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A Methodology To Stabilize The Supply Chain

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A METHODOLOGY TO STABILIZE THE SUPPLY CHAIN

by

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ABSTRACT

In today's world, supply chains are facing market dynamics dominated by strong global competition, high labor costs, shorter product life cycles, and environmental regulations. Supply chains have evolved to keep pace with the rapid growth in these business dynamics, becoming longer and more complex. As a result, supply chains are systems with a great number of network connections among their multiple components. The interactions of the network components with respect to each other and the environment cause these systems to behave in a highly nonlinear dynamic manner.

Ripple effects that have a huge, negative impact on the behavior of the supply chain (SC) are called instabilities. They can produce oscillations in demand forecasts, inventory levels, and employment rates and, cause unpredictability in revenues and profits. Instabilities amplify risk, raise the cost of capital, and lower profits. To reduce these negative impacts, modern enterprise managers must be able to change policies and plans quickly when those consequences can be detrimental.

This research proposes the development of a methodology that, based on the concepts of asymptotic stability and accumulated deviations from equilibrium (ADE) convergence, can be used to stabilize a great variety of supply chains at the aggregate levels of decision making that correspond to strategic and tactical decision levels. The general applicability and simplicity of this method make it an effective tool for practitioners specializing in the stability analysis of systems with complex dynamics, especially those with oscillatory behavior.

This methodology captures the dynamics of the supply chain by using system dynamics

(SD) modeling. SD was the chosen technique because it can capture the complex relationships, feedback processes, and multiple time delays that are typical of systems in which oscillations are present. If the behavior of the supply chain shows instability patterns, such as ripple effects, the methodology solves an optimization problem to find a stabilization policy to remove instability or minimize its impact. The policy optimization problem relies upon a theorem which states that ADE convergence of a particular state variable of the system, such as inventory, implies asymptotic stability for that variable. The stabilization based on the ADE requires neither linearization of the system nor direct knowledge of the internal structure of the model. Moreover, the ADE concept can be incorporated easily in any SD modeling language.

The optimization algorithm combines the advantage of particle swarm optimization (PSO) to determine good regions of the search space with the advantage of local optimization to quickly find the optimal point within those regions. The local search uses a Powell hill-climbing (PHC) algorithm as an improved procedure to the solution obtained from the PSO algorithm, which assures a fast convergence of the ADE. The experiments showed that solutions generated by this hybrid optimization algorithm were robust.

A framework built on the premises of this methodology can contribute to the analysis of planning strategies to design robust supply chains. These improved supply chains can then effectively cope with significant changes and disturbances, providing companies with the corresponding cost savings.

Dedicated to my parents, for their love and continuous support.

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LIST OF ACRONYMS/ABBREVIATIONS

ADE	Accumulated Deviations from Equilibrium
AEDE	Accumulated Exponential Deviations from Equilibrium
BDW	Behavioral Decomposition Weights
DE	Deviations from the Equilibrium Point
EEA	Eigenvalue Elasticity Analysis
EP	Equilibrium Point
GA	Genetic Algorithms
MSA	Model Structural Analysis
I-W	Inventory-Workforce
PHC	Powell Hill-Climbing
PMO	Precision Molded Optics
PPM	Pathway Participation Metric
PSO	Particle Swarm Optimization
SADE	Stabilization based on the Accumulated Deviations from Equilibrium
SC	Supply Chain
SCM	Supply Chain Management
SD	System Dynamics
WIP	Work in Process

CHAPTER ONE: INTRODUCTION

During the last decade, manufacturing enterprises have been under pressure to compete in a market that is rapidly changing due to global competition, shorter product life cycles, dynamic changes of demand patterns and product varieties and environmental standards. In these global markets, competition is ever increasing and companies are widely adopting customer-focused strategies in integrated-system approaches. In addition, push manufacturing concepts are being replaced by pull concepts and notions of quality systems are getting more and more significant.

Globalization of products and services and the rapid changes in technology have also resulted in fast-growing dynamic markets and greater uncertainty in customer demand. The process of managing and controlling the supply chain has become increasingly complex due to the geographic extension of the global operations between facilities. Moreover, competition has evolved from one company against other companies to one supply chain against other supply chains.

Supply chain management (SCM) is seen as a mechanism that will allow companies to respond to these environmental changes and has become one of the top priorities on the strategic agenda of industrial and service businesses. The objective of SCM activities is to provide right quality of the right product at the right time. The attempt is to improve responsiveness, understand customer demand, control production or service processes, and align together the objectives of all partners in the supply chain. To achieve this goal, companies need the ability to provide improved management policies in order to react quickly to unexpected events taking place in the supply chain, eliminate the most undesirable effects if possible, and, minimize the impact of those that can not be eliminated.

In order to make investigations and to support decision-making about the impact of supply chain dynamics, system dynamics simulation models are suitable tools. "System dynamics is an approach for the modeling and simulation of nonlinear dynamic systems that aims at the understanding of a system's structure and the deduction of the behavior from it. This focus on understanding is a great advantage of the system dynamics methodology as it is a requirement for the development of policies that lead to the improvement of the system's performance. One important advantage of system dynamics is the possibility to deduce the occurrence of a specific behavior mode because the structure that leads to systems' behavior is made transparent" (Schieritz and Größler 2003).

Policy analysis[1] as a method to generate stabilization policies in SCM can be addressed by getting a better understanding of the model structure that determines the SC behavior. The main idea behind this structural investigation is that the behavior of a SC model is obtained by adding elementary behavior modes. For linear models the eigenvalues[2] represent these different behavior modes the superposition of which gives rise to the observed behavior of the system. For nonlinear systems the model has to be linearized at any point in time. Finding the connection between structure and behavior provides a way to discover pieces of the model where to apply policies to eliminate instabilities. However, other techniques are required to determine the best values of the parameters related to the stabilization policy.

[1] In policy analysis, decisions are represented by a set of parameters, referred to as "policy parameters" (Grossmann 2002).

SD models coupled with policy optimization techniques have proven to be a very powerful means for improving the behavior of dynamic systems. These methods are based on the optimization of a certain objective function to find the parameter values of the improved policy (Mohapatra and Sharma 1985). Choosing the objective function appropriately is critical for the effective change of the system behavior (Chen and Jeng 2004).

The objective of this research is to develop a methodology that models and manages supply chains as dynamic systems[3] and uses a policy optimization approach to modify the behavior of entire supply chains in order to achieve stability.

1.1. Statement of the Problem

As the world changes, supply chains have evolved to keep pace with the changing business dynamics, becoming longer and more complex. Today, supply chains are networks with an overwhelming number of interactions and interdependencies among different entities, processes and resources. These interactions of the system components with respect to each other and the environment create a highly nonlinear dynamic system.

