networks. In this study we seek to model the fully quantized excitation and spontaneous decay from these levels through an optical emission of a photon in the non-Markovian limit of the photon bath. We propose the use of discretized central-difference approximations of space and time partial derivatives to describe the interaction between single photon and quantum dot states. In the future scope of this model, we seek results in the Purcell and Rabi regimes for spontaneous emission events from quantum dots embedded in dielectric cavities. In this chapter we discuss in detail the motivation for studying this system and the work plan that has been carried out while developing a computational model that calculate results that can be directly compared to experimental results.

1.2 Background

The field of quantum computation (QC) and quantum information technology (QIT) has recently experienced escalated activity in the search for a physically legitimate description of photonic states coupled to their quantum sources [1, 2]. The observed increase in activity is in part due to the suggestion that quantum information processing based on the electron spins of quantum dots (QDs), coupled through optical modes of a dielectric cavity [1], could improve on schemes based on the energy states of trapped ions [3] and nuclear spins in chemical solution [4, 5]. Some advantages of a QC scheme based on the suggested semiconductor quantum dot arrays may include greater scalability, longer spin decoherence and long distance, fast interactions by means of photons [1].

We propose a model for describing this coupling between a photonic state to the electron spin state of a quantum source (such as a QD) through optical modes present in a dielectric cavities. Additional applications of this model may include the design of devices aimed at single photon emission [6], single photon detection [7, 8], quantum teleportation, and quantum computing within a quantum network [2]. In order to successfully describe the coupling between spontaneously emitted photons and their quantum sources in this model, we show that high computational resolution is required to describe whispering gallery modes [9] and other optical modes available in dielectric cavities such as micro-disks and photonic crystals.

The proposed model for spontaneous photonic emission of radiation describes the evolution of photonic states by means of a Dirac like equation in 3+1 dimensions. The rigorous description of the interaction between a quantum source and the generated Photonic Wave Function (PWF) [10, 11] is initially made within the formalism of relativistic quantum field theory (QFT) [12, 13]. In this procedure, a connection between the PWF and the four vector potential for a Maxwell Field is drawn. This canonical quantization procedure leads to two important results, a complex tensor introduced into the photon field Lagrangian in place of the Faraday tensor and the coupled electron-photon field equations in terms of a field equation for the PWF. We describe the coupling between quantum sources and the PWF it generates. We extend these results to show how to model the emission of photonic states from a QD embedded in dielectric cavities. Experimentally, the quadratures of photonic states [14, 15, 16, 17, 18] have been measured and used to calculate quantum state's Wigner and $g^{(2)}$ correlation functions [16]. The models of photonic states presented herein can be extended and used to generate such photonic states for verification or comparison directly to experimental observations of this type.

1.3   Introduction
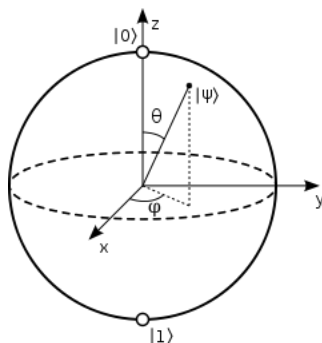
Figure 1.1: Bloch Sphere (used from Wikipedia)

As transistors begin to near the angstrom scale, in order for the computing industry to continue following Moore's law it has become increasingly important for computer chip manufacturers to consider quantum mechanical effects when designing computing devices. Currently, classical computers are being manufactured via processes that manipulate matter at the order of 45 nm or less; such as Intel i7 (Nehalem) [19], AMD Opteron (Shanghai) [20], and IBM Cell Broad Band Engine [21] processors. In order to understand how quantum mechanical effects will affect computing in coming years computer chip manufacturers will have to focus on the differences between classical and quantum computers. Both industry and government have started initiatives over the last few decades to investigate the viability of quantum computing (QC). In this study we discuss the motivation for designing quantum computers, outline differences between quantum and classical computers, and focus on how we contribute to the design of a quantum computing network. We also extend our contribution to the areas of single photon emission and detection.

Classical computers differ from quantum computers at the fundamental level of information storage and communication. Since the conception of the Touring-complete general purpose computer ENIAC by John Von Neumann the principal unit of information in computing technologies has been a binary digit (bit). To this day computers and all the information stored within them operate on this basic unit of information. However, since the advent of computers, the need to access information and process it has increased at ever increasing rates. This growth in information production

and processing has been accompanied by a need for ever faster computers. One such need is presented by the armed forces requirement for secure communication and the ability to decode mission critical information while intercepting potentially dangerous communications between possible enemies of the state. In addition, large amounts of information lead to large databases that need to be sorted through when seeking particular entries in a vast sea of information. For example industry giants like Google perpetually generate and parallelize databases replete with information in hopes of speeding up search algorithms and facilitating nearly instantaneous access to an ever increasing and newly generated set of information. Note that we have omitted the entire banking sector, economic industry, and sciences from this discussion for brevity. The main point revolves around the necessity that computer manufacturers have to stack millions of transistors into ever smaller volumes in order to create ever faster computer chips, storage units, and memory necessary for processing and storing copious amounts of information.

Where classical computers depend on bits to store information, quantum computers depend on quantum bits (qubits) for this purpose. As the name suggests quantum binary digits are a quantum representation of the on and off state as interpreted in machines like ENIAC and those in existence today. It is possible to visualize the relationship between bits and qubits by means of what is known as the Bloch sphere in figure 1.1 in terms of the on $|1\rangle$ and off states $|0\rangle$ of a bit and a superposition state of a qubit $|\Psi\rangle$. One major advantage of quantum computers is that they do not alter the Church-Turing thesis. This implies that QC does not allow the computation of functions which are not theoretically computable. So far it has been shown through the discoveries of Shor and others that it is possible to develop quantum algorithms for important problems like prime factorization [22], protocols for quantum error correction (QEC) [23], and fault-tolerant QC [23]. Other algorithms in QC, that if physically implemented could be of immediate use, include Grover's Algorithm for database searches [24] and the quantum Fourier Transform [25].

Central to the discussion on QC and QEC is the decoherence rate of qubits. It is imperative to QC to find an implementation where qubits that are part of the QC are well isolated from their environment. Among suggested implementations for QC are Raman coupled low-energy states of

trapped ions [3] and nuclear spins in chemical solution [4, 5]. Implementations of qubits based on these implementations could provide the first examples of QC up to the 5 through 10 qubits level. However, these implementations may not be scalable to more than 100 qubits [1]. One proposed implementation that could be scalable to more than 100 coupled qubits, with long spin decoherence times and long distance, fast interactions between qubits may be based on quantum dot (QD) electron spins[26] coupled through an optical mode of a photonic crystal or dielectric micro-cavity [1].

A theory and model focused on the interaction between quantum sources and photonic states may be generated on the basis of the consideration of quantum dots [27, 28, 29, 30, 31], molecules with sharp absorption and emission peaks [32, 33, 34, 35, 36], or N-V (single nitrogen vacancy) centers in diamond [37, 38, 39, 40, 41, 42, 43, 44] as artificial few level systems [15]. The selection rules of these few-level systems are subject to the interactions dictated by physical properties the system. For a QD [2] which specify σ_+ or σ_- transitions during the spontaneous creation of a photonic state. In this contribution we model the spontaneous creation of photonic states coupled to the electronic states of a quantum dot beyond the Markovian limit. We avoid this limit in order to describe, from first principles, how a possibly infinite set of photonic frequencies ν_k present in a dielectric-cavity can be excited from the vacuum state by a QD that is physically represented by the state to state transition frequency ω_{nm} and dipole moments $\wp_{nm}$.

The model presented herein describes the spatio-temporal evolution of photonic states by means of a Dirac-like equation for the PWF [10, 45, 11, 46, 47]. It has been to shown that the quantum state of a photon can be experimentally reconstructed using optical homodyne tomography techniques by measuring quantum noise statistics of field amplitudes at different optical phases [14, 17, 18]. In this work, the rigorous description of the interaction between a quantum source and the generated PWF draws a reference point for relating its $g^{(2)}$ function to those available from experimental measurements. In the theoretical description presented in this contribution, we follow the canonical quantization procedure presented by the Gupta and Beuler method [48, 12, 13] to arrive at a resulting interaction between the PWF and the electronic states of a QD. To do so, we adopt the

radiation gauge for interacting fields[1] and seek a connection between the PWF [49, 50, 51, 47], and the four vector potential for a Maxwell Field. This leads to two important results, the complex photonic tensor (not to be confused with the textbook Faraday or Maxwell Field Tensor) and the coupled electron-photon field equations in terms of a field equation for the PWF. Where it has been previously proposed that the PWF can not yield an interaction between photons [52] we show that the prescription presented here yields the possibility to produce such an interaction as mediated via few level systems. This complex photonic tensor gives rise to the PWF which directly satisfies and yields equations of motion equivalent to the generalized Maxwell equations [51, 11, 46, 45] (not to be confused with the textbook Maxwell's equations for real electric and magnetic components). In the second result we show the coupling equations of motion between a two level quantum source and the PWF it generates and extend these results to show how these can be applied to model the emission of a single photon from a dielectric micro-/nano-cavity using finite differencing schemes. Throughout the relativistic treatment of these fields we will maintain the Minkowski Metric $\eta_{\mu\nu}$ to have the signature $(+, -, -, -)$, and adopt the 4-notation consistent with x_μ which corresponds to the position four-vector (ct, x) and the product $\eta_{\mu\nu}x_\nu = x_\mu = (ct, -x)$.

Additional applications of this model could include the design of devices aimed at single photon emission [6], single photon detection [7, 8], quantum teleportation, and quantum computing within a quantum network [2]. Preliminary results implementing the MIT Electromagnetic Equation Propagation (MEEP) [53] and MIT Photonics Bands (MPB)[54] computer codes to model said whispering gallery modes in dielectric micro-cavities and optical modes in photonic crystals are encouraging. The preliminary results presented in figures (1.2), (1.3), and (1.4) show that Finite Difference Time Domain (FDTD) techniques could possibly be used to model a whispering gallery mode or photonic crystal mode coupled to an embedded quantum source.

1.4 Preliminary Theory & Work Plan

The objectives carried out as part of this dissertation revolve around the construction of a theory and model describing the spontaneous emission of a photon coupled to the electronic state of a

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[1]In the relativistic regime we set $\vec{\nabla} \cdot \vec{A} + \frac{1}{c}\partial_t \Phi = 0$
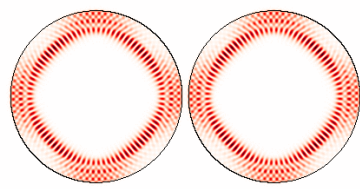
(a) Electromagnetic Field Energy
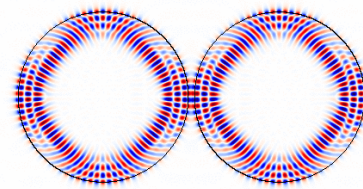
(b) Electric field x component
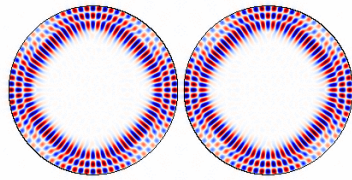
(c) Magnetic field z component
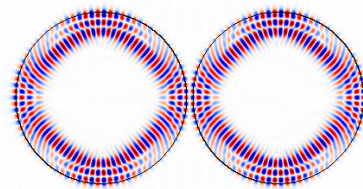
(d) Electric field y component

Figure 1.2: Preliminary results for 2-dimensional whispering gallery mode predictions in cylindrical dielectric micro-cavities.
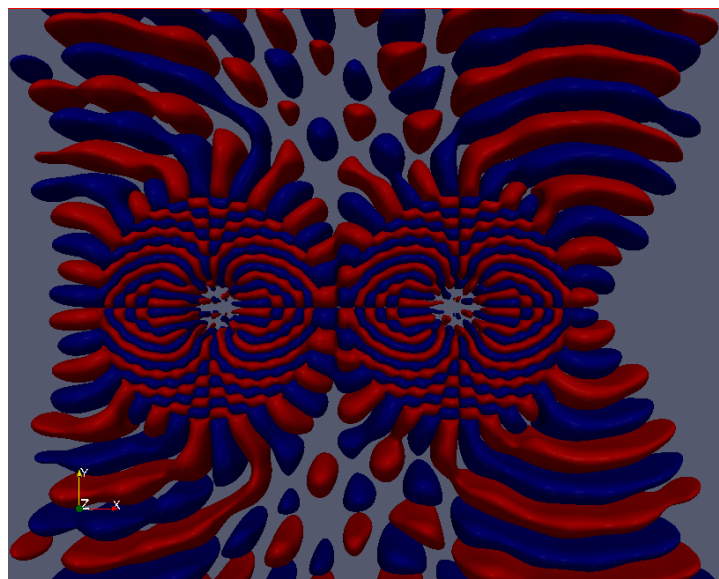
Figure 1.3: Preliminary results for x component of electric field whispering gallery mode in 3-dimensions
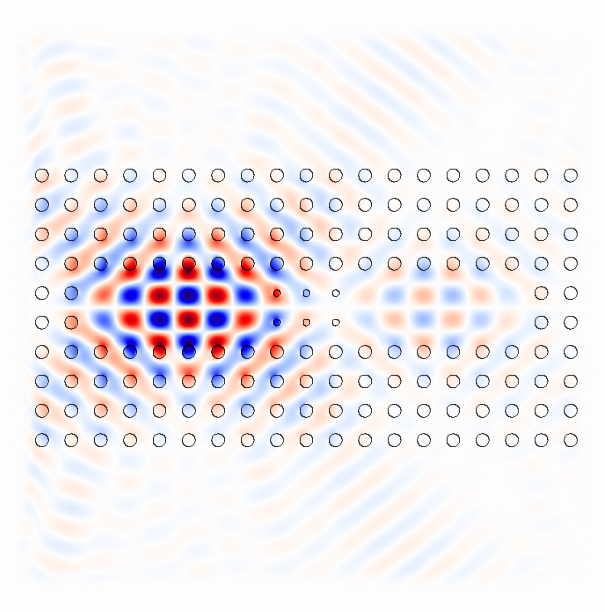
(a) Large air pockets at an initial time
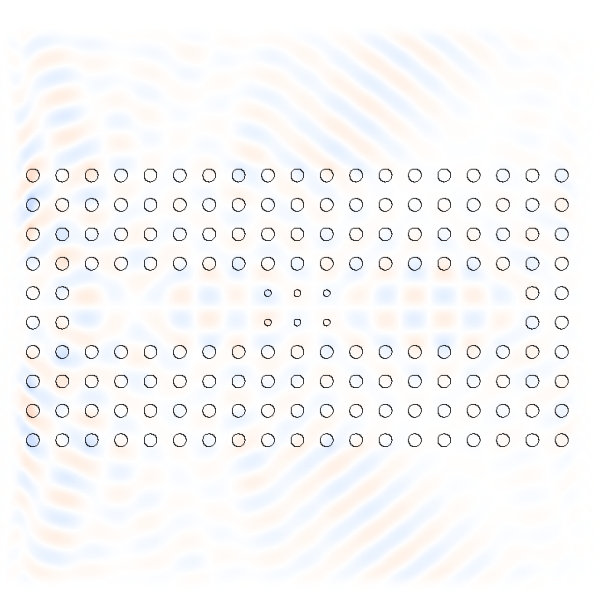
(b) Large air pockets at a later time
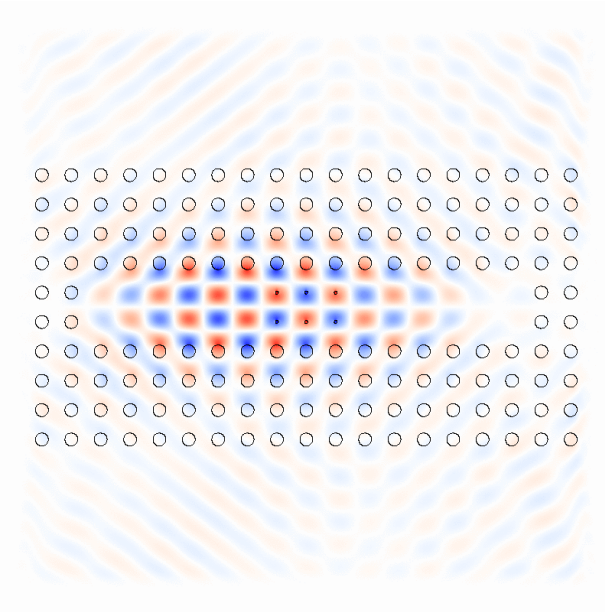
(c) Small air pockets at an initial time
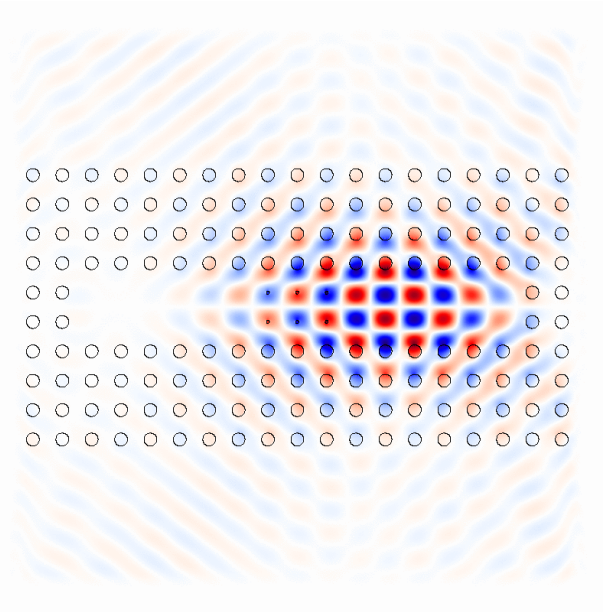
(d) Small air pockets at a later time

Figure 1.4: Electric Field x projection for photonic crystal with air pockets of variable size at set times. Preliminary results for 2-dimensional optical mode predictions in photonic crystal microcavities.

quantum dot. The aim of this study is to understand and implement the quantum interactions between the quantum source and its emitted photons in dielectric media. These interaction are carried out in the non-Markovian limit which <u>does not</u> make use of the Weisskopf-Wigner approximation. Applications of the theory and model may include single photon emission [6], single photon detection [7, 8], quantum teleportation, and quantum computing within a quantum network [2].

- Preliminary results presented in figures (1.2), (1.3), and (1.4) were produced by means of the existing MEEP [53] FDTD program for electromagnetic modeling. These model electromagnetic wave propagation in dielectric media. We extend these results by developing a theory that models spontaneously created photonic states coupled to few level systems embedded in such cavities. We additionally extend this model to an array of two couple few level systems required for modeling a quantum computing network.

- We develop a theoretical construct by following the procedure presented in the field of Quantum Optics. We use this theoretical construct for describing the coupling of an atomic-like state to that of a photonic state emitted by a two and six level system. Currently, approximations to the solution of this problem make use of the Weisskopf-Wigner approximation. This approxiamtion fixes the value of accessible modes from a possibly infinite set of frequencies ν_k to a single mode of the cavity ω_γ. We do not make this approximation. We model spontaneous emission events from quantum sources embedded in micro-cavities which enter regimes inclusive of possible collapse and revival phenomena. Within the context of a full quantum source interacting with it's emitted electromagnetic field at the quantum level these are known as the Purcell and Rabi regimes.

- Within this theoretical construct we show how models of spontaneously created photonics states couple PWFs via the few level systems such as QDs.

- We develop a new computational scheme that implements these models two and six level systems. The numerical methods implemented depend on a finite differencing of time and space similar to Finite Difference Time Domain (FDTD) methods. We suggest that future efforts

should focus Finite Element Methods (FEM), Spectral Methods (SM), & Quasi-Spectral Methods (QSM).

- Finite differencing computational methodologies proved to be too computationally time consuming and required parallelization of the algorithm. Parellization was carried out through the use of the Message Passing Interface (MPI). An effort was made to adopt FEM or SM computational techniques for computational speed up, but proved to be beyond the scope of time assigned for this study.

We assume that the interaction between a quantum source and photon can be fully described by a PWF Lagrangian for this system. This Lagrangian is composed of the quantum source, the free field term (in terms creation and annihilation operators $a_{\vec{k},\lambda}^{\dagger}$, $a_{\vec{k},\lambda}$), and the interaction introduced by means of a minimal substitution replacing the canonical momentum p_μ with $p_\mu \to p_\mu - \frac{e}{c}A_\mu$, where e is the electric charge, c is the speed of light, and A_μ is the electromagnetic four vector potential. Subsequently, such a self dual Lagrangian leads to a complete Hamiltonian $\mathcal{H}$

$$\mathcal{H} \;=\; \mathcal{H}_\sigma + \mathcal{H}_\gamma + \mathcal{H}_{\mathrm{Int}} \tag{1.4.1}$$

where $\mathcal{H}_\gamma$, $\mathcal{H}_\sigma$, and $\mathcal{H}_{\mathrm{Int}}$ represent the photonic, electronic and interacting terms respectively. The interaction term depends on both the dipole and rotating wave approximations (RWA) and is presented in the interaction picture. We solve the set of these coupled differential equations first analytically for a two level system and then numerically for more than two levels.

1.4.1 Plane Wave Dipole Approximation

The interaction and quantum source terms of the Hamiltonian density can be found by making the dipole approximation and applying it to the equations of motion for the various systems. We start by taking the proposed PWF Lagrangian expressing it in terms of the conjugate momentum $p_\mu \to p_\mu - \dfrac{e}{c}A_\mu$. In textbooks [15], the dipole approximation traditionally assumes that the entire

quantum source is immersed in a plane electromagnetic wave described by a vector potential[2] $\vec{A}$ which satisfies

$$\vec{A}(\vec{r}_0 + \vec{r}, t) = \vec{A}(t) e^{i\vec{k}\cdot(\vec{r}_0+\vec{r})} \tag{1.4.2}$$

$$\approx \vec{A}(t) e^{i\vec{k}\cdot(\vec{r}_0)} \tag{1.4.3}$$

to first order in $\vec{r}$, where $\vec{r}$ represents some position in space with respect to a globally defined coordinate system, t represents the time, e is the natural number, i represents the imaginary number, $\vec{k}$ represents the wave vector, and $\vec{r}_0$ is taken to be the spatial origin of the quantization axes of the quantum source state. In this approximation we additionally require the possibility to redefine the wave-function $\varphi(\vec{r}, t)$ of the quantum source by

$$\varphi(\vec{r}, t) = e^{\frac{i}{\hbar}\frac{e}{c}\vec{A}(\vec{r}_0,t)\cdot\vec{r}} \phi(\vec{r}, t) \tag{1.4.4}$$

where $\hbar$ represents Plank's constant. Substituting these two approximations back into (1.4.6) and solving yields

$$i\hbar\partial_t \left[e^{\frac{i}{\hbar}\frac{e}{c}\vec{A}(\vec{r}_0,t)\cdot\vec{r}} \phi \right] = \frac{\left(\vec{p} - \frac{e}{c}\vec{A}\right)^2}{2m} \left[e^{\frac{i}{\hbar}\frac{e}{c}\vec{A}(\vec{r}_0,t)\cdot\vec{r}} \phi \right]$$

$$i\hbar \left[\frac{i}{\hbar}\frac{e}{c}\left(\dot{\vec{A}}\cdot\vec{r}\right)\phi + \dot{\phi} \right] e^{i\frac{e}{\hbar}\vec{A}(\vec{r}_0,t)\cdot\vec{r}} = \frac{\left(\vec{p} - \frac{e}{c}\vec{A}\right)^2}{2m} \left[e^{\frac{i}{\hbar}\frac{e}{c}\vec{A}(\vec{r}_0,t)\cdot\vec{r}} \phi \right]$$

where ∂_t represents the partial derivative with respect to time, m represents the mass of a particle. Remembering that the vector potential in the radiation gauge requires that the electric field $\vec{E}$ satisfy $\vec{E} = -\frac{1}{c}\partial_t\vec{A}$. We can find an equation describing the approximated quantum source state ϕ

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[2]Here we switch from four notation to 3+1 notation.

given by

$$i\hbar\partial_t\phi = \left[\frac{\left(\vec{p} - \frac{e}{c}\vec{A}\right)^2}{2m} - e\left(\vec{E}\cdot\vec{r}\right)\right]\phi$$

$$= \left[\frac{\vec{p}^2 - \frac{e}{c}\vec{p}\cdot\vec{A} - \frac{e}{c}\vec{A}\cdot\vec{p} + \left(\frac{e}{c}\vec{A}\right)^2}{2m} - e\left(\vec{E}\cdot\vec{r}\right)\right]\phi\,.$$

Making a final assumption by approximating that $\left(e\vec{A}\right)^2 \ll 1$ and again remembering that the divergence, written in terms of the nabla operator $\vec{\nabla}$, $\vec{\nabla}\cdot\vec{A} = 0$ implies that the commutator between the momentum and electromagnetic vector potential yields $\left[\vec{p},\vec{A}\right] = 0$ in the radiation gauge, we arrive at

$$i\hbar\partial_t\phi = \left[\frac{\vec{p}^2}{2m} - e\left(\vec{E}\cdot\vec{r}\right)\right]\phi\,. \tag{1.4.5}$$

Note that if we had included any external influence on the quantum source wave-function by including an additive term

$$i\hbar\partial_t\varphi = \frac{\left(\vec{p} - \frac{e}{c}\vec{A}\right)^2}{2m}\varphi + V\left(\vec{r},t\right)\varphi \tag{1.4.6}$$

we would arrive at a similar result

$$i\hbar\partial_t\phi = \left[\frac{\vec{p}^2}{2m} - \frac{e}{c}\left(\vec{E}\cdot\vec{r}\right)\right]\phi + V\left(\vec{r},t\right)\phi\,. \tag{1.4.7}$$

This leads to the conclusion that we can re-express this equation as

$$i\hbar\partial_t\phi = \mathcal{H}_\sigma\phi + \mathcal{H}_{Int}\phi\,. \tag{1.4.8}$$

Regrouping these terms we recognize that $\mathcal{H}_\sigma$ represents the Hamiltonian that fully describes the energy eigenstates of the quantum source in the absence of the vector potential $\vec{A}$ and that the interaction term is given by

$$\mathcal{H}_{Int} = -e\left(\vec{E}\cdot\vec{r}\right). \tag{1.4.9}$$

However, in this dissertation we choose to follow the derivation which calculates the dipole approximation from the commutation of the total Hamiltonian with the position operator [55, 56], $\left[\mathcal{H}, \hat{\vec{r}}\right] = \frac{im_0}{\hbar}\vec{p}$. This approach is described with the PWF interpretation later in the text.

1.4.2 Energy & Fock Representation

To complete this section we define the product energy and Fock eigenstates $|a\rangle$ (the excited state) and $|b\rangle$(the ground state). In the general case where we are interested in state to state transitions of the quantum source, we can represent the quantum source Hamiltonian in terms of the quantum source transition operators by

$$\mathcal{H}_\sigma = \sum_n E_n |n\rangle\langle n|, \tag{1.4.10}$$

where $|n\rangle$ is an energy eigenstate of the quantum source and E_n is it's corresponding energy. Restricting our discussion to a two state system implies that

$$\mathcal{H}_\sigma = E_a |a\rangle\langle a| + E_b |b\rangle\langle b|. \tag{1.4.11}$$

Here we simplify the notation by introducing $\sigma_{nm} = |n\rangle\langle m|$ to represent the transition operators.

$$\mathcal{H}_\sigma = E_a\sigma_{aa} + E_b\sigma_{bb} \tag{1.4.12}$$

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We recognize that these operators σ_{nm} satisfy the algebra of the spin-$\frac{1}{2}$ Pauli spin matrices

$$[\sigma_{nm}, \sigma_{n'm'}] = \sigma_{nm'}\delta_{mn'} - \sigma_{n'm}\delta_{nm'}, \qquad (1.4.13)$$

where δ_{nm} is the Kronecker delta. As such σ_{ab} and σ_{ba} can be considered to be the creation and annihilation operators of the two level quantum source states $\sigma_- \equiv \sigma_{ba}$ and $\sigma_+ \equiv \sigma_{ab}$ (which are different from the optical spin transitions) respectively [15].

To describe the interaction between a photon and a two level quantum source, the interaction Hamiltonian is expanded in terms of the energy eigenstates of the quantum source and summed over the two accessible states. The coupling of the source with the optical field is then mitigated by the transition dipole operator $\vec{\wp}_{nm}$ between the sates enumerated by n and m, $\vec{\wp}_{nm} = e \langle n| \vec{r} |m \rangle$

$$e\vec{r} = \vec{\wp}_{ab}\sigma_+ + \vec{\wp}_{ba}\sigma_- . \qquad (1.4.14)$$

Assuming that the dipole operators of the quantum source states are real, these satisfy the equation $\vec{\wp}_{ab} = \vec{\wp}_{ba}$

$$e\vec{r} = \vec{\wp}_{ab} (\sigma_+ + \sigma_-) \qquad (1.4.15)$$

and the interaction component of the total Hamiltonian will be given by

$$\mathcal{H}_{Int} = -\vec{E} \cdot \vec{\wp}_{ab} (\sigma_+ + \sigma_-) . \qquad (1.4.16)$$

To complete the total Hamiltonian that describes the single photon/quantum source system one can make use of the single photon Hamiltonian $\mathcal{H}_\gamma$ given by

$$\mathcal{H}_\gamma = \sum_{\vec{k},\lambda} \hbar\nu_k \left(a^\dagger_{\vec{k},\lambda} a_{\vec{k},\lambda} + \frac{1}{2} \right), \qquad (1.4.17)$$

(where $\lambda \in \{-, +\}$ represents positive or negative spin or helicities) and the electric field operator

$$\vec{E} \;=\; \sum_{\vec{k},\lambda} \sqrt{\frac{\hbar\nu_k}{2\varepsilon_0 V}} \left[\hat{\epsilon}_{\vec{k},\lambda} a_{\vec{k},\lambda} e^{-i\nu_k t} e^{i\vec{k}\cdot\vec{r}} + \hat{\epsilon}^*_{\vec{k},\lambda} a^{\dagger}_{\vec{k},\lambda} e^{i\nu_k t} e^{-i\vec{k}\cdot\vec{r}} \right] , \qquad (1.4.18)$$

where $a^{\dagger}_{\vec{k},\lambda}$ and $a_{\vec{k},\lambda}$ are the creation and annihilation operators for single photon states and satisfy the algebra of $\left[a_{\vec{k},\lambda}, a^{\dagger}_{\vec{k}',\lambda'} \right] = \delta_{\vec{k},\vec{k}'} \delta_{\lambda,\lambda'}$. In addition this introduces the traditional notion of a spin polarized unit vector $\hat{\epsilon}_{\vec{k},\lambda}$. Taking all of these contribution into consideration the total Hamiltonian $\mathcal{H} = \mathcal{H}_\sigma + \mathcal{H}_\gamma + \mathcal{H}_{Int}$ can be represented in the dipole approximation by

$$\mathcal{H} \;=\; E_a \sigma_{aa} + E_b \sigma_{bb} + \sum_{\vec{k},\lambda} \hbar\nu_k \left(a^{\dagger}_{\vec{k},\lambda} a_{\vec{k},\lambda} + \frac{1}{2} \right) - \vec{E} \cdot \vec{\wp}_{ab} \left(\sigma_+ + \sigma_- \right) . \qquad (1.4.19)$$

1.4.3 Interaction Picture

Table 1.1: Interaction Picture Operators

Free Operators	Interacting Operators
$a_{\vec{k},\sigma_\lambda} e^{-i\nu_k t}$	$e^{\frac{i}{\hbar}(\mathcal{H}_\sigma + \mathcal{H}_\gamma)t} a_{\vec{k},\sigma_\lambda} e^{-\frac{i}{\hbar}(\mathcal{H}_\sigma + \mathcal{H}_\gamma)t}$
$a^{\dagger}_{\vec{k},\sigma_\lambda} e^{i\nu_k t}$	$e^{\frac{i}{\hbar}(\mathcal{H}_\sigma + \mathcal{H}_\gamma)t} a^{\dagger}_{\vec{k},\sigma_\lambda} e^{-\frac{i}{\hbar}(\mathcal{H}_\sigma + \mathcal{H}_\gamma)t}$
$\sigma_- e^{-i\omega_\sigma t}$	$e^{\frac{i}{\hbar}(\mathcal{H}_\sigma + \mathcal{H}_\gamma)t} \sigma_- e^{-\frac{i}{\hbar}(\mathcal{H}_\sigma + \mathcal{H}_\gamma)t}$
$\sigma_+ e^{i\omega_\sigma t}$	$e^{\frac{i}{\hbar}(\mathcal{H}_\sigma + \mathcal{H}_\gamma)t} \sigma_+ e^{-\frac{i}{\hbar}(\mathcal{H}_\sigma + \mathcal{H}_\gamma)t}$

It is well known that one can change to the interaction picture by following the transcription originally proposed by Dirac [57]. To switch to the interaction picture we first find the initial state of the Hamiltonian density by evaluating it at $t = t_0 = 0$ and continue by following the diction presented in table (1.1) which can be constructed from the discussion presented in [55]. The resulting Hamiltonian in the interaction picture $\mathcal{V}$ is given by

$$\mathcal{V} = \sum_{\vec{k},\sigma_\lambda} \sqrt{\frac{\hbar\nu_k}{2\varepsilon_0 V}} \left[\hat{\epsilon}_{\vec{k},\lambda} a_{\vec{k},\lambda} e^{-i\nu_k t} e^{i\vec{k}\cdot\vec{x}} + \hat{\epsilon}^*_{\vec{k},\lambda} a^\dagger_{\vec{k},\lambda} e^{i\nu_k t} e^{-i\vec{k}\cdot\vec{x}} \right] \cdot \vec{\wp}_{ab} \left(\sigma_+ e^{i\omega_\sigma t} + \sigma_- e^{-i\omega_\sigma t} \right) . \quad (1.4.20)$$

Assuming that the quantum source is initially in the state $|a\rangle$ at $t_0 = 0$ and photon is in the vacuum state $|0\rangle$ we can define the state vector [15] for the combined quantum source/photon system by the product state $|\gamma\sigma\rangle$, given by the product $|\gamma\sigma\rangle = |\gamma\rangle \otimes |\sigma\rangle$ as

$$|\gamma\sigma\rangle = c_a(t)|a0\rangle + \sum_{\vec{k},\lambda} c_{b,\vec{k},\lambda}(t) |b1_{\vec{k}}\rangle , \quad (1.4.21)$$

with $c_a(0) = 1$ and $c_{b,\vec{k}}(0) = 0$ which represent the absence or existence of a photon respectively. The equations of motion for these probability amplitudes [15] are found from the Schrödinger equation in the interaction picture

$$i\hbar\partial_t |\gamma\sigma\rangle = \mathcal{V}|\gamma\sigma\rangle , \quad (1.4.22)$$

by applying the rotating wave approximation [58], summing over all positive and negative values of $\vec{k}$, expanding the state vector, substituting into the expression above, and grouping like terms to be

$$\partial_t c_a(t) = \frac{i}{\hbar} \sum_{\vec{k}} \sqrt{\frac{\hbar\nu_k}{2\varepsilon_0 V}} \vec{\wp}_{ab} \cdot \hat{\epsilon}^*_{\vec{k}} e^{i(\omega_\sigma - \nu_k)t} c_{b,\vec{k}}(t) \quad (1.4.23)$$

$$\partial_t c_{b,\vec{k}}(t) = \frac{i}{\hbar} \sqrt{\frac{\hbar\nu_k}{2\varepsilon_0 V}} \vec{\wp}_{ab} \cdot \hat{\epsilon}_{\vec{k}} e^{-i(\omega_\sigma - \nu_k)t} c_a(t) . \quad (1.4.24)$$

Note that these equations of motion lack the spin resolved components of the unit helicity vectors. We will re-incorporate and consider the spatial propagation of these by working with the PWF interpretation.

1.4.4   Propagate and Couple to the Electric Field

The final expression for the state coefficient corresponding to the probability density of measuring a photon through the term $c_{b,\vec{k}}$ that is coupled to the quantum source state c_a at $\vec{r}_0$ may be related to the electric component of the photon. This is evident by formally integrating the expression for $c_{b,\vec{k}}$ and applying the Weisskopf-Wigner approximation discussed previously to get an expression for the generated electric field [15].

Once the electric field is know it should be possible to propagate it via Maxwell's Equations. In order to solve the problem without making the Weisskopf-Wigner approximation for more than 2 levels it is necessary to computationally implement an algorithm that can model the propagation of electromagnetic waves and couple to the equations of motion. As a first approach we focused on modifying existing FDTD code to include interaction and came to the conclusion that it was required that a new new code to solve this problem. As presented within our preliminary results we had proposed the use of MEEP [53] and MPB [54] to model electromagnetic field propagation for implementing the FDTD scheme and searching for optical cavity modes both in dielectric micro-cavities or photonic crystals. To incorporate the inclusion of the quantum source and electric field coupling we attempted to modify this code and investigate the possible use of different numerical algorithms [59]. Since the modification of the MEEP and MPB suites proved intractable we developed a new program in C++ that implements the Yee grid [60] to model single photon emission and propagation away from a quantum source.

1.4.5   Visualize and Calculate Spatial Profiles

A scheme for visualizing photon states can be implemented by incorporating similar methods used by MEEP and MPB. This visualization scheme follows existing visualization techniques such as those presented in preliminary results and extend these to incorporate the visualization of spatial interference between fock states. The visualization of the electromagnetic field modes out of resonance with the quantum source may be extracted by extending this mode to use existing Harminv routines [61, 62]. Through the analysis of this model it is be possible to view study

any polarization state associated with these photon Fock states. The measure and visualization of the amount quantum entanglement between quantum states can be computed from the density matrix by means of quantum entropy calculations [63, 64]. Visualization of results for these quantities may be explicitly done by means of existing programs such as gnuplot or a simple real time OpenGL implementation. The developed model, analysis of results, and visualization that are part of this work plan provide the ability to compare first principles calculations to the experimental development of a scalable quantum computing networks composed of photonic crystals and/or dielectric micro-cavities.

1.5 Dissertation Time Table

Table 1.2: Dissertation Time line

Milestone	Fall '09	Spring '10	Summer '10	Fall '10	Spring/Summer '11
Dissertation Proposal	X				
Theoretical Model	X	X	X	X	
Numerical Techniques	X	X	X	X	
Single Photon Source	X				
Interaction of Q. Sources				X	
Multilevel Q. Sources				X	
Writing Dissertation	X	X	X	X	X
Defend Dissertation					X

2   DESCRIBING THE PHOTON VIA THE PHOTONIC WAVE FUNCTION

In the 1930s Ettore Majorana and J. Robert Oppenheimer had already constructed an approach that could be used to express the photon by means of a photonic wave function PWF. Both Majorana and Oppenheimer eventually went on to describe a plausible Lagrangian formalism for that wave function. Since that time, most studies that have worked with the PWF have expressed it terms of its electric and magnetic field components [65, 66]. Even more impressively, the Maxwellians, such as Heaviside and Fitzgerald, [67], may have in the 1800s unknowingly found the Dirac-like equation for the relativistic photon by expression Maxwell's laws in operator form. Since those times, Quantum Optics (QO) and Quantum Electro-Dynamics (QED) have matured to their current state. Currently in these subjects, the quantum nature of the light and the PWF can be recorded at detector by means of a procedure known as Homodyne Tomography [14]. Current efforts in the field of QO are focused on extending these and other methods to improve on the lifetime and coherence of quantum states. In this way the interaction between quadratures of the PWF and quantum sources can be manipulated in the weak field limit. Quantum sources that are popular in today's search for components of a quantum computer include quantum dots [27, 28, 29, 30, 31], molecules with sharp absorption and emission peaks [32, 33, 34, 35, 36], as well as N-V centers in diamond[37, 38, 39, 40, 41, 42, 43, 44]. These are of interest because they can be modeled as artificial few level atoms[15]. It is remarkable to note that additional efforts in the 1950s focused on finding a particle theory for a photon which was compatible with Dirac's theory[50]. In fact, these efforts were aimed at further investigating the similarity between the Dirac equation, Maxwell's field equations, and their corresponding wave-functions [49].

There is a stark contrast between the motivation that led researchers in the early 1900's to the presently controversial subject of the simple contemplation of the existence of a photonic wave function. The study that occurred in the early 1900's stemmed from an interest to find a fundamen-

tal description for a photon that incorporated all the general considerations of field theory [49]. In describing the state of a photon in terms of ordinary differential operators, a compromise between the complexities associated with spinor components and the discussion of solutions could be found [50].

The current controversy that exists around the PWF stems from the fact that PWFs are said to not have all the properties which are traditionally associated with wave functions in non-relativistic wave-mechanics [11]. In fact, standard textbooks in the subject of quantum optics are careful to point out that one may not think of the photon in the same context as the a massive non-relativistic particle [15]. The key here is the term <u>non-relativistic</u>. In fact, studies point out, that this is due to the fact that photons can have positive and negative frequencies where as Schrödinger-like particles can only have positive ones [15]. Furthermore, it is said that since the $\vec{E}$ and $\vec{B}$ fields associated with Maxwell's equations on free-space are real and can therefore not be eigen-functions of the energy operator $i\hbar\partial_t$. That this is true would depend on the idea that Schrödinger-like particles have to be complex [68]. In this chapter we present our motivation for introducing the PWF in our studies and re-iterate for posterity the definition of the photonic wave function that can be found in the literature. In this contribution we introduce a new single photon operator and use it to motivate the form of the PWF we use throughout the text and in out new calculations.

2.1   From Maxwell's Equations to the PWF

In this section we re-iterate in summary the common approach presented in the literature that is used to define the PWF [69, 10, 70, 15]. A good point of departure stems from the second quantized description of the electric and magnetic fields [15] working within the vacuum

$$\vec{E}\left(\vec{x}, t\right) = \sum_{\vec{k},\lambda} \hat{\epsilon}_{\vec{k},\lambda} \mathcal{E}_{\vec{k}} a_{\vec{k},\lambda} e^{-i\nu_k t} e^{i\vec{k}\cdot\vec{x}} + \text{H.c.} \tag{2.1.1}$$

$$\vec{B}\left(\vec{x}, t\right) = \sum_{\vec{k},\lambda} \frac{\vec{k} \times \hat{\epsilon}_{\vec{k},\sigma_\lambda}}{\nu_k} \mathcal{E}_{\vec{k}} a_{\vec{k},\lambda} e^{-i\nu_k t} e^{i\vec{k}\cdot\vec{x}} + \text{H.c.} . \tag{2.1.2}$$

In the expressions above the field field strengths are traditionally give by the quantity $\mathcal{E}_{\vec{k}} = \sqrt{\dfrac{\hbar \nu_k}{2\varepsilon_0 V}}$ and the expansion functions $e^{-i\nu_k t} e^{i\vec{k}\cdot\vec{x}} + \text{H.c.}$ are chosen to satisfy the D'Alembertian operator $(\Box = c^{-2}\partial_t^2 - \nabla^2)$ [71] in the following equation of motion

$$\Box \left[e^{-i\nu_k t} e^{i\vec{k}\cdot\vec{x}} + \text{H.c.} \right] = 0. \tag{2.1.3}$$

The definition of the positive and negative frequency parts of the magnetic field, which correspond to their creation and annihilation operators, can be re-written in terms of the stray or demagnetizing field from the definitions [71] $\vec{B}^{(+)}(\vec{x},t) \equiv \mu_0 \vec{H}^{(+)}(\vec{x},t)$, $c^{-2} = \mu_0 \varepsilon_0$, & $k = \nu_k c$ as

$$\vec{H}^{(+)}(\vec{x},t) = \sum_{\vec{k},\lambda} \frac{\vec{k}}{k} \times \hat{\epsilon}_{\vec{k},\lambda} \sqrt{\frac{\hbar \nu_k}{2\mu_0}} a_{\vec{k}} e^{-i\nu_k t} e^{i\vec{k}\cdot\vec{x}}. \tag{2.1.4}$$

By rewriting the quantized electric and magnetic field operators in terms of the their positive and negative energy parts

$$\vec{E}(\vec{x},t) = \vec{E}^{(+)}(\vec{x},t) + \vec{E}^{(-)}(\vec{x},t) \tag{2.1.5}$$

$$\vec{H}(\vec{x},t) = \vec{H}^{(+)}(\vec{x},t) + \vec{H}^{(-)}(\vec{x},t), \tag{2.1.6}$$

their interpretation can be understood to lead to the probability of exciting the state of a detector (atom, QD, etc) in terms of these. As such, at a given position $\vec{x}$, the probability of measuring these (for example the electric field strength) is governed by the relationship

$$P_\psi(\vec{x},t) \propto \langle \psi | \vec{E}^{(-)}(\vec{x},t) \vec{E}^{(+)}(\vec{x},t) | \psi \rangle, \tag{2.1.7}$$

which may be written in terms of the vacuum state as

$$P_\psi(\vec{x},t) \propto \langle \psi | \vec{E}^{(-)}(\vec{x},t) | 0 \rangle \langle 0 | \vec{E}^{(+)}(\vec{x},t) | \psi \rangle. \tag{2.1.8}$$

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Taking care to carefully consider the definition of the state vector $|\psi\rangle$, where the probability is dependent on the creation and annihilation operators of the electric field [15], the state-vectors that we will be working with through this dissertation a the tradition Fock states. These states are eigenstates of the creation, annihilation, and Hamiltonian operators. Since the creation and annihilation operators we will be working with must obey the bosonic algebra, as was stated before, these states obey the following rules:

$$a_{\vec{k},\lambda}^{\dagger} |0\rangle = \sqrt{n+1} \left| (n+1)_{\vec{k},\lambda} \right\rangle \tag{2.1.9}$$

$$a_{\vec{k},\lambda} |n\rangle = \sqrt{n} \left| (n-1)_{\vec{k},\lambda} \right\rangle . \tag{2.1.10}$$

From the first definition it is clear that any state $\left| n_{\vec{k},\lambda} \right\rangle$ can be expressed in terms of the vacuum state as

$$\left| n_{\vec{k},\lambda} \right\rangle = \sqrt{\frac{1}{n!}} \left(a_{\vec{k},\lambda}^{\dagger} \right)^{n} |0\rangle .$$

In fact, by operating on either the vacuum or $|\psi\rangle$ states with a single electric or magnetic field operator will lead to the fact that this state must be

$$|\psi\rangle \equiv |1\rangle . \tag{2.1.11}$$

Leading to the conclusion that since the probability is given by

$$P_{\psi}(\vec{x},t) \propto \langle 1| \vec{E}^{(-)}(\vec{x},t) |0\rangle \langle 0| \vec{E}^{(+)}(\vec{x},t) |1\rangle , \tag{2.1.12}$$

there must exists a probability density of measuring a single photon state $|1\rangle$ sharply peaked about a frequency ω through a detector $|\psi\rangle$ that leads to the definition of an "electric field" by [15]

$$\vec{\Psi}_E(\vec{x},t) = \langle 0| \vec{E}^{(+)}(\vec{x},t) |1\rangle \tag{2.1.13}$$

$$\vec{\Psi}_E^\dagger(\vec{x},t) = \langle 1| \vec{E}^{(-)}(\vec{x},t) |0\rangle . \tag{2.1.14}$$

Traditionally the probability amplitude is taken to be sharply peaked about the transition frequency of the detector ω. This means that the frequency of the optical field ν_k is often expressed as a slowly varying frequency represented by ω. Therefore, normalizing $\vec{\Psi}_E$ in these cases will yields the photo-detection probability density due to the electric field without any contribution from the additional modes $\nu_k \neq \omega$. One can always retract this assumption and not necessitate the frequency ν_k to vary slowly by again including it in the sum. In the case where the approximation $\nu_k = \omega$ is not made, the normalized photo-detection probability amplitude for the electric field is given by the expressions[1]

$$\vec{\varphi}_\gamma(\vec{x},t) \equiv \sqrt{\frac{2\epsilon_0}{\hbar\omega}}\vec{\Psi}_E(\vec{x},t) \tag{2.1.15}$$

$$= \langle 0| \frac{1}{\sqrt{V}} \sum_{\vec{k},\lambda} \sqrt{\frac{\nu_k}{\omega}}\hat{\epsilon}_{\vec{k},\lambda} a_{\vec{k},\lambda} e^{-i\nu_k t} e^{i\vec{k}\cdot\vec{x}} |1\rangle \tag{2.1.16}$$

$$\vec{\varphi}_\gamma^\dagger(\vec{x},t) = \langle 1| \frac{1}{\sqrt{V}} \sum_{\vec{k},\lambda} \sqrt{\frac{\nu_k}{\omega}}\hat{\epsilon}_{\vec{k},\lambda}^* a_{\vec{k},\lambda}^\dagger e^{i\nu_k t} e^{-i\vec{k}\cdot\vec{x}} |0\rangle . \tag{2.1.17}$$

A similar probability amplitude for the magnetic field may be written which is also sharply peaked about the transition frequency of the detector

$$\vec{\Psi}_H(\vec{r},t) = \langle 0| \vec{H}^{(+)}(\vec{r},t) |1\rangle \tag{2.1.18}$$

$$\vec{\Psi}_H^\dagger(\vec{r},t) = \langle 1| \vec{H}^{(-)}(\vec{r},t) |0\rangle \tag{2.1.19}$$

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[1]The volume term V was kept as part of the normalization requirement. This is due to the fact that when we reiterate any observable associated with an eigenstate of $\vec{\Gamma}_\gamma$ and integrate over all space we will inevitably introduce a contribution from the volume.

and normalized to give a probability density due to a "magnetic field"

$$\vec{\chi}_{\gamma}\left(\vec{r}, t\right) \equiv \sqrt{\frac{2\mu_0}{\hbar\omega}}\vec{\Psi}_H\left(\vec{r}, t\right) \tag{2.1.20}$$

$$= \langle 0| \frac{1}{\sqrt{V}} \sum_{\vec{k},\lambda} \sqrt{\frac{\nu_k}{\omega}} \frac{\vec{k}}{k} \times \hat{\epsilon}_{\vec{k},\lambda} a_{\vec{k},\lambda} e^{-i\nu_k t} e^{i\vec{k}\cdot\vec{x}} |1\rangle \tag{2.1.21}$$

$$\vec{\chi}_{\gamma}^{\dagger}\left(\vec{r}, t\right) = \langle 1| \frac{1}{\sqrt{V}} \sum_{\vec{k},\lambda} \sqrt{\frac{\nu_k}{\omega}} \frac{\vec{k}}{k} \times \hat{\epsilon}_{\vec{k},\lambda}^* a_{\vec{k},\lambda}^{\dagger} e^{i\nu_k t} e^{-i\vec{k}\cdot\vec{x}} |0\rangle \; . \tag{2.1.22}$$

In terms of (2.1.16), (2.1.17), (2.1.21), and (2.1.22) these energy-normalized probability densities can be shown to satisfy a normalized form of Maxwell's equations also found in [15]

$$\vec{\nabla} \times \vec{\varphi}_{\gamma} = -c^{-1}\partial_t \vec{\chi}_{\gamma} \tag{2.1.23}$$

$$\vec{\nabla} \times \vec{\chi}_{\gamma} = c^{-1}\partial_t \vec{\varphi}_{\gamma} \, , \tag{2.1.24}$$

along with their complex conjugates

$$\vec{\nabla} \times \vec{\varphi}_{\gamma}^{\dagger} = -c^{-1}\partial_t \vec{\chi}_{\gamma}^{\dagger} \tag{2.1.25}$$

$$\vec{\nabla} \times \vec{\chi}_{\gamma}^{\dagger} = c^{-1}\partial_t \vec{\varphi}_{\gamma}^{\dagger} \, . \tag{2.1.26}$$

From the literature it is known that these can be expressed as a Dirac-like equation [49, 50], both in free space and with sources, by representing the probability densities discussed above in terms of normalized spinors. The fact that this analogy is immediately made clear comes about from the interpretation of these as the quantum form of Maxwell's equations in matrix notation. This form

is given by [15]

$$\partial_t \begin{pmatrix} \vec{\varphi}_\gamma \\ \vec{\chi}_\gamma \end{pmatrix} = \begin{pmatrix} 0 & c\vec{\nabla}\times \\ -c\vec{\nabla}\times & 0 \end{pmatrix} \begin{pmatrix} \vec{\varphi}_\gamma \\ \vec{\chi}_\gamma \end{pmatrix} \tag{2.1.27}$$

$$\partial_t \begin{pmatrix} \vec{\varphi}_\gamma^\dagger \\ \vec{\chi}_\gamma^\dagger \end{pmatrix} = \begin{pmatrix} 0 & c\vec{\nabla}\times \\ -c\vec{\nabla}\times & 0 \end{pmatrix} \begin{pmatrix} \vec{\varphi}_\gamma^\dagger \\ \vec{\chi}_\gamma^\dagger \end{pmatrix}. \tag{2.1.28}$$

2.1.1 Definition of the Photonic Wave-function

In this section we present the counter argument to the assertion present in the literature for the non-existence of a photonic wave-function. In the discussion above it was presented that the electric and magnetic components of an electromagnetic field could be represented in terms of complex probability densities. In addition it was discussed that these satisfy a normalized quantum mechanical form of Maxwell's equations. That discussion enforces that these wave-functions could satisfy a complex energy operator $i\partial_t$. Further, the additional presentation of the argument associated with the real nature of the $\vec{E}$ and $\vec{B}$ fields does not constrain the possibility of introducing a new complex vector which can eventually be recognized as the PWF[51, 70, 11, 46, 45, 72]. In addition, the problem with the positive and negative frequencies in the non-relativistic limit can be addressed by means of an argument that already exists in relativistic quantum mechanics. In this argument the positive and negative frequency parts are associated with particle and anti-particle solutions respectively which have their own corresponding equations of motion [10, 70].

It is here that we will slightly depart from the description given in Scully and Zubairy and include the photonic wave function (PWF) formalism [51, 10, 11, 73, 46, 50, 49, 45]. We adopt this formalism for two reasons.

1. The PWF is a spin (helicity) resolved wave-function that will facilitate the representation and interpretation of results dependent on the interaction that arises due to pure σ_+ and σ_- transitions.

2. The equations of motion of the PWF are of first order and therefore lend themselves to computational techniques derived from algorithms that are stable in 3+1 dimensions.

Here we reiterate in summary the definition of the wave-function and introduce the notation that will be used throughout this dissertation. The notation that will be adopted to describe the un-normalized form of the PWF in terms of the Riemann-Silberstein vector[10] is given by

$$\vec{\mathcal{F}}_\gamma \equiv \begin{pmatrix} \vec{\mathcal{F}}_\gamma^+ \\ \vec{\mathcal{F}}_\gamma^- \end{pmatrix}, \tag{2.1.29}$$

where the Riemann-Silberstein vector depends on the classical (and real) displacement and magnetic fields as given by the definition $\vec{\mathcal{F}}_\gamma^\pm \equiv \dfrac{1}{\sqrt{2}} \left(\dfrac{\vec{D}}{\sqrt{\varepsilon}} \pm \dfrac{i\vec{B}}{\sqrt{\mu}} \right)$. In media that is homogeneous we distinguish the speed of light in vacuum c or c_0 from the speed of light in that medium v via the definition $v = \dfrac{1}{\sqrt{\varepsilon\mu}}$. Therefore, using the "spin-1 matrices" [66] (here expressed in terms of the Levi-Civita symbol [71]) $\overleftrightarrow{s} = -i\epsilon_{ijk}$

$$i\hbar\partial_t \vec{\mathcal{F}}_\gamma^\pm = \pm v \left(\overleftrightarrow{s} \cdot \vec{p} \right) \vec{\mathcal{F}}_\gamma^\pm . \tag{2.1.30}$$

When written in matrix form this complex set of Maxwell's Equations is indeed a sort of Dirac-like (relativistic wave) equation for the free propagation of the electromagnetic field [74, 50, 49, 11, 51]

$$i\hbar\partial_t \begin{pmatrix} \vec{\mathcal{F}}_\gamma^+ \\ \vec{\mathcal{F}}_\gamma^- \end{pmatrix} = \begin{pmatrix} v\overleftrightarrow{s} \cdot \vec{p} & 0 \\ 0 & -v\overleftrightarrow{s} \cdot \vec{p} \end{pmatrix} \begin{pmatrix} \vec{\mathcal{F}}_\gamma^+ \\ \vec{\mathcal{F}}_\gamma^- \end{pmatrix}. \tag{2.1.31}$$

Noting that a PWF that satisfies (2.1.31) and can be represented as a superposition of plane waves, their spin or helicity polarization vectors[2] $\hat{\epsilon}_{\vec{k},\lambda}$ must depend on the propagation vector $\vec{k}$ [11]. This is analogous to the discussion presented in the quantization of the electric and magnetic fields. A useful property that exists between the wave-vector and the polarization vectors is that the cross-

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[2]Note that these are not the transverse polarization vectors, but instead, these are spin polarization vectors.

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product between these yields the same polarization vector with an added complex phase[3] of $\frac{\pi}{2}$

$$\frac{\vec{k}}{k} \times \hat{\epsilon}_{\vec{k},\lambda} = -\lambda i \hat{\epsilon}_{\vec{k},\lambda}. \tag{2.1.32}$$

In the remainder of this document the polarization vector $\hat{\epsilon}_{\vec{k},-}$ will be referred to as carrying positive helicity or left handed polarization (by convention $\hat{\epsilon}_{\vec{k},-} = \frac{1}{\sqrt{2}}(x_i - ix_j)$), and $\hat{\epsilon}_{\vec{k},+}$ will be referred to as carrying negative helicity or right handed polarization (by convention $\hat{\epsilon}_{\vec{k},+} = \frac{1}{\sqrt{2}}(x_i + ix_j)$). We will use the property 2.1.32 many times while searching for the PWF in Maxwell's equations and a more general Lagrangian formalism.

The relationship between un-normalized Riemann-Silberstein vector and the equations of motion for the normalized PWF, as defined in this approach, depend on the ability to normalize the electric and magnetic fields simultaneously. By expressing the complex form of Maxwell's equations, along with its conjugate, in terms of the quantized electric and stray fields (instead of the displacement and magnetic fields) one arrives at the classical equations of motion

$$i\hbar\partial_t \left(\frac{\varepsilon \vec{E}}{\sqrt{2\varepsilon}} \pm \frac{i\mu \vec{H}}{\sqrt{2\mu}} \right) = \pm v \overleftrightarrow{s} \cdot \vec{p} \left(\frac{\varepsilon \vec{E}}{\sqrt{2\varepsilon}} \pm \frac{i\mu \vec{H}}{\sqrt{2\mu}} \right), \tag{2.1.33}$$

by using the constitutive relations for linear homogeneous media [10, 11, 75]. In a different approach than the one which is presented in the literature, we recognize the above in terms of the associated second-quantized operators for the electric $\hat{\vec{\Psi}}_E(\vec{r},t)$ and stray $\hat{\vec{\Psi}}_H(\vec{r},t)$ fields respectively

$$i\hbar\partial_t \left[\frac{\varepsilon \left(\hat{\vec{\Psi}}_E + \hat{\vec{\Psi}}_E^\dagger \right)}{\sqrt{2\varepsilon}} \pm \frac{i\mu \left(\hat{\vec{\Psi}}_H + \hat{\vec{\Psi}}_H^\dagger \right)}{\sqrt{2\mu}} \right] =$$

$$\pm v \overleftrightarrow{s} \cdot \vec{p} \left[\frac{\varepsilon \left(\hat{\vec{\Psi}}_E + \hat{\vec{\Psi}}_E^\dagger \right)}{\sqrt{2\varepsilon}} \pm \frac{i\mu \left(\hat{\vec{\Psi}}_H + \hat{\vec{\Psi}}_H^\dagger \right)}{\sqrt{2\mu}} \right].$$

---

[3]Please see (C.5)

Taking advantage of the definitions (2.1.17), (2.1.16) & (2.1.22), (2.1.21) and using the same constitutive relations as before, we can substitute these definitions into the equations of motion above and find the respective equations of motion for the energy-normalized spinor operators to be

$$
i\hbar\partial_t \left[\sqrt{\frac{1}{4}} \left(\hat{\vec{\varphi}}_\gamma + \hat{\vec{\varphi}}_\gamma^\dagger \right) \pm i\sqrt{\frac{1}{4}} \left(\hat{\vec{\chi}}_\gamma + \hat{\vec{\chi}}_\gamma^\dagger \right) \right] =
$$
$$
\pm v \overleftrightarrow{s} \cdot \vec{p} \left[\sqrt{\frac{1}{4}} \left(\hat{\vec{\varphi}}_\gamma + \hat{\vec{\varphi}}_\gamma^\dagger \right) \pm i\sqrt{\frac{1}{4}} \left(\hat{\vec{\chi}}_\gamma + \hat{\vec{\chi}}_\gamma^\dagger \right) \right] .
$$

The definition of the energy normalized PWF operator then stems from the energy-normalized electric and magnetic field operators that satisfy the Dirac-like equation outlined above and is given by

$$
\hat{\vec{\Psi}}_{\gamma,\pm} \equiv \frac{1}{2} \left[\left(\hat{\vec{\varphi}}_\gamma + \hat{\vec{\varphi}}_\gamma^\dagger \right) \pm i \left(\hat{\vec{\chi}}_\gamma + \hat{\vec{\chi}}_\gamma^\dagger \right) \right] . \tag{2.1.34}
$$

Making use of the property (2.1.32) and expressing each polarization of $\vec{\varphi}_\gamma$ & $\vec{\chi}_\gamma$ in terms of their creation and annihilation operators $a_{\vec{k},\lambda}^\dagger$ & $a_{\vec{k},\lambda}$ respectively, we can see that the second quantization form of $\hat{\vec{\Psi}}_{\gamma,+}$ & $\hat{\vec{\Psi}}_{\gamma,-}$ are given by the sums

$$
\hat{\vec{\Psi}}_{\gamma,+} \equiv \frac{1}{2} \frac{1}{\sqrt{V}} \left[\left(\sum_{\vec{k},\lambda} \sqrt{\frac{\nu_k}{\omega}} \hat{\epsilon}_{\vec{k},\lambda} a_{\vec{k},\lambda} e^{-i\nu_k t} e^{i\vec{k}\cdot\vec{x}} + \text{H.c.} \right) \right.
$$
$$
\left. + i \left(\sum_{\vec{k},\lambda} -i\lambda \sqrt{\frac{\nu_k}{\omega}} \hat{\epsilon}_{\vec{k},\lambda} a_{\vec{k},\lambda} e^{-i\nu_k t} e^{i\vec{k}\cdot\vec{x}} + \text{H.c.} \right) \right]
$$
$$
\hat{\vec{\Psi}}_{\gamma,-} \equiv \frac{1}{2} \frac{1}{\sqrt{V}} \left[\left(\sum_{\vec{k},\lambda} \sqrt{\frac{\nu_k}{\omega}} \hat{\epsilon}_{\vec{k},\lambda} a_{\vec{k},\lambda} e^{-i\nu_k t} e^{i\vec{k}\cdot\vec{x}} + \text{H.c.} \right) \right.
$$
$$
\left. - i \left(\sum_{\vec{k},\lambda} -i\lambda \sqrt{\frac{\nu_k}{\omega}} \hat{\epsilon}_{\vec{k},\lambda} a_{\vec{k},\lambda} e^{-i\nu_k t} e^{i\vec{k}\cdot\vec{x}} - \text{H.c.} \right) \right] .
$$

In later chapters we will use these expressions for describing the σ_+ and σ_- transitions in a new light.

Please note that taking the Hermitian conjugate of either of these expressions will yield one another. This does not imply the existence of a negative energy solution. It means that one could consider that these expressions describe particle and antiparticle pairs of the photon. This property of the photon has peen previously studied and its consequences addressed in the vacuum [10, 11]. In those studies it was pointed out that a distinction between a photon and anti-photon does not exist and that there is no need for a photon conservation law[10, 11]. In those studies it is also noted that the operators $\hat{\vec{\Psi}}_{\gamma,+}$ and $\hat{\vec{\Psi}}_{\gamma,-}$ correspond to left-handed and right-handed photons that can be freely superposed. Further, it is suggested that electromagnetic interactions are invariant under the parity transformation[4] and that a change of polarization should be associated with the parity transformation and not with a particle-antiparticle transition. The rationale for the assertions regarding "antiparticle polarization" rely on the self-evident property that the Hermitian conjugate of the polarization vectors have to satisfy the equation $\hat{\epsilon}^*_{\vec{k},\sigma_+} = \hat{\epsilon}_{\vec{k},\sigma_-}$.

A couple of interesting properties of the Riemann-Silberstein vector worth mentioning are as follows. The energy carried by an electromagnetic field expressed by the Riemann-Silberstein vector is given by

$$\mathcal{H}_\gamma = \vec{\mathbf{F}}^\dagger_\gamma \vec{\mathbf{F}}_\gamma \tag{2.1.35}$$

Also, there exists the possibility to describe the coupling between Riemann-Silberstein vectors of opposing spin, $\vec{F}^+_\gamma$ and $\vec{F}^-_\gamma$, by working with inhomogeneous media. In media that is inhomogeneous the definition of the equations of motion for the un-normalized PWF leads to a Dirac-like equation for a photon that is dressed in an effective mass [11, 10]

$$i\hbar\partial_t\vec{\mathbf{F}}_\gamma = \begin{pmatrix} \sqrt{v\left(\vec{r}\right)}\overleftrightarrow{s}\cdot\vec{p} & -i\hbar\frac{v(\vec{r})}{2h(\vec{r})}\left(\overleftrightarrow{s}\cdot\nabla h\left(\vec{r}\right)\right) \\ i\hbar\frac{v(\vec{r})}{2h(\vec{r})}\left(\overleftrightarrow{s}\cdot\nabla h\left(\vec{r}\right)\right) & -\sqrt{v\left(\vec{r}\right)}\overleftrightarrow{s}\cdot\vec{p} \end{pmatrix}\vec{\mathbf{F}}_\gamma. \tag{2.1.36}$$

---

[4]These are transformations that interchange states with opposite helicities

This is analogous to a relativistic theory of a massive particles. For example and in comparison, in relativistic quantum mechanics the Dirac equation for a neutrino is given below [13, 12, 10, 11]

$$\psi_\eta\left(\vec{r},t\right) \equiv \begin{pmatrix} \vec{\varphi}_\eta \\ \vec{\chi}_\eta \end{pmatrix} \tag{2.1.37}$$

$$i\hbar\partial_t\psi_\eta\left(\vec{r},t\right) = \begin{pmatrix} c\overleftrightarrow{\sigma}\cdot\vec{p} & mc^2 \\ mc^2 & -c\overleftrightarrow{\sigma}\cdot\vec{p} \end{pmatrix} \psi_\eta\left(\vec{r},t\right). \tag{2.1.38}$$

2.2 Definition of the PWF Operator from Maxwell's Equations

The normalized electric and magnetic field operators defined in expression (2.1.34) are operators that can create a photon with any bandwidth that is dependent only on the state vector that they operate on. For the case of a vacuum state, all of the modes contribute towards the definition of a single photon state with a definite line-width. As was mentioned in the previous section the real advantage of using the wave function arises due to the clarity of interpretation that can be derived of the interactions we seek to describe in this dissertation. In this section we focus on this advantage and seek an operator that will lead to an accessible interpretation of the interaction between the quantum systems we will be studying. We note that this approach can in general be applied to any charged massive field that interacting with numbered photon states. A detailed discussion on this interaction will be presented in later chapters.

We start the definition of this operator from the expression (2.1.27) which we found from the expressions for the energy-normalized electric (2.1.17) and magnetic (2.1.22) field operators. Carrying out the full sum over the spin states, λ find that this is in fact a single photon creation operator given by the expressions

$$\hat{\vec{\Psi}}_{\gamma,+} = \frac{1}{\sqrt{V}}\sum_{\vec{k}}\sqrt{\frac{\nu_k}{\omega}}\left[\hat{\epsilon}_{\vec{k},+}a_{\vec{k},+}e^{-i\nu_k t}e^{i\vec{k}\cdot\vec{x}} + \hat{\epsilon}^*_{\vec{k},-}a^\dagger_{\vec{k},-}e^{i\nu_k t}e^{-i\vec{k}\cdot\vec{x}}\right] \tag{2.2.1}$$

$$\hat{\vec{\Psi}}_{\gamma,-} = \frac{1}{\sqrt{V}}\sum_{\vec{k}}\sqrt{\frac{\nu_k}{\omega}}\left[\hat{\epsilon}_{\vec{k},-}a_{\vec{k},-}e^{-i\nu_k t}e^{i\vec{k}\cdot\vec{x}} + \hat{\epsilon}^*_{\vec{k},+}a^\dagger_{\vec{k},+}e^{i\nu_k t}e^{-i\vec{k}\cdot\vec{x}}\right]. \tag{2.2.2}$$

30

These operators carry all of the information about momentum, number, energy, and spin that we can possibly associate with the state of a single photon. Please note that by taking the adjoint of either yields the relations

$$\hat{\vec{\Psi}}_{\gamma,+}^{\dagger} = \frac{1}{\sqrt{V}} \sum_{\vec{k}} \sqrt{\frac{\nu_k}{\omega}} \left[\hat{\epsilon}_{\vec{k},+}^{*} a_{\vec{k},+}^{\dagger} e^{i\nu_k t} e^{-i\vec{k}\cdot\vec{x}} + \hat{\epsilon}_{\vec{k},-} a_{\vec{k},-} e^{-i\nu_k t} e^{i\vec{k}\cdot\vec{x}} \right] \qquad (2.2.3)$$

$$= \hat{\vec{\Psi}}_{\gamma,-} \qquad (2.2.4)$$

and vice-verse. We therefore define operator to be the notation that we will use throughout this dissertation as the energy normalized single photon operator[5]

$$\hat{\vec{\Gamma}}_{\gamma} = \frac{1}{\sqrt{V}} \sum_{\vec{k}} \sqrt{\frac{\nu_k}{\omega}} \left[\hat{\epsilon}_{\vec{k},+} a_{\vec{k},+} e^{-i\nu_k t} e^{i\vec{k}\cdot\vec{x}} + \hat{\epsilon}_{\vec{k},-}^{*} a_{\vec{k},-}^{\dagger} e^{i\nu_k t} e^{-i\vec{k}\cdot\vec{x}} \right] . \qquad (2.2.5)$$

This operator, being a spin-resolved operator also carries with it all the properties of a spinor[13, 11, 10, 51, 70]

$$\vec{\Psi}_{\gamma}(\vec{r}, t) = \begin{pmatrix} \hat{\vec{\Psi}}_{\gamma,+} \\ \hat{\vec{\Psi}}_{\gamma,-} \end{pmatrix}$$

and we will refer to $\hat{\vec{\Gamma}}_{\gamma}$ as the normalized Riemann-Silberstein Field Operator for historical reference (or photonic wave function operator for short). Similarly the un-normalized Riemann-Silberstein vector can also be associated with a similar spinor [11, 10, 51, 70]

$$\vec{F}_{\gamma}(\vec{r}, t) = \begin{pmatrix} \hat{\vec{F}}_{\gamma,+} \\ \hat{\vec{F}}_{\gamma,-} \end{pmatrix} .$$

---

[5]To rewrite this expression as an integral over an infinite number of accessible modes or as a discrete sum of modes, it is necessary to use the transition from a continuum of states to a discrete space via

$$\frac{V}{(2\pi)^3} \int d^3 k \rightarrow \sum_{\vec{k}}$$

Though these expressions are already sufficient to construct the formalism we will be using for coupling low intensity Maxwell Fields to charged matter fields, in the next chapter and in the discussion present in the appendices[6], we will describe in detail all of the conserved quantities that are associated with the newly defined single photon operator.

To interpret the meaning of the single photon operator we will follow a discussion similar to the one presented in the previous section and operate with these on numbered Fock states. The full correspondence between QED and this PWF formalism has been studied previously [52] and we therefore constrain our discussion to a summary of the interpretation of single-photon states. Though previous studies have argued that the PWF picture can not be used to introduce an interaction between photonic states [52, 66], we show, as part of later chapters of this dissertation, that the interaction between photonic states can be mitigated by charged matter fields.

We start the interpretation of the single photon wave function similarly to our discussion in the previous section. We therefore break up the PWF operator into positive and negative **frequency** (not to be confused with spin, energy, or anti-particle) parts.

$$\vec{\Gamma}_\gamma \;=\; \vec{\Gamma}_\gamma^{(+)} + \vec{\Gamma}_\gamma^{(-)} \,. \tag{2.2.6}$$

We limit our discussion to the terminology of **frequency** to avoid the debate over the interpretation of these as corresponding to positive or negative **energy** [51, 52, 66, 45, 73, 50, 74, 49, 10, 11].