The classical way of managing a supply chain was to observe and analyze sales, demand, and inventory levels at the end of a certain pre-defined time and fill the required gap in it. That was based on the assumption that the supply and demand would remain linear, or at least stable, with no drastic fluctuations. This assumption was valid in a market dominated by the supplier's

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[2] Eigenvalues (λ) are special set of scalars (real or complex numbers) associated with a linear system $\dot{\boldsymbol{x}} = \boldsymbol{J}\boldsymbol{x}$. They are the roots of the characteristic equation $\boldsymbol{J}\boldsymbol{r} = \lambda \boldsymbol{r}$, where $\boldsymbol{J} = \partial \dot{x}_i / \partial x_j$ is a square matrix known as the Jacobian matrix, and $\mathbf{r} \neq \mathbf{0}$ is called the eigenvector (AbdelGawad *et al*. 2005).

perspective, not the consumer's (Rabelo *et al.* 2004). However, due to the complexity of the current supply chains, for example, small fluctuations in customer demand can lead to instabilities that quickly ripple through the entire supply chain. These ripple effects can cause excessive inventory buildup, poor customer service, unnecessary capital investment, and dangerously low profits (Sterman 2006).

This research recognizes the difficulties and challenges for developing a methodology that will be based on a general concept that can be used to stabilize a great variety of supply chains, with emphasis on the ones with complex dynamic behaviors arising from nonlinearities and complicated interdependencies.

1.2. Motivation of this Research

This research is motivated by the large negative impacts of supply chain instabilities. Those impacts occur because instabilities can cause (1) oscillations in demand forecasts, inventory levels, and employment rates and (2) unpredictability in revenues and profits. These impacts amplify risk, raise the cost of capital, and lower profits. Modern enterprise managers can minimize these negative impacts by having the ability to determine alternative policies and plans quickly.

This work proposes a methodology to reduce or eliminate undesirable behaviors by generating stabilization policies that focus on the aggregate levels of decision making that

[3] A dynamic system is a model that captures the relevant changes among variables and parameters over time. For example, a simple pendulum is a system that actually evolves over time and can be modeled as a dynamic system.

correspond to the strategic and tactical levels[4] of the SC. These policies will be obtained by redefining the relevant parameters of the SC at these levels. The proposed approach first uses SD to capture current dynamics. Then, a parameter optimization problem will produce policies that will remain stable for small variations[5] in the system, providing the managers with an instrument to generate robust policies that eliminate instabilities in the SC.

A supply chain model can be described by its structure and its parameters. Traditional approaches that relate model structure and behavior have relied upon sensitivity analysis and linearized models when exploring possible changes in complex systems. Although it is possible from the structure of the system to identify relevant parameters responsible for generating specific behaviors, it is hard to determine how much these parameters have to change simultaneously in order to obtain a desired overall behavior. While these methods can be used to obtain stable policies of the SC, the complexity of the associated mathematics makes them difficult to use for managers and practitioners.

On the other hand, policy optimization methods have been used to optimally modify the parameters of dynamic systems to achieve certain objectives. However, these objectives are defined to meet particular characteristics of the system, implying that different systems require different objectives and settings.

For the reasons presented above, there is a necessity for a methodology that, based on stability conditions obtained from the structure of a generic SC model, can produce robust policies to eliminate or reduce the impact of instabilities. It is the hope of the author of this

[4] The different decision levels of the supply chain are explained in chapter two.
[5] Small variations represent a region close to the equilibrium state

research that this methodology will be able to provide a powerful and simple tool that can be used by practitioners and academics.

1.3. Research Question

Due to the non-existence and the need for a general methodology that can assist in the stabilization capabilities for supply chains, it is the primary focus of this research to answer the following question:

Can a methodology be developed that extends the current research findings in the engineering field to form the components of a framework that allows to eliminate or minimize the impact of supply chain instabilities?

1.4. Research Objectives

The objectives of this research include the following:

1) Proposition of generic stability conditions, based on the accumulated deviations from equilibrium[6], to produce robust policies that can be applied to a great variety of supply chain models.
2) Formulation of an optimization problem, based on the conditions proposed in objective 1, to eliminate or minimize instability of the SC.
3) Use of a simulation optimization method that combines SD with a hybrid search engine based on PSO and PHC algorithms, to model and solve the optimization problem stated

[6] For a state variable of a SC model, the deviations from equilibrium at time "t" represent the absolute value of the difference between its value at time "t" and its value in the equilibrium state. The ADE are the summation of these

in objective 2. The efficiency of this hybrid method relies on the advantage of the PSO algorithm to provide a global view of the search space and the ability of the PHC to find the local optimum with high accuracy.

4) Development and demonstration of the benefits of a computerized framework for modifying the behavior of SC models in order to achieve stability. The framework will use the conditions and methods presented in objectives 1 through 3.

1.5. Research Contribution

Companies are discovering that effective SCM is having a tremendous impact to increase profit and market share. On the contrary, inefficient SCM can cause numerous problems, such as ineffective production and transportation schedules, poor customer service and excessive inventory investment. These problems can cost companies millions of dollars. Here are some facts that show the magnitude of the costs:

- The Wall Street Journal published an article (Chozick 2007) that explains that due to a delay in making deliveries of a piston ring costing $1.50 from a mayor supplier, nearly 70% of Japan's auto production was temporarily paralyzed. For instance, Toyota stopped production in its Japanese plants for at least one day and a half, causing a loss of output of at least 25,000 vehicles.
- Instability around the world has cost U.S companies more than $300 billion in SC disruptions, according to a recent study by Aon Trade Credit (Karrenbauer 2006).

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deviations for the time horizon considered in the SC model. The mathematical equation for the ADE is introduced in chapter 3.

- A fire in 2000 at a chip plant of the Dutch electronics giant, Phillips N.V., caused the shortage of millions of chips needed for the cell phone manufacturer Ericsson. It took a \$2.34 billion loss in its mobile phone division (Bartholomew 2006).
- According to the State of Logistics Report (Delaney and Wilson 2000), in 2000, the US companies spent \$1 trillion (10% of GNP) on supply-related activities (movement, storage, and control of products across supply chains).
- Compaq Computer estimated that it lost \$500 million to \$1 billion in sales in 1994 because its laptops and desktops were not available when and where customers were ready to buy them (Henkoff 1994).
- One study suggested that inefficiencies within a supply chain increase costs by as much as 25% (Kurt Salmon Associates 1993).
- It is estimated that the grocery industry could save \$30 billion per year (10% of operating cost) by using effective logistics strategies (Kurt Salmon Associates 1993).
- Carlsson and Fullér (1999) claimed that demand variability along the SC would cost \$17-34 million euros per year to the forest products industry.

Due to the dynamic changes in the business environment, managers today rely on decision technology[7] more than ever to make decisions. In the area of supply chain, the top projected activities where decision technology applications have great potential of development are planning, forecasting, and scheduling (Poirier and Quinn 2006).

This research work is proposing a methodology that from now on will be called stabilization based on the accumulated deviations from equilibrium (SADE). A framework built on the base of this methodology will allow the analysis of planning strategies to design robust supply chains that can effectively cope with significant changes and disturbances, with the corresponding cost savings to the companies.