2.3 Working with the PWF $\vec{\Gamma}_\gamma$ operator

The commutation relations associated with the positive and negative frequency parts of the PWF operator are[7][15, 13, 12]

$$\left[\vec{\Gamma}_\gamma^{(+)}, \vec{\Gamma}_\gamma^{(-)} \right] \;=\; 0 \,,$$

---

[6]Refer to alpha and beta matrix representation of the PWF Dirac Equation in the appendix

[7]The polarization vectors satisfy the identity $\hat{\epsilon}^*_{\vec{k},\sigma_\lambda} \cdot \hat{\epsilon}_{\vec{k}',\sigma_{\lambda'}} = \delta_{\vec{k}.\vec{k}'} \delta_{\sigma_\lambda,\sigma'_\lambda}$. See (C.5)

$$\left[\vec{\Gamma}_\gamma^{(\pm)}, \vec{\Gamma}_\gamma^{\dagger(\pm)} \right] = 0 \,,$$

$$\left[\vec{\Gamma}_\gamma^{(\pm)}, \vec{\Gamma}_\gamma^{\dagger(\mp)} \right] = \frac{1}{V} \sum_{\vec{k}} \frac{\nu_k}{\omega} \hat{\epsilon}_{\vec{k},\pm} \hat{\epsilon}_{\vec{k},\pm}^* \,.$$

From these we can obtain the the commutation relations for the PWF operator and its Hermitian conjugate. These commutation relations yield the relation

$$
\begin{aligned}
\left[\vec{\Gamma}_\gamma, \vec{\Gamma}_\gamma^\dagger \right] &= \left[\vec{\Gamma}_\gamma^{(+)}, \vec{\Gamma}_\gamma^{\dagger(+)} \right] + \left[\vec{\Gamma}_\gamma^{(-)}, \vec{\Gamma}_\gamma^{\dagger(-)} \right] + \left[\vec{\Gamma}_\gamma^{(-)}, \vec{\Gamma}_\gamma^{\dagger(+)} \right] + \left[\vec{\Gamma}_\gamma^{(+)}, \vec{\Gamma}_\gamma^{\dagger(-)} \right] \\
&= \frac{1}{V} \sum_{\vec{k}} \frac{\nu_k}{\omega} e^{-2i\nu_k t} e^{2i\vec{k}\cdot\vec{x}} \left(I - \frac{\vec{k}\vec{k}}{|k|^2} \right) \,.
\end{aligned}
$$

Considering a continuum of states we can re-write this expression as an integral over an infinite number of modes yields the commutation relation to be

$$
\begin{aligned}
\frac{1}{V} \sum_{\vec{k}} \frac{\nu_k}{\omega} e^{-2i\nu_k t} e^{2i\vec{k}\cdot\vec{x}} \left(I - \frac{\vec{k}\vec{k}}{|k|^2} \right) &\rightarrow \int \frac{d^3k}{(2\pi)^3} \frac{\nu_k}{\omega} e^{-2i\nu_k t} e^{2i\vec{k}\cdot\vec{x}} \left(I - \frac{\vec{k}\vec{k}}{|k|^2} \right) \\
&= \frac{c}{\omega} \int \frac{d^3k \, |k|}{(2\pi)^3} e^{2i|k|(|x|\cos\theta - ct)} \left(I - \frac{\vec{k}\vec{k}}{|k|^2} \right)
\end{aligned}
$$

which require the definition of a coordinate system in order to be evaluated.

We can gain a further understanding of these operators by interpreting them in terms of their corresponding wave-functions. We accomplish this by operating on a single photon state $\langle 1| = \langle 0| \sum_{\vec{k},\lambda} a_{\vec{k},\lambda}$. However, since we will be coupling this state to that of a quantum source, we will no longer have free particle states. We are therefore introduce a probability amplitude $c_{\vec{k},\sigma_\lambda}^*$ associated with this state $\langle 1| = \langle 0| \sum_{\vec{k},\lambda} c_{\vec{k},\lambda}^* a_{\vec{k},\lambda}$. In later chapters we will show how this probability amplitude couples to the state of a quantum source. However, in the meantime, we work with the vacuum state in the absence of matter. Proceeding with operation of $\vec{\Gamma}_\gamma^\dagger$ on this state $\langle 1| = \langle 0| \sum_{\vec{k},\lambda} c_{\vec{k},\lambda}^* a_{\vec{k},\lambda}$,

the probability density is determined to be

$$\langle 1| \vec{\Gamma}^\dagger_\gamma |0\rangle = \sqrt{\frac{1}{V}} \sum_{\vec{k}} \sqrt{\frac{\nu_k}{\omega}} c^*_{\vec{k},+} \hat{\epsilon}^*_{\vec{k},+} e^{i\nu_k t} e^{-i\vec{k}\cdot\vec{x}} \, .$$

Conversely, operating on $|1\rangle = \sum_{\vec{k},\lambda} c_{\vec{k},\lambda} a^\dagger_{\vec{k},\lambda} |0\rangle$ yields this probability density to be

$$\langle 0| \vec{\Gamma}^\dagger_\gamma |1\rangle = \sqrt{\frac{1}{V}} \sum_{\vec{k}} \sqrt{\frac{\nu_k}{\omega}} c_{\vec{k},-} \hat{\epsilon}_{\vec{k},-} e^{-i\nu_k t} e^{i\vec{k}\cdot\vec{x}} \, .$$

Operating with $\vec{\Gamma}$ on $|1\rangle = \sum_{\vec{k},\lambda} c_{\vec{k},\lambda} a^\dagger_{\vec{k},\lambda} |0\rangle$ yields the probability density to be

$$\langle 0| \vec{\Gamma}_\gamma |1\rangle = \sqrt{\frac{1}{V}} \sum_{\vec{k}} \sqrt{\frac{\nu_k}{\omega}} c_{\vec{k},+} \hat{\epsilon}_{\vec{k},+} e^{-i\nu_k t} e^{i\vec{k}\cdot\vec{x}} \, .$$

Again, conversely operating on $\langle 1| = \langle 0| \sum_{\vec{k},\lambda} c^*_{\vec{k},\lambda} a_{\vec{k},\lambda}$ yields the probability density to be

$$\langle 1| \vec{\Gamma}_\gamma |0\rangle = \sqrt{\frac{1}{V}} \sum_{\vec{k}} \sqrt{\frac{\nu_k}{\omega}} c^*_{\vec{k},-} \hat{\epsilon}_{\vec{k},-} e^{i\nu_k t} e^{-i\vec{k}\cdot\vec{x}} \, .$$

These probability densities may be interpreted similarly to the probability of measuring an Electric Field as before. Following this interpretation, the probability of measuring a photon with negative spin (positive helicity or left handed polarization), as given by the discussion above, implies that the PWF operators $\vec{\Gamma}^{(+)}_\gamma$, $\vec{\Gamma}^{(-)}_\gamma$ & $\vec{\Gamma}^{\dagger(+)}_\gamma$, $\vec{\Gamma}^{\dagger(-)}_\gamma$ yield

$$P_{\gamma,+} \propto \langle 1| \left(\vec{\Gamma}^{(+)}_\gamma\right)^\dagger |0\rangle \langle 0| \vec{\Gamma}^{(+)}_\gamma |1\rangle \, .$$

Recalling that the Hermitian conjugate of the positive frequency part of the PWF operator is the negative frequency part of the Hermitian conjugate of the PWF operator $\left(\vec{\Gamma}^{(+)}_\gamma\right)^\dagger = \vec{\Gamma}^{\dagger(-)}_\gamma$. Using this identity we can define the notation for the probability density of the PWF, which has a positive

spin, by dropping the hat of the symbol $\vec{\hat{\Psi}}_{\gamma,+}$ according to the following expressions

$$\vec{\Psi}_{\gamma,+} = \langle 0| \vec{\Gamma}^{(+)}_{\gamma} |1\rangle$$
$$\vec{\Psi}^{\dagger}_{\gamma,+} = \langle 1| \vec{\Gamma}^{\dagger(-)}_{\gamma} |0\rangle \, .$$

Similarly, we define the notation for describing the probability density for the PWF, this time negative spin according to the following expressions

$$P_{\gamma,\sigma_-} \propto \langle 1| \left(\vec{\Gamma}^{\dagger(+)}_{\gamma} \right)^{\dagger} |0\rangle \langle 0| \vec{\Gamma}^{\dagger(+)}_{\gamma} |1\rangle \, .$$

This time we take advantage of the converse argument for the Hermitian conjugate of the Hermitian conjugate of the positive frequency part of the PWF operator $\left(\vec{\Gamma}^{\dagger(+)}_{\gamma} \right)^{\dagger} = \vec{\Gamma}^{(-)}_{\gamma}$. Through this identity we define the notation for the probability density of the PWF, which has a negative spin, by dropping the hat of the symbol $\vec{\hat{\Psi}}_{\gamma,-}$ according to the following expressions

$$\vec{\Psi}^{\dagger}_{\gamma,-} = \langle 0| \vec{\Gamma}^{\dagger(+)}_{\gamma} |1\rangle$$
$$\vec{\Psi}_{\gamma,-} = \langle 1| \vec{\Gamma}^{(-)}_{\gamma} |0\rangle \, .$$

The overall probability of measuring a photon with any helicity must therefore be given as a the sum of the probabilities for either case. The normalization condition must therefore be

$$\int d^3x \, \vec{\boldsymbol{\Psi}}^{\dagger}_{\gamma} \vec{\boldsymbol{\Psi}}_{\gamma} = \int d^3x \left(\vec{\Psi}^{\dagger}_{\gamma,+} \vec{\Psi}_{\gamma,+} + \vec{\Psi}^{\dagger}_{\gamma,-} \vec{\Psi}_{\gamma,-} \right) = .1 \tag{2.3.1}$$

Consequently the interpretation of these photonic wave functions is take to be the following

- $\vec{\Psi}_{\gamma,+}(\vec{r},t)$ represents an energy normalized photonic wave function which propagates with $\lambda \to +$ (referred to as positive spin, negative helicity, or right handed polarization), when

operating on a single photon state, which can be described via a PWF given by

$$\vec{\Psi}_{\gamma,+}\left(\vec{r},t\right) = \sum_{\vec{k}} \zeta_{\vec{k},+}\hat{\epsilon}_{\vec{k},+}e^{-i\nu_k t}e^{i\vec{k}\cdot\vec{x}} \tag{2.3.2}$$

$$\vec{\Psi}_{\gamma,+}^{\dagger}\left(\vec{r},t\right) = \sum_{\vec{k}} \zeta_{\vec{k},+}^{*}\hat{\epsilon}_{\vec{k},+}^{*}e^{i\nu_k t}e^{-i\vec{k}\cdot\vec{x}} \tag{2.3.3}$$

- $\vec{\Psi}_{\gamma,-}\left(\vec{r},t\right)$ represents an energy normalized photonic wave function propagating with $\lambda \rightarrow -$ (referred to as negative spin, positive helicity, or left handed polarization), when operating on a single photon state, which can be described via a PWF given by

$$\vec{\Psi}_{\gamma,-}\left(\vec{r},t\right) = \sum_{\vec{k}} \zeta_{\vec{k},-}\hat{\epsilon}_{\vec{k},-}e^{-i\nu_k t}e^{i\vec{k}\cdot\vec{x}} \tag{2.3.4}$$

$$\vec{\Psi}_{\gamma,-}^{\dagger}\left(\vec{r},t\right) = \sum_{\vec{k}} \zeta_{\vec{k},-}^{*}\hat{\epsilon}_{\vec{k},-}^{*}e^{i\nu_k t}e^{-i\vec{k}\cdot\vec{x}} \tag{2.3.5}$$

Therefore, a unit-normalized operator, associated with the PWF, is given by

$$\psi_{\gamma}\left(\vec{r},t\right) = \sum_{\vec{k}} \left(\zeta_{\vec{k},+}\hat{\epsilon}_{\vec{k},+}a_{\vec{k},+}e^{-i\nu_k t}e^{i\vec{k}\cdot\vec{x}} + \zeta_{\vec{k},-}^{*}\hat{\epsilon}_{\vec{k},-}^{*}a_{\vec{k},-}^{\dagger}e^{i\nu_k t}e^{-i\vec{k}\cdot\vec{x}} \right) \tag{2.3.6}$$

$$\psi_{\gamma}^{\dagger}\left(\vec{r},t\right) = \sum_{\vec{k}} \left(\zeta_{\vec{k},+}^{*}\hat{\epsilon}_{\vec{k},+}^{*}a_{\vec{k},+}^{\dagger}e^{i\nu_k t}e^{-i\vec{k}\cdot\vec{x}} + \zeta_{\vec{k},-}\hat{\epsilon}_{\vec{k},-}a_{\vec{k},-}e^{-i\nu_k t}e^{i\vec{k}\cdot\vec{x}} \right). \tag{2.3.7}$$

Again, we stress that these operators represent two possible polarizations of the photon. The interpretation corresponding to these as a particle anti-particle pair is not investigated and is considered beyond the scope of the discussion presented in this dissertation.[8] Comparing to (2.1.16), (2.1.17), (2.1.21), and (2.1.22) we note that these electric and magnetic wave-functions are indeed composed of single photon wave functions. Moreover, to represent an electric field wave-function it is now evident that more than one PWF is required.

---

[8] Please refer to the previous section for a discussion on the interpretation and meaning of the particle anti-particle pair of photons

2.3.1 Spatial Representation of the ψ_γ Operator

Since we have employed a full sum over accessible states denoted by $\vec{k}$ in the definition (2.2.5) for the second quantized single photon operators of ψ_γ, we can express the spatial representation of this operator by defining the coordinate system in which it exists. Since there are an infinite number of accessible states to this operator we can once again make the transition from a discrete number of $\vec{k}$- states to a continuum[9]

$$\psi_\gamma\left(\vec{r}, t\right) = \sqrt{\frac{1}{V} \frac{V}{(2\pi)^3}} \int d^3\vec{k} \left[\zeta_{\vec{k},+} \hat{\epsilon}_{\vec{k},+} a_{\vec{k},+} e^{-i\nu_k t} e^{i\vec{k}\cdot\vec{x}} + \zeta^*_{\vec{k},-} \hat{\epsilon}^*_{\vec{k},-} a^\dagger_{\vec{k},-} e^{i\nu_k t} e^{-i\vec{k}\cdot\vec{x}} \right] \qquad (2.3.8)$$

$$\psi^\dagger_\gamma\left(\vec{r}, t\right) = \sqrt{\frac{1}{V} \frac{V}{(2\pi)^3}} \int d^3\vec{k} \left[\zeta^*_{\vec{k},+} \hat{\epsilon}^*_{\vec{k},+} a^\dagger_{\vec{k},+} e^{i\nu_k t} e^{-i\vec{k}\cdot\vec{x}} + \zeta_{\vec{k},-} \hat{\epsilon}_{\vec{k},-} a_{\vec{k},-} e^{-i\nu_k t} e^{i\vec{k}\cdot\vec{x}} \right] . \qquad (2.3.9)$$

Setting the coordinate system such that $\vec{k}$ always points along some radial direction with respect to the quantization axis associated with it's emitting source[10] from (2.1) we can define the unit wave vector as

$$\hat{k} \equiv \hat{r}. \qquad (2.3.10)$$

This fixes the polarization vectors to point along orientations that are orthogonal to $\hat{r}$. We choose these so as to retain the required spin nature of the photon by following the identity (2.1.32) and setting the polarization vectors,

$$\hat{\epsilon}_{\vec{k},+} = \frac{1}{\sqrt{2}} \left(\hat{\theta} + i\hat{\phi} \right) \qquad (2.3.11)$$

$$\hat{\epsilon}_{\vec{k},-} = \frac{1}{\sqrt{2}} \left(\hat{\theta} - i\hat{\phi} \right) \qquad (2.3.12)$$

---

[9]To rewrite the traditional integral form of the 4-vector as a discrete sum of modes which are available to an optical, cavity we use

$$\frac{V}{(2\pi)^3} \int d^3k \;\; \rightarrow \;\; \sum_{\vec{k}}$$

[10]Here we assume that any emitted photons will always tend to move away from their sources.
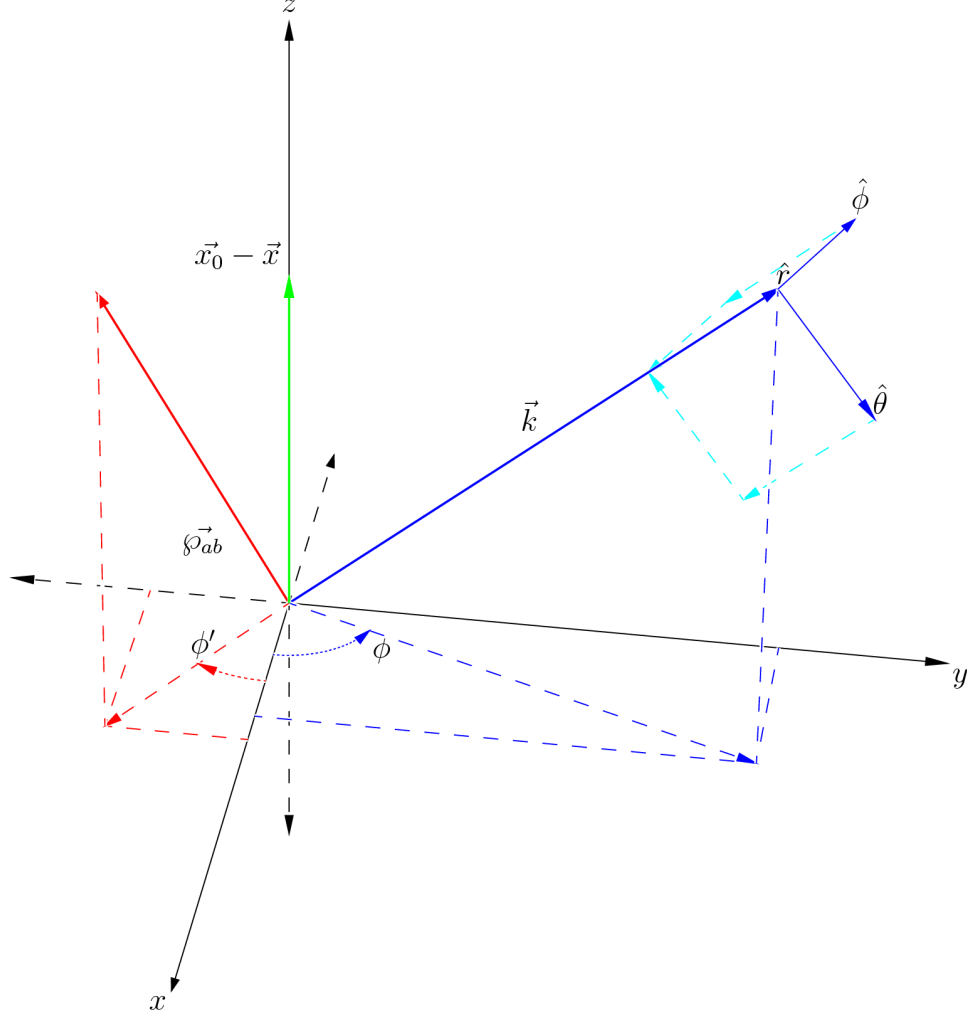
Figure 2.1: Coordinate System for $\vec{\Gamma}_\gamma$

in this coordinate system[11]. Expressing the wave and polarization vectors in terms of Cartesian unit vectors yields the relationship between these to be give by the transformations

$$\hat{k} = \sin\theta\cos\phi\hat{x} + \sin\theta\sin\phi\hat{y} + \cos\theta\hat{z}$$

$$\hat{\epsilon}_{\vec{k},+} = \frac{1}{\sqrt{2}}\left[(\cos\theta\cos\phi\hat{x} + \cos\theta\sin\phi\hat{y} - \sin\theta\hat{z}) + i\left(-\sin\phi\hat{x} + \cos\phi\hat{y} + 0\hat{z}\right)\right]$$

$$\hat{\epsilon}_{\vec{k},-} = \frac{1}{\sqrt{2}}\left[(\cos\theta\cos\phi\hat{x} + \cos\theta\sin\phi\hat{y} - \sin\theta\hat{z}) - i\left(-\sin\phi\hat{x} + \cos\phi\hat{y} + 0\hat{z}\right)\right] .$$

---

[11]By setting $\theta = 0$ and $\phi = 0$ to get $\hat{r}(\theta = 0, \phi = 0) = \hat{z}$, $\hat{\theta}(\theta = 0, \phi = 0) = \hat{x}$, and $\hat{\phi}(\theta = 0, \phi = 0) = \hat{y}$, and substituting these into the derivation (C.5), one can verify that the definition of these polarization and wave vectors indeed satisfy the derivation presented in (C.5).

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These can be simplified by grouping unit vectors to get

$$\hat{k} = \sin\theta\cos\phi\hat{x} + \sin\theta\sin\phi\hat{y} + \cos\theta\hat{z} \tag{2.3.13}$$

$$\hat{\epsilon}_{\vec{k},+} = \frac{1}{\sqrt{2}}\left[(\cos\theta\cos\phi - i\sin\phi)\,\hat{x} + (\cos\theta\sin\phi + i\cos\phi)\,\hat{y} - \sin\theta\hat{z}\right] \tag{2.3.14}$$

$$\hat{\epsilon}_{\vec{k},-} = \frac{1}{\sqrt{2}}\left[(\cos\theta\cos\phi + i\sin\phi)\,\hat{x} + (\cos\theta\sin\phi - i\cos\phi)\,\hat{y} - \sin\theta\hat{z}\right]. \tag{2.3.15}$$

By making a substitution of these coordinate transformations into the equation for the unit normalized PWF operators (2.3.9) we can express each component of the PWF independently. These components take the form of integrals that have to be evaluated as dependent upon unit normalized probability amplitudes which are expressed as

$$\psi_{\gamma,\vec{x}}^{(+)} = \sqrt{\frac{1}{V}\frac{V}{(2\pi)^3}}\int d^3\vec{k}\zeta_{\vec{k},+}\hat{\epsilon}_{\vec{k},+}a_{\vec{k},+}e^{-i\nu_k t}e^{i\vec{k}\cdot\vec{x}} \tag{2.3.16}$$

$$\psi_{\gamma,\vec{x}}^{(-)} = \sqrt{\frac{1}{V}\frac{V}{(2\pi)^3}}\int d^3\vec{k}\zeta_{\vec{k},-}^*\hat{\epsilon}_{\vec{k},-}^*a_{\vec{k},-}^\dagger e^{i\nu_k t}e^{-i\vec{k}\cdot\vec{x}} \tag{2.3.17}$$

$$\psi_{\gamma,\vec{x}}^{\dagger(+)} = \sqrt{\frac{1}{V}\frac{V}{(2\pi)^3}}\int d^3\vec{k}\zeta_{\vec{k},-}\hat{\epsilon}_{\vec{k},-}a_{\vec{k},-}e^{-i\nu_k t}e^{i\vec{k}\cdot\vec{x}} \tag{2.3.18}$$

$$\psi_{\gamma,\vec{x}}^{\dagger(-)} = \sqrt{\frac{1}{V}\frac{V}{(2\pi)^3}}\int d^3\vec{k}\zeta_{\vec{k},+}^*\hat{\epsilon}_{\vec{k},+}^*a_{\vec{k},+}^\dagger e^{i\nu_k t}e^{-i\vec{k}\cdot\vec{x}}. \tag{2.3.19}$$

Since these integrals have the general form

$$\vec{\Xi}_{\vec{x},\sigma_\pm} = \sqrt{2}\int d^3\vec{k}\hat{\epsilon}_{\vec{k},\pm}a_{\vec{k},\pm}\zeta_{\vec{k},\pm}e^{\mp i\nu_k t}e^{\pm i\vec{k}\cdot\vec{x}} \tag{2.3.20}$$

$$\vec{\Xi}_{\vec{x},\sigma_\pm}^\dagger = \sqrt{2}\int d^3\vec{k}\hat{\epsilon}_{\vec{k},\pm}^*a_{\vec{k},\pm}^\dagger\zeta_{\vec{k},\pm}^*e^{\pm i\nu_k t}e^{\mp i\vec{k}\cdot\vec{x}}, \tag{2.3.21}$$

we can expand out each of these components in terms of the three orthogonal unit vectors $\hat{x}$, $\hat{y}$, and $\hat{z}$. This yields 12 integrals need to be evaluated in order to acquire the spatial representation of $\vec{\Gamma}_\gamma$

and $\vec{\Gamma}_\gamma^\dagger$. These twelve integrals are

$$\vec{\Xi}_{\hat{x},\pm} = \hat{x} \int d^3\vec{k} \left(\cos\theta\cos\phi \pm i\sin\phi\right) a_{\vec{k},\sigma_\pm} \zeta_{\vec{k},\pm} e^{\mp i\nu_k t} e^{\pm i\vec{k}\cdot\vec{x}} \qquad (2.3.22)$$

$$\vec{\Xi}_{\hat{y},\pm} = \hat{y} \int d^3\vec{k} \left(\cos\theta\sin\phi \pm i\cos\phi\right) a_{\vec{k},\sigma_\pm} \zeta_{\vec{k},\pm} e^{\mp i\nu_k t} e^{\pm i\vec{k}\cdot\vec{x}} \qquad (2.3.23)$$

$$\vec{\Xi}_{\hat{z},\pm} = -\hat{z} \int d^3\vec{k} \left(\sin\theta\right) a_{\vec{k},\pm} \zeta_{\vec{k},\pm} e^{\mp i\nu_k t} e^{\pm i\vec{k}\cdot\vec{x}} \qquad (2.3.24)$$

and

$$\vec{\Xi}_{\hat{x},\sigma_\pm}^\dagger = \hat{x} \int d^3\vec{k} \left(\cos\theta\cos\phi \mp i\sin\phi\right) a_{\vec{k},\sigma_\pm}^\dagger \zeta_{\vec{k},\pm}^* e^{\pm i\nu_k t} e^{\mp i\vec{k}\cdot\vec{x}} \qquad (2.3.25)$$

$$\vec{\Xi}_{\hat{y},\sigma_\pm}^\dagger = \hat{y} \int d^3\vec{k} \left(\cos\theta\sin\phi \mp i\cos\phi\right) a_{\vec{k},\sigma_\pm}^\dagger \zeta_{\vec{k},\pm}^* e^{\pm i\nu_k t} e^{\mp i\vec{k}\cdot\vec{x}} \qquad (2.3.26)$$

$$\vec{\Xi}_{\hat{z},\sigma_\pm}^\dagger = -\hat{z} \int d^3\vec{k} \left(\sin\theta\right) a_{\vec{k},\sigma_\pm}^\dagger \zeta_{\vec{k},\pm}^* e^{\pm i\nu_k t} e^{\mp i\vec{k}\cdot\vec{x}} \,. \qquad (2.3.27)$$

We do not yet have any explicit expressions for $a_{\vec{k},\pm}^\dagger$, $a_{\vec{k},\pm}$, or $\zeta_{\vec{k},\pm}$. These will be determined in a later chapter. In the meantime, these integrals can not be evaluated, but these expressions can and will be used when coupling a quantum source to the photonic states corresponding to the Dirac-like equation of motion for this field. Finally we note that the integrals $\vec{\Xi}_{i,\pm}$ and $\vec{\Xi}_{i,\pm}^\dagger$ can be understood to be the Cartesian projections for the creation or annihilation operators of a photon in the spatial representation. Expressing the unit normalized PWF operators ψ_γ in terms of these integrals we can see that there is no longer a wave-vector $\vec{k}$ dependence on these.

$$\psi_\gamma = \frac{\sqrt{V}}{(2\pi)^3} \sum_{i=1}^3 \frac{1}{\sqrt{2}} \left(\vec{\Xi}_{i,\sigma_+} + \vec{\Xi}_{i,\sigma_-}^\dagger\right) \qquad (2.3.28)$$

$$\psi_\gamma^\dagger = \frac{\sqrt{V}}{(2\pi)^3} \sum_{i=1}^3 \frac{1}{\sqrt{2}} \left(\vec{\Xi}_{i,\sigma_-} + \vec{\Xi}_{i,\sigma_+}^\dagger\right) \,. \qquad (2.3.29)$$

2.4   Electric and Magnetic Field operators from the PWF $\vec{\Gamma}_\gamma$ Operator

In the previous section we defined and motivated the use of the PWF operator. In this section we define the relationship between this operator and the original electric and magnetic field operators.

We start by looking at the independent parts of the fields and then generalize the discussion to incorporate all the necessary components of all the required fields. From the operators defined in the previous section we can describe the Electric and Magnetic ones through definitions (2.1.1) and (2.1.2). These are

$$
\vec{E}^{(+)}\left(\vec{r},t\right) = \sum_{\vec{k},\sigma_\lambda} \hat{\epsilon}_{\vec{k},\sigma_\lambda} a_{\vec{k},\sigma_\lambda} \sqrt{\frac{\hbar\nu_k}{2\varepsilon_0 V}} e^{-i\nu_k t} e^{i\vec{k}\cdot\vec{x}}
$$

$$
\vec{H}^{(+)}\left(\vec{r},t\right) = \sum_{\vec{k},\sigma_\lambda} \frac{\vec{k}}{k} \times \hat{\epsilon}_{\vec{k},\sigma_\lambda} \sqrt{\frac{\hbar\nu_k}{2\mu_0 V}} a_{\vec{k},\sigma_\lambda} e^{-i\nu_k t} e^{i\vec{k}\cdot\vec{x}} .
$$

Using the identity (2.1.32) and rewriting in terms of $\vec{\Gamma}_\gamma^{(+)}$ yields,

$$
\vec{E}^{(+)}\left(\vec{r},t\right) = \sqrt{\frac{\hbar\omega}{2\varepsilon_0}} \left(\vec{\Gamma}_\gamma^{(+)} + \vec{\Gamma}_\gamma^{\dagger(+)}\right) \tag{2.4.1}
$$

$$
\vec{H}^{(+)}\left(\vec{r},t\right) = -i\sqrt{\frac{\hbar\omega}{2\mu_0}} \left(\vec{\Gamma}_\gamma^{(+)} - \vec{\Gamma}_\gamma^{\dagger(+)}\right) . \tag{2.4.2}
$$

Since the same is true for $\vec{E}^{(-)}\left(\vec{r},t\right)$ and $\vec{H}^{(-)}\left(\vec{r},t\right)$, the general case is then given by

$$
\vec{E}\left(\vec{r},t\right) = \sum_{\vec{k},\sigma_\lambda} \sqrt{\frac{\hbar\nu_k}{2\varepsilon_0 V}} \left[\hat{\epsilon}_{\vec{k},\sigma_\lambda} a_{\vec{k},\sigma_\lambda} e^{-i\nu_k t} U_{\vec{k}}\left(\vec{r}\right) + \hat{\epsilon}_{\vec{k},\sigma_\lambda}^* a_{\vec{k},\sigma_\lambda}^\dagger e^{i\nu_k t} U_{\vec{k}}^*\left(\vec{r}\right)\right]
$$

$$
\vec{H}\left(\vec{r},t\right) = \sum_{\vec{k},\sigma_\lambda} \sqrt{\frac{\hbar\nu_k}{2\mu_0 V}} \left[\frac{\vec{k}}{k} \times \hat{\epsilon}_{\vec{k},\sigma_\lambda} a_{\vec{k},\sigma_\lambda} e^{-i\nu_k t} U_{\vec{k}}\left(\vec{r}\right) + \frac{\vec{k}}{k} \times \hat{\epsilon}_{\vec{k},\sigma_\lambda}^* a_{\vec{k},\sigma_\lambda}^\dagger e^{i\nu_k t} U_{\vec{k}}^*\left(\vec{r}\right)\right] .
$$

Again making the same substitution and using the identity (2.1.32) as above yields the complete set of positive and negative frequencies of the electromagnetic fields to be related to the PWF operator via

$$
\vec{E}\left(\vec{r},t\right) = \sqrt{\frac{\hbar\omega}{2\varepsilon_0}} \left(\vec{\Gamma}_\gamma + \vec{\Gamma}_\gamma^\dagger\right) \tag{2.4.3}
$$

$$
\vec{H}\left(\vec{r},t\right) = -i\sqrt{\frac{\hbar\omega}{2\mu_0}} \left(\vec{\Gamma}_\gamma - \vec{\Gamma}_\gamma^\dagger\right) . \tag{2.4.4}
$$

This discussion can be extended to incorporate the displacement and magnetic fields $\vec{D} \equiv \varepsilon_0 \vec{E}$ and $\vec{B} \equiv \mu_0 \vec{H}$ by solving for $\vec{\Gamma}_\gamma$ and $\vec{\Gamma}_\gamma^\dagger$. From this problem it can be found that, at least in media that is homogeneous, the energy normalized PWF operator can be expressed in terms of the displacement and magnetic fields by means of the expressions

$$\vec{\Gamma}_\gamma = \frac{1}{\sqrt{2}} \frac{1}{\sqrt{\hbar\omega}} \left(\frac{\vec{D}}{\sqrt{\varepsilon_0}} + i \frac{\vec{B}}{\sqrt{\mu_0}} \right) \tag{2.4.5}$$

$$\vec{\Gamma}_\gamma^\dagger = \frac{1}{\sqrt{2}} \frac{1}{\sqrt{\hbar\omega}} \left(\frac{\vec{D}}{\sqrt{\varepsilon_0}} - i \frac{\vec{B}}{\sqrt{\mu_0}} \right). \tag{2.4.6}$$

Conversely, the displacement vector and magnetic fields can also be expressed in terms of the PWF operator by means of the following expressions

$$\frac{1}{\sqrt{\hbar\omega}} \frac{\vec{D}}{\sqrt{2\varepsilon_0}} = \vec{\Gamma}_\gamma + \vec{\Gamma}_\gamma^\dagger \tag{2.4.7}$$

$$\frac{1}{\sqrt{\hbar\omega}} \frac{i\vec{B}}{\sqrt{2\mu_0}} = \vec{\Gamma}_\gamma - \vec{\Gamma}_\gamma^\dagger. \tag{2.4.8}$$

The treatment presented above can be generalized to any linear homogeneous media by multiplying the vacuum dielectric constants by their corresponding dielectric constants such that $\varepsilon \equiv (1 + \chi_e) \varepsilon_0$ and $\mu \equiv (1 + \chi_m) \mu_0$ and following the same procedure above. It is a major advantage of this quantum mechanical picture that, even if these dielectric factors depend on the number of photons that are present in the media, any change in these dielectric factors can be immediately taken into consideration by means of the following expressions

$$\frac{1}{\sqrt{\hbar\omega}} \frac{\vec{D}}{\sqrt{2\varepsilon}} = \vec{\Gamma}_\gamma + \vec{\Gamma}_\gamma^\dagger \tag{2.4.9}$$

$$\frac{1}{\sqrt{\hbar\omega}} \frac{\vec{B}}{\sqrt{2\mu}} = \vec{\Gamma}_\gamma - \vec{\Gamma}_\gamma^\dagger. \tag{2.4.10}$$

The treatment of the photon as a PWF comes full circle to the analogy of the quantum oscillator. Here we can see that if we treat the energy normalized PWF operator as the creation and annihilation operators of spin resolved photonic Fock states, the Magnetic component correspond

to a momentum like operator, and the Electric field to position-like operator. One final and important identity that relates the energy density between the formulation of the photon in terms of electromagnetic fields and that of the PWF is given in the following short derivation

$$
\begin{aligned}
\frac{\varepsilon}{2}\vec{E}^2 + \frac{1}{2\mu}\vec{B}^2 &= \frac{1}{4}\left(\vec{\Gamma}_\gamma + \vec{\Gamma}_\gamma^\dagger\right)\left(\vec{\Gamma}_\gamma^\dagger + \vec{\Gamma}_\gamma\right) + \frac{1}{4}\left(-i\vec{\Gamma}_\gamma + i\vec{\Gamma}_\gamma^\dagger\right)\left(i\vec{\Gamma}_\gamma^\dagger - i\vec{\Gamma}_\gamma\right) \\
&= \frac{1}{4}\left(\vec{\Gamma}_\gamma\vec{\Gamma}_\gamma^\dagger + \vec{\Gamma}_\gamma\vec{\Gamma}_\gamma + \vec{\Gamma}_\gamma^\dagger\vec{\Gamma}_\gamma^\dagger + \vec{\Gamma}_\gamma^\dagger\vec{\Gamma}_\gamma\right) \\
&+ \frac{1}{4}\left(\vec{\Gamma}_\gamma\vec{\Gamma}_\gamma^\dagger - \vec{\Gamma}_\gamma\vec{\Gamma}_\gamma - \vec{\Gamma}_\gamma^\dagger\vec{\Gamma}_\gamma^\dagger + \vec{\Gamma}_\gamma^\dagger\vec{\Gamma}_\gamma\right) \\
&= \frac{1}{2}\left(\vec{\Gamma}_\gamma^\dagger\vec{\Gamma}_\gamma + \vec{\Gamma}_\gamma\vec{\Gamma}_\gamma^\dagger\right) \\
&= \frac{1}{V}\sum_{\vec{k},\lambda}\hbar\nu_k\left(a_{\vec{k},\lambda}^\dagger a_{\vec{k},\lambda} + \frac{1}{2}\right) \ .
\end{aligned}
$$

This shows that defining the PWF does not violate energy conservation. In the next chapter we will derive in full detail the conserved quantities associated with this formalism to show that it is a good candidate for yielding new insight into the problems we are interested in. To do so we will have to introduce a new PWF Lagrangian which will be proposed purely from the requirements that photons have no mass and that they carry unit spin.

3   RELATIVISTIC QUANTUM FIELD FORMULATION OF THE PHOTONIC WAVE FUNCTION AND SELF DUAL ELECTROMAGNETIC TENSOR

It is evident from previous studies that a fully co/contra-variant Relativistic Quantum Field Theory describing a photon in a PWF formalism should be possible [10, 11]. The arguments presented in those studies were focused on using Maxwell's equations to justify the definition of the PWF in terms of the Riemann-Silberstein vector. In this chapter we use the ideas presented in additional studies that look at the possibility of defining a Lagrangian for the PWF formalism [73, 46, 47, 45, 66]. We start from the fully relativistic limit due to the fact that we recognize the photon as a fully relativistic particle which needs to be described by means of a complex wave function. This is due to the previously mentioned constraints that were imposed on such a PWF. In summary, these constraints state that a non-relativistic and non-complex photonic wave function does not exist [68, 15]. Our intent is to arrive at an interaction between a PWF formulated to describe a photon and a quantum source in any approximation.

In the proposed theory, which describes the photon by means of a Dirac-like equation in 3+1 dimensions, we begin the rigorous description of the PWF by defining the Casimir invariants of the Poincaré group which we will use as motivation. We will then propose a PWF Lagrangian and within the a canonical quantization procedure, like the Gupta and Beuler method [13, 12] and determine the quantized form of the resulting field [76]. In later chapters we will use this quantized field for the definition of a Lagrangian and gauge of the interacting fields. During this procedure, we will draw a connection between the photon wave function (PWF) and the gauge field that is necessary to describe the interaction between charged massive particles and photons. The canonical quantization procedure will lead to two important results, the complex Maxwell Field Tensor (not to be confused with the Faraday or real Maxwell Field tensor) and the coupled electron-photon field equations in terms of a field equation for the PWF. This field tensor gives rise

to a PWF whose equations of motion and real and imaginary parts, when separated and treated as real vectors, satisfy Maxwell's equations. In the second result, which will be discussed in the next chapter, we show the coupling between a quantum source and the Maxwell Field it generates.

The Gauge for the PWF will be shown to correspond to the radiation gauge[1]. In this gauge, the correspondence between the quantized four-vector potential and the PWF has been shown to be given by the creation and annihilation operators of the photon [51]. Throughout the relativistic treatment we will maintain the Minkowski Metric to have the signature $(+,-,-,-)$ and to be represented in terms of the Dirac Gamma matrices (D.2) by

$$
\begin{aligned}
\eta^{\mu\nu} &= \frac{1}{2}\{\gamma^\mu, \gamma^\nu\} \\
&= \frac{1}{2}\left(\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu\right).
\end{aligned}
$$

The co-variant gamma matrices that will be used throughout the chapter are defined by[2]

$$
\begin{aligned}
\gamma_\mu &= \eta_{\mu\nu}\gamma^\nu \\
&= \left(\begin{array}{cccc} \gamma^0 & -\gamma^1 & -\gamma^2 & -\gamma^3 \end{array}\right).
\end{aligned}
$$

These gamma matrices are explicitly stated in the appendix (D.2). We present the convention that we will be using throughout the chapter and which is normally used in textbooks[13, 77, 12, 78]. For clarity we itemize the signature and definition of symbols used in this chapter.

- Greek indices such as μ, ν refer to four notation, while latin indices such as i, j refer to three notation

---

[1]

- Coulomb Gauge - $\vec{\nabla} \cdot \vec{A} = 0$ & $-\nabla^2 \Phi = 4\pi\rho$
- Radiation Gauge - $\vec{\nabla} \cdot \vec{A} = 0$ & $\Phi = 0$
- Lorenz Gauge - $\vec{\nabla} \cdot \vec{A} + \frac{1}{c}\partial_t \Phi = 0$ (Originally used and proposed by George F. Fitzgerald [citation needed])

[2]Some may prefer to use the alternative signature $(-,+,+,+)$, here we keep this signature in order to keep the traditional definition of the Dirac gamma matrices in the Dirac basis.

- Signature $(+, -, -, -)$

- Minkowski-Metric $\eta^{\mu\nu} = \eta_{\mu\nu} \equiv \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$

- Four-vectors

  - Standard Basis

    $x^\mu \equiv (ct, \vec{x}), \eta_{\mu\nu} x^\nu = x_\mu = (ct, -\vec{x})$, where $\vec{x} = (x, y, z)$

  - Derivatives in Einstein Notation

    $\partial_{x^\mu} \equiv \partial_\mu \equiv \left(c^{-1} \partial_t, \vec{\nabla} \right), \partial_{x_\mu} \equiv \eta^{\mu\nu} \partial_\nu = \partial^\mu = \left(c^{-1} \partial_t, -\vec{\nabla} \right)$

  - Momentum

    $p^\mu \equiv \left(\frac{E}{c}, \vec{p} \right), \eta_{\mu\nu} p^\nu = p_\mu = \left(\frac{E}{c}, -\vec{p} \right)$, where $\vec{p} = (p_x, p_y, p_z)$

    $p^\mu \equiv i\hbar \partial^\mu = \left(i\hbar c^{-1} \partial_t, \frac{\hbar}{i} \vec{\nabla} \right), \eta_{\mu\nu} p^\nu = p_\mu = i\hbar \partial_\mu = \left(i\hbar c^{-1} \partial_t, -\frac{\hbar}{i} \vec{\nabla} \right)$

- Pseudo-tensors

  - Permutation/Levi-Civita Symbol

    * $\varepsilon^{\mu\nu\alpha\beta}$ is 0 for any repeated indices and 1 for cyclic -1 for all non-cyclic permutations of $\mu\nu\alpha\beta$

    * ε_{ijk} is 0 for any repeated indices and 1 for cyclic -1 for all non-cyclic permutations of ijk

3.1 Defining the PWF from the Lagrangian Formalism

In this section we will start by finding the invariants we need to satisfy when developing a theory that relies on the Lagrangian formalism for PWF. By carrying out the definition of the PWF in this fashion we will no longer be constrained to the argument that necessitates Maxwell's equations and new insight about the photon may be discerned. Previous studies have sought to find such

a formalism,but have not arrived at a PWF Lagrangian that can be included with the interaction [46, 73, 45, 66]. In this discussion we contribute and yield a naturally feasible and straight forward definition for what will be single and multiphoton states interacting with quantum sources. We seek the aid of the Poincaré group, which is the group of isometries of Minkowski space-time [79], to determine what are the invariants we can rely on when approaching this discussion. Since all elementary particles fall in representations of this group, we expect that if we satisfy these invariants we will have the description for an elementary particle in this formalism. The Poincaré algebra is the Lie algebra of the Poincaré group. In component form, the Poincaré algebra is given by the commutation relations for the generator of translations P_μ and the generator of Lorentz transforms $M_{\mu\nu}$ [79, 80].

- $[P_\mu, P_\nu] = 0$

- $\frac{1}{i}[M_{\mu\nu}, P_\nu] = \eta_{\mu\rho}P_\nu - \eta_{\nu\rho}P_\mu$

- $\frac{1}{i}[M_{\mu\nu}, M_{\rho\sigma}] = \eta_{\mu\rho}M_{\nu\sigma} - \eta_{\mu\sigma}M_{\nu\rho} - \eta_{\nu\rho}M_{\mu\sigma} + \eta_{\nu\sigma}M_{\mu\rho}$

In this algebra, Casimir invariants or Casimir operators are distinguished elements of the center of the universal enveloping algebra of a Lie algebra which commutes with all elements of the Lie algebra [78]. Two Casimir invariants of the Poincaré algebra are:

- Norm square of the Momentum Four-Vector - $P^\mu \equiv p^\mu = \left(\frac{E}{c}, \vec{p}\right)$

  $P^2 = p^\mu p_\mu = \frac{E^2}{c^2} - \vec{p} \cdot \vec{p}$

- Norm square of the Pauli-Lubanski Four Vector [46, 73, 75, 81] $W^\mu \equiv \frac{\varepsilon_{\mu\nu\rho\sigma}}{2} P^\nu M^{\rho\sigma}$

  $W^2 = W^\mu W_\mu = \frac{\varepsilon_{\mu\nu\rho\sigma}\varepsilon^{\mu\nu\rho\sigma}}{4} P^\nu M^{\rho\sigma} P_\nu M_{\rho\sigma} = -\frac{1}{2}P^2 M^{\nu\rho}M_{\nu\rho} + P_\rho P^\nu M^{\rho\mu}M_{\nu\mu}$

These Casimir invariants are important to previous studies since they govern the divergence conditions of massless particles. These have the consequence that the photon has unit spin and no mass [73]. We present the definition of a photonic wave-function as motivated from the norm square of the momentum four vector and use the same conditions of the Pauli-Lubanski four-vector [73] to satisfy the divergence condition in the vacuum.

3.1.1 Quantization of the 4-Potential corresponding to a Maxwell Field in QFT

The discussion of the interaction of an electromagnetic field with any charged massive particle depends on the possibility of defining a gauge field that can be used in the minimal substitution for a canonical momentum [78]. Therefore, we define the quantized 4 vector of the electromagnetic field which we will later use for the interaction. This exercise is not only useful for definition purposes, but also for direct comparison and interpretation of the relationship between any object defined in terms of this vector potential and the PWF we are after.

We refer the quantization of the 4-potential in the radiation Gauge to the canonical quantization procedure outlined by the Gupta-Beuler method for an electromagnetic field presented in textbooks [12, 78, 13, 77]. In those discussions the classical analog to the 4-vector potentials lead to Maxwell's Equations and can therefore be expanded in terms of any function $\mathfrak{U}_{\vec{k}}(\vec{x})$ that satisfies the wave-equation for the D'Alembertian operator $\Box \mathfrak{U}_{\vec{k}}(x) = 0$. That expansion is given by the definition of the 3 vector potential in terms of these functions $\mathfrak{U}_{\vec{k}}(\vec{x}, t)$ and their corresponding polarization $\hat{\epsilon}_{\vec{k},\lambda}$, and annihilation operator $a_{\vec{k},\lambda}$

$$\vec{A}(\vec{x}, t) = \sum_{\vec{k},\lambda} \left(\hat{\epsilon}_{\vec{k},\lambda} a_{\vec{k},\lambda} \mathfrak{U}_{\vec{k}}(\vec{x}, t) + \hat{\epsilon}^*_{\vec{k},\lambda} a^\dagger_{\vec{k},\lambda} \mathfrak{U}^*_{\vec{k}}(\vec{x}, t) \right). \tag{3.1.1}$$

One set of functions that satisfy these restrictions are plane waves. Therefore the choice is made to define $\mathfrak{U}_{\vec{k}}(\vec{x}) = \frac{1}{ck^0} \mathcal{E}_{\vec{k}} e^{-ik^\mu x_\mu} e^{-i\phi}$ with a normalization factor $\mathcal{E}_{\vec{k}} = \sqrt{\frac{\hbar \nu_k}{2\varepsilon_0 V}}$ that guarantees units of energy for this object. The wave vector $\vec{k}$ is taken to be part of a uniform k-space where the norm of the 4 wave vector is analogous to the norm of the 4-momentum $k^\mu k_\mu = c^{-2}\nu_k^2 t^2 - k^2 = 0$, $k^\mu \equiv \left(c^{-1}\nu_k, \vec{k} \right)$, $k_\mu \equiv \left(c^{-1}\nu_k, -\vec{k} \right)$, and ϕ can be considered a global Gauge parameter. Please note the meaning of $\vec{x}$ as the position vector in Cartesian coordinates $\vec{x} \equiv x\hat{x} + y\hat{y} + z\hat{z}$ since we plan to use the notation $\vec{x} = \sum_{i=1}^{3} x^i \hat{x}^i$ and $-\vec{x} = \sum_{i=1}^{3} x_i \hat{x}_i$.

In terms of this plane wave expansion, the vector potential is[3]

$$\vec{A}\left(x_{\mu}\right) = \sum_{\vec{k},\lambda} \frac{1}{ck^0} \left(\hat{\epsilon}_{\vec{k},\lambda} \mathcal{E}_{\vec{k}} a_{\vec{k},\lambda} e^{-ik^{\mu}x_{\mu}} e^{-i\phi} + \hat{\epsilon}^*_{\vec{k},\lambda} \mathcal{E}^*_{\vec{k}} a^{\dagger}_{\vec{k},\lambda} e^{ik^{\mu}x_{\mu}} e^{i\phi} \right) . \qquad (3.1.2)$$

Assuming that there are an infinite number of accessible states with wave-vector $\vec{k}$, the vector potential can be represented as a Fourier Integral through[4]

$$\vec{A}\left(\vec{x},t\right) = \frac{V}{(2\pi)^3} \int \frac{d^3k}{\nu_k} \sum_{\lambda} \left(\hat{\epsilon}_{\vec{k},\lambda} \mathcal{E}_{\vec{k}} a_{\vec{k},\lambda} e^{-i\nu_k t} e^{i\vec{k}\cdot\vec{x}} e^{-i\phi} + \hat{\epsilon}^*_{\vec{k},\lambda} \mathcal{E}^*_{\vec{k}} a^{\dagger}_{\vec{k},\lambda} e^{i\nu_k t} e^{-i\vec{k}\cdot\vec{x}} e^{i\phi} \right) . \qquad (3.1.3)$$

The textbook Lagrangian density for this vector potential of the quantized Maxwell field is then given in terms of the Faraday tensor rate of change of the vector potential as [72, 78, 13, 12]

$$\begin{aligned}
\mathfrak{L} &= -\frac{1}{4} \left(\partial_{\mu} A_{\nu} - \partial_{\nu} A_{\mu} \right) \left(\partial^{\mu} A^{\nu} - \partial^{\nu} A^{\mu} \right) \\
&= -\frac{1}{4} \mathfrak{F}_{\mu\nu} \mathfrak{F}^{\mu\nu} ,
\end{aligned}$$

where $\partial_{\mu} \equiv \left(\frac{1}{c}\partial_t, \nabla \right)$ is the four-gradient and $\mathfrak{F}^{\mu\nu}$ is said to be the quantized electromagnetic or Maxwell field tensor or Faraday tensor. In terms of the electric and magnetic fields, the Faraday

---

[3]Though this notation is suggestive and similar to that which has been presented so far in the text, at this point please consider this expression as <u>not</u> written in terms of the traditional circular polarization vectors $\hat{\epsilon}_{\vec{k},\lambda}$ defined above or the same creation/annihilation operators $a_{\vec{k},\lambda}$ & $a^{\dagger}_{\vec{k},\lambda}$ we have been using so far. That these are the same is still in need of proof which we will present herein.

[4]To rewrite the traditional integral form of the 4-vector as a discrete sum of modes which are available to an optical, cavity we use

$$\frac{V}{(2\pi)^3} \int d^3k \quad \rightarrow \quad \sum_{\vec{k}}$$

tensor is then

$$\mathfrak{F}^{\mu\nu} = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{pmatrix}. \tag{3.1.4}$$

In the absence of sources this tensor satisfies $\Box\mathfrak{F}^{\mu\nu} = 0$ and can be derived from the 4-potential by $\mathfrak{F}^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$.

3.1.2 *Electric, Magnetic Fields, and Maxwell's Equations*

The comparison and correspondence of the PWF as a Maxwell Field in QED has previously been discussed in detail [52]. Those discussions revolve around rewriting Maxwell's equations to arrive at a set of equations for the Riemann-Silberstein vector [10, 11, 51]. The definition of Maxwell's equations from the Faraday tensor will be instructive later when we seek the equations of motion for the PWF. From the Lagrangian formalism, Maxwell's equations are the equations of motion and are written in terms of the Faraday tensor $\mathfrak{F}^{\mu\nu}$. However, a sort of symmetry breaking occurs in this formalism since in order to be able to get back all of Maxwell's equations it becomes necessary to introduce the dual of the Faraday tensor $\mathfrak{F}_D^{\mu\nu}$. This dual tensor can be calculated from the vector potential A^μ via the definition of the dual tensor [82]

$$\mathfrak{F}_{\mu\nu}^D = \frac{1}{2}\varepsilon_{\mu\nu\alpha\beta}\mathfrak{F}^{\alpha\beta} \tag{3.1.5}$$

$$\mathfrak{F}_{\nu\mu}^D = -\mathfrak{F}_{\mu\nu}^D. \tag{3.1.6}$$

Rewriting this definition in terms of the 4-vector potential shows that there is a type of curling of this vector that gives rise to the dual

$$\mathfrak{F}_D^{\mu\nu} = \frac{1}{2}\varepsilon^{\mu\nu\alpha\beta}\left(\partial_\alpha A_\beta - \partial_\beta A_\alpha\right) \tag{3.1.7}$$

Explicitly writing this tensor in terms of the electric and magnetic fields yields that indeed this object refers to

$$\mathfrak{F}_D^{\mu\nu} = \begin{pmatrix} 0 & -B_x & -B_y & -B_z \\ B_x & 0 & E_z & -E_y \\ B_y & -E_z & 0 & E_x \\ B_z & E_y & -E_x & 0 \end{pmatrix} \tag{3.1.8}$$

The corresponding electric and magnetic fields can then be extrapolated in terms of the two tensors $\mathfrak{F}^{\mu\nu}$ and its dual $\mathfrak{F}_D^{\mu\nu}$ to be (3.1.3)

$$\begin{aligned} \vec{E} &= \mathfrak{F}^{\mu 0} \\ &= -\mathfrak{F}^{0\nu} \\ \vec{B} &= \mathfrak{F}_D^{\mu 0} \\ &= \mathfrak{F}_D^{0\nu}. \end{aligned}$$

Taking the time derivative and curl of $\vec{A}(\vec{r}, t)$ respectively yields these expressions (given in natural units) to be

$$\begin{aligned} \vec{E}(\vec{r}, t) &= -\partial_t \vec{A} \\ &= i\frac{V}{(2\pi)^3} \int d^3k \sum_\lambda \hat{\epsilon}_{\vec{k},\lambda} \mathcal{E}_{\vec{k}} a_{\vec{k},\lambda} e^{-i\nu_k t} e^{i\vec{k}\cdot\vec{x}} e^{-i\phi} + \text{H.c.} \end{aligned}$$

$$\begin{aligned} \vec{B}(\vec{r}, t) &= \vec{\nabla} \times \vec{A} \\ &= i\frac{V}{(2\pi)^3} \int \frac{d^3k}{\nu_k} \sum_\lambda \vec{k} \times \hat{\epsilon}_{\vec{k},\lambda} \mathcal{E}_{\vec{k}} a_{\vec{k},\lambda} e^{-i\nu_k t} e^{i\vec{k}\cdot\vec{x}} e^{-i\phi} + \text{H.c.} . \end{aligned}$$

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Re-expressing these as a discrete sum[5] and fixing the global Gauge parameter as [51] $\phi = \dfrac{\pi}{2}$ indeed yields the second quantization form for the electric and magnetic fields that we used in the previous chapters

$$\vec{E} = \sum_{\vec{k},\lambda} \hat{\epsilon}_{\vec{k},\lambda} \mathcal{E}_{\vec{k}} a_{\vec{k},\lambda} e^{-i\nu_k t} e^{i\vec{k}\cdot\vec{x}} + \text{H.c.} \tag{3.1.9}$$

$$\vec{B} = \sum_{\vec{k},\lambda} \frac{1}{\nu_k} \vec{k} \times \hat{\epsilon}_{\vec{k},\lambda} \mathcal{E}_{\vec{k}} a_{\vec{k},\lambda} e^{i\nu_k t} e^{-i\vec{k}\cdot\vec{x}} + \text{H.c.} . \tag{3.1.10}$$

Note that in this section we have not yet defined any properties of the polarization vectors we are using. This leaves their interpretation open to either transverse linear vectors or helical spin resolved left or right handed vectors.

3.1.3 Relativistic Quantum Field Equation of the Free Photon

To propose a new Lagrangian we begin by trying to guess the equations of motion for the elementary particle we are looking for and from these work backwards. Much of this discussion has already been determined in previous studies and we will use the idea that proposes a new Lagrangian which will directly lead to a vector similar to the Riemann-Silberstein; vector up to a phase factor of $\phi = \frac{\pi}{2}$ [73, 46, 45, 49, 80, 47, 52, 10, 11, 51]. Starting from the four momentum Lorentz invariant $P^\mu P_\mu = \frac{E^2}{c^2} - \vec{p}\cdot\vec{p} = m^2 c^2$ we follow an approach similar to that taken by Dirac in hopes of finding the "square root" of this Lorentz invariant.

$$E = c\sqrt{(\vec{p}\cdot\vec{p} + m^2 c^2)} . \tag{3.1.11}$$

---

[5]To rewrite the traditional integral form of the 4-vector as a discrete sum of modes which are available to an optical, cavity we use

$$\frac{V}{(2\pi)^3} \int d^3 k \quad \rightarrow \quad \sum_{\vec{k}}$$

Upon substituting $E = i\hbar c^{-1}\partial_t$ and $\vec{p} = \frac{\hbar}{i}\vec{\nabla}$, this energy relation yields an operator equation which should factor as

$$\left[\hbar^2\vec{\nabla}^2 - \hbar^2 c^{-2}\partial_t\right] = \left(A\partial_x + B\partial_y + C\partial_z + ic^{-1}D\partial_t\right)\left(A\partial_x + B\partial_y + C\partial_z + ic^{-1}D\partial_t\right),$$

(3.1.12)

where A, B, C, and D are matrices. One suggestions for the definition of these matrices which has been previously studied [10, 11] is given below

$$\begin{pmatrix} \frac{i\hbar}{c}\partial_t + \hbar\vec{\nabla}\times & 0 \\ 0 & -\frac{i\hbar}{c}\partial_t + \hbar\vec{\nabla}\times \end{pmatrix} \begin{pmatrix} \vec{F}_+ \\ \vec{F}_- \end{pmatrix} = 0$$

and is subject to the divergence conditions

$$\begin{pmatrix} \frac{\hbar}{i}\vec{\nabla}\cdot & 0 \\ 0 & \frac{\hbar}{i}\vec{\nabla}\cdot \end{pmatrix} \begin{pmatrix} \vec{F}_+ \\ \vec{F}_- \end{pmatrix} = 0 .$$

Those studies have taken advantage of the fact that the curl operator $\vec{\nabla}\times$ may be written as $i\partial_k\left(-i\varepsilon_{ijk}\right)$, or in the matrix notation of an SO(3) group, $\vec{\sigma}^{(3)} = -i\varepsilon_{ijk}$ (where the three denotes that these are three, 9 component (3×3) matrices). In that notation the expression above takes on the following form

$$\left[\frac{i\hbar}{c}\partial_t \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} - \frac{\hbar}{i}\partial_k \begin{pmatrix} \sigma_k^{(3)} & 0 \\ 0 & \sigma_k^{(3)} \end{pmatrix}\right] \begin{pmatrix} \vec{F}_+ \\ \vec{F}_- \end{pmatrix} = 0 .$$

However, that expression can be shown not to satisfy the invariance condition for the four momentum P^2 above. Instead, following the discussion presented in other studies [45, 73, 46, 75], those

matrices can be better represented by the equation of motion

$$\left[\frac{i\hbar}{c} \partial_t \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} - \frac{\hbar}{i} \partial_k \begin{pmatrix} 0 & \sigma_k^{(3)} \\ \sigma_k^{(3)} & 0 \end{pmatrix} \right] \begin{pmatrix} \vec{F}_- \\ \vec{F}_+ \end{pmatrix} = 0 \,. \tag{3.1.13}$$

For the case where we also consider sources this expression takes the form[6]

$$\left[\frac{i}{c} \partial_t \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} - \frac{1}{i} \begin{pmatrix} 0 & \vec{\nabla} \cdot \vec{\sigma}^{(3)} \\ \vec{\nabla} \cdot \vec{\sigma}^{(3)} & 0 \end{pmatrix} \right] \begin{pmatrix} \vec{F}_- \\ \vec{F}_+ \end{pmatrix} = \begin{pmatrix} \psi \vec{\alpha} \psi^\dagger \\ \psi^\dagger \vec{\alpha} \psi \end{pmatrix} \tag{3.1.14}$$

We study the invariance condition for the four momentum P^2 associated with this equation of motion equally closely. The notation of this expression may be simplified further by adopting definitions of the SO(3) gamma matrices [45]

$$\Gamma^0 \equiv \begin{pmatrix} 0 & I^{(3)} \\ -I^{(3)} & 0 \end{pmatrix} \tag{3.1.15}$$

$$\Gamma^k \equiv \begin{pmatrix} 0 & \vec{\sigma}^{(3)} \\ \vec{\sigma}^{(3)} & 0 \end{pmatrix} \,. \tag{3.1.16}$$

Using the index raising and lowering Minkowski we can see that these satisfy the equation $\eta_{\mu\nu}\Gamma^\nu = \Gamma_\mu$. For brevity we also introduce the spinor $\Psi \equiv \begin{pmatrix} \vec{F}_- \\ \vec{F}_+ \end{pmatrix}$ that yields these equations in the condensed form

$$\Gamma^\mu p_\mu \Psi = 0 \,. \tag{3.1.17}$$

This condensed notation will be used throughout the remainder of the chapter.

---

[6]The abscence of $\frac{\mu_0}{\sqrt{\mu_0}}$ is due to the fact that we work in Gaussian units.

3.1.4   Energy-Momentum relation

The energy-momentum relation for a free photon that is satisfied by equation 3.1.17 above can be found by multiplying the that expression with its conjugate. Doing so leads to the expression

$$
\Psi^\dagger \left(\Gamma^\mu p_\mu \right) \left(\Gamma^\mu p_\mu \right) \Psi = \Psi^\dagger \left(\frac{i\hbar}{c}\partial_t \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} - \begin{pmatrix} 0 & \sigma_k^{(3)} \\ \sigma_k^{(3)} & 0 \end{pmatrix} \frac{\hbar}{i}\nabla_k \right)^2 \Psi
$$

$$
= \Psi^\dagger \left[\frac{i\hbar}{c}\partial_t \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} \right]^2 \Psi - \Psi^\dagger \left[\begin{pmatrix} 0 & \sigma_k^{(3)} \\ \sigma_k^{(3)} & 0 \end{pmatrix} \frac{\hbar}{i}\nabla_k \right]^2 \Psi \, .
$$

Using the identity which states that the square of the matrix $\begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$ is the identity matrix, we can replace

$$
\left[\frac{i\hbar}{c}\partial_t \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} \right]^2 = -\frac{\hbar^2}{c^2}\partial_t^2 \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}
$$

$$
= \left(\frac{i\hbar}{c}\partial_t \right)^2 \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}
$$

into the expression above. Doing so shows that we are left to evaluate the square of the SO(3) matrices $\sigma_k^{(3)}$ of the expression

$$
\left[\begin{pmatrix} 0 & \sigma_k^{(3)} \\ \sigma_k^{(3)} & 0 \end{pmatrix} \frac{\hbar}{i}\nabla_k \right]^2 = \begin{pmatrix} \left(\sigma_k^{(3)}\frac{\hbar}{i}\nabla_k \right)^2 & 0 \\ 0 & \left(\sigma_k^{(3)}\frac{\hbar}{i}\nabla_k \right)^2 \end{pmatrix} .
$$

Before evaluating these however, we also substitute these into the expression above. Doing so tells us that we simply have to prove that $\left(\frac{\hbar}{i}\vec{\nabla}\cdot\vec{\sigma}^{(3)} \right)^2$ is equivalent to $\hbar^2\vec{\nabla}^2$ as evident from the

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equation of the energy-momentum relation we have so far, given by

$$\left(\frac{i\hbar}{c}\partial_t\right)^2 - \left(\frac{\hbar}{i}\vec{\nabla}\cdot\vec{\sigma}^{(3)}\right)^2 = 0\,.$$

We are therefore required to define the term $\left(\frac{\hbar}{i}\vec{\nabla}\cdot\vec{\sigma}^{(3)}\right)^2$. To prove that the equation above yields the correct energy momentum relation we will have to work with the commutator and anti-commutators of $\vec{\sigma}^{(3)}$. The commutation relation of these matrices are defined by

$$\begin{aligned}
\left[\sigma_i^{(3)}, \sigma_j^{(3)}\right]_- &= \left(\sigma_{ak}^{(3)}\right)_i \left(\sigma_{kb}^{(3)}\right)_j - \left(\sigma_{ak}^{(3)}\right)_j \left(\sigma_{kb}^{(3)}\right)_i \\
&= (-i\varepsilon_{aki})(-i\varepsilon_{kbj}) - (-i\varepsilon_{akj})(-i\varepsilon_{kbi}) \\
&= -\left[(\delta_{ib}\delta_{aj} - \delta_{ij}\delta_{ab}) - (\delta_{jb}\delta_{ai} - \delta_{ji}\delta_{ab})\right] \\
&= -(\delta_{ib}\delta_{aj} - \delta_{jb}\delta_{ai}) = i\varepsilon_{ijk}(-i\varepsilon_{abk}) \\
&= i\varepsilon_{ijk}\sigma_k^{(3)}\,.
\end{aligned}$$

These commutation relations assert that the matrices $\sigma^{(3)}$ are indeed an irreducible representations of the $SO(3)$ group [83]. The anti-commutators for these matrices are

$$\begin{aligned}
\left[\sigma_i^{(3)}, \sigma_j^{(3)}\right]_+ &= \left(\sigma_{ak}^{(3)}\right)_i \left(\sigma_{kb}^{(3)}\right)_j + \left(\sigma_{ak}^{(3)}\right)_j \left(\sigma_{kb}^{(3)}\right)_i \\
&= (-i\varepsilon_{aki})(-i\varepsilon_{kbj}) + (-i\varepsilon_{akj})(-i\varepsilon_{kbi}) \\
&= -\left[(\delta_{ib}\delta_{aj} - \delta_{ij}\delta_{ab}) + (\delta_{jb}\delta_{ai} - \delta_{ji}\delta_{ab})\right] \\
&= -(\delta_{ia}\delta_{jb} + \delta_{ib}\delta_{ja} - 2\delta_{ij}\delta_{ab}) \\
&= 2\delta_{ij}\delta_{ab} - \delta_{ia}\delta_{jb} - \delta_{ib}\delta_{ja}\,.
\end{aligned}$$

Substituting these in for $\sigma_i^{(3)}\sigma_j^{(3)}$ in $\left(\frac{\hbar}{i}\vec{\nabla}\cdot\vec{\sigma}^{(3)}\right)^2$ and making use of the identity $\sigma_i^{(3)}\sigma_j^{(3)} = \frac{1}{2}\left(\left[\sigma_i^{(3)},\sigma_j^{(3)}\right]_+ + \left[\sigma_i^{(3)}\right.\right.$

yields

$$
\begin{aligned}
\left(\frac{\hbar}{i}\vec{\nabla}\cdot\vec{\sigma}^{(3)}\right)^2 &= \frac{\hbar}{i}\nabla_i\sigma_i^{(3)}\frac{\hbar}{i}\nabla_j\sigma_j^{(3)} \\
&= -\hbar^2\frac{\nabla_i\nabla_j}{2}\left(\left[\sigma_i^{(3)},\sigma_j^{(3)}\right]_+ + \left[\sigma_i^{(3)},\sigma_j^{(3)}\right]_-\right) \\
&= -\hbar^2\frac{\nabla_i\nabla_j}{2}\left([2\delta_{ij}\delta_{ab} - \delta_{ia}\delta_{jb} - \delta_{ib}\delta_{ja}] - [\delta_{ib}\delta_{aj} - \delta_{jb}\delta_{ai}]\right) \\
&= -\hbar^2\frac{\nabla_i\nabla_j}{2}\left(\delta_{ij}\delta_{ab} - \delta_{ib}\delta_{ja}\right) \\
&= (i\hbar\nabla_i)(i\hbar\nabla_i) - (i\hbar\nabla_i)(i\hbar\nabla_j)\delta_{ib}\delta_{ja}\,.
\end{aligned}
$$

Representing the identify matrix as a Kronecker delta yields the energy-momentum relation in operator form to be

$$
\left(\frac{i\hbar}{c}\partial_t\right)^2\delta_{ab} - \left(\frac{\hbar}{i}\nabla_i\right)^2\delta_{ab} = -\left(\frac{\hbar}{i}\nabla_i\delta_{ib}\right)\left(\frac{\hbar}{i}\nabla_j\delta_{ja}\right)\,.
$$

Comparing this expression against the energy-momentum relation for a massive particle

$$
\frac{E^2}{c^2}\delta_{ab} - |\vec{p}|^2\,\delta_{ab} = m^2c^2\delta_{ab}\,, \tag{3.1.18}
$$

suggests that photons with longitudinal components in the vacuum have an effective mass. This can be more explicitly shown by expressing each component of Ψ as Ψ_a to find

$$
\begin{aligned}
&\Psi_b^\dagger\left(\frac{i\hbar}{c}\partial_t\right)^2\delta_{ab}\Psi_a - \Psi_b^\dagger\left(\frac{\hbar}{i}\nabla_i\right)^2\delta_{ab}\Psi_a + \Psi_b^\dagger\left(\frac{\hbar}{i}\nabla_i\delta_{ib}\right)\left(\frac{\hbar}{i}\nabla_j\delta_{ja}\right)\Psi_a \\
&= \Psi_a^\dagger\left(\frac{i\hbar}{c}\partial_t\right)^2\Psi_a - \Psi_a^\dagger\left(\frac{\hbar}{i}\nabla_i\right)^2\Psi_a + \Psi_b^\dagger\left(\frac{\hbar}{i}\nabla_i\delta_{ib}\right)\left(\frac{\hbar}{i}\nabla_j\delta_{ja}\right)\Psi_a \\
&= \frac{E^2}{c^2} - |\vec{p}|^2 + \left(p_i\cdot\Psi_i^\dagger\right)(p_j\cdot\Psi_j) \\
&= \frac{E^2}{c^2} - |\vec{p}|^2 - m_{eff}^2c^2\,.
\end{aligned}
$$

That effective mass would then be given by the scalar quantity

$$m_{eff}^2 \equiv -\frac{1}{c^2} \left(p_i \cdot \Psi_i^\dagger \right) \left(p_j \cdot \Psi_j \right) . \tag{3.1.19}$$

Note that if the photon field is divergence-less we regain the free photon Energy-momentum relation

$$\frac{E^2}{c^2} - |\vec{p}|^2 = 0 .$$

This also implies that the norm squared of the Pauli-Lubanski vector for such an elementary particle is independent of mass and due only to the spin of the system as governed by the relation[73]

$$W^\mu W_\mu = -s \left(s + 1 \right) p_\mu p^\mu I^{(s)} ,$$

where s corresponds to the total spin carried by the system and $I^{(s)}$ is the $(2s+1)$ unit matrix. In contrast, this relation would be mass dependent in the case of a massive particle as reported in previous studies [84]

3.1.5 General Lagrangian Formalism

We now adopt Feynman notation for brevity, where $\vec{\not{p}}$ is known as Feynman dagger notation for any operator $\not{\Lambda} \equiv \gamma^\mu \Lambda_\mu$, and γ^μ replace the commonly used Dirac gamma matrices matrices(D.2)with the definitions for Γ^μ above. Additionally we adopt the use of the notation

$$\bar{\psi} \equiv \psi^\dagger \gamma_0 \tag{3.1.20}$$

as the adjoint spinor of ψ instead with Γ^0 instead of the Dirac gamma matrix γ_0. In this way general Lagrangian formalism for the photon can be determined from the equation of motion

$$\not{p}\Psi = 0 .$$

We construct the Lagrangian by left multiplying by $\delta\bar{\Psi}$, where $\bar{\Psi} \equiv \Psi^\dagger\Gamma_0$ and integrating over all space-time

$$0 = \int d^4x\, \delta\bar{\Psi}\slashed{p}\Psi$$
$$= \delta\left(\int d^4x\, \bar{\Psi}\slashed{p}\Psi\right)$$

this leads us to identify the Lagrangian density that is in agreement with those defined in previous studies [46, 45]

$$\mathfrak{L}_{ph} = \bar{\Psi}\slashed{p}\Psi$$
$$= \Psi^\dagger\Gamma_0\left(\frac{i\hbar}{c}\partial_t\Gamma^0 - p_k\Gamma^k\right)\Psi$$

where $\vec{\Gamma} \equiv \Gamma^k$. The main difference here, is that we have not assigned an object to this Lagrangian. Namely, we have not stated what Ψ is and at this point leave it open to interpretation. In the following discussion we will fix this Lagrangian to an object which can be interpreted in terms of the unit normalized PWF discussed in the previous chapter.

3.1.5.1 Equations of Motion

Before moving onto the discussion of the relationship between this Lagrangian and the PWF, we first check that we do in fact get the same equations of motion that we started out with. The equations of motion arrived at by varying this Lagrangian with respect to Ψ are given by

$$\partial_\mu\left[\frac{\partial\mathfrak{L}_{ph}}{\partial\left(\partial_\mu\Psi\right)}\right] - \frac{\partial\mathfrak{L}_{ph}}{\partial\Psi}$$
$$= \partial_\mu\left[\bar{\Psi}i\hbar\Gamma^\mu\right] - 0$$

and are recognized as the adjoint equation for the photon that we started with.

3.1.5.2 Conjugate momentum and conservation laws

This Lagrangian density yields the momentum conjugate

$$\pi = \frac{\partial \mathfrak{L}_{ph}}{\partial \left(\partial_0 \Psi \right)}$$

$$= i\hbar \Psi^{\dagger}$$

from which we can define most conservation laws.