1.6. Thesis Outline

The remainder of this thesis is organized as follows. Chapter Two discusses the literature review of SD modeling in SCM, model structural analysis (MSA), policy optimization, stability analysis of the supply chain, PSO and PHC algorithms. Chapter Three presents the steps of the research methodology which considers the logic and the validation of the SADE methodology and its integration into a framework. Chapter Four provides the definitions and theorems that support the theoretical concepts of the stabilization methodology. Chapter Five discusses the application of the framework to several case studies and presents the results of the experimental analysis for the SADE methodology. Finally, Chapter Six presents the conclusions and contributions of this work and suggests directions for further research.

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[7] Decision technology adds value to network infrastructure and applications by making them smarter.

CHAPTER TWO: LITERATURE REVIEW

This chapter discusses the definitions and technical aspects that are necessary for a conceptualization of a methodology of general applicability for stabilizing supply chains. Much of the literature that can be related to SCM control systems presents techniques for separated analysis of the SC related to specific problems, but does not discuss wider conceptual methodologies for stability analysis.

This literature review focuses on presenting the scope and approaches behind the concepts of stability, system dynamics, eigenvalue analysis and optimization, as a facilitator to link some of these concepts to one another. Applications of these concepts in the area of supply chain are provided when found in the literature; otherwise similar lines of development in engineering and business fields are presented. This chapter covers the following topics:

1. *System dynamics modeling in supply chain management*. This topic presents the different decision levels of the supply chain and explains the use of SD modeling at the top level of the management hierarchy.
2. *Model structural analysis*. This topic shows the use of the model structural analysis for identifying the connection between behavior and structure of dynamic systems.
3. *Policy optimization*. This topic illustrates the use of optimization methods to find policies that modify the system behavior and when combined with simulation optimization represents the most general mean for achieving stability.
4. *Stability analysis of the supply chain*. This topic shows how the stability of supply chains can be analyzed using different approaches such as control theory methods,

MSA and policy optimization.

5. *Particle swarm optimization*. This topic presents the advantages of using a global search algorithm based on PSO to find optimal policies that can stabilize linear and nonlinear SD models.
6. *Powell hill-climbing algorithm*. This topic shows how the benefits of hill-climbing algorithms can help to improve the searching capabilities of global search methods.

As the final result of this review, the research gaps that justify the development of the proposed methodology are identified.

2.1. System Dynamics Modeling in Supply Chain Management

This topic is divided in two parts. The first introduces the definition of SCM and the decision levels involved in the SC. The second presents the SD methodology and its applications in supply chain management.

2.1.1. Supply Chain Management and Decision Levels

In today's business environment, companies can not expect to build a successful product, process, or service advantage if their strategies are not integrated with those of the supply chain systems in which they are interconnected (Ross 2003). Therefore, supply chain management is a mayor component of competitive strategy to enhance organizational productivity and profitability (Gunasekaran *et al.* 2004).

In order to understand the concept of supply chain management, firstly, it is necessary to define what a supply chain is. Several authors have proposed the following definitions for SC and SCM:

Handfield and Nichols (1999) define supply chain as encompassing:

> "all activities with the flow and transformation of goods from the raw materials stage (extraction), through to the end user, as well as the associated information flows. Material and information flow both up and down the supply chain".

After that, supply chain management is defined as:

> "the integration of these activities through improved supply chain relationships to achieve a sustainable competitive advantage".

According to Ayers (2001), the supply chain is "more than the physical move of goods". It is also information, money movement, and the creation and deployment of intellectual capital. He defines the supply chain as:

> "life cycle processes comprising physical, information, financial, and knowledge flows for moving products and services from suppliers to end users".

Ayers then defines supply chain management as the:

> "design, maintenance and operation of supply chain processes for satisfaction of end user needs".

Simchi-Levi *et al.* (2002) propose the idea of supply chain network in their definition of SCM. They state that "supply chain management is a discipline that focuses on the integration of suppliers, factories, warehouses, distribution centers, and retail outlets so that the items are produced and distributed to the right customers, at the right time, at the right place, and at the right price. It is important to do this in a way that minimizes costs while satisfying a certain level of service".

Having defined the concept of SCM, it is necessary to categorize the decisions associated with the planning and control activities of the supply chain.

Anthony (1965) proposes a category where the decision process can be partitioned, to select adequate models and techniques to deal with the individual decisions at different hierarchical levels. He classifies decisions in three categories: strategic planning, tactical planning, and operations control. This hierarchical classification recognizes the distinct level of aggregation of the required information, and the time frame in which the decision is to be made. This classification has been incorporated into the SCM to support integrated decision making (Chang and Harris 2001; Surana *et al.* 2005; Huang *et al.* 2003; Beamon and Chen 2001; Gunasekaran *et al.* 2004).

Strategic planning is concerned mainly with establishing managerial policies and with developing the necessary resources the enterprise needs to satisfy its external requirements in a manner consistent with its specific goals (Hax 1974). Strategic decisions consider the long term (time horizons up to several years in length), and the scope is corporate wide, requiring information to be processed in a very aggregate form. At this level, the performance of the SC is measured against corporate goals often reflecting broad based policies, financial plans, and competitiveness. Strategic level analysis includes location and capacity of warehouses and manufacturing plants, the flow of material through the supply network, inventory management policies, distribution strategies, outsourcing and procurement strategies, product design, etc. (Georgiadis *et al.* 2005).

Tactical planning is concerned with the effective allocation of resources to satisfy demand and technological requirements. Tactical decisions correspond to the medium term (time horizons up to several months in length), and the scope is at least regional or corporate wide. Some resources, such as the location of manufacturing facilities, are fixed. Tactical level analysis considers demand forecast, inventory control, production/distribution coordination, material handling and layout design.

Operational planning is concerned with the very short term decisions made from day to day. This requires the complete disaggregation of the information generated at higher levels into the details consistent with the managerial procedures followed in daily activities. Resources are typically known and fixed. Analysis at this level considers routing, scheduling, workload balancing and inventory control. Performance measures of the SC at the operational level require accurate data and assess the results of low level managers' decisions.

2.1.2. System Dynamics and its Application in Supply Chain Management

Jay Forrester (1958, 1961) introduced SD in the late 50s as a model and simulation methodology for the analysis and long-term decision making of dynamic industrial management problems. System Dynamics has its origins in control engineering and management; the approach uses a perspective based on information feedback and delays to understand the dynamic behavior of complex physical, biological, and social systems (Angerhofer and Angelides 2000). The essential idea in SD is that all the objects in a system interact through causal relationships. These relationships are represented by feedback loops, which control the interactions between the system objects and cause the system behavior (Rabelo *et al.* 2003).

According to Lane (1997), Forrester (1958) proposes a whole new way to understand and model management problems. He summarizes that Forrester claims:

> "… systems should be modeled as flow rates and accumulations linked by information feedback loops involving delays and nonlinear relationships. Computer simulation is then the means of inferring the time evolutionary dynamics endogenously created by such system structures. The purpose is to learn about their modes of behavior and to design policies which improve performance".