3.1.5.3 Hamiltonian Density

The Hamiltonian density resulting from this conjugate momentum is given by

$$\mathcal{H}_{ph} = \pi \partial_0 \Psi - \mathfrak{L}_{ph}$$

$$= i\hbar \Psi^{\dagger} \partial_0 \Psi - \bar{\Psi} \not{p} \Psi$$

$$= \Psi^{\dagger} \frac{i\hbar}{c} \partial_t \Psi - \bar{\Psi} \frac{i\hbar}{c} \partial_t \Gamma^0 \Psi + \bar{\Psi} \frac{\hbar}{i} \partial_k \Gamma^k \Psi$$

which follows from the photon wave equation

$$\mathcal{H}_{ph} = \bar{\Psi} \frac{\hbar}{i} \partial_k \Gamma^k \Psi = \bar{\Psi} \frac{i\hbar}{c} \partial_t \Gamma^0 \Psi$$

$$\mathcal{H}_{ph} \rightarrow \bar{\Psi} \frac{i\hbar}{c} \partial_t \Gamma^0 \Psi - \bar{\Psi} \frac{\hbar}{i} \partial_k \Gamma^k \Psi = 0 \, .$$

This is useful for two major reasons. Recalling the discussion on the non-existence of the PWF, the two main arguments posed against its existence were dependent on the fact that the operator $i\partial_t$ could not be satisfied by real electromagnetic fields and that the negative frequency solution could not be satisfied by a non-relativistic theory. Herein lies the success of this Lagrangian. It is not evident that with this Lagrangian we can now define a PWF that satisfied the energy operator $i\partial_t$ and which, being explicitly relativistic, can yield positive and negative energy solutions. One major challenge that still needs to be overcome depends on the need to couple said PWF to a

charged matter field. To do so we will need to re-express this Lagrangian in terms of a new object that is not the Faraday tensor. This is discussion is left for the next chapter.

3.1.5.4 Comparison between PWF and Standard Lagrangian formulations

The question of how this Lagrangian compares to the standard Lagrangian still remains. To answer this questions it is necessary to compare them both. From the standard Lagrangian for the Maxwell Field [78] we know that this Lagrangian is

$$
\begin{aligned}
\mathfrak{L}_{ph} &= -\frac{1}{4}\mathfrak{F}_{\mu\nu}\mathfrak{F}^{\mu\nu} \\
&= -\frac{1}{2}\left(\left|\vec{E}\right|^2 - \left|\vec{B}\right|^2\right) \\
&= -\frac{1}{2}\left(\left\langle -\frac{1}{c}\partial_t\vec{A}\right\rangle^2 - \left\langle \sigma^{k(3)}p_k\vec{A}\right\rangle\right).
\end{aligned}
$$

In contrast the PWF Lagrangian yields the following differential relations

$$
\begin{aligned}
\mathfrak{L}_{ph} &= \psi^\dagger\Gamma_0\left(\frac{i\hbar}{c}\partial_t\Gamma^0 - p_k\Gamma^k\right)\psi \\
&= \left[\psi^\dagger I^{(2)}\frac{i\hbar}{c}\partial_t\psi + \psi^\dagger\begin{pmatrix} 0 & -\sigma^{k(3)} \\ \sigma^{k(3)} & 0 \end{pmatrix}p_k\psi\right],
\end{aligned}
$$

, where here we are compelled to use the SU(2) representation of the spin matrices $\sigma^{k(3)}$ as suggested in the literature [45]. Since we started from the assumption that ψ is an eigenfunction of $\frac{i\hbar}{c}\partial_t$ and p_k we find that this description corresponds to the same Lagrangian so long as the eigen-values satisfy certain constraints. As stated before, the eigen value of the operator $\frac{i\hbar}{c}\partial_t$ must correspond to the energy, and the eigen value of p_k must correspond to the momentum. Also from these relations it is understood that, where $\vec{E} = -\frac{1}{c}\partial_t\vec{A}$ and $\vec{B} = \sigma^{k(3)}p_kA_k$, we are not introducing a new degree of freedom for what would in the future be a Lagrangian for the interaction. This degree of freedom is the some one that is introduced by means of a gauge field when treating the charge matter field.

3.1.6   Gauge Invariance of the Riemann-Silberstein vector

In the discussion presented in the previous sections we have determined a new Lagrangian for the quantum description of the photon field. As such, we have to show that this formalism produces a field and it's Lagrangian which are gauge invariant. Note that we have arrived at this formalism without need for the discussion presented in previous chapters where we made the substitution $\vec{\psi}_{\pm} = \vec{E} \pm i\vec{B}$, for transverse fields. Extending, this discussion to include longitudinal fields would not affect this formalism since, as we saw in the previous sections, these are inherently treated in the discussion of Lorentz and Translational invariants (recall $P_{\mu}P^{\mu}$). Taking $\vec{\psi}_{\pm}$ to yield any transverse or longitudinal electromagnetic vector, and recalling the definition of these we find that

$$\vec{\psi}_{\pm} = \left(-\vec{\nabla}\varphi - \frac{1}{c}\partial_t\vec{A}\right) \pm i\left(\vec{\nabla} \times \vec{A}\right)$$

and upon fixing a gauge which transforms $\vec{A}$ & φ simultaneously via

$$\vec{A} \to \vec{A} + \vec{\nabla}\xi$$
$$\varphi \to \varphi - \frac{1}{c}\partial_t\xi\,,$$

where ξ is a scalar function of $\vec{x}$ and t, yields

$$\vec{\psi}_{\pm} = \left(-\vec{\nabla}\left[\varphi - \frac{1}{c}\partial_t\xi\right] - \frac{1}{c}\partial_t\left[\vec{A} + \vec{\nabla}\xi\right]\right) \pm i\left(\vec{\nabla} \times \left[\vec{A} + \vec{\nabla}\xi\right]\right)$$
$$= \left(-\vec{\nabla}\varphi - \frac{1}{c}\partial_t\vec{A}\right) \pm i\left(\vec{\nabla} \times \vec{A}\right) + \vec{\nabla}\left[\frac{1}{c}\partial_t\xi\right] - \frac{1}{c}\partial_t\left[\vec{\nabla}\xi\right] \pm i\vec{\nabla} \times \left[\vec{\nabla}\xi\right]$$
$$= \left(-\vec{\nabla}\varphi - \frac{1}{c}\partial_t\vec{A}\right) \pm i\left(\vec{\nabla} \times \vec{A}\right)\,,$$

such that both the Lagrangian density and all equations of motion for this interpretation of $\vec{\psi}_{\pm}$ remain gauge invariant. The same is true for the spinor Ψ. However, to show that these wave functions correspond to the gauge field we will have to work first with the charged matter field that is susceptible to these phase and gauge transformations.

3.2 Lagrangian for Dirac Fields Interacting with Photons

In the previous section we found a Lagrangian form for a free photon. In this section we suggest an interaction Lagrangian and additionally include it into the sum of the Lagrangian for the interacting fields. For a spin $\frac{1}{2}$ massive Dirac particle, using the traditional Dirac matrices, the Lagrangian is given by [12, 78]

$$\mathfrak{L}_{\frac{1}{2}, m=m_0} = \bar{\Psi}_{\frac{1}{2}} \left[\not{p} - m_0 c^2 \right] \Psi_{\frac{1}{2}} . \tag{3.2.1}$$

From the section above, the Lagrangian for free photons, using the SO(3) matrices, is given by

$$\mathfrak{L}_{1, m=0} = \bar{\Psi}_1 \left[\not{p} \right] \Psi_1 . \tag{3.2.2}$$

However, the interaction mitigated by these two fields has to be given by a Lagrangian which includes the minimal coupling.

Interaction Lagrangian $\mathcal{L}_{\frac{1}{2}, m_0, A^\mu}$ from Phase/Gauge Invariance

The interaction between a bound Photon and a Dirac field can be shown to be mitigated by the 4-vector potential by requiring the phase and gauge invariance of these respectively [78, 77, 13]. Here we introduce the constant g_{PWF} to satisfy the unit requirements of some scalar field Θ, where $\Theta = \xi g_{\mathrm{PWF}}$. We do not suggest a value for this constant. We only suggest that it is necessary to insure any necessary unit conversions if ξ is unit-less. For now, we focus on the phase and gauge freedoms as given by

- Phase Freedom

$$\Psi'_{\frac{1}{2}} \to e^{-\frac{i}{\hbar} \frac{e}{c} (g_{\mathrm{PWF}}) \xi} \Psi_{\frac{1}{2}} \tag{3.2.3}$$

- Gauge Freedom

$$\Psi_1' \to [\Psi_1 + \partial_\nu \xi] \tag{3.2.4}$$

Applying these substitutions to an interaction Lagrangian of the form

$$\mathcal{L}'_{\frac{1}{2},m_0,\psi_{\mathrm{PWF}}} = \bar{\Psi}'_{\frac{1}{2}} \left[\not{p} - m_0 c^2 \right] \Psi'_{\frac{1}{2}} - (g_{\mathrm{PWF}}) \frac{e}{c} \bar{\Psi}'_{\frac{1}{2}} \gamma^\mu \Psi'_1 \Psi'_{\frac{1}{2}} \tag{3.2.5}$$

yields

$$\mathcal{L}'_{\frac{1}{2},m_0,\psi_{\mathrm{PWF}}} = \mathcal{L}_{\frac{1}{2},m_0,\psi_{\mathrm{PWF}}} - (g_{\mathrm{PWF}}) \frac{e}{c} \bar{\Psi}_{\frac{1}{2}} \left(\gamma^\mu \partial_\mu \xi \right) \Psi_{\frac{1}{2}} - (g_{\mathrm{PWF}}) \frac{e}{c} \bar{\Psi}_{\frac{1}{2}} \gamma^\mu \Psi_1 \Psi_{\frac{1}{2}}$$

$$= \mathcal{L}_{\frac{1}{2},m_0,\psi_{\mathrm{PWF}}} \, .$$

Therefore, these expressions retain the invariance of the Lagrangian as required by the phase and gauge freedom conditions above [13]. Most importantly, we can see that the relation between Ψ_1 and A_μ. This is given by resolving A_μ into two possible helicity states each with four components. Unfortunately, the development of the consequences of such a discussion is beyond the scope of this dissertation. However, it is satisfying to see that the gauge freedom can also be introduced by means of a PWF [74, 50]. In this formalism, gauge freedom is satisfied by the photonic wave function and its equations of motion. This is evident from making the same gauge substitution as the one above into the EOMs for the PWF, $\partial_\mu \left[\bar{\Psi}_1 i\hbar \Gamma^\mu \right] = 0$. By making the gauge substitution into the conjugate of this equation of motion, it is evident from

$$\Gamma^\mu p_\mu \Psi_1' = \Gamma^\mu p_\mu \Psi_1 + \Gamma^\mu p_\mu \partial_\nu \xi$$

$$= 0 \,,$$

that the only way these equations of motion can be satisfied is if we are in a gauge where $\Gamma^\mu p_\mu \partial_\nu \xi = 0$ and Γ^μ are the 4×4 matrices introduced by Moses [74, 50]. Please note that since we are not

constrained to a zero field, these constraints do not break the U(1) symmetry associated with the phase freedom of the massive field. In fact, these specifically state that the phase symmetry can be localized in space or time, leading to an associated bandwidth for any spontaneously created photon. Any detailed discussions on the consequences of these constraints in the high energy limit are beyond the scope of this dissertation.

Interaction Lagrangian $\mathcal{L}_{\frac{1}{2},m_0,A^\mu,1}$ for a Photon with Dirac Field

The Lagrangian for a charged massive field interacting with a relativistic photon is finally at hand. This interaction can be given by adding the Lagrangian of the massive field to that of a PWF and fixing the constant g_{PWF} to unity, while retaining the appropriate units of $\frac{[J \cdot s]}{[C \cdot m^2]}$. The resulting Lagrangian is then given by the expression

$$
\begin{aligned}
\mathfrak{L}_{\text{Int}} &= \mathfrak{L}_{\frac{1}{2},m_0,\psi_{\text{PWF}}} + \mathfrak{L}_{1,m=0} \\
&= \bar{\Psi}_{\frac{1}{2}}\left[\Gamma^{\mu(2)}_{\frac{1}{2}} p_\mu - m_0 c^2\right]\Psi_{\frac{1}{2}} + \frac{e}{c}g_{\text{PWF}}\bar{\Psi}_{\frac{1}{2}}\Gamma^{\mu(2)}_{\frac{1}{2}}\Psi_1\Psi_{\frac{1}{2}} + \bar{\Psi}_1\Gamma^{\mu(2)}_1 p_\mu\Psi_1 \, ,
\end{aligned}
$$

where $\Gamma^{\mu(2)}_{\frac{1}{2}}$ and $\Gamma^{\mu(2)}_1$ represent the corresponding gamma matrices for a spin $\frac{1}{2}$ and spin 1 particle respectively. Then, the primary challenge posed is to know how to switch from the $SO(3)$ representation of the PWF to the $SU(2)$ representation of the charged massive field. Another approach to the interaction leads to a tensorial representation of the photon term. The interaction in that approach depends on the complex Maxwell Field Tensor and keeps the definition of the lagrangian in terms of the electromagnetic vector potential. This approach is discussed in detail throughout the remainder of this dissertation.

3.3   Tensorial Representation of the Photon

Though we are after an equation for a single photon and its corresponding wave function, there are still many questions that still have to be answered before a full theory of the PWF formalism can be used to model experimental results. For this reason we constrain our discussion in the remainder of this dissertation to the traditional QED Lagrangian. We use Maxwell's equations

which have been shown in the discussion above to be sufficient for coupling the single photon state to a quantum source to develop our model further. Maxwell's equations can be defined as the equations of motion for the Lagrangian

$$\mathfrak{L}_{ph} \;=\; -\frac{1}{4}\mathfrak{F}_{\mu\nu}\mathfrak{F}^{\mu\nu}\,.$$

In tensor form, Maxwell's equations are give by

$$\partial_\mu \mathfrak{F}^{\mu\nu} \;=\; 0$$

$$\partial_\mu \mathfrak{F}_D^{\mu\nu} \;=\; 0\,.$$

The equations of motion for the PWF vector $\vec{F} \;=\; \vec{B} + i\vec{E}$ can then be defined from these by multiplying the second by $-i$ and summing them together

$$\partial_\mu\left[\mathfrak{F}^{\mu\nu} - i\mathfrak{F}_D^{\mu\nu}\right] \;=\; 0\,.$$

3.3.1 Correspondence between the Maxwell Field Tensor and the Photon Field Tensor

In defining $\mathfrak{G}^{\mu\nu} \equiv \mathfrak{F}^{\mu\nu} - i\mathfrak{F}_D^{\mu\nu}$ we have constructed a complex tensor that has been previously studied under a different Minkowski signature [46]

$$\mathfrak{G}^{\mu\nu} \equiv \begin{pmatrix} 0 & -\frac{\varepsilon_0 E_x}{\sqrt{\varepsilon_0}} + i\frac{\mu_0 H_x}{\sqrt{\mu_0}} & -\frac{\varepsilon_0 E_y}{\sqrt{\varepsilon_0}} + iH_y & -\frac{\varepsilon_0 E_z}{\sqrt{\varepsilon_0}} + iH_z \\ \frac{\varepsilon_0 E_x}{\sqrt{\varepsilon_0}} - i\frac{\mu_0 H_x}{\sqrt{\mu_0}} & 0 & -\frac{\mu_0 H_z}{\sqrt{\mu_0}} - i\frac{\varepsilon_0 E_z}{\sqrt{\varepsilon_0}} & \frac{\mu_0 H_y}{\sqrt{\mu_0}} + i\frac{\varepsilon_0 E_y}{\sqrt{\varepsilon_0}} \\ \frac{\varepsilon_0 E_y}{\sqrt{\varepsilon_0}} - i\frac{\mu_0 H_y}{\sqrt{\mu_0}} & \frac{\mu_0 H_z}{\sqrt{\mu_0}} + i\frac{\varepsilon_0 E_z}{\sqrt{\varepsilon_0}} & 0 & -\frac{\mu_0 H_x}{\sqrt{\mu_0}} - i\frac{\varepsilon_0 E_x}{\sqrt{\varepsilon_0}} \\ \frac{\varepsilon_0 E_z}{\sqrt{\varepsilon_0}} - i\frac{\mu_0 H_z}{\sqrt{\mu_0}} & -\frac{\mu_0 H_y}{\sqrt{\mu_0}} - i\frac{\varepsilon_0 E_y}{\sqrt{\varepsilon_0}} & \frac{\mu_0 H_x}{\sqrt{\mu_0}} + i\frac{\varepsilon_0 E_x}{\sqrt{\varepsilon_0}} & 0 \end{pmatrix}\,.$$

The definition of this vector is presented in detail in the appendix C.1. An energy normalized version of $\mathfrak{G}^{\mu\nu}$ can be expressed in terms of the the vector

$$\vec{F} \quad = \quad \frac{1}{\sqrt{\hbar\omega_\sigma}} \left[\frac{\mu_0 \vec{H}}{\sqrt{\mu_0}} + i \frac{\varepsilon_0 \vec{E}}{\sqrt{\varepsilon_0}} \right] \tag{3.3.1}$$

as

$$\mathfrak{G}^{\mu\nu} \quad = \quad \begin{pmatrix} 0 & iF_x & iF_y & iF_z \\ -iF_x & 0 & -F_z & F_y \\ -iF_y & F_z & 0 & -F_x \\ -iF_z & -F_y & F_x & 0 \end{pmatrix}. \tag{3.3.2}$$

Therefore, the generalized and energy normalized Maxwell's equations can be expressed in terms of an single tensor $\mathfrak{G}^{\mu\nu}$ in the following form

$$\partial_\mu \mathfrak{G}^{\mu\nu} \quad = \quad 0. \tag{3.3.3}$$

In the vacuum, the equations of motion for this vector are given as before via the expressions

$$\begin{pmatrix} -i\vec{\nabla} \cdot \vec{F} \\ i\frac{1}{c}\partial_t \vec{F} + \vec{\nabla} \times \vec{F} \end{pmatrix} \quad = \quad \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{3.3.4}$$

These are the energy normalized and complex form of $\partial_\nu \mathfrak{F}^{\mu\nu} = 0$ and $\partial_\nu \mathfrak{F}_D^{\mu\nu} = 0$ from discussion above.

3.3.1.1 Self-dual electromagnetic tensor

We have shown in the discussion above that in the literature there exists a tensor $\mathfrak{G}^{\mu\nu}$ which can be used to find Maxwell's equations. In the Minkowski metric we have adopted it is straight forward this is a self dual tensor. To prove this, we first mention that this tensor can be purely defined in the language of geometric algebra [85]. Its duality is then given by the Hodge dual [85] via the

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definition

$$
\mathfrak{G}_D^{\mu\nu} = \frac{(-i)}{2}\varepsilon^{\mu\nu\alpha\beta}\mathfrak{G}_{\alpha\beta}. \tag{3.3.5}
$$

Re-writing this expression in terms of the energy normalized Faraday tensor, $\mathfrak{F}^{\mu\nu}$ and its dual $\mathfrak{F}_D^{\mu\nu}$, the proof is completed by following a series of algebraic manipulations and use of the following identity of the Levi-Civita symbol associated with our Minkowski metric,

$$
\varepsilon^{\alpha\beta\mu\nu}\varepsilon_{\alpha\beta\eta\gamma} = -2!\left(\eta_\eta^\mu\eta_\gamma^\nu - \eta_\gamma^\mu\eta_\eta^\nu\right) \tag{3.3.6}
$$

$$
\begin{aligned}
\frac{(-i)}{2}\varepsilon^{\mu\nu\alpha\beta}\mathfrak{G}_{\alpha\beta} &= \frac{(-i)}{2}\varepsilon^{\mu\nu\alpha\beta}\left(\mathfrak{F}_{\alpha\beta} - i\mathfrak{F}_{\alpha\beta D}\right) \\
&= \left(-i\mathfrak{F}_D^{\mu\nu} - i\frac{(-i)}{2}\varepsilon^{\mu\nu\alpha\beta}\varepsilon_{\alpha\beta\eta\gamma}\mathfrak{F}^{\eta\gamma}\right) \\
&= \left(-i\mathfrak{F}_D^{\mu\nu} + \mathfrak{F}^{\mu\nu}\right) \\
&= \mathfrak{G}^{\mu\nu}.
\end{aligned}
$$

This proves the self-duality of the complex, generalized, and energy normalized electromagnetic field tensor,

$$
\mathfrak{G}^{\mu\nu} = \mathfrak{G}_D^{\mu\nu} \tag{3.3.7}
$$

This tensor additionally facilitates a description of the free field in terms of a self-dual Lagrangian for the free electromagnetic field of the form,

$$
\mathfrak{L}_{ph} = -\frac{1}{8}\hbar\omega_\sigma\mathfrak{G}_{\mu\nu}\mathfrak{G}^{\mu\nu}. \tag{3.3.8}
$$

This is a self-dual Lagrangian that yields the Dirac-like equations of motion associated with the PWF. We herevy conclude the discussion on the relativistic quantum field formulation of the La-

grangian formalism for the PWF and self-dual electromagnetic tensor.

4 INTERACTION BETWEEN THE MAXWELL FIELD AND A QUANTUM SOURCE

In the previous chapters it was established that the creation and annihilation operators implemented in the quantization of the PWF correspond to those of the EM field as presented by Quantum Field Theory (QFT), Quantum Electrodynamics (QED), and applied in Quantum Optics and Photonics (QO). In this chapter different cases in modern application requiring the coupling between a quantum source and the Maxwell Field it generates are presented. The first couple of sections describe the general formalism and general approximations underlying these interactions. Later sections will describe the possible cases these approximations may be applied to. Recent studies have been conducted that require the need to model an interaction that couples relativistic single photon states with those of quantum sources in devices [14, 86, 18, 87]. Therefore, in this chapter we model a theory which can lead to parameters which would allow the calculation of revival and sudden death phenomena associated with revival and sudden death of entanglement [88, 89, 90, 91] as well as enhanced emission times in optical cavities [27, 28, 92].

4.1 Interaction Lagrangian Density

In previous chapters we defined the interaction Lagrangian used in textbooks to quantize, by means of the Gupta-Beuler method, a classical interaction that identifies the canonical momenta which allows equal time commutation relations for the electromagnetic field to be written down[12, 78, 77]. These Lagrangians, which correspond to the charged quantum source and photon respectively, are given by the expressions

$$\mathfrak{L}_{\text{qs}} \;=\; \bar{\psi}\left(c\slashed{p} - m_0 c^2\right)\psi \tag{4.1.1}$$

$$\mathfrak{L}_{\text{ph}} \;=\; -\frac{1}{4}\mathfrak{F}_{\mu\nu}\mathfrak{F}^{\mu\nu}. \tag{4.1.2}$$

You may recall from the previous chapter that $\not{p}$ is referred to as Feynman slash notation. In this notation any operator $\not{\Lambda} \equiv \gamma^\mu \Lambda_\mu$ is expressed as product of the Dirac gamma matrices γ^μ (D.2) and itself. The bar notation $\bar\psi$ defines the adjoint spinor of the field ψ and is given by the expression

$$\bar\psi \equiv \psi^\dagger \gamma_0 . \tag{4.1.3}$$

The electromagnetic interaction can be included in these Lagrangians by means of a "minimal" gauge invariant substitution $c\not{p} \to c\left(\not{p} - \frac{e_0}{c}\not{A}\right)$ and summing the two independent Lagrangians [12, 78, 77]. This operation is accomplished in the expressions

$$\mathfrak{L}_{\text{ph,qs}} = \bar\psi\left(\not{p} - \frac{e}{c}\not{A} - m_0 c\right)\psi - \frac{1}{4}\mathfrak{F}_{\mu\nu}\mathfrak{F}^{\mu\nu} . \tag{4.1.4}$$

These expressions tell us that the interaction Lagrangian density is given by the term corresponding to

$$\mathfrak{L}_{\text{int}} = -\frac{e}{c}\bar\psi\not{A}\psi . \tag{4.1.5}$$

From here on out we will work in the radiation gauge when propagating the EM field far from electron sources. In regions near the sources we may use the Coulomb or Lorenz gauges. By studying the interaction distance associated with the charge density given by the field spinor ψ that bound to the quantum source we will determine explicitly if are we are either in the Coulomb or Lorenz gauge as long as. If we find that our optical fields give rise to a divergence or gradient term we will assume that we are near a region that satisfies the expressions

$$\vec{\nabla} \cdot \vec{E} = -\vec{\nabla} \cdot \dot{\vec{A}} - \nabla^2 \Phi\left(x\right) \neq 0$$
$$\partial_t \vec{E} = -\ddot{\vec{A}} - \partial_t\left(\nabla\Phi\right) \neq 0 ,$$

where $\Phi = e_0 \int d^3x' \frac{\rho(x')}{4\pi|x'-x|}$, and will return to the coulomb gauge. If we find that we do have a divergent field associated with the Poisson equation $-\nabla^2 \Phi(x) = e_0 \bar{\psi}(x) \psi(x)$, then we will also report the charge density by means of the operator $\rho(x) \equiv e_0 \bar{\psi}(x) \psi(x) \neq 0$. In addition, if we find that we are in the Lorenz gauge we will report a current density or Zitterbewegung by calculating the therm $j^k = c\bar{\psi} \alpha^k \psi \neq 0$, where α^k is a Dirac matrix (D.2).

Substituting the self-dual Lagrangian, $\mathfrak{L}_{ph} = -\frac{1}{8}\hbar\omega_\sigma \mathfrak{G}_{\mu\nu}\mathfrak{G}^{\mu\nu}$ which we defined in the previous chapter, into the full Lagrangian for the interacting fields $\mathfrak{L}_{\mathrm{ph,qs}}$, the coupled field equations are defined by minimizing the variation of the action of the Maxwell and Dirac fields [12, 78, 13]

$$\partial_\psi \mathfrak{L}_{\mathrm{ph,qs}} - \partial_\mu \left[\frac{\partial \mathfrak{L}_{\mathrm{ph,qs}}}{\partial (\partial_\mu \psi)} \right] = 0 \tag{4.1.6}$$

$$\partial_{A_i} \mathfrak{L}_{\mathrm{ph,qs}} - \partial_\mu \left[\frac{\partial \mathfrak{L}_{\mathrm{ph,qs}}}{\partial (\partial_\mu A_i)} \right] = 0. \tag{4.1.7}$$

These yield the coupled field equations which we will be studying for the remainder of the chapter as given by the expressions

$$\left(c\slashed{p} - m_0 c^2 \right) \psi(x) = \frac{e}{c}\slashed{A}\psi(x) \tag{4.1.8}$$

$$\partial_\nu \mathfrak{G}^{\mu\nu} = \frac{e}{c}\bar{\psi}(x)\gamma^\mu\psi(x). \tag{4.1.9}$$

The first equation is the equation of motion for a quantum source interacting with an electromagnetic field. The second equation 4.1.9 can be shown to correspond to the generalized form of Maxwell's equations with sources, as described in the previous chapter.

4.1.1 Interaction Hamiltonian

The Hamiltonian density for the interacting fields if given by the definition [13, 12]

$$\mathcal{H}_{\mathrm{ph,qs}} = \partial_{\dot{\psi}} \mathfrak{L}_{\mathrm{ph,qs}} \dot{\psi} + \partial_{\dot{A}_i} \mathfrak{L}_{\mathrm{ph,qs}} \dot{A}_i - \mathfrak{L}_{\mathrm{ph,qs}}. \tag{4.1.10}$$

The explicitly calculated Hamiltonian is known from textbooks to be given by the expression [12]

$$\mathcal{H}_{\text{ph,qs}} = \psi^\dagger \left(c\vec{\alpha} \cdot \vec{p} + \beta m_0 c^2 \right) \psi + \frac{1}{2} \left(\vec{E}^2 + \vec{B}^2 \right) + \frac{1}{c} \vec{E} \cdot \vec{\nabla}\Phi + \frac{e}{c} \bar{\psi}\gamma_\mu A^\mu \psi. \quad (4.1.11)$$

In this expression $\frac{e}{c}\bar{\psi}\gamma_0 A^0 \psi = \frac{e}{c}\psi^\dagger\psi\Phi$ and $\frac{1}{c}\vec{E}\cdot\vec{\nabla}\Phi - \frac{e}{c}\psi^\dagger\psi\Phi = \frac{e}{c}\vec{\nabla}\cdot\left(\vec{E}\Phi - \vec{E}\Phi\right) = 0$. Making these substitutions in the expression above, the Hamiltonian density simplifies to

$$\mathcal{H}_{\text{ph,qs}} = \psi^\dagger \left(c\vec{\alpha} \cdot \left[\vec{p} + \frac{e}{c}\vec{A} \right] + \beta m_0 c^2 \right) \psi + \frac{1}{2} \left(\vec{E}^2 + \vec{B}^2 \right), \quad (4.1.12)$$

where $\vec{\alpha}$ and β are Dirac matrices discussed in the appendix (D.2). In terms of the self-dual Lagrangian formalism for the PWF the Hamiltonian takes on the form

$$\mathcal{H}_{\text{ph,qs}} = \psi^\dagger \left(c\vec{\alpha} \cdot \vec{p} + \beta m_0 c^2 \right) \psi + \frac{\hbar\omega_\sigma}{2} F_k F_k^\dagger - \frac{i}{c}\sqrt{\hbar\omega_\sigma} F_k \partial_k \Phi + \frac{e}{c}\bar{\psi}\gamma_\mu A^\mu \psi. \quad (4.1.13)$$

In terms of the elctric and magnetic fields this expression is given by

$$\mathcal{H}_{\text{ph,qs}} = \psi^\dagger \left(c\vec{\alpha} \cdot \vec{p} + \beta m_0 c^2 \right) \psi + \frac{1}{2} \left(\mu_0 \vec{H}^2 + \varepsilon_0 \vec{E}^2 \right) - \frac{i}{c} \left(\frac{\mu_0 \vec{H}}{\sqrt{\mu_0}} + i\frac{\varepsilon_0 \vec{E}}{\sqrt{\varepsilon_0}} \right) \cdot \vec{\nabla}\Phi + \frac{e}{c}\bar{\psi}\gamma_\mu A^\mu \psi.$$

From this expression we can begin to develop an approximate interaction Hamiltonian (4.2.1) which we will use throughout the dissertation. In addition, from the formalism above we can see that we do not need to go the Hamiltonian description to solve this problem. Therefore, we will be defining the interaction directly from the coupled field equations of motion for he systems of interest as derived from interaction Lagrangian density $\mathfrak{L}_{\text{ph,el}}$. Further, we will not focus on solving the complex non-linear problem for the spatial dependence of the wave-functions for the matter field. Instead, we look at the applications that are of interest and see what approximations are appropriate to simplify avoid that part of the calculations and make the problem more manageable.

4.1.2 Conservation Laws for Interacting Fields

Throughout our calculations we check conserved quantities to make sure that we do not violate conservation laws. The conservation laws that we rely on during computational checks rely on the traditional Lagrangian formalism. As discussed in previous chapters, we rely on invariants of the formalism to define these conservation laws.

Translational Invariance of the Lagrangian

The conservation laws associated with translational invariance of the interacting Dirac and Maxwell fields naturally arise form Noether's Theorem [12]. Under an infinitesimal displacement,

$$x'_\mu = x_\mu + \epsilon_\mu$$

the Lagrangian $\mathfrak{L}$ can be shown to change by the amount

$$\delta\mathfrak{L} = \mathfrak{L}' - \mathfrak{L} = \epsilon_\mu \frac{\partial\mathfrak{L}}{\partial x_\mu}\,.$$

From textbook discussions it is known that in the Lagrangian formalism if the Lagrangian $\mathfrak{L}$ is translationally invariant, then the Noether current $\mathfrak{J}^{\mu\nu}$ [12, 13] will satisfy the expression

$$\partial_\mu\mathfrak{J}^{\mu\nu} = 0\,.$$

This Noether current is explicitly given by the expression

$$\mathfrak{J}^{\mu\nu} = \left[\frac{\partial\mathfrak{L}_{\mathrm{ph,qs}}}{\partial\left(\partial_\mu\psi\right)}\right]\partial^\nu\psi + \left[\frac{\partial\mathfrak{L}_{\mathrm{ph,qs}}}{\partial\left(\partial_\mu A_i\right)}\right]\partial^\nu A_i - \eta^{\mu\nu}\mathfrak{L}\,,$$

where $\eta^{\mu\nu}$ corresponds to the traditional Minkowski metric defined in the previous chapter. From this conservation law, the conserved quantities are given in general by

$$P_\nu \equiv \int d^3x \mathfrak{J}_{0\nu} = \int d^3x \left\{ \partial_{\dot\psi} \mathcal{L}_{\text{ph,qs}} \partial^\nu \psi + \partial_{\dot A_i} \mathcal{L}_{\text{ph,qs}} \partial^\nu A_i - \eta_{0\nu} \mathcal{L} \right\} .$$

It is therefore the case that the Noether current gives rise to the conservation laws must satisfy the equality

$$\partial_0 P_\nu = 0 .$$

Since the conjugate momenta are given by taking the partial with respect to the time derivative of the field of interest, as show by $\partial_{\dot\psi} \mathcal{L}_{\text{ph,qs}} = i\psi^\dagger$ and $\partial_{\dot{\vec A}} \mathcal{L}_{\text{ph,qs}} = -\partial_0 \vec A$, it is clear that the first conserved quantity is the Hamiltonian of the system

$$\partial_0 P_0 = \partial_0 \int d^3x \mathfrak{J}_{00} = \partial_0 \int d^3x \mathcal{H} = 0 .$$

In general, one of the main checks that we will conduct during our calculations corresponds to the commutator of the Hamiltonian with the conserved quantities as given by

$$[H, \mathbf{P}] = 0 .$$

The additional components left to be calculated correspond to the quantized analog of Poynting's theorem

$$\partial_0 P_i = \partial_0 \int d^3x J_{0i} = \partial_0 \int d^3x \left\{ \partial_{\dot\psi} \mathcal{L}_{\text{ph,qs}} \partial^i \psi + \partial_{\dot A_i} \mathcal{L}_{\text{ph,qs}} \partial^i A_i - \eta_{0i} \mathcal{L} \right\} = 0 .$$

From these expressions we find that the conserved momentum is

$$\partial_{\dot{\psi}}\mathfrak{L}_{\mathrm{ph,qs}}\partial^i\psi + \partial_{\dot{\vec{A}}}\mathfrak{L}_{\mathrm{ph,qs}}\partial^i\vec{A} - \eta_{0i}\mathfrak{L}$$

$$= i\psi^\dagger\partial^i\psi - \partial_0\vec{A}\partial^i\vec{A} - \eta_{0i}\bar{\psi}\left(c\gamma^0\vec{p} - \frac{e_0}{c}\gamma^0\vec{A} - m_0c^2\right)\psi + \eta_{0i}\frac{1}{4}\mathfrak{F}_{0i}\mathfrak{F}^{0i}$$

$$= i\psi^\dagger\partial^i\psi - \partial_0\vec{A}\partial^i\vec{A} - \eta_{0i}\psi^\dagger\left(ci\hbar\vec{\nabla} - \frac{e_0}{c}\gamma^0\vec{A} - m_0c^2\right)\psi + \eta_{0i}\frac{1}{4}\mathfrak{F}_{0i}\mathfrak{F}^{0i}\,.$$

However, since we do not calculate the properties of the matter field, we are left with only the properties of the photon field. As such, we assume that the quantum source is stationary and that the linear momentum of the matter field is conserved, namely,

$$\partial_0\int d^3x\left[i\psi^\dagger\partial^i\psi - \partial_0\vec{A}\partial^i\vec{A} - \eta_{0i}\psi^\dagger\left(ci\hbar\vec{\nabla} - \frac{e_0}{c}\gamma^0\vec{A} - m_0c^2\right)\psi\right] = 0\,.$$

The quantity that we expect to be conserved, and which we do calculate is given by

$$\partial_0 P_i = \partial_0\int d^3x\eta_{0i}\frac{1}{4}\mathfrak{F}_{0i}\mathfrak{F}^{0i}\,.$$

4.2   Hamiltonian of the PWF

The textbook approach to determining the equations of motion for the coupled system revolves around first identifying the Hamiltonian. In this case, the Hamiltonian density derived in the previous section can be expressed more compactly as

$$\mathcal{H} = \mathcal{H}_\sigma + \mathcal{H}_\gamma + \mathcal{H}_{Int}\,, \tag{4.2.1}$$

where $\mathcal{H}_\sigma$ can be thought of as representing the quantum source, $\mathcal{H}_\gamma$ can be thought of as expressing the free Maxwell Field in terms of the PWF creation and annihilation operators $a^\dagger_{\vec{k},\lambda}$, $a_{\vec{k},\lambda}$, and $\mathcal{H}_{Int}$ as the interaction part. Since the equations of motion and the interaction are now known for this system, the next step requires that we show that the definition of Hamiltonian for both yield equivalent results. This can be accomplished by expressing $\mathcal{H}_\gamma$ in terms of $\vec{E}$ & $\vec{B}$ or the unit

normalized operator $\vec{\Gamma}_\gamma$.

$\mathcal{H}_\gamma$ in terms of $\vec{E}$ & $\vec{B}$

We first express $\mathcal{H}_\gamma$ in terms of Electric and Magnetic operators. The plane wave expansion of this Hamiltonian in terms of the single photon creation and annihilation operators is well known from textbooks [15]. We define this expansion again as an exercise that will allow us to draw parallels when working with the PWF operators. Using the definition of $\vec{E}$ & $\vec{B}$ in terms of the creation an annihilation operators and recalling the identity given by the expression $\dfrac{\vec{k}}{k} \times \hat{\epsilon}_{\vec{k},\sigma_\lambda} = -i\lambda\hat{\epsilon}_{\vec{k},\sigma_\lambda}$ [1], the Hamiltonian for the free photon is given by

$$
\begin{aligned}
\mathcal{H}_\gamma &\equiv \frac{\varepsilon_0}{2}\vec{E}^2 + \frac{1}{2\mu_0}\vec{B}^2 && (4.2.2)\\
&= \frac{\hbar\omega}{4}\left(\vec{\varphi}_\gamma + \vec{\varphi}_\gamma^\dagger\right)^2 + \frac{\hbar\omega}{4}\left(\vec{\chi}_\gamma + \vec{\chi}_\gamma^\dagger\right)^2 && (4.2.3)\\
&= \frac{1}{V}\sum_{\vec{k},\lambda}\hbar\nu_k\left(a_{\vec{k},\lambda}^\dagger a_{\vec{k},\lambda} + \frac{1}{2}\right). && (4.2.4)
\end{aligned}
$$

$\mathcal{H}_\gamma$ in terms of $\vec{\Gamma}_\gamma$

The expression for $\mathcal{H}_\gamma$ as written in terms of the PWF operators has also been previously studied [10, 11]. We also define it again here to show its equivalence to the expression above. The expression for the expanded form of the Hamiltonian in this expression can be arrived at by using the definition of $\vec{\Psi}_\gamma$ in terms of $\vec{\Gamma}_\gamma$ & $\vec{\Gamma}_\gamma^\dagger$, that we worked with in previous chapters. In this instance we will again have to make use of the identity for the helically polarization vectors as before. The result for this Hamiltonian, given by the expression below

$$
\begin{aligned}
\mathcal{H}_\gamma &= \frac{\hbar\omega}{2}\vec{\Psi}_\gamma^\dagger\vec{\Psi}_\gamma && (4.2.5)\\
&= \frac{1}{V}\sum_{\vec{k},\lambda}\hbar\nu_k\left(a_{\vec{k},\lambda}^\dagger a_{\vec{k},\lambda} + \frac{1}{2}\right), && (4.2.6)
\end{aligned}
$$

---

[1]Please see (B.1)

is equivalent to the expression derived from representing $\mathcal{H}_\gamma$ in terms of $\vec{E}$ & $\vec{B}$.

4.3   Non-relativistic Approximation

Having established that the conservation laws we sought are the same no matter the formalism we are working in, we can now move forward with establishing the interaction. We work here with the coupled Dirac and Maxwell field equations. In particular we will make some approximations based on the ideas that, in the domain of condensed matter, electronic or quantum source states in semi-conductors can be well described within a non-relativistic approximation. Therefore , the only states that will retain any relativistic properties[2] will correspond to the states of the Maxwell field. Using the equations of motion that were defined above, for the electronic part

$$\left[i\hbar\gamma^\mu\partial_\mu + \frac{e}{c}\gamma^\mu\left(A_\mu + A_\mu^{Ext}\right) + m_0 c \right]\bar{\psi} = 0 \tag{4.3.1}$$

$$\left[i\hbar\gamma^\mu\partial_\mu - \frac{e}{c}\gamma^\mu\left(A_\mu + A_\mu^{Ext}\right)^\dagger - m_0 c \right](\bar{\psi})^\dagger = 0 \tag{4.3.2}$$

and photonic part

$$-\frac{e}{c}\bar{\psi}\gamma^\nu\psi + \partial_\mu\mathfrak{G}^{\mu\nu} = 0 \tag{4.3.3}$$

$$-\frac{e}{c}\psi\gamma^\nu\bar{\psi} + \partial_\mu(\mathfrak{G}^{\mu\nu})^\dagger = 0 \,, \tag{4.3.4}$$

we define what would be the exact picture for the interaction. Following the traditional discussion also presented in text books [56, 12, 13], the first step we will make focuses on the matter equations of motion. There, we fix the representation of the Dirac matrices and express the equations for the coupled spinor states as follows[3]

---

[2] Except for the energies and intensities which will correspond to the that which is available from the quantum source.

[3] $A_{k,Tot} \equiv \left(A_k + A_k^{Ext}\right)$ and $\Phi_{Tot} \equiv \left(\Phi + \Phi^{Ext}\right)$

$$\left[I \frac{i\hbar}{c} \partial_t - \begin{pmatrix} 0 & \sigma_k \\ \sigma_k & 0 \end{pmatrix} \left[\frac{\hbar}{i} \partial_k + \frac{e}{c} A^\dagger_{k,Tot} \right] - \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} m_0 c + \frac{e}{c} \Phi^\dagger_{Tot} \right] \begin{pmatrix} \varphi \\ \chi \end{pmatrix} = 0 \,. \quad (4.3.5)$$

The equation of motion for the term corresponding to the "large" component of the spinor, $\partial_t \varphi$, is then given by the expression

$$\frac{i\hbar}{c} \partial_t \varphi = \sigma_k \left(\frac{\hbar}{i} \partial_k + \frac{e}{c} A^\dagger_{k,Tot} \right) \chi + \left[m_0 c - \frac{e}{c} \Phi^\dagger_{Tot} \right] \varphi \,. \quad (4.3.6)$$

In contrast, the equation of motion for the "small" component of the spinor $\partial_t \chi$ is given by

$$\frac{i\hbar}{c} \partial_t \chi = \sigma_k \left(\frac{\hbar}{i} \partial_k + \frac{e}{c} A^\dagger_{k,Tot} \right) \varphi - \left[m_0 c + \frac{e}{c} \Phi^\dagger_{Tot} \right] \chi \,. \quad (4.3.7)$$

The difference between "large" and "small" will be made evident in from the following discussion. The Pauli equation arises naturally from the Dirac field equation by making the non-relativistic approximation[56]. In this non-relativistic limit the speed of a massive particle is much less than that of the speed of light, $\frac{|\vec{v}_{el}|}{c} \ll 1$. Working in the radiation gauge, and taking the limit where $m_0 c^2$ is considered that largest component of the energy[4], $\psi \approx e^{-\frac{i}{\hbar} m_0 c^2 t} \psi'$, the equation of motion corresponding to the evolution of large partner is given by, $\partial_t \varphi$

$$i\hbar \partial_t \varphi' + e\Phi_{\mathrm{Ext}} \varphi' = \sigma_k c \pi_k \chi' \,. \quad (4.3.8)$$

Similarly applying the same approximation to the equation of motion which has the evolution of the weaker term $\partial_t \chi$ yields

$$\frac{1}{2m_0 c^2} \left[i\hbar \partial_t + e\Phi_{\mathrm{Ext}} \right] \chi' = \frac{\sigma_k c \pi_k}{2m_0 c^2} \varphi' - \chi' \,. \quad (4.3.9)$$

---

[4] $\pi = \frac{\hbar}{i} \partial_k + \frac{e}{c} A^\dagger_{k,Tot}$, $A^\dagger = A, \vec{\nabla} \cdot \vec{A} = 0$, and $\Phi = 0$

We also assume that the potential field and kinetic energies present in the system are much less than the energy contained in the rest mass of the particle we are working with, $[i\hbar\partial_t + e\Phi_{Ext}]\chi' \ll 2m_0c^2$. Therefore, the equations of motion simplify significantly to yield the expressions

$$\chi' \approx \frac{\sigma_k c \pi_k}{2m_0 c^2}\varphi' \tag{4.3.10}$$

and

$$i\hbar\partial_t\varphi' + e\Phi_{Ext}\varphi' = \frac{(\sigma_k c \pi_k)^2}{2m_0 c^2}\varphi'. \tag{4.3.11}$$

The identity $\vec{\sigma} \cdot \vec{\pi}\vec{\sigma} \cdot \vec{\pi} = \vec{\pi}^2 + i\vec{\sigma} \cdot \vec{\pi} \times \vec{\pi}$ is necessary to evaluate the square of the term $\sigma_k c \pi_k$. Substituting this identity back into the first "large" component of the spinor equation yields that

$$i\hbar\partial_t\varphi = \left[\frac{1}{2m_0}(\pi_k\pi_k + i\sigma_k\varepsilon_{kij}\pi_i\pi_j) - e\Phi_{Ext}\right]\varphi. \tag{4.3.12}$$

Where we have used the Levi-Civita symbol [83, 82]. The term in the parentheses can be evaluated by using the expansion

$$\varepsilon_{kij}\pi_i\pi_j\phi = \varepsilon_{kij}\left[A_i\left(\partial_j\phi\right) + \left(\partial_i\phi\right)A_j + \left(\partial_i A_j\right)\phi\right] \tag{4.3.13}$$

$$i\hbar\partial_t\varphi = \left[\frac{1}{2m_0}\left(\left[\frac{\hbar}{i}\partial_k + \frac{e}{c}A_{k,Tot}\right]^2 + \frac{e\hbar}{c}\sigma_k B_{k,Tot}\right) - e\Phi_{Ext}\right]\varphi. \tag{4.3.14}$$

At this point we have a spin resolved interaction that can be modeled via this equation and the equations of motion for the photonic operators. However, since we will not be taking into consideration contributions due to external weak magnetic fields, we split the discussion once again by making another approximation. Before making the next approximation however, we express the equation above in terms of energy normalized PWF operators, $\vec{\Gamma}_\gamma$. To do so , we use the form of

the magnetic field in terms of $\vec{\Gamma}_\gamma$ which we determined in a previous chapter.

$$i\hbar\partial_t\varphi = \left[\frac{1}{2m_0}\left(\pi^2 + \frac{e\hbar}{2c}\sigma_k\left(-i\sqrt{\frac{\hbar\omega}{2}}\left(\vec{\Gamma}_\gamma - \vec{\Gamma}_\gamma^\dagger\right) + B_{k,Ext}\right)\right) - e\Phi_{Ext}\right]\varphi. \qquad (4.3.15)$$

We expect that in this equation we will be working with very weak fields spontaneously created fields. Therefore, we can neglect $\vec{\Gamma}_\gamma$ in the term that coupled to an external magnetic field, since it is small by comparison to the external field. The result of making that approximation yields that the only contribution from the spontaneously emitted field is due to the canonical momentum,

$$i\hbar\partial_t\varphi = \left[\frac{1}{2m_0}\left(\pi^2 + \frac{e\hbar}{2c}\sigma_k B_{k,Ext}\right) - e\Phi_{Ext}\right]\varphi. \qquad (4.3.16)$$

4.3.1   The Dipole Approximation

Please note that since we are working in the low intensity limit, we expect that this model can not describe non-linear inhomogeneous media. Therefore, we assume that the material we are working with is well behaved in the region of the quantum source and that the quantum source responds quickly to the optical field. From the discussion in the previous section we know that the square of the canonical momentum yields a coupling to the momentum of the massive particle in the presence of some external field. As before, we split the contributions due to the electromagnetic influence to two parts. The first part is due to some external field and the other to the spontaneously emitted field. Therefore, the canonical momentum for the quantum source in the absence of the contribution from the spontaneously created field is given by $\vec{\pi}' = \vec{p} + \vec{A}_{Ext}^\dagger$. Further, we expect that the emitted field will have no longitudinal components and therefore expect that its divergence will be null $\vec{\nabla} \cdot \vec{A}^\dagger = 0$. The corresponding square of the total canonical momentum is therefore given as a sum of three terms. The first term is the contribution from some external field and the momentum of the massive particle, the second is the coupling between the two, and the third is the

contribution due to the free field as given in the expression

$$\vec{\pi}^2 \varphi = \left[\vec{\pi}'^2 + 2\frac{e}{c}\vec{A} \cdot \vec{\pi}' + \left(\frac{e}{c}\vec{A}\right)^2 \right] \varphi . \qquad (4.3.17)$$

For simplicity we define what we will refer to as the electronic Hamiltonian. This term contains all of the contributions to the state of the system which are associated with the matter field. All other contributions will be taken to come from the spontaneously emitted field of the quantum source. Therefore, this *a priori* component to the equations of motion $\mathcal{H}_0$ is given by the expression

$$\mathcal{H}_0 \equiv \frac{1}{2m_0} \left(\pi'^2 + \frac{e\hbar}{2c}\sigma_k B_{k,Ext} \right) - e\Phi_{Ext} . \qquad (4.3.18)$$

Please note that we do not refer to this expression as the Hamiltonian of the system since we have not shown it to be so. In fact, the Hamiltonian is given by a similar expression which has some differences in signs. In addition, as before, we assume that the spontaneously emitted field is weak, As a result the square of this field should be even weaker as shown by the approximation give by $\left(\frac{e}{c}\vec{A}\right)^2 \approx 0$. Therefore, rewriting the canonical momentum of the electronic system as $\vec{\pi}' \rightarrow \vec{p}$, we arrive at its equation of motion prior to making the dipole approximation as given by

$$i\hbar\partial_t\varphi = \left[\mathcal{H}_0 + \frac{e}{m_0c}\vec{A} \cdot \vec{p} \right] \varphi . \qquad (4.3.19)$$

Here we note that there are two possible ways of taking the dipole approximation. The differences, similarities, and correspondence is outlined in detail in textbooks [15]. We will address both of these and show that for your system both will provide an adequate description.

4.3.1.1   A source completely immersed in the optical field interacting with an electric field, $\vec{E} \cdot \vec{r}$

One textbook approach to describing the dipole approximation assumes that the entire quantum source is immersed in a plane electromagnetic wave [15]. Here, the electromagnetic field is de-

scribed by a vector potential which satisfies the approximation

$$\vec{A}\left(\vec{r}_0 + \vec{r}, t\right) = \vec{A}\left(t\right) e^{i\vec{k}\cdot(\vec{r}_0+\vec{r})}$$

$$\approx \vec{A}\left(t\right) e^{i\vec{k}\cdot(\vec{r}_0)} .$$

This approximation attributes the scalar potential Φ only to the description of the quantum source state in the absence of the Maxwell field. As a consequence this constrains the electronic wave function to be the product between some spatio temporal evolution driven by the field and another part dependent on the other components of the system.

$$\varphi\left(\vec{r}, t\right) = e^{i\frac{e}{\hbar}\vec{A}(\vec{r}_0,t)\cdot\vec{r}}\phi\left(\vec{r}, t\right) . \tag{4.3.20}$$

Substituting this expression into the traditional, non-relativistic, Schödinger equation yields the dipole approximation for the interaction to be

$$i\hbar\partial_t \left[e^{i\frac{e}{\hbar}\vec{A}(\vec{r}_0,t)\cdot\vec{r}}\phi\right] = \frac{\left(\vec{p} - \frac{e}{c}\vec{A}\right)^2}{2m}\left[e^{i\frac{e}{\hbar}\vec{A}(\vec{r}_0,t)\cdot\vec{r}}\phi\right]$$

$$i\hbar\left[i\frac{e}{\hbar}\left(\dot{\vec{A}}\cdot\vec{r}\right)\phi + \dot{\phi}\right]e^{i\frac{e}{\hbar}\vec{A}(\vec{r}_0,t)\cdot\vec{r}} = \frac{\left(\vec{p} - \frac{e}{c}\vec{A}\right)^2}{2m}\left[e^{i\frac{e}{\hbar}\vec{A}(\vec{r}_0,t)\cdot\vec{r}}\phi\right] .$$

Using the fact that in the radiation gauge we required that $\vec{E} = -\dot{\vec{A}}$, we can find an equation that describes the quantum source state ϕ as given by

$$i\hbar\partial_t\phi = \left[\frac{\left(\vec{p} - \frac{e}{c}\vec{A}\right)^2}{2m} - e\left(\vec{E}\cdot\vec{r}\right)\right]\phi$$

$$= \left[\frac{\vec{p}^2 + \frac{e}{c}\vec{p}\cdot\vec{A} + \frac{e}{c}\vec{A}\cdot\vec{p} + \left(\frac{e}{c}\vec{A}\right)^2}{2m} - e\left(\vec{E}\cdot\vec{r}\right)\right]\phi .$$

Making the same assumption as before by approximating that $\left(\frac{e}{c}\vec{A}\right)^2 \ll 1$ and again remembering that $\vec{\nabla} \cdot \vec{A} = 0$ implies that we arrive at the electronic equation of motion

$$i\hbar\partial_t\phi \;\;=\;\; \left[\frac{\vec{p}^2}{2m} - e\left(\vec{E}\cdot\vec{r}\right)\right]\phi\,. \tag{4.3.21}$$

Next we will look at an approach that does not require an explicit expression in terms of the electric field or any derivative of the vector potential.

4.3.1.2 A source interacting with the vector potential via its coupling to the canonical momentum of a charged particle, $\vec{A}\cdot\vec{p}$

Many textbooks use the alternative approach that we adopt for this interaction [15]. It is also part of textbook discussions that these independent approached yield a difference in measurable quantities. These differ by a factor of the ratio $\frac{\omega}{v}$, where ω refers to the transition frequency and v to the speed of the electronic term $\mathcal{H}_0$ [15]. This implies that the wave-functions defined for each approach needs to be treated differently. A detailed discussion on this correspondence is given in Section 5.A of [15] as an appendix. One of the affected factors in making the transition from one description to the other is the decay rate of the excited state [93]. We therefore constrain the remainder of this discussion to the coupling associated with the canonical momentum of the few level system. In this description, the canonical momentum of the electronic term is replaced with the commutator between it and the position operator through the relation

$$[\mathcal{H}_0,\vec{r}] = \frac{im_0}{\hbar}\vec{p}\,. \tag{4.3.22}$$

Then the equation of motion for the coupled charged particle and optical field can be describe through the equation

$$i\hbar\partial_t\varphi = \left[\mathcal{H}_0 + \frac{ie}{\hbar c}\vec{A}\cdot[\mathcal{H}_0,\vec{r}]\right]\varphi\,. \tag{4.3.23}$$

Expanding this electronic component in terms of its energy eigen basis [56],

$$\mathcal{H}_0 \;=\; \sum_n E_n \left| n \right\rangle \left\langle n \right| \tag{4.3.24}$$

and the vector potential, as before, in terms of plane-waves,

$$\vec{A} = \sum_{\vec{k},\lambda} \frac{c}{\nu_k} \sqrt{\frac{\hbar \nu_k}{2V}} \left(\hat{\epsilon}_{\vec{k},\lambda} a_{\vec{k},\lambda} e^{-i\left(\nu_k t - i\vec{k}\cdot\vec{x}\right)} e^{-i\phi_+} + \hat{\epsilon}^*_{\vec{k},\lambda} a^\dagger_{\vec{k},\lambda} e^{i\left(\nu_k t - i\vec{k}\cdot\vec{x}\right)} e^{i\phi_-} \right) \;,$$

will allow us to draw a comparison point between this approach and that of the single photon wave-function. For brevity, we also introducing the notation $\sigma_{nm} \equiv \left| n \right\rangle \left\langle m \right|$ to denote transitions between states.

4.3.2 The Quantum Source & PWF coupling in the interaction picture

Separating what we can express as the description of the quantum source state in the absence of the photon and the contribution due to the photon by including again the free Maxwell field terms into (4.3.21) we have an equation describing an approximate interaction between the Maxwell Field and its source given by equation 4.3.23. This equation can be abbreviated in terms of an interaction, $\mathcal{H}_{\mathrm{Int}}$, and free field term $\mathcal{H}_\gamma$, to read

$$i\hbar \partial_t \phi \;=\; \mathcal{H}_\sigma + \mathcal{H}_{\mathrm{Int}}\,. \tag{4.3.25}$$

Please note that in light of the newly introduced notation in terms of σ_{nm}, we are relabeling $\mathcal{H}_0$ as $\mathcal{H}_\sigma$. In summary, this is the interaction between a quantum source state given by $\mathcal{H}_\sigma$ interacting with a coupled field via the non-relativistic dipole approximation $\mathcal{H}_{\mathrm{Int}}$. Note that since these are not describing the full Hamiltonian of the system there is no contribution from a free field, $\mathcal{H}_\gamma$.

4.3.2.1 Interaction Hamiltonian for two Polarizations in Linear Homogeneous Media

The full interaction of this coupled system can therefore be expressed by expanding it in terms of the energy-eigen basis and plane wave contributions of the electromagnetic field. This expansion

is explicitly outlined in the equation

$$\left[\mathcal{H}_0 + \frac{ie}{\hbar c}\vec{A}\cdot[\mathcal{H}_0,\vec{r}]\right] = \sum_n \left\{E_n\sigma_{nn} + \frac{ie}{\hbar c}\vec{A}\cdot[E_n\sigma_{nn},\vec{r}]\right\} . \tag{4.3.26}$$

From the commutator above, we can then define a term for the transition dipole operator, $\vec{\wp}_{nm} \equiv \mathcal{E}\langle n|\,\vec{r}\,|m\rangle$ which will contain all the physical factors [56, 13], $\mathcal{E} \equiv \frac{e}{\hbar}\sqrt{\frac{\hbar}{2}}$ The simplified interaction is then given in terms of this operator by means of the expression

$$\frac{e}{\hbar}\sqrt{\frac{\hbar}{2}}\sum_n [E_n\sigma_{nn},\vec{r}] = \sum_{n,m} E_n\vec{\wp}_{nm}\sigma_{nm} - \sum_{n,m} E_m\vec{\wp}_{nm}\sigma_{nm} . \tag{4.3.27}$$

We fix part of the gauge freedom originally introduced in the definition of the vector potential to $\phi_\pm = \pm\frac{\pi}{2}$ and then take advantage of the assumption that the transition energy of the electronic state is driven by that of the optical field it generates, $E_n = E_m + \hbar\nu_k$. From these conditions, in a region near the location of the quantum source $\vec{x}_0$, the interaction component of the equation of motion then takes on the form of the expression

$$\frac{ie}{\hbar c}\vec{A}\cdot[\mathcal{H}_0,\vec{r}] = \sum_{n,m}\sum_{\vec{k},\lambda}\sqrt{\frac{\hbar\nu_k}{2\varepsilon_0 V}}\left(\hat{\epsilon}_{\vec{k},\lambda}a_{\vec{k},\lambda}e^{-i\left(\nu_k t - i\vec{k}\cdot\vec{x}_0\right)} + \text{H.c.}\right)\cdot\left(\vec{\wp}_{nm}\sigma_{nm}\right) . \tag{4.3.28}$$

This interaction has been previously studied extensively with significant success and is the traditional interaction covered in many textbooks [56, 13, 12, 93, 15, 94, Berman].

4.3.2.2 Interaction Hamiltonian for a Two Level Quantum Source and a Maxwell Field in Linear Homogeneous Media

For a two level quantum source with energy eigenstates $|a\rangle$ (the excited state) and $|b\rangle$ (the ground state), $\mathcal{H}_\sigma$ may be expressed by

$$\mathcal{H}_\sigma = E_a|a\rangle\langle a| + E_b|b\rangle\langle b| . \tag{4.3.29}$$

The state vector for these levels, which incorporates the Fock states of the optical field of a single mode $\vec{k}$ with two possible spin states (polarizations) $\lambda \in \{+, -\}$, is defined by through the expression

$$|\sigma\gamma\rangle = c_a(t)\,|a0\rangle + \sum_{\vec{k},\lambda} c_{b,\vec{k},\lambda}(t)\left|b1_{\vec{k},\lambda}\right\rangle . \tag{4.3.30}$$

This state vector implies that a source in the excited state is in a state described by the direct product of it with the vacuum state. Similarly, for the case when the source is in the ground state we associate this state with its direct product to a single photon Fock state resolved for the spin of the spontaneously generated photon. This is similar to traditional textbook discussions [15], but differs in that the state we are now working with can resolve the contributions of integer spin states associate with those of the photon. Additionally, we consider σ_{ab} and σ_{ba} as the annihilation and creation operators of the two level quantum source states σ_+ and σ_- respectively.

Table 4.1 outlines the transformations [12] necessary to change to the interaction picture [56, 15, 13, 12]. There, we will solve the coupled systems of equations that arise from this two level system.

Table 4.1: Operator Expansion Coefficients

Free Fields	Interacting Fields
$a_{\vec{k},\lambda}e^{-i\nu_k t}$	$e^{\frac{i}{\hbar}\mathcal{H}_\sigma t}a_{\vec{k},\sigma_\lambda}e^{-\frac{i}{\hbar}\mathcal{H}_\sigma t}$
$a_{\vec{k},\lambda}^{\dagger}e^{i\nu_k t}$	$e^{\frac{i}{\hbar}\mathcal{H}_\sigma t}a_{\vec{k},\sigma_\lambda}^{\dagger}e^{-\frac{i}{\hbar}\mathcal{H}_\sigma t}$
$\sigma_{-}e^{-i\omega_\sigma t}$	$e^{\frac{i}{\hbar}\mathcal{H}_\sigma t}\sigma_{-}e^{-\frac{i}{\hbar}\mathcal{H}_\sigma t}$
$\sigma_{+}e^{i\omega_\sigma t}$	$e^{\frac{i}{\hbar}\mathcal{H}_\sigma t}\sigma_{+}e^{-\frac{i}{\hbar}\mathcal{H}_\sigma t}$

4.3.2.2.1   Single photon interacting with a two level quantum source in the interaction picture

In the interaction picture equation 4.3.23 reduces to the expression

$$i\hbar\partial_t \left|\sigma\gamma\right\rangle = \sum_{\vec{k},\lambda} \sqrt{\frac{\hbar\nu_k}{V}} \left(\hat{\epsilon}_{\vec{k},\lambda} a_{\vec{k},\lambda} e^{-i\left(\nu_k t - i\vec{k}\cdot\vec{x}_0\right)} + \text{H.c.}\right) \cdot \left(\vec{\wp}_{ab}\sigma_+ e^{i\omega_\sigma t} + \vec{\wp}_{ba}\sigma_- e^{-i\omega_\sigma t}\right) \left|\sigma\gamma\right\rangle ,$$

(4.3.31)

where we have introduced state to state transition dipole moments $\vec{\wp}_{ab}$ and transition frequencies ω_σ as parameters of the calculation. In theory we could solve the entire system of coupled spatio-temporal differential equations established above, but such an effort is beyond the scope of this dissertation. Instead, these parameters will be referenced from multiple methods, depending on the material and geometry that is being modeled. Fortunately there have been my studies aimed and the measurement of these values in the literature [95, 96, 97, 98]. When values for these parameters are not available from the literature, their values will be calculated by means of previously established methods [99, 100, 101] and those values suggested for experimental comparison.

In terms of the energy normalized photon creation and annihilation operators defined in equation 2.2.5, the interaction takes on the form

$$\mathcal{V} = \sqrt{\hbar\omega_\sigma} \left(\vec{\Gamma}_\gamma + \vec{\Gamma}_\gamma^\dagger\right) \cdot \left(\sigma_+ e^{i\omega_\sigma t} \vec{\wp}_{ba} + \sigma_- e^{-i\omega_\sigma t} \vec{\wp}_{ab}\right) .$$

(4.3.32)

Already we can see several cases arising for strong and weak coupling between the quantum source states and the state of a Maxwell Field. There can be modes of the Maxwell Field where the interaction Hamiltonian is in exact resonance with respect to the quantum source $\omega_\sigma = \nu_k$, intermediate cases where it is close to resonance, and others where it is far off resonance. The simplest expression for the interaction is then

$$i\hbar\partial_t \left|\gamma\sigma\right\rangle = \mathcal{V}\left|\gamma\sigma\right\rangle .$$

(4.3.33)

4.3.2.2.2   Interpretation of the interaction Hamiltonian represented in terms of the $\vec{\Gamma}_\gamma$ operator.

Using the electronic $|a\rangle$, $|b\rangle$ and electromagnetic $|0\rangle$, $a^\dagger_{\vec{k},\lambda}|0\rangle$ states that correspond to the product state

$$|\gamma\rangle \otimes |\sigma\rangle \equiv |\gamma\sigma\rangle \ ,$$

and their Hermitian conjugates, conversely $\langle a|$, $\langle b|$ and $\langle 0|$, $\langle 0| a_{\vec{k},\lambda}$, that correspond to

$$\langle\sigma| \otimes \langle\gamma| \equiv \langle\sigma\gamma| \ ,$$

we interpret the meaning of the interaction by taking the sandwich of the interaction $i\hbar\partial_t |\gamma\sigma\rangle = \mathcal{V} |\gamma\sigma\rangle$. We will independently compute the 8 possible cases that are left after making the rotating wave approximation (RWA) [58, 15]for each component of the interaction. Within the formalism of the PWF, the four terms that are left after the RWA in the interaction can be interpreted by operating on the direct product of a single photon state and the quantum source states. Since we are interested being able to describe revival phenomena implicitly we will not invoke that Weisskopf-Wigner approximation in the derivation below:

For the product$\vec{\Gamma}_\gamma\sigma_-$ the following sandwich represents an unavailable transition in the quantum source by the operator σ_- as can be shown by

$$
\begin{aligned}
\langle a0| \vec{\Gamma}_\gamma\sigma_- \sum_{\vec{k},\lambda} c_{b,\vec{k},\lambda}(t) \left|b1_{\vec{k},\lambda}\right\rangle &= \sum_{\vec{k},\lambda} c_{b,\vec{k},\lambda}(t) \langle a0| \vec{\Gamma}_\gamma\sigma_- \left|b1_{\vec{k},\lambda}\right\rangle \\
&= 0 \ ,
\end{aligned}
$$

since the quantum source annihilation operator σ_- satisfies $\left[\vec{\Gamma}_\gamma, \sigma_-\right] = 0$, operating on the states as done above yields

$$
\begin{aligned}
\vec{\Gamma}_\gamma \sigma_- \left|b0\right\rangle &= \vec{\Gamma}_\gamma \left|b\right\rangle \left\langle a\right| \left|b0\right\rangle \\
&= 0 \\
\left\langle a0\right| \sigma_- \vec{\Gamma}_\gamma &= 0 \,.
\end{aligned}
$$

This term represents the processes where the quantum source is taken from is upper state a to its lower state b

$$
\begin{aligned}
\sum_{\vec{k},\lambda} \left\langle b1_{\vec{k},\lambda}\right| \vec{\Gamma}_\gamma \sigma_- c_a\left(t\right) \left|0a\right\rangle &= \sum_{\vec{k},\lambda} c_a\left(t\right) \left\langle b0\right| a_{\vec{k},\lambda} \vec{\Gamma}_\gamma \sigma_- \left|0a\right\rangle \\
&= \sum_{\vec{k},\lambda} \sum_{\vec{k}',\lambda'} \sqrt{\frac{1}{4V}} \sqrt{\frac{\nu_k}{\omega_\sigma}} c_a\left(t\right) \hat{\epsilon}^*_{\vec{k}',\lambda'} e^{i\nu_{k'}t} e^{-i\vec{k}'\cdot\vec{x}_0} \left(1-\lambda\right) \left\langle b0\right| a_{\vec{k},\lambda} \sigma_- a^\dagger_{\vec{k}',\lambda'} \left|0a\right\rangle \\
&= \sum_{\vec{k},\lambda} \sum_{\vec{k}',\lambda'} \left\langle a0\right| \sqrt{\frac{1}{4V}} \sqrt{\frac{\nu_k}{\omega_\sigma}} c_a\left(t\right) \hat{\epsilon}^*_{\vec{k}',\lambda'} e^{i\nu_{k'}t} e^{-i\vec{k}'\cdot\vec{x}_0} \left(1-\lambda\right) \delta_{\vec{k},\vec{k}'} \delta_{\lambda,\lambda'} \left|0a\right\rangle \\
&= \sum_{\vec{k},\lambda} \left\langle a0\right| \sqrt{\frac{1}{4V}} \sqrt{\frac{\nu_k}{\omega_\sigma}} c_a\left(t\right) \hat{\epsilon}^*_{\vec{k},\lambda} e^{i\nu_k t} e^{-i\vec{k}\cdot\vec{x}_0} \left(1-\lambda\right) \left|0a\right\rangle \,.
\end{aligned}
$$

Forcing the sum over the two available polarizations λ yields

$$
\begin{aligned}
\vec{\Psi}^{(-)}_{\gamma,+,a} &= \sum_{\vec{k}} \sqrt{\frac{1}{4V}} \sqrt{\frac{\nu_k}{\omega_\sigma}} c_a\left(t\right) \hat{\epsilon}^*_{\vec{k},-} e^{i\nu_k t} e^{-i\vec{k}\cdot\vec{x}_0} \left(1+1\right) \\
&= \sum_{\vec{k}} \sqrt{\frac{1}{V}} \sqrt{\frac{\nu_k}{\omega_\sigma}} c_a\left(t\right) \hat{\epsilon}_{\vec{k},+} e^{i\nu_k t} e^{-i\vec{k}\cdot\vec{x}_0} \,.
\end{aligned}
$$

This corresponds to the creation of a photon with a negative helicity or right handed polarization. For the product $\vec{\Gamma}_\gamma \sigma_+$ the following sandwich represents the process where the quantum source is

take from its lower state b to is upper state a

$$\langle a0|\, \vec{\Gamma}_\gamma \sigma_+ \sum_{\vec{k},\lambda} c_{b,\vec{k},\lambda}\,(t)\left|b1_{\vec{k},\lambda}\right\rangle = \sum_{\vec{k},\lambda} c_{b,\vec{k},\lambda}\,(t)\,\langle a0|\, \vec{\Gamma}_\gamma \sigma_+ a^\dagger_{\vec{k},\lambda}\,|b0\rangle$$

$$= \sum_{\vec{k},\lambda} \sum_{\vec{k}',\lambda'} \sqrt{\frac{1}{4V}}\sqrt{\frac{\nu_k}{\omega_\sigma}} c_{b,\vec{k},\lambda}\,(t)\,\hat{\epsilon}_{\vec{k}',\lambda'} e^{-i\nu_{k'}t}e^{i\vec{k}'\cdot\vec{x}_0}\,(1+\lambda)\,\langle a0|\, a_{\vec{k}',\lambda'}\sigma_+ a^\dagger_{\vec{k},\lambda}\,|b0\rangle$$

$$= \sum_{\vec{k},\lambda} \sum_{\vec{k}',\lambda'} \left\langle b1_{\vec{k},\lambda'}\right| \sqrt{\frac{1}{4V}}\sqrt{\frac{\nu_k}{\omega_\sigma}} c_{b,\vec{k},\lambda}\,(t)\,\hat{\epsilon}_{\vec{k}',\lambda'} e^{-i\nu_{k'}t}e^{i\vec{k}'\cdot\vec{x}_0}\,(1+\lambda)\,\delta_{\vec{k},\vec{k}'}\delta_{\lambda,\lambda'}\left|b1_{\vec{k},\lambda}\right\rangle$$

$$= \sum_{\vec{k},\lambda} \left\langle b1_{\vec{k},\lambda}\right| \sqrt{\frac{1}{4V}}\sqrt{\frac{\nu_k}{\omega_\sigma}} c_{b,\vec{k},\lambda}\,(t)\,\hat{\epsilon}_{\vec{k},\lambda} e^{-i\nu_k t}e^{i\vec{k}\cdot\vec{x}_0}\,(1+\lambda)\left|b1_{\vec{k},\lambda}\right\rangle$$

forcing the sum over the two available polarizations λ yields

$$\vec{\Psi}^{(+)}_{\gamma,+,b} = \sum_{\vec{k}} \sqrt{\frac{1}{4V}}\sqrt{\frac{\nu_k}{\omega_\sigma}} c_{b,\vec{k},+}\,(t)\,\hat{\epsilon}_{\vec{k},+} e^{-i\nu_k t}e^{i\vec{k}\cdot\vec{x}_0}\,(1+1)$$

$$\vec{\Psi}^{(+)}_{\gamma,+,b} = \sum_{\vec{k}} \sqrt{\frac{1}{V}}\sqrt{\frac{\nu_k}{\omega_\sigma}} c_{b,\vec{k},+}\,(t)\,\hat{\epsilon}_{\vec{k},+} e^{-i\nu_k t}e^{i\vec{k}\cdot\vec{x}_0}$$

which corresponds to the creation of a photon with a negative helicity or right handed polarization. The following term represents an unavailable transition in the quantum source by the operator σ_+ as can be shown by

$$\sum_{\vec{k},\lambda} \langle b1_{\vec{k}}|\, \vec{\Gamma}_\gamma \sigma_+ c_a\,(t)\,|a0\rangle = \sum_{\vec{k},\lambda} c_a\,(t)\,\langle b1_{\vec{k}}|\, \vec{\Gamma}_\gamma \sigma_+\,|a0\rangle$$

$$= 0$$

since the quantum source annihilation operator σ_+ satisfies $\left[\vec{\Gamma}_\gamma, \sigma_+\right] = 0$, operating on the states as done above yields

$$\vec{\Gamma}_\gamma \sigma_+\,|a0\rangle = 0$$

$$\langle b0|\, \sigma_+ \vec{\Gamma}_\gamma = 0\,.$$

For the product $\vec{\Gamma}_\gamma^\dagger \sigma_-$, the following term represents an unavailable transition in the quantum source by the operator σ_- as can be shown by

$$\langle a0| \vec{\Gamma}_\gamma^\dagger \sigma_- \sum_{\vec{k},\lambda} c_{b,\vec{k},\lambda}(t) \left| b1_{\vec{k},\lambda} \right\rangle = \sum_{\vec{k},\lambda} c_{b,\vec{k},\lambda}(t) \langle a0| \vec{\Gamma}_\gamma^\dagger \sigma_- \left| b1_{\vec{k},\lambda} \right\rangle$$
$$= 0$$

since the quantum source annihilation operator σ_- satisfies $\left[\vec{\Gamma}_\gamma^\dagger, \sigma_- \right] = 0$, operating on the states as done above yields

$$\vec{\Gamma}_\gamma^\dagger \sigma_- \left| b0 \right\rangle = 0$$
$$\langle a0| \sigma_- \vec{\Gamma}_\gamma^\dagger = 0$$

This term represents the process where the quantum source is taken from its upper state to its lower state

$$\sum_{\vec{k},\lambda} \left\langle b1_{\vec{k},\lambda} \right| \vec{\Gamma}_\gamma^\dagger \sigma_- c_a(t) \left| a0 \right\rangle = \sum_{\vec{k},\lambda} c_a(t) \langle b0| a_{\vec{k},\lambda} \vec{\Gamma}_\gamma^\dagger \sigma_- \left| a0 \right\rangle$$

$$= \sum_{\vec{k},\lambda} \sum_{\vec{k}',\lambda'} \sqrt{\frac{1}{4V}} \sqrt{\frac{\nu_k}{\omega_\sigma}} c_a(t) \hat{\epsilon}_{\vec{k}',\lambda'}^* e^{i\nu_{k'} t} e^{-i\vec{k}'\cdot\vec{x}_0} (1+\lambda) \langle b0| a_{\vec{k},\lambda} a_{\vec{k}',\lambda'}^\dagger \sigma_- \left| a0 \right\rangle$$

$$= \sum_{\vec{k},\lambda} \sum_{\vec{k}',\lambda'} \langle a0| \sqrt{\frac{1}{4V}} \sqrt{\frac{\nu_k}{\omega_\sigma}} c_a(t) \hat{\epsilon}_{\vec{k}',\lambda'}^* e^{i\nu_{k'} t} e^{-i\vec{k}'\cdot\vec{x}_0} (1+\lambda) \delta_{\vec{k},\vec{k}'} \delta_{\lambda,\lambda'} \left| a0 \right\rangle$$

$$= \sum_{\vec{k},\lambda} \langle 0a| \sqrt{\frac{1}{4V}} \sqrt{\frac{\nu_k}{\omega_\sigma}} c_a(t) \hat{\epsilon}_{\vec{k},\lambda}^* e^{i\nu_k t} e^{-i\vec{k}\cdot\vec{x}_0} (1+\lambda) \left| a0 \right\rangle$$

forcing the sum over the two available polarizations λ yields

$$\vec{\Psi}_{\gamma,-,a}^{\dagger(-)} = \sum_{\vec{k}} \sqrt{\frac{1}{4V}} \sqrt{\frac{\nu_k}{\omega_\sigma}} c_a(t) \hat{\epsilon}_{\vec{k},+}^* e^{i\nu_k t} e^{-i\vec{k}\cdot\vec{x}_0} (1+1)$$

$$= \sum_{\vec{k}} \sqrt{\frac{1}{V}} \sqrt{\frac{\nu_k}{\omega_\sigma}} c_a(t) \hat{\epsilon}_{\vec{k},-}^* e^{i\nu_k t} e^{-i\vec{k}\cdot\vec{x}_0}$$

which corresponds to the creation of a photon with a positive helicity or left handed polarization.