Supply chain is a dynamic process and involves the complex flow of information, material, and funds across multiple functional areas both within and among companies (Ahn *et al.* 2003). Surana *et al.* (2005) present some characteristics of supply chains:

- Coexistence of competition and cooperation. The entities in a SC often have conflicting objectives. Competition abounds in the form of sharing and contention of resources. Global control over nodes is an exception rather than a rule; more likely is a localized cooperation out of which a global order emerges, which is itself unpredictable.
- Nonlinear dynamics. Customers can initiate transactions at any time with little or not regard for existing load. The coordination protocols in the SC attempt to arbitrate among entities with resource conflicts, generating over- and under-corrections which contribute to the nonlinear character of the network.
- Quasi-equilibrium: Supply chains can experience a structural change when they are stretched from equilibrium. At such a point, a small event can trigger a cascade of changes that eventually can lead to system-wide reconfiguration. One of the causes of unstable phenomena is that the information feedback in the system is slow relative to

the rate of changes that occur in the SC.

- Emergent behavior: From the interaction of the simple entities, the organization of the overall SC emerges through a natural process order and spontaneity. Demand amplification and inventory swing are two undesirable emergent phenomena that can also arise.

SD models use a system of differential equations to capture interactions between different subsystems and the impacts of delays in the SC. Because of the complexity of the system with nonlinearity, many times it is not possible to solve it analytically. In such cases, continuous simulation must be used to provide the solution. Therefore, simulation is the most versatile tool for dealing with complex dynamic systems like the supply chain.

Since differential equations produce *smooth outputs*, they are not suited to the modeling of all levels of the supply chain. The system must be considered at an aggregate level, in which individual entities in the system (products) are not considered. Consequently, SD is not an appropriate technique to be used in production processes in which each individual entity has an impact on the fundamental state of the system, such as lot sizing and job sequencing problems (Riddalls *et al*. 2000).

Strategic supply chain management deals with a wide spectrum of issues and includes several types of decision-making problems that affect the long term development and operations of a firm. The data required at this stage is more aggregate than at subsequent levels (tactical and operational) and there are not fixed resources. Mathematical programming (optimization techniques) and SD have been two approaches used for the analysis and study of the strategic SCM models. However, SC optimization models may produce an optimal solution for a static point in time, but this solution may not prove to be robust in dynamic environments (Blackhurst *et al.* 2005).

Conversely, simulation is a well suited technique for studying dynamics in supply chains and generally proceeds based on SD models (Surana *et al.* 2005). Riddalls *et al.* (2000) conclude that global behavior of a supply chain can only be assessed by using dynamic simulation. Ashayeri *et al.* (1998) consider that SD is an adequate technique for the modeling and simulation of strategic systems. The reason proposed is that detailed information is not required to represent the relationships of the feedback loops used in SD models to represent the behavior of the system. Akkermans and Bertrand (1997) affirm that SD models are commonly characterized as especially successful in capturing strategic issues. However, this does not mean that SD models contain no links to tactical or operational processes in the SC (Baines and Harrison 1999). To build a SD model it is necessary to identify the main operational flows in an organization and the main stages in these flows: the flow of customer orders, of goods or services, of employees, etc. (Forrester 1961; Richmond 1994; Kleijnen 2005).

System dynamic models represent the frequency domain more naturally than optimization models, providing a framework particularly suited to the study of systems in which oscillations are a main attribute. Through these models it is possible to investigate which factors determine how demand fluctuations may be amplified as they are passed along the supply chain (Riddalls *et al.* 2000).

According to Akkermans (2005), SD is an approach that is able to model "implicit system boundaries explicitly". For instance, "the presence of competitors is often only noticeable in elements like shortage costs (a customer who is not served well might go somewhere else)".

However, “only in SD are these competitors considered to be within the system boundary when this is relevant for the problem at stake”. Moreover, modeling causal structures in feedback loops helps to provide an endogenous explanation of real system behavior. For example, the customer demand can be considered part of the SD model, something that is very rare in other quantitative approaches. Customer demand is not exogenous but endogenous, and is determined at least in part by the supply network (Akkermans and Dellaert 2005; Mandal and Sohal 1998).

The application of SD modeling to SCM has its roots in *Industrial Dynamics* (Forrester 1958, 1961). Forrester presents a three-level SC model consisting of a factory with a warehouse, a distributor and a retailer. He suggests that the main task of managers is the understanding and control of five types of flows that occur in industrial companies: “information, materials, money, manpower, and capital equipment”. Forrester examines how production and distribution procedures in a supply chain may result in an inadequate assessment of perceived demand, creating a *demand amplification* effect (see Figure 2.1). This effect, also know as the *bullwhip effect*, is the process by which small fluctuations in demand at the retailer end of SC are amplified as they proceed throughout the chain, causing increased inventory, irregular capacity utilization, and reduced service level (Chu 2003).

More examples of practical applications of SD modeling to SCM will be presented in the following lines.

Sterman (1989) uses a SD model of the *Beer Distribution Game*, which is a realistic simplification of the SC for beer manufacturer, to rigorously test the existence of the bullwhip effect in an experimental context. He provides evidence that the bullwhip effect exists and may be caused by chain member’s tendency to underweight inventory in the SC. Later, Sterman

(2000) introduces a generic SD model of the stock management structure which is used to explain the origin of oscillations in supply chains. He concludes that SC distortions can be amplified due to the existence of hard safety stock policies. Oscillations arise from the combination of time delays in negative feedbacks[8] and failure of the decision maker to take the time delays into account. Villegas and Smith (2006) extend Sterman's work by considering in the analysis the trade-off between production quantity oscillations and inventory oscillations. They show that this trade-off can be managed by a change to the planning policy to give more relevance to the forecast rather than the safety stock policy.

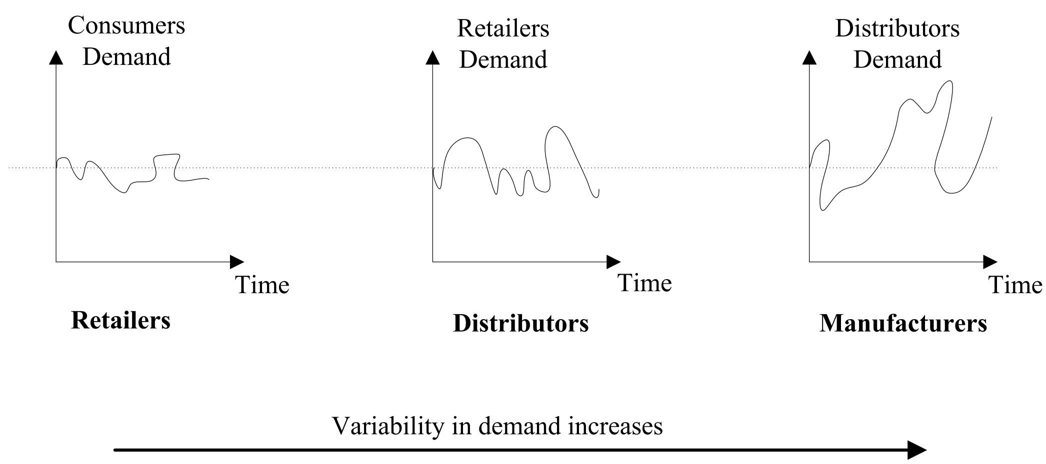

Figure 2.1. Bullwhip effect in a supply chain

Anderson Jr. *et al.* (2000) investigate *demand amplification effects* in the machine tool industry through a SD model. The SD methodology allows them to incorporate typical features of the capital equipment supply chains, such as feedback loops, delays and nonlinearities. Unlike other modeling studies which only concentrate on logistical decisions, these authors also

[8] In SD theory, all dynamics arise from the interaction of just two types of feedback loops, positive (or self-reinforcing) and negative (or self-correcting) loops. Positive feedbacks tend to reinforce or amplify whatever is

investigate the effect of factors such as work force learning on supply chain dynamics. The study demonstrate that: (1) the (observed and simulated) extreme amplification is primarily due to the machine tool industry capacity in conjunction with investment accelerator effect, (2) the machine maker's employee productivity decreases with increasing volatility, and (3) smoothing employment and product order policies can improve machine maker operations. They also identify the machine tool customers' order forecast rules as important point for reducing volatility, which could be improved through closer collaboration between customers and suppliers in the machine tool industry (Angerhofer and Angelides 2000).

Ashayeri *et al.* (1998) create a model for the distribution chain of Edisco – the European distribution arm of the US Company Abbott Laboratories. They propose a new conceptual framework for conducting a structured business process reengineering supported by SD simulation. The SD model is simulated in order to find out which strategies will result in the highest performance improvements and help a company to change toward its vision. Important conclusions obtained from the experiments are: (1) increase of production capacity (a structural change) does not guarantee a stable supply chain, (2) the higher the total lead-time, the bigger the degree of instability, and (3) although sufficient production capacity does not guarantee a stable supply chain, full scheduling of capacity is disastrous.

Joshi (2000) builds a framework for improving visibility of information in the SC by reducing the delays in information flow. He analyzes the dynamics of a SC under different scenarios of information visibility and forecasting decisions with the help of SD simulation.

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happening in the system. On the contrary, negative feedbacks counteract and oppose change (Sterman 2000).

The SD methodology has also been applied to the food industry supply chain. Minegishi and Thiel (2000) develop a model to improve the understanding of the complex logistic behavior of an integrated food industry. The model allows them to study the influence of different policies applied to the poultry production and processing, and to show the phenomena of instabilities and system controls in supply chains confronted with serious hazards in their customer demand. Georgiadis *et al.* (2005) present a holistic model to capture the extended food supply chain at a strategic level. They adopt the SD methodology as a modeling and analytical tool to tackle strategic issues for food supply chains. The model is used to identify effective policies and optimal parameters for various strategic decision making problems of single and multi-echelon supply chains. Finally, they demonstrate the applicability of the developed methodology on a multi-echelon network of a major Greek fast food chain.

Design and development of close-loop supply chains[9] have been analyzed using SD models. Spengler and Schröter (2003) use SD to model and evaluate different scenarios of a SC for the recovery of spare parts in the electronics industry. The scenario analysis allowed taking managerial decisions based on cost and production capacity. Georgiadis and Vlachos (2004) study long-term behavior of reverse SC with product recovery under various *ecological awareness* influences capacity planning policies. The behavior of the system is analyzed through a dynamic simulation model based on the principles of the SD methodology. They examine two main environmental issues, the *green image* effect on customer demand, and the effect of state

[9] Close-loop supply chains include the return process in addition to the conventional activities of forward supply chains, where the customer is the end of the process. (Guide Jr. *et al.* 2003).

environmental protection policies, such as the state campaigns for proper disposal of used products.

Higuchi and Troutt (2004) simulate the SC for the Japanese pet-toy called Tamagotchi using SD technique. They use a multi-echelon model that considers the simultaneous influences of several phenomena, such as the bullwhip effect, boom and bust, and multi-echelon decisions. The model contributes to decision-making such as the levels of manufacturing capacity and advertisement, as well as the timing to foreign market. Recommendations are derived about three important issues in the SC: the control of diffusion speed of new products, the importance of repeat purchasers as a buffer, and the identification of phantom demand.

From the literature presented above, it is clear that SD is a well-proven technique for the modeling and analysis of supply chains in different industries. Because SD uses simulation to evaluate SC strategies, it provides more flexibility to deal with nonlinear dynamic systems than the mathematical analytical methods. However, the advantage of SD over other approaches that study the behavior of supply chains is that it uses feedback loops to capture the complex relationships of the system. Although SD has its origins in control engineering, it models feedback loops differently. SD uses causal loop diagrams, which makes it easy to identify and understand the causal-effect relationships that drive system behavior, extending the comprehension of the system from the engineering to the management levels.

2.2. Model Structural Analysis

One of the most challenging tasks in understanding the behavior of the supply chain model is uncovering the components (structure) responsible for generating such behavior, and to what extent. In SD modeling, these important components are called dominant loops. Three analytical methods have been identified that help to find this relationship between the structure and behavior of the SD model. They are the following: the eigenvalue elasticity analysis (EEA), the pathway participation metric (PPM) method and the behavioral decomposition weights (BDW) analysis.

The first method, the eigenvalue elasticity analysis, was introduced by Nathan Forrester (1982) in the context of analyzing stabilization policies in a macro-economic model. The method consists of linearizing the model under study at any point in time, calculating its eigenvalues and then noting how the eigenvalues change as link gains change[10] (Güneralp 2005), that is, link elasticities[11]. The eigenvalues characterize different behavior modes (exponential growth, exponential decay, expanding oscillations, sustained oscillations, dampened oscillations) the superposition of which gives the rise to the observed behavior of the system. A large elasticity would indicate that the link gain (structural component) is in some sense "important" on generating the behavior mode associated with that eigenvalue. Forrester then extends this concept considering the loops elasticies[12], which measure the overall importance of a loop to a behavior mode. Therefore, EEA, by forming a connection between the model structure and behavior, provides a means to figure out the dominant structure in the model. By governing these structural components, it is possible to influence the modes of behavior that govern the model behavior and thus manage the model (Saleh and Davidsen 2001).

[10] The gain of the link between two variables is defined as the partial derivative of the output variable with respect to the input variable (Saleh 2006).

[11] $\varepsilon = (\partial\lambda_i / \lambda_i)/(\partial g_k / g_k)$, where ε is the elasticity of the eigenvalue λ_i with respect to the link gain g_k

In a typical EEA study, only one behavior mode is regarded as dominant at each analysis time step. Thus, the resulting explanation on loop dominance would be based on that behavior mode. This approach fails when there is more than a single dominant behavior mode to explain the behavior of the selected variable. The EEA has been extended to consider several behavior modes affecting simultaneously the decision variables. The procedure proposed by Güneralp (2005) considers all behavior modes to the contribution of the behavior of the variable of interest at each time step and a conglomerate measure for loop dominance is devised based on this method.