For the product $\vec{\Gamma}_\gamma^\dagger \sigma_+$, this term represents the processes where the quantum source is taken from is lower state b to its upper state a and a Maxwell Field with a negative helicity or right handed polarization is created.

$$
\langle a0| \vec{\Gamma}_\gamma^\dagger \sigma_+ \sum_{\vec{k},\lambda} c_{b,\vec{k},\lambda}(t) \left| b1_{\vec{k},\lambda} \right\rangle = \sum_{\vec{k},\lambda} c_{b,\vec{k},\lambda}(t) \langle a0| \vec{\Gamma}_\gamma^\dagger \sigma_+ a_{\vec{k},\lambda}^\dagger |b0\rangle
$$

$$
= \sum_{\vec{k},\lambda} \sum_{\vec{k}',\lambda'} \sqrt{\frac{1}{4V}} \sqrt{\frac{\nu_k}{\omega_\sigma}} c_{b,\vec{k},\lambda}(t) \hat{\epsilon}_{\vec{k}',\lambda'} e^{-i\nu_{k'}t} e^{i\vec{k}'\cdot\vec{x}_0} (1-\lambda) \langle a0| a_{\vec{k}',\lambda'} \sigma_+ a_{\vec{k},\lambda}^\dagger |b0\rangle
$$

$$
= \sum_{\vec{k},\lambda} \sum_{\vec{k}',\lambda'} \left\langle b1_{\vec{k},\lambda'} \right| \sqrt{\frac{1}{4V}} \sqrt{\frac{\nu_k}{\omega_\sigma}} c_{b,\vec{k},\lambda}(t) \hat{\epsilon}_{\vec{k}',\lambda'} e^{-i\nu_{k'}t} e^{i\vec{k}'\cdot\vec{x}_0} (1-\lambda) \delta_{\vec{k},\vec{k}'} \delta_{\lambda,\lambda'} \left| b1_{\vec{k},\lambda} \right\rangle
$$

$$
= \sum_{\vec{k},\lambda} \left\langle b1_{\vec{k},\lambda} \right| \sqrt{\frac{1}{4V}} \sqrt{\frac{\nu_k}{\omega_\sigma}} c_{b,\vec{k},\lambda}(t) \hat{\epsilon}_{\vec{k},\lambda} e^{-i\nu_k t} e^{i\vec{k}\cdot\vec{x}_0} (1-\lambda) \left| b1_{\vec{k},\lambda} \right\rangle
$$

forcing the sum over the two available polarizations λ yields

$$
\vec{\Psi}_{\gamma,-,b}^{\dagger(+)} = \sum_{\vec{k}} \sqrt{\frac{1}{4V}} \sqrt{\frac{\nu_k}{\omega_\sigma}} c_{b,\vec{k},-}(t) \hat{\epsilon}_{\vec{k},-} e^{-i\nu_k t} e^{i\vec{k}\cdot\vec{x}_0} (1+1)
$$

$$
= \sum_{\vec{k}} \sqrt{\frac{1}{V}} \sqrt{\frac{\nu_k}{\omega_\sigma}} c_{b,\vec{k},-}(t) \hat{\epsilon}_{\vec{k},-} e^{-i\nu_k t} e^{i\vec{k}\cdot\vec{x}_0}
$$

which corresponds to the creation of a photon with a positive helicity or left handed polarization. The following term represents an unavailable transition in the quantum source by the operator σ_+ as can be shown by

$$
\sum_{\vec{k},\lambda} \left\langle b1_{\vec{k},\lambda} \right| \vec{\Gamma}_\gamma^\dagger \sigma_+ c_a(t) |0a\rangle = \sum_{\vec{k},\lambda} c_a(t) \left\langle b1_{\vec{k},\lambda} \right| \vec{\Gamma}_\gamma^\dagger \sigma_+ |a0\rangle
$$

$$
= 0
$$

since the quantum source annihilation operator σ_+ satisfies $\left[\vec{\Gamma}_\gamma^\dagger, \sigma_+ \right] = 0$, operating on the states as done above yields

$$\vec{\Gamma}_\gamma^\dagger \sigma_+ \left| a0 \right\rangle = 0$$

$$\left\langle b0 \right| \sigma_+ \vec{\Gamma}_\gamma^\dagger = 0 \, .$$

These operations can be captured in the following set of relations, where $\chi_{\pm,n}^{(\pm)}$ is only used as a place holder, $(\pm)$ represents the sign of frequency part, and $\pm, n$ represent the helicity of the produced photon and the eigen-state affected. n can therefore only take on one of the two values E for excited or g for ground. Explicitly these are given by the scheme $\vec{\Gamma}_\gamma \sigma_- \to \chi_{+,E}^{(-)}$, $\vec{\Gamma}_\gamma \sigma_+ \to \chi_{+,g}^{(+)}$, $\vec{\Gamma}_\gamma^\dagger \sigma_- \to \chi_{-,E}^{(-)}$, and $\vec{\Gamma}_\gamma^\dagger \sigma_+ \to \chi_{-,g}^{(+)}$. These cases translate into the coupled equations of motion in the Interaction picture through two cases.

First, taking the sandwich of the single photon state/lowest quantum source state

$$\sum_{\vec{k},\lambda} \left\langle b1_{\vec{k},\lambda} \right| i\hbar\partial_t \left| \gamma\sigma \right\rangle = \sum_{\vec{k},\lambda} \left\langle b1_{\vec{k},\lambda} \right| \mathcal{V} \left| \gamma\sigma \right\rangle$$

the left hand side yields

$$\sum_{\vec{k},\lambda} \left\langle b1_{\vec{k},\lambda} \right| i\hbar\partial_t \left| \gamma\sigma \right\rangle = i\hbar \sum_{\vec{k},\lambda} \dot{c}_{b,\vec{k},\sigma_\lambda}(t)$$

and the right hand side yields

$$\sum_{\vec{k},\lambda} \left\langle b1_{\vec{k},\lambda} \right| \mathcal{V} \left| \gamma\sigma \right\rangle = \sum_{\vec{k},\lambda} \left\langle b1_{\vec{k},\lambda} \right| \left(\vec{\Gamma}_\gamma \sigma_+ + \vec{\Gamma}_\gamma^\dagger \sigma_+ \right) \left(c_a(t) \left| a0 \right\rangle + \sum_{\vec{k},\lambda} c_{b,\vec{k},\sigma_\lambda}(t) \left| b1_{\vec{k},\lambda} \right\rangle \right) e^{i\omega_\sigma t} \cdot \vec{\wp}_{ba}$$

$$+ \sum_{\vec{k},\lambda} \left\langle b1_{\vec{k},\lambda} \right| \left(\vec{\Gamma}_\gamma \sigma_- + \vec{\Gamma}_\gamma^\dagger \sigma_- \right) \left(c_a(t) \left| a0 \right\rangle + \sum_{\vec{k},\lambda} c_{b,\vec{k},\lambda}(t) \left| b1_{\vec{k},\lambda} \right\rangle \right) e^{-i\omega_\sigma t} \cdot \vec{\wp}_{ab} \, .$$

In terms of our interpretation above this expression leaves two of the 8 terms we originally started with. The terms that are left are given by the expressions

$$
\sum_{\vec{k},\lambda} \langle b1_{\vec{k}}| \, \mathcal{V} \, |\gamma\sigma\rangle = \sum_{\vec{k},\lambda} \langle b1_{\vec{k}}| \left(\vec{\Gamma}_\gamma \sigma_- + \vec{\Gamma}_\gamma^\dagger \sigma_- \right) c_a(t) \, |a0\rangle \, e^{-i\omega_\sigma t} \cdot \vec{\wp}_{ab}
$$

$$
= \left(\sum_{\vec{k},\lambda} \langle b1_{\vec{k}}| \, \vec{\Gamma}_\gamma \sigma_- c_a(t) \, |a0\rangle + \sum_{\vec{k},\lambda} \langle b1_{\vec{k}}| \, \vec{\Gamma}_\gamma^\dagger \sigma_- c_a(t) \, |a0\rangle \right) e^{-i\omega_\sigma t} \cdot \vec{\wp}_{ab}
$$

$$
= \sqrt{\hbar\omega_\sigma} \left(\vec{\Psi}_{\gamma,+}^{(-)} + \vec{\Psi}_{\gamma,-}^{\dagger(-)} \right)_{,a} e^{-i\omega_\sigma t} \cdot \vec{\wp}_{ab} \, .
$$

Since the probability amplitude for the excited state is subject to the condition that it must not be a function of the mode of the optical field, $\forall a, \vec{k}, \lambda \rightarrow \tilde{c}_{a,\vec{k},\lambda}(t) \equiv c_a(t)$, these expressions simplify to the following equation of motion for the ground state of the two level system,

$$
i\hbar \sum_{\vec{k},\lambda} \dot{c}_{b,\vec{k},\lambda}(t) = \sqrt{\hbar\omega_\sigma} \left(\vec{\Psi}_{\gamma,+}^{(-)} + \vec{\Psi}_{\gamma,-}^{\dagger(-)} \right)_{,a} e^{-i\omega_\sigma t} \cdot \vec{\wp}_{ab} \, . \tag{4.3.34}
$$

Second, taking the sandwich with respect to the excited quantum source state and the vacuum state $\langle a0| \, i\hbar\partial_t \, |\gamma\sigma\rangle$. For the left hand side, we find the rate of change of the excited state probability amplitude

$$
\langle a0| \, i\hbar\partial_t \, |\gamma\sigma\rangle = i\hbar\dot{c}_a(t) \, .
$$

Looking at the right hand side we find that

$$
\langle a0| \, \mathcal{V} \, |\gamma\sigma\rangle = \langle a0| \left(\vec{\Gamma}_\gamma \sigma_+ + \vec{\Gamma}_\gamma^\dagger \sigma_+ \right) \left(c_a(t) \, |a0\rangle + \sum_{\vec{k},\lambda} c_{b,\vec{k},\lambda}(t) \, \Big| b1_{\vec{k},\lambda} \Big\rangle \right) e^{i\omega_\sigma t} \cdot \vec{\wp}_{ba}
$$

$$
+ \langle a0| \left(\vec{\Gamma}_\gamma \sigma_- + \vec{\Gamma}_\gamma^\dagger \sigma_- \right) \left(c_a(t) \, |a0\rangle + \sum_{\vec{k},\lambda} c_{b,\vec{k},\sigma_\lambda}(t) \, \Big| b1_{\vec{k},\lambda} \Big\rangle \right) e^{-i\omega_\sigma t} \cdot \vec{\wp}_{ab}
$$

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which in terms of our interpretation above leaves two of the 8 terms

$$\langle a0|\,\mathcal{V}\,|\gamma\sigma\rangle = \sqrt{\hbar\omega_\sigma}\,\langle a0|\left(\vec{\Gamma}_\gamma\sigma_+ + \vec{\Gamma}_\gamma^\dagger\sigma_+\right)\left(\sum_{\vec{k},\lambda}c_{b,\vec{k},\lambda}\,(t)\left|b1_{\vec{k},\lambda}\right\rangle\right)e^{i\omega_\sigma t}\cdot\vec{\wp}_{ba}$$

$$= \sqrt{\hbar\omega_\sigma}\left(\langle a0|\,\vec{\Gamma}_\gamma\sigma_+\sum_{\vec{k},\lambda}c_{b,\vec{k},\lambda}\,(t)\left|b1_{\vec{k}}\right\rangle + \langle a0|\,\vec{\Gamma}_\gamma^\dagger\sigma_+\sum_{\vec{k},\lambda}c_{b,\vec{k},\lambda}\,(t)\left|b1_{\vec{k},\lambda}\right\rangle\right)e^{i\omega_\sigma t}\cdot\vec{\wp}_{ba}$$

$$= \sqrt{\hbar\omega_\sigma}\left(\vec{\Psi}^{(+)}_{\gamma,+} + \vec{\Psi}^{\dagger(+)}_{\gamma,-}\right)_{,b}e^{i\omega_\sigma t}\cdot\vec{\wp}_{ba}$$

and simplifies to

$$i\hbar\dot{c}_a\,(t) \quad = \quad \sqrt{\hbar\omega_\sigma}\left(\vec{\Psi}^{(+)}_{\gamma,+} + \vec{\Psi}^{\dagger(+)}_{\gamma,-}\right)_{,b}e^{i\omega_\sigma t}\cdot\vec{\wp}_{ba} \qquad (4.3.35)$$

For clarity, we again present the coupled equations below and make the substitutions which subjects the excited state to the condition $\forall a,\vec{k},\lambda \to \tilde{c}_{a,\vec{k},\lambda}\,(t)\equiv c_a\,(t)$ or in terms of the PWF

$$i\hbar\dot{c}_a\,(t) \quad = \quad \sqrt{\hbar\omega_\sigma}\left(\vec{\Psi}^{(+)}_{\gamma,+} + \vec{\Psi}^{\dagger(+)}_{\gamma,-}\right)_{,b}e^{i\omega_\sigma t}\cdot\vec{\wp}_{ba} \qquad (4.3.36)$$

$$i\hbar\sum_{\vec{k},\lambda}\dot{c}_{b,\vec{k},\lambda}\,(t) \quad = \quad \sqrt{\hbar\omega_\sigma}\left(\vec{\Psi}^{(-)}_{\gamma,+} + \vec{\Psi}^{\dagger(-)}_{\gamma,-}\right)_{,a}e^{-i\omega_\sigma t}\cdot\vec{\wp}_{ab}\,.$$

After taking the dot product with $\vec{\wp}_{ab}$ and $\vec{\wp}_{ba}$, solving these expressions for $\dot{c}_a\,(t)$ and $\dot{c}_{b,\vec{k},\lambda}\,(t)$ provides the possibility of propagating a single photon with circular or linear polarizations. One can show that it is possible to describe both types of polarizations, either transverse or circular, or any combination there or. To do so, one simply has to fix the probability amplitudes and polarization vectors to the desired configuration. This implies that all the information of the quantum system is contained in these energy normalized wave-functions $\vec{\Psi}^{(+)}_{\gamma,+,b}$ and $\vec{\Psi}^{\dagger(+)}_{\gamma,-,b}$.

4.3.2.2.3   *Analytical Integration of Equations of Motion*

This two level system lends itself to analytic integration by carrying out the formal integration of c_a and $c_{b,\vec{k},\lambda}$ over the interval $[t_0,t_f]$. As discussed in (D.10), this procedure yields the integral

equations

$$\sum_{\vec{k},\lambda} \left(c_{b,\vec{k},\lambda}(t_f) - c_{b,\vec{k},\lambda}(t_0) \right) = -\frac{i}{\hbar}\sqrt{\hbar\omega_\sigma} \int_{t_0}^{t_f} dt' \left(\vec{\Psi}_{\gamma,+}^{(-)} + \vec{\Psi}_{\gamma,-}^{\dagger(-)} \right)_{,a} e^{-i\omega_\sigma t'} \cdot \vec{\wp}_{ab}$$

$$= -\frac{i}{\hbar}\sqrt{\hbar\omega_\sigma} \int_{t_0}^{t_f} dt' \sum_{\vec{k},\lambda} c_a(t') \left(\left\langle b1_{\vec{k},\lambda} \middle| \vec{\Gamma}_\gamma \sigma_- |a0\rangle + \left\langle b1_{\vec{k},\lambda} \middle| \vec{\Gamma}_\gamma^{\dagger} \sigma_- |a0\rangle \right)_{,a} e^{-i\omega_\sigma t'} \cdot \vec{\wp}_{ab}$$

$$= -\frac{i}{\hbar} \int_{t_0}^{t_f} dt' \sum_{\vec{k}} \sqrt{\frac{\hbar\nu_k}{V}} c_a(t') \left\{ \hat{\epsilon}_{\vec{k},-}^* + \hat{\epsilon}_{\vec{k},+}^* \right\} e^{i\nu_k t'} e^{-i\vec{k}\cdot\vec{x}_0} e^{-i\omega_\sigma t'} \cdot \vec{\wp}_{ab} .$$

Taking the sum on a term by term basis over $\vec{k}$, each term can be expressed independently of the sum by

$$\sum_{\lambda} \left(c_{b,\vec{k},\lambda}(t_f) - c_{b,\vec{k},\lambda}(t_0) \right)$$

$$= -\frac{i}{\hbar} \int_{t_0}^{t_f} dt' \sqrt{\frac{\hbar\nu_k}{V}} c_a(t') \left\{ \hat{\epsilon}_{\vec{k},-}^* + \hat{\epsilon}_{\vec{k},+}^* \right\} e^{i\nu_k t'} e^{-i\vec{k}\cdot\vec{x}_0} e^{-i\omega_\sigma t'} \cdot \vec{\wp}_{ab} .$$

4.3.2.2.4   Solving for $c_a(t)$

Working with the equation of motion for $c_a(t_f)$ we already know that the rate of change of this term depends on every possible mode

$$i\hbar \dot{c}_a(t_f) = \sqrt{\hbar\omega_\sigma} \left(\vec{\Psi}_{\gamma,+}^{(+)}(t_f) + \vec{\Psi}_{\gamma,-}^{\dagger(+)}(t_f) \right)_{,b} e^{i\omega_\sigma t_f} \cdot \vec{\wp}_{ba}$$

$$= \sqrt{\hbar\omega_\sigma} \sum_{\vec{k},\lambda} c_{b,\vec{k},\lambda}(t_f) \left(\langle a0| \vec{\Gamma}_\gamma \sigma_+ \middle| b1_{\vec{k},\lambda} \right\rangle + \langle a0| \vec{\Gamma}_\gamma^{\dagger} \sigma_+ \middle| b1_{\vec{k},\lambda} \right\rangle \right) e^{i\omega_\sigma(t_f)} \cdot \vec{\wp}_{ba}$$

$$= \sum_{\vec{k}} \sqrt{\frac{\hbar\nu_k}{V}} \left(c_{b,\vec{k},+}(t_f) \hat{\epsilon}_{\vec{k},+} + c_{b,\vec{k},-}(t_f) \hat{\epsilon}_{\vec{k},-} \right) e^{-i\nu_k t_f} e^{i\vec{k}\cdot\vec{x}_0} e^{i\omega_\sigma t_f} \cdot \vec{\wp}_{ba} .$$

Substituting $c_{b,\vec{k},\lambda}(t_f)$ into this expression from the formal integration conducted above tells us that in fact, it also depends on its value in some small time interval.

$$
\begin{aligned}
i\hbar\dot{c}_a(t_f) = &-\frac{i}{\hbar}\sum_{\vec{k}}\frac{\hbar\nu_k}{V}\int_{t_0}^{t_f}dt'c_a(t')\left\{\left(\hat{\epsilon}_{\vec{k},+}^*\cdot\vec{\wp}_{ab}\right)\left(\hat{\epsilon}_{\vec{k},+}\cdot\vec{\wp}_{ba}\right)\right. \\
&\left.+\left(\hat{\epsilon}_{\vec{k},-}^*\cdot\vec{\wp}_{ab}\right)\left(\hat{\epsilon}_{\vec{k},-}\cdot\vec{\wp}_{ba}\right)\right\}e^{i\nu_k(t'-t_f)}e^{-i\omega_\sigma(t'-t_f)} \\
&+\sum_{\vec{k}}\sqrt{\frac{\hbar\nu_k}{V}}\left(c_{b,\vec{k},+}(t_0)\,\hat{\epsilon}_{\vec{k},+}+c_{b,\vec{k},-}(t_0)\,\hat{\epsilon}_{\vec{k},-}\right)e^{-i\nu_k t_f}e^{i\vec{k}\cdot\vec{x}_0}e^{i\omega_\sigma t_f}\cdot\vec{\wp}_{ba}\,.
\end{aligned}
$$

Therefore, by assuming that the interaction takes place around some time τ between t_f and t_0, we can approximate this time to be averaged such that $t_f - t_0 = \delta t$ and $t_f = \tau + \frac{1}{2}\delta t$ and $t_0 = \tau - \frac{1}{2}\delta t$. Then, substituting into the above, yields that the we have but to evaluate the sum of the series over all modes of $\vec{k}$ in order to find a solution. This approximation is shown in the expression below,

$$
\sum_{\vec{k}}\sqrt{\frac{\hbar\nu_k}{V}}\left(c_{b,\vec{k},\sigma+}\left(\tau-\frac{1}{2}\delta t\right)\hat{\epsilon}_{\vec{k},\sigma+}+c_{b,\vec{k},\sigma-}\left(\tau-\frac{1}{2}\delta t\right)\hat{\epsilon}_{\vec{k},\sigma-}\right)e^{-i\nu_k\left(\tau+\frac{1}{2}\delta t\right)}e^{i\vec{k}\cdot\vec{x}_0}e^{i\omega_\sigma\left(\tau+\frac{1}{2}\delta t\right)}\cdot\vec{\wp}_{ba}\,.
$$

By substituting this approximation into the above, for time scales where $\delta t^{-1}\gg(\nu_k-\omega_\sigma)$ we do not require memory effects which establish a non-Markovian limit for this model[5] [102, 103]. The expression below outlines the difference between this approach and the approach traditionally found in text books [15]. Here, instead of neglecting the additional term contributed by the photonic state, we retain it. This corresponds to the assumption that at a previous time step we need not be purely in the excited state. This is shown by the last term of the following expression

$$
\begin{aligned}
i\hbar\dot{c}_a(t_f) = &-\frac{i}{\hbar}\sum_{\vec{k}}\frac{\hbar\nu_k}{V}\int_{t_0}^{t_f}dt'c_a(t')\sum_\lambda\left|\hat{\epsilon}_{\vec{k},\lambda}\cdot\vec{\wp}_{ba}\right|^2 e^{i(\nu_k-\omega_\sigma)(t'-t_f)} \\
&+\sum_{\vec{k},\lambda}\sqrt{\frac{\hbar\nu_k}{V}}\left(c_{b,\vec{k},\lambda}(\tau)\,\hat{\epsilon}_{\vec{k},\lambda}\right)e^{-i\nu_k\tau}e^{i\vec{k}\cdot\vec{x}_0}e^{i\omega_\sigma\tau}\cdot\vec{\wp}_{ba}\,.
\end{aligned}
$$

---

[5]This limit is important for describing quantum error effects and has been applied previously to particle production at relativistic energies. Here we are interested in both, though not in the range of relativistic energies.

By comparing that term to the sum of the positive frequency parts of the energy normalized PWF and its complex conjugate, we see that it corresponds to a state of the field already present in the region of interaction

$$\vec{\Psi}_{\gamma,b,+}^{(+)}(\tau) + \vec{\Psi}_{\gamma,b,-}^{*(+)}(\tau) = \frac{1}{\sqrt{V}}\sum_{\vec{k},\lambda}\sqrt{\frac{\nu_k}{\omega_\sigma}}\left(c_{b,\vec{k},\lambda}(\tau)\,\hat{\epsilon}_{\vec{k},\lambda}\right)e^{-i\nu_k\tau}e^{i\vec{k}\cdot\vec{x}_0}\,.$$

Making that substitution, the expression for the rate of change of the excited state probability amplitude becomes,

$$i\hbar\dot{c}_a(t_f) = -\frac{i}{\hbar}\sum_{\vec{k}}\frac{\hbar\nu_k}{V}\int_{t_0}^{t_f}dt'c_a(t')\sum_\lambda\left|\hat{\epsilon}_{\vec{k},\lambda}\cdot\vec{\wp}_{ba}\right|^2 e^{i(\nu_k-\omega_\sigma)(t'-t_f)}$$
$$+\sqrt{\hbar\omega_\sigma}e^{i\omega_\sigma\tau}\left[\vec{\Psi}_{\gamma,b,+}^{(+)}(\tau) + \vec{\Psi}_{\gamma,b,-}^{*(+)}(\tau)\right]\cdot\vec{\wp}_{ba}\,.$$

We next divide through by $i\hbar$ and change the order of the sum over modes and the integration over time[6]. By doing so we can sum over all the accessible modes of the field and then investigate the effects that one time-step yields on later ones.

$$\dot{c}_a(t_f) = -\frac{1}{\hbar}\int_{t_0}^{t_f}dt'c_a(t')\sum_{\vec{k}}\frac{\nu_k}{V}\sum_\lambda\left|\hat{\epsilon}_{\vec{k},\lambda}\cdot\vec{\wp}_{ba}\right|^2 e^{i(\nu_k-\omega_\sigma)(t'-t_f)}$$
$$-\frac{i}{\hbar}\sqrt{\hbar\omega_\sigma}e^{i\omega_\sigma\tau}\left[\vec{\Psi}_{\gamma,b,+}^{(+)}(\tau) + \vec{\Psi}_{\gamma,b,-}^{*(+)}(\tau)\right]\cdot\vec{\wp}_{ba}\,.$$

To carry out the infinite sum over all accessible modes we change from a discrete sum over modes to a continuum of modes via the substitution $\frac{1}{V}\sum_{\vec{k}} \rightarrow \frac{1}{(2\pi)^3}\int d\vec{k}$. Then we are left to evaluate the integral

$$\frac{1}{(2\pi)^3}\int d\vec{k}\nu_k\left\{\left|\hat{\epsilon}_{\vec{k},\sigma_+}\cdot\vec{\wp}_{ba}\right|^2 + \left|\hat{\epsilon}_{\vec{k},\sigma_-}\cdot\vec{\wp}_{ba}\right|^2\right\}e^{i(\nu_k-\omega_\sigma)(t'-t_f)}\,. \qquad (4.3.37)$$

---

[6]In order to swap the integration over time with the sum over modes, $\sum_{\vec{k}}$ we make use of the assumption that we are in a region with linear dispersion relations that are not time-dependent.

Making use of the coordinate system defined in figure (C.1), where for brevity we define $\mathcal{C}_\eta \equiv \cos\eta$, $\mathcal{S}_\eta \equiv \sin\eta$, we can see that the helical polarization vectors dotted by the transition dipole moments can be expressed of these coordinates according to the relation below

$$\hat{\epsilon}_{\vec{k},\pm} \cdot \vec{\wp}_{nm} = \frac{1}{\sqrt{2}} \left(\begin{array}{ccc} \mathcal{C}_\theta\mathcal{C}_\phi \mp i\mathcal{S}_\phi & \mathcal{C}_\theta\mathcal{S}_\phi \pm i\mathcal{C}_\phi & -\mathcal{S}_\theta \end{array} \right) \left(\begin{array}{c} \vec{\wp}_{nm,x} \\ \vec{\wp}_{nm,y} \\ \vec{\wp}_{nm,z} \end{array} \right)$$

$$= \frac{1}{\sqrt{2}} \left\{ \left(\mathcal{C}_\theta\mathcal{C}_\phi \mp i\mathcal{S}_\phi \right) \vec{\wp}_{nm,x} + \left(\mathcal{C}_\theta\mathcal{S}_\phi \pm i\mathcal{C}_\phi \right) \vec{\wp}_{nm,y} - \mathcal{S}_\theta\vec{\wp}_{nm,z} \right\} .$$

These terms are important for the evaluation of the integral of equation (4.3.37). We will analyze each of these terms independently and evaluate them under the assumption that we are working with a linear dispersion relation $\nu_k = v_0 k$. In addition we will make use of the definition of the derivative of a delta function which is presented in the appendix (D.5.2). Please note that in text book discussions it is here that the Weisskopf-Wigner approximation is traditionally made. By not fixing the value of the amplitude of $\vec{k}$ to that of the transition frequency we are in a non-Markovian regime which allows us to evaluate the emission lifetime of a spontaneously generated Fock state as well as any revival effects associated with it. The resulting integrals can be evaluated by separating the sum. For $\frac{\vec{\wp}_{ab,x}^2}{2\left(2\pi\right)^3} \int d\vec{k} \nu_k e^{i\left(\nu_k - \omega_\sigma\right)\left(t' - t_f\right)} \left(\mathcal{C}_\theta^2\mathcal{C}_\phi^2 + \mathcal{S}_\phi^2 \right)$ the two indpendent terms of are as follows: $v_0 \int dk d\theta d\phi \left[k^3 \mathcal{S}_\theta e^{-i\left(v_0 k - \omega_\sigma\right)\left(t_f - t'\right)} \mathcal{C}_\theta^2\mathcal{C}_\phi^2 \right]$ yields

$$\frac{2\pi v_0}{3} \int_0^\infty dk \left[k^3 e^{-i\left(v_0 k - \omega_\sigma\right)\left(t_f - t'\right)} \right] = \frac{2\pi v_0}{3} \int_0^\infty du \left[\frac{1}{v_0^4} \left(u + \omega_\sigma\right)^3 e^{-iu\left(t_f - t'\right)} \right]$$

$$= \frac{2\pi}{3v_0^3} \int_0^\infty du \left[\left(u^3 + 3u^2\omega_\sigma + 3u\omega_\sigma^2 + \omega_\sigma^3\right) e^{-iu\left(t_f - t'\right)} \right]$$

$$= \frac{2\pi}{3v_0^3} \pi \left\{ \left(-1\right)^3 i^3 \delta^{(3)}\left(t_f - t'\right) + 3\omega_\sigma \left(-1\right)^2 i^2 \delta^{(2)}\left(t_f - t'\right) \right.$$

$$\left. + 3\omega_\sigma^2 \left(-1\right) i\delta^{(1)}\left(t_f - t'\right) + \omega_\sigma^3 \delta^{(0)}\left(t_f - t'\right) \right\} .$$

and $v_0 \int dk d\theta d\phi \left[k^3 \mathcal{S}_\theta e^{-i(v_0 k - \omega_\sigma)\left(t_f - t'\right)} \mathcal{S}_\phi^2 \right]$ yields

$$\frac{2\pi}{v_0^3} \pi \left\{ (-1)^3 \, i^3 \delta^{(3)} \left(t_f - t'\right) + 3\omega_\sigma \, (-1)^2 \, i^2 \delta^{(2)} \left(t_f - t'\right) \right.$$
$$\left. + \, 3\omega_\sigma^2 \, (-1) \, i \delta^{(1)} \left(t_f - t'\right) + \omega_\sigma^3 \delta^{(0)} \left(t_f - t'\right) \right\} \, .$$

For $\dfrac{\vec{\wp}_{ab,y}^2}{2 \, (2\pi)^3} \int d\vec{k} \nu_k e^{i(\nu_k - \omega_\sigma)\left(t_f - t'\right)} \left(\mathcal{C}_\theta^2 \mathcal{S}_\phi^2 + \mathcal{C}_\phi^2 \right)$ the two indpendent terms of are as follows: $v_0 \int dk d\theta d\phi \left[k^3 \mathcal{S}_\theta e^{-i(v_0 k - \omega_\sigma)} \right.$ yields

$$\frac{2\pi}{3v_0^3} \pi \left\{ (-1)^3 \, i^3 \delta^{(3)} \left(t_f - t'\right) + 3\omega_\sigma \, (-1)^2 \, i^2 \delta^{(2)} \left(t_f - t'\right) \right.$$
$$\left. + \, 3\omega_\sigma^2 \, (-1) \, i \delta^{(1)} \left(t_f - t'\right) + \omega_\sigma^3 \delta^{(0)} \left(t_f - t'\right) \right\}$$

and $v_0 \int dk d\theta d\phi \left[k^3 \mathcal{S}_\theta e^{-i(v_0 k - \omega_\sigma)\left(t_f - t'\right)} \mathcal{C}_\phi^2 \right]$ yields

$$\frac{2\pi}{v_0^3} \pi \left\{ (-1)^3 \, i^3 \delta^{(3)} \left(t_f - t'\right) + 3\omega_\sigma \, (-1)^2 \, i^2 \delta^{(2)} \left(t_f - t'\right) \right.$$
$$\left. + \, 3\omega_\sigma^2 \, (-1) \, i \delta^{(1)} \left(t_f - t'\right) + \omega_\sigma^3 \delta^{(0)} \left(t_f - t'\right) \right\} \, .$$

For $\dfrac{\vec{\wp}_{ab,z}^2}{2 \, (2\pi)^3} \int d\vec{k} \nu_k e^{i(\nu_k - \omega_\sigma)\left(t' - t_f\right)} \mathcal{S}_\theta^2$ the substitutiongive: $v_0 \int dk d\theta d\phi \left[k^3 \mathcal{S}_\theta e^{-i(v_0 k - \omega_\sigma)\left(t_f - t'\right)} \mathcal{S}_\theta^2 \right]$ which yields

$$\frac{4\pi}{3v_0^3} \pi \left\{ (-1)^3 \, i^3 \delta^{(3)} \left(t_f - t'\right) + 3\omega_\sigma \, (-1)^2 \, i^2 \delta^{(2)} \left(t_f - t'\right) \right.$$
$$\left. + \, 3\omega_\sigma^2 \, (-1) \, i \delta^{(1)} \left(t_f - t'\right) + \omega_\sigma^3 \delta^{(0)} \left(t_f - t'\right) \right\} \, .$$

From the integrations conducted above we are left with the equation of motion for c_a as in terms of the spatial integration given by

$$\dot{c}_a \left(t_f\right) \;\; = \;\; -\frac{1}{\hbar} \int_{t_0}^{t_f} dt' c_a \left(t'\right) h \left(t_f - t'\right) \, .$$

This expression relates the rate of change of the excited to state to the accessible modes which can describe a spontaneously emitted field. In that expression the term $h\left(t_f - t'\right)$ contains all the information associated with the contributions of the transition dipole moments and any constraints imposed on the modes, $\vec{k}$

$$
\begin{aligned}
h\left(t_f - t'\right) = \frac{1}{4\pi}\frac{1}{3v_0^3} &\left\{-i^3\delta^{(3)}\left(t_f - t'\right) + 3\omega_\sigma i^2\delta^{(2)}\left(t_f - t'\right)\right. \\
&\left. - 3\omega_\sigma^2 i\delta^{(1)}\left(t_f - t'\right) + \omega_\sigma^3\delta^{(0)}\left(t_f - t'\right)\right\}\left[2\vec{\wp}_{ab,x}^2 + 2\vec{\wp}_{ab,y}^2 + \vec{\wp}_{ab,z}^2\right] \ .
\end{aligned}
$$

The evaluation of the time integration of the product of this expression with the excited state requires that we use some properties of the Dirac Delta function and its derivative as defined in the appendix through equation (D.4.1) [82, 83]. For clarity, we define the substitution $\lambda \equiv \frac{1}{4\pi}\frac{1}{3v_0^3}\left[\frac{1}{2}\vec{\wp}_{ab,x}^2 + \frac{1}{2}\vec{\wp}_{ab,y}^2 + \vec{\wp}\right.$ do not confuse this parameter with the helicities or spin states of the excited optical fields). We use this definition to express that in reality c_a satisfies a third order equation of motion. This is a new result prescribed by the expression

$$
\begin{aligned}
c_a^{(1)}\left(t_f\right) + \frac{i}{\hbar}e^{i\omega_\sigma\tau}&\left[\vec{\Psi}_{\gamma,b,+}^{(+)}\left(\tau\right) + \vec{\Psi}_{\gamma,b,-}^{*(+)}\left(\tau\right)\right]\cdot\vec{\wp}_{ba} = -\frac{1}{\hbar}\int_{t_0}^{t_f} dt'\, c_a\left(t'\right) h\left(t_f - t'\right) \\
&= -\frac{\lambda}{\hbar}\left\{-i^3\partial_{t_f}^{(3)} + 3\omega_\sigma i^2\partial_{t_f}^{(2)} - 3\omega_\sigma^2 i\partial_{t_f}^{(1)} + \omega_\sigma^3\right\}\left[c_a\left(t_f\right)\right] \quad\quad (4.3.38)\\
&= -\frac{i\lambda}{\hbar}\left\{c_a^{(3)}\left(t_f\right) + 3i\omega_\sigma c_a^{(2)}\left(t_f\right) - 3\omega_\sigma^2 c_a^{(1)}\left(t_f\right) - i\omega_\sigma^3 c_a\left(t_f\right)\right\} \ .
\end{aligned}
$$

In that expression $c_a^{(n)}$ represents the nth derivative with respect to time of $c_a(t)$. The corresponding ordinary differential equation is then of the form [7]

$$c_a^{(3)} + 3i\omega_\sigma c_a^{(2)} - \left(3\omega_\sigma^2 + i\frac{\hbar}{\lambda}\right)c_a^{(1)} - i\omega_\sigma^3 c_a = -\frac{\sqrt{\hbar\omega_\sigma}}{\lambda}e^{i\omega_\sigma\tau}\left[\vec{\Psi}_{\gamma,b,+}^{(+)}(\tau) + \vec{\Psi}_{\gamma,b,-}^{*(+)}(\tau)\right] \quad (4.3.39)$$

This ODE may be solved analytically for fixed values of $\vec{\Psi}_{\gamma,b,+}^{(+)}(\tau)$ and $\vec{\Psi}_{\gamma,b,-}^{*(+)}(\tau)$. However, since we do not have these values, we solve the problem computationally in later chapters. Using the cubic formula and standard mathematical techniques, it is possible to find a closed form solution for this expression. The inhomogeneous term depends on the form of $\vec{\Psi}_{\gamma,b,+}^{(+)}(\tau) + \vec{\Psi}_{\gamma,b,-}^{*(+)}(\tau)$, which we do not have. Therefore, we also delay the discussion on the evaluation of the inhomogeneous solution to the chapters on the computational methods applied. Finding the homogeneous solution is much more straight forward. We find the homogeneous solution using standard techniques of ODEs. First, by guessing a solution of the form $c_{a,h}(t) = A_h e^{zt}$ and substituting it into (4.3.39) yields the characteristic equation

$$z^3 + a_2 z^2 + a_1 z + a_0 = 0,$$

---

[7] From the expression (4.3.38) derived for $c_a^{(1)}$ above, we now show that neglecting all the higher order derivatives and keeping only $-\frac{\lambda\omega_\sigma^3}{\hbar}c_a$ can be said to correspond to making the Weisskopf-Wigner approximation presented in [15]. In our discussion, by fixing $\theta' = 0$ we can orient the transition dipole moment along a relative z-axis to get $-\frac{\lambda\omega_\sigma^3}{\hbar}\Big|_{\theta'=0} = -\frac{1}{2\pi}\frac{\omega_\sigma^3(\vec{\wp}_{ba})_z^2}{3\hbar v_0^3}$. Rewriting $\vec{\wp}_{ba}$ in the units of textbooks [15] as $\vec{\wp}_{ba} = \frac{\vec{\wp}_{ba}}{\sqrt{2\varepsilon_0}}$, this expression yields $-\frac{\lambda\omega_\sigma^3}{\hbar}\Big|_{\theta'=0} = -\frac{1}{4\pi\varepsilon_0}\frac{\omega_\sigma^3|\vec{\wp}_{ba}|^2}{3\hbar v_0^3}$. Rewriting $v_0 \to c$, we find that in particular expression (6.3.13) of [15], which defines $-\frac{\Gamma}{2} \equiv \frac{1}{2}\frac{1}{4\pi\epsilon_0}\frac{4\omega_\sigma|\vec{\wp}_{ba}|^2}{3\hbar c^3}$ is in fact given in our notation by $-\frac{\Gamma}{2} = -2\frac{\lambda\omega_\sigma^3}{\hbar}\Big|_{\theta'=0}$. Careful consideration shows that the fact that these two expressions differ by a factor of 2 makes perfect sense. In the approach that we have presented above, we are taking into consideration only one polarization at a time. We are therefore forced to use the transformation $\frac{1}{V}\sum_{\vec{k}} \to \frac{1}{(2\pi)^3}\int d\vec{k}$ when converting from discrete sum modes to an integration over a continuum of modes. In contrast, the discussion in [15] immediately considers a sum over 2 polarizations by making the transformation $\sum_{\vec{k}} \to 2\frac{V}{(2\pi)^3}\int d^3k$ when computing $-\frac{\Gamma}{2}$. We can therefore, once again, assert that neglecting the higher order derivatives of c_a corresponds to making the Weisskopf-Wigner approximation discussed in textbooks [15].

where $a_2 = (3i\omega_\sigma)$, $a_1 = -\left(3\omega_\sigma^2 + i\frac{\hbar}{\lambda}\right)$, and $a_0 = (-i\omega_\sigma^3)$. Note that since a_0 is complex, all the characteristic roots z_n will also be complex. Since the cubic equation produces 3 roots and none of them are assumed to be repeated, we have a characteristic solution of the form

$$c_{a,h}(t_f) = A_1 e^{z_1 t_f} + A_2 e^{z_2 t_f} + A_3 e^{z_3 t_f}.$$

The roots for z_n can be analytically calculated by implementing the cubic equation (D.8) or using an algorithm that makes some approximation [83, 82, 59]. We do not neglect exponential terms that can give rise to un-inhibited growth in the solution. We instead propose an analogy with the solutions of the Schrödinger equation in the presence of a potential barrier. This analogy can be intuited as arising due to the discretization of time in the interaction region of the few-level system. We assume that the terms that exhibit exponential growth are analogous to tunneling components associated with the near-field interaction of the photon with the few-level system where we have centered the interaction at a time t' exactly half way between t_f and t_0.

Next, we use the cubic equation to find the characteristic roots for the characteristic equation since we are searching for an analytical solution. In this analytical calculation we first define all of the parameters required by the traditional treatment of the cubic equation

$$Q = -i\frac{\hbar}{3\lambda} \tag{4.3.40}$$

$$R = \frac{\hbar\omega_\sigma}{2\lambda}. \tag{4.3.41}$$

These parameters define the next term,

$$D = \frac{\hbar^2\omega_\sigma^2}{2^2\lambda^2} + i\left(\frac{\hbar^3}{3^3\lambda^3}\right). \tag{4.3.42}$$

Finally, the resulting roots are given by the expressions

$$S = \left(\frac{\hbar\omega_\sigma}{2\lambda}\right)^{\frac{1}{3}} \sqrt[3]{1 + \sqrt{1 + \frac{4i}{27}\frac{\hbar}{\lambda\omega_\sigma^2}}} \tag{4.3.43}$$

$$T = \left(\frac{\hbar\omega_\sigma}{2\lambda}\right)^{\frac{1}{3}} \sqrt[3]{1 - \sqrt{1 + \frac{4i}{27}\frac{\hbar}{\lambda\omega_\sigma^2}}} . \tag{4.3.44}$$

In order to evaluate and interpret these roots we must take advantage of the properties of high order roots of complex numbers outlined in section (D.9). There we outline how we can use Demoivre's theorem to express these roots in the complex form $\alpha + i\beta$. Since the expressions for these roots are very cumbersome, it suffices to say that the three characteristic roots z_n are

$$z_1 = i\omega_\sigma + (S + T) \tag{4.3.45}$$

$$z_2 = i\omega_\sigma - \frac{1}{2}(S + T) + \frac{1}{2}i\sqrt{3}(S - T) \tag{4.3.46}$$

$$z_3 = i\omega_\sigma - \frac{1}{2}(S + T) - \frac{1}{2}i\sqrt{3}(S - T) . \tag{4.3.47}$$

In complex notation these roots can be expressed as in terms of $\varsigma \equiv \frac{\lambda\omega_\sigma^2}{\hbar}$ by the phasors

$$\alpha_1 = \left(1 - 4\varsigma^2\right) + 4i\left(\frac{1}{9\varsigma} - \varsigma\right) \tag{4.3.48}$$

$$\alpha_{2,\pm} = \left(1 \pm |\alpha_1|^{\frac{1}{2}}\cos\frac{\theta_1}{2}\right) \pm i\left(|\alpha_1|^{\frac{1}{2}}\sin\frac{\theta_1}{2} \mp 2\varsigma\right) . \tag{4.3.49}$$

The characteristic roots given in this form are then explicitly

Root 1:

$$z_1 = i\omega_\sigma - \left(\frac{\hbar\omega_\sigma}{2\lambda}\right)^{\frac{1}{3}} \sum_{l=\pm} |\alpha_{2,\pm}|^{\frac{1}{3}}\left[\cos\frac{\theta_{2,l}}{3} + i\sin\frac{\theta_{2,l}}{3}\right] \tag{4.3.50}$$

Root 2:

$$z_2 = i\omega_\sigma + \left(\frac{\hbar\omega_\sigma}{2\lambda}\right)^{\frac{1}{3}} \sum_{l=\pm} \frac{|\alpha_{2,\pm}|^{\frac{1}{3}}}{2} \left[\left(\cos\frac{\theta_{2,l}}{3} - l\sqrt{3}\sin\frac{\theta_{2,l}}{3}\right) + il\left(\sqrt{3}\cos\frac{\theta_{2,l}}{3} + l\sin\frac{\theta_{2,l}}{3}\right)\right]$$

(4.3.51)

Root 3:

$$z_3 = i\omega_\sigma + \left(\frac{\hbar\omega_\sigma}{2\lambda}\right)^{\frac{1}{3}} \sum_{l=\pm} \frac{|\alpha_{2,\pm}|^{\frac{1}{3}}}{2} \left[\left(\cos\frac{\theta_{2,l}}{3} + l\sqrt{3}\sin\frac{\theta_{2,l}}{3}\right) - il\left(\sqrt{3}\cos\frac{\theta_{2,l}}{3} + \sin\frac{\theta_{2,l}}{3}\right)\right]$$

(4.3.52)

A more detailed discussion and the full analytical form of the roots z_n for various systems will be presented in (C.6.1).

4.3.2.2.5 *Solving for* $\vec{\Psi}^{(+)}_{\gamma,+,b}$ *and* $\vec{\Psi}^{\dagger(+)}_{\gamma,-,b}$

Similarly to the procedure carried out above, we can substitute the solution of (4.3.39) for $c_a(t_f)$ into the integral expression for $c_{b,\vec{k},\lambda}(t_f)$. Here we arrive at the photonic wave function for a single photon interacting with a two level quantum source in terms of $c_{b,\vec{k},\lambda}(t_f)$. We reiterate energy normalized wave-functions and make an averaged time phase approximation, where $e^{-i\nu_k t} \approx e^{-i\nu_k \frac{1}{2}(t_f + t_0)}$, for these expressions

$$\vec{\Psi}^{\dagger(+)}_{\gamma,-,b}(\vec{x},t) = \sqrt{\frac{1}{V}} \sum_{\vec{k}} \sqrt{\frac{\nu_k}{\omega_\sigma}} c_{b,\vec{k},-}(t)\, \hat{\epsilon}_{\vec{k},\sigma_-} e^{-i\nu_k \frac{1}{2}(t_f+t_0)} e^{i\vec{k}\cdot\vec{x}}$$

$$\vec{\Psi}^{(+)}_{\gamma,+,b}(\vec{x},t) = \sqrt{\frac{1}{V}} \sum_{\vec{k}} \sqrt{\frac{\nu_k}{\omega_\sigma}} c_{b,\vec{k},+}(t)\, \hat{\epsilon}_{\vec{k},\sigma_+} e^{-i\nu_k \frac{1}{2}(t_f+t_0)} e^{i\vec{k}\cdot\vec{x}}.$$

For readability we also restate the spin resolved terms corresponding to the photonic state probability amplitudes and the spin resolved PWF's above

$$
c_{b,\vec{k},-}\left(t_f\right) - c_{b,\vec{k},-}\left(t_0\right) = -\frac{i}{\hbar}\int_{t_0}^{t_f} dt' \sqrt{\frac{\hbar\nu_k}{V}} c_a\left(t'\right) \hat{\epsilon}_{\vec{k},-}^{*} e^{i\nu_k t'} e^{-i\vec{k}\cdot\vec{x}_0} e^{-i\omega_\sigma t'} \cdot \vec{\wp}_{ab}
$$

$$
c_{b,\vec{k},+}\left(t_f\right) - c_{b,\vec{k},+}\left(t_0\right) = -\frac{i}{\hbar}\int_{t_0}^{t_f} dt' \sqrt{\frac{\hbar\nu_k}{V}} c_a\left(t'\right) \hat{\epsilon}_{\vec{k},+}^{*} e^{i\nu_k t'} e^{-i\vec{k}\cdot\vec{x}_0} e^{-i\omega_\sigma t'} \cdot \vec{\wp}_{ab} \,.
$$

Previously we assumed that we could let $t_f = t_0 + \delta t$, and $\delta t^{-1} \ll (\nu_k - \omega_\sigma)$. In this limit we can make the approximation $e^{-i(\nu_k - \omega_\sigma)\delta t} \approx 1$, such that for these short times there isn't much change in the exponential functions $e^{-i(\nu_k - \omega_\sigma)t_f} \approx e^{-i(\nu_k - \omega_\sigma)t_0}$. Within these constraints we can yield the relationship between the excited state c_a and the produced single photon wave function associated with the probability amplitude $c_{b,\vec{k},\pm}$ to be

$$
\vec{\Psi}_{\gamma,-,b}^{\dagger(+)}\left(t_f\right) - \vec{\Psi}_{\gamma,-,b}^{\dagger(+)}\left(t_0\right) = \frac{-i}{(2\pi)^3}\frac{1}{\sqrt{\hbar\omega_\sigma}}\int d\vec{k}\int_{t_0}^{t_f} dt'\nu_k \left(\hat{\epsilon}_{\vec{k},\sigma_-}\hat{\epsilon}_{\vec{k},\sigma_-}^{*}\cdot\vec{\wp}_{ab}\right) e^{-i\nu_k\frac{1}{2}(t_f+t_0)} e^{-i\vec{k}\cdot(\vec{x}_0-\vec{x})} c_a\left(t'\right) e^{i\nu_k t'} e^{-i\omega_\sigma t'}
$$

$$
\vec{\Psi}_{\gamma,+,b}^{(+)}\left(t_f\right) - \vec{\Psi}_{\gamma,+,b}^{(+)}\left(t_0\right) = \frac{-i}{(2\pi)^3}\frac{1}{\sqrt{\hbar\omega_\sigma}}\int d\vec{k}\int_{t_0}^{t_f} dt'\nu_k \left(\hat{\epsilon}_{\vec{k},+}\hat{\epsilon}_{\vec{k},+}^{*}\cdot\vec{\wp}_{ab}\right) e^{-i\nu_k\frac{1}{2}(t_f+t_0)} e^{-i\vec{k}\cdot(\vec{x}_0-\vec{x})} c_a\left(t'\right) e^{i\nu_k t'} e^{-i\omega_\sigma t'} \,.
$$

By separating these wave-functions we again deviate from the traditional approach found in text books [15], which specify the use of the identity[8] $\sum_\lambda \hat{\epsilon}_{\vec{k},\lambda}^{*}\hat{\epsilon}_{\vec{k},\lambda} = \vec{1} - \frac{\vec{k}\vec{k}}{k^2}$ for the evaluation of the integral.

In fact, it is this separation of the wave-function into independent spin states that will have significant consequences down the road. This separation will allow us to suggest future experiments. In addition we will be able to propose numerical models and results that can predict and/or provide experimental benchmarks for a spin resolved experimental set-up. These would be benchmarks that would be essential when studying quantum networks that depend on the helical (spin) nature of the photon to transport or teleport spin encoded quantum information from one qubit to another. Going back to the analysis of this separation, we next substitute the solution for $c_a\left(t_f\right)$ into this expression and again make use of the approximation mentioned earlier, where $t_f = t_0 + \delta t$, and $\delta t^{-1} \gg (\nu_k - \omega_\sigma)$, to the term corresponding to the initial optical wave-function

$$
\frac{i}{(2\pi)^3}\sum_{n=1}^{3}\int d\vec{k}\nu_k \left(\hat{\epsilon}_{\vec{k},-}\hat{\epsilon}_{\vec{k},-}^{*}\cdot\vec{\wp}_{ab}\right) e^{-i\nu_k\frac{1}{2}(t_f+t_0)} e^{-i\vec{k}\cdot(\vec{x}_0-\vec{x})} \left(-\frac{i\sqrt{\hbar\omega_\sigma}}{\lambda\omega_\sigma^3}\vec{\Psi}_{\gamma,\pm}^{(+)}\left(t_0\right)\cdot\vec{\wp}_{ba} e^{i\omega_\sigma t_0}\right)\int_{t_0}^{t_f} dt' e^{i\nu_k t'} e^{-i\omega_\sigma t'} \,.
$$

It is evident from this approximation that this term drops off. This is due to the fact that any history associated with the wave-function is assumed to be contained purely in the complete solution of

---

[8]This identity is proved in section (C.5) of the appendix.

the excited state probability amplitude c_a. This is explicitly shown in the expression below

$$\int_{t_0}^{t_f} dt' e^{i\nu_k t'} e^{-i\omega_\sigma t'} = \frac{\left[e^{i(\nu_k - \omega_\sigma)t_f} - e^{i(\nu_k - \omega_\sigma)t_0}\right]}{i\left(\nu_k - \omega_\sigma\right)}$$
$$\approx \frac{\left[e^{i(\nu_k - \omega_\sigma)t_0} - e^{i(\nu_k - \omega_\sigma)t_0}\right]}{i\left(\nu_k - \omega_\sigma\right)} = 0\,.$$

This does not adversely affect the average time phase approximation. In fact, this result also relies on the average time phase approximation since it implies that if $e^{i(\nu_k - \omega_\sigma)t_f} \approx e^{i(\nu_k - \omega_\sigma)t_0}$, then $e^{i(\nu_k - \omega_\sigma)(2t - t_0)} \approx e^{i(\nu_k - \omega_\sigma)(2t - t_f)}$, which is exactly true when t_0 and t_f are equally spaced from t. Note that this does not constrain how big or small δt must be. The full expression of the wave functions $\vec{\Psi}^{\dagger(+)}_{\gamma,\pm,b}(t_f)$ are then given by the equations

$$\vec{\Psi}^{\dagger(+)}_{\gamma,-,b}(t_f) - \vec{\Psi}^{\dagger(+)}_{\gamma,-,b}(t_0) = \frac{-i}{(2\pi)^3} \frac{1}{\sqrt{\hbar\omega_\sigma}} \sum_{n=1}^{3} \int d\vec{k} \int_{t_0}^{t_f} dt' \nu_k \left(\hat{\epsilon}_{\vec{k},-} \hat{\epsilon}^*_{\vec{k},-} \cdot \vec{\wp}_{ab}\right) e^{-i\nu_k \frac{1}{2}(t_f + t_0)} e^{-i\vec{k}\cdot(\vec{x_0} - \vec{x})} A_n e^{z_n t'} e^{i\nu_k t'} e^{-i\omega_\sigma t'}$$

$$\vec{\Psi}^{(+)}_{\gamma,+,b}(t_f) - \vec{\Psi}^{(+)}_{\gamma,+,b}(t_0) = \frac{-i}{(2\pi)^3} \frac{1}{\sqrt{\hbar\omega_\sigma}} \sum_{n=1}^{3} \int d\vec{k} \int_{t_0}^{t_f} dt' \nu_k \left(\hat{\epsilon}_{\vec{k},+} \hat{\epsilon}^*_{\vec{k},+} \cdot \vec{\wp}_{ab}\right) e^{-i\nu_k \frac{1}{2}(t_f + t_0)} e^{-i\vec{k}\cdot(\vec{x_0} - \vec{x})} A_n e^{z_n t'} e^{i\nu_k t'} e^{-i\omega_\sigma t'}$$

and simplifies to the evaluation of the vector integrals

$$\vec{\Psi}^{\dagger(+)}_{\gamma,-,b}(t_f) - \vec{\Psi}^{\dagger(+)}_{\gamma,-,b}(t_0) = \frac{-i}{(2\pi)^3} \frac{1}{\sqrt{\hbar\omega_\sigma}} \sum_{n=1}^{3} A_n \int d\vec{k} e^{-i\nu_k \frac{1}{2}(t_f + t_0)} e^{-i\vec{k}\cdot(\vec{x_0} - \vec{x})} \nu_k \left(\hat{\epsilon}_{\vec{k},-} \hat{\epsilon}^*_{\vec{k},-} \cdot \vec{\wp}_{ab}\right) \int_{t_0}^{t_f} dt' e^{z_n t'} e^{i\nu_k t'} e^{-i\omega_\sigma t'}$$

$$\vec{\Psi}^{(+)}_{\gamma,+,b}(t_f) - \vec{\Psi}^{(+)}_{\gamma,+,b}(t_0) = \frac{-i}{(2\pi)^3} \frac{1}{\sqrt{\hbar\omega_\sigma}} \sum_{n=1}^{3} A_n \int d\vec{k} e^{-i\nu_k \frac{1}{2}(t_f + t_0)} e^{-i\vec{k}\cdot(\vec{x_0} - \vec{x})} \nu_k \left(\hat{\epsilon}_{\vec{k},+} \hat{\epsilon}^*_{\vec{k},+} \cdot \vec{\wp}_{ab}\right) \int_{t_0}^{t_f} dt' e^{z_n t'} e^{i\nu_k t'} e^{-i\omega_\sigma t'}\,.$$

We first carry out the integration over t'. This integration yields the term

$$\int_{t_0}^{t_f} dt' e^{i(\nu_k - \omega_\sigma - iz_n)t'} = \frac{\left[e^{i(\nu_k - \omega_\sigma - iz_n)t_f} - e^{i(\nu_k - \omega_\sigma - iz_n)t_0}\right]}{i\left(\nu_k - \omega_\sigma - iz_n\right)}\,.$$

Another difference between our approach and the discussion present in textbooks [104, 15, 105] is that we do not take the long time limit, and instead focus on small times due to the nature of our computational model. Therefore the integration over $\vec{k}$ is more involved. This integration is discussed in detail in the appendix (C.7). The integrals that are evaluated in the appendix are of the form

$$\vec{I}_{-,n}(|\vec{x} - \vec{x_0}|, t_f) = \int d\vec{k} \frac{e^{-i\vec{k}\cdot(\vec{x_0} - \vec{x})} e^{-i\nu_k \frac{1}{2}(t_f + t_0)} \left[e^{i(\nu_k - \omega_\sigma - iz_n)t_f} - e^{i(\nu_k - \omega_\sigma - iz_n)t_0}\right]}{(\nu_k - \omega_\sigma - iz_n)} \nu_k \left(\hat{\epsilon}_{\vec{k},\sigma_-} \hat{\epsilon}^*_{\vec{k},\sigma_-} \cdot \vec{\wp}_{ab}\right)$$

$$\vec{I}_{+,n}(|\vec{x} - \vec{x_0}|, t_f) = \int d\vec{k} \frac{e^{-i\vec{k}\cdot(\vec{x_0} - \vec{x})} e^{-i\nu_k \frac{1}{2}(t_f + t_0)} \left[e^{i(\nu_k - \omega_\sigma - iz_n)t_f} - e^{i(\nu_k - \omega_\sigma - iz_n)t_0}\right]}{(\nu_k - \omega_\sigma - iz_n)} \nu_k \left(\hat{\epsilon}_{\vec{k},\sigma_+} \hat{\epsilon}^*_{\vec{k},\sigma_+} \cdot \vec{\wp}_{ab}\right)\,.$$

Through the assumption of a linear dispersion relation $\nu_k = v_0 k$ and defining $|\vec{x} - \vec{x}_0| \equiv z$, these can be computed through the expressions discussed in appendix section (C.7). These yield that the wave functions have the form[9]

$$\vec{\Psi}^{\dagger(+)}_{\gamma,-,b}\left(z,t_f\right) = \frac{-1}{(2\pi)^3}\frac{1}{\sqrt{\hbar\omega_\sigma}}\sum_{n=1}^{3} A_n \vec{I}_{-,n}\left(z,t_f\right) + \vec{\Psi}^{\dagger(+)}_{\gamma,-,b}\left(z,t_0\right)$$

$$\vec{\Psi}^{(+)}_{\gamma,+,b}\left(z,t_f\right) = \frac{-1}{(2\pi)^3}\frac{1}{\sqrt{\hbar\omega_\sigma}}\sum_{n=1}^{3} A_n \vec{I}_{+,n}\left(z\right) + \vec{\Psi}^{(+)}_{\gamma,+,b}\left(z,t_0\right) ,$$

where $\vec{I}_{\pm,n}\left(z\right)$ is given by the respective expressions below. For compactness and clarity, we define the following causal functions for all three cases

$$\zeta_+ = \pi^2\Theta\left(t + \frac{z}{v_0}\right) e^{-i(\omega_\sigma + iz_n)\left(t + \frac{z}{v_0}\right)}$$

$$\zeta_- = \pi^2\Theta\left(t - \frac{z}{v_0}\right) e^{-i(\omega_\sigma + iz_n)\left(t - \frac{z}{v_0}\right)}$$

These objects define the integrals to be calculated with poles in the lower half plane.

$$\mathcal{I}_{\pm,n,x}\left(z\right) =$$
$$\left[\left(\pm 2i\vec{\wp}_{ab,y} + (1\pm 1)\vec{\wp}_{ab,x}\right)\frac{(\omega_\sigma + iz_n)^2}{z v_0^2} + \left(\vec{\wp}_{ab,x} \pm \vec{\wp}_{ab,y}\right)\frac{2i(\omega_\sigma + iz_n)}{z^2 v_0} + \frac{2\vec{\wp}_{ab,x}}{z^3}\right]\zeta_-$$
$$-\left[\left(\pm 2i\vec{\wp}_{ab,y} - (1\pm 1)\vec{\wp}_{ab,x}\right)\frac{(\omega_\sigma + iz_n)^2}{z v_0^2} - \left(\vec{\wp}_{ab,x} \mp \vec{\wp}_{ab,y}\right)\frac{2i(\omega_\sigma + iz_n)}{z^2 v_0} - \frac{2\vec{\wp}_{ab,x}}{z^3}\right]\zeta_+$$

$$\mathcal{I}_{\pm,n,y}\left(z\right) =$$
$$\left[\left(\mp 2i\vec{\wp}_{ab,x} - (1\mp 1)\vec{\wp}_{ab,y}\right)\frac{(\omega_\sigma + iz_n)^2}{z v_0^2} + \left(\vec{\wp}_{ab,y} \mp \vec{\wp}_{ab,x}\right)\frac{2i(\omega_\sigma + iz_n)}{z^2 v_0} + \frac{2\vec{\wp}_{ab,y}}{z^3}\right]\zeta_-$$
$$+\left[\left(\pm 2i\vec{\wp}_{ab,x} - (1\mp 1)\vec{\wp}_{ab,y}\right)\frac{(\omega_\sigma + iz_n)^2}{z v_0^2} - \left(\vec{\wp}_{ab,y} \pm \vec{\wp}_{ab,x}\right)\frac{2i(\omega_\sigma + iz_n)}{z^2 v_0} + \frac{2\vec{\wp}_{ab,y}}{z^3}\right]\zeta_+$$

---

[9]Please note that the units of this integral are $[eV]^{\frac{1}{2}}[m]^{-\frac{3}{2}}$ and that those of the prefactor are $[eV]^{-\frac{1}{2}}$. This implies that indeed this wave-function has the correct units of probability density.

$$\mathcal{I}_{\sigma_\pm,n,z}\left(\imath\right) =$$

$$\left[\frac{1}{\imath^3} - \frac{i\left(\omega_\sigma + iz_n\right)}{\imath^2 v_0}\right] 4\vec{\wp}_{ab,z}\zeta_-$$

$$+ \left[\frac{1}{\imath^3} + \frac{i\left(\omega_\sigma + iz_n\right)}{\imath^2 v_0}\right] 4\vec{\wp}_{ab,z}\zeta_+$$

These objects define the integrals to be calculated for poles in the upper half plane.

$$\mathcal{I}_{\pm,n,x}\left(\imath\right) =$$

$$\left[\left(\pm 2i\vec{\wp}_{ab,y} - \left(1 \pm 1\right)\vec{\wp}_{ab,x}\right)\frac{\left(\omega_\sigma + iz_n\right)^2}{\imath v_0^2} + \left(\vec{\wp}_{ab,x} \mp \vec{\wp}_{ab,y}\right)\frac{2i\left(\omega_\sigma + iz_n\right)}{\imath^2 v_0} - \frac{2\vec{\wp}_{ab,x}}{\imath^3}\right]\zeta_-$$

$$+ \left[\left(\pm 2i\vec{\wp}_{ab,y} + \left(1 \pm 1\right)\vec{\wp}_{ab,x}\right)\frac{\left(\omega_\sigma + iz_n\right)^2}{\imath v_0^2} + \left(\vec{\wp}_{ab,x} \pm \vec{\wp}_{ab,y}\right)\frac{2i\left(\omega_\sigma + iz_n\right)}{\imath^2 v_0} + \frac{2\vec{\wp}_{ab,x}}{\imath^3}\right]\zeta_+$$

$$\mathcal{I}_{\pm,n,y}\left(\imath\right) =$$

$$\left[\left(\mp 2i\vec{\wp}_{ab,x} - \left(1 \mp 1\right)\vec{\wp}_{ab,y}\right)\frac{\left(\omega_\sigma + iz_n\right)^2}{\imath v_0^2} + \left(\vec{\wp}_{ab,y} \pm \vec{\wp}_{ab,x}\right)\frac{2i\left(\omega_\sigma + iz_n\right)}{\imath^2 v_0} + \frac{2\vec{\wp}_{ab,y}}{\imath^3}\right]\zeta_-$$

$$+ \left[\left(\mp 2i\vec{\wp}_{ab,x} + \left(1 \mp 1\right)\vec{\wp}_{ab,y}\right)\frac{\left(\omega_\sigma + iz_n\right)^2}{\imath v_0^2} + \left(\vec{\wp}_{ab,y} \mp \vec{\wp}_{ab,x}\right)\frac{2i\left(\omega_\sigma + iz_n\right)}{\imath^2 v_0} - \frac{2\vec{\wp}_{ab,y}}{\imath^3}\right]\zeta_+$$

$$\mathcal{I}_{\sigma_\pm,n,z}\left(\imath\right) =$$

$$\left[\frac{1}{\imath^3} - \frac{i\left(\omega_\sigma + iz_n\right)}{\imath^2 v_0}\right] 4\vec{\wp}_{ab,z}\zeta_-$$

$$- \left[\frac{1}{\imath^3} + \frac{i\left(\omega_\sigma + iz_n\right)}{\imath^2 v_0}\right] 4\vec{\wp}_{ab,z}\zeta_+$$

4.3.2.2.6   Initial Conditions and Conservation Laws for Equations of Motion

The coefficients A_n may be determined by applying initial and normalization conditions to c_a. The first condition requires that the initial state of $c_a\,(t_0)$ be known in the region of the quantum dot

$$c_a\,(t_0 = 0) = \sum_n A_n\,.$$

The second condition requires only that $\left(\vec{\Psi}_{\gamma,\sigma_+}^{(+)} + \vec{\Psi}_{\gamma,\sigma_-}^{\dagger(+)}\right)_{,b}$ be known. Here we have explicitly used the equations of motion to arrive at this condition.

$$\dot{c}_a\,(t_0 = 0) = -\frac{i}{\hbar}\left(\vec{\Psi}_{\gamma,\sigma_+}^{(+)}\,(t_0) + \vec{\Psi}_{\gamma,\sigma_-}^{*(+)}\,(t_0)\right)_{,b}\cdot\vec{\wp}_{ba}$$

$$= \sum_n A_n z_n\,.$$

The last condition is given by the second derivative of c_a

$$\ddot{c}_a\,(t_0 = 0) = -\frac{i}{\hbar}\frac{d}{dt}\left[\left(\vec{\Psi}_{\gamma,\sigma_+}^{(+)} + \vec{\Psi}_{\gamma,\sigma_-}^{\dagger(+)}\right)_{,b}e^{i\omega_\sigma t}\cdot\vec{\wp}_{ba}\right]_{t_0=0}$$

$$= \sum_n A_n z_n^2$$

where we expect that the total energy will remain bounded. Here we make explicit use of the invariance we calculated in previous chapters. This conservation law is given by the expression,

$$\partial_t \mathcal{H}\,(\vec{x},t) = 0\,.$$

In general, the coefficients A_n may be determined from the following system of equations, derived for some initial state of the quantum dot and a known photonic wave-function via the substitution

$$\sum_n A_n = |c_{a,0}|\,e^{i\theta_0}$$

$$\sum_n A_n z_n = -\frac{i}{\hbar} \left(\vec{\Psi}^{(+)}_{\gamma,\sigma_+} (0) + \vec{\Psi}^{*(+)}_{\gamma,\sigma_-} (0) \right)_{,b} \cdot \vec{\wp}_{ba}$$

$$\sum_n A_n z_n^2 = -\frac{i}{\hbar} \frac{d}{dt} \left[\left(\vec{\Psi}^{(+)}_{\gamma,\sigma_+} + \vec{\Psi}^{\dagger(+)}_{\gamma,\sigma_-} \right)_{,b} e^{i\omega_\sigma t} \cdot \vec{\wp}_{ba} \right]_{t_0=0}$$

where we can verify their correspondence with physical reality by checking that the Hamiltonian densities satisfy the following restrictions

$$\partial_t \left[\mathcal{H}_e + \frac{1}{2} \hbar \omega_\sigma \vec{\Psi}_b \cdot \vec{\Psi}_b^* \right] = 0$$
$$= \partial_t \left[\hbar \omega_\sigma |c_a(t)|^2 + \frac{1}{2} \hbar \omega_\sigma \left| \vec{\Psi}_b(t) \right|^2 \right] .$$

Note that the interaction term has been absorbed into the term $\mathcal{H}_e$. This expression also can be satisfied equally well by factoring out any coefficients and expressed as a probability conservation law . In the expression below the integration over all space is understood to be carried out over all these terms.

$$1 = |c_a(t)|^2 + \frac{1}{2} \left| \vec{\Psi}_b(t) \right|^2$$

In this way we have three cases, fixing $\tau = 0$, defining $\tilde{\Psi}_b = -\frac{i}{\hbar} \left(\vec{\Psi}^{(+)}_{\gamma,\sigma_+} + \vec{\Psi}^{*(+)}_{\gamma,\sigma_-} \right)_{,b} (0) \cdot \vec{\wp}_{ba}$ and $\zeta_k = \left[\tilde{\Psi}_b - |c_{a,0}| e^{i\theta_0} z_k \right]$ for compactness and ease of calculation. For the case of A_i, the first two equations yield

$$A_j + A_k = |c_{a,0}| e^{i\theta_0} - A_i$$
$$A_j z_j + A_k z_k = \tilde{\Psi}_b - A_i z_i$$

which simplify to

$$A_j = \frac{1}{(z_j - z_k)} \left[\tilde{\Psi}_b - |c_{a,0}| e^{i\theta_0} z_k + A_i (z_k - z_i) \right]$$

$$A_k = \frac{1}{(z_k - z_j)} \left[\tilde{\Psi}_b - |c_{a,0}| e^{i\theta_0} z_j + A_i (z_j - z_i) \right]$$

where upon using our definitions can be more compactly expressed as

$$A_j = \frac{\zeta_k + A_i (z_k - z_i)}{(z_j - z_k)}$$

$$A_k = -\frac{\zeta_j + A_i (z_j - z_i)}{(z_j - z_k)}$$

Substituting into the condition for the second derivative, as given by the third expression, we find that

$$A_i = \left[z_i^2 + \frac{z_j^2 (z_k - z_i) - z_k^2 (z_j - z_i)}{(z_j - z_k)} \right]^{-1} \left[\dot{\tilde{\Psi}}_b - \frac{\zeta_k z_j^2 - \zeta_j z_k^2}{(z_j - z_k)} \right]$$

4.3.2.3 Interaction Hamiltonian for a Six Level Quantum Source in Linear Homogeneous Media with positive-spin and negative-spin transitions