Saleh *et al.* (2005) focus the eigenvalue analysis on the contribution of both eigenvalues and eigenvectors on model behavior. They provide a computational method (implemented in Matlab) to calculate such influence. Recently, an analytical method to incorporate eigenvectors to the more traditional eigenvalue analysis has been proposed by Gonçalves (2006). His work identifies the significant role of the eigenvector in the short term behavior of the system, while the behavior mode is more influenced by the eigenvalue in the long term. Eigenvalue and eigenvector sensitivities (i.e., the partial derivatives with respect to a link gain) are incorporated in the analysis to show how they work together to influence system behavior. A shortcoming to the method is that solutions to the system behavior equations are required to obtain the analytical results.

Mojtahedzadeh (1997) proposes a second method that would aid in understanding structure behavior linkages. His method uses the so-called pathway participation metric to find

[12] Similar to the link elasticity but instead of a link gain it uses a loop gain. The loop gain is given by the product of all link gains forming the loop (Gonçalves *et al*. 2000).

the structure that most influences the time path of a state variable (variable of interest). This measure is based on the loop dominance work proposed by Richardson (1995)[13]. PPM relies on the analysis of linkages or pathways between two state variables of a model as the primary building blocks of influential structure. Using a recursive heuristic systematic analysis, the PPM calculations always yield a reduced structure of a key feedback loop plus one or more pathways that contribute most to a given mode of behavior for the selected variable. PPM stands as the only approach whose features are implemented in an experimental piece of software, *Digest* (Mojtahedzadeh *et al.* 2004). A limitation of the current implementation of PPM is that it identifies only the single most influential pathway for a variable. The pathway searching algorithm does not capture the situation when more than one structure may contribute significantly to the model behavior and may miss alternative paths that could prove to yield a larger total value of the metric (Kampmann and Oliva 2006). Another problem about the PPM method is its somewhat myopic approach to structure-behavior relationship. In other words, by confining itself to a single path of dominance of the selected variable, the method could lead to "localized" explanations of the variable's behavior. Thus, while the method has the advantage of being computationally simple it is not well suited for systems that oscillate, since the analysis is local and cannot capture global modes of behavior.

The third approach, the behavioral decomposition weights analysis, is proposed by Saleh *et al.* (2007). This method explores the *significance* that each behavior mode has on the system state variables. This is achieved by decomposing the behavior of a variable into a sum of

---

[13] Richarson proposed that the net time derivative of a state variable with respect to the state variable itself, i.e.

weighted reference modes observed in a linear system (represented by the eigenvalues). They propose that focusing on the weights, rather than on the eigenvalues, is a more efficient way to develop policy recommendations. In other words, instead of aiming to change the *characteristic* of the behavior mode, the authors suggest that a more effective method to identify policies is adjusting the *significance* of a particular behavior mode. In order to identify the elements of model structure more responsible for the observed behavior, they define the elasticity of a weight to gain[14]. A routine developed in *Mathematica* is provided to calculate the eigenvalues and weights. The fact that weights and eigenvalues are not independently determined is perhaps one the shortcomings of the method.

The literature shows that MSA methods provide a powerful mechanism for identifying the structural parts of the SD model that are responsible for certain behaviors of interest. This can help to concentrate the focus of the analysis on specific pieces of the model, reducing the number of parameters considered to stabilize the system. However, these approaches require the linearization of the system and they rely on the sensitivity analysis to determine the parameter values of the stabilization policy.

2.3. Policy Optimization

The policy optimization process uses methods based on mathematical programming and algorithmic search to find an improved policy. Several optimization methods have been used to obtain policies that modify system behavior. Burns and Malone (1974) express the required

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$\partial(\partial x / \partial t) / \partial x = \partial \dot{x} / \partial x$, can be an important measure of when a loop shifts dominance. The PPM approach calls $\partial \dot{x} / \partial x$ the Total Pathway Participation Measure.

policy as an open-loop solution (i.e., the solution function has not the variables from the system). The drawback of this method is that if the system fluctuates by some little impact, the open loop solution without information feedback can not adjust itself to the new state. Keloharju (1982) proposed a method of iterative simulation where each iteration consists of a parameter optimization. He suggests predefining the policy structure by allowing certain parameters of the model to be variables and by adding new parameters. However, the policies obtained with Keloharju's method are not robust when subject to variations of external inputs because the policy structure was predefined and thereafter optimized (Macedo 1989). Coyle (1985) includes structural changes to the model, and applies the method to a production system.

Kleijnen (1995) presents a method that includes design of experiments and response surface methodology for optimizing the parameters of a model. The approach treats SD as a black box, creating a set of regression equations to approximate the simulation model. The statistical design of experiments is applied to determine which parameters are significant. After dropping the insignificant parameters, the objective function is optimized by using the Lagrange multiplier method. The parameter values obtained through the procedure are the final solution. Bailey *et al.* (2000) extend Kleijnen's method by using response surfaces not to replace the simulation models with analytic equations, but instead to direct attention to regions within the design space with the most desirable performance. Their approach identifies the exploration points surrounding the solution of Kleijnen's method and the finds a set of real best-combination of parameters from them (Chen and Jeng 2004).

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[14] It is the ratio of the fractional change in the weight to the fractional change in the gain.

Grossmann (2002) uses genetic algorithms (GA) to find optimal policies. He demonstrates his approach in the Information Society Integrated System Model where he evaluates different objective functions. Another method that uses genetic algorithms to search the solution space is the one proposed by Chen and Jeng (2004). First, they transform the SD model into a recurrent neural network. Next, they use a genetic algorithm to generate policies by fitting the desired system behavior to patterns established in the neural network. Chen and Jeng claim their approach is flexible in the sense that it can find policies for a variety of behavior patterns including stable trajectories. However, the transformation stage might become difficult when SD models reach real-world sizes.

In the area of optimal control applied to system dynamics, Macedo (1989) introduces a mixed approach in which optimal control and traditional optimization are sequentially applied in the improvement of the SD model. Macedo's approach consists principally of two models: a reference model and a control model. The reference model is an optimization model whose main objective is to obtain the desired trajectories of the variables of interest. The control model is an optimal linear-quadratic control model whose fundamental goal is to reduce the difference between the desired trajectories (obtained by solving the reference model) and the observed trajectories (obtained by simulation of the system dynamic model).

The drawback of the methods presented above is that the objective function has to be defined for each particular model and it is not easy to choose.

2.4. Stability Analysis of the Supply Chain

The main objective in stability analysis is to determine whether a system that is pushed slightly from an equilibrium state (system variables do not change over time) will return to that state. If for small perturbations or disturbances from the equilibrium state the system always remains within a finite region surrounding that state, then this equilibrium state is stable. However, if a system tends to continue to move away from its original equilibrium state when perturbed from it, the system is unstable.

Sterman (2006) states that "supply chain instability is a persistent and enduring characteristic of market economies". As a result, company indicators such as demand forecast, inventory level, and employment rate show an irregular and constant fluctuation. Supply chain instability is costly because it creates "excessive inventories, poor customer service, and unnecessary capital investment" (Sterman 2006).

In dynamic complex systems like supply chains, a small deviation from the equilibrium state can cause disproportionately large changes in the system behavior, such as oscillatory behavior of increasing magnitude over time. The four main contribution factors to instability in SC have been identified by Lee *et al.* (1997), which are:

- Demand forecast updating: when companies throughout the SC do not share information about demand, this have to be forecasted with the possible cause of information distortion.
- Order batching: this means a company ordering a large quantity of a product in one week and not ordering any for many weeks, which will cause distortion on the demand forecast of other members of the SC, because it is based on orders rather than actual sales.
- Shortage gaming: when a product demand exceeds supply, a manufacturer often rations

its product to customers, which will cause that customers exaggerate their orders to ensure that they receive enough amount of the required product.

- Price fluctuations: when the price of a product changes significantly, customers will purchase the product when it is cheapest, causing them to buy in bulk (order batching problem).

The stability of supply chains models can be analyzed using the vast theory of linear and nonlinear dynamic systems control. Disney *et al.* (2000) describe a procedure for optimizing the performance of an industrially design inventory control system. They quantify five desirable characteristics of a production distribution system by drawing in classical control techniques for use in a modern optimization procedure based on GA. They demonstrate that their procedure can improve the performance of a production or distribution control system by fully understanding the trade-off between inventory levels and factory orders. Riddalls and Bennett (2002) study the stability properties of a continuous time version of the *Beer Distribution Game*. They demonstrate the importance of robust stability, i.e. stability for a range a production/distribution delays, and how stock outs in lower echelons can create vicious circle of unstable influences in the supply chain. Nagatani and Helbing (2004) study several production strategies to stabilize supply chains, which is expressed by different specifications of the management function controlling the production speed in dependence of the stock levels. They derive linear stability conditions and carry out simulations for different control strategies. Ortega and Lin (2004) show that control theory can be applied to the production-inventory problem to address issues such as reduction of inventory variation, demand amplification, and ordering rules optimization.

Linearization is frequently the quickest and easiest way to determine stability of an equilibrium point (EP) for a nonlinear system. The linearization approach of nonlinear systems can be used to extend the stability concepts for linear systems (eigenvalue analysis[15]) to equilibrium points of nonlinear systems in which deviation from linear behavior can be presumed small. Mohapatra and Sharma (1985) apply modal control to analyze and improve a SD model of a manufacturing company that has two departments: manufacturing and distribution. The eigenvalues of the motion equations are used to synthesize new policy options. The main strength of using modal control theory is that new policy structures can be generated mathematically. Drawbacks of modal control theory include the amount of computation, and the design of realistic policies from the synthetically generated policies.

Control theory has been combined with other approaches to determine stability conditions. Daganzo (2004) examines the stability of decentralized, multistage supply chains under arbitrary demand conditions. He uses numerical analysis for conservation laws to design stable policies. His research looks for intrinsic properties of the inventory replenishment policies that hold for all customer demand processes and for policies with desirable properties. He discovers that a simple necessary condition for the bullwhip avoidance is identified in terms of a policy's *gain*. Gain is defined as the marginal change in average inventory induced by a policy where there is a small but sustained change in demand rate. It is shown that all policies with positive gain produce the bullwhip effect if they do not use future order commitments. Perea *et al.* (2000) propose an approach for SCM that relies on dynamic modeling and control theory. The approach is based on two elements, a framework to capture the dynamics of the SC, and on

[15] Eigenvalues in the right half of the complex plane cause instability, whereas eigenvalues in the left half of the complex plane determine stable systems.

the design of methodical procedures defined by control laws to manage the SC. They test several heuristic control laws and analyze their impact on the behavior of the SC.

Model structural analysis methods have also been used to eliminate oscillatory behavior in SC models. Lertpattarapong (2002) and Gonçalves (2003) use EEA to identify the loops that are responsible for the oscillatory behavior of the inventory in the SC. Then they use the insights about the impact of feedback structures on model behavior to propose policies for stabilizing the system. These policies are based on inventory buffers or safety stock. Saleh *et al.* (2006) use the BDW analysis to identify relevant parameters that stabilize the inventory fluctuations in a linear inventory-force model. To explore the utility of the method in a SD nonlinear model they choose a medium-size economic model. In order to perform the BDW analysis, they linearize the model at a point in time, once the eigenvalues have become stable. The method provides a partial policy analysis as it studies the effects of changing individual policy parameters. Currently, the method does not consider the interactions due to changes in several parameters simultaneously.

Forrester (1982) presents several policies for stabilizing dynamic systems. The first two approaches, reduction of the frequency of oscillations and increment in the rate decay of oscillations, represent a measure of behavior of the whole system and are covered by the linear system control theory. Other methods such as variance reduction and gain reduction are focused on the stability of a particular variable of the system. Therefore, they have to be extended to implement stabilizing policies of the entire system.

Policy optimization provides an efficient method for obtaining SC stabilization policies. O'Donnell *et al.* (2006) employ GA to reduce the bullwhip effect and cost in the MIT Beer Distribution Game. The GA is used to determine the optimal ordering policy for members of the

SC. Lakkoju (2005) uses a methodology for minimizing the oscillations in the SC based on SD and GA. He applies the variance reduction criterion proposed by Forrester to stabilize the finished goods inventory of an electronics manufacturing company.

The literature review on stability analysis of the SC shows that several techniques have been used to generate stabilization policies. Model structural analysis methods can provide some insights into how to tackle the behaviors that generate instability of supply chains modeled as dynamic systems through the identification of the loops responsible for them. However, these methods rely on sensitivity analysis to design the stabilization policies. Control theory can support the stabilization methodologies by providing theoretical concepts to stabilize dynamics systems. One problem with the approaches based on control theory is the mathematics involved in order to determine the analytical solution. Moreover, similar to the model structural analysis methods, they can require certain simplifications, such as the linearization of the system (Dangerfield and Roberts 1996). On the other hand, policy optimization based on algorithmic search methods that use simulation represent the most general mean for stability analysis of nonlinear systems, due to its effectiveness in handling the general cases and most of special problems that arise from nonlinearity. However, the objective functions are chosen to represent the stability conditions particular to each model. The use of a generic objective function applied to stabilize SC models independent of their linear or nonlinear structure has not been found in the literature surveyed so far.

2.5. Particle Swarm Optimization

Optimization techniques based on evolutionary algorithms belong to the class of direct search strategies, where every considered solution is rated using the objective function values only. Therefore, no closed form of the problem and no further analytical information is required to direct the search process towards good or preferably optimal elements of the search space. For that reason, evolutionary search strategies are well suited for simulation optimization problems. Additionally, because of their flexibility, ease of operation, minimal requirements and global perspective, evolutionary algorithms have been successfully used in a wide range of combinatorial and continuous problems.

Evolutionary algorithms differ from conventional nonlinear optimization techniques, such as tabu search and simulated annealing, in that they search by maintaining a population of solutions from which better solutions are created rather than making incremental changes to a single solution to the problem. In other words, they do not carry out examinations sequentially, but search in parallel mode using a multi-individual population (O'Donnell *et al.* 2006).