Here we extend the our approach for generating a theory of the interaction between multiple levels of a quantum source and photonic states. For the case of a quantum source modeled after an effective six level system, the selection rules presented in figure (4.1) can yield an interaction via the dipole approximation. By implementing the state to state transition dipole moments typically found in QDs we expect to yield compute results that can be directly compared to experimental ones. All the accessible levels of the system can be expressed by means of the state vector

(a) Heavy-Heavy hole   (b) Light-Light hole
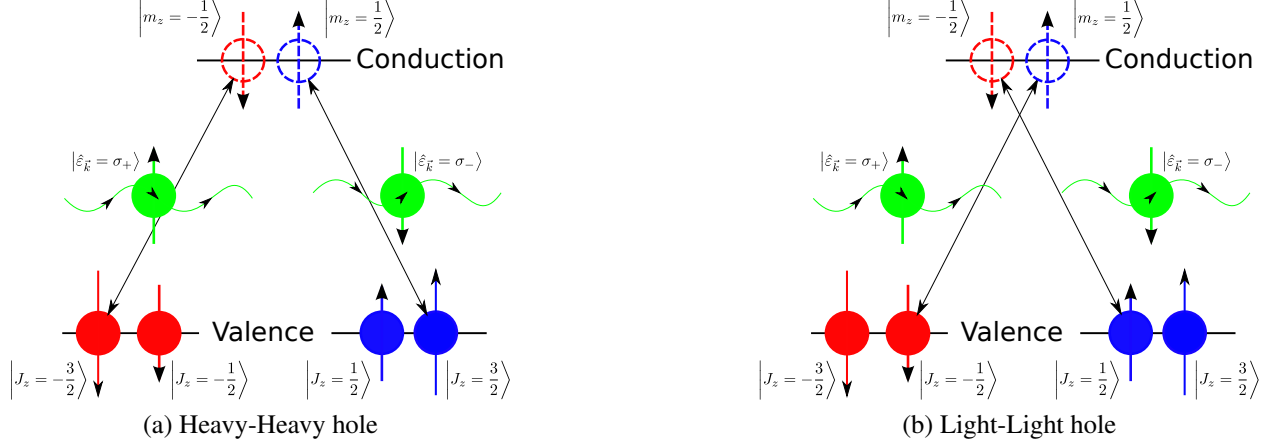
Figure 4.1: Selection rules in a two-level quantum source coupling to single photon states

$$
\begin{aligned}
|\sigma\lambda\rangle = {} & c_{\frac{1}{2}\mathrm{c}}\left|\frac{1}{2}c, \mathrm{vac}\right\rangle + c_{-\frac{1}{2}\mathrm{c}}\left|-\frac{1}{2}c, \mathrm{vac}\right\rangle \\
& + c_{\frac{3}{2}\mathrm{v},\vec{k}-}\left|\frac{3}{2}v, 1_{\vec{k}-}\right\rangle + c_{-\frac{3}{2}\mathrm{v},\vec{k}+}\left|-\frac{3}{2}v, 1_{\vec{k}+}\right\rangle \\
& + c_{\frac{1}{2}\mathrm{v},\vec{k}-}\left|\frac{1}{2}v, 1_{\vec{k}-}\right\rangle + c_{-\frac{1}{2}\mathrm{v},\vec{k}+}\left|-\frac{1}{2}v, 1_{\vec{k}+}\right\rangle .
\end{aligned}
$$

Following the prescription defined above, we can couple these states by means of the interaction outlined in equation (4.3.32). Since we have already described in detail how we achieve the coupling between the states, we will not be as verbose here as before. Instead, we specify in summary the result of the definitions of the equations of motion for each state. For this six-level system, the interaction is given by

$$
\mathcal{V} = \sqrt{\hbar\omega_\sigma}\left(\vec{\Gamma}_\gamma + \vec{\Gamma}_\gamma^\dagger\right) \cdot \sum_{n,m}\left(\vec{\wp}_{nm}\sigma_{nm}e^{i\omega_{nm}t}\right) . \tag{4.3.53}
$$

Once expanded in terms of the transitions that we will consider, this interaction takes on the form, where $\mathcal{V} = \mathcal{V}_\gamma \cdot \mathcal{V}_e$,

$$
\mathcal{V}_e = \sum_{n=1}^{4}\left[\vec{\wp}_+^{(n)}\sigma_+^{(n)}e^{i\omega_+^{(n)}t} + \vec{\wp}_-^{(n)}\sigma_-^{(n)}e^{i\omega_-^{(n)}t}\right]
$$

and the terms $\wp_\pm^{(n)}\sigma_\pm^{(n)}e^{i\omega_\pm^{(n)}t}$ are evaluated according to the expression 4.3.54 . From the structure for the allowed transitions presented in that table we proceed by constructing the explicit interaction with the field. The six-level terms $\wp_\pm^{(n)}\sigma_\pm^{(n)}e^{i\omega_\pm^{(n)}t}$, the contribute to the interaction are therefore tabulated as follows,

	$-\frac{1}{2}c$	$-\frac{1}{2}v$	$-\frac{3}{2}v$	$\frac{3}{2}v$	$\frac{1}{2}v$	$\frac{1}{2}c$
$-\frac{1}{2}c$	0	0	$\wp_-^{(1)}\sigma_-^{(1)}e^{i\omega_-^{(1)}t}$	0	$\wp_-^{(2)}\sigma_-^{(2)}e^{i\omega_-^{(2)}t}$	0
$-\frac{1}{2}v$	0	0	0	0	0	$\wp_+^{(3)}\sigma_+^{(3)}e^{i\omega_+^{(3)}t}$
$-\frac{3}{2}v$	$\wp_+^{(1)}\sigma_+^{(1)}e^{i\omega_+^{(1)}t}$	0	0	0	0	0
$\frac{3}{2}v$	0	0	0	0	0	$\wp_+^{(4)}\sigma_+^{(4)}e^{i\omega_+^{(4)}t}$
$\frac{1}{2}v$	$\wp_+^{(2)}\sigma_+^{(2)}e^{i\omega_+^{(2)}t}$	0	0	0	0	0
$\frac{1}{2}c$	0	$\wp_-^{(3)}\sigma_-^{(3)}e^{i\omega_-^{(3)}t}$	0	$\wp_-^{(4)}\sigma_-^{(4)}e^{i\omega_-^{(4)}t}$	0	0

$$(4.3.54)$$

This is given by the expression

$$\mathcal{V} \;=\; \sum_n^4 \sqrt{\hbar\omega^{(n)}}\left(\vec{\Gamma}_\gamma^{(n)} + \vec{\Gamma}_\gamma^{(n)\dagger}\right)\cdot\left[\wp_+^{(n)}\sigma_+^{(n)}e^{i\omega^{(n)}t} + \wp_-^{(n)}\sigma_-^{(n)}e^{-i\omega^{(n)}t}\right] .$$

By operating from the left and right with the the state vector and recalling the schematic relations $\vec{\Gamma}_\gamma\sigma_- \to \chi_{+,E}^{(-)}$, $\vec{\Gamma}_\gamma\sigma_+ \to \chi_{+,g}^{(+)}$, $\vec{\Gamma}_\gamma^\dagger\sigma_- \to \chi_{-,E}^{(-)}$, and $\vec{\Gamma}_\gamma^\dagger\sigma_+ \to \chi_{-,g}^{(+)}$ one can show that this sandwich yields the following equations of motion after dropping all non-contributing terms,

$$i\hbar\dot{c}_{-\frac{3}{2}v,1_{\vec{k}+}} \;=\; \left\langle -\frac{3}{2}v, 1_{\vec{k}+}\left|\sqrt{\hbar\omega^{(1)}}\vec{\Gamma}_\gamma^{\dagger(1)}\xi_-^{(1)}c_{-\frac{1}{2}c}\right| -\frac{1}{2}c, \text{vac}\right\rangle \qquad (4.3.55)$$

$$i\hbar\dot{c}_{-\frac{1}{2}v,1_{\vec{k}+}} \;=\; \left\langle -\frac{1}{2}v, 1_{\vec{k}+}\left|\sqrt{\hbar\omega^{(3)}}\vec{\Gamma}_\gamma^{\dagger(3)}\xi_-^{(3)}c_{\frac{1}{2}c}\right| \frac{1}{2}c, \text{vac}\right\rangle \qquad (4.3.56)$$

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$$i\hbar\dot{c}_{-\frac{1}{2}c} = \left\langle -\frac{1}{2}c, \text{vac} \left| \sqrt{\hbar\omega^{(1)}}\vec{\Gamma}_\gamma^{(1)}\xi_+^{(1)}c_{-\frac{3}{2}v,1_{\vec{k}+}} \right| -\frac{3}{2}v, 1_{\vec{k}+} \right\rangle$$
$$+ \left\langle -\frac{1}{2}c, \text{vac} \left| \sqrt{\hbar\omega^{(2)}}\vec{\Gamma}_\gamma^{\dagger(2)}\xi_+^{(2)}c_{-\frac{3}{2}v,1_{\vec{k}+}} \right| \frac{1}{2}v, 1_{\vec{k}-} \right\rangle \tag{4.3.57}$$

$$i\hbar\dot{c}_{\frac{1}{2}c} = \left\langle \frac{1}{2}c, \text{vac} \left| \sqrt{\hbar\omega^{(3)}}\vec{\Gamma}_\gamma^{(3)}\xi_+^{(3)}c_{-\frac{1}{2}v,1_{\vec{k}+}} \right| -\frac{1}{2}v, 1_{\vec{k}+} \right\rangle$$
$$+ \left\langle \frac{1}{2}c, \text{vac} \left| \sqrt{\hbar\omega^{(4)}}\vec{\Gamma}_\gamma^{\dagger(4)}\xi_+^{(4)}c_{\frac{3}{2}v,1_{\vec{k}-}} \right| \frac{3}{2}v, 1_{\vec{k}-} \right\rangle \tag{4.3.58}$$

$$i\hbar\dot{c}_{\frac{1}{2}v,1_{\vec{k}-}} = \left\langle \frac{1}{2}v, 1_{\vec{k}-} \left| \sqrt{\hbar\omega^{(3)}}\vec{\Gamma}_\gamma^{(3)}\xi_-^{(3)} \right| -\frac{1}{2}c, \text{vac} \right\rangle \tag{4.3.59}$$

$$i\hbar\dot{c}_{\frac{3}{2}v,1_{\vec{k}-}} = \left\langle \frac{3}{2}v, 1_{\vec{k}-} \left| \sqrt{\hbar\omega^{(4)}}\vec{\Gamma}_\gamma^{(4)}\xi_-^{(4)} \right| \frac{1}{2}c, \text{vac} \right\rangle , \tag{4.3.60}$$

where we have made the substitution $\xi_\pm^{(n)} = \wp_\pm^{(n)}\sigma_\pm^{(n)}e^{i\omega_\pm^{(n)}t}$ for brevity. Explicitly in terms of the PWF for each corresponding state, these are then given by

$$i\hbar\dot{c}_{-\frac{3}{2}v,1_{\vec{k}+}} = \sqrt{\hbar\omega^{(1)}}\left(\vec{\Psi}_{\gamma,-}^{(-)}\right)_{\frac{1}{2}c}^{(1)} \cdot \wp_-^{(1)}e^{i\omega_-^{(1)}t} \tag{4.3.61}$$

$$i\hbar\dot{c}_{-\frac{1}{2}v,1_{\vec{k}+}} = \sqrt{\hbar\omega^{(3)}}\left(\vec{\Psi}_{\gamma,-}^{(-)}\right)_{\frac{1}{2}c}^{(3)} \cdot \wp_-^{(3)}e^{i\omega_-^{(3)}t} \tag{4.3.62}$$

$$i\hbar\dot{c}_{-\frac{1}{2}c} = \sqrt{\hbar\omega^{(1)}}\left(\vec{\Psi}_{\gamma,+}^{(+)}\right)_{-\frac{3}{2}v,1_{\vec{k}+}}^{(1)} \cdot \wp_+^{(1)}e^{i\omega_+^{(1)}t} + \sqrt{\hbar\omega^{(2)}}\left(\vec{\Psi}_{\gamma,-}^{(+)}\right)_{\frac{1}{2}v,1_{\vec{k}-}}^{(2)} \cdot \wp_+^{(2)}e^{i\omega_+^{(2)}t} \tag{4.3.63}$$

$$i\hbar\dot{c}_{\frac{1}{2}c} = \sqrt{\hbar\omega^{(3)}}\left(\vec{\Psi}_{\gamma,+}^{(+)}\right)_{-\frac{1}{2}v,1_{\vec{k}+}}^{(3)} \cdot \wp_+^{(3)}e^{i\omega_+^{(3)}t} + \sqrt{\hbar\omega^{(4)}}\left(\vec{\Psi}_{\gamma,-}^{(+)}\right)_{\frac{3}{2}v,1_{\vec{k}-}}^{(4)} \cdot \wp_+^{(4)}e^{i\omega_+^{(4)}t} \tag{4.3.64}$$

$$i\hbar\dot{c}_{\frac{1}{2}v,1_{\vec{k}_-}} = \sqrt{\hbar\omega^{(3)}} \left(\vec{\Psi}_{\gamma,+}^{(-)}\right)_{-\frac{1}{2}c}^{(3)} \cdot \vec{\wp}_-^{(3)} e^{i\omega_-^{(3)}t} \tag{4.3.65}$$

$$i\hbar\dot{c}_{\frac{3}{2}v,1_{\vec{k}_-}} = \sqrt{\hbar\omega^{(4)}} \left(\vec{\Psi}_{\gamma,+}^{(-)}\right)_{\frac{1}{2}c}^{(4)} \cdot \vec{\wp}_-^{(4)} e^{i\omega_-^{(4)}t} . \tag{4.3.66}$$

It is evident that where we had a cubic equation to solve before, we now have a much more complex problem. We do not attempt to solve this problem analytically, though we can benchmark against our previous analytical result. The best approach to solving this problem is the implementation of an algorithm that can handle fast matrix diagonalization and fast Fourier transforms. Such packages for computational methods can be found in suites such as LAPACK, Super-LU, and FFTW [106, 107, 108]. In the next chapter we will apply some numerical techniques to study the questions we have encountered in this chapter.

5 COMPUTATIONAL METHODS, RESULTS, AND CONCLUSIONS

In this chapter we discuss in detail all the computational methods and results that were completed throughout this dissertation. We start by studying an explicit example for quantum sources composed of a mixture of GaAs. All experimental parameters are referenced from the literature and are used to calculate the computational parameters needed to carry out the the model. In order to discuss the analytical solutions to the interacting light matter few level systems the first set of parameters that have to be calculated the three characteristic roots associated with the probability amplitude of the excited state $c_a(t)$. These three poles, which are presented in figure (5.1), need to be carefully considered when solving for the mixed electron-photon probability densities $c_{b,\vec{k},\lambda}(t)$. Each of these poles correspond to emission and/or re-absorption, and model the Rabi oscillations associated with revival phenomena in near field regions. These poles can be evaluated numerically or analytically by application of Demoivre's theorem. In figure (5.1) we present solutions for poles z_n corresponding to transition wavelengths and transition dipole moments in the ranges of 750 - 1300 nm and 20 to 100 Debye [95, 96, 97]. The poles of the contour integrals arrived at from these results and which are associated with the evaluation of $c_a(t)$ are also presented in figure (5.1). Their physical meaning is interpreted from their location in the complex plane, by means of their arguments and absolute magnitudes. The contours for the poles are closed by imposing further constraints as discussed in the previous chapter. The coupling between the equations of motion for the light matter interaction as well as the propagation of the free field was modeled computationally. This calculation relied on the use of the following algorithm as implemented in a leap-frogging scheme [59]. In this scheme we alternate between real and imaginary parts of the wave-functions at half time step intervals. The propagation was run both within and beyond the interaction regions, please see figure (5.2a). This algorithm was implemented in a 3 step process defined as,

1. Determine analytic approximation for c_a and use its leading coefficients and roots to determine the state of the photonic wave function $\vec{\Psi}^{(+)}_{\gamma,+,b}$ & $\vec{\Psi}^{*(+)}_{\gamma,-,b}$ at the next time step

2. Update the state of the quantum dot from the current state of the photonic wave function to

3. Propagate $\vec{\Psi}^{(+)}_{\gamma,+,b}$ & $\vec{\Psi}^{*(+)}_{\gamma,-,b}$ and repeat from step 1

The figure (5.2b) shows a stable near-field exchange of energy between the quantum dot and single photon states. These results are derived from two computational experiments set to have the same ratio of spatial finite differences to temporal ones, $\frac{\Delta x}{\Delta t}$. It is interesting to note that the envelope functions for both of these oscillations of the probability amplitudes are very similar and both exhibit a seemingly linear decay of energy shortly after an initial revival of the quantum dot state. This seems to suggest that though it is worth while to retain a time resolution small enough to observe the initial revival event, this resolution can be made coarser almost immediately following the first revival. In the next couple of sections we will outline, the scale of the calculation conducted, the parameters used during the execution of the model, and the explicit algorithm of the model. We will conclude the chapter and this dissertation in the section dedicated to future work that can extend the work carried out herein.

5.1 The problem of scale

Determining the scale of the problem is imperative when seeking stability conditions for this computational model while retaining physical reality. For the case of the poles which determine the time scales of the interaction we find that these are dependent on the strength of the transition dipole moments and the transition frequencies associated with the energy band gap of the system under study. The time scales associated with the characteristic roots of the problem are close to those time scales that can be found in the literature. These correspond to the order of $10^{-15}s$ for quantum dots composed of InGaAs and GaAs. As mentioned previously, experimentally measured values for transition wavelengths and dipole moments are in the range of 750 - 1300 nm and 20 to 100 Debye[95, 96, 97]. In our model the units of the poles z_n are arrived at from the definition of

the parameter λ, which has units of $\frac{[D]^2}{[m/s]^3}$. For clarity we reiterate the expression for this parameter, which is given by the expression $\lambda \equiv \frac{1}{4\pi} \frac{1}{3v_0^3} \left[2\vec{\wp}_{ab,x}^2 + 2\vec{\wp}_{ab,y}^2 + \vec{\wp}_{ab,z}^2 \right]$. These can be translated to our time-scale by making a series of conversions. To start off, we decide to work in of two sets of units, either SI or cgs units. for the cast of cgs units, a Debye is given by [71] the definition of the unit as

$$[D] = 10^{-18}\, statC{\cdot}cm$$
$$= \frac{1}{c} \times 10^{-21}\, [C] \left[\frac{m^2}{s} \right]$$

By taking the square of this unit we can avoid any future fractional power of units. Doing so we are left with the following set of units.

$$[D]^2 = 10^{-22}\, statC^2 \cdot nm^2$$

Since the stat-Coulomb, statC is defined in cgs units by a unity of the product of the three basic units in the unit system we find that,

$$statC^2 = cm^3 g^1 s^{-2}$$
$$= 10^{-9}\, m \cdot \frac{kg m^2}{s^2}$$
$$= nm \cdot J$$
$$= \left(\frac{1}{1.602176487 \times 10^{-19}} \right) nm \cdot eV\,.$$

In the calculation carried out above we were seeking units of electron-volt nano-meters. By using those units of measurement we can directly compare our results, as well as use parameters derived from the literature. In this way we use the standard of the experimental papers that provide the experimental parameters necessary for our model. Finishing our the unit conversion in the desired

units, we see that the basic unit of the Debye in electron-volt nanometers is given by,

$$[D]^2 = 10^{-22} \left(\frac{1}{1.602176487 \times 10^{-19}} \right) eV \cdot nm^3$$

$$= 6.24150965 \times 10^{-4} \, eV \cdot nm^3 \, .$$

Optionally we could also work in SI units. In these units we use the formal definition of the coulomb to check our result above. We follow pretty much the same approach as before, except this time expanding the units in terms of Coulombs, meters, and seconds. This calculation is given below,

$$[D]^2 = \left(2.99792458 \times 10^8 \left[\frac{m}{s} \right] \right)^{-2} \times 10^{-42} \, [C]^2 \left[\frac{m^2}{s} \right]^2$$

$$= 11.1265006 \times 10^{-60} \, [C]^2 \left[m^2 \right]$$

$$= 11.1265006 \times 10^{-42} \, [C]^2 \, [nm]^2 \, .$$

A third possibility is to convert between Coulombs and stat-Coulombs to take advantage of the conversion for a coulomb squared, where $statC \leftrightarrow 3.33564095 \times 10^{-10} C = cm^{\frac{3}{2}} g^{\frac{1}{2}} s^{-1}$. This will allow a triple check to make sure that these unit conversion match the standard. We now focus purely on the calculation of a Coulomb squared, as given in cgs units by,

$$\left(3.33564095 \times 10^{-10} C \right)^2 = cm^3 g^1 s^{-2}$$

$$= 10^{-9} m \cdot \frac{kg m^2}{s^2}$$

$$= nm \cdot J$$

$$= \left(\frac{1}{1.602176487 \times 10^{-19}} \right) nm \cdot eV \, .$$

In terms of the basic units we are after, the Coulomb squared is given by the unit conversion,

$$1 C^2 = 5.60958911 \times 10^{37} nm \cdot eV \, .$$

Therefore, we can represent the square of a Debye in the desired units,

$$[D]^2 = 11.1265006 \times 10^{-42} * 5.60958911 \times 10^{37} nm^3 \cdot eV$$

$$= (11.1265006)(5.60958911) \times 10^{-5} [nm]^3 \cdot eV$$

$$= 6.24150966 \times 10^{-4} nm^3 \cdot eV.$$

This calculation gives almost an exact agreement to 7 decimal places. Even though the numbers are not exactly the same, we are with the desired level of accuracy. We therefore take the unit of a Debye from the square root of its square to be given by the constant unit conversion,

$$[D] = 2.498301353 \times 10^{-2} nm^{\frac{3}{2}} \cdot eV^{\frac{1}{2}}.$$

In the next step we calculate the units of the parameter λ in anticipation of calculating the characteristic roots we are after. Using the definition of this parameter from previous sections, we know that this parameter has units of,

$$\frac{[D]^2}{[m/s]^3} = \frac{6.24150966 \times 10^{-4} nm^3 \cdot eV}{1 \times 10^{27} nm^3} s^3$$

$$= 6.24150966 \times 10^{-31} eV \cdot s^3.$$

Since there is an additional factor of $\hbar$ associated with the calculation, we present plots in units of $\frac{1}{\hbar} \frac{[D]^2}{[m/s]^3}$. These units are of the order defined by,

$$\frac{1}{\hbar} \frac{[D]^2}{[m/s]^3} = \frac{6.24150966 \times 10^{-31} eV \cdot s^3}{6.58211899 \times 10^{-16} eV \cdot s}$$

$$= 0.948252329 \times 10^{-15} s^2$$

$$= 948.252329 \times 10^{-18} s^2$$

$$= (30.7937060\, ns)^2.$$

For a range transition dipole moment values between $20 - 30\ Debye$, we find that our λ parameter lies in the range of the following values,

$$\frac{\lambda}{\hbar} = \frac{1}{4\pi}\frac{1}{3}\frac{1}{(2.99792458 \times 10^8)^3} \begin{cases} (20)^2 * 948.252329 \times 10^{-18}s^2, & @20Debye \\ (100)^2 * 948.252329 \times 10^{-18}s^2, & @100Debye \end{cases}$$

$$\begin{cases} 9.84480000 \times 10^{-28} * (20)^2 * 948.252329 \times 10^{-18}s^2, & @20Debye \\ 9.84480000 \times 10^{-28} * (100)^2 * 948.252329 \times 10^{-18}s^2, & @35Debye \end{cases}$$

$$= \begin{cases} 3.73414181 \times 10^{-40}s^2, & @20Debye \\ 9.33535453 \times 10^{-39}s^2, & @35Debye \end{cases}.$$

Additionally, the poles z_n also depend on the time scales of ω_σ. For transitions associated with the transition dipole moments mentioned above, these are transition frequencies are in the range of 750 - to 1300 nm. Assuming that the wavelength measurements are made near vacuum, we can use the corresponding relationship between the wave-length and transition frequency as given by, $\omega_\sigma = 2\pi\frac{c}{\lambda_\sigma}$

$$\omega_\sigma = \begin{cases} 2.51153540 \times 10^{15}\frac{1}{s}, & @\lambda_\sigma = 750nm \\ 1.44896274 \times 10^{15}\frac{1}{s}, & @\lambda_\sigma = 1300nm \end{cases}.$$

This implies that our transition frequency is in the range of the values above. In addition, the time-scales that correspond to these frequencies are in following range of values, $\tau = \frac{1}{\omega_\sigma}$

$$\tau = \begin{cases} 0.398162813\,fs, & @\lambda_\sigma = 750nm \\ 0.69014887\,fs, & @\lambda_\sigma = 1300nm \end{cases}.$$

From the previous chapter you may recall that the energy also strongly dominated the results of the mode. In this case, the corresponding to energies ,in eV, for the transition frequencies above

are within the range,

$$\hbar\omega_\sigma = \begin{cases} 1.65312249eV, & @\lambda_\sigma = 750nm \\ \\ 0.95372452eV, & @\lambda_\sigma = 1300nm \end{cases}.$$

To test the feasibility of these times scales we compare to the lifetime of a state described in the Weisskopf-Wigner approximation[15]. The approximate lifetime Γ related purely to the first order derivative of c_a is defined by neglecting all higher order derivatives and assuming the complete absence of an initial E-M field as given by

$$\dot{c}_a = -\frac{\Gamma}{2}c_a$$
$$= -\frac{\lambda_{Dip}}{\hbar}\omega_\sigma^3 c_a .$$

As such, the lifetime $\tau = \frac{1}{\Gamma}$ is explicitly given by

$$\Gamma = \frac{2\lambda_{Dip}}{\hbar}\omega_\sigma^3$$

$$\approx \begin{cases} 2 * \left(1.44896274 \times 10^{15}\frac{1}{s}\right)^3 * 3.73414181 \times 10^{-40}s^2, & @20Debye; 1300nm \\ \\ 2 * \left(2.51153540 \times 10^{15}\frac{1}{s}\right)^3 * 9.33535453 \times 10^{-39}s^2, & @100Debye; 1000nm \end{cases}$$

$$= \begin{cases} 2.27191697 \times 10^6\frac{1}{s}, & @20Debye; 1300nm \\ \\ 2.95786755 \times 10^8\frac{1}{s}, & @100Debye; 750nm \end{cases}.$$

or

$$\tau \approx \begin{cases} 440.156931\,ns, & @20Debye; 1300nm \\ \\ 3.38081399\,ns, & @100Debye; 750nm \end{cases}.$$

This is a wide range of times, but it still puts our calculations where we expected them to be from previous studies [109, 110]. The poles of the contour integrals arrived at from these results and

which are associated with the evaluation of $c_a(t)$ are presented in figure (5.1). Their physical meaning is interpreted from their location in the complex plane, by means of their arguments and absolute magnitudes. The contours for the poles are closed by imposing further causal constraints. We interpret these poles to have the following meaning:

- z_0: Corresponds to product state of an incoming photonic state with the QD in the ground state at $t = -\infty$

- z_2: Corresponds to product state of in the absence of a photonic state with the QD in the excited state at $t = 0$

- z_1: Corresponds to product state of an emitted photonic state with the QD in the ground state at $t = \infty$

These interpretations for z_n tell us that for a given time t the solutions of the equations of motion consist of three eigenstates and three energy eigenvalues that corresponding to our three poles. From these characteristic roots we can also calculate what the initial contribution to the free electromagnetic field should be.

5.2   Finite Difference algorithm for propagating a free Maxwell field

In this section we focus on the algorithm that we will use to propagate the free field coupled to the quantum source we are interested in. In many studies, Finite Difference Time Domain (FDTD) models have been used to accurately and efficiently solve Maxwell's equations. This method is very popular for computing and finding physical interpretations for electromagnetic scattering and wave propagation. FDTD has also become common applied to the analyze of microwave circuits [111]. In the model we have studied, tested, and used to develop a theory on the near-field emission effects of few level systems, a scheme similar to FDTD was implemented. In this approach, the FDTD like program was used to propagate a photonic wave-function that was coupled to a quantum source via (C.1.17). These equations of motion for the free fields correspond to the non-relativistic

(a) $\theta\left(z_0 + i\omega_\sigma\right) \approx -15°$

(b) $\theta\left(z_1 + i\omega_\sigma\right) \approx 165°$

(c) $\theta\left(z_2 + i\omega_\sigma\right) \approx -150°$

(d) $\left|z_{0,1} + i\omega_\sigma\right|$

(e) $\left|z_2 + i\omega_\sigma\right|$
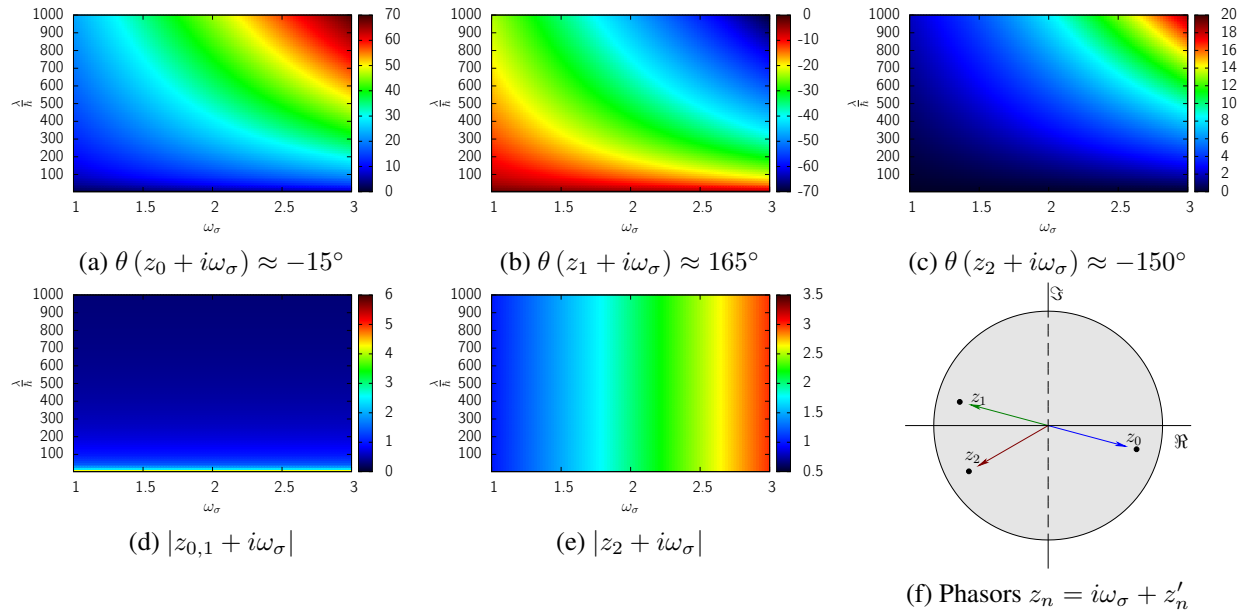
(f) Phasors $z_n = i\omega_\sigma + z'_n$

Figure 5.1: Analytically evaluated poles z_n through use of Demoivre's theorem. Plots present the relative change of the argument θ in arc-seconds $\Delta\theta$" for z_0 & z_1. For z_2 the change in the argument is of the order $\Delta\theta$" $\times 10^{-2}$. Radial length phasors $z_n + i\omega_\sigma$ for z_0 and z_1 are of the order of 10^{19}s^{-1} and for z_2 these are of the order of 10^{15}s^{-1} as plotted with respect to transition frequency ω_σ in 10^{15}s^{-1} and $\frac{\lambda}{\hbar}$ in 10^{-40}s^2. Phasors with negative real components correspond to emission and those with positive real components correspond to revival in near field regions.

approximation through the expressions

$$\left(-ic^{-1}\partial_t + \vec{\nabla}\times \right) \vec{\mathcal{F}}^{(+)} \;=\; 0 \tag{5.2.1}$$

$$\left(-ic^{-1}\partial_t - \vec{\nabla}\times \right) \vec{\mathcal{F}}^{\dagger(+)} \;=\; 0\,. \tag{5.2.2}$$

From the discussion in previous chapters, the interaction is coupled to these fields by means of the general electronic equation of motion 4.3.33.

$$i\hbar\partial_t \left| \gamma\sigma \right\rangle \;=\; \mathcal{V} \left| \gamma\sigma \right\rangle . \tag{5.2.3}$$

Within this formalism it has been presented in previous chapters that these expressions can be applied to solve classes of problems which include, but are not limited to

- a two level quantum source emitting a single photon in linear homogeneous media

- an 6-level quantum source emitting a Maxwell Fields in linear homogeneous media

We now define the explicit finite-differences algorithm. First we will define the expressions that we will use to propagate a free PWF and later we will introduce how to entangle the the PWF to a quantum source.

5.2.1   Free PWF Propagation

In regions where the field is free or in the cases where the field is not coupled to a quantum source, the propagation of the positive energy part of the photonic wave-function takes the form of:

$$i\partial_t \vec{\Psi}_\gamma \;=\; v\nabla \times \vec{\Psi}_\gamma\,,$$

where $\vec{\Psi}_\gamma = \dfrac{1}{\sqrt{2}}\dfrac{1}{\sqrt{\hbar\omega_\sigma}} \left(\dfrac{\varepsilon\vec{\Psi}_{\vec{E}}}{\sqrt{\varepsilon}} + i\dfrac{\mu\vec{\Psi}_{\vec{H}}}{\sqrt{\mu}} \right)$ and $v^{-1} = \sqrt{\epsilon\mu}$. These expressions may be solved computationally by expressing them in a leap-frog finite differences algorithm as is shown herein. Other computational schemes are beyond the scope of this dissertation. The derivation of the

leap-frog algorithm follows from separating the real and imaginary parts of $\vec{\Psi}_\gamma$ and discretizing them over a Cartesian grid. With these objects defined, the leap-frog algorithm , in the absence of a quantum source, is given from $\partial_t \left(-\Im \left\{ \vec{\Psi}_\gamma \right\} \right) = v \vec{\nabla} \times \left(\Re \left\{ \vec{\Psi}_\gamma \right\} \right)$ and $\partial_t \left(i \Re \left\{ \vec{\Psi}_\gamma \right\} \right) = v \vec{\nabla} \times \left(i \Im \left\{ \vec{\Psi}_\gamma \right\} \right)$ which can be explicitly stated in terms of the iterative expressions:

$$
\Im \{\Psi_{\gamma,x}\}_{i,j,k}^{n+\frac{1}{2}} = \Im \{\Psi_{\gamma,x}\}_{i,j,k}^{n-\frac{1}{2}} + \frac{v\Delta t}{2} \left[\frac{1}{\Delta x_j} \left(\Re \{\Psi_{\gamma,z}\}_{i,j+1,k}^{n} - \Re \{\Psi_{\gamma,z}\}_{i,j-1,k}^{n} \right) \right.
$$
$$
\left. - \frac{1}{\Delta x_k} \left(\Re \{\Psi_{\gamma,y}\}_{i,j,k+1}^{n} - \Re \{\Psi_{\gamma,y}\}_{i,j,k-1}^{n} \right) \right]
$$
$$
\Im \{\Psi_{\gamma,y}\}_{i,j,k}^{n+\frac{1}{2}} = \Im \{\Psi_{\gamma,y}\}_{i,j,k}^{n-\frac{1}{2}} - \frac{v\Delta t}{2} \left[\frac{1}{\Delta x_i} \left(\Re \{\Psi_{\gamma,z}\}_{i+1,j,k}^{n} - \Re \{\Psi_{\gamma,z}\}_{i-1,j,k}^{n} \right) \right.
$$
$$
\left. - \frac{1}{\Delta x_k} \left(\Re \{\Psi_{\gamma,x}\}_{i,j,k+1}^{n} - \Re \{\Psi_{\gamma,x}\}_{i,j,k-1}^{n} \right) \right]
$$
$$
\Im \{\Psi_{\gamma,z}\}_{i,j,k}^{n+\frac{1}{2}} = \Im \{\Psi_{\gamma,z}\}_{i,j,k}^{n-\frac{1}{2}} + \frac{v\Delta t}{2} \left[\frac{1}{\Delta x_i} \left(\Re \{\Psi_{\gamma,y}\}_{i+1,j,k}^{n} - \Re \{\Psi_{\gamma,y}\}_{i-1,j,k}^{n} \right) \right.
$$
$$
\left. - \frac{1}{\Delta x_j} \left(\Re \{\Psi_{\gamma,x}\}_{i,j+1,k}^{n} - \Re \{\Psi_{\gamma,x}\}_{i,j-1,k}^{n} \right) \right]
$$

and:

$$
\Re \{\Psi_{\gamma,x}\}_{i,j,k}^{n+1} = \Re \{\Psi_{\gamma,x}\}_{i,j,k}^{n} - v\Delta t \left[\frac{1}{\Delta x_j} \left(\Im \{\Psi_{\gamma,z}\}_{i,j+1,k}^{n+\frac{1}{2}} - \Im \{\Psi_{\gamma,z}\}_{i,j-1,k}^{n+\frac{1}{2}} \right) \right.
$$
$$
\left. - \frac{1}{\Delta x_k} \left(\Im \{\Psi_{\gamma,y}\}_{i,j,k+1}^{n+\frac{1}{2}} - \Im \{\Psi_{\gamma,y}\}_{i,j,k-1}^{n+\frac{1}{2}} \right) \right]
$$
$$
\Re \{\Psi_{\gamma,y}\}_{i,j,k}^{n+1} = \Re \{\Psi_{\gamma,y}\}_{i,j,k}^{n} + v\Delta t \left[\frac{1}{\Delta x_i} \left(\Im \{\Psi_{\gamma,z}\}_{i+1,j,k}^{n+\frac{1}{2}} - \Im \{\Psi_{\gamma,z}\}_{i-1,j,k}^{n+\frac{1}{2}} \right) \right.
$$
$$
\left. - \frac{1}{\Delta x_k} \left(\Im \{\Psi_{\gamma,x}\}_{i,j,k+1}^{n+\frac{1}{2}} - \Im \{\Psi_{\gamma,x}\}_{i,j,k-1}^{n+\frac{1}{2}} \right) \right]
$$
$$
\Re \{\Psi_{\gamma,z}\}_{i,j,k}^{n+1} = \Re \{\Psi_{\gamma,z}\}_{i,j,k}^{n} - v\Delta t \left[\frac{1}{\Delta x_i} \left(\Im \{\Psi_{\gamma,y}\}_{i+1,j,k}^{n+\frac{1}{2}} - \Im \{\Psi_{\gamma,y}\}_{i-1,j,k}^{n+\frac{1}{2}} \right) \right.
$$
$$
\left. - \frac{1}{\Delta x_j} \left(\Im \{\Psi_{\gamma,x}\}_{i,j+1,k}^{n+\frac{1}{2}} - \Im \{\Psi_{\gamma,x}\}_{i,j-1,k}^{n+\frac{1}{2}} \right) \right] .
$$

A similar finite differencing procedure can be implemented for

$$
i\partial_t \vec{\Psi}_\gamma^\dagger = -v\nabla \times \vec{\Psi}_\gamma^\dagger ,
$$

where $\vec{\Psi}_\gamma^\dagger = \dfrac{1}{\sqrt{2}} \dfrac{1}{\sqrt{\hbar \omega_\sigma}} \left(\dfrac{\varepsilon \vec{\Psi}_{\vec{E}}}{\sqrt{\varepsilon}} - i \dfrac{\mu \vec{\Psi}_{\vec{H}}}{\sqrt{\mu}} \right)$, except that the sign in front of v is opposite. Apply-ing this procedure yields another leapfrog algorithm. In the absence of a quantum source the iterative scheme can be defined from the relations $\partial_t \left(-\Im \left\{ \vec{\Psi}_\gamma^\dagger \right\} \right) = -v \vec{\nabla} \times \left(\Re \left\{ \vec{\Psi}_\gamma^\dagger \right\} \right)$ and $\partial_t \left(i \Re \left\{ \vec{\Psi}_\gamma^\dagger \right\} \right) = -v \vec{\nabla} \times \left(i \Im \left\{ \vec{\Psi}_\gamma^\dagger \right\} \right)$. These iterative expressions are explicitly stated below as:

$$\Im \left\{ \Psi_{\gamma,x}^\dagger \right\}_{i,j,k}^{n+\frac{1}{2}} = \Im \left\{ \Psi_{\gamma,x}^\dagger \right\}_{i,j,k}^{n-\frac{1}{2}} - \frac{v \Delta t}{2} \left[\frac{1}{\Delta x_j} \left(\Re \left\{ \Psi_{\gamma,z}^\dagger \right\}_{i,j+1,k}^n - \Re \left\{ \Psi_{\gamma,z}^\dagger \right\}_{i,j-1,k}^n \right) \right.$$
$$\left. - \frac{1}{\Delta x_k} \left(\Re \left\{ \Psi_{\gamma,y}^\dagger \right\}_{i,j,k+1}^n - \Re \left\{ \Psi_{\gamma,y}^\dagger \right\}_{i,j,k-1}^n \right) \right]$$
$$\Im \left\{ \Psi_{\gamma,y}^\dagger \right\}_{i,j,k}^{n+\frac{1}{2}} = \Im \left\{ \Psi_{\gamma,y}^\dagger \right\}_{i,j,k}^{n-\frac{1}{2}} + \frac{v \Delta t}{2} \left[\frac{1}{\Delta x_i} \left(\Re \left\{ \Psi_{\gamma,z}^\dagger \right\}_{i+1,j,k}^n - \Re \left\{ \Psi_{\gamma,z}^\dagger \right\}_{i-1,j,k}^n \right) \right.$$
$$\left. - \frac{1}{\Delta x_k} \left(\Re \left\{ \Psi_{\gamma,x}^\dagger \right\}_{i,j,k+1}^n - \Re \left\{ \Psi_{\gamma,x}^\dagger \right\}_{i,j,k-1}^n \right) \right]$$
$$\Im \left\{ \Psi_{\gamma,z}^\dagger \right\}_{i,j,k}^{n+\frac{1}{2}} = \Im \left\{ \Psi_{\gamma,z}^\dagger \right\}_{i,j,k}^{n-\frac{1}{2}} - \frac{v \Delta t}{2} \left[\frac{1}{\Delta x_i} \left(\Re \left\{ \Psi_{\gamma,y}^\dagger \right\}_{i+1,j,k}^n - \Re \left\{ \Psi_{\gamma,y}^\dagger \right\}_{i-1,j,k}^n \right) \right.$$
$$\left. - \frac{1}{\Delta x_j} \left(\Re \left\{ \Psi_{\gamma,x}^\dagger \right\}_{i,j+1,k}^n - \Re \left\{ \Psi_{\gamma,x}^\dagger \right\}_{i,j-1,k}^n \right) \right]$$

and:

$$\Re \left\{ \Psi_{\gamma,x}^\dagger \right\}_{i,j,k}^{n+1} = \Re \left\{ \Psi_{\gamma,x}^\dagger \right\}_{i,j,k}^{n} + v \Delta t \left[\frac{1}{\Delta x_j} \left(\Im \left\{ \Psi_{\gamma,z}^\dagger \right\}_{i,j+1,k}^{n+\frac{1}{2}} - \Im \left\{ \Psi_{\gamma,z}^\dagger \right\}_{i,j-1,k}^{n+\frac{1}{2}} \right) \right.$$
$$\left. - \frac{1}{\Delta x_k} \left(\Im \left\{ \Psi_{\gamma,y}^\dagger \right\}_{i,j,k+1}^{n+\frac{1}{2}} - \Im \left\{ \Psi_{\gamma,y}^\dagger \right\}_{i,j,k-1}^{n+\frac{1}{2}} \right) \right]$$
$$\Re \left\{ \Psi_{\gamma,y}^\dagger \right\}_{i,j,k}^{n+1} = \Re \left\{ \Psi_{\gamma,y}^\dagger \right\}_{i,j,k}^{n} - v \Delta t \left[\frac{1}{\Delta x_i} \left(\Im \left\{ \Psi_{\gamma,z}^\dagger \right\}_{i+1,j,k}^{n+\frac{1}{2}} - \Im \left\{ \Psi_{\gamma,z}^\dagger \right\}_{i-1,j,k}^{n+\frac{1}{2}} \right) \right.$$
$$\left. - \frac{1}{\Delta x_k} \left(\Im \left\{ \Psi_{\gamma,x}^\dagger \right\}_{i,j,k+1}^{n+\frac{1}{2}} - \Im \left\{ \Psi_{\gamma,x}^\dagger \right\}_{i,j,k-1}^{n+\frac{1}{2}} \right) \right]$$
$$\Re \left\{ \Psi_{\gamma,z}^\dagger \right\}_{i,j,k}^{n+1} = \Re \left\{ \Psi_{\gamma,z}^\dagger \right\}_{i,j,k}^{n} + v \Delta t \left[\frac{1}{\Delta x_i} \left(\Im \left\{ \Psi_{\gamma,y}^\dagger \right\}_{i+1,j,k}^{n+\frac{1}{2}} - \Im \left\{ \Psi_{\gamma,y}^\dagger \right\}_{i-1,j,k}^{n+\frac{1}{2}} \right) \right.$$
$$\left. - \frac{1}{\Delta x_j} \left(\Im \left\{ \Psi_{\gamma,x}^\dagger \right\}_{i,j+1,k}^{n+\frac{1}{2}} - \Im \left\{ \Psi_{\gamma,x}^\dagger \right\}_{i,j-1,k}^{n+\frac{1}{2}} \right) \right] .$$

The stability of this approach is identical to the Courant condition [59, 60]. The computational limit is then determined by the amount of physical memory and computational cycles available to the program. A rule of thumb when modeling devices requires that the finite differences should

be at least tone tow orders of magnitude smaller than the smallest feature in the computation. Since our smallest feature is of the order of one nano-meter $1\,\mathrm{nm}$, using the Courant condition $\Delta t < c^{-1}\Delta x$. This implies that the largest time step we can take is of the order of $\Delta t_{\mathrm{Max}} = [3 \times 10^8]^{-1} [0.06 \times 10^{-9}]\,\mathrm{s}$, or $\Delta t_{\mathrm{Max}} = 2.0 \times 10^{-19}\mathrm{s}$. In our models we traditionally use time scales of the order of $\Delta t_{\mathrm{Max}} = 1.0 \times 10^{-20}\mathrm{s}$. This way we can satisfy both the fast oscillations of the optical field, which are of the order of $\Delta t = 10^{-15}s$ and the characteristic roots which are also of the order of $\Delta t = 10^{-19}s$.

5.2.2   PWF and Quantum Source Entanglement

In previous chapters we couple the PWF to a quantum source analytically. By applying another computational scheme similar to the one presented above, we now show that it is possible to work in the interaction picture and define an equation of motion for the interaction 4.3.33 to be evaluated computationally. To couple the FDTD description of the PWF propagation defined above to a quantum source it is now necessary to have an explicit definition of this interaction term. In the discussion presented below we describe how to numerically solve/integrate the expression

$$\left(-ic^{-1}\partial_t + \vec{\nabla}\times \right) \vec{\Psi}_\gamma = 0 \tag{5.2.4}$$

$$\left(-ic^{-1}\partial_t - \vec{\nabla}\times \right) \vec{\Psi}_\gamma^\dagger = 0\,, \tag{5.2.5}$$

when it is coupled to a quantum source. The main idea depends on the same procedure we used previously to develop a leap-frog algorithm.

5.2.2.1   One Two Level Quantum Source emitting a Maxwell Field in Linear Homogeneous Media

We begin by separating the real and imaginary parts of this expression. In the case of a dipole non-relativistic approximation, the interaction Hamiltonian for this system was shown to yield the

coupled equations

$$
\begin{aligned}
i\hbar \dot{c}_a(t) &= \sqrt{\hbar\omega_\sigma}\left(\vec{\Psi}_{\gamma,+}^{(+)} + \vec{\Psi}_{\gamma,-}^{\dagger(+)}\right)_{,b} e^{i\omega_\sigma t} \cdot \vec{\wp}_{ba} \\
i\hbar \sum_{\vec{k},\lambda} \dot{c}_{b,\vec{k},\lambda}(t) &= \sqrt{\hbar\omega_\sigma}\left(\vec{\Psi}_{\gamma,+}^{(-)} + \vec{\Psi}_{\gamma,-}^{\dagger(-)}\right)_{,a} e^{-i\omega_\sigma t} \cdot \vec{\wp}_{ab}\,.
\end{aligned}
$$

These coupled expression can be numerically expressed by separating them into their real and imaginary components and discretizing them over a Cartesian grid. With these defined the algorithm is given from

- Excited State Probability Density:

  - $\partial_t\left(-\Im\{c_a\}\right) = \frac{1}{\hbar}\Re\left\{\sqrt{\hbar\omega_\sigma}\left(\vec{\Psi}_{\gamma,+}^{(+)} + \vec{\Psi}_{\gamma,-}^{\dagger(+)}\right)_{,b} e^{i\omega_\sigma t} \cdot \vec{\wp}_{ba}\right\}$

  - $\partial_t\left(i\Re\{c_a\}\right) = i\frac{1}{\hbar}\Im\left\{\sqrt{\hbar\omega_\sigma}\left(\vec{\Psi}_{\gamma,+}^{(+)} + \vec{\Psi}_{\gamma,-}^{\dagger(+)}\right)_{,b} e^{i\omega_\sigma t} \cdot \vec{\wp}_{ba}\right\}$

- Optical State Probability Density:

  - $\partial_t\left(-\Im\left\{\sum_{\vec{k},\lambda} c_{b,\vec{k},\lambda}\right\}\right) = \frac{1}{\hbar}\Re\left\{\sqrt{\hbar\omega_\sigma}\left(\vec{\Psi}_{\gamma,+}^{(-)} + \vec{\Psi}_{\gamma,-}^{\dagger(-)}\right)_{,a} e^{-i\omega_\sigma t} \cdot \vec{\wp}_{ab}\right\}$

  - $\partial_t\left(i\Re\left\{\sum_{\vec{k},\lambda} c_{b,\vec{k},\lambda}\right\}\right) = i\frac{1}{\hbar}\Im\left\{\sqrt{\hbar\omega_\sigma}\left(\vec{\Psi}_{\gamma,+}^{(-)} + \vec{\Psi}_{\gamma,-}^{\dagger(-)}\right)_{,a} e^{-i\omega_\sigma t} \cdot \vec{\wp}_{ab}\right\}$

The time discretization for the coupling between the quantum source and Maxwell Field states are given for the excited and vacuum states are

$$
\begin{aligned}
\Im\{c_a\}_{i,j,k}^{n+1} &= \Im\{c_a\}_{i,j,k}^{n-1} - \frac{\Delta t}{\hbar}\Re\left\{\sqrt{\hbar\omega_\sigma}\left(\vec{\Psi}_{\gamma,+}^{(+)} + \vec{\Psi}_{\gamma,-}^{\dagger(+)}\right)_{,b} e^{i\omega_\sigma t} \cdot \vec{\wp}_{ba}\right\}_{i,j,k}^{n} \\
\Re\{c_a\}_{i,j,k}^{n+1} &= \Re\{c_a\}_{i,j,k}^{n-1} + \frac{\Delta t}{\hbar}\Im\left\{\sqrt{\hbar\omega_\sigma}\left(\vec{\Psi}_{\gamma,+}^{(+)} + \vec{\Psi}_{\gamma,-}^{\dagger(+)}\right)_{,b} e^{i\omega_\sigma t} \cdot \vec{\wp}_{ba}\right\}_{i,j,k}^{n}\,.
\end{aligned}
$$

The time discretization for the coupling between the quantum source and the Maxwell Field state given for the ground state and photonic state are

$$\Im\left\{\sum_{\vec{k},\lambda} c_{b,\vec{k},\lambda}\right\}_{i,j,k}^{n+1} = \Im\left\{\sum_{\vec{k},\lambda} c_{b,\vec{k},\lambda}\right\}_{i,j,k}^{n-1} - \frac{\Delta t}{\hbar}\Re\left\{\sqrt{\hbar\omega_\sigma}\left(\vec{\Psi}_{\gamma,+}^{(-)} + \vec{\Psi}_{\gamma,-}^{\dagger(-)}\right)_{,a} e^{-i\omega_\sigma t} \cdot \vec{\wp}_{ab}\right\}_{i,j,k}^{n}$$

$$\Re\left\{\sum_{\vec{k},\lambda} c_{b,\vec{k},\lambda}\right\}_{i,j,k}^{n+1} = \Re\left\{\sum_{\vec{k},\lambda} c_{b,\vec{k},\lambda}\right\}_{i,j,k}^{n-1} + \frac{\Delta t}{\hbar}\Im\left\{\sqrt{\hbar\omega_\sigma}\left(\vec{\Psi}_{\gamma,+}^{(-)} + \vec{\Psi}_{\gamma,-}^{\dagger(-)}\right)_{,a} e^{-i\omega_\sigma t} \cdot \vec{\wp}_{ab}\right\}_{i,j,k}^{n},$$

(each component must be set independently). Studying the expressions that determine c_a and $\sum_{\vec{k},\lambda} c_{b,\vec{k},\lambda}$ one can show that tone way to start the procedure is by either setting an initial value for c_a or by bringing an incident field near c_a. Since this state will act as the initial quantum source of any photon/s, then we just need to let the system evolve naturally. Since c_a is initially determined then all the terms $\sum_{\vec{k},\lambda} c_{b,\vec{k},\lambda}$ will be subsequently computed and updated from that term. Moreover, the update of $\sum_{\vec{k},\lambda} c_{b,\vec{k},\lambda}$ is a direct update to the PWF that is being emitted or absorbed by the quantum source. Similarly, this term is also coupled to the ground state of the few-level system and would, if required, enter into a more complicated calculation of the modified electronic structure. We can show that from the definitions $\vec{\Psi}_{\gamma,+,a}^{(-)} = \sum_{\vec{k},\lambda} \langle b1_{\vec{k}}| \vec{\Gamma}_\gamma \sigma_- c_a(t)|a0\rangle$ and $\vec{\Psi}_{\gamma,-,a}^{\dagger(-)} = \sum_{\vec{k},\lambda} \langle b1_{\vec{k}}| \vec{\Gamma}_\gamma^\dagger \sigma_- c_a(t)|a0\rangle$, any change to $c_{b,\vec{k},\gamma}$ will result in a delayed change to c_a. However, the discussion on recalculating the electronic spatial wave-function from changes to the spatial profile of the electromagnetic field is beyond the scope of this dissertation.
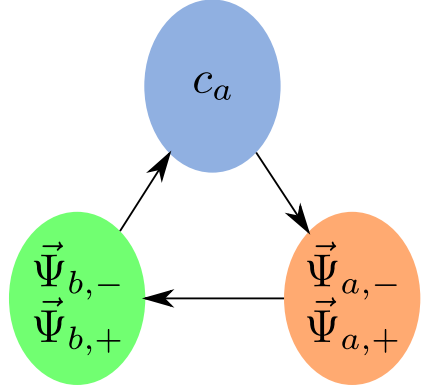
(a) Computational Algorithm Diagram
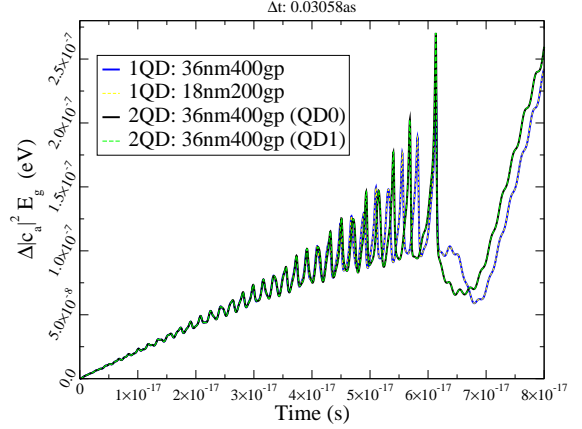
(b) Near field revival

Figure 5.2: Computational algorithm and near field revival phenomena. Near field oscillations are represented in terms of energy exchange $\Delta E \equiv E_g \left(1 - |c_a|^2\right)$ between the two and four QD states and their spontaneously emitted PWF. Envelope functions bounding the region of coherent oscillation behave as low order polynomials. The polynomial behavior of the envelope functions is contradictory to the expectation of exponential behavior with characteristic times of the order of the roots z_n for c_a. For the case of two QDs a phase shift of $\frac{\pi}{2}$ in revival oscillations is observed with respect to those present in one QD.

5.2.2.2 Determining Coupled Maxwell Field & Quantum Source at times between the fixed time grid values

This difficulty can be circumvented by interpolating the missing values at the desired time. One such method which we implement here is simple linear interpolation.

$$\begin{aligned}
\Im\{z\}^n &= \Im\{z\}^{n-\frac{1}{2}} + \frac{1}{2\Delta t}\left(\Im\{z\}^{n+\frac{1}{2}} - \Im\{z\}^{n-\frac{1}{2}}\right) \\
\Re\{z\}^{n+\frac{1}{2}} &= \Re\{z\}^n + \frac{1}{2\Delta t}\left(\Re\{z\}^{n+1} - \Re\{z\}^n\right).
\end{aligned}$$

5.2.2.3 Updating the excited state probability density without the Computational Fourier Transform

In describing the interaction between a quantum source its spontaneously generated Maxwell Field we found that the evolution of the excited state is given for a known field according to equation (4.3.39). Discretizing this expression we find that we can compute the next time step value of

the excited state $c_a (t + \delta t)$. Therefore, we will compute the time evolution of c_a over a small time interval δt according to the discrete form of time derivative expressions where the first order derivative is centered in the interval $t \in \left[t + \frac{\delta t}{2}, t - \frac{\delta t}{2} \right]$. To do so we will need to make use the discretized forms of the first-

$$f^{(1)} (t) = \frac{f \left(t + \frac{\delta t}{2} \right) - f \left(t - \frac{\delta t}{2} \right)}{\delta t},$$

second-

$$\begin{aligned} f^{(2)} (t) &= \frac{f^{(1)} \left(t + \frac{\delta t}{2} \right) - f^{(1)} \left(t - \frac{\delta t}{2} \right)}{\delta t} \\ &= \frac{f (t + \delta t) - 2 f (t) + f (t - \delta t)}{(\delta t)^2}, \end{aligned}$$

and third- order derivatives

$$\begin{aligned} f^{(3)} (t) &= \frac{f^{(2)} (t + \delta t) - f^{(2)} (t - \delta t)}{\delta t} \\ &= \frac{f \left(t + \frac{3}{2} \delta t \right) - 3 f \left(t + \frac{\delta t}{2} \right) + 3 f \left(t - \frac{\delta t}{2} \right) - f \left(t - \frac{3}{2} \delta t \right)}{(\delta t)^3}. \end{aligned}$$

We then have to substitute these expressions into (4.3.39) (repeated here for convenience)[1]

$$c_a^{(3)} + 3 i \omega_\sigma c_a^{(2)} - \left(3 \omega_\sigma^2 + i \frac{\hbar}{\lambda} \right) c_a^{(1)} - i \omega_\sigma^3 c_a = -\frac{1}{\lambda} e^{i \omega_\sigma \tau} \left[\vec{\Psi}_{\gamma,b,+}^{(+)} (\tau) + \vec{\Psi}_{\gamma,b,-}^{*(+)} (\tau) \right] \cdot \vec{\wp}_{ba}.$$

From that substitution we find that discretized form of the LHS of this expression is given by

$$\begin{aligned} &\frac{c_a \left(t + \frac{3}{2} \delta t \right) - 3 c_a \left(t + \frac{\delta t}{2} \right) + 3 c_a \left(t - \frac{\delta t}{2} \right) - c_a \left(t - \frac{3}{2} \delta t \right)}{(\delta t)^3} \\ &+ 3 i \omega_\sigma \left(\frac{c_a (t + \delta t) - 2 c_a (t) + c_a (t - \delta t)}{(\delta t)^2} \right) \\ &- \left(3 \omega_\sigma^2 + i \frac{\hbar}{\lambda} \right) \left(\frac{c_a \left(t + \frac{\delta t}{2} \right) - c_a \left(t - \frac{\delta t}{2} \right)}{\delta t} \right) - i \omega_\sigma^3 c_a (t). \end{aligned}$$

---

[1]$\lambda = \frac{1}{4\pi} \frac{1}{3 v_0^3} \left[2 \vec{\wp}_{ab,x}^2 + 2 \vec{\wp}_{ab,y}^2 + \vec{\wp}_{ab,z}^2 \right]$

However, since we do not have information at the time $t + \frac{3}{2}\delta t$, we have to shift the location where we evaluated these derivatives back by δt. The discrete form of these shifted derivatives is given by the expression below.

$$
\frac{c_a\left(t + \frac{\delta t}{2}\right) - 3c_a\left(t - \frac{\delta t}{2}\right) + 3c_a\left(t - \frac{3}{2}\delta t\right) - c_a\left(t - \frac{5}{2}\delta t\right)}{(\delta t)^3}
$$
$$
+ 3i\omega_\sigma\left(\frac{c_a(t) - 2c_a(t - \delta t) + c_a(t - 2\delta t)}{(\delta t)^2}\right)
$$
$$
- \left(3\omega_\sigma^2 + i\frac{\hbar}{\lambda}\right)\left(\frac{c_a\left(t - \frac{\delta t}{2}\right) - c_a\left(t - \frac{3\delta t}{2}\right)}{\delta t}\right) - i\omega_\sigma^3 c_a(t - \delta t) .
$$

In order to be able to couple this solution to the field propagation defined before we need to solving for a time that corresponds to the time when that propagation is set to begin again. The term we need to solve for is given at the time $c_a\left(t + \frac{\delta t}{2}\right)$. Once we have been able to isolate that term we can complete discretized form of (4.3.39) which is given by,

$$
\frac{1}{(\delta t)^3}c_a\left(t + \frac{\delta t}{2}\right) - \frac{3}{(\delta t)^3}c_a\left(t - \frac{\delta t}{2}\right) + \frac{3}{(\delta t)^3}c_a\left(t - \frac{3}{2}\delta t\right) - \frac{1}{(\delta t)^3}c_a\left(t - \frac{5}{2}\delta t\right)
$$
$$
+ \left(\frac{3i\omega_\sigma}{(\delta t)^2}c_a(t) - \frac{6i\omega_\sigma}{(\delta t)^2}c_a(t - \delta t) + \frac{3i\omega_\sigma}{(\delta t)^2}c_a(t - 2\delta t)\right)
$$
$$
- \frac{\left(3\omega_\sigma^2 + i\frac{\hbar}{\lambda}\right)}{\delta t}c_a\left(t - \frac{\delta t}{2}\right) + \frac{\left(3\omega_\sigma^2 + i\frac{\hbar}{\lambda}\right)}{\delta t}c_a\left(t - \frac{3\delta t}{2}\right)
$$
$$
- i\omega_\sigma^3 c_a(t - \delta t) = -\frac{1}{\lambda}e^{i\omega_\sigma\tau}\left[\vec{\Psi}_{\gamma,b,+}^{(+)}(\tau) + \vec{\Psi}_{\gamma,b,-}^{*(+)}(\tau)\right] \cdot \vec{\wp}_{ba} .
$$

This procedure is computationally cheap since it only requires that we retain the history of at least 5 previous time steps. From the previously conducted time steps we can then update the most current time step. This is evident from the fact that algorithmically

$$
c_a^{n+\frac{1}{2}} = 3c_a^{n-\frac{1}{2}} - 3c_a^{n-\frac{3}{2}} + c_a^{n-\frac{5}{2}}
$$
$$
- \left[c_a^n - 2c_a^{n-\frac{2}{2}} + c_a^{n-\frac{4}{2}}\right]3i\omega_\sigma\Delta t
$$
$$
+ \left[c_a^{n-\frac{1}{2}} - c_a^{n-\frac{3}{2}}\right]\left(3\omega_\sigma^2 + i\frac{\hbar}{\lambda}\right)\Delta t^2
$$
$$
+ \left[i\omega_\sigma^3 c_a^{n-\frac{2}{2}} - \frac{1}{\lambda}e^{i\omega_\sigma\tau}\left[\vec{\Psi}_{\gamma,b,+}^{(+)} + \vec{\Psi}_{\gamma,b,-}^{*(+)}\right]^n \cdot \vec{\wp}_{ba}\right]\Delta t^3 .
$$

5.2.2.4 Evaluating other components for comparison to experimental results

Computationally the coupling between the interaction and the propagation of the Maxwell Field occurs through direct substitution of the newly computed and energy normalized PWF vectors $\vec{\mathcal{F}}_b^{(+)}$ and $\vec{\mathcal{F}}_b^{\dagger(+)}$ into their corresponding equations of motion. Subsequent computations of $\vec{\mathcal{F}}_b^{(+)}$, $\vec{\mathcal{F}}_b^{\dagger(+)}$, $\vec{\wp}_{ab}$, ω_σ, and $\vec{\mathcal{F}}_a^{(-)}$, $\vec{\mathcal{F}}_a^{\dagger(-)}$ then depend on their previously computed values. The computation cycle presented above for one two level quantum source emitting a Maxwell Field in linear homogeneous media can be represented by (5.3).
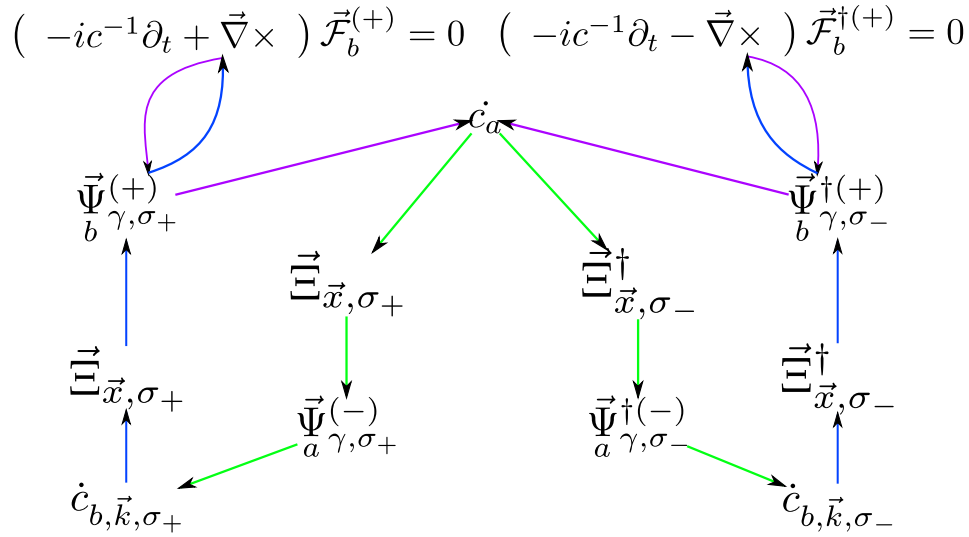
Figure 5.3: Computational Cycle for modeling one - two level quantum source emitting a Maxwell Field in linear homogeneous media.

From these values we can then compute electric, magnetic, and Riemann-Silberstein wave functions. Referring back to the discussion on the PWF, we know that:

- Electric Components are given by:

$$- \vec{\Psi}_{E,b}^{(+)} = \sqrt{\frac{\hbar\omega_\sigma}{2\varepsilon_0}} \left(\vec{\Psi}_{\gamma,+}^{(+)} + \vec{\Psi}_{\gamma,-}^{\dagger(+)} \right)_b$$

$$- \vec{\Psi}_{E,a}^{(-)} = \sqrt{\frac{\hbar\omega_\sigma}{2\varepsilon_0}} \left(\vec{\Psi}_{\gamma,+}^{(-)} + \vec{\Psi}_{\gamma,-}^{\dagger(-)} \right)_a$$

- Magnetic Components are given by:

$$- \vec{\Psi}_{H,b}^{(+)} = -i\sqrt{\frac{\hbar\omega_\sigma}{2\mu_0}} \left(\vec{\Psi}_{\gamma,+}^{(+)} - \vec{\Psi}_{\gamma,-}^{\dagger(+)} \right)_b$$

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(a) 1QD $\Re\{F_x\}_+$ (b) 1QD $\Re\{F_y\}_+$ (c) 1QD $\Re\{F_z\}_+$
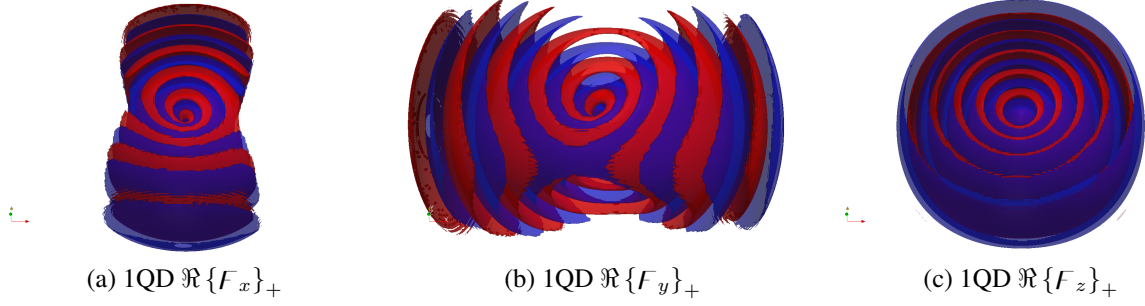
Figure 5.4: Finite difference models of the spontaneous emission of a single photon from a two-level system of quantum dot states. Color intensity in red (positive) and blue (negative) represent the amplitude of the photonic wave-function field strengths.

(a) 2QDs $\Re\{F_x\}_+$ (b) 2QDs $\Re\{F_y\}_+$ (c) 2QDs $\Re\{F_z\}_+$
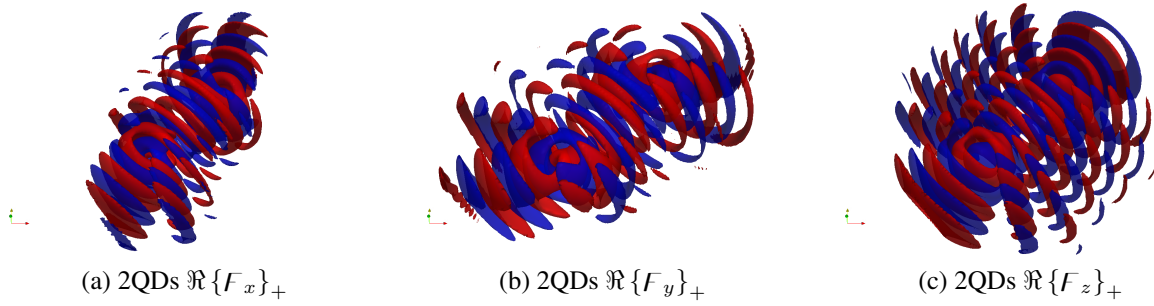
Figure 5.5: Finite difference models of the spontaneous emission of two photons from two 2-level systems of quantum dot states. Color intensity in red (positive) and blue (negative) represent the amplitude of the photonic wave-function field strengths.

$$- \quad \vec{\Psi}_{H,a}^{(-)} = -i\sqrt{\frac{\hbar\omega_\sigma}{2\mu_0}}\left(\vec{\Psi}_{\gamma,+}^{(-)} - \vec{\Psi}_{\gamma,-}^{\dagger(-)}\right)_a$$

- Riemann-Silberstein Vector components are given by:

$$- \quad \vec{\mathcal{F}}_b^{(+)} = \sqrt{\hbar\omega_\sigma}\left(\vec{\Psi}_{\gamma,+}^{(+)}\right)_b$$

$$- \quad \vec{\mathcal{F}}_b^{\dagger(+)} = \sqrt{\hbar\omega_\sigma}\left(\vec{\Psi}_{\gamma,-}^{\dagger(+)}\right)_b$$

5.3 Computational Results and Conclusions

For the case of two quantum sources, the influence of the adjacent QD states is evident in the phase shift of the non-Markovian revival effects observed. Fig. (5.4) presents spatial single photon state components $F_{x,+}$, $F_{y,+}$, and F_{z+} following the establishment of coherent oscillations within the polynomial envelope functions, at a time of 91.74 as, of Fig. (5.2b). The dipolar structure of the isosurfaces presented in Fig. (5.4) was expected due to the initial orientation of the transition dipole

moment solely along the z quantization axis. Further, the interference present in the isosurfaces of Fig. (5.5) were also expected due to the out-of-phase emission associated with the second QD. From these computational results we have demonstrated near-field non-Markovian effects required for the study of single- and multi-photon emission from quantum sources embedded in dielectric structures by means of the Riemann-Silberstein wave-function and finite-difference methods that go beyond the Markovian limit; both analytically and computationally. The bandwidth characteristic frequencies ν_c, of order $\nu_c = 10^{17}$Hz, observed in Fig. (5.2b) are due to revival phenomena in near field regions. These fast oscillations can be understood to arise from the confinement of the photon in the spontaneous emission region. A run-of-the-mill calculation using the Heisenberg uncertainty principle yields that these are correct, since $\Delta x \Delta p > \frac{\hbar}{2}$. The explicit calculation yields that for a quantum source of diameter $\Delta x \approx 10^{-9}$nm the following conditions mus be satisfied,

$$\Delta \nu_k > \frac{1}{2} \frac{c}{\Delta x} \ .$$

Therefore, attributing the near-field oscillations to this bandwith implies that $\nu_c \equiv \Delta \nu_k$, which is[2] $\Delta \nu_k \approx \frac{3}{2} 10^{17}$Hz. It was further demonstrated that the product QD-photon state is highly localized during spontaneous emission while energy is being injected and exchanged between both photon and quantum dot states. Test cases were directly compared for different values of Δx & Δt . From these it was determined that the polynomial envelope functions for coherent oscillations were in agreement within the initial revival period of the quantum dot excited state. Furthermore, the theoretical approximations made provide further analytic and interpretative insight to the periodicity of the initial decay and revival phenomena present in the near field limit. This work provides a recipe for computationally designing and evolving photonic states to be emitted and detected by solid-state quantum sources embedded within dielectric structures and to compare them to experimental results by means of their corresponding density matrix, Wigner functions, and $g^{(2)}$ functions. The model developed for single-photon emission from QFT principles yields the expected emis-

---

[2] It is interesting to note that this bandwidth is of the same order as the oscillatory components (imaginary parts) of the large characteristic roots, with fast decay times, presented in the analysis of the revival terms in Fig. 5.1.

sion profile for a dipole oriented along the quantization axis. The adopted representation of the photon in terms of the Riemann-Silberstein vector was shown to yield a generalized form of Maxwell's equations that facilitates the interpretation of numerical results and convolution of localized photon-exciton fields within near field and free field regions. Results from these models can be used to calculate density matrix elements $\rho_{\sigma\gamma} = |\sigma\gamma\rangle \langle\sigma\gamma|$ for the coupled photon-matter field. These values can then be directly compared to experimental ones. To calculate the density matrix from the results of the mode only the time evolution of the probability amplitudes is necessary. The density matrix for the case of the two level system, with two possible photonic spin states, is shown in the following expression,

$$
\rho_{\sigma\gamma} =
\begin{array}{c|ccc}
 & |a0\rangle & \left|b1_{\vec{k},+}\right\rangle & \left|b1_{\vec{k},-}\right\rangle \\
\hline
\langle 0a| & |c_a|^2 & c_a^* c_{b,\vec{k},+} & c_a^* c_{b,\vec{k},-} \\
\left\langle 1_{\vec{k},+}b\right| & c_{b,\vec{k},+}^* c_a & \left|c_{b,\vec{k},+}\right|^2 & c_{b,\vec{k},-} c_{b,\vec{k},+}^* \\
\left\langle 1_{\vec{k},-}b\right| & c_{b,\vec{k},-}^* c_a & c_{b,\vec{k},-}^* c_{b,\vec{k},+} & \left|c_{b,\vec{k},-}\right|^2
\end{array}
\tag{5.3.1}
$$

The density matrix for the case of the 5 level system, with 4 possible photonic spin states, this is given by the expression

$\rho_{\sigma\gamma} =$

$$
\begin{array}{c|cccccc}
 & \left|\tfrac{3}{2}{}_v 1_{\vec{k},-}\right\rangle & \left|\tfrac{1}{2}{}_v 1_{\vec{k},-}\right\rangle & \left|\tfrac{1}{2}{}_c 0\right\rangle & \left|-\tfrac{1}{2}{}_c 0\right\rangle & \left|-\tfrac{1}{2}{}_v 1_{\vec{k},+}\right\rangle & \left|-\tfrac{3}{2}{}_v 1_{\vec{k},+}\right\rangle \\
\hline
\left\langle \tfrac{3}{2}{}_v 1_{\vec{k},-}\right| & \left|c_{\tfrac{3}{2}{}_v,\vec{k},-}\right|^2 & c_{\tfrac{3}{2}{}_v,\vec{k},-}^* c_{\tfrac{1}{2}{}_v,\vec{k},-} & c_{\tfrac{3}{2}{}_v,\vec{k},-}^* c_{\tfrac{1}{2}{}_c} & c_{\tfrac{3}{2}{}_v,\vec{k},-}^* c_{-\tfrac{1}{2}{}_c} & c_{\tfrac{3}{2}{}_v,\vec{k},-}^* c_{-\tfrac{1}{2}{}_v,\vec{k},+} & c_{\tfrac{3}{2}{}_v,\vec{k},-}^* c_{-\tfrac{3}{2}{}_v,\vec{k},+} \\
\left\langle \tfrac{1}{2}{}_v 1_{\vec{k},-}\right| & c_{\tfrac{1}{2}{}_v,\vec{k},-}^* c_{\tfrac{3}{2}{}_v,\vec{k},-} & \left|c_{\tfrac{1}{2}{}_v,\vec{k},-}\right|^2 & c_{\tfrac{1}{2}{}_v,\vec{k},-}^* c_{\tfrac{1}{2}{}_c} & c_{\tfrac{1}{2}{}_v,\vec{k},-}^* c_{-\tfrac{1}{2}{}_c} & c_{\tfrac{1}{2}{}_v,\vec{k},-}^* c_{-\tfrac{1}{2}{}_v,\vec{k},+} & c_{\tfrac{1}{2}{}_v,\vec{k},-}^* c_{-\tfrac{3}{2}{}_v,\vec{k},+} \\
\left\langle \tfrac{1}{2}{}_c 0\right| & c_{\tfrac{1}{2}{}_c}^* c_{\tfrac{3}{2}{}_v,\vec{k},-} & c_{\tfrac{1}{2}{}_c}^* c_{\tfrac{1}{2}{}_v,\vec{k},-} & \left|c_{\tfrac{1}{2}{}_c}\right|^2 & c_{\tfrac{1}{2}{}_c}^* c_{-\tfrac{1}{2}{}_c} & c_{\tfrac{1}{2}{}_c}^* c_{-\tfrac{1}{2}{}_v,\vec{k},+} & c_{\tfrac{1}{2}{}_c}^* c_{-\tfrac{3}{2}{}_v,\vec{k},+} \\
\left\langle -\tfrac{1}{2}{}_c 0\right| & c_{-\tfrac{1}{2}{}_c}^* c_{\tfrac{3}{2}{}_v,\vec{k},-} & c_{-\tfrac{1}{2}{}_c}^* c_{\tfrac{1}{2}{}_v,\vec{k},-} & c_{-\tfrac{1}{2}{}_c}^* c_{\tfrac{1}{2}{}_c} & \left|c_{-\tfrac{1}{2}{}_c}\right|^2 & c_{-\tfrac{1}{2}{}_c}^* c_{-\tfrac{1}{2}{}_v,\vec{k},+} & c_{-\tfrac{1}{2}{}_c}^* c_{-\tfrac{3}{2}{}_v,\vec{k},+} \\
\left\langle -\tfrac{1}{2}{}_v 1_{\vec{k},+}\right| & c_{-\tfrac{1}{2}{}_v,\vec{k},+}^* c_{\tfrac{3}{2}{}_v,\vec{k},-} & c_{-\tfrac{1}{2}{}_v,\vec{k},+}^* c_{\tfrac{1}{2}{}_v,\vec{k},-} & c_{-\tfrac{1}{2}{}_v,\vec{k},+}^* c_{\tfrac{1}{2}{}_c} & c_{-\tfrac{1}{2}{}_v,\vec{k},+}^* c_{-\tfrac{1}{2}{}_c} & \left|c_{-\tfrac{1}{2}{}_v,\vec{k},+}\right|^2 & c_{-\tfrac{1}{2}{}_v,\vec{k},+}^* c_{-\tfrac{3}{2}{}_v,\vec{k},+} \\
\left\langle -\tfrac{3}{2}{}_v 1_{\vec{k},+}\right| & c_{-\tfrac{3}{2}{}_v,\vec{k},+}^* c_{\tfrac{3}{2}{}_v,\vec{k},-} & c_{-\tfrac{3}{2}{}_v,\vec{k},+}^* c_{\tfrac{1}{2}{}_v,\vec{k},-} & c_{-\tfrac{3}{2}{}_v,\vec{k},+}^* c_{\tfrac{1}{2}{}_c} & c_{-\tfrac{3}{2}{}_v,\vec{k},+}^* c_{-\tfrac{1}{2}{}_c} & c_{-\tfrac{3}{2}{}_v,\vec{k},+}^* c_{-\tfrac{1}{2}{}_v,\vec{k},+} & \left|c_{-\tfrac{3}{2}{}_v,\vec{k},+}\right|^2
\end{array}
$$

$$(5.3.2)$$

These density matrices represent the main interpretation of calculated results. From these we can compare to experimental results and further compute the entanglement present in these few-

level systems. It has been suggested that a photon interacting with two electron spins can establish Greenberger-Horne-Zeilinger entanglement between spin-photon-spin systems [112, 113, 2]. Though there are various measures for entanglement [114, 115], we follow the prescription of previous studies which suggest the use of the quantum von Neumann entropy [116] to calculate quantum entanglement. The von Neumann entropy is given by the expressions [117]

$$S\left(\rho_{\sigma\gamma}\right) \quad = \quad -Tr\left(\rho_{\sigma\gamma}\ln\rho_{\sigma\gamma}\right),$$ (5.3.3)

or

$$S\left(\rho_{\sigma\gamma}\right) \quad = \quad = -\sum_{i}\left(\lambda_i\ln\lambda_i\right),$$ (5.3.4)

where the values λ_i represent the eigen-values of the density matrix, $\rho_{\sigma\gamma}$. Additionally, our results can be used to calculate the Wigner-Weyl, quasi-probability distributions from the probability amplitudes that result from our model. The Wigner-Weyl quasi-distributions can be compared to experimental results by using the same method utilized in experimental studies [118, 119, 14, 86, 17, 18, 87, 51, 16, 120], where the tomography kernel is given by $K\left(x\right) = \frac{1}{2}\int_{-\infty}^{\infty}\left|\xi\right|e^{i\xi x}d\xi$ & $\mathrm{pr}\left(X_\theta,\theta\right) \equiv \langle X_\theta\rangle$

$$W\left(X,P\right) = \frac{1}{2\pi^2}\int_0^\pi\int_{-\infty}^{\infty}d\theta dX_\theta\mathrm{pr}\left(X_\theta,\theta\right)\times K\left(X\cos\theta + P\sin\theta - X_\theta\right).$$ (5.3.5)

5.3.1 Homodyne Tomographic Experimental Comparison

The parameters X_θ in these types of experiments are known as the quadratures of the field and are designated by the rotation matrix[104, 15, 14]

$$\begin{pmatrix} X_\theta \\ P_\theta \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} X \\ P \end{pmatrix},$$ (5.3.6)

where X and P are probability amplitudes of the field, and are given by [104, 15, 14],

$$X = \frac{1}{\sqrt{2}} \left(\hat{b} + \hat{b}^\dagger \right) \tag{5.3.7}$$

$$P = \frac{1}{\sqrt{2}} e^{-i\frac{\pi}{2}} \left(\hat{b} - \hat{b}^\dagger \right) . \tag{5.3.8}$$

The correspondence of these quadratures to the signal field emitted from a weak source (such as a quantum source), is given by,

$$\hat{\vec{E}}_S = i \sum_{\vec{k}} \hat{\epsilon}_{\vec{k}} \mathcal{E}_{\vec{k}} \hat{a}_{\vec{k}} e^{-i\left(\nu_k t - \vec{k} \cdot \vec{r}\right)} + \text{h.c.}$$

In the in the presence of a 50/50 beam splitter, experimentalists mix the signal with a local-oscillator field, given by

$$\vec{E}_{LO} = i \hat{\epsilon}_{LO} \mathcal{E}_{LO} \alpha_{LO} g\left(x, y\right) h\left(t\right) e^{-i\left(\nu_{LO} t - k_{LO} z\right)} + \text{c.c.},$$

and which oscillates at the same frequency as a signal. At the beam splitter the fields undergo the transformation,

$$\begin{pmatrix} \hat{\vec{E}}_1 \\ \hat{\vec{E}}_2 \end{pmatrix} = \begin{pmatrix} \cos 45° & \sin 45° \\ -\sin 45° & \cos 45° \end{pmatrix} \begin{pmatrix} \hat{\vec{E}}_S \\ \vec{E}_{LO} \end{pmatrix} .$$

This transformation yields two beams incident on two detectors at $90°$ from one another. The expression for the field incident on detector 1 is then $\hat{\vec{E}}_1 = \frac{1}{\sqrt{2}} \left(\hat{\vec{E}}_S + \vec{E}_{LO} \right)$ and the one incident on detector 2 is is given by $\hat{\vec{E}}_2 = \frac{1}{\sqrt{2}} \left(\vec{E}_{LO} - \hat{\vec{E}}_S \right)$. By counting the number of clicks at each detector, as expected according to the expressions, where $\hat{N}_1$ corresponds to detector 1 and $\hat{N}_2$ to detector 2,

$$\hat{N}_1 = \int_{V_{\text{det}}} \int_{\Delta t} \frac{\varepsilon_0 V}{c\hbar\omega} \frac{1}{2} \left(\hat{\vec{E}}_S^2 + \vec{E}_{LO}^2 + 2\hat{\vec{E}}_S \cdot \vec{E}_{LO} \right) dt dV ,$$

$$\hat{N}_2 = \int_{V_{\text{det}}} \int_{\Delta t} \frac{\varepsilon_0 V}{c\hbar\omega} \frac{1}{2} \left(\hat{\vec{E}}_S^2 + \vec{E}_{LO}^2 - 2\hat{\vec{E}}_S \cdot \vec{E}_{LO} \right) dt dV \ .$$

Taking the difference of these counts by means of a difference aggregator yields that the measurement will provide information about the quantum statistics of the signal field. The expression for this is given below,

$$\hat{N}_- = \hat{N}_1 - \hat{N}_2 = \int_{V_{\text{det}}} \int_{\Delta t} \frac{\varepsilon_0 V}{c\hbar\omega} \left(2\hat{\vec{E}}_S \cdot \vec{E}_{LO} \right) dt dV \ .$$

In fact, the quadrature it self will by proportional to $\hat{N}_-$ as is evident from the expression [14],

$$\hat{\vec{E}}_S \cdot \vec{E}_{LO} = \left(i \sum_{\vec{k}} \hat{\epsilon}_{\vec{k}} \mathcal{E}_{\vec{k}} \hat{a}_{\vec{k}} e^{-i\left(\nu_k t - \vec{k}\cdot\vec{r}\right)} + \text{h.c} \right) \cdot \left(i\hat{\epsilon}_{LO} \mathcal{E}_{LO} \alpha_{LO} g\left(x,y\right) h\left(t\right) e^{-i\left(\nu_{LO} t - k_{LO} z\right)} + \text{c.c} \right) \ .$$

Simplifying that expression, yields the result

$$\hat{N}_- = \sum_{\vec{k}} C_{\vec{k}} \left(\hat{a}_k \alpha_{LO}^* + \hat{a}_k^\dagger \alpha_{LO} \right) = |\alpha_{LO}| \left(\hat{b} e^{-i\theta} + \hat{b}^\dagger e^{i\theta} \right)$$

$$X_\theta = \frac{1}{\sqrt{2}} \left(\hat{b} e^{-i\theta} + \hat{b}^\dagger e^{i\theta} \right) = \frac{1}{\sqrt{2}} \frac{\hat{N}_-}{|\alpha_{LO}|} \ .$$

This is given by the traditional normalization condition imposed on measurements so that the quadratures satisfy the commutation relation

$$[X, P] = i$$

If the probability amplitude of the signal field is given by the complex quantity $\alpha = |\alpha| e^{i\theta_\alpha}$, the for completeness and clarity we reiterate that the X quadrature is traditionally associated with the real
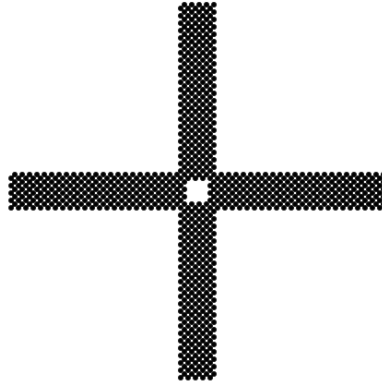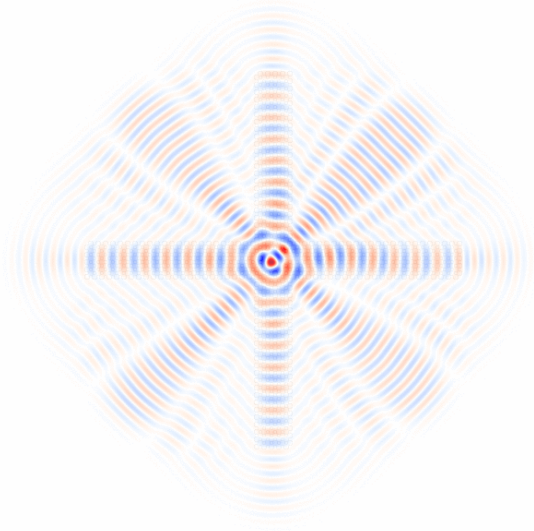
(a) Photonic crystal structure   (b) Photonic crystal structure

Figure 5.6: Quantum source beam splitter embedded in a photonic crystal nano-cavity.

part of the probability amplitude and the P quadrature with the imaginary part [15, 104, 17, 14],

$$
\begin{aligned}
X &= \Re\{\alpha\} \\
P &= \Im\{\alpha\} \ .
\end{aligned}
$$

This concludes the discussion on experimental comparison.

5.4 Future Work

Future work associated with this dissertation should revolve around the implementation of a more realistic model that can describe the interaction directly from electronic structure calculations such as hole entrapment at STP while taking into consideration selection rules of multiple QD states simultaneously. The model for single photon emission should also be improved to provide more realistic effects such as multi-photon states and optical mode generation explicitly in micro- or nano-cavities. However, such calculations will be computationally expensive and therefore necessitate benchmarks to insure code scalability. The description presented in this section will facilitate the proposal of new and enhanced devices that could, for example, rely on electron-hole pair re-

combination for a quantum wire embedded in a cavity. Overall, these effects will determine the entanglement and teleportation of quantum information between QD (or Qubit) states via optical modes by means of a single-photon or multi-photon state. Additionally, future studies along this area could look at Maxwell Field states that carry orbital angular momenta as well as spin. Those conservation laws associated with orbital angular momentum quantum numbers can be studied in detail based on the formalism presented as part of this dissertation. Finally, calculations that model detectors can be build on effective quantum dot beam splitters embedded in photonic crystal nano-cavities and it is the sincere interest of the authors that such an are of research be studied.

6 COMPUTATIONAL IMPLEMENTATION OF QUANTUM SOURCE

In this chapter we describe detail the computational design and structure of the program developed to calculate the integration of the equations of motion for the coupled quantum source and electromagnetic fields. The design of this program implements a finite differencing scheme which requires high resolution finite differences to describe features that can be up to three orders of magnitude smaller than the wavelength of the modeled electromagnetic field. In order to address the high memory and computational requirements associated with the model described in previous chapters, we incorporate a domain decomposition strategy that divides the complete computational domain into a Cartesian grid. This strategy is required to reduce the total wall time associated with job execution at run-time. Without this strategy in would be intractable to solve this problem in a single modern processor. We implement the use of parallel interfaces [121, 122] for managing inter node communication and parallel input and output to storage files [123, 124]. The principal classes of the program are the Maxwell field and the quantum dot. However, these classes depend on secondary classes that define computational parametrization of physical constraints, grid decomposition, and presentation of results.

6.1 Finite Differencing Strategy

It is well known that finite differencing integration techniques are memory and compute efficient and so have been widely applied to computational models of physical electromagnetic phenomena [60]. However, the computational resource requirements for physically viable models depend strongly on the scale of the features of the model under study. In this study the expected and applied difference in scale from the smallest feature to the largest feature spans 3 to 4 orders of magnitude. As mentioned previously, the time scale associated with the model requires spatio-temporal resolutions of the order of $\Delta t \leq 10^{-20}s$ and $\Delta x \leq 10^{-10}m$. These conditions automatically satisfy the Courant condition associated with spatio-temporal numerical techniques [59]. However, the

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discretization of a space which requires a 4 order of magnitude difference results in grid point requirements of the order of $50,000$ grid points per spatial direction. The memory requirement associated with this grid point value is of the order of $50,000^3$ or 125×10^{12} grid points.

Since the field values associated with this type of calculation are complex, each memory component requires 16 bits to store. This implies that any instantiation of the Maxwell field class, in the full calculation, will require approximately $16 \times 125 \times 10^{12}$ bits of storage. This is equivalent to 2×10^{15} bits or ~ 1.78 peta bytes. In a linpack benchmark [125] we conducted of a modern Intel processor, see figure 6.1a, it was determined that the computational ability is of the order of 10^8 floating point operations per second. Since fixed floating point values are represented by 6 bits, this translates to a performance of the order of 10^9 bits or approximately 931 mega bytes per second. This implies that it would take a single node of this type approximately 2×10^6 seconds, or 12 days to complete one iteration of this size. Since we would need 10^{12} iterations, this would be equivalent to approximately 1.2×10^{10} years to complete one run.

Simulating a model of this type of physical phenomena at these scales is completely intractable with the computational hardware configuration discussed. Therefore, the only way to avoid the problem associated with these large time scales is to constrain the domain of the simulation only to the regions of interest and decompose the domain so that multiple computers can work in unison to solve the problem. The domain of the simulation can be constrained by specifying special boundary conditions that behave as perfectly absorbing media and which do not at reflections from the boundaries. These types of boundary conditions, known as perfectly matched layers (PMLs) or absorbing boundary conditions (ABCs) [126, 127] have been widely studied and previously applied.

6.2   Domain Decomposition

As mentioned in the previous section, accurate and precise computational models of physical phenomena require the main domain of the model to be decomposed in to sub-domains so that multiple computers can work in unison to achieve a timely solution of the model. Many computational libraries exist that are designed primarily for the purpose of decomposing structured or unstructured
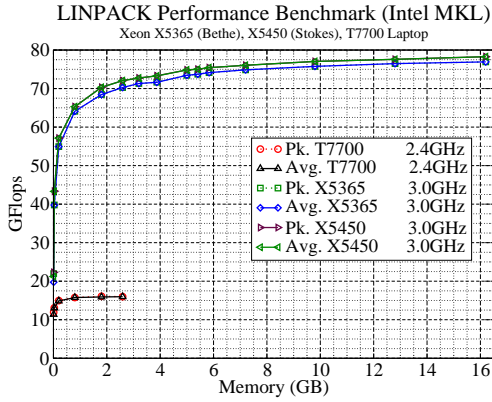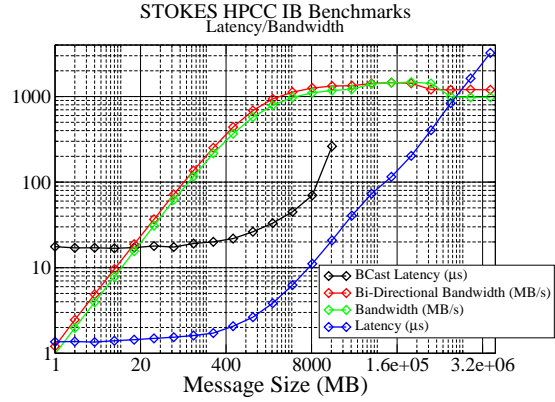
(a) Floating point operations per second   (b) Communication bandwidth and latency

Figure 6.1: Computational hardware benchmarks

grids [128, 129]. In this study we constrained the program to a structured Cartesian grid decomposition of the computational domain. To implement this type of domain decomposition we utilized built in routines MPI_CART_CREATE available in the message passing interface (MPI) standard [130, 122, 121]. The grid decompositions that are utilized with this program are presented in figure 6.2. The type of decomposition is determined by the number of processes that are requested for use at run time. For the case of 1, 2, 3, 4, 5, 6, 7, and 8 processes the grid is decomposed as follows:

1. One large cubic block that encompasses the entire domain

2. Two planar sheets with half the length of the total domain

3. Three planar sheets with one third the length of the total domain

4. Four rectangular prisms with sides $s = \frac{1}{2}L$ and height $h = L$, where L is the total length of the domain

5. Five planar sheets with one fifth the length of the total domain

6. Six planar sheets with one sixth the length of the total domain

7. Seven planar sheets with one seventh the length of the total domain

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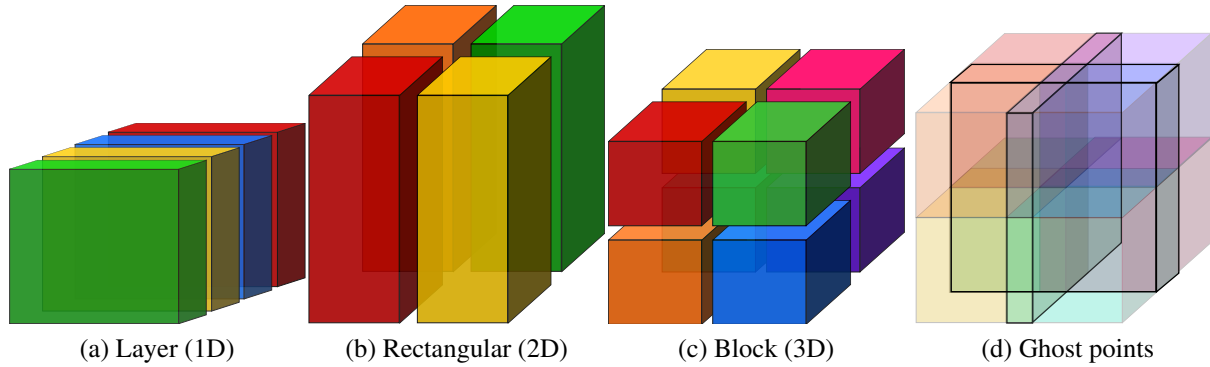
Figure 6.2: Computational grid decomposition types & update skin

8. Eight cubic blocks with sides $s = \frac{1}{2}L$.

As part of the domain decomposition for modeling physical phenomena it is imperative that a continuity equation be satisfied between domains. If this is not the case, un-physical effects, such as artificial reflections, can be observed at the boundaries. This means that at the boundary between sub-domains of the total domain there has to be an update skin which will manage to insure that all the information required for the sub-domain computation is present. One example that helps insure the availability of this information takes advantage of a technique know as the update skin where ghost points are defined in the sub-domain calculation, see figure 6.2d.

Another type of domain decomposition specifies grid regions with higher point densities. In these instances interpolation techniques are used to insure that the information in the sub-grid is not lost to the larger grid [131, 132]. These techniques, though not yet incorporated in this computational implementation, are considered in the mesh class of the design of the program.

6.3 File Handling

The domain decomposition adopted in our computational approach assigns a section of the computational tasks to discrete sets of processing units. To supply the required information to these tasks we are therefore required to distribute computational data across multiple computational nodes. In order to analyze results of the computation it is, in many cases, necessary to unify the distributed data into contiguous files. In the program we have developed we utilize the HDF5 file format [123] to handle parallel input and output from files stored on disk. We use the H5Dwrite routine,

available from the HDF5 libraries, to write the hyper dimensional data structure that represents the electromagnetic into a set of contiguous files. The hyper-dimensional data structure is four dimensional in that it captures the three dimensional spatial distribution of the fields as well as the polarization in terms of the vector components associated with the two possible helicities of the field. The data structure can be expressed mathematically in terms of the following object

$$\vec{\Psi}_\gamma \;\; \equiv \;\; \Psi_{ijk,x}.$$

The dimensions of the object are therefore dependent on the set definitions $i \in \{0, N_x\}$, $j \in \{0, N_y\}$, $k \in \{0, N_z\}$, $x \in \{0, 6\}$. The first three indices represent the number of grid points along the x, y, and z axis. The last index represents the field component associated with the electromagnetic field.

The HDF5 file format can be converted to various file formats. In our case, we utilized the h5utils package, developed by the AbInit group which also develops MEEP and MPB [53], to translate our results from this file format to the commonly used visualization toolkit (VTK) file format [133]. Using this scheme allows visualization of isosurfaces and analysis of data corresponding to the modeled electromagnetic phenomena. In addition to the visualization class implemented in our program, we additionally plotted and quantitatively studied the generated data using the data analysis and visualization Paraview suite [134]. From these isosurface plots we were able to exactly determine interference regions and 3 dimensional field profiles. The data gathered from those results is essential for the calculation of spatially resolved probability densities and temporally resolved probability amplitudes as presented in previous chapters.

6.4   Program Structure

The physical classes of the model are the Maxwell Field, Boundary Conditions, and Quantum Dot. The physical classes of the algorithm represent the implementation of the finite differencing schemes presented in previous chapters. The coupling of these physical classes can be seen in figures 6.4 and 6.5. The quantum dot class incorporates several methods which define how the
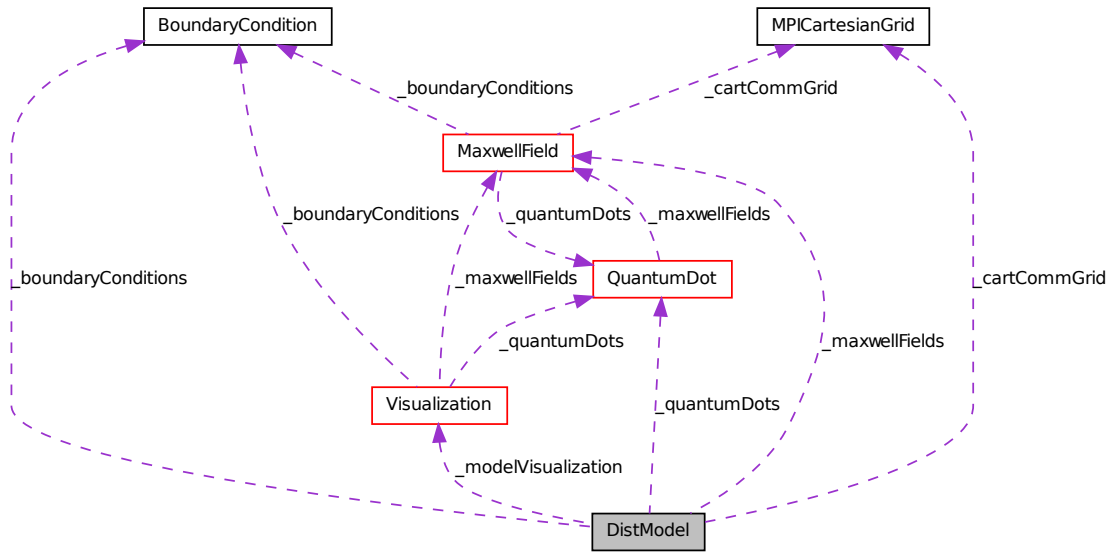
Figure 6.3: Distributed Model
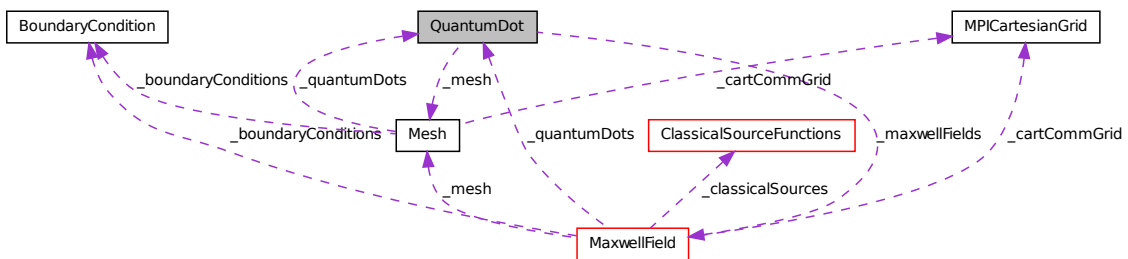
Figure 6.4: Quantum Dot

150
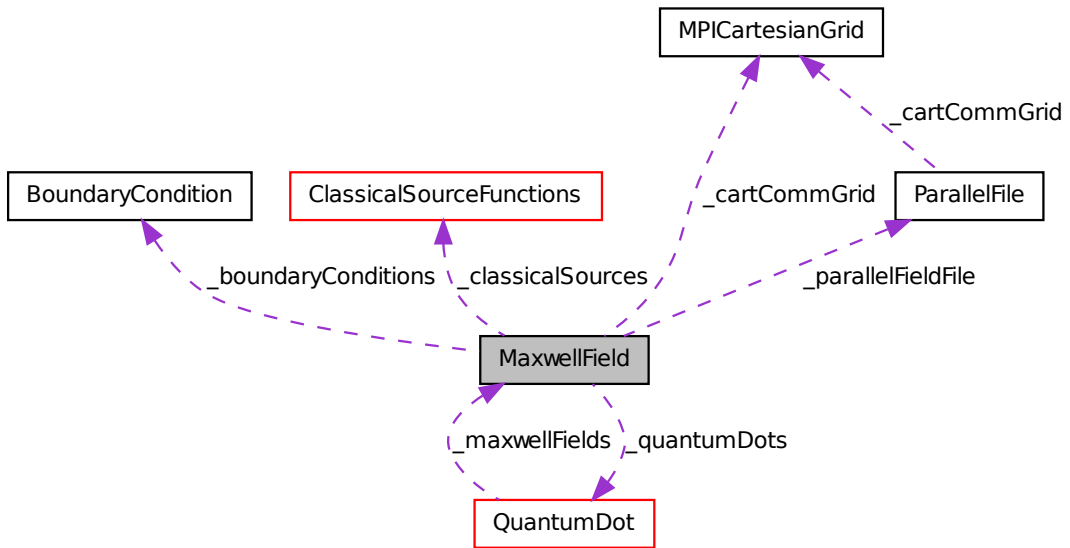
Figure 6.5: Maxwell Field
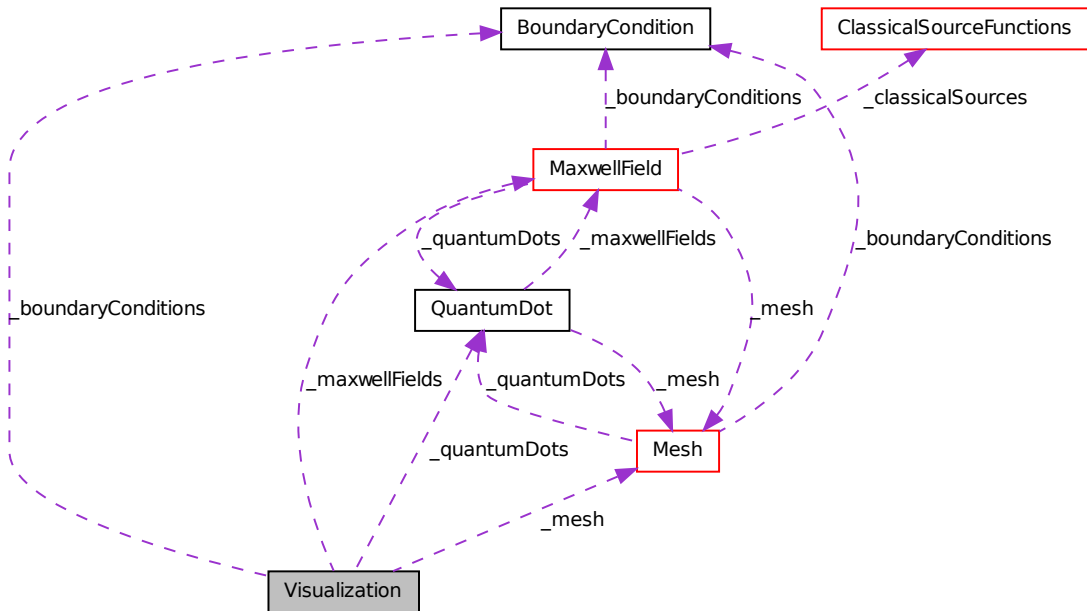
Figure 6.6: Visualization

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Maxwell field defines the quantum source states and vice verse. One such class is the class that takes the Fourier transform of the Maxwell field and couples it to the probability amplitude of the quantum source in the interaction region. As evident from this discussion, the physical classes can not exist independent of other computational and structural classes of the program. For example, the definition of the "interaction region" is defined else where in the program, though attributed to the physical parameters of the quantum source. The Boundary Conditions class defines these essential physical parameters of scale, coupling strengths, and dielectric properties associated with the model. In addition it defines the methods which allow other classes access to this information for construction of the simulation.

The primary computational and structural classes of the model are the distribution grid, the mesh, the visualization, and the distributed model. The mesh defines the differentiation and number of grid points in three dimensions that we associate with the quantization axis of the model. This mesh represents the fixed information that the Maxwell field and Quantum dot use to calculate updates and adjacent spatial values. In addition, the mesh class defines what type of decomposition topology the distribution grid will define. The distribution grid has the primary purpose of using all the information available to it, including memory limits, number of processors, mesh grid points, and specified decomposition topology, to generate a series of sub domains which encompass the original mesh and all its geometric properties. The methods available in the grid distribution class facilitate the access to shared skin updates and recalculation in the Maxwell Field class.

The distributed model class is effectively the complete model which constructs all of these classes at run time and coordinates their call sequence. It starts by populating the configuration variables such as max size and calculating minimum differences from computational constraints such as the Courant condition. Then, it creates the boundary conditions instance which determine the mesh characteristics of the model. Once the mesh is constructed it creates the quantum sources, initializes the available quantum states and incorporates that additional information into the mesh. After the mesh is finalized, the information contained in its instance will be used by the distributed model to construct the distributed grid. The Maxwell field is created only after the distributed grid has

been constructed. Since the Maxwell Field has the largest requirements for memory in the model it can not be instantiated in a single node. The distribution grid class therefore creates all of the distributed instances of the Maxwell Field instance simultaneously and in parallel. This implies then that the quantum sources are also distributed to those instances. Therefore, the interaction regions are perfectly contained within these instances and the only cross-talk that can occur is governed by the transfer of information to the instances of the Maxwell Field class. During development and initial testing of the proposed models, the visualization class provides real time coarse results of the evolution of the model. This technique is invaluable for saving time during development and when first starting the design of boundary conditions associated with the model. The results presented throughout this dissertation depend strongly on the presented program structure.

All conclusions drawn about the nature of the creation of photonic states coupled to few level systems in near field regions could not have been arrived at without the development of these computational techniques. It is the sincere hope of the authors that in the near future this program will be extended from its current form to include additional computational methods that can incorporate hundreds of few level systems coupled by means of highly occupied photonic states.

APPENDIX: MATHEMATICAL TRACTS

A Quantization of Maxwell's Equations

The relation between Maxwell's Equations for these and their classical counter parts are best given in the tabular format of the table below. (Please see (C.4)) To construct the PWF we take the quantum form of Maxwell's equations from the table and re-write

$$\vec{\nabla} \times \left[\langle \gamma | \left(\vec{E}^{(+)}(\vec{r},t) + \vec{E}^{(-)}(\vec{r},t) \right) | \gamma \rangle \right] = -\mu_0 \partial_t \left[\langle \gamma | \left(\vec{H}^{(+)}(\vec{r},t) + \vec{H}^{(-)}(\vec{r},t) \right) | \gamma \rangle \right] \tag{A.0.1}$$

$$\vec{\nabla} \times \left[\langle \gamma | \left(\vec{H}^{(+)}(\vec{r},t) + \vec{H}^{(-)}(\vec{r},t) \right) | \gamma \rangle \right] = \varepsilon_0 \partial_t \left[\langle \gamma | \left(\vec{E}^{(+)}(\vec{r},t) + \vec{E}^{(-)}(\vec{r},t) \right) | \gamma \rangle \right] \tag{A.0.2}$$

which are two expressions relating the probability densities defined above.

$$\vec{\nabla} \times \left[\langle \gamma | \vec{E}^{(+)}(\vec{r},t) | \gamma \rangle + \langle \gamma | \vec{E}^{(-)}(\vec{r},t) | \gamma \rangle \right] = -\mu_0 \partial_t \left[\langle \gamma | \vec{H}^{(+)}(\vec{r},t) | \gamma \rangle + \langle \gamma | \vec{H}^{(-)}(\vec{r},t) | \gamma \rangle \right] \tag{A.0.3}$$

$$\vec{\nabla} \times \left[\langle \gamma | \vec{H}^{(+)}(\vec{r},t) | \gamma \rangle + \langle \gamma | \vec{H}^{(-)}(\vec{r},t) | \gamma \rangle \right] = \varepsilon_0 \partial_t \left[\langle \gamma | \vec{E}^{(+)}(\vec{r},t) | \gamma \rangle + \langle \gamma | \vec{E}^{(-)}(\vec{r},t) | \gamma \rangle \right] \tag{A.0.4}$$

This can be explicitly shown by taking advantage that $| \gamma \rangle$ is a complete set of states occupied by a single photon $| \gamma \rangle = \sum_n c_n(t) | n \rangle$, such that the identity is given by $\mathbf{I} = \sum_n | n \rangle \langle n |$, and introduce

Table A.1: Classical Wave Optics vs Quantum Optics

Classical and Quantum form of Maxwell's Equations	
Classical	$\vec{\nabla} \times \vec{E} = -\partial_t \vec{B}$ $\vec{\nabla} \times \vec{H} = \partial_t \vec{D}$
Quantum	$\vec{\nabla} \times \left[\langle \gamma | \left(\vec{E}^{(+)}(\vec{r},t) + \vec{E}^{(-)}(\vec{r},t) \right) | \gamma \rangle \right] = -\mu_0 \left[\partial_t \langle \gamma | \left(\vec{H}^{(+)}(\vec{r},t) + \vec{H}^{(-)}(\vec{r},t) \right) | \gamma \rangle \right]$ $\vec{\nabla} \times \left[\langle \gamma | \left(\vec{H}^{(+)}(\vec{r},t) + \vec{H}^{(-)}(\vec{r},t) \right) | \gamma \rangle \right] = \varepsilon_0 \partial_t \left[\langle \gamma | \left(\vec{E}^{(+)}(\vec{r},t) + \vec{E}^{(-)}(\vec{r},t) \right) | \gamma \rangle \right]$

it into these expressions

$$\vec{\nabla} \times \left[\sum_n \langle n| c_n^* \vec{E}^{(+)} |\gamma\rangle + \langle\gamma| \vec{E}^{(-)} \sum_n c_n |n\rangle \right] = -\mu_0 \partial_t \left[\sum_n \langle n| c_n^* \vec{H}^{(+)} |\gamma\rangle + \langle\gamma| \vec{H}^{(-)} \sum_n c_n |n\rangle \right] \quad \text{(A.0.5)}$$

$$\vec{\nabla} \times \left[\sum_n \langle n| c_n^* \vec{H}^{(+)} |\gamma\rangle + \langle\gamma| \vec{H}^{(-)} \sum_n c_n |n\rangle \right] = \varepsilon_0 \partial_t \left[\sum_n \langle n| c_n^* \vec{E}^{(+)} |\gamma\rangle + \langle\gamma| \vec{E}^{(-)} \sum_n c_n |n\rangle \right] \quad \text{(A.0.6)}$$

These can be expressed as four equations for the propagation of a <u>single photon</u> with components $\left\langle \vec{E}(\vec{r},t) \right\rangle$ and $\left\langle \vec{B}(\vec{r},t) \right\rangle$. This is due to the textbook argument [15] which states that "... since there is only one photon in $|\gamma\rangle$ ... only the vacuum term will contribute ...". Explicitly, these four expressions are

$$\vec{\nabla} \times \langle 0| \vec{E}^{(+)}(\vec{r},t) |\gamma\rangle = -\mu_0 \partial_t \langle 0| \vec{H}^{(+)}(\vec{r},t) |\gamma\rangle \quad \text{(A.0.7)}$$

$$\vec{\nabla} \times \langle 0| \vec{H}^{(+)}(\vec{r},t) |\gamma\rangle = \varepsilon_0 \partial_t \langle 0| \vec{E}^{(+)}(\vec{r},t) |\gamma\rangle \quad \text{(A.0.8)}$$

and

$$\vec{\nabla} \times \langle\gamma| \vec{E}^{(-)}(\vec{r},t) |0\rangle = -\mu_0 \partial_t \langle\gamma| \vec{H}^{(-)}(\vec{r},t) |0\rangle$$

$$\vec{\nabla} \times \langle\gamma| \vec{H}^{(-)}(\vec{r},t) |0\rangle = \varepsilon_0 \partial_t \langle\gamma| \vec{E}^{(-)}(\vec{r},t) |0\rangle$$

These are Maxwell's equations for the probability of measuring the electric or magnetic fields. The expressions for $\vec{\varphi}_\gamma$ and $\vec{\chi}_\gamma$ can be expected to respectively represent the electronic and magnetic wave function (EWF $\left\langle \vec{E} \right\rangle = \vec{\Psi}_E(\vec{r},t) + \vec{\Psi}_E^\dagger(\vec{r},t)$ & MWF $\left\langle \vec{H} \right\rangle = \vec{\Psi}_H(\vec{r},t) + \vec{\Psi}_H^\dagger(\vec{r},t)$) components of the PWF, as will be shown below. Therefore we close this chapter with the relationship

between the Magnetic, Electric, and PWF

$$\left\langle \vec{D} \right\rangle \; = \; \sqrt{\frac{1}{2\varepsilon}} \left(\vec{F}_\gamma + \vec{F}_\gamma^\dagger \right) \tag{A.0.9}$$

$$\left\langle \vec{B} \right\rangle \; = \; -i\sqrt{\frac{1}{2\mu}} \left(\vec{F}_\gamma - \vec{F}_\gamma^\dagger \right) \tag{A.0.10}$$

$$\vec{F}_\gamma \; = \; \frac{1}{\sqrt{2}} \left(\frac{\varepsilon \left\langle \vec{D} \right\rangle}{\sqrt{\varepsilon}} + i\frac{\mu \left\langle \vec{B} \right\rangle}{\sqrt{\mu}} \right) \tag{A.0.11}$$

$$\vec{F}_\gamma^\dagger \; = \; \frac{1}{\sqrt{2}} \left(\frac{\varepsilon \left\langle \vec{D} \right\rangle}{\sqrt{\varepsilon}} + i\frac{\mu \left\langle \vec{B} \right\rangle}{\sqrt{\mu}} \right) \tag{A.0.12}$$

where $\vec{F}_\gamma$, $\left\langle \vec{D} \right\rangle$, and $\left\langle \vec{B} \right\rangle$ represent the Riemann-Silberstein photonic wave-function, electric field wave function, and magnetic field wave-function respectively.

B   Energy in the PWF Formalism

B.1   *The Hamiltonian from the EWF and MWF*

Using the creation and annihilation operators of the EWF and MWF as well as making use of $\frac{\vec{k}}{k} \times \hat{\epsilon}_{\vec{k},\sigma_\lambda} = -i\hat{\epsilon}_{\vec{k},\sigma_\lambda}$ [1]

$$
\begin{aligned}
\mathcal{H}_\gamma &\equiv \frac{1}{2}\left(\varepsilon \vec{E}^2 + \frac{1}{\mu}\vec{B}^2\right) &&\text{(B.1.1)} \\
&= \frac{1}{2}\left[\frac{\hbar\omega}{2V}\left(\vec{\varphi}_\gamma^{(+)} + \vec{\varphi}_\gamma^{(-)}\right)^2 + \frac{\hbar\omega}{2}\left(\vec{\chi}_\gamma^{(+)} + \vec{\chi}_\gamma^{(-)}\right)^2\right] &&\text{(B.1.2)} \\
&= \frac{\hbar\omega}{4V}\left[\left(\vec{\varphi}_\gamma^{(+)} + \vec{\varphi}_\gamma^{(-)}\right)^2 + \left(\vec{\chi}_\gamma^{(+)} + \vec{\chi}_\gamma^{(-)}\right)^2\right] &&\text{(B.1.3)}
\end{aligned}
$$

We focus on

$$
\left(\vec{\varphi}_\gamma^{(+)} + \vec{\varphi}_\gamma^{(-)}\right)^2 = \vec{\varphi}_\gamma^{(+)}\vec{\varphi}_\gamma^{(+)} + \vec{\varphi}_\gamma^{(+)}\vec{\varphi}_\gamma^{(-)} + \vec{\varphi}_\gamma^{(-)}\vec{\varphi}_\gamma^{(+)} + \vec{\varphi}_\gamma^{(-)}\vec{\varphi}_\gamma^{(-)} \qquad\text{(B.1.4)}
$$

$$
\left(\vec{\chi}_\gamma^{(+)} + \vec{\chi}_\gamma^{(-)}\right)^2 = \vec{\chi}_\gamma^{(+)}\vec{\chi}_\gamma^{(+)} + \vec{\chi}_\gamma^{(+)}\vec{\chi}_\gamma^{(-)} + \vec{\chi}_\gamma^{(-)}\vec{\chi}_\gamma^{(+)} + \vec{\chi}_\gamma^{(-)}\vec{\chi}_\gamma^{(-)} \qquad\text{(B.1.5)}
$$

Substituting

$$
\vec{\varphi}_\gamma^{(+)} = \frac{1}{\sqrt{V}}\sum_{\vec{k},\sigma_\lambda}\hat{\epsilon}_{\vec{k},\sigma_\lambda}a_{\vec{k},\sigma_\lambda}e^{-i\nu_k t}U_{\vec{k}}(\vec{r}) \qquad\text{(B.1.6)}
$$

$$
\vec{\varphi}_\gamma^{(-)} = \frac{1}{\sqrt{V}}\sum_{\vec{k},\sigma_\lambda}\hat{\epsilon}^*_{\vec{k},\sigma_\lambda}a^\dagger_{\vec{k},\sigma_\lambda}e^{i\nu_k t}U^*_{\vec{k}}(\vec{r}) \qquad\text{(B.1.7)}
$$

---

[1]Please see (C.5)

and

$$\vec{\chi}_\gamma^{(+)} \;=\; \frac{1}{\sqrt{V}} \sum_{\vec{k},\sigma_\lambda} -i\hat{\epsilon}_{\vec{k},\sigma_\lambda} a_{\vec{k},\sigma_\lambda} e^{-i\nu_k t} U_{\vec{k}}(\vec{r}) \tag{B.1.8}$$

$$\vec{\chi}_\gamma^{(-)} \;=\; \frac{1}{\sqrt{V}} \sum_{\vec{k},\sigma_\lambda} i\hat{\epsilon}_{\vec{k},\sigma_\lambda}^* a_{\vec{k},\sigma_\lambda}^\dagger e^{i\nu_k t} U_{\vec{k}}^*(\vec{r}) \tag{B.1.9}$$

we get that

$$\vec{\varphi}_\gamma^{(+)} \vec{\varphi}_\gamma^{(+)} \;=\; 0 \tag{B.1.10}$$

$$\vec{\varphi}_\gamma^{(+)} \vec{\varphi}_\gamma^{(-)} \;=\; \frac{1}{V} \sum_{\vec{k},\sigma_\lambda} a_{\vec{k},\sigma_\lambda} a_{\vec{k},\sigma_\lambda}^\dagger \tag{B.1.11}$$

$$\vec{\varphi}_\gamma^{(-)} \vec{\varphi}_\gamma^{(+)} \;=\; \frac{1}{V} \sum_{\vec{k},\sigma_\lambda} a_{\vec{k},\sigma_\lambda}^\dagger a_{\vec{k},\sigma_\lambda} \tag{B.1.12}$$

$$\vec{\varphi}_\gamma^{(-)} \vec{\varphi}_\gamma^{(-)} \;=\; 0 \tag{B.1.13}$$

and

$$\vec{\chi}_\gamma^{(+)} \vec{\chi}_\gamma^{(+)} \;=\; 0 \tag{B.1.14}$$

$$\vec{\chi}_\gamma^{(+)} \vec{\chi}_\gamma^{(-)} \;=\; \frac{1}{V} \sum_{\vec{k}} a_{\vec{k},\sigma_+} a_{\vec{k},\sigma_+}^\dagger \tag{B.1.15}$$

$$\vec{\chi}_\gamma^{(-)} \vec{\chi}_\gamma^{(+)} \;=\; \frac{1}{V} \sum_{\vec{k}} a_{\vec{k},\sigma_+}^\dagger a_{\vec{k},\sigma_+} \tag{B.1.16}$$

$$\vec{\chi}_\gamma^{(-)} \vec{\chi}_\gamma^{(-)} \;=\; 0 \tag{B.1.17}$$

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which yield

$$\vec{\varphi}_\gamma^{(+)}\vec{\varphi}_\gamma^{(+)} + \vec{\chi}_\gamma^{(+)}\vec{\chi}_\gamma^{(+)} = 0 \qquad\qquad (B.1.18)$$

$$\vec{\varphi}_\gamma^{(-)}\vec{\varphi}_\gamma^{(-)} + \vec{\chi}_\gamma^{(-)}\vec{\chi}_\gamma^{(-)} = 0 \qquad\qquad (B.1.19)$$

$$\vec{\varphi}_\gamma^{(+)}\vec{\varphi}_\gamma^{(-)} + \vec{\chi}_\gamma^{(+)}\vec{\chi}_\gamma^{(-)} = \frac{2}{V}\sum_{\vec{k},+} a_{\vec{k},\sigma_+} a_{\vec{k},\sigma_+}^\dagger \qquad\qquad (B.1.20)$$

$$\vec{\varphi}_\gamma^{(-)}\vec{\varphi}_\gamma^{(+)} + \vec{\chi}_\gamma^{(-)}\vec{\chi}_\gamma^{(+)} = \frac{2}{V}\sum_{\vec{k},+} a_{\vec{k},\sigma_+}^\dagger a_{\vec{k},\sigma_+} \qquad\qquad (B.1.21)$$

and we can use these to get

$$\mathcal{H}_\gamma = \frac{\hbar\omega}{2V}\sum_{\vec{k}}\left(a_{\vec{k},\sigma_+} a_{\vec{k},\sigma_+}^\dagger + a_{\vec{k},\sigma_+}^\dagger a_{\vec{k},\sigma_+} \right) \qquad\qquad (B.1.22)$$

where by using the identity $\left[a_{\vec{k},\sigma_\lambda}, a_{\vec{k}',\lambda}^\dagger \right] = \delta_{\vec{k},\vec{k}'}\delta_{\lambda,\lambda'}$

$$\mathcal{H}_\gamma = \frac{\hbar\omega}{2V}\sum_{\vec{k},\lambda}\left[a_{\vec{k},\sigma_+} a_{\vec{k},\sigma_+}^\dagger + \left(a_{\vec{k},\sigma_+}^\dagger a_{\vec{k},\sigma_+} - a_{\vec{k},\sigma_+}^\dagger a_{\vec{k},\sigma_+} + a_{\vec{k},\sigma_+}^\dagger a_{\vec{k},\sigma_+} \right) \right] \qquad (B.1.23)$$

$$= \frac{\hbar\omega}{2V}\sum_{\vec{k},\lambda}\left[a_{\vec{k},\sigma_+} a_{\vec{k},\sigma_+}^\dagger + \left(a_{\vec{k},\sigma_+}^\dagger a_{\vec{k},\sigma_+} - a_{\vec{k},\sigma_+} a_{\vec{k},\sigma_+}^\dagger + 1 + a_{\vec{k},\sigma_+}^\dagger a_{\vec{k},\sigma_+} \right) \right] \qquad (B.1.24)$$

$$= \frac{\hbar\omega}{2V}\sum_{\vec{k}}\left(2a_{\vec{k},\sigma_+}^\dagger a_{\vec{k},\sigma_+} + 1 \right) \qquad\qquad (B.1.25)$$

which is more commonly expressed as

$$\mathcal{H}_\gamma = \frac{\hbar\omega}{V}\sum_{\vec{k},\sigma_\lambda}\left(a_{\vec{k},\sigma_\lambda}^\dagger a_{\vec{k},\sigma_\lambda} + \frac{1}{2} \right) \qquad\qquad (B.1.26)$$

B.2 Energy in the the Riemann-Silberstein PWF

Birula and others claim that the total energy of the photon is contained in the Riemann-Silberstein photonic wave function (RSPWF). Here we check this and expect to get the same $\mathcal{H}_\gamma$ as above.

Starting from the definition of the energy according present in previous studies [10, 11]

$$\langle \mathcal{H}_\gamma \rangle = \int_V d^3\vec{r} \left(\vec{F}_\gamma^\dagger \vec{F}_\gamma \right) \tag{B.2.1}$$

and the RSPWF

$$\vec{F}_\gamma^{(\pm)} \equiv \frac{1}{\sqrt{2}} \left(\frac{\varepsilon \vec{E}}{\sqrt{\varepsilon}} \pm \frac{i\mu \vec{H}}{\sqrt{\mu}} \right) \tag{B.2.2}$$

The computation of the energy requires the evaluation of

$$\vec{F}_\gamma^\dagger \vec{F}_\gamma = \vec{F}_\gamma^{(-)} \vec{F}_\gamma^{(+)} \tag{B.2.3}$$

where

$$\vec{F}_\gamma^{(-)} \vec{F}_\gamma^{(+)} = \frac{1}{2} \left(\frac{\varepsilon \vec{E}^*}{\sqrt{\varepsilon}} - \frac{i\mu \vec{H}^*}{\sqrt{\mu}} \right) \left(\frac{\varepsilon \vec{E}}{\sqrt{\varepsilon}} + \frac{i\mu \vec{H}}{\sqrt{\mu}} \right) \tag{B.2.4}$$

$$= \frac{1}{2} \left(\varepsilon \vec{E}^* \vec{E} + \frac{\varepsilon \vec{E}^*}{\sqrt{\varepsilon}} \frac{i\mu \vec{H}}{\sqrt{\mu}} - \frac{i\mu \vec{H}^*}{\sqrt{\mu}} \frac{\varepsilon \vec{E}}{\sqrt{\varepsilon}} + \mu \vec{H}^* \vec{H} \right) \tag{B.2.5}$$

which at first glance we expect to give

$$\mathcal{H}_\gamma = \frac{1}{2} \int_V d^3\vec{r} \left(\vec{F}_\gamma^\dagger \vec{F}_\gamma \right) \tag{B.2.6}$$

$$= \frac{1}{2} \int_V d^3\vec{r} \left(\varepsilon \vec{E}^2 + \mu \vec{H}^2 \right) \tag{B.2.7}$$

$$= \frac{1}{2} \int_V d^3\vec{r} \left(\varepsilon \vec{E}^2 + \frac{1}{\mu} \vec{B}^2 \right) \tag{B.2.8}$$

and get the same expression for the energy which we started from above. Inspecting the term

$$\frac{\varepsilon \vec{E}^*}{\sqrt{\varepsilon}} \frac{i\mu \vec{H}}{\sqrt{\mu}} - \frac{i\mu \vec{H}^*}{\sqrt{\mu}} \frac{\varepsilon \vec{E}}{\sqrt{\varepsilon}} = \frac{i\varepsilon\mu}{\sqrt{\varepsilon\mu}} \left[\left(\vec{\Psi}_E^{(+)} + \vec{\Psi}_E^{(-)} \right) \left(\vec{\Psi}_H^{(+)} + \vec{\Psi}_H^{(-)} \right) - \left(\vec{\Psi}_H^{(+)} + \vec{\Psi}_H^{(-)} \right) \left(\vec{\Psi}_E^{(+)} + \vec{\Psi}_E^{(-)} \right) \right] \tag{B.2.9}$$

$$= i \frac{\hbar\omega}{2V} \left[\left(\vec{\varphi}_\gamma^{(+)} + \vec{\varphi}_\gamma^{(-)} \right) \left(\vec{\chi}_\gamma^{(+)} + \vec{\chi}_\gamma^{(-)} \right) - \left(\vec{\chi}_\gamma^{(+)} + \vec{\chi}_\gamma^{(-)} \right) \left(\vec{\varphi}_\gamma^{(+)} + \vec{\varphi}_\gamma^{(-)} \right) \right] \tag{B.2.10}$$

which upon expansion yields

$$i\left(\vec{\varphi}_\gamma^{(+)} + \vec{\varphi}_\gamma^{(-)}\right)\left(\vec{\chi}_\gamma^{(+)} + \vec{\chi}_\gamma^{(-)}\right) = i\left[\vec{\varphi}_\gamma^{(+)}\vec{\chi}_\gamma^{(+)} + \vec{\varphi}_\gamma^{(-)}\vec{\chi}_\gamma^{(+)} + \vec{\varphi}_\gamma^{(+)}\vec{\chi}_\gamma^{(-)} + \vec{\varphi}_\gamma^{(-)}\vec{\chi}_\gamma^{(-)}\right] \tag{B.2.11}$$

$$-i\left(\vec{\chi}_\gamma^{(+)} + \vec{\chi}_\gamma^{(-)}\right)\left(\vec{\varphi}_\gamma^{(+)} + \vec{\varphi}_\gamma^{(-)}\right) = -i\left[\vec{\chi}_\gamma^{(+)}\vec{\varphi}_\gamma^{(+)} + \vec{\chi}_\gamma^{(+)}\vec{\varphi}_\gamma^{(-)} + \vec{\chi}_\gamma^{(-)}\vec{\varphi}_\gamma^{(+)} + \vec{\chi}_\gamma^{(-)}\vec{\varphi}_\gamma^{(-)}\right] \tag{B.2.12}$$

such that this term is expected to be zero

$$\frac{i\varepsilon\mu}{\sqrt{\varepsilon\mu}}\left(\vec{E}^*\vec{H} - \vec{H}^*\vec{E}\right) = i\frac{\hbar\omega}{2V}\left(\left[\vec{\varphi}_\gamma^{(+)}, \vec{\chi}_\gamma^{(+)}\right] + \left[\vec{\varphi}_\gamma^{(-)}, \vec{\chi}_\gamma^{(+)}\right] + \left[\vec{\varphi}_\gamma^{(+)}, \vec{\chi}_\gamma^{(-)}\right] + \left[\vec{\varphi}_\gamma^{(-)}, \vec{\chi}_\gamma^{(-)}\right]\right) \tag{B.2.13}$$

where using the identity $\left[a_{\vec{k},\sigma_\lambda}, a_{\vec{k}',\lambda}^\dagger\right] = \delta_{\vec{k},\vec{k}'}\delta_{\lambda,\lambda'}$

$$\left[\vec{\varphi}_\gamma^{(+)}, \vec{\chi}_\gamma^{(+)}\right] = 0 \tag{B.2.14}$$

$$\left[\vec{\varphi}_\gamma^{(-)}, \vec{\chi}_\gamma^{(+)}\right] = i \tag{B.2.15}$$

$$\left[\vec{\varphi}_\gamma^{(+)}, \vec{\chi}_\gamma^{(-)}\right] = i \tag{B.2.16}$$

$$\left[\vec{\varphi}_\gamma^{(-)}, \vec{\chi}_\gamma^{(-)}\right] = 0 \tag{B.2.17}$$

which unexpectedly results in

$$\frac{i\varepsilon\mu}{\sqrt{\varepsilon\mu}}\left(\vec{E}^*\vec{H} - \vec{H}^*\vec{E}\right) = -\frac{\hbar\omega}{V} \tag{B.2.18}$$

and conflicts with the previous result giving

$$\begin{aligned}
\mathcal{H}_\gamma &= \frac{1}{2}\int_V d^3\vec{r}\left(\varepsilon\vec{E}^2 + \frac{1}{\mu}\vec{B}^2 - \frac{\hbar\omega}{V}\right) \\
&= \hbar\omega\sum_{\vec{k},+} a_{\vec{k},\sigma_+}^\dagger a_{\vec{k},\sigma_+}
\end{aligned}$$

Taking the RS vector in terms of creation and annihilation operators, this result can be quickly verified from

$$\langle \mathcal{H}_\gamma \rangle = \int_V d^3\vec{r} \left(\vec{F}_\gamma^{(-)} \vec{F}_\gamma^{(+)} \right) \tag{B.2.19}$$

$$= \hbar\omega \int_V d^3\vec{r} \left(\vec{\Psi}_\gamma^{(-)} \vec{\Psi}_\gamma^{(+)} \right) \tag{B.2.20}$$

such that the Hamiltonian operator is given by

$$\mathcal{H}_\gamma = \hbar\omega \int_V d^3\vec{r} \left(\vec{\mathcal{F}}_\gamma^{(-)} \vec{\mathcal{F}}_\gamma^{(+)} \right) \tag{B.2.21}$$

where

$$\vec{\mathcal{F}}_\gamma^{(+)} \equiv \frac{1}{\sqrt{V}} \sum_{\vec{k}} \hat{\epsilon}_{\vec{k},\sigma_+} a_{\vec{k},\sigma_+} e^{-i\nu_k t} U_{\vec{k}}(\vec{r}) \tag{B.2.22}$$

$$\vec{\mathcal{F}}_\gamma^{(-)} \equiv \frac{1}{\sqrt{V}} \sum_{\vec{k}} \hat{\epsilon}_{\vec{k},\sigma_+} a_{\vec{k},\sigma_+}^\dagger e^{i\nu_k t} U_{\vec{k}}^*(\vec{r}) \tag{B.2.23}$$

yields

$$\mathcal{H}_\gamma = \hbar\omega \sum_{\vec{k}} a_{\vec{k},\sigma_+}^\dagger a_{\vec{k},\sigma_+} \tag{B.2.24}$$

Which is not what was expected. This requires an alternative representation of the PWF than the RS vector.

Starting directly from the operator form of the RSPWF and the expression for the energy which we expect to include the vacuum fluctuation energy

$$
\mathcal{H}_\gamma = \frac{\hbar\omega}{2} \int_V d^3\vec{r} \left(\vec{\mathcal{F}}_\gamma^{(-)} \ \ \vec{\mathcal{F}}_\gamma^{(+)} \right) \begin{pmatrix} \vec{\mathcal{F}}_\gamma^{(+)} \\ \vec{\mathcal{F}}_\gamma^{(-)} \end{pmatrix} \tag{B.3.1}
$$

$$
= \frac{\hbar\omega}{2} \int_V d^3\vec{r} \left(\vec{\mathcal{F}}_\gamma^{(-)} \vec{\mathcal{F}}_\gamma^{(+)} + \vec{\mathcal{F}}_\gamma^{(+)} \vec{\mathcal{F}}_\gamma^{(-)} \right) \tag{B.3.2}
$$

where

$$
\vec{\mathcal{F}}_\gamma^{(+)} \equiv \frac{1}{\sqrt{V}} \sum_{\vec{k},+} \hat{\epsilon}_{\vec{k},\sigma_+} a_{\vec{k},\sigma_+} e^{-i\nu_k t} U_{\vec{k}}(\vec{r}) \tag{B.3.3}
$$

$$
\vec{\mathcal{F}}_\gamma^{(-)} \equiv \frac{1}{\sqrt{V}} \sum_{\vec{k},+} \hat{\epsilon}_{\vec{k},\sigma_+} a_{\vec{k},\sigma_+}^\dagger e^{i\nu_k t} U_{\vec{k}}^*(\vec{r}) \tag{B.3.4}
$$

and from the above we already know that

$$
\vec{\mathcal{F}}_\gamma^{(-)} \vec{\mathcal{F}}_\gamma^{(+)} = \frac{1}{V} \sum_{\vec{k}} a_{\vec{k},\sigma_+}^\dagger a_{\vec{k},\sigma_+} \tag{B.3.5}
$$

$$
\vec{\mathcal{F}}_\gamma^{(+)} \vec{\mathcal{F}}_\gamma^{(-)} = \frac{1}{\sqrt{V}} \sum_{\vec{k}} a_{\vec{k},\sigma_+} a_{\vec{k},\sigma_+}^\dagger \tag{B.3.6}
$$

we can use similar results from above to derive that

$$
\vec{\mathcal{F}}_\gamma^{(-)} \vec{\mathcal{F}}_\gamma^{(+)} + \vec{\mathcal{F}}_\gamma^{(+)} \vec{\mathcal{F}}_\gamma^{(-)} = \sum_{\vec{k}} \left(a_{\vec{k},\sigma_+}^\dagger a_{\vec{k},\sigma_+} + a_{\vec{k},\sigma_+} a_{\vec{k},\sigma_+}^\dagger \right) \tag{B.3.7}
$$

$$
= \sum_{\vec{k},+} \left(2 a_{\vec{k},\sigma_+}^\dagger a_{\vec{k},\sigma_+} + 1 \right) \tag{B.3.8}
$$

such that the Hamiltonian is given by

$$\mathcal{H}_\gamma \;=\; \hbar\omega \sum_{\vec{k}} \left(a^\dagger_{\vec{k},\sigma_+} a_{\vec{k},\sigma_+} + \frac{1}{2} \right) \tag{B.3.9}$$

B.4 *Alpha and Beta Matrix Representation of the Dirac Equation for a Maxwell Field*

It is beneficial to represent the

$$\vec{\boldsymbol{\Psi}}_\gamma \;=\; \begin{pmatrix} \vec{\Psi}_\gamma \\[4pt] \vec{\Psi}^\dagger_\gamma \end{pmatrix}$$

where each component of this spinor can be represented as

$$\vec{\Psi}_\gamma \;=\; \begin{pmatrix} \psi^{(+)}_{\gamma,\sigma_+} \\[6pt] \psi^{(-)}_{\gamma,\sigma_-} \end{pmatrix}$$

$$\vec{\Psi}^\dagger_\gamma \;=\; \begin{pmatrix} \psi^{\dagger(-)}_{\gamma,\sigma_+} \\[6pt] \psi^{\dagger(+)}_{\gamma,\sigma_-} \end{pmatrix}$$

where

$$\psi^{(+)}_{\gamma,\sigma_+} \;=\; \langle 0 | \sqrt{\frac{\hbar\omega}{V}} \sum_{\vec{k}} \hat{\epsilon}_{\vec{k},\sigma_+} a_{\vec{k},\sigma_+} e^{-i\nu_k t} U_{\vec{k}}(\vec{r}) | \gamma \rangle$$

$$\psi^{(-)}_{\gamma,\sigma_-} \;=\; \langle \gamma | \sqrt{\frac{\hbar\omega}{V}} \sum_{\vec{k}} \hat{\epsilon}^*_{\vec{k},\sigma_-} a^\dagger_{\vec{k},\sigma_-} e^{i\nu_k t} U^*_{\vec{k}}(\vec{r}) | 0 \rangle$$

$$\psi^{\dagger(-)}_{\gamma,\sigma_+} \;=\; \langle \gamma | \sqrt{\frac{\hbar\omega}{V}} \sum_{\vec{k}} \hat{\epsilon}^*_{\vec{k},\sigma_+} a^\dagger_{\vec{k},\sigma_+} e^{i\nu_k t} U^*_{\vec{k}}(\vec{r}) | 0 \rangle$$

$$\psi^{\dagger(+)}_{\gamma,\sigma_-} \;=\; \langle 0 | \sqrt{\frac{\hbar\omega}{V}} \sum_{\vec{k}} \hat{\epsilon}_{\vec{k},\sigma_-} a_{\vec{k},\sigma_-} e^{-i\nu_k t} U_{\vec{k}}(\vec{r}) | \gamma \rangle$$

A clear comparison between Maxwell's equations written in Dirac form and the Wave Equation of the Riemann-Silberstein wave-function for a free particle shows that [10, 11]

$$
i\hbar\partial_t \begin{pmatrix} \vec{\Psi}_\gamma\left(\vec{r},t\right) \\ \vec{\Psi}_\gamma^\dagger\left(\vec{r},t\right) \end{pmatrix} = \begin{pmatrix} c\hbar\vec{\nabla}\times & 0 \\ 0 & -c\hbar\vec{\nabla}\times \end{pmatrix} \begin{pmatrix} \vec{\Psi}_\gamma\left(\vec{r},t\right) \\ \vec{\Psi}_\gamma^\dagger\left(\vec{r},t\right) \end{pmatrix}
$$

(The Wave Equation for the Riemann-Silberstein wave-function) will lend itself well to coupling between states of the Maxwell Field (on the off-diagonal terms) and contains all the information of the following two equations [15]

$$
i\hbar\partial_t \begin{pmatrix} \vec{\varphi}_\gamma \\ \vec{\chi}_\gamma \end{pmatrix} = \begin{pmatrix} 0 & -c\hbar\vec{\nabla}\times \\ c\hbar\vec{\nabla}\times & 0 \end{pmatrix} \begin{pmatrix} \vec{\varphi}_\gamma \\ \vec{\chi}_\gamma \end{pmatrix}
$$

$$
i\hbar\partial_t \begin{pmatrix} \vec{\varphi}_\gamma^\dagger \\ \vec{\chi}_\gamma^\dagger \end{pmatrix} = \begin{pmatrix} 0 & c\hbar\vec{\nabla}\times \\ -c\hbar\vec{\nabla}\times & 0 \end{pmatrix} \begin{pmatrix} \vec{\varphi}_\gamma^\dagger \\ \vec{\chi}_\gamma^\dagger \end{pmatrix}
$$

It is beneficial to have this representation when referring to single photon states or entangled single photon states since these fully represent these states when coupling the propagation of this field to other fields or describing a single photon propagating through inhomogeneous media.

C   Mathematical Disambiguation

C.1   Derivation of the Photon Field Tensor

Here we show that where $c^{-1} = \sqrt{\varepsilon_0 \mu_0}$, Maxwell's equations with sources, in four notation through the Maxwell Field Tensor for a Minkowski Metric with signature $(i, -, -, -)$, can be rewritten in terms of a self-dual photon field tensor $\partial_\mu \mathfrak{G}^{\mu\nu} = \mathfrak{J}^\nu$. In previous studies it has been shown that it is possible to define a self dual tensor from the Maxwell Field Tensor and its dual, in terms of the PWF operators in the vacuum [46]. For a photonic wave function propagating through a linear homogeneous medium it is the case that one must satisfy the expression

$$\partial_\mu \mathfrak{F}^{\mu\nu} = \frac{e}{c} \bar{\psi}(x) \gamma^\nu \psi(x)$$

where $\partial_\mu \equiv \frac{\partial}{\partial x^\mu} = \begin{pmatrix} c^{-1} i \partial_t & \partial_x & \partial_y & \partial_z \end{pmatrix}$

$$\mathfrak{F}^{\mu\nu} = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{pmatrix}$$

and γ^μ are the Dirac gamma matrices. The dual [83] of $\mathfrak{F}^{\mu\nu}$ is defined as

$$\mathfrak{F}_D^{\mu\nu} = \frac{1}{2} \varepsilon^{\mu\nu\alpha\beta} \mathfrak{F}_{\alpha\beta}$$

$$= \begin{pmatrix} 0 & -B_x & -B_y & -B_z \\ B_x & 0 & E_z & -E_y \\ B_y & -E_z & 0 & E_x \\ B_z & E_y & -E_x & 0 \end{pmatrix}$$

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To define the self-dual photon field tensor we begin with

$$\partial_\mu \mathfrak{F}^{\mu\nu} = J^\nu$$

$$\partial_\mu \mathfrak{F}^{\mu\nu}_D = 0$$

which are indeed Maxwell's equations in four notation .

C.1.1 Gaussian Units

In Gaussian Units Maxwell's Equations read

$$\begin{pmatrix} \vec{\nabla} \cdot \vec{E} \\ \vec{\nabla} \times \vec{B} - \frac{1}{c}\partial_t \vec{E} \end{pmatrix} = \begin{pmatrix} 4\pi\rho \\ \frac{4\pi}{c}\vec{J} \end{pmatrix}$$
$$\begin{pmatrix} \vec{\nabla} \cdot \vec{B} \\ \vec{\nabla} \times \vec{E} + \frac{1}{c}\partial_t \vec{B} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

multiplying the second set by i and subtracting $\vec{\nabla} \cdot \vec{B}$ from the $\vec{\nabla} \cdot \vec{E}$ and adding $\vec{\nabla} \times \vec{E} + \frac{1}{c}\partial_t \vec{B}$ to $\vec{\nabla} \times \vec{B} - \frac{1}{c}\partial_t \vec{E}$ we find

$$\begin{pmatrix} \vec{\nabla} \cdot \vec{E} - \vec{\nabla} \cdot i\vec{B} \\ \vec{\nabla} \times \vec{B} - \frac{1}{c}\partial_t \vec{E} + \vec{\nabla} \times i\vec{E} + \frac{1}{c}\partial_t i\vec{B} \end{pmatrix} = \begin{pmatrix} 4\pi\rho \\ \frac{4\pi}{c}\vec{J} \end{pmatrix}$$

which simplifies to

$$\begin{pmatrix} -i\vec{\nabla} \cdot \left(\vec{B} + i\vec{E} \right) \\ \vec{\nabla} \times \left(\vec{B} + i\vec{E} \right) + i\frac{1}{c}\partial_t \left(\vec{B} + i\vec{E} \right) \end{pmatrix} = \begin{pmatrix} 4\pi\rho \\ \frac{4\pi}{c}\vec{J} \end{pmatrix}$$

which in terms of the vector $\vec{F} \equiv \vec{B} + i\vec{E}$ can be expressed as

$$\begin{pmatrix} -i\vec{\nabla} \cdot (\vec{F}) \\ \vec{\nabla} \times (\vec{F}) + i\frac{1}{c}\partial_t (\vec{F}) \end{pmatrix} = \begin{pmatrix} 4\pi\rho \\ \frac{4\pi}{c}\vec{J} \end{pmatrix}$$

C.1.2 SI Units

In SI units, Maxwell's equations in linear media can now be re-expressed by using the definition of $c^{-1} = \sqrt{\varepsilon\mu}$ as

$$\begin{pmatrix} \sqrt{\varepsilon\mu}\vec{\nabla} \cdot \vec{E} \\ -c^{-1}\left(\sqrt{\varepsilon\mu}\partial_t\vec{E} - c\vec{\nabla} \times \vec{B}\right) \end{pmatrix} = \begin{pmatrix} \mu c\rho_{free} \\ \mu\vec{J}_{free} \end{pmatrix}$$

$$\begin{pmatrix} \vec{\nabla} \cdot \vec{B} \\ -c^{-1}\left(\partial_t\vec{B} + c\vec{\nabla} \times \left[\sqrt{\varepsilon\mu}\vec{E}\right]\right) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

which may be simplified to get

$$\begin{pmatrix} \sqrt{\mu}\vec{\nabla} \cdot \left[\frac{\varepsilon\vec{E}}{\sqrt{\varepsilon}}\right] \\ -c^{-1}\sqrt{\mu}\left(\partial_t\left[\frac{\varepsilon\vec{E}}{\sqrt{\varepsilon}}\right] - c\vec{\nabla} \times \left[\frac{\mu\vec{H}}{\sqrt{\mu}}\right]\right) \end{pmatrix} = \begin{pmatrix} \mu c\rho_{free} \\ \mu\vec{J}_{free} \end{pmatrix} \tag{C.1.1}$$

$$\begin{pmatrix} \sqrt{\mu}\vec{\nabla} \cdot \left[\frac{\mu\vec{H}}{\sqrt{\mu}}\right] \\ -c^{-1}\sqrt{\mu}\left(\partial_t\left[\frac{\mu\vec{H}}{\sqrt{\mu}}\right] + c\vec{\nabla} \times \left[\frac{\varepsilon\vec{E}}{\sqrt{\varepsilon}}\right]\right) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{C.1.2}$$

where now defining $\tilde{\vec{D}} \equiv \frac{\varepsilon\vec{E}}{\sqrt{\varepsilon}}$ and $\tilde{\vec{B}} \equiv \frac{\mu\vec{H}}{\sqrt{\mu}}$ and simplifying again gives

$$\begin{pmatrix} \mu^{\frac{1}{2}}\vec{\nabla} \cdot \tilde{\vec{D}} \\ -c^{-1}\mu^{\frac{1}{2}}\left(\partial_t\tilde{\vec{D}} - c\vec{\nabla} \times \tilde{\vec{B}}\right) \end{pmatrix} = \begin{pmatrix} \mu c\rho_{free} \\ \mu\vec{J}_{free} \end{pmatrix} \tag{C.1.3}$$

$$\begin{pmatrix} \mu^{\frac{1}{2}}\vec{\nabla} \cdot \tilde{\vec{B}} \\ -c^{-1}\mu^{\frac{1}{2}}\left(\partial_t\tilde{\vec{B}} + c\vec{\nabla} \times \tilde{\vec{D}}\right) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{C.1.4}$$

at this point one can multiply the bottom expression by i or $-i$ and subtract or add to get

$$
\begin{pmatrix}
\mu^{\frac{1}{2}} \left(\vec{\nabla} \cdot \tilde{\vec{D}} - \vec{\nabla} \cdot i\tilde{\vec{B}} \right) \\
c^{-1}\mu^{\frac{1}{2}} \left(-\partial_t \tilde{\vec{D}} + c\vec{\nabla} \times \tilde{\vec{B}} \right) + c^{-1}\mu^{\frac{1}{2}} \left(\partial_t i\tilde{\vec{B}} + c\vec{\nabla} \times i\tilde{\vec{D}} \right)
\end{pmatrix}
=
\begin{pmatrix}
\mu c \rho_{free} \\
\mu \vec{J}_{free}
\end{pmatrix}
$$

which upon simplification becomes

$$
\begin{pmatrix}
-i\vec{\nabla} \cdot \left(\tilde{\vec{B}} + i\tilde{\vec{D}} \right) \\
i\frac{1}{c}\partial_t \left(\tilde{\vec{B}} + i\tilde{\vec{D}} \right) + \vec{\nabla} \times \left(\tilde{\vec{B}} + i\tilde{\vec{D}} \right)
\end{pmatrix}
=
\begin{pmatrix}
\sqrt{\mu}c\rho_{free} \\
\sqrt{\mu}\vec{J}_{free}
\end{pmatrix}
$$

where now substituting $\vec{F} \equiv \tilde{\vec{B}} + i\tilde{\vec{D}} = \frac{\mu_0 \vec{H}}{\sqrt{\mu_0}} + i\frac{\varepsilon_0 \vec{E}}{\sqrt{\varepsilon_0}}$ yields

$$
\begin{pmatrix}
-i\vec{\nabla} \cdot \vec{F} \\
i\frac{1}{c}\partial_t \vec{F} + \vec{\nabla} \times \vec{F}
\end{pmatrix}
=
\begin{pmatrix}
\sqrt{\mu}c\rho_{free} \\
\sqrt{\mu}\vec{J}_{free}
\end{pmatrix}
\tag{C.1.5}
$$

C.1.3 General Description of the Photon Field Tensor

In order to re-write this expression in terms of a properly normalized single photon field tensor we expand the four notation for such an expected tensor

$$
\partial_\mu \mathfrak{G}^{\mu\nu} = \mathfrak{J}^\nu \tag{C.1.6}
$$

such that for each component μ

$$
\partial_\mu \mathfrak{G}^{\mu 0} = c^{-1}\partial_t \mathfrak{G}^{00} + \partial_x \mathfrak{G}^{10} + \partial_y \mathfrak{G}^{20} + \partial_z \mathfrak{G}^{30} \tag{C.1.7}
$$

$$
\partial_\mu \mathfrak{G}^{\mu 1} = c^{-1}\partial_t \mathfrak{G}^{01} + \partial_x \mathfrak{G}^{11} + \partial_y \mathfrak{G}^{21} + \partial_z \mathfrak{G}^{31} \tag{C.1.8}
$$

$$
\partial_\mu \mathfrak{G}^{\mu 2} = c^{-1}\partial_t \mathfrak{G}^{02} + \partial_x \mathfrak{G}^{12} + \partial_y \mathfrak{G}^{22} + \partial_z \mathfrak{G}^{32} \tag{C.1.9}
$$

$$
\partial_\mu \mathfrak{G}^{\mu 3} = c^{-1}\partial_t \mathfrak{G}^{03} + \partial_x \mathfrak{G}^{13} + \partial_y \mathfrak{G}^{23} + \partial_z \mathfrak{G}^{33} \tag{C.1.10}
$$

which comparing to the expression for $\vec{F}$ in either of assigned coordinate systems above yields

$$-i\vec{\nabla} \cdot \vec{F} = \left(-ic^{-1}\partial_t 0 - i\partial_x F_x - i\partial_y F_y - i\partial_z F_z\right) \tag{C.1.11}$$

$$\left(i\frac{1}{c}\partial_t\vec{F} + \vec{\nabla} \times \vec{F}\right)^1 = \left(ic^{-1}\partial_t F_x + \partial_x 0 + \partial_y F_z - \partial_z F_y\right) \tag{C.1.12}$$

$$\left(i\frac{1}{c}\partial_t\vec{F} + \vec{\nabla} \times \vec{F}\right)^2 = \left(ic^{-1}\partial_t F_y - \partial_x F_z + \partial_y 0 + \partial_z F_x\right) \tag{C.1.13}$$

$$\left(i\frac{1}{c}\partial_t\vec{F} + \vec{\nabla} \times \vec{F}\right)^3 = \left(ic^{-1}\partial_t F_z + \partial_x F_y - \partial_y F_x + \partial_z 0\right) \tag{C.1.14}$$

which by direct comparison allows us to construct the self-dual tensor [1]

$$\mathfrak{G}^{\mu\nu} = \begin{pmatrix} 0 & iF_x & iF_y & iF_z \\ -iF_x & 0 & -F_z & F_y \\ -iF_y & F_z & 0 & -F_x \\ -iF_z & -F_y & F_x & 0 \end{pmatrix} \tag{C.1.15}$$

which indeed satisfies the field equation

$$\partial_\mu \mathfrak{G}^{\mu\nu} = \mathfrak{J}^\nu \tag{C.1.16}$$

and $\mathfrak{J}^\nu$ is given the appropriate units.

C.1.4 Maxwell Field Interacting with Quantum Sources

Working solely in Gaussian units throughout this subsection, we reiterate that we have found how to work $\vec{F}$ and can now recognize that there are two possible ways to define it. In the discussion above we were in fact working with with $\vec{F}_+ \equiv i\vec{\mathcal{F}}_-$ from the discussion in the text, we still have to consider the case of $\vec{F}_- = -i\vec{\mathcal{F}}_+$, where respectively $\vec{\mathcal{F}}_\pm \equiv \vec{E} \pm i\vec{B}$. Previous studies have focused on investigating how this tensor equation can be expressed in terms of a Dirac like

---

[1]Note that here the advantage of using a Minkowski Metric with the signature $(-i, +, +, +)$, implemented in the derivation by Gersten and defining complex time, becomes evident in the definition of $\mathfrak{G}^{\mu\nu}$. However, for simplicity of comparison to the rest of the optics, condensed matter, and particle physics literature we remain in the continued use of a Minkowski Metric with the signature $(+, -, -, -)$.

equation [75, 66, 45, 73, 46, 47, 52, 10, 11, 50, 74, 49, 80]. To do so we have to define integer spin Dirac matrices within a representation given for the Minkowski metric of signature $(+, -, -, -)$. This set of matrices need to satisfy the following expressions (where $\bar{\psi}_{el} \equiv \psi_{el}^{\dagger}\gamma^0$)

$$\hbar\partial_\mu\mathfrak{G}^{\mu\nu} = \hbar\frac{e}{c}\bar{\psi}_{el}\gamma^\nu\psi_{el} \tag{C.1.17}$$

for both of the cases which arise from the definition of $\vec{\mathcal{F}}_\pm$

$$\begin{pmatrix} \vec{\nabla}\cdot \\ \frac{1}{c}\partial_t + \vec{\nabla}\times \end{pmatrix} i\vec{\mathcal{F}}_- = \begin{pmatrix} 4\pi\rho \\ \frac{4\pi}{c}\vec{J} \end{pmatrix} \tag{C.1.18}$$

$$\begin{pmatrix} \vec{\nabla}\cdot \\ -\frac{1}{c}\partial_t + \vec{\nabla}\times \end{pmatrix} - i\vec{\mathcal{F}}_+ = -\begin{pmatrix} 4\pi\rho \\ \frac{4\pi}{c}\vec{J} \end{pmatrix} \tag{C.1.19}$$

where γ^ν and α_{el}^i are the traditional Gamma and Dirac Matrices defined in (D.2) and for simplicity we adopted the convention $\vec{\alpha}_{el} \equiv \begin{pmatrix} \alpha_{el}^1 \\ \alpha_{el}^2 \\ \alpha_{el}^3 \end{pmatrix}$. Here we can recognize that since $\vec{\mathcal{F}}_+ = \vec{\mathcal{F}}_-^*$, these expressions can in fact be fully described by (C.1.17) and their complex conjugate. Therefore, the set of matrices γ_{ph}^ν that fully describe these expressions have been found [50, 74] to be

$$\gamma_{ph}^0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\gamma_{ph}^{1} = \begin{pmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \end{pmatrix}$$

$$\gamma_{ph}^{2} = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & i \\ -1 & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \end{pmatrix}$$

$$\gamma_{ph}^{3} = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}$$

where they guessed that the correct four notation for this vector was $F^{\nu} \equiv \begin{pmatrix} 0 \\ F_1 \\ F_2 \\ F_3 \end{pmatrix}$ allowed them to rewrite (C.1.17) as

$$i\hbar\gamma_{ph}^{\mu}\partial_{\mu}F^{\nu} = \hbar\mathfrak{J}^{\nu} \tag{C.1.20}$$

C.1.5   Conservation of Probability by Continuity Equation

C.1.5.1   Dirac Field Continuity Equation

From (C.1.17) the explicit continuity equation for probability flowing out of a quantum source can be found through

$$\begin{pmatrix} i\vec{\nabla}\cdot \\ -ic^{-1}\partial_t + \vec{\nabla}\times \end{pmatrix} \vec{\mathcal{F}} = ie_0 \frac{\mu_0}{\sqrt{\mu_0}} c\psi_{el}^{\dagger} \begin{pmatrix} I \\ \alpha_{el}^i \end{pmatrix} \psi_{el} \tag{C.1.21}$$

by first finding the divergence of

$$\nabla \cdot \left[\begin{pmatrix} -ic^{-1}\partial_t + \vec{\nabla}\times \end{pmatrix} \vec{\mathcal{F}} \right] = \nabla \cdot \left[ie_0 \frac{\mu_0}{\sqrt{\mu_0}} c\psi_{el}^{\dagger} \begin{pmatrix} \alpha_{el}^i \end{pmatrix} \psi_{el} \right] \tag{C.1.22}$$

$$-ic^{-1}\partial_t \left[\nabla \cdot \vec{\mathcal{F}} \right] = ie_0 \frac{\mu_0}{\sqrt{\mu_0}} c \left(\nabla \cdot \left[\psi_{el}^{\dagger} \begin{pmatrix} \alpha_{el}^i \end{pmatrix} \psi_{el} \right] \right) \tag{C.1.23}$$

and substituting $\nabla \cdot \vec{\mathcal{F}} = e_0 \frac{\mu_0}{\sqrt{\mu_0}} c\psi_{el}^{\dagger} I \psi_{el}$, to get a conservation law for information flowing into and out of the quantum source in the form of the continuity equation:

$$-c^{-1}\partial_t \left[\psi_{el}^{\dagger} I \psi_{el} \right] = \nabla \cdot \left[\psi_{el}^{\dagger} \begin{pmatrix} \alpha_{el}^i \end{pmatrix} \psi_{el} \right] \tag{C.1.24}$$

which upon integration over all space, using Green's Theorem, yields

$$\partial_t \left[\int d^3x \psi_{el}^{\dagger} \psi_{el} \right] = 0 \tag{C.1.25}$$

C.1.5.2   Maxwell Field Continuity Equation

As opposed to searching for the Divergence of the current $\vec{J}$, we now look for the rate of change of $\left|\vec{\mathcal{F}}\right|^2$ with respect to to time. We therefore make the following operations:

$$\vec{\mathcal{F}}_\pm^\dagger \cdot \left(\mp ic^{-1}\partial_t + \vec{\nabla}\times \right) \vec{\mathcal{F}}_\pm = \pm ie_0 \frac{\mu_0}{\sqrt{\mu_0}} c\vec{\mathcal{F}}_\pm^\dagger \cdot \psi_{el}^\dagger\ \vec{\alpha}_{el}\ \psi_{el} \tag{C.1.26}$$

$$\vec{\mathcal{F}}_\pm \cdot \left(\pm ic^{-1}\partial_t + \vec{\nabla}\times \right) \vec{\mathcal{F}}_\pm^\dagger = \mp ie_0 \frac{\mu_0}{\sqrt{\mu_0}} c\vec{\mathcal{F}}_\pm \cdot \psi_{el}^\dagger\ \vec{\alpha}_{el}\ \psi_{el} \tag{C.1.27}$$

Where subtracting one from two, we get

$$\pm ic^{-1}\left(\vec{\mathcal{F}}_\pm \cdot \partial_t\vec{\mathcal{F}}_\pm^\dagger + \vec{\mathcal{F}}_\pm^\dagger \cdot \partial_t\vec{\mathcal{F}}_\pm \right) \pm \left(\vec{\mathcal{F}}_\pm \cdot \vec{\nabla}\times \vec{\mathcal{F}}_\pm^\dagger - \vec{\mathcal{F}}_\pm^\dagger \cdot \vec{\nabla}\times \vec{\mathcal{F}}_\pm \right) \tag{C.1.28}$$

$$= \mp ie_0 \frac{\mu_0}{\sqrt{\mu_0}} c\left(\vec{\mathcal{F}}_\pm + \vec{\mathcal{F}}_\pm^\dagger \right) \cdot \psi_{el}^\dagger\ \vec{\alpha}_{el}\ \psi_{el} \tag{C.1.29}$$

which simplifies to

$$\mp ic^{-1}\partial_t\left(\vec{\mathcal{F}}_\pm \cdot \vec{\mathcal{F}}_\pm^\dagger \right) \mp \vec{\nabla}\cdot \left(\vec{\mathcal{F}}_\pm \times \vec{\mathcal{F}}_\pm^\dagger \right) = ie_0 \frac{\mu_0}{\sqrt{\mu_0}} c\left(\vec{\mathcal{F}}_\pm + \vec{\mathcal{F}}_\pm^\dagger \right) \cdot \psi_{el}^\dagger\ \vec{\alpha}_{el}\ \psi_{el} \tag{C.1.30}$$

C.2   Spinor Formulation of Electrodynamics and it's correspondence to the PWF

We begin using the conversion rules from vector notation to matrix notation

$$\vec{a}\times\vec{b} \;=\; -i\left(\vec{a}\cdot\overleftrightarrow{s}\right)\vec{b} \tag{C.2.1}$$

where $\overleftrightarrow{s}$ is given by from

$$s_x \;=\; \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}$$

$$s_y = \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{pmatrix}$$

$$s_z = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Given the original set of equations prior to the substitution $\dfrac{\vec{k}}{k} \times \hat{\epsilon}_{\vec{k},\sigma_\lambda} = -i\hat{\epsilon}_{\vec{k},\sigma_\lambda}$ in the original form

$$\vec{\nabla} \times \vec{\varphi}_\gamma = -c^{-1}\partial_t \vec{\chi}_\gamma \tag{C.2.2}$$

$$\vec{\nabla} \times \vec{\chi}_\gamma = c^{-1}\partial_t \vec{\varphi}_\gamma \tag{C.2.3}$$

we rewrite these in terms of their matrix form

$$-i\left(\vec{\nabla} \cdot \overleftrightarrow{s}\right)\vec{\varphi}_\gamma = -c^{-1}\partial_t \vec{\chi}_\gamma \tag{C.2.4}$$

$$-i\left(\vec{\nabla} \cdot \overleftrightarrow{s}\right)\vec{\chi}_\gamma = c^{-1}\partial_t \vec{\varphi}_\gamma \tag{C.2.5}$$

where recognizing that $\vec{p} = \dfrac{\hbar}{i}\vec{\nabla}$ rearranging terms yields

$$i\hbar\partial_t \vec{\chi}_\gamma = -c\left(\overleftrightarrow{s} \cdot \vec{p}\right)\vec{\varphi}_\gamma \tag{C.2.6}$$

$$i\hbar\partial_t \vec{\varphi}_\gamma = c\left(\overleftrightarrow{s} \cdot \vec{p}\right)\vec{\chi}_\gamma \tag{C.2.7}$$

which defining $\vec{\Phi}_\gamma \equiv \begin{pmatrix} \vec{\varphi}_\gamma \\ \vec{\chi}_\gamma^\dagger \end{pmatrix}$ yields the spinor formulation for the freely propagating PWF as

$$i\hbar\partial_t\vec{\Phi}_\gamma = \begin{pmatrix} 0 & -c\overleftrightarrow{s}\cdot\vec{p} \\ c\overleftrightarrow{s}\cdot\vec{p} & 0 \end{pmatrix}\vec{\Phi}_\gamma \tag{C.2.8}$$

C.3 Proof that $\frac{1}{\sqrt{2}}(1\pm i\lambda) = e^{\pm\lambda\frac{\pi}{4}}$

Since $\lambda \in \{-1, 1\}$

$$\begin{aligned}
\frac{1}{\sqrt{2}}(1\pm i\lambda) &= \frac{1}{\sqrt{2}}\left(e^{i0} + e^{\pm i\frac{\pi}{2}}e^{\pi i\lambda}\right) \\
&= \frac{e^{\pm i\frac{\pi}{4}}}{\sqrt{2}}\left(e^{\mp i\frac{\pi}{4}} + e^{\pm i\frac{\pi}{4}}e^{\pi i\lambda}\right) \\
&= \frac{e^{\pm i\frac{\pi}{4}}}{\sqrt{2}}\begin{cases} e^{\mp i\frac{\pi}{4}} + e^{\pm i\frac{\pi}{4}}, & \lambda = 1 \\ e^{\mp i\frac{\pi}{4}} - e^{\pm i\frac{\pi}{4}}, & \lambda = -1 \end{cases}
\end{aligned}$$

introducing the manipulation $\frac{1}{2i}2i$ for the case where $\lambda = -1$ and $\frac{1}{2}2$ for the case where $\lambda = 1$ yields:

$$\begin{aligned}
\frac{1}{\sqrt{2}}(1\pm i\lambda) &= \frac{e^{\pm i\frac{\pi}{4}}}{\sqrt{2}}\begin{cases} 2\frac{1}{2}\left(e^{\mp i\frac{\pi}{4}} + e^{\pm i\frac{\pi}{4}}\right), & \lambda = 1 \\ 2i\frac{1}{2i}\left(e^{\mp i\frac{\pi}{4}} - e^{\pm i\frac{\pi}{4}}\right), & \lambda = -1 \end{cases} \\
&= \frac{e^{\pm i\frac{\pi}{4}}}{\sqrt{2}}\begin{cases} 2\cos\left(\frac{\pi}{4}\right), & \lambda = 1 \\ \mp 2i\sin\left(\frac{\pi}{4}\right), & \lambda = -1 \end{cases} \\
&= e^{\pm i\frac{\pi}{4}}\begin{cases} 1, & \lambda = 1 \\ \mp i, & \lambda = -1 \end{cases}
\end{aligned}$$

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again expressing $\mp i = e^{\mp i \frac{\pi}{2}}$ yields

$$\frac{1}{\sqrt{2}} (1 \pm i\lambda) = \begin{cases} e^{\pm i \frac{\pi}{4}}, \ \lambda = 1 \\ e^{\mp i \frac{\pi}{2}} e^{\pm i \frac{\pi}{4}}, \ \lambda = -1 \end{cases}$$

$$= \begin{cases} e^{\pm i \frac{\pi}{4}}, \ \lambda = 1 \\ e^{\mp i \frac{\pi}{4}}, \ \lambda = -1 \end{cases}$$

$$= e^{\pm i\lambda \frac{\pi}{4}}$$

where we have what we set out to prove

$$\frac{1}{\sqrt{2}} (1 \pm i\lambda) = e^{\pm i\lambda \frac{\pi}{4}}$$

in fact this comes from the fact that the square root of any complex number is also a complex number. This means that our problem is equivalent to finding the complex form of the $\sqrt{i}$. From Demoivre's formula

$$\sqrt{i} = \cos\left(\frac{\pi}{4}\right) \pm i \sin\left(\frac{\pi}{4}\right)$$

$$= \frac{1}{\sqrt{2}} (1 \pm i)$$

$$= e^{\pm i \frac{\pi}{4}}$$

the lambda can be introduces seamlessly by setting

$$\cos\left(\lambda \frac{\pi}{4}\right) \pm i \sin\left(\lambda \frac{\pi}{4}\right) = \cos\left(\frac{\pi}{4}\right) \pm i\lambda \sin\left(\frac{\pi}{4}\right)$$

$$= \frac{1}{\sqrt{2}} (1 \pm i\lambda)$$

$$= e^{\pm i\lambda \frac{\pi}{4}}$$

Starting from the expressions

$$
\begin{aligned}
\vec{E}\left(\vec{r},t\right) &= \sum_{\vec{k},\lambda} \hat{\epsilon}_{\vec{k},\sigma_\lambda} \mathcal{E}_{\vec{k}} a_{\vec{k},\sigma_\lambda} e^{-i\nu_k t} U_{\vec{k}}\left(\vec{r}\right) + \text{H.c.}\\
\vec{B}\left(\vec{r},t\right) &= \sum_{\vec{k},\lambda} \frac{\vec{k}\times\hat{\epsilon}_{\vec{k},\sigma_\lambda}}{\nu_k} \mathcal{E}_{\vec{k}} a_{\vec{k},\sigma_\lambda} e^{-i\nu_k t} U_{\vec{k}}\left(\vec{r}\right) + \text{H.c.}
\end{aligned}
$$

where

$$
\mathcal{E}_{\vec{k}} = \sqrt{\frac{\hbar\nu_k}{2\varepsilon_0 V}}
$$

and

$$
\vec{B}\left(\vec{r},t\right) = \mu_0 \vec{H}\left(\vec{r},t\right)
$$

we may rewrite the expressions for $\vec{E}\left(\vec{r},t\right)$ & $\vec{B}\left(\vec{r},t\right)$

$$
\begin{aligned}
\vec{E}\left(\vec{r},t\right) &= \sum_{\vec{k},\lambda} \hat{\epsilon}_{\vec{k},\sigma_\lambda} \sqrt{\frac{\hbar\nu_k}{2\varepsilon_0 V}} a_{\vec{k},\sigma_\lambda} e^{-i\nu_k t} U_{\vec{k}}\left(\vec{r}\right) + \text{H.c.}\\
\vec{H}\left(\vec{r},t\right) &= \frac{1}{\mu_0}\sum_{\vec{k},\lambda} \frac{\vec{k}\times\hat{\epsilon}_{\vec{k},\sigma_\lambda}}{\nu_k} \sqrt{\frac{\hbar\nu_k}{2\varepsilon_0 V}} a_{\vec{k},\sigma_\lambda} e^{-i\nu_k t} U_{\vec{k}}\left(\vec{r}\right) + \text{H.c.}
\end{aligned}
$$

Taking advantage of the fact that $\varepsilon_0\mu_0 = \dfrac{1}{c^2}$ and $k = \dfrac{\nu_k}{c}$ making the substitution

$$
\begin{aligned}
\frac{1}{\mu_0}\frac{1}{\nu_k}\sqrt{\frac{\hbar\nu_k}{2\varepsilon_0 V}} &= \frac{1}{\nu_k}\sqrt{\frac{1}{\mu_0}}\sqrt{\frac{\hbar\nu_k}{2\mu_0\varepsilon_0 V}}\\
&= \frac{c}{\nu_k}\sqrt{\frac{1}{\mu_0}}\sqrt{\frac{\hbar\nu_k}{2V}}\\
&= \frac{1}{k}\sqrt{\frac{\hbar\nu_k}{2\mu_0 V}}
\end{aligned}
$$

yields

$$\vec{E}\left(\vec{r},t\right) = \sum_{\vec{k},\lambda} \hat{\epsilon}_{\vec{k},\sigma_\lambda} \sqrt{\frac{\hbar\nu_k}{2\varepsilon_0 V}} a_{\vec{k},\sigma_\lambda} e^{-i\nu_k t} U_{\vec{k}}\left(\vec{r}\right) + \text{H.c.}$$

$$\vec{H}\left(\vec{r},t\right) = \sum_{\vec{k},\lambda} \frac{\vec{k} \times \hat{\epsilon}_{\vec{k},\sigma_\lambda}}{k} \sqrt{\frac{\hbar\nu_k}{2\mu_0 V}} a_{\vec{k},\sigma_\lambda} e^{-i\nu_k t} U_{\vec{k}}\left(\vec{r}\right) + \text{H.c.}$$

Following the construction of a photonic state (2.1.11), assumed to be created by the decay of an atomic or QD state, leads to a sharply peaked single photon state about a frequency ω [15]. This allows us to treat ν_k as a slowly varying frequency and replace it by ω in the square root factor. This leaves us with

$$\left\langle \vec{E}\left(\vec{r},t\right)\right\rangle = \sqrt{\frac{\hbar\omega}{2\varepsilon_0 V}} \left\langle \sum_{\vec{k},\lambda} \hat{\epsilon}_{\vec{k},\sigma_\lambda} a_{\vec{k},\sigma_\lambda} e^{-i\nu_k t} U_{\vec{k}}\left(\vec{r}\right) + \text{H.c.} \right\rangle$$

$$\left\langle \vec{H}\left(\vec{r},t\right)\right\rangle = \sqrt{\frac{\hbar\omega}{2\mu_0 V}} \left\langle \sum_{\vec{k},\lambda} \frac{\vec{k} \times \hat{\epsilon}_{\vec{k},\sigma_\lambda}}{k} a_{\vec{k},\sigma_\lambda} e^{-i\nu_k t} U_{\vec{k}}\left(\vec{r}\right) + \text{H.c.} \right\rangle$$

which under the construction of (2.1.11) as a single photon state lead directly to the Electric and Magnetic wave functions

$$\vec{\varphi}_\gamma\left(\vec{r},t\right) \equiv \sqrt{\frac{2\epsilon_0}{\hbar\omega}} \vec{\Psi}_E\left(\vec{r},t\right)$$

$$\vec{\varphi}_\gamma\left(\vec{r},t\right) = \langle 0 | \frac{1}{\sqrt{V}} \sum_{\vec{k},\lambda} \hat{\epsilon}_{\vec{k},\sigma_\lambda} a_{\vec{k},\sigma_\lambda} e^{-i\nu_k t} U_{\vec{k}}\left(\vec{r}\right) |\gamma\rangle$$

and

$$\vec{\chi}_\gamma\left(\vec{r},t\right) \equiv \sqrt{\frac{2\mu_0}{\hbar\omega}} \vec{\Psi}_H\left(\vec{r},t\right)$$

$$= \langle 0 | \frac{1}{\sqrt{V}} \sum_{\vec{k},\lambda} \frac{\vec{k}}{k} \times \hat{\epsilon}_{\vec{k},\sigma_\lambda} a_{\vec{k},\sigma_\lambda} e^{-i\nu_k t} U_{\vec{k}}\left(\vec{r}\right) |\gamma\rangle$$

C.5 Proof that $\dfrac{\vec{k}}{k} \times \hat{\epsilon}_{\vec{k},\sigma_\lambda} = -i\lambda\hat{\epsilon}_{\vec{k},\sigma_\lambda}$ up to a phase factor

The definition of these helical polarization vectors has been proposed in previous studies [10, 11] and filling in intermediate steps we start by multiplying (C.2.2) by i & adding (C.2.3) yields

$$i\partial_t\left(\vec{\varphi}_\gamma + i\vec{\chi}_\gamma\right) = c\vec{\nabla} \times \left(\vec{\varphi}_\gamma + i\vec{\chi}_\gamma\right) \tag{C.5.1}$$

where adding the conjugate equations yields

$$-i\partial_t\left(\vec{\varphi}_\gamma^\dagger - i\vec{\chi}_\gamma^\dagger\right) = c\vec{\nabla} \times \left(\vec{\varphi}_\gamma^\dagger - i\vec{\chi}_\gamma^\dagger\right) \tag{C.5.2}$$

which can be rewritten in terms of $\vec{\xi}_{\gamma,\pm} \equiv \left(\vec{\varphi}_\gamma^\dagger \pm i\vec{\chi}_\gamma^\dagger\right)$ as general complex wave equation

$$\pm i\partial_t\vec{\xi}_{\gamma,\pm} = c\vec{\nabla} \times \vec{\xi}_{\gamma,\pm} \tag{C.5.3}$$

which by

$$\vec{\nabla} \cdot \vec{\chi}_\gamma = 0 \tag{C.5.4}$$

$$\vec{\nabla} \cdot \vec{\varphi}_\gamma = 0 \tag{C.5.5}$$

is shown to also satisfy

$$\vec{\nabla} \cdot \vec{\xi}_{\gamma,\pm} = 0 \tag{C.5.6}$$

and therefore satisfies

$$\Box^2\vec{\xi}_{\gamma,\pm} = 0$$

where $\square^2 \equiv \nabla^2 - c^{-2}\partial_t^2$ is the D'Alembertian operator and $\vec{\xi}_{\gamma,\pm} = \vec{\xi}_{\gamma,\mp}^\dagger$. This means that the vector wave functions $\vec{\xi}_{\gamma,\pm}$ can be expressed as superpositions of plane waves

$$\vec{\xi}_{\gamma,\pm}(\vec{r},t) = \int \frac{d^3k}{(2\pi)^3}\left\{\vec{\eta}_{\gamma,\pm}\left(\vec{k}\right)e^{-i\omega t + i\vec{k}\cdot\vec{r}} + \vec{\eta}_{\gamma,\mp}^*\left(\vec{k}\right)e^{i\omega t - i\vec{k}\cdot\vec{r}}\right\} \qquad (\text{C.5.7})$$

with $\omega = ck$. Using the fact that $\vec{\xi}_{\gamma,\pm}(\vec{r},t)$ must satisfy (C.5.1) & (C.5.2) respectively, substituting (C.5.7) into these gives

$$\pm i\partial_t \int \frac{d^3k}{(2\pi)^3}\left\{\vec{\eta}_{\gamma,\pm}\left(\vec{k}\right)e^{-i\omega t + i\vec{k}\cdot\vec{r}} + \text{H.c.}\right\} = c\vec{\nabla}\times\int \frac{d^3k}{(2\pi)^3}\left\{\vec{\eta}_{\gamma,\pm}\left(\vec{k}\right)e^{-i\omega t + i\vec{k}\cdot\vec{r}} + \text{H.c.}\right\}$$

$$(\text{C.5.8})$$

which moving the differential operators into the integrals (assuming $\vec{k} \nrightarrow \vec{k}(\vec{r},t)$) require the evaluations

$$\partial_t\left[\vec{\eta}_{\gamma,\pm}\left(\vec{k}\right)e^{-i\omega t + i\vec{k}\cdot\vec{r}}\right] = -i\omega\vec{\eta}_{\gamma,\pm}\left(\vec{k}\right)e^{-i\omega t + i\vec{k}\cdot\vec{r}} \qquad (\text{C.5.9})$$

$$\partial_t\left[\vec{\eta}_{\gamma,\mp}^*\left(\vec{k}\right)e^{i\omega t - i\vec{k}\cdot\vec{r}}\right] = i\omega\vec{\eta}_{\gamma,\mp}^*\left(\vec{k}\right)e^{i\omega t - i\vec{k}\cdot\vec{r}} \qquad (\text{C.5.10})$$

$$\vec{\nabla}\times\left[\vec{\eta}_{\gamma,\pm}\left(\vec{k}\right)e^{-i\omega t + i\vec{k}\cdot\vec{r}}\right] = i\vec{k}\times\vec{\eta}_{\gamma,\pm}\left(\vec{k}\right)e^{-i\omega t + i\vec{k}\cdot\vec{r}} \qquad (\text{C.5.11})$$

$$\vec{\nabla}\times\left[\vec{\eta}_{\gamma,\mp}^*\left(\vec{k}\right)e^{i\omega t - i\vec{k}\cdot\vec{r}}\right] = -i\vec{k}\times\vec{\eta}_{\gamma,\mp}^*\left(\vec{k}\right)e^{i\omega t - i\vec{k}\cdot\vec{r}} \qquad (\text{C.5.12})$$

substituting and using $k = \frac{\omega}{c}$

$$\pm i\int \frac{d^3k}{(2\pi)^3}\left\{-ik\vec{\eta}_{\gamma,\pm}\left(\vec{k}\right)e^{-i\omega t + i\vec{k}\cdot\vec{r}} + \text{H.c.}\right\} = \int \frac{d^3k}{(2\pi)^3}\left\{i\vec{k}\times\vec{\eta}_{\gamma,\pm}\left(\vec{k}\right)e^{-i\omega t + i\vec{k}\cdot\vec{r}} + \text{H.c.}\right\} \qquad (\text{C.5.13})$$

yields

$$\int \frac{d^3k}{8\pi^3}\left[\pm k\vec{\eta}_{\gamma,\pm}\left(\vec{k}\right) - i\vec{k}\times\vec{\eta}_{\gamma,\pm}\left(\vec{k}\right)\right]e^{-i\omega t + i\vec{k}\cdot\vec{r}} = \int \frac{d^3k}{8\pi^3}\left[\pm k\vec{\eta}_{\gamma,\mp}^*\left(\vec{k}\right) - i\vec{k}\times\vec{\eta}_{\gamma,\mp}^*\left(\vec{k}\right)\right]e^{i\omega t - i\vec{k}\cdot\vec{r}}$$

which is true only when

$$\frac{\vec{k}}{k} \times \vec{\eta}_{\gamma,\pm}\left(\vec{k}\right) = \mp i\vec{\eta}_{\gamma,\pm}\left(\vec{k}\right) \tag{C.5.14}$$

$$\frac{\vec{k}}{k} \times \vec{\eta}_{\gamma,\mp}^{*}\left(\vec{k}\right) = \pm i\vec{\eta}_{\gamma,\mp}^{*}\left(\vec{k}\right) \tag{C.5.15}$$

Normalizing these, proves that there exists a unit vector that satisfies

$$\frac{\vec{k}}{k} \times \hat{\epsilon}_{\vec{k},\sigma_\lambda} = -i\lambda\hat{\epsilon}_{\vec{k},\sigma_\lambda} \tag{C.5.16}$$

$$\frac{\vec{k}}{k} \times \hat{\epsilon}_{\vec{k},\sigma_\lambda}^{*} = i\lambda\hat{\epsilon}_{\vec{k},\sigma_\lambda}^{*} \tag{C.5.17}$$

and

$$\hat{\epsilon}_{\vec{k},\sigma_\lambda}^{*} \cdot \hat{\epsilon}_{\vec{k}',\sigma_{\lambda'}} = \delta_{\vec{k}.\vec{k}'}\delta_{\sigma_\lambda,\sigma_\lambda'}$$

These relations can be clearly represented by expressing $\hat{\epsilon}_{\vec{k},\sigma_\lambda}$ in terms of the orthonormal vectors $\hat{\epsilon}_{\vec{k},i} \cdot \hat{\epsilon}_{\vec{k},j} = \delta_{ij}$ through

$$\hat{\epsilon}_{\vec{k},\sigma_+} \equiv \frac{1}{\sqrt{2}}\left(\hat{\epsilon}_{\vec{k},i} + i\hat{\epsilon}_{\vec{k},j}\right) \tag{C.5.18}$$

$$\hat{\epsilon}_{\vec{k},\sigma_-} \equiv \frac{1}{\sqrt{2}}\left(\hat{\epsilon}_{\vec{k},i} - i\hat{\epsilon}_{\vec{k},j}\right) \tag{C.5.19}$$

$$\hat{\epsilon}_{\vec{k},i} \times \hat{\epsilon}_{\vec{k},j} \equiv \frac{\vec{k}}{k} \tag{C.5.20}$$

some useful identities of these applied in the text include

$$\hat{\epsilon}_{\vec{k},\sigma_+} = \hat{\epsilon}_{\vec{k},\sigma_-}^{*} \tag{C.5.21}$$

$$\hat{\epsilon}_{\vec{k},\sigma_-} = \hat{\epsilon}_{\vec{k},\sigma_+}^{*} \tag{C.5.22}$$

$$\hat{\epsilon}_{\vec{k},i} = \frac{1}{\sqrt{2}}\left(\hat{\epsilon}_{\vec{k},\sigma_+} + \hat{\epsilon}_{\vec{k},\sigma_-}\right) \tag{C.5.23}$$

$$\hat{\epsilon}_{\vec{k},j} = \frac{i}{\sqrt{2}}\left(\hat{\epsilon}_{\vec{k},\sigma_-} - \hat{\epsilon}_{\vec{k},\sigma_+}\right) \tag{C.5.24}$$

$$\hat{\epsilon}_{\vec{k},i} + \hat{\epsilon}_{\vec{k},j} = \frac{1}{\sqrt{2}}\left[\hat{\epsilon}_{\vec{k},\sigma_+}\left(1-i\right) + \hat{\epsilon}_{\vec{k},\sigma_-}\left(1+i\right)\right] \tag{C.5.25}$$

$$= \left[\hat{\epsilon}_{\vec{k},\sigma_+}e^{-i\frac{\pi}{4}} + \hat{\epsilon}_{\vec{k},\sigma_-}e^{i\frac{\pi}{4}}\right] \tag{C.5.26}$$

$$\hat{\epsilon}^*_{\vec{k},\sigma_+}\hat{\epsilon}_{\vec{k},\sigma_+} = \frac{1}{2}\left(\hat{\epsilon}_{\vec{k},i} - i\hat{\epsilon}_{\vec{k},j}\right)\left(\hat{\epsilon}_{\vec{k},i} + i\hat{\epsilon}_{\vec{k},j}\right) \tag{C.5.27}$$

$$= \frac{1}{2}\left(\hat{\epsilon}_{\vec{k},i}\hat{\epsilon}_{\vec{k},i} + \hat{\epsilon}_{\vec{k},j}\hat{\epsilon}_{\vec{k},j} + i\hat{\epsilon}_{\vec{k},i}\hat{\epsilon}_{\vec{k},j} - i\hat{\epsilon}_{\vec{k},j}\hat{\epsilon}_{\vec{k},i}\right) \tag{C.5.28}$$

$$\hat{\epsilon}^*_{\vec{k},\sigma_-}\hat{\epsilon}_{\vec{k},\sigma_-} = \frac{1}{2}\left(\hat{\epsilon}_{\vec{k},i} + i\hat{\epsilon}_{\vec{k},j}\right)\left(\hat{\epsilon}_{\vec{k},i} - i\hat{\epsilon}_{\vec{k},j}\right) \tag{C.5.29}$$

$$= \frac{1}{2}\left(\hat{\epsilon}_{\vec{k},i}\hat{\epsilon}_{\vec{k},i} + \hat{\epsilon}_{\vec{k},j}\hat{\epsilon}_{\vec{k},j} - i\hat{\epsilon}_{\vec{k},i}\hat{\epsilon}_{\vec{k},j} + i\hat{\epsilon}_{\vec{k},j}\hat{\epsilon}_{\vec{k},i}\right) \tag{C.5.30}$$

$$\hat{\epsilon}^*_{\vec{k},\sigma_+}\hat{\epsilon}_{\vec{k},\sigma_+} + \hat{\epsilon}^*_{\vec{k},\sigma_-}\hat{\epsilon}_{\vec{k},\sigma_-} = \frac{1}{2}\left(\hat{\epsilon}_{\vec{k},i}\hat{\epsilon}_{\vec{k},i} + \hat{\epsilon}_{\vec{k},j}\hat{\epsilon}_{\vec{k},j} + i\hat{\epsilon}_{\vec{k},i}\hat{\epsilon}_{\vec{k},j} - i\hat{\epsilon}_{\vec{k},j}\hat{\epsilon}_{\vec{k},i}\right)$$

$$+ \frac{1}{2}\left(\hat{\epsilon}_{\vec{k},i}\hat{\epsilon}_{\vec{k},i} + \hat{\epsilon}_{\vec{k},j}\hat{\epsilon}_{\vec{k},j} - i\hat{\epsilon}_{\vec{k},i}\hat{\epsilon}_{\vec{k},j} + i\hat{\epsilon}_{\vec{k},j}\hat{\epsilon}_{\vec{k},i}\right)$$

$$= \hat{\epsilon}_{\vec{k},i}\hat{\epsilon}_{\vec{k},i} + \hat{\epsilon}_{\vec{k},j}\hat{\epsilon}_{\vec{k},j}$$

$$= \vec{1} - \frac{\vec{k}\vec{k}}{k^2}$$

C.6 *Discussion on the Cubic Roots of the Excited State Characteristic Equations*

In this section we discuss in detail the results and properties pertaining to the roots of the characteristic equations for the evolution of the excited state/s of quantum sources beyond the Weisskopf Wigner Approximation.

C.6.1   Single Photon interacting with Two level sources

We left the discussion for the evolution of this system in (4.3.2.2.3) after expressing the roots of the characteristic equation, though we did not give the explicit result of these. We omitted these details to avoid any additional confusion due to obfuscated notation. In this subsection we explicitly outline what these roots will be explicitly. We left of with the definitions

$$
S = \left(\frac{\hbar\omega_\sigma}{2\lambda}\right)^{\frac{1}{3}} \sqrt[3]{1 + \sqrt{1 + \frac{4i}{27}\frac{\hbar}{\lambda\omega_\sigma^2}}}
$$

$$
T = \left(\frac{\hbar\omega_\sigma}{2\lambda}\right)^{\frac{1}{3}} \sqrt[3]{1 - \sqrt{1 + \frac{4i}{27}\frac{\hbar}{\lambda\omega_\sigma^2}}}
$$

Referring to the discussion (D.9), using Demoivre's Theorem, it becomes possible to first rewrite S and T in terms of the dimensionless constant $\varsigma \equiv \frac{\hbar}{\lambda\omega_\sigma^2}$ and

$$
\Re\{\alpha_1\} = 1
$$

$$
\Im\{\alpha_1\} = \frac{4\varsigma}{27}
$$

$$
\theta_1 = \arg(\alpha_1)
$$

as

$$
S = \left(\frac{\hbar\omega_\sigma}{2\lambda}\right)^{\frac{1}{3}} \sqrt[3]{1 + |\alpha_1|^{\frac{1}{2}}\cos\frac{\theta_1}{2} + i\,|\alpha_1|^{\frac{1}{2}}\sin\frac{\theta_1}{2}}
$$

$$
= \left(\frac{\hbar\omega_\sigma}{2\lambda}\right)^{\frac{1}{3}} \sqrt[3]{\left(1 + |\alpha_1|^{\frac{1}{2}}\cos\frac{\theta_1}{2}\right) + i\left(|\alpha_1|^{\frac{1}{2}}\sin\frac{\theta_1}{2}\right)}
$$

and

$$
\begin{aligned}
T &= \left(\frac{\hbar \omega_\sigma}{2\lambda} \right)^{\frac{1}{3}} \sqrt[3]{1 - |\alpha_1|^{\frac{1}{2}} \cos \frac{\theta_1}{2} - i \, |\alpha_1|^{\frac{1}{2}} \sin \frac{\theta_1}{2}} \\
&= \left(\frac{\hbar \omega_\sigma}{2\lambda} \right)^{\frac{1}{3}} \sqrt[3]{\left(1 - |\alpha_1|^{\frac{1}{2}} \cos \frac{\theta_1}{2} \right) - i \left(|\alpha_1|^{\frac{1}{2}} \sin \frac{\theta_1}{2} \right)}
\end{aligned}
$$

Referring again to the discussion (D.9) and using Demoivre's Theorem, we can now completely separate real and imaginary parts under the cube root to find that by defining

$$
\Re \left\{ \alpha_{2,\pm} \right\} = \left(1 \pm |\alpha_1|^{\frac{1}{2}} \cos \frac{\theta_1}{2} \right)
$$

$$
\Im \left\{ \alpha_{2,\pm} \right\} = \pm \left(|\alpha_1|^{\frac{1}{2}} \sin \frac{\theta_1}{2} \right)
$$

$$
\theta_{2,\pm} = \arg \left(\alpha_{2,\pm} \right)
$$

we get

$$
S = \left(\frac{\hbar \omega_\sigma}{2\lambda} \right)^{\frac{1}{3}} |\alpha_{2,+}|^{\frac{1}{3}} \left(\cos \frac{\theta_{2,+}}{3} + i \sin \frac{\theta_{2,+}}{3} \right)
$$

and

$$
T = \left(\frac{\hbar \omega_\sigma}{2\lambda} \right)^{\frac{1}{3}} |\alpha_{2,-}|^{\frac{1}{3}} \left(\cos \frac{\theta_{2,-}}{3} + i \sin \frac{\theta_{2,-}}{3} \right)
$$

where clearly $|\alpha_{2,+}| = |\alpha_{2,-}| = |\alpha_2|$. Finally though it is impractical to write down all 3 roots, we will still do so for completeness, though we limit ourselves to writing them down in terms of $\alpha_{2,\pm}$ and $\theta_{2,\pm}$ as

- z_1

$$
-\frac{1}{3} a_2 + (S + T) = i\omega_\sigma + \left(\frac{\hbar \omega_\sigma}{2\lambda} \right)^{\frac{1}{3}} \sum_{l=\pm} |\alpha_{2,\pm}|^{\frac{1}{3}} \left[\cos \frac{\theta_{2,l}}{3} + i \sin \frac{\theta_{2,l}}{3} \right]
$$

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- z_2

$$-\frac{1}{3}a_2 - \left(1 - i\sqrt{3}\right)\frac{S}{2} - \left(1 + i\sqrt{3}\right)\frac{T}{2} = i\omega_\sigma$$

$$-\left(\frac{\hbar\omega_\sigma}{2\lambda}\right)^{\frac{1}{3}}\sum_{l=\pm}\frac{|\alpha_{2,\pm}|^{\frac{1}{3}}}{2}\left[\left(\cos\frac{\theta_{2,l}}{3} - l\sqrt{3}\sin\frac{\theta_{2,l}}{3}\right) + il\left(\sqrt{3}\cos\frac{\theta_{2,l}}{3} + l\sin\frac{\theta_{2,l}}{3}\right)\right]$$

- z_3

$$-\frac{1}{3}a_2 - \left(1 + i\sqrt{3}\right)\frac{S}{2} - \left(1 - i\sqrt{3}\right)\frac{T}{2} = i\omega_\sigma$$

$$-\left(\frac{\hbar\omega_\sigma}{2\lambda}\right)^{\frac{1}{3}}\sum_{l=\pm}\frac{|\alpha_{2,\pm}|^{\frac{1}{3}}}{2}\left[\left(\cos\frac{\theta_{2,l}}{3} + l\sqrt{3}\sin\frac{\theta_{2,l}}{3}\right) - il\left(\sqrt{3}\cos\frac{\theta_{2,l}}{3} + \sin\frac{\theta_{2,l}}{3}\right)\right]$$

C.7 Equation of Motion Integrals in k-space

We can asses the value of the vector

$$\vec{\mathcal{I}}_{\pm,n}(x) = \int d\vec{k}\,\frac{e^{-i\nu_k\frac{1}{2}(t_f+t_0)}\left[e^{i(\nu_k-\omega_\sigma-iz_n)t_f} - e^{i(\nu_k-\omega_\sigma-iz_n)t_0}\right]}{(\nu_k - \omega_\sigma - iz_n)}\nu_k e^{-i\vec{k}\cdot(\vec{x_0}-\vec{x})}\left(\hat{\epsilon}_{\vec{k},\sigma_\pm}\hat{\epsilon}^*_{\vec{k},\sigma_\pm}\cdot\vec{\wp}_{ab}\right) \quad \text{(C.7.1)}$$

by setting the coordinate system such that $\vec{k}$ always points along some radial direction from the origin of a quantum source. We start by defining the unit wave vector as

$$\hat{k} \equiv \hat{r} \quad \text{(C.7.2)}$$

Defining $\vec{k}$ radially fixes the polarization vectors to point along orientations that are orthogonal to $\hat{r}$. We choose the polarization vectors so as to retain the required spin nature of the photon by following the identity (2.1.32) and setting

$$\hat{\epsilon}_{\vec{k},\sigma_+} = \frac{1}{\sqrt{2}}\left(\hat{\theta} + i\hat{\phi}\right) \quad \text{(C.7.3)}$$

$$\hat{\epsilon}_{\vec{k},\sigma_-} = \frac{1}{\sqrt{2}}\left(\hat{\theta} - i\hat{\phi}\right) \quad \text{(C.7.4)}$$

where for completeness we also reiterate

$$\hat{\epsilon}_{\vec{k},i} \equiv \hat{\theta}$$

$$\hat{\theta} = (\cos\theta\cos\phi\hat{x} + \cos\theta\sin\phi\hat{y} - \sin\theta\hat{z})$$

$$\hat{\epsilon}_{\vec{k},j} \equiv \hat{\phi}$$

$$\hat{\phi} = (-\sin\phi\hat{x} + \cos\phi\hat{y} + 0\hat{z})$$

In this coordinate system[2]. Expressing the wave and polarization vectors in terms of Cartesian unit vectors yields (for compactness we define $\mathcal{C}_\eta \equiv \cos\eta$, $\mathcal{S}_\eta \equiv \sin\eta$)

$$\hat{k} = \mathcal{S}_\theta\mathcal{C}_\phi\hat{x} + \mathcal{S}_\theta\mathcal{S}_\phi\hat{y} + \mathcal{C}_\theta\hat{z}$$

$$\hat{\epsilon}_{\vec{k},\sigma_+} = \frac{1}{\sqrt{2}} \left[(\mathcal{C}_\theta\mathcal{C}_\phi\hat{x} + \mathcal{C}_\theta\mathcal{S}_\phi\hat{y} - \mathcal{S}_\theta\hat{z}) + i \left(-\mathcal{S}_\phi\hat{x} + \mathcal{C}_\phi\hat{y} + 0\hat{z} \right) \right]$$

$$\hat{\epsilon}_{\vec{k},\sigma_-} = \frac{1}{\sqrt{2}} \left[(\mathcal{C}_\theta\mathcal{C}_\phi\hat{x} + \mathcal{C}_\theta\mathcal{S}_\phi\hat{y} - \mathcal{S}_\theta\hat{z}) - i \left(-\mathcal{S}_\phi\hat{x} + \mathcal{C}_\phi\hat{y} + 0\hat{z} \right) \right]$$

which simplify into

$$\hat{k} = \mathcal{S}_\theta\mathcal{C}_\phi\hat{x} + \mathcal{S}_\theta\mathcal{S}_\phi\hat{y} + \mathcal{C}_\theta\hat{z} \tag{C.7.5}$$

$$\hat{\epsilon}_{\vec{k},\sigma_\pm} = \frac{1}{\sqrt{2}} \left[(\mathcal{C}_\theta\mathcal{C}_\phi \mp i\mathcal{S}_\phi) \hat{x} + (\mathcal{C}_\theta\mathcal{S}_\phi \pm i\mathcal{C}_\phi) \hat{y} - \mathcal{S}_\theta\hat{z} \right] \tag{C.7.6}$$

and imply

---

[2]By setting $\theta = 0$ and $\phi = 0$ to get $\hat{r}(\theta = 0, \phi = 0) = \hat{z}$, $\hat{\theta}(\theta = 0, \phi = 0) = \hat{x}$, and $\hat{\phi}(\theta = 0, \phi = 0) = \hat{y}$, and substituting these into the derivation (C.5), one can verify that the definition of these polarization and wave vectors indeed satisfy the derivation presented in (C.5).

$$\hat{\epsilon}^*_{\vec{k},\sigma_\pm} \cdot \hat{\epsilon}_{\vec{k},\sigma_\pm} = \frac{1}{2} \left[\left(\mathcal{C}_\theta \mathcal{C}_\phi \pm i\mathcal{S}_\phi \right) \hat{x} + \left(\mathcal{C}_\theta \mathcal{S}_\phi \mp i\mathcal{C}_\phi \right) \hat{y} - \mathcal{S}_\theta \hat{z} \right] \cdot \left[\left(\mathcal{C}_\theta \mathcal{C}_\phi \mp i\mathcal{S}_\phi \right) \hat{x} + \left(\mathcal{C}_\theta \mathcal{S}_\phi \pm i\mathcal{C}_\phi \right) \hat{y} - \mathcal{S}_\theta \hat{z} \right] \quad \text{(C.7.7)}$$

$$= \frac{1}{2} \left[\left(\mathcal{C}_\theta \mathcal{C}_\phi^2 + \mathcal{S}_\phi^2 \right) + \left(\mathcal{C}_\theta \mathcal{S}_\phi^2 + \mathcal{C}_\phi^2 \right) + \mathcal{S}_\theta^2 \right] \quad \text{(C.7.8)}$$

$$= \frac{1}{2} \left[\mathcal{C}_\theta \left(\mathcal{C}_\phi^2 + \mathcal{S}_\phi^2 \right) + \mathcal{S}_\theta^2 + \mathcal{S}_\phi^2 + \mathcal{C}_\phi^2 \right] \quad \text{(C.7.9)}$$

$$= 1 \quad \text{(C.7.10)}$$

With the polarization vectors defined we can now evaluate (C.7.1). Taking advantage of this coordinate system and defining its orientation such that the displacement vector is oriented along the $z - axis$, or $\vec{x} - \vec{x}_0 = |\vec{x} - \vec{x}_0| \, \hat{z}$ (see C.1) then $\vec{k} \cdot (\vec{x} - \vec{x}_0) \equiv \left| \vec{k} \right| |\vec{x} - \vec{x}_0| \cos \theta$. Using the identities (C.5.27) and (C.5.29) we can then restate the problem in terms of the tensors

$$\hat{\epsilon}^*_{\vec{k},\sigma_\pm} \hat{\epsilon}_{\vec{k},\sigma_\pm} = \frac{1}{2} \begin{pmatrix} \left(\mathcal{C}_\theta^2 \mathcal{C}_\phi^2 \pm \mathcal{S}_\phi^2 \right) & \left([\mathcal{C}_\theta^2 \mp 1] S_\phi \mathcal{C}_\phi \pm i\mathcal{C}_\theta \right) & -S_\theta \left(\mathcal{C}_\theta \mathcal{C}_\phi \pm iS_\phi \right) \\ \left([\mathcal{C}_\theta^2 \pm 1] \mathcal{S}_\phi \mathcal{C}_\phi \mp i\mathcal{C}_\theta \right) & \left(\mathcal{C}_\theta^2 S_\phi^2 \mp \mathcal{C}_\phi^2 \right) & -S_\theta \left(\mathcal{C}_\theta S_\phi \mp i\mathcal{C}_\phi \right) \\ -\mathcal{C}_\theta \left(\mathcal{C}_\theta \mathcal{C}\phi \mp i\mathcal{S}_\phi \right) & -S_\theta \left(\mathcal{C}_\theta S_\phi \pm i\mathcal{C}_\phi \right) & S_\theta^2 \end{pmatrix}$$

At this point the definition of $\vec{\wp}_{ab}$ becomes essential for the evaluation of the integrals. Approximating that the orientation of $\vec{\wp}_{ab}$ is fixed in time with respect to this coordinate system and independent of $\vec{k}$, we can define $\vec{\wp}_{ab}$ in general as

$$\vec{\wp}_{ab} \equiv |\vec{\wp}_{ba}| \left(\iota_x \sin \theta' \cos \phi' \hat{x} + \iota_y \sin \theta' \sin \phi' \hat{y} + \iota_z \cos \theta' \hat{z} \right)$$

$$\equiv |\vec{\wp}_{ba}| \begin{pmatrix} \iota_x \sin \theta' \cos \phi' \\ \iota_y \sin \theta' \sin \phi' \\ \iota_z \cos \theta' \end{pmatrix}$$

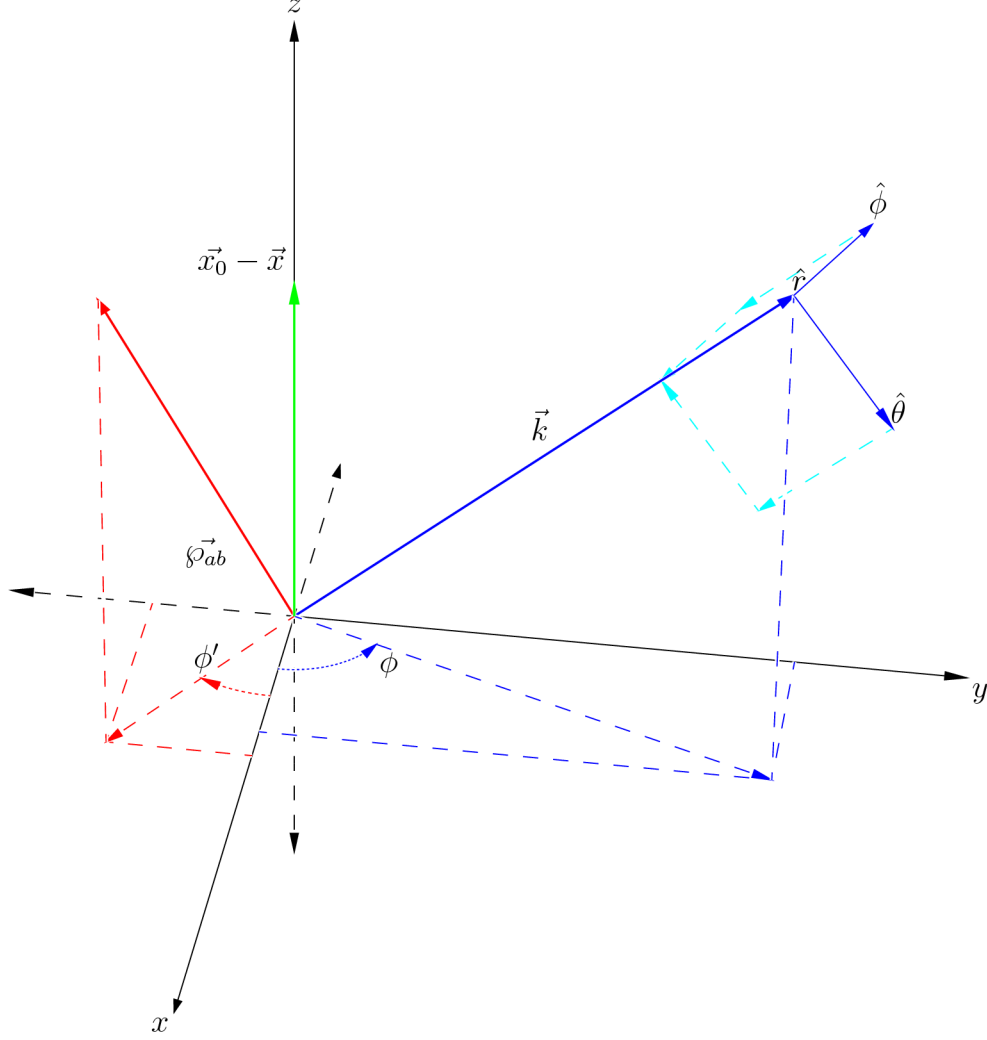where α, β, γ are all complex. Using the arguments above, we can use simple linear algebra to

Figure C.1: Coordinate system orientation of QD transition dipole moment

show that

$$
\left(\hat{\epsilon}^{*}_{\vec{k},\sigma_{\pm}} \cdot \vec{\wp}_{ab}\right) \hat{\epsilon}_{\vec{k},\sigma_{\pm}} = \frac{|\vec{\wp}_{ba}|}{2}
\begin{pmatrix}
\left(\mathcal{C}_\theta^2 \mathcal{C}_\phi^2 \pm \mathcal{S}_\phi^2\right) & \left([\mathcal{C}_\theta^2 \mp 1] S_\phi \mathcal{C}_\phi \pm i\mathcal{C}_\theta\right) & -S_\theta \left(\mathcal{C}_\theta \mathcal{C}_\phi \pm i S_\phi\right) \\
\left([\mathcal{C}_\theta^2 \pm 1] \mathcal{S}_\phi \mathcal{C}_\phi \mp i\mathcal{C}_\theta\right) & \left(\mathcal{C}_\theta^2 S_\phi^2 \mp \mathcal{C}_\phi^2\right) & -S_\theta \left(\mathcal{C}_\theta S_\phi \mp i\mathcal{C}_\phi\right) \\
-\mathcal{C}_\theta \left(\mathcal{C}_\theta \mathcal{C}\phi \mp i\mathcal{S}_\phi\right) & -S_\theta \left(\mathcal{C}_\theta S_\phi \pm i\mathcal{C}_\phi\right) & S_\theta^2
\end{pmatrix}
\begin{pmatrix}
\iota_x S_{\theta'} \mathcal{C}_{\phi'} \\
\iota_y S_{\theta'} \mathcal{S}_{\phi'} \\
\iota_z \mathcal{C}_{\theta'}
\end{pmatrix}
$$

$$
= \frac{|\vec{\wp}_{ba}|}{2}
\begin{pmatrix}
\iota_x \left(\mathcal{C}_\theta^2 \mathcal{C}_\phi^2 \pm \mathcal{S}_\phi^2\right) \mathcal{S}_{\theta'} \mathcal{C}_{\phi'} + \iota_y \left([\mathcal{C}_\theta^2 \mp 1] \mathcal{S}_\phi \mathcal{C}_\phi \pm i\mathcal{C}_\theta\right) \mathcal{S}_{\theta'} \mathcal{S}_{\phi'} - \iota_z S_\theta \left(\mathcal{C}_\theta \mathcal{C}_\phi \pm i S_\phi\right) \mathcal{C}_{\theta'} \\
\iota_x \left([\mathcal{C}_\theta^2 \pm 1] \mathcal{S}_\phi \mathcal{C}_\phi \mp i\mathcal{C}_\theta\right) \mathcal{S}_{\theta'} \mathcal{C}_{\phi'} + \iota_y \left(\mathcal{C}_\theta^2 S_\phi^2 \mp \mathcal{C}_\phi^2\right) \mathcal{S}_{\theta'} \mathcal{S}_{\phi'} - \iota_z S_\theta \left(\mathcal{C}_\theta S_\phi \mp i\mathcal{C}_\phi\right) \mathcal{C}_{\theta'} \\
-\iota_x \mathcal{S}_\theta \left(\mathcal{C}_\theta \mathcal{C}_\phi \mp i\mathcal{S}_\phi\right) \mathcal{S}_{\theta'} \mathcal{C}_{\phi'} - \iota_y \mathcal{S}_\theta \left(\mathcal{C}_\theta S_\phi \pm i\mathcal{C}_\phi\right) \mathcal{S}_{\theta'} \mathcal{S}_{\phi'} + \iota_z S_\theta^2 \mathcal{C}_{\theta'}
\end{pmatrix}
$$

Where will define $\vec{\wp}_{ab,x} \equiv \iota_x |\vec{\wp}_{ba}| \mathcal{S}_{\theta'} \mathcal{C}_{\phi'}$, $\vec{\wp}_{ab,y} \equiv \iota_y |\vec{\wp}_{ba}| \mathcal{S}_{\theta'} \mathcal{S}_{\phi'}$, $\vec{\wp}_{ab,z} \equiv \iota_z |\vec{\wp}_{ba}| \mathcal{C}_{\theta'}$ to more easily keep track of indices. We can now evaluate the integrals defined in terms of these by approximating to first order that $d\nu_k = v_0 d \left|\vec{k}\right|$ is linear and yields $\nu_k = v_0 \left|\vec{k}\right|$. Additionally, we make the

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substitution $\vec{k} \cdot (\vec{x} - \vec{x}_0) \equiv \left|\vec{k}\right| \left|\vec{x} - \vec{x}_0\right| \cos\theta = k \, z \cos\theta$, where we have defined $z = \left|\vec{x} - \vec{x}_0\right|$.

C.7.1 Integral Expressions Evaluated for integration over k-space

$$I_{1,\pm}^x = \frac{v_0 \vec{\wp}_{ab,x}}{2} \int d^3k \left(\cos^2\theta \cos^2\phi \pm \sin^2\phi\right) k e^{ikz\cos\theta} e^{-i\nu_k \frac{1}{2}(t_f+t_0)} \frac{\left[e^{i(\nu_k-\omega_\sigma-iz_n)t_f} - e^{i(\nu_k-\omega_\sigma-iz_n)t_0}\right]}{(v_0 k - \omega_\sigma - iz_n)}$$

This integral is of the form

$$\int dk d\theta d\phi \left(\cos^2\theta \cos^2\phi \pm \sin^2\phi\right) \sin\theta k^3 e^{ika\cos\theta} e^{-i\nu_k \frac{1}{2}(t_f+t_0)} \frac{\left[e^{i(\nu_k-\omega_\sigma-iz_n)t_f} - e^{i(\nu_k-\omega_\sigma-iz_n)t_0}\right]}{(v_0 k-\omega_\sigma-iz_n)} \quad \text{(C.7.11)}$$

Performing the integration first over $\phi \in [0, 2\pi]$ yields

$$\pi \int dk d\theta \left(\cos^2\theta \pm 1\right) \sin\theta k^3 e^{ika\cos\theta} e^{-i\nu_k \frac{1}{2}\left(t_f+t_0\right)} \frac{\left[e^{i(\nu_k-\omega_\sigma-iz_n)t_f} - e^{i(\nu_k-\omega_\sigma-iz_n)t_0}\right]}{(v_0 k-\omega_\sigma-iz_n)}$$

next, performing over $\theta \in [0, \pi]$ yields

$$\pi \int dk k^3 e^{-i\nu_k \frac{1}{2}\left(t_f+t_0\right)} \frac{\left[e^{i(\nu_k-\omega_\sigma-iz_n)t_f} - e^{i(\nu_k-\omega_\sigma-iz_n)t_0}\right]}{(v_0 k-\omega_\sigma-iz_n)} \left[-\left(\frac{[ika\cos\theta(ika\cos\theta-2)+2]}{(ika)^3} e^{ika\cos\theta} \pm \frac{e^{ika\cos\theta}}{ika}\right)\right]_0^\pi$$

$$= \pi \int dk k^3 e^{-i\nu_k \frac{1}{2}\left(t_f+t_0\right)} \frac{\left[e^{i(\nu_k-\omega_\sigma-iz_n)t_f} - e^{i(\nu_k-\omega_\sigma-iz_n)t_0}\right]}{(v_0 k-\omega_\sigma-iz_n)} \left[\frac{4i\sin(ka)}{(iak)^3} - \frac{4\cos(ka)}{(ika)^2} + \frac{2i\sin(ka)}{(ika)}(1\pm1)\right]$$

therefore, using the results in (D.11), this integral has the solution:
For poles in the lower half plane

$$\pi \int dk \left[\frac{4i\sin(ka)}{(ia)^3} - \frac{4k\cos(ka)}{(ia)^2} + \frac{2ik^2\sin(ka)}{(ia)}(1\pm1)\right] \frac{\left[e^{-i(\omega_\sigma+iz_n)t_f - i\frac{v_0 k}{2}(t_0-t_f)} - e^{-i(\omega_\sigma+iz_n)t_0 - i\frac{v_0 k}{2}(t_f-t_0)}\right]}{(v_0 k - \omega_\sigma - iz_n)}$$

$$= \frac{4\pi^2}{a^3 v_0}\left[\Theta\left(t - \frac{a}{v_0}\right) e^{-i(\omega_\sigma+iz_n)\left(t-\frac{a}{v_0}\right)} + \Theta\left(t + \frac{a}{v_0}\right) e^{-i(\omega_\sigma+iz_n)\left(t+\frac{a}{v_0}\right)}\right]$$

$$+ \frac{4i\pi^2}{a^2 v_0^2}(\omega_\sigma + iz_n)\left[\Theta\left(t - \frac{a}{v_0}\right) e^{-i(\omega_\sigma+iz_n)\left(t-\frac{a}{v_0}\right)} - \Theta\left(t + \frac{a}{v_0}\right) e^{-i(\omega_\sigma+iz_n)\left(t+\frac{a}{v_0}\right)}\right]$$

$$+ (1\pm1)\frac{2\pi^2}{a v_0^3}(\omega_\sigma + iz_n)^2\left[\Theta\left(t - \frac{a}{v_0}\right) e^{-i(\omega_\sigma+iz_n)\left(t-\frac{a}{v_0}\right)} + \Theta\left(t + \frac{a}{v_0}\right) e^{-i(\omega_\sigma+iz_n)\left(t+\frac{a}{v_0}\right)}\right]$$

For poles in the upper half plane

$$
\pi \int dk \left[\frac{4i \sin{(ka)}}{(ia)^3} - \frac{4k \cos{(ka)}}{(ia)^2} + k^2 \frac{2i \sin{(ka)}}{(ia)} (1 \pm 1) \right] \frac{\left[e^{-i(\omega_\sigma + iz_n)t_f - i\frac{v_0 k}{2}(t_0 - t_f)} - e^{-i(\omega_\sigma + iz_n)t_0 - i\frac{v_0 k}{2}(t_f - t_0)} \right]}{(v_0 k - \omega_\sigma - iz_n)}
$$

$$
= \frac{4\pi^2}{a^3 v_0} \left[\Theta \left(t + \frac{a}{v_0} \right) e^{-i(\omega_\sigma + iz_n)\left(t + \frac{a}{v_0} \right)} - \Theta \left(t - \frac{a}{v_0} \right) e^{-i(\omega_\sigma + iz_n)\left(t - \frac{a}{v_0} \right)} \right]
$$

$$
+ \frac{4i\pi^2}{a^2 v_0^2} (\omega_\sigma + iz_n) \left[\Theta \left(t + \frac{a}{v_0} \right) e^{-i(\omega_\sigma + iz_n)\left(t + \frac{a}{v_0} \right)} + \Theta \left(t - \frac{a}{v_0} \right) e^{-i(\omega_\sigma + iz_n)\left(t - \frac{a}{v_0} \right)} \right]
$$

$$
(1 \pm 1) \frac{2\pi^2}{a v_0^3} (\omega_\sigma + iz_n)^2 \left[\Theta \left(t + \frac{a}{v_0} \right) e^{-i(\omega_\sigma + iz_n)\left(t + \frac{a}{v_0} \right)} - \Theta \left(t - \frac{a}{v_0} \right) e^{-i(\omega_\sigma + iz_n)\left(t - \frac{a}{v_0} \right)} \right]
$$

$$
I_{2,\pm}^x = \frac{v_0 \vec{\rho}_{ab,y}}{2} \int d^3 k \left([\cos^2 \theta \mp 1] \sin \phi \cos \phi \pm i \cos \theta \right) k e^{ik\imath \cos \theta} e^{-i\nu_k \frac{1}{2}(t_f + t_0)} \frac{\left[e^{i(\nu_k - \omega_\sigma - iz_n)t_f} - e^{i(\nu_k - \omega_\sigma - iz_n)t_0} \right]}{(v_0 k - \omega_\sigma - iz_n)}
$$

This integral is of the form

$$
\int dk d\theta d\phi \left([\cos^2 \theta \mp 1] \sin \phi \cos \phi \pm i \cos \theta \right) \sin \theta k^3 e^{ika \cos \theta} e^{-i\nu_k \frac{1}{2}(t_f + t_0)} \frac{\left[e^{i(\nu_k - \omega_\sigma - iz_n)t_f} - e^{i(\nu_k - \omega_\sigma - iz_n)t_0} \right]}{(v_0 k - \omega_\sigma - iz_n)}
$$

$$
\text{(C.7.12)}
$$

Performing the integration first over $\phi \in [0, 2\pi]$ yields

$$
\pm 2\pi i \int dk d\theta \cos \theta \sin \theta k^3 e^{ika \cos \theta} e^{-i\nu_k \frac{1}{2}(t_f + t_0)} \frac{\left[e^{i(\nu_k - \omega_\sigma - iz_n)t_f} - e^{i(\nu_k - \omega_\sigma - iz_n)t_0} \right]}{(v_0 k - \omega_\sigma - iz_n)}
$$

next, performing over $\theta \in [0, \pi]$ yields

$$
\pm 2\pi i \int dk k^3 \left[\frac{-(ika \cos \theta - 1)}{(ika)^2} e^{ika \cos \theta} \right]_0^\pi e^{-i\nu_k \frac{1}{2}(t_f + t_0)} \frac{\left[e^{i(\nu_k - \omega_\sigma - iz_n)t_f} - e^{i(\nu_k - \omega_\sigma - iz_n)t_0} \right]}{(v_0 k - \omega_\sigma - iz_n)}
$$

$$
= \pm 2\pi i \int dk k^3 \left[\frac{2 \cos{(ka)}}{ika} - \frac{2 \sin{(ka)}}{(ika)^2} \right] e^{-i\nu_k \frac{1}{2}(t_f + t_0)} \frac{\left[e^{i(\nu_k - \omega_\sigma - iz_n)t_f} - e^{i(\nu_k - \omega_\sigma - iz_n)t_0} \right]}{(v_0 k - \omega_\sigma - iz_n)}
$$

therefore, using the results in (D.11), this integral has the solution:

For poles in the lower half plane

$$
\pm 2\pi i \int dk \left[\frac{2k^2 \cos{(ka)}}{ia} - \frac{2k \sin{(ka)}}{i^2 a^2} \right] \frac{\left[e^{-i(\omega_\sigma + iz_n)t_f - i\frac{v_0 k}{2}(t_0 - t_f)} - e^{-i(\omega_\sigma + iz_n)t_0 - i\frac{v_0 k}{2}(t_f - t_0)} \right]}{(v_0 k - \omega_\sigma - iz_n)}
$$

$$
= \pm \frac{4i\pi^2}{a v_0^3} (\omega_\sigma + iz_n)^2 \left[\Theta \left(t - \frac{a}{v_0} \right) e^{-i(\omega_\sigma + iz_n)\left(t - \frac{a}{v_0} \right)} - \Theta \left(t + \frac{a}{v_0} \right) e^{-i(\omega_\sigma + iz_n)\left(t + \frac{a}{v_0} \right)} \right]
$$

$$
\pm \frac{4i\pi^2}{a^2 v_0^2} (\omega_\sigma + iz_n) \left[\Theta \left(t - \frac{a}{v_0} \right) e^{-i(\omega_\sigma + iz_n)\left(t - \frac{a}{v_0} \right)} + \Theta \left(t + \frac{a}{v_0} \right) e^{-i(\omega_\sigma + iz_n)\left(t + \frac{a}{v_0} \right)} \right]
$$

For poles in the upper half plane

$$\pm 2\pi i \int dk \left[\frac{2k^2 \cos(ka)}{ia} - \frac{2k \sin(ka)}{i^2 a^2} \right] \frac{\left[e^{-i(\omega_\sigma + iz_n)t_f - i\frac{v_0 k}{2}(t_0 - t_f)} - e^{-i(\omega_\sigma + iz_n)t_0 - i\frac{v_0 k}{2}(t_f - t_0)} \right]}{(v_0 k - \omega_\sigma - iz_n)}$$

$$= \pm \frac{4i\pi^2}{av_0^3} (\omega_\sigma + iz_n)^2 \left[\Theta\left(t + \frac{a}{v_0} \right) e^{-i(\omega_\sigma + iz_n)\left(t + \frac{a}{v_0} \right)} + \Theta\left(t - \frac{a}{v_0} \right) e^{-i(\omega_\sigma + iz_n)\left(t - \frac{a}{v_0} \right)} \right]$$

$$\pm \frac{4i\pi^2}{a^2 v_0^2} (\omega_\sigma + iz_n) \left[\Theta\left(t + \frac{a}{v_0} \right) e^{-i(\omega_\sigma + iz_n)\left(t + \frac{a}{v_0} \right)} - \Theta\left(t - \frac{a}{v_0} \right) e^{-i(\omega_\sigma + iz_n)\left(t - \frac{a}{v_0} \right)} \right]$$

$$I_{3,\pm}^x = -\frac{v_0 \vec{\wp}_{ab,z}}{2} \int d^3k \sin\theta (\cos\theta\cos\phi \pm i\sin\phi) k e^{ik\imath \cos\theta} e^{-i\nu_k \frac{1}{2}(t_f + t_0)} \frac{\left[e^{i(\nu_k - \omega_\sigma - iz_n)t_f} - e^{i(\nu_k - \omega_\sigma - iz_n)t_0} \right]}{(v_0 k - \omega_\sigma - iz_n)}$$

This integral is of the form

$$\int dk d\theta d\phi (\cos\theta\cos\phi \pm i\sin\phi) \sin^2\theta k^3 e^{ika\cos\theta} \frac{\sin[(v_0 k - \omega_\sigma - iz_n)\Delta t]}{(v_0 k - \omega_\sigma - iz_n)} \tag{C.7.13}$$

therefore it has the solution:

For $\pm$

$$I_{x,3,\pm} = 0$$

$$I_{1,\pm}^y = \frac{v_0 \vec{\wp}_{ab,x}}{2} \int d^3k \left([\cos^2\theta \pm 1] \sin\phi\cos\phi \mp i\cos\theta \right) k e^{ik\imath \cos\theta} e^{-i\nu_k \frac{1}{2}(t_f + t_0)} \frac{\left[e^{i(\nu_k - \omega_\sigma - iz_n)t_f} - e^{i(\nu_k - \omega_\sigma - iz_n)t_0} \right]}{(v_0 k - \omega_\sigma - iz_n)}$$

This integral is of the form

$$\int dk d\theta d\phi \left([\cos^2\theta \pm 1] \sin\phi\cos\phi \mp i\cos\theta \right) \sin\theta k^3 e^{ika\cos\theta} e^{-i\nu_k \frac{1}{2}(t_f + t_0)} \frac{\left[e^{i(\nu_k - \omega_\sigma - iz_n)t_f} - e^{i(\nu_k - \omega_\sigma - iz_n)t_0} \right]}{(v_0 k - \omega_\sigma - iz_n)}$$

Performing the integration first over $\phi \in [0, 2\pi]$ yields

$$\mp 2\pi i \int dk d\theta \cos\theta \sin\theta k^3 e^{ika\cos\theta} e^{-i\nu_k \frac{1}{2}(t_f + t_0)} \frac{\left[e^{i(\nu_k - \omega_\sigma - iz_n)t_f} - e^{i(\nu_k - \omega_\sigma - iz_n)t_0} \right]}{(v_0 k - \omega_\sigma - iz_n)}$$

next, performing over $\theta \in [0, \pi]$ yields

$$\mp 2\pi i \int dk k^3 \left[\frac{-(iak\cos\theta - 1)}{(iak)^2} e^{ika\cos\theta} \right]_0^\pi e^{-i\nu_k \frac{1}{2}(t_f + t_0)} \frac{\left[e^{i(\nu_k - \omega_\sigma - iz_n)t_f} - e^{i(\nu_k - \omega_\sigma - iz_n)t_0} \right]}{(v_0 k - \omega_\sigma - iz_n)}$$

$$= \mp 2\pi i \int dk k^3 \left[\frac{2\cos(ka)}{(ika)} - \frac{2\sin(ka)}{(ika)^2} \right] e^{-i\nu_k \frac{1}{2}(t_f + t_0)} \frac{\left[e^{i(\nu_k - \omega_\sigma - iz_n)t_f} - e^{i(\nu_k - \omega_\sigma - iz_n)t_0} \right]}{(v_0 k - \omega_\sigma - iz_n)}$$

therefore, using the results in (D.11), this integral has the solution:

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For poles in the lower half plane

$$\mp 2\pi i \int dk \left[\frac{2k^2 \cos(ka)}{ia} - \frac{2k \sin(ka)}{i^2 a^2}\right] \frac{\left[e^{-i(\omega_\sigma + iz_n)t_f - i\frac{v_0 k}{2}(t_0 - t_f)} - e^{-i(\omega_\sigma + iz_n)t_0 - i\frac{v_0 k}{2}(t_f - t_0)}\right]}{(v_0 k - \omega_\sigma - iz_n)}$$

$$= \mp \frac{4i\pi^2}{av_0^3} (\omega_\sigma + iz_n)^2 \left[\Theta\left(t - \frac{a}{v_0}\right) e^{-i(\omega_\sigma + iz_n)\left(t - \frac{a}{v_0}\right)} - \Theta\left(t + \frac{a}{v_0}\right) e^{-i(\omega_\sigma + iz_n)\left(t + \frac{a}{v_0}\right)}\right]$$

$$\mp \frac{4i\pi^2}{a^2 v_0^2} (\omega_\sigma + iz_n) \left[\Theta\left(t - \frac{a}{v_0}\right) e^{-i(\omega_\sigma + iz_n)\left(t - \frac{a}{v_0}\right)} + \Theta\left(t + \frac{a}{v_0}\right) e^{-i(\omega_\sigma + iz_n)\left(t + \frac{a}{v_0}\right)}\right]$$

For poles in the upper half plane

$$\mp 2\pi i \int dk \left[\frac{2k^2 \cos(ka)}{ia} - \frac{2k \sin(ka)}{i^2 a^2}\right] \frac{\left[e^{-i(\omega_\sigma + iz_n)t_f - i\frac{v_0 k}{2}(t_0 - t_f)} - e^{-i(\omega_\sigma + iz_n)t_0 - i\frac{v_0 k}{2}(t_f - t_0)}\right]}{(v_0 k - \omega_\sigma - iz_n)}$$

$$= \mp \frac{4i\pi^2}{av_0^3} (\omega_\sigma + iz_n)^2 \left[\Theta\left(t + \frac{a}{v_0}\right) e^{-i(\omega_\sigma + iz_n)\left(t + \frac{a}{v_0}\right)} + \Theta\left(t - \frac{a}{v_0}\right) e^{-i(\omega_\sigma + iz_n)\left(t - \frac{a}{v_0}\right)}\right]$$

$$\mp \frac{4i\pi^2}{a^2 v_0^2} (\omega_\sigma + iz_n) \left[\Theta\left(t + \frac{a}{v_0}\right) e^{-i(\omega_\sigma + iz_n)\left(t + \frac{a}{v_0}\right)} - \Theta\left(t - \frac{a}{v_0}\right) e^{-i(\omega_\sigma + iz_n)\left(t - \frac{a}{v_0}\right)}\right]$$

$$I_{2,\pm}^y = \frac{v_0 \vec{\wp}_{ab,y}}{2} \int d^3 k \left(\cos^2\theta \sin^2\phi \mp \cos^2\phi\right) k e^{ik \wr \cos\theta} e^{-i\nu_k \frac{1}{2}(t_f + t_0)} \frac{\left[e^{i(\nu_k - \omega_\sigma - iz_n)t_f} - e^{i(\nu_k - \omega_\sigma - iz_n)t_0}\right]}{(v_0 k - \omega_\sigma - iz_n)}$$

This integral is of the form

$$\int dk d\theta d\phi \left(\cos^2\theta \sin^2\phi \mp \cos^2\phi\right) \sin\theta k^3 e^{ika \cos\theta} e^{-i\nu_k \frac{1}{2}(t_f + t_0)} \frac{\left[e^{i(\nu_k - \omega_\sigma - iz_n)t_f} - e^{i(\nu_k - \omega_\sigma - iz_n)t_0}\right]}{(v_0 k - \omega_\sigma - iz_n)} \quad \text{(C.7.14)}$$

Performing the integration first over $\phi \in [0, 2\pi]$ yields

$$\pi \int dk d\theta \left(\cos^2\theta \mp 1\right) \sin\theta k^3 e^{ika \cos\theta} e^{-i\nu_k \frac{1}{2}(t_f + t_0)} \frac{\left[e^{i(\nu_k - \omega_\sigma - iz_n)t_f} - e^{i(\nu_k - \omega_\sigma - iz_n)t_0}\right]}{(v_0 k - \omega_\sigma - iz_n)}$$

next, performing over $\theta \in [0, \pi]$ yields

$$\pi \int dk k^3 \left[-\left(\frac{[ika \cos\theta (ika \cos\theta - 2) + 2]}{(ika)^3} e^{ika \cos\theta} \mp \frac{e^{ika \cos\theta}}{ika}\right)\right]_0^\pi e^{-i\nu_k \frac{1}{2}(t_f + t_0)} \frac{\left[e^{i(\nu_k - \omega_\sigma - iz_n)t_f} - e^{i(\nu_k - \omega_\sigma - iz_n)t_0}\right]}{(v_0 k - \omega_\sigma - iz_n)}$$

$$= \pi \int dk k^3 \left[\frac{4i \sin(ka)}{(ika)^3} - \frac{4 \cos(ka)}{(ika)^2} + \frac{2i \sin(ka)}{(ika)} (1 \mp 1)\right] e^{-i\nu_k \frac{1}{2}(t_f + t_0)} \frac{\left[e^{i(\nu_k - \omega_\sigma - iz_n)t_f} - e^{i(\nu_k - \omega_\sigma - iz_n)t_0}\right]}{(v_0 k - \omega_\sigma - iz_n)}$$

therefore, using the results in (D.11), this integral has the solution:

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For poles in the lower half plane

$$\pi \int dk \left[\frac{4i\sin(ka)}{(ia)^3} - \frac{4k\cos(ka)}{(ia)^2} + \frac{2ik^2\sin(ka)}{(ia)} (1\mp 1) \right] \frac{\left[e^{-i(\omega_\sigma+iz_n)t_f - i\frac{v_0 k}{2}(t_0-t_f)} - e^{-i(\omega_\sigma+iz_n)t_0 - i\frac{v_0 k}{2}(t_f-t_0)} \right]}{(v_0 k - \omega_\sigma - iz_n)}$$

$$= \frac{4\pi^2}{a^3 v_0} \left[\Theta\left(t - \frac{a}{v_0}\right) e^{-i(\omega_\sigma+iz_n)\left(t-\frac{a}{v_0}\right)} + \Theta\left(t + \frac{a}{v_0}\right) e^{-i(\omega_\sigma+iz_n)\left(t+\frac{a}{v_0}\right)} \right]$$

$$+ \frac{i4\pi^2}{a^2 v_0^2} (\omega_\sigma+iz_n) \left[\Theta\left(t - \frac{a}{v_0}\right) e^{-i(\omega_\sigma+iz_n)\left(t-\frac{a}{v_0}\right)} - \Theta\left(t + \frac{a}{v_0}\right) e^{-i(\omega_\sigma+iz_n)\left(t+\frac{a}{v_0}\right)} \right]$$

$$- (1\mp 1) \frac{2\pi^2}{a v_0^3} (\omega_\sigma+iz_n)^2 \left[\Theta\left(t - \frac{a}{v_0}\right) e^{-i(\omega_\sigma+iz_n)\left(t-\frac{a}{v_0}\right)} + \Theta\left(t + \frac{a}{v_0}\right) e^{-i(\omega_\sigma+iz_n)\left(t+\frac{a}{v_0}\right)} \right]$$

For poles in the upper half plane

$$\pi \int dk \left[\frac{4i\sin(ka)}{(ia)^3} - \frac{4k\cos(ka)}{(ia)^2} + \frac{2ik^2\sin(ka)}{(ia)} (1\mp 1) \right] \frac{\left[e^{-i(\omega_\sigma+iz_n)t_f - i\frac{v_0 k}{2}(t_0-t_f)} - e^{-i(\omega_\sigma+iz_n)t_0 - i\frac{v_0 k}{2}(t_f-t_0)} \right]}{(v_0 k - \omega_\sigma - iz_n)}$$

$$= -\frac{4\pi^2}{a^3 v_0} \left[\Theta\left(t + \frac{a}{v_0}\right) e^{-i(\omega_\sigma+iz_n)\left(t+\frac{a}{v_0}\right)} - \Theta\left(t - \frac{a}{v_0}\right) e^{-i(\omega_\sigma+iz_n)\left(t-\frac{a}{v_0}\right)} \right]$$

$$+ \frac{4i\pi^2}{a^2 v_0^2} (\omega_\sigma+iz_n) \left[\Theta\left(t + \frac{a}{v_0}\right) e^{-i(\omega_\sigma+iz_n)\left(t+\frac{a}{v_0}\right)} + \Theta\left(t - \frac{a}{v_0}\right) e^{-i(\omega_\sigma+iz_n)\left(t-\frac{a}{v_0}\right)} \right]$$

$$+ (1\mp 1) \frac{2\pi^2}{a v_0^3} (\omega_\sigma+iz_n)^2 \left[\Theta\left(t + \frac{a}{v_0}\right) e^{-i(\omega_\sigma+iz_n)\left(t+\frac{a}{v_0}\right)} - \Theta\left(t - \frac{a}{v_0}\right) e^{-i(\omega_\sigma+iz_n)\left(t-\frac{a}{v_0}\right)} \right]$$

$$I_{3,\pm}^y = -\frac{v_0 \vec{\wp}_{ab,z}}{2} \int d^3k \sin\theta \left(\cos\theta\sin\phi \mp i\cos\phi\right) k e^{ik\imath\cos\theta} e^{-i\nu_k \frac{1}{2}(t_f+t_0)} \frac{\left[e^{i(\nu_k-\omega_\sigma-iz_n)t_f} - e^{i(\nu_k-\omega_\sigma-iz_n)t_0} \right]}{(v_0 k - \omega_\sigma - iz_n)}$$

This integral is of the form

$$\int dk\, d\theta\, d\phi \left(\cos\theta\sin\phi \mp i\cos\phi\right) \sin^2\theta\, k^3 e^{ika\cos\theta} e^{-i\nu_k \frac{1}{2}(t_f+t_0)} \frac{\left[e^{i(\nu_k-\omega_\sigma-iz_n)t_f} - e^{i(\nu_k-\omega_\sigma-iz_n)t_0} \right]}{(v_0 k - \omega_\sigma - iz_n)} \quad \text{(C.7.15)}$$

therefore it has the solution:

For $\pm$

$$I_{y,3,\pm} = 0$$

$$I_{1,\pm}^z = -\frac{v_0 \vec{\wp}_{ab,x}}{2} \int d^3k \sin\theta \left(\cos\theta\cos\phi \mp i\sin\phi\right) k e^{ik\imath\cos\theta} e^{-i\nu_k \frac{1}{2}(t_f+t_0)} \frac{\left[e^{i(\nu_k-\omega_\sigma-iz_n)t_f} - e^{i(\nu_k-\omega_\sigma-iz_n)t_0} \right]}{(v_0 k - \omega_\sigma - iz_n)}$$

This integral is of the form

$$\int dk\, d\theta\, d\phi \left(\cos\theta\cos\phi \mp i\sin\phi\right) \sin^2\theta\, k^3 e^{ika\cos\theta} e^{-i\nu_k \frac{1}{2}(t_f+t_0)} \frac{\left[e^{i(\nu_k-\omega_\sigma-iz_n)t_f} - e^{i(\nu_k-\omega_\sigma-iz_n)t_0} \right]}{(v_0 k - \omega_\sigma - iz_n)} \quad \text{(C.7.16)}$$

therefore it has the solution:

For $\pm$

$$I_{z,1,\pm} \;=\; 0$$

$$I_{2,\pm}^{z} = -\frac{v_0 \vec{\wp}_{ab,y}}{2} \int d^3k \sin\theta \left(\cos\theta \sin\phi \pm i \cos\phi\right) k e^{ik\imath\cos\theta} e^{-i\nu_k \frac{1}{2}(t_f+t_0)} \frac{\left[e^{i(\nu_k-\omega_\sigma-iz_n)t_f} - e^{i(\nu_k-\omega_\sigma-iz_n)t_0}\right]}{(v_0 k - \omega_\sigma - iz_n)}$$

This integral is of the form

$$\int dk d\theta d\phi \left(\cos\theta \sin\phi \pm i \cos\phi\right) \sin^2\theta k^3 e^{ika\cos\theta} e^{-i\nu_k \frac{1}{2}(t_f+t_0)} \frac{\left[e^{i(\nu_k-\omega_\sigma-iz_n)t_f} - e^{i(\nu_k-\omega_\sigma-iz_n)t_0}\right]}{(v_0 k - \omega_\sigma - iz_n)} \quad \text{(C.7.17)}$$

therefore it has the solution:

For $\pm$

$$I_{z,2,\pm} \;=\; 0$$

$$I_{3,\pm}^{z} = \frac{v_0 \vec{\wp}_{ab,z}}{2} \int d^3k \sin^2\theta k e^{ik\imath\cos\theta} \frac{\sin\left[(v_0 k - \omega_\sigma - iz_n)\Delta t\right]}{(v_0 k - \omega_\sigma - iz_n)}$$

This integral is of the form

$$\int dk d\theta d\phi \sin^3\theta k^3 e^{ika\cos\theta} e^{-i\nu_k \frac{1}{2}(t_f+t_0)} \frac{\left[e^{i(\nu_k-\omega_\sigma-iz_n)t_f} - e^{i(\nu_k-\omega_\sigma-iz_n)t_0}\right]}{(v_0 k - \omega_\sigma - iz_n)} \quad \text{(C.7.18)}$$

Performing the integration first over $\phi \in [0, 2\pi]$ yields

$$2\pi \int dk d\theta \sin^3\theta k^3 e^{ika\cos\theta} e^{-i\nu_k \frac{1}{2}(t_f+t_0)} \frac{\left[e^{i(\nu_k-\omega_\sigma-iz_n)t_f} - e^{i(\nu_k-\omega_\sigma-iz_n)t_0}\right]}{(v_0 k - \omega_\sigma - iz_n)}$$

next, performing over $\theta \in [0, \pi]$ yields

$$2\pi \int dk k^3 \left[-\left(\frac{e^{ika\cos\theta}}{(ika)} - \frac{[ika\cos\theta(ika\cos\theta-2)+2]}{(ika)^3} e^{ika\cos\theta}\right)\right]_0^\pi e^{-i\nu_k \frac{1}{2}(t_f+t_0)} \frac{\left[e^{i(\nu_k-\omega_\sigma-iz_n)t_f} - e^{i(\nu_k-\omega_\sigma-iz_n)t_0}\right]}{(v_0 k - \omega_\sigma - iz_n)}$$

$$= 2\pi \int dk k^3 \left[\frac{4\cos(ka)}{(ika)^2} - \frac{4i\sin(ka)}{(ika)^3}\right] e^{-i\nu_k \frac{1}{2}(t_f+t_0)} \frac{\left[e^{i(\nu_k-\omega_\sigma-iz_n)t_f} - e^{i(\nu_k-\omega_\sigma-iz_n)t_0}\right]}{(v_0 k - \omega_\sigma - iz_n)}$$

therefore, using the results in (D.11), this integral has the solution:

For poles in the lower half plane

$$2\pi \int dk \left[\frac{4k\cos{(ka)}}{(ia)^2} - \frac{4i\sin{(ka)}}{(ia)^3} \right] \frac{\left[e^{-i(\omega_\sigma+iz_n)t_f - i\frac{v_0 k}{2}(t_0-t_f)} - e^{-i(\omega_\sigma+iz_n)t_0 - i\frac{v_0 k}{2}(t_f-t_0)} \right]}{(v_0 k - \omega_\sigma - iz_n)}$$

$$= -\frac{8i\pi^2}{a^2 v_0^2}(\omega_\sigma + iz_n)\left[\Theta\left(t - \frac{a}{v_0}\right) e^{-i(\omega_\sigma+iz_n)\left(t-\frac{a}{v_0}\right)} - \Theta\left(t + \frac{a}{v_0}\right) e^{-i(\omega_\sigma+iz_n)\left(t+\frac{a}{v_0}\right)} \right]$$

$$+ \frac{8\pi^2}{a^3 v_0}\left[\Theta\left(t - \frac{a}{v_0}\right) e^{-i(\omega_\sigma+iz_n)\left(t-\frac{a}{v_0}\right)} + \Theta\left(t + \frac{a}{v_0}\right) e^{-i(\omega_\sigma+iz_n)\left(t+\frac{a}{v_0}\right)} \right]$$

For poles in the upper half plane

$$2\pi \int dk \left[\frac{4k\cos{(ka)}}{(ia)^2} - \frac{4i\sin{(ka)}}{(ia)^3} \right] \frac{\left[e^{-i(\omega_\sigma+iz_n)t_f - i\frac{v_0 k}{2}(t_0-t_f)} - e^{-i(\omega_\sigma+iz_n)t_0 - i\frac{v_0 k}{2}(t_f-t_0)} \right]}{(v_0 k - \omega_\sigma - iz_n)}$$

$$= -\frac{8i\pi^2}{a^2 v_0^2}(\omega_\sigma + iz_n)\left[\Theta\left(t + \frac{a}{v_0}\right) e^{-i(\omega_\sigma+iz_n)\left(t+\frac{a}{v_0}\right)} + \Theta\left(t - \frac{a}{v_0}\right) e^{-i(\omega_\sigma+iz_n)\left(t-\frac{a}{v_0}\right)} \right]$$

$$- \frac{8\pi^2}{a^3 v_0}\left[\Theta\left(t + \frac{a}{v_0}\right) e^{-i(\omega_\sigma+iz_n)\left(t+\frac{a}{v_0}\right)} - \Theta\left(t - \frac{a}{v_0}\right) e^{-i(\omega_\sigma+iz_n)\left(t-\frac{a}{v_0}\right)} \right]$$

C.7.2 Two Level Analytical Solutions to Vector Integrals

After carrying out the integrations above, we now apply them in order to find the vector expressions we are after. These are given as $\vec{\mathcal{I}}_{\pm,n}(\imath)$ with the respective expressions given below. For compactness and clarity, we define the following causal functions for all three cases

$$\zeta_+ \quad = \quad \pi^2 \Theta\left(t + \frac{\imath}{v_0}\right) e^{-i(\omega_\sigma+iz_n)\left(t+\frac{\imath}{v_0}\right)} \tag{C.7.19}$$

$$\zeta_- \quad = \quad \pi^2 \Theta\left(t - \frac{\imath}{v_0}\right) e^{-i(\omega_\sigma+iz_n)\left(t-\frac{\imath}{v_0}\right)} \tag{C.7.20}$$

These terms represent the directionality and causal conditions associated with the reported solutions. The components of the solutions associated with ζ_+ represent an incoming wave constrained to travel at the speed of light in the medium. The components of the solutions associated with ζ_- represent an outgoing wave constrained to travel at the speed of light in the medium. With this model we therefore have the ability to describe both incoming and outgoing waves in the non-Markovian limit. Additionally, the results we present contrast with text-book discussions in that we do not neglect higher order terms than $O^{(2)}\{\imath\}$. Therefore, we are also including near field

effects in the solutions presented below.

- These objects define the integrals to be calculated with poles in the lower half plane.

$$\mathcal{I}_{\pm,n,x}\left(\imath\right) =$$

$$\left[\left(\pm 2i\vec{\wp}_{ab,y} + \left(1 \pm 1\right)\vec{\wp}_{ab,x}\right)\frac{\left(\omega_\sigma + iz_n\right)^2}{w_0^2} + \left(\vec{\wp}_{ab,x} \pm \vec{\wp}_{ab,y}\right)\frac{2i\left(\omega_\sigma + iz_n\right)}{\imath^2 v_0} + \frac{2\vec{\wp}_{ab,x}}{\imath^3}\right]\zeta_-$$

$$-\left[\left(\pm 2i\vec{\wp}_{ab,y} - \left(1 \pm 1\right)\vec{\wp}_{ab,x}\right)\frac{\left(\omega_\sigma + iz_n\right)^2}{w_0^2} - \left(\vec{\wp}_{ab,x} \mp \vec{\wp}_{ab,y}\right)\frac{2i\left(\omega_\sigma + iz_n\right)}{\imath^2 v_0} - \frac{2\vec{\wp}_{ab,x}}{\imath^3}\right]\zeta_+$$

$$\text{(C.7.21)}$$

$$\mathcal{I}_{\pm,n,y}\left(\imath\right) =$$

$$\left[\left(\mp 2i\vec{\wp}_{ab,x} - \left(1 \mp 1\right)\vec{\wp}_{ab,y}\right)\frac{\left(\omega_\sigma + iz_n\right)^2}{w_0^2} + \left(\vec{\wp}_{ab,y} \mp \vec{\wp}_{ab,x}\right)\frac{2i\left(\omega_\sigma + iz_n\right)}{\imath^2 v_0} + \frac{2\vec{\wp}_{ab,y}}{\imath^3}\right]\zeta_-$$

$$+\left[\left(\pm 2i\vec{\wp}_{ab,x} - \left(1 \mp 1\right)\vec{\wp}_{ab,y}\right)\frac{\left(\omega_\sigma + iz_n\right)^2}{w_0^2} - \left(\vec{\wp}_{ab,y} \pm \vec{\wp}_{ab,x}\right)\frac{2i\left(\omega_\sigma + iz_n\right)}{\imath^2 v_0} + \frac{2\vec{\wp}_{ab,y}}{\imath^3}\right]\zeta_+$$

$$\text{(C.7.22)}$$

$$\mathcal{I}_{\sigma_\pm,n,z}\left(\imath\right) =$$

$$\left[\frac{1}{\imath^3} - \frac{i\left(\omega_\sigma + iz_n\right)}{\imath^2 v_0}\right]4\vec{\wp}_{ab,z}\zeta_-$$

$$+\left[\frac{1}{\imath^3} + \frac{i\left(\omega_\sigma + iz_n\right)}{\imath^2 v_0}\right]4\vec{\wp}_{ab,z}\zeta_+ \qquad \text{(C.7.23)}$$

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- These objects define the integrals to be calculated for poles in the upper half plane.

$$\mathcal{I}_{\pm,n,x}\left(\imath\right) =$$

$$\left[\left(\pm 2i\vec{\wp}_{ab,y} - (1\pm 1)\vec{\wp}_{ab,x}\right)\frac{\left(\omega_\sigma + iz_n\right)^2}{w_0^2} + \left(\vec{\wp}_{ab,x} \mp \vec{\wp}_{ab,y}\right)\frac{2i\left(\omega_\sigma + iz_n\right)}{\imath^2 v_0} - \frac{2\vec{\wp}_{ab,x}}{\imath^3} \right]\zeta_-$$

$$+ \left[\left(\pm 2i\vec{\wp}_{ab,y} + (1\pm 1)\vec{\wp}_{ab,x}\right)\frac{\left(\omega_\sigma + iz_n\right)^2}{w_0^2} + \left(\vec{\wp}_{ab,x} \pm \vec{\wp}_{ab,y}\right)\frac{2i\left(\omega_\sigma + iz_n\right)}{\imath^2 v_0} + \frac{2\vec{\wp}_{ab,x}}{\imath^3} \right]\zeta_+$$

$$\text{(C.7.24)}$$

$$\mathcal{I}_{\pm,n,y}\left(\imath\right) =$$

$$\left[\left(\mp 2i\vec{\wp}_{ab,x} - (1\mp 1)\vec{\wp}_{ab,y}\right)\frac{\left(\omega_\sigma + iz_n\right)^2}{w_0^2} + \left(\vec{\wp}_{ab,y} \pm \vec{\wp}_{ab,x}\right)\frac{2i\left(\omega_\sigma + iz_n\right)}{\imath^2 v_0} + \frac{2\vec{\wp}_{ab,y}}{\imath^3} \right]\zeta_-$$

$$+ \left[\left(\mp 2i\vec{\wp}_{ab,x} + (1\mp 1)\vec{\wp}_{ab,y}\right)\frac{\left(\omega_\sigma + iz_n\right)^2}{w_0^2} + \left(\vec{\wp}_{ab,y} \mp \vec{\wp}_{ab,x}\right)\frac{2i\left(\omega_\sigma + iz_n\right)}{\imath^2 v_0} - \frac{2\vec{\wp}_{ab,y}}{\imath^3} \right]\zeta_+$$

$$\text{(C.7.25)}$$

$$\mathcal{I}_{\sigma_\pm,n,z}\left(\imath\right) =$$

$$\left[\frac{1}{\imath^3} - \frac{i\left(\omega_\sigma + iz_n\right)}{\imath^2 v_0} \right] 4\vec{\wp}_{ab,z}\zeta_-$$

$$- \left[\frac{1}{\imath^3} + \frac{i\left(\omega_\sigma + iz_n\right)}{\imath^2 v_0} \right] 4\vec{\wp}_{ab,z}\zeta_+ \qquad \text{(C.7.26)}$$

D   Mathematical Definitions and Identities

Unless otherwise stated, the definitions presented herein are referenced from [83, 82].

D.1   Pauli Matrices

The Pauli Spin matrices are defined for half-integer spin particles via

$$\sigma_1 \equiv \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$\sigma_2 \equiv \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

$$\sigma_3 \equiv \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

particles with integer spin have the following integer spin matrices [10, 11, 46, 72] in the $SO\,(3)$ representation

$$s_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}$$

$$s_2 = \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{pmatrix}$$

$$s_3 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

in the $SU(2)$ representation

$$s_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

$$s_2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix}$$

$$s_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

D.2   Dirac and Gamma Matrices

The Dirac Gamma Matrices for a half-integer spin particle come from the definition of α^i and $\overleftrightarrow{\beta}$ for a Minkowski Metric with signature $(+,-,-,-)$ which are defined by

$$\overleftrightarrow{\beta} \equiv \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}$$

$$\alpha^i \equiv \begin{pmatrix} 0 & \sigma^i \\ \sigma^i & 0 \end{pmatrix}$$

The Gamma Matrices are defined in terms of the Dirac Matrices as

$$\gamma^0 \equiv \overleftrightarrow{\beta}$$

$$\gamma^i \equiv \alpha^i \overleftrightarrow{\beta}$$

and, in turn, the Pauli matrices as follows

$$\gamma^0 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}$$

$$\gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}$$

for the case of integer spin particles the Dirac Gamma matrices correspond the matrices α^i and $\overleftrightarrow{\beta}$ matrices defined as

$$\overleftrightarrow{\beta} \equiv \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}$$

$$\alpha^i \equiv \begin{pmatrix} 0 & s^i \\ s^i & 0 \end{pmatrix}$$

D.3   *Used Representations of the Dirac Delta Function*

[83, 82]

$$\delta(x) = \int_{-\infty}^{\infty} e^{-2\pi ikx} dk$$

D.4   *Useful Properties of the Dirac Delta Function*

In this section we express identities and definitions of the Dirac Delta Distribution (Function) used throughout the text.

Definition in terms of a Fourier Integral:

$$\delta(x) = \int_{-\infty}^{\infty} e^{-2\pi ikx} dk$$
$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-ikx} dk$$

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The fundamental equation that defines the derivatives of the "Dirac Delta Function" is:

$$\int f(x)\,\delta^{(n)}(x)\,dx \;\equiv\; (-1)^n \int \partial_x\,[f(x)]\,\delta^{(n-1)}(x)\,dx \tag{D.4.1}$$

Additionally and more generally,

$$\int f(x)\,\delta^{(n)}(x\pm a) \;=\; (-1)^n\,f^{(n)}(\pm a)$$

For example the nth derivative of this distribution satisfies:

$$\int [x^n f(x)]\,\delta^{(n)}(x) \;=\; (-1)^n \int \frac{\partial^n\,[x^n f(x)]}{\partial x^n}\,\delta(x)\,dx$$

$$x^n\,\delta^{(n)}(x) \;=\; (-1)^n\,n!\,\delta(x) \tag{D.4.2}$$

In general, it is straight forward to prove

$$(x\mp a)^n\,\delta^{(n)}(x\pm a) \;=\; (-1)^n\,n!\,\delta(x\pm a)$$

D.5 Useful Properties of Fourier Transforms

In this section we express properties of Fourier Transforms that are used through this text.

The Fourier transform is defined by

$$\mathcal{F}_x\,[f(x)]\,(k) \;\equiv\; \int_{-\infty}^{\infty} f(x)\,e^{-2\pi i k x}\,dx$$

$$f(x) \;\equiv\; \int_{-\infty}^{\infty} \mathcal{F}\,[f(x)]\,(k)\,e^{2\pi i k x}\,dk$$

Similarly it can be defined by a change of variables as

$$\mathcal{F}_k\,[f(k)]\,(x) \;\equiv\; \int_{-\infty}^{\infty} f(k)\,e^{-2\pi i k x}\,dk$$

$$f(k) \;\equiv\; \int_{-\infty}^{\infty} \mathcal{F}\,[f(k)]\,(x)\,e^{2\pi i k x}\,dx$$

Additionally, these may be defined via another change of variables

$$\hat{\mathcal{F}}_x\left[f\left(x\right)\right]\left(k\right) \equiv \frac{1}{2\pi}\int_{-\infty}^{\infty} f\left(x\right)e^{-ikx}dx$$

$$g\left(x\right) \equiv \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathcal{F}\left[f\left(x\right)\right]\left(k\right)e^{ikx}dk$$

and

$$\hat{\mathcal{F}}_k\left[f\left(k\right)\right]\left(x\right) \equiv \frac{1}{2\pi}\int_{-\infty}^{\infty} f\left(k\right)e^{-ikx}dk$$

$$g\left(k\right) \equiv \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathcal{F}\left[f\left(k\right)\right]\left(x\right)e^{ikx}dx$$

D.5.1 The nth Derivative of a Fourier Transform

Taking the nth derivative of $\hat{\mathcal{F}}_k\left[f\left(k\right)\right]\left(x\right)$ with respect to x yields:

$$\partial_x^{(n)}\left[\hat{\mathcal{F}}_k\left[f\left(k\right)\right]\left(x\right)\right] = \left(\frac{1}{2\pi}\right)\int_{-\infty}^{\infty} f\left(k\right)\left(-i\right)^n k^n e^{-ikx}dk$$

This result yields the following property of the Fourier transform

$$\int_{-\infty}^{\infty} k^n f\left(k\right)e^{-ikx}dk = 2\pi i^n \partial_x^{(n)}\left[\hat{\mathcal{F}}_k\left[f\left(k\right)\right]\left(x\right)\right]$$

D.5.1.1 Evaluating $\int_{-\infty}^{\infty} k^n e^{-ikx}dk$

This term arises commonly in QFT and is known to give rise to the UV- Catastrophe. Using the properties of the Fourier transform defined above this integral yields

$$\int_{-\infty}^{\infty} k^n e^{-ikx}dk = 2\pi i^n \partial_x^{(n)}\left[\hat{\mathcal{F}}_k\left[f\left(k\right)=1\right]\left(x\right)\right]$$

$$= 2\pi i^n \partial_x^{(n)}\left[\delta\left(x\right)\right] \tag{D.5.1}$$

$$= 2\pi i^n \delta^{(n)}\left(x\right) \tag{D.5.2}$$

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D.6   *Some Useful Identities of Complex Numbers*

Defining a, b, c, d, e, and f as complex numbers where

$$e \equiv [\Re\{a\}\Re\{b\} - \Im\{a\}\Im\{b\}] + i[\Re\{a\}\Im\{b\} + \Im\{a\}\Re\{b\}]$$

$$f \equiv [\Re\{c\}\Re\{d\} - \Im\{c\}\Im\{d\}] + i[\Re\{c\}\Im\{d\} + \Im\{c\}\Re\{d\}]$$

D.6.1   Multiplication of a, b, c, and d

The multiplication of a, b, c, and d is given by

$$
\begin{aligned}
a*b*c*d &= ([\Re\{a\}\Re\{b\} - \Im\{a\}\Im\{b\}] + i[\Re\{a\}\Im\{b\} + \Im\{a\}\Re\{b\}]) \\
&* ([\Re\{c\}\Re\{d\} - \Im\{c\}\Im\{d\}] + i[\Re\{c\}\Im\{d\} + \Im\{c\}\Re\{d\}]) \\
&= [\Re\{e\}\Re\{f\} - \Im\{e\}\Im\{f\}] + i[\Re\{e\}\Im\{f\} + \Im\{e\}\Re\{f\}]
\end{aligned}
$$

which in terms of a, b, c, and d is given by

$$
\begin{aligned}
[\Re\{e\}\Re\{f\} - \Im\{e\}\Im\{f\}] &= [\Re\{a\}\Re\{b\} - \Im\{a\}\Im\{b\}] * [\Re\{c\}\Re\{d\} - \Im\{c\}\Im\{d\}] \\
&- [\Re\{a\}\Im\{b\} + \Im\{a\}\Re\{b\}] * [\Re\{c\}\Im\{d\} + \Im\{c\}\Re\{d\}] \\
[\Re\{e\}\Im\{f\} + \Im\{e\}\Re\{f\}] &= [\Re\{a\}\Re\{b\} - \Im\{a\}\Im\{b\}] * [\Re\{c\}\Im\{d\} + \Im\{c\}\Re\{d\}] \\
&+ [\Re\{a\}\Im\{b\} + \Im\{a\}\Re\{b\}] * [\Re\{c\}\Re\{d\} - \Im\{c\}\Im\{d\}]
\end{aligned}
$$

and can be simplified to

$$
\begin{aligned}
[\Re\{e\}\Re\{f\} - \Im\{e\}\Im\{f\}] &= \Re\{a\}\Re\{b\}\Re\{c\}\Re\{d\} + \Im\{a\}\Im\{b\}\Im\{c\}\Im\{d\} \\
&- \Re\{a\}\Re\{b\}\Im\{c\}\Im\{d\} - \Im\{a\}\Im\{b\}\Re\{c\}\Re\{d\} \\
&- \Re\{a\}\Im\{b\}\Re\{c\}\Im\{d\} - \Im\{a\}\Re\{b\}\Im\{c\}\Re\{d\} \\
&- \Re\{a\}\Im\{b\}\Im\{c\}\Re\{d\} - \Im\{a\}\Re\{b\}\Re\{c\}\Im\{d\}
\end{aligned}
$$

and

$$
\begin{aligned}
[\Re\{e\}\,\Im\{f\} + \Im\{e\}\,\Re\{f\}] \;=\;\; & \Re\{a\}\,\Re\{b\}\,\Re\{c\}\,\Im\{d\} - \Im\{a\}\,\Im\{b\}\,\Im\{c\}\,\Re\{d\} \\
+ \;\; & \Re\{a\}\,\Re\{b\}\,\Im\{c\}\,\Re\{d\} - \Im\{a\}\,\Im\{b\}\,\Re\{c\}\,\Im\{d\} \\
+ \;\; & \Re\{a\}\,\Im\{b\}\,\Re\{c\}\,\Re\{d\} - \Im\{a\}\,\Re\{b\}\,\Re\{c\}\,\Im\{d\} \\
- \;\; & \Re\{a\}\,\Im\{b\}\,\Im\{c\}\,\Im\{d\} + \Im\{a\}\,\Re\{b\}\,\Re\{c\}\,\Re\{d\}
\end{aligned}
$$

D.7 Useful Identities of Complex Numbers

In this section we express some identities of complex numbers which were used in this dissertation.

$$
(a + ib)^{c+id} \;=\; (a + ib)^{\frac{c+id}{2}}\, e^{i(c+id)\,\arg(a+ib)}
$$

D.8 Roots of the Cubic Equation

In the text we utilize roots of the cubic equation for various purposes. Since this equation has analytic solutions we find no need to waste compute time in calculating these solutions and present their analytic representation herein. Given that a cubic equation has the general form

$$
z^3 + a_2 z^2 + a_1 z + a_0 \;=\; 0
$$

The analytic roots for this equation are given by defining a series of coefficients and substitutions (such as Vieta's substitution). We start by defining

$$
\begin{aligned}
Q &\equiv \frac{3a_1 - a_2^2}{9} \\
R &\equiv \frac{9a_1 a_2 - 27a_0 - 2a_2^3}{54}
\end{aligned}
$$

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and in terms of these

$$
\begin{aligned}
D &\equiv Q^3 + R^2 \\
S &= \sqrt[3]{R + \sqrt{D}} \\
T &= \sqrt[3]{R - \sqrt{D}}
\end{aligned}
$$

such that the roots are then

$$
\begin{aligned}
z_1 &= -\frac{1}{3}a_2 + (S + T) \\
z_2 &= -\frac{1}{3}a_2 - \frac{1}{2}(S + T) + \frac{1}{2}i\sqrt{3}(S - T) \\
z_3 &= -\frac{1}{3}a_2 - \frac{1}{2}(S + T) - \frac{1}{2}i\sqrt{3}(S - T)
\end{aligned}
$$

or

$$
\begin{aligned}
z_1 &= -\frac{1}{3}a_2 + (S + T) \\
z_2 &= -\frac{1}{3}a_2 - \left(1 - i\sqrt{3}\right)\frac{S}{2} - \left(1 + i\sqrt{3}\right)\frac{T}{2} \\
z_3 &= -\frac{1}{3}a_2 - \left(1 + i\sqrt{3}\right)\frac{S}{2} - \left(1 - i\sqrt{3}\right)\frac{T}{2}
\end{aligned}
$$

In terms of a_n the roots are then

- z_1

$$
\begin{aligned}
z_1 = -\frac{1}{3}a_2 \\
+ \sqrt[3]{\frac{9a_1a_2 - 27a_0 - 2a_2^3}{54} + \sqrt{\left(\frac{3a_1 - a_2^2}{9}\right)^3 + \left(\frac{9a_1a_2 - 27a_0 - 2a_2^3}{54}\right)^2}} \\
+ \sqrt[3]{\frac{9a_1a_2 - 27a_0 - 2a_2^3}{54} - \sqrt{\left(\frac{3a_1 - a_2^2}{9}\right)^3 + \left(\frac{9a_1a_2 - 27a_0 - 2a_2^3}{54}\right)^2}}
\end{aligned}
$$

- z_2

$$z_2 = -\frac{1}{3}a_2$$

$$-\frac{1}{2}\sqrt[3]{\frac{9a_1a_2 - 27a_0 - 2a_2^3}{54} + \sqrt{\left(\frac{3a_1 - a_2^2}{9}\right)^3 + \left(\frac{9a_1a_2 - 27a_0 - 2a_2^3}{54}\right)^2}}$$

$$-\frac{1}{2}\sqrt[3]{\frac{9a_1a_2 - 27a_0 - 2a_2^3}{54} - \sqrt{\left(\frac{3a_1 - a_2^2}{9}\right)^3 + \left(\frac{9a_1a_2 - 27a_0 - 2a_2^3}{54}\right)^2}}$$

$$+\frac{1}{2}i\sqrt{3}\sqrt[3]{\frac{9a_1a_2 - 27a_0 - 2a_2^3}{54} + \sqrt{\left(\frac{3a_1 - a_2^2}{9}\right)^3 + \left(\frac{9a_1a_2 - 27a_0 - 2a_2^3}{54}\right)^2}}$$

$$-\frac{1}{2}i\sqrt{3}\sqrt[3]{\frac{9a_1a_2 - 27a_0 - 2a_2^3}{54} - \sqrt{\left(\frac{3a_1 - a_2^2}{9}\right)^3 + \left(\frac{9a_1a_2 - 27a_0 - 2a_2^3}{54}\right)^2}}$$

- z_3

$$z_3 = -\frac{1}{3}a_2$$

$$-\frac{1}{2}\sqrt[3]{\frac{9a_1a_2 - 27a_0 - 2a_2^3}{54} + \sqrt{\left(\frac{3a_1 - a_2^2}{9}\right)^3 + \left(\frac{9a_1a_2 - 27a_0 - 2a_2^3}{54}\right)^2}}$$

$$-\frac{1}{2}\sqrt[3]{\frac{9a_1a_2 - 27a_0 - 2a_2^3}{54} - \sqrt{\left(\frac{3a_1 - a_2^2}{9}\right)^3 + \left(\frac{9a_1a_2 - 27a_0 - 2a_2^3}{54}\right)^2}}$$

$$-\frac{1}{2}i\sqrt{3}\sqrt[3]{\frac{9a_1a_2 - 27a_0 - 2a_2^3}{54} + \sqrt{\left(\frac{3a_1 - a_2^2}{9}\right)^3 + \left(\frac{9a_1a_2 - 27a_0 - 2a_2^3}{54}\right)^2}}$$

$$+\frac{1}{2}i\sqrt{3}\sqrt[3]{\frac{9a_1a_2 - 27a_0 - 2a_2^3}{54} - \sqrt{\left(\frac{3a_1 - a_2^2}{9}\right)^3 + \left(\frac{9a_1a_2 - 27a_0 - 2a_2^3}{54}\right)^2}}$$

D.9   High Order Roots of Complex Numbers

High order roots of complex numbers can be written in terms of complex numbers.

D.9.1   Square Roots of Complex numbers

For example:

$$
\begin{aligned}
(a+ib)^2 &= \left(a^2 - b^2\right) + 2i\left(ab\right) \\
&= c + id
\end{aligned}
$$

therefore

$$
\sqrt{c+id} = a + ib
$$

where

$$
\begin{aligned}
c &= a^2 - b^2 \\
d &= 2ab
\end{aligned}
$$

and inverting it we find that a and b must satisfy

$$
\begin{aligned}
a^2 &= \frac{c + \sqrt{c^2 + d^2}}{2} \\
b &= \frac{d}{2a}
\end{aligned}
$$

D.9.2   Cubic Roots of Complex numbers

For example

$$
\begin{aligned}
(a+ib)^3 &= a^3 + 3ia^2 b - 3ab^2 - ib^3 \\
&= \left(a^3 - 3ab^2\right) + i\left(3a^2 b - b^3\right) \\
&= c + id
\end{aligned}
$$

therefore

$$\sqrt[3]{c+id} \ = \ a+ib$$

where

$$c \ = \ a^3 - 3ab^2$$

$$d \ = \ 3a^2b - b^3$$

Since working with cubic terms is not so pretty we will work with 4th order terms

$$ac \ = \ a^4 - 3a^2b^2$$

$$bd \ = \ 3a^2b^2 - b^4$$

this yields the expressions

$$-a^4 + 3a^2b^2 + ac \ = \ 0$$

$$b^4 - 3a^2b^2 + bd \ = \ 0$$

adding these two expressions yields

$$b^4 + db + ca - a^4 \ = \ 0$$

and

$$a^4 - ca - db - b^4 \ = \ 0$$

solving the quadratic formula for a and b respectively yields

$$b^2 = \frac{-d \pm \sqrt{d^2 - 4(ac - a^4)}}{2}$$

$$a^2 = \frac{c \pm \sqrt{c^2 + 4(bd + b^4)}}{2}$$

substituting these back into the expressions for c & d yields

$$c = a^3 - 3a\frac{-d \pm \sqrt{d^2 - 4(ac - a^4)}}{2}$$

$$d = 3\frac{c \pm \sqrt{c^2 + 4(bd + b^4)}}{2}b - b^3$$

such that each can be inverted for a and b respectively to yield

$$10a^6 - 3a^4 d - 11a^3 c + 3cda + c^2 = 0$$

$$32b^6 + 12cb^4 + 32db^3 + 12db + 4d^2 = 0$$

D.9.3 n^{th} order roots of complex numbers from Demoivre's Theorem

optionally, we could have used Demoivre's theorem to find

$$(c + id)^{\frac{1}{n}} = r^{\frac{1}{n}}\left(\cos\frac{\theta}{n} + i\sin\frac{\theta}{n}\right)$$

where $r = \sqrt{c^2 + d^2}$, and $\theta = \tan^{-1}\frac{d}{c}$. This simplifies matters greatly. For example, for the case of the square root we use $n = 2$

$$(c + id)^{\frac{1}{2}} = \sqrt[4]{c^2 + d^2}\left(\cos\left[\frac{\tan^{-1}\frac{d}{c}}{2}\right] + i\sin\left[\frac{\tan^{-1}\frac{d}{c}}{2}\right]\right)$$

For the case of the cubic root where $n = 3$

$$(c + id)^{\frac{1}{3}} = \sqrt[6]{c^2 + d^2} \left(\cos \left[\frac{\tan^{-1} \frac{d}{c}}{3} \right] + i \sin \left[\frac{\tan^{-1} \frac{d}{c}}{3} \right] \right)$$

As an example we show the case for $n = 2$ and $n = 3$

D.10   Solving Ordinary Differential Equations by Formal Integration

It is some times the case that is useful to solve ordinary differential equations through formal integration. A simple example of how to go about this integration is given by

$$\frac{dN(t)}{dt} = \lambda N(t)$$

where, λ is a constant. Formally integrating this expression on the interval $[t_0, t]$ yields

$$\int_{t_0}^{t} \frac{dN(t')}{dt'} dt' = \lambda \int_{t_0}^{t} N(t') \, dt'$$

which upon carrying out the formal integration on the LHS yields

$$N(t) - N(t_0) = \lambda \int_{t_0}^{t} N(t') \, dt'$$

It is evident that the interval $[t_0, t]$ may be shifted in any direction, even, for example $[t - \Delta t, t + \Delta t]$. This would imply that for any time t this expression may be rewritten as

$$N(t + \Delta t) = \lambda \int_{t-\Delta t}^{t+\Delta t} N(t') \, dt' + N(t - \Delta t)$$

D.11   Useful Countour Integrals

Since in several cases we consider the conversion of infinite discrete sums in k space to continuous ones, we are often faced with the necessity to evaluate improper integrals. Some of these can be evaluated by means of Countour Integration which take advantage of Cauchy's Integral Theorem.

D.11.1 For those improper integrals which are of the form

$$\int_{-\infty}^{\infty} dk \left[k^n \cos\left(ka\right) \frac{\left[e^{-i(\omega_\sigma + iz_n)t_f - i\frac{v_0 k}{2}\left(t_0 - t_f\right)} - e^{-i(\omega_\sigma + iz_n)t_0 - i\frac{v_0 k}{2}\left(t_f - t_0\right)} \right]}{(v_0 k - \omega_\sigma - iz_n)} \right]$$

and satisfy the contours found in (D.1), we can find their physically sensible closed forms in terms of a, Δt, and z_n so long as we constrain ourselves to causal results. To do this, we first expand $\cos ka \equiv \frac{1}{2}\left[e^{ika} + e^{-ika} \right]$.

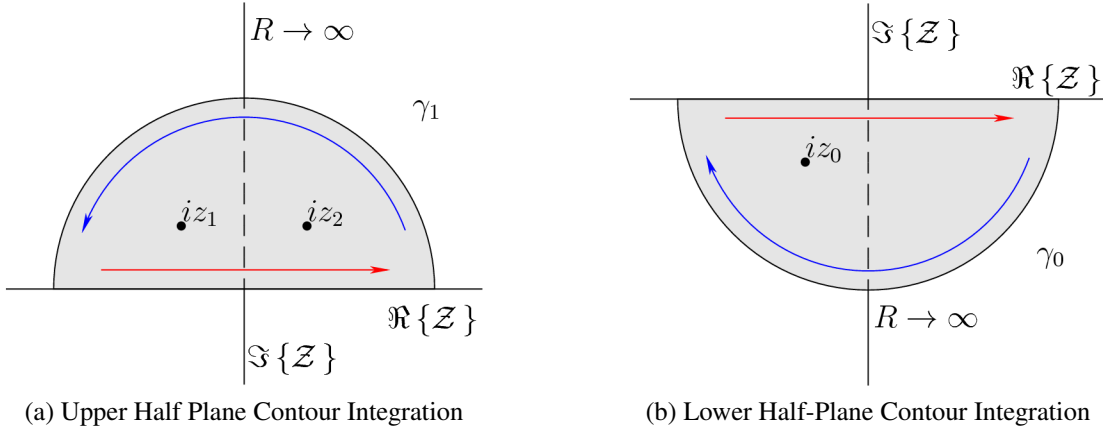(a) Upper Half Plane Contour Integration   (b) Lower Half-Plane Contour Integration

Figure D.1: Contour Integration Paths

Upon making the expansion, and defining $\Delta t = t_f - t_0$ we are left to evaluate the integrals:

$$I_1 = \frac{1}{2} e^{-i(\omega_\sigma + iz_n)t_f} \int_{-\infty}^{\infty} dk \left[k^n \frac{e^{ik\left(\frac{v_0 \Delta t}{2} + a\right)}}{(v_0 k - \omega_\sigma - iz_n)} \right]$$

$$I_2 = \frac{1}{2} e^{-i(\omega_\sigma + iz_n)t_f} \int_{-\infty}^{\infty} dk \left[k^n \frac{e^{ik\left(\frac{v_0 \Delta t}{2} - a\right)}}{(v_0 k - \omega_\sigma - iz_n)} \right]$$

$$I_3 = -\frac{1}{2} e^{-i(\omega_\sigma + iz_n)t_0} \int_{-\infty}^{\infty} dk \left[k^n \frac{e^{-ik\left(\frac{v_0 \Delta t}{2} - a\right)}}{(v_0 k - \omega_\sigma - iz_n)} \right]$$
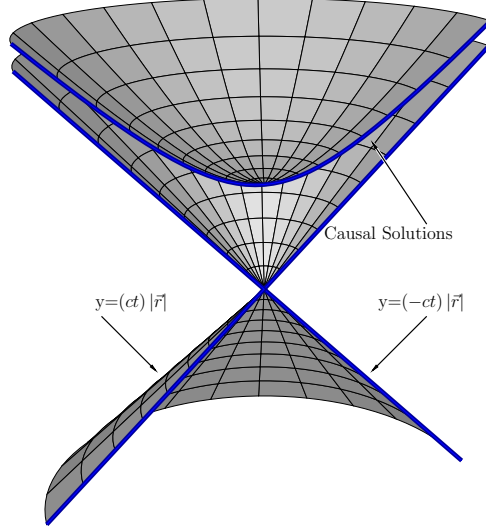
Figure D.2: Causality Conditions

$$I_4 = -\frac{1}{2}e^{-i(\omega_\sigma + iz_n)t_0} \int_{-\infty}^{\infty} dk \left[k^n \frac{e^{-ik\left(\frac{v_0 \Delta t}{2} + a\right)}}{(v_0 k - \omega_\sigma - iz_n)} \right]$$

For the case where we consider the cases in (D.1) corresponding to poles $(\omega_\sigma + iz_n) = a \pm ib$ in the lower or upper half plane, when mapping $v_0 k \equiv z$ to the complex plane, the possible combinations which give causal results are tabulated in (D.1). Causality is determined according to values of $|\vec{r}|$ that satisfy the conditions associated with the light cone presented in figure D.2. Integrating yields the following results, (here $\Theta(t)$ is the traditional step function which conserves causality)

1. $\frac{1}{2}e^{-i(\omega_\sigma + iz_n)t_f} \int_{-\infty}^{\infty} dk \left[k^n \frac{e^{ik\left(\frac{v_0 \Delta t}{2} + a\right)}}{(v_0 k - \omega_\sigma - iz_n)} \right]$:

   (a) Upper half plane requires $\Im\{k\} > 0$ and $\frac{v_0 \Delta t}{2} - (-a) \leq 0$

$$\begin{aligned} I_1 =& \frac{1}{2}e^{-i(\omega_\sigma + iz_n)t_f} \int_{-\infty}^{\infty} dk \left[k^n \frac{e^{ik\left(\frac{v_0 \Delta t}{2} + a\right)}}{(v_0 k - \omega_\sigma - iz_n)} \right] \\ =& \frac{1}{2}e^{-i(\omega_\sigma + iz_n)t_f} \left[\frac{2\pi i}{v_0^{n+1}} (\omega_\sigma + iz_n)^n \, e^{i(\omega_\sigma + iz_n)\left(\frac{\Delta t}{2} + \frac{a}{v_0}\right)} \right] \end{aligned}$$

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Table D.1: Causal & Non-Causal Contours

Exponential	$\Im\{k\}$	Constraints	Evaluation
$e^{ik\left(\frac{v_0\Delta t}{2}+a\right)}$	+, upper half	$\frac{v_0\Delta t}{2}-(-a)\geq 0$	Not Causal, Incoming Wave
$e^{ik\left(\frac{v_0\Delta t}{2}+a\right)}$	+, upper half	$\frac{v_0\Delta t}{2}-(-a)\leq 0$	Causal, Incoming Wave
$e^{ik\left(\frac{v_0\Delta t}{2}-a\right)}$	+, upper half	$\frac{v_0\Delta t}{2}-a\geq 0$	Causal, Outgoing Wave
$e^{ik\left(\frac{v_0\Delta t}{2}-a\right)}$	-, lower half	$\frac{v_0\Delta t}{2}-a\leq 0$	Not Causal, Outgoing Wave
$e^{-ik\left(\frac{v_0\Delta t}{2}-a\right)}$	+,upper half	$\frac{v_0\Delta t}{2}-a\leq 0$	Not Causal, Outgoing Wave
$e^{-ik\left(\frac{v_0\Delta t}{2}-a\right)}$	-,lower half	$\frac{v_0\Delta t}{2}-a\geq 0$	Causal, Outgoing Wave
$e^{-ik\left(\frac{v_0\Delta t}{2}+a\right)}$	-,lower half	$\frac{v_0\Delta t}{2}-(-a)\leq 0$	Causal, Incoming Wave
$e^{-ik\left(\frac{v_0\Delta t}{2}+a\right)}$	-,lower half	$\frac{v_0\Delta t}{2}-(-a)\geq 0$	Not Causal, Incoming Wave

We therefore conclude that the result of this contour integral is

$$I_1 = \frac{i\pi}{v_0^{n+1}}\left(\omega_\sigma + iz_n\right)^n \Theta\left(t_f - \frac{\Delta t}{2} - \frac{a}{v_0}\right) e^{-i(\omega_\sigma + iz_n)\left(t_f - \frac{\Delta t}{2} - \frac{a}{v_0}\right)}$$

2. $\frac{1}{2}e^{-i(\omega_\sigma + iz_n)t_f}\int_{-\infty}^{\infty} dk\left[k^n \frac{e^{ik\left(\frac{v_0\Delta t}{2}-a\right)}}{(v_0 k - \omega_\sigma - iz_n)}\right]$:

(a) Upper half plane requires $\Im\{k\}>0$ and $\frac{v_0\Delta t}{2}-a>0$:

$$I_2 = \frac{1}{2}e^{-i(\omega_\sigma + iz_n)t_f}\int_{-\infty}^{\infty} dk\left[k^n \frac{e^{ik\left(\frac{v_0\Delta t}{2}-a\right)}}{(v_0 k - \omega_\sigma - iz_n)}\right]$$
$$= \frac{1}{2}e^{-i(\omega_\sigma + iz_n)t_f}\left[\frac{2\pi i}{v_0^{n+1}}\left(\omega_\sigma + iz_n\right)^n e^{i(\omega_\sigma + iz_n)\left(\frac{\Delta t}{2}-\frac{a}{v_0}\right)}\right]$$

We therefore conclude that the result of this contour integral is

$$I_2 = \frac{i\pi}{v_0^{n+1}}\left(\omega_\sigma + iz_n\right)^n \Theta\left(t_f - \frac{\Delta t}{2} + \frac{a}{v_0}\right) e^{-i(\omega_\sigma + iz_n)\left(t_f - \frac{\Delta t}{2} + \frac{a}{v_0}\right)}$$

3. $-\frac{1}{2}e^{-i(\omega_\sigma + iz_n)t_0}\int_{-\infty}^{\infty} dk\left[k^n \frac{e^{-ik\left(\frac{v_0\Delta t}{2}-a\right)}}{(v_0 k - \omega_\sigma - iz_n)}\right]$ over the

(a) Lower half plane requires $\Im\{k\} < 0$ and $\frac{v_0\Delta t}{2} - a > 0$, yields

$$I_3 = -\frac{1}{2}e^{-i(\omega_\sigma+iz_n)t_0}\int_{-\infty}^{\infty}dk\left[k^n\frac{e^{-ik\left(\frac{v_0\Delta t}{2}-a\right)}}{(v_0k-\omega_\sigma-iz_n)}\right]$$

$$= -\frac{1}{2}e^{-i(\omega_\sigma+iz_n)t_0}\left[\frac{-2\pi i}{v_0^{n+1}}(\omega_\sigma+iz_n)^n\,e^{-i(\omega_\sigma+iz_n)\left(\frac{\Delta t}{2}-\frac{a}{v_0}\right)}\right]$$

We therefore conclude that the result of this contour integral is

$$I_3 = \frac{i\pi}{v_0^{n+1}}(\omega_\sigma+iz_n)^n\,\Theta\left(t_0+\frac{\Delta t}{2}-\frac{a}{v_0}\right)e^{-i(\omega_\sigma+iz_n)\left(t_0+\frac{\Delta t}{2}-\frac{a}{v_0}\right)}$$

4. $\frac{1}{2}e^{-i(\omega_\sigma+iz_n)t_0}\int_{-\infty}^{\infty}dk\left[k^n\frac{e^{-ik\left(\frac{v_0\Delta t}{2}+a\right)}}{(v_0k-\omega_\sigma-iz_n)}\right]$

(a) Lower half plane requires $\Im\{v_0k\} < 0$ and $\frac{v_0\Delta t}{2}-(-a) \le 0$, yields

$$I_4 = \frac{1}{2}e^{-i(\omega_\sigma+iz_n)t_0}\int_{-\infty}^{\infty}dk\left[k^n\frac{e^{-ik\left(\frac{v_0\Delta t}{2}+a\right)}}{(v_0k-\omega_\sigma-iz_n)}\right]$$

$$= \frac{1}{2}e^{-i(\omega_\sigma+iz_n)t_0}\left[\frac{-2\pi i}{v_0^{n+1}}(\omega_\sigma+iz_n)^n\,e^{-i(\omega_\sigma+iz_n)\left(\frac{\Delta t}{2}+\frac{a}{v_0}\right)}\right]$$

We therefore conclude that the result of this contour integral is

$$I_4 = -\frac{\pi i}{v_0^{n+1}}(\omega_\sigma+iz_n)^n\,\Theta\left(t_0+\frac{\Delta t}{2}+\frac{a}{v_0}\right)e^{-i(\omega_\sigma+iz_n)\left(t_0+\frac{\Delta t}{2}+\frac{a}{v_0}\right)}$$

The final result, where $t = \frac{t_f+t_0}{2}$ and $\Delta t = t_f - t_0$, then $t = t_f - \frac{\Delta t}{2}$, or $t = t_0 + \frac{\Delta t}{2}$, is given for the following conditions

- for poles in the lower half plane is

$$\int_{-\infty}^{\infty}dk\left[k^n\cos(ka)\frac{\left[e^{-i(\omega_\sigma+iz_n)t_f-i\frac{v_0k}{2}\left(t_0-t_f\right)}-e^{-i(\omega_\sigma+iz_n)t_0-i\frac{v_0k}{2}\left(t_f-t_0\right)}\right]}{(v_0k-\omega_\sigma-iz_n)}\right]$$

$$= \frac{i\pi}{v_0^{n+1}}(\omega_\sigma+iz_n)^n\left[\Theta\left(t-\frac{a}{v_0}\right)e^{-i(\omega_\sigma+iz_n)\left(t-\frac{a}{v_0}\right)}-\Theta\left(t+\frac{a}{v_0}\right)e^{-i(\omega_\sigma+iz_n)\left(t+\frac{a}{v_0}\right)}\right]$$

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- for poles in the upper half plane is

$$\int_{-\infty}^{\infty} dk \left[k^n \cos\left(ka\right) \frac{\left[e^{-i(\omega_\sigma+iz_n)t_f - i\frac{v_0 k}{2}\left(t_0-t_f\right)} - e^{-i(\omega_\sigma+iz_n)t_0 - i\frac{v_0 k}{2}\left(t_f-t_0\right)}\right]}{\left(v_0 k - \omega_\sigma - iz_n\right)} \right]$$

$$= \frac{i\pi}{v_0^{n+1}} \left(\omega_\sigma + iz_n\right)^n \left[\Theta\left(t + \frac{a}{v_0}\right) e^{-i(\omega_\sigma+iz_n)\left(t+\frac{a}{v_0}\right)} + \Theta\left(t - \frac{a}{v_0}\right) e^{-i(\omega_\sigma+iz_n)\left(t-\frac{a}{v_0}\right)}\right]$$

D.11.2 For those improper integrals which are of the form

$$\int_{-\infty}^{\infty} dk \left[k^n \sin\left(ka\right) \frac{\left[e^{-i(\omega_\sigma+iz_n)t_f - i\frac{v_0 k}{2}\left(t_0-t_f\right)} - e^{-i(\omega_\sigma+iz_n)t_0 - i\frac{v_0 k}{2}\left(t_f-t_0\right)}\right]}{\left(v_0 k - \omega_\sigma - iz_n\right)} \right]$$

and satisfy the contours found in (D.1), we can find their physically sensible closed forms in terms of a, Δt, and z_n so long as we constrain ourselves to causal results. To do this, we first expand $\sin ka \equiv \frac{1}{2i}\left[e^{ika} - e^{-ika}\right]$. Upon making the expansion, and defining $\Delta t = t_f - t_0$ we are left to evaluate the integrals:

$$I_1 = \frac{1}{2i} e^{-i(\omega_\sigma+iz_n)t_f} \int_{-\infty}^{\infty} dk \left[k^n \frac{e^{ik\left(\frac{v_0 \Delta t}{2}+a\right)}}{\left(v_0 k - \omega_\sigma - iz_n\right)} \right]$$

$$I_2 = -\frac{1}{2i} e^{-i(\omega_\sigma+iz_n)t_f} \int_{-\infty}^{\infty} dk \left[k^n \frac{e^{ik\left(\frac{v_0 \Delta t}{2}-a\right)}}{\left(v_0 k - \omega_\sigma - iz_n\right)} \right]$$

$$I_3 = -\frac{1}{2i} e^{-i(\omega_\sigma+iz_n)t_0} \int_{-\infty}^{\infty} dk \left[k^n \frac{e^{-ik\left(\frac{v_0 \Delta t}{2}-a\right)}}{\left(v_0 k - \omega_\sigma - iz_n\right)} \right]$$

$$I_4 = \frac{1}{2i} e^{-i(\omega_\sigma+iz_n)t_0} \int_{-\infty}^{\infty} dk \left[k^n \frac{e^{-ik\left(\frac{v_0 \Delta t}{2}+a\right)}}{\left(v_0 k - \omega_\sigma - iz_n\right)} \right]$$

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For the case where we consider the cases in (D.1) corresponding to poles $(\omega_\sigma + iz_n) = a \pm ib$ in the lower or upper half plane, when mapping $v_0 k \equiv z$ to the complex plane, the possible combinations which give physical results are again tabulated in (D.1). It is clear that these are of the same form as the previous 4 integrals, off by a phase of $e^{i\frac{\pi}{2}}$ or $e^{-i\frac{\pi}{2}}$. Therefore, the results follow the same constraints in the complex plane as before. We evaluate these from inspection of the leading coefficients of the results above as divided by the appropriate leading factor corresponding tho the

- for poles in the lower half plane is

$$
\int_{-\infty}^{\infty} dk \left[k^n \sin(ka) \frac{\left[e^{-i(\omega_\sigma + iz_n)t_f - i\frac{v_0 k}{2}\left(t_0 - t_f\right)} - e^{-i(\omega_\sigma + iz_n)t_0 - i\frac{v_0 k}{2}\left(t_f - t_0\right)} \right]}{(v_0 k - \omega_\sigma - iz_n)} \right]
$$
$$
= \frac{\pi}{v_0^{n+1}} (\omega_\sigma + iz_n)^n \left[\Theta\left(t - \frac{a}{v_0}\right) e^{-i(\omega_\sigma + iz_n)\left(t - \frac{a}{v_0}\right)} + \Theta\left(t + \frac{a}{v_0}\right) e^{-i(\omega_\sigma + iz_n)\left(t + \frac{a}{v_0}\right)} \right]
$$

- for poles in the upper half plane is

$$
\int_{-\infty}^{\infty} dk \left[k^n \sin(ka) \frac{\left[e^{-i(\omega_\sigma + iz_n)t_f - i\frac{v_0 k}{2}\left(t_0 - t_f\right)} - e^{-i(\omega_\sigma + iz_n)t_0 - i\frac{v_0 k}{2}\left(t_f - t_0\right)} \right]}{(v_0 k - \omega_\sigma - iz_n)} \right]
$$
$$
= \frac{\pi}{v_0^{n+1}} (\omega_\sigma + iz_n)^n \left[\Theta\left(t + \frac{a}{v_0}\right) e^{-i(\omega_\sigma + iz_n)\left(t + \frac{a}{v_0}\right)} - \Theta\left(t - \frac{a}{v_0}\right) e^{-i(\omega_\sigma + iz_n)\left(t - \frac{a}{v_0}\right)} \right]
$$

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