Particle swarm optimization was invented in the mid 1990s by Kennedy and Eberhart (1995) as an alternative to genetic algorithms. PSO is based on a social simulation of the movement of flocks of birds. PSO performs a population-based search to optimize the objective function. The population is composed by a swarm of particles that represent potential solutions to the problem. These particles, which are a metaphor of birds in flocks, fly through the search space updating their positions and velocities based on the best experience of their own and the swarm. The swarm moves in the direction of "the region with the higher objective function value, and eventually all particles will gather around the point with the highest objective value" (Jones 2005).

Among the advantages of PSO, it can be mentioned that PSO is conceptually simple and can be implemented in a few lines of code. In comparison with other stochastic optimization techniques like GA or simulated annealing, PSO has fewer complicated operations and fewer defining parameters (Cui and Weile 2005). PSO has been shown to be effective in optimizing difficult multidimensional discontinuous problems in a variety of fields (Eberhart and Shi 1998), and it is also very effective in solving minimax problems (Laskari *et al.* 2002). According to Schutte and Groenwold (2005), a drawback of the original PSO algorithm proposed by Kennedy and Eberhart lies in that although the algorithm is known to quickly converge to the approximate region of the global minimum; however, it does not maintain this efficiency when entering the stage where a refined local search is required to find the minimum exactly. To overcome this shortcoming, variations of the original PSO algorithm that employ methods with adaptive parameters have been proposed (Shi and Eberhart 1998, 2001; Clerk 1999).

Comparison on the performance of GA and PSO, when solving different optimization problems, is mentioned in the literature. Hassan *et al.* (2005) compare the performance of both algorithms using a benchmark test of problems. The analysis shows that PSO is more efficient than GA in terms of computational effort when applied to unconstrained nonlinear problems with continuous variables. The computational savings offered by PSO over GA are not very significant when used to solve constrained nonlinear problems with discrete or continuous variables. Jones (2005) chooses the identification of model parameters for control systems as the problem area for the comparison. He indicates that in terms of computational effort, the GA approach is faster, although it should be noted that neither algorithm takes an unacceptably long time to determine their results. With respect to accuracy of model parameters, the GA determines values which are closer to the known ones than does the PSO. Moreover, the GA seems to arrive at its final parameter values in fewer generations that the PSO. Lee *et al.* (2005) select the return evaluation in stock market as the scenario for comparing GA and PSO. They show that PSO shares the ability of GA to handle arbitrary nonlinear functions, but PSO can reach the global optimal value with less iteration that GA. When finding technical trading rules, PSO is more efficient than GA too. Clow and White (2004) compare the performance of GA and PSO when used to train artificial neural networks (weight optimization problem). They show that PSO is superior for this application, training networks faster and more accurately than GA does, once properly optimized.

From the literature presented above, it is shown that PSO combined with simulation optimization is a very efficient technique that can be implemented and applied easily to solve various function optimization problems. Thus, this approach can be extended to the SCM area to search for policies using an objective function defined on a general stabilization concept like the one that is proposed in this research.

2.6. Powell Hill-Climbing Algorithm

Hill-climbing methods are heuristics that use an iterative improvement technique and are based on a single solution search strategy. These methods can only provide local optimum values, and they depend on the selection of the starting point (Michalewicz and Fogel 2000). Some advantages of hill-climbing-based approaches include: (1) very easy to use (Michalewicz and Fogel 2000), (2) do not require extensive parameter tuning, and (3) very effective in producing good solutions in a moderate amount of time (DeRonne and Karypis 2007).

The Powell hill-climbing algorithm was developed by Powell (1964) and it is a hill-climbing optimization approach that searches the objective in a multidimensional space by repeatedly using single dimensional optimization. The method finds an optimum in one search direction before moving to a perpendicular direction in order to find an improvement (Press *et al*. 1992). The main advantage of this algorithm lies in not requiring the calculation of derivatives to find an unconstraint minimum of a function of several variables (Powell 1964). This allows using the method to optimize highly nonlinear problems where it can be laborious or practically impossible to calculate the derivatives. Moreover, it has been shown that a hybrid strategy that uses a local search method such as hill-climbing can accelerate the search towards the global optimum, improving the performance of the searching algorithm (Yin *et al.* 2006; Özcan and Yilmaz 2007).

2.7. Discussion of Research Gaps

This chapter presented a review of the literatures that are related to the proposed methodology for stabilizing the SC. The following research gaps that require further research and implementation have been identified:

1. The lack of a methodology that uses SD modeling, generic stability conditions and simulation optimization to eliminate instability of the SC and produce robust policies. This methodology has the potential to solve a wide variety of complex stabilization problems not only in SCM but also in many other fields. Previous attempts have been focused on few variables of interest and selected parameters for the optimization problem. Moreover, analytical methods in control theory have been restricted to particular cases. However, due to the complexity of generalizing the stability criterion for nonlinear dynamic systems, still

there is not a methodology capable of optimize the behavior of any SD model.

2. The lack of a fully computerized framework for policy optimization that uses a hybrid PSO-PHC based searching engine and works with SD models.

Table 2.1 summarizes the literature research done in the field of supply chain management for each of the four surveyed areas. It is clear from the table that methods based on a general stabilization concept (useful for linear and nonlinear models) that use an optimization engine based on PSO and PHC have not been applied to the supply chain.

Table 2.1. Literature review for surveyed areas related to SCM

Researches	**SD modeling in SCM**	**Model structural analysis**	**Stability analysis**		**Optimization**	
			Particular condition	**General condition**	**PSO + PHC**	**Other**
Akkermans and Dellaert (2005)	√					
Anderson Jr. *et al.* (2000)	√					
Angerhofer and Angelides (2000)	√					
Ashayeri *et al.* (1998)	√					
Daganzo (2004)			√			
Disney *et al.* (2000)			√			√
Forrester (1958, 1961)	√					
Georgiadis and Vlachos (2004)	√					
Georgiadis *et al.* (2005)	√					
Gonçalves (2003)	√	√	√			
Higuchi and Troutt (2004)	√					
Huang *et al.* (2003)	√					
Joshi (2000)	√					
Lakkoju (2005)	√			√		√
Lee *et al.* (1997)			√			
Lertpattarapong (2002)	√	√	√			
Minegishi and Thiel (2000)	√					
Mohapatra and Sharma (1985)	√		√			√
Nagatani and Helbing (2004)			√			
O'Donnell *et al.* (2006)	√		√			√
Perea *et al.* (2000)			√			
Riddalls and Bennett (2002)			√			
Riddalls *et al.* (2000)	√					

Researches	**SD modeling in SCM**	**Model structural analysis**	**Stability analysis**		**Optimization**	
			Particular condition	**General condition**	**PSO + PHC**	**Other**
Spengler and Schröter (2003)	√					
Saleh (2007)	√	√	√			
Sterman (1989, 2000)	√					
Surana *et al.* (2005)	√					
Villegas and Smith (2006)	√					
Sarmiento (2010)	√			√	√	

CHAPTER THREE: METHODOLOGY

This chapter presents an overview of the research methodology, which is divided in three phases as shown in Figure 3.1.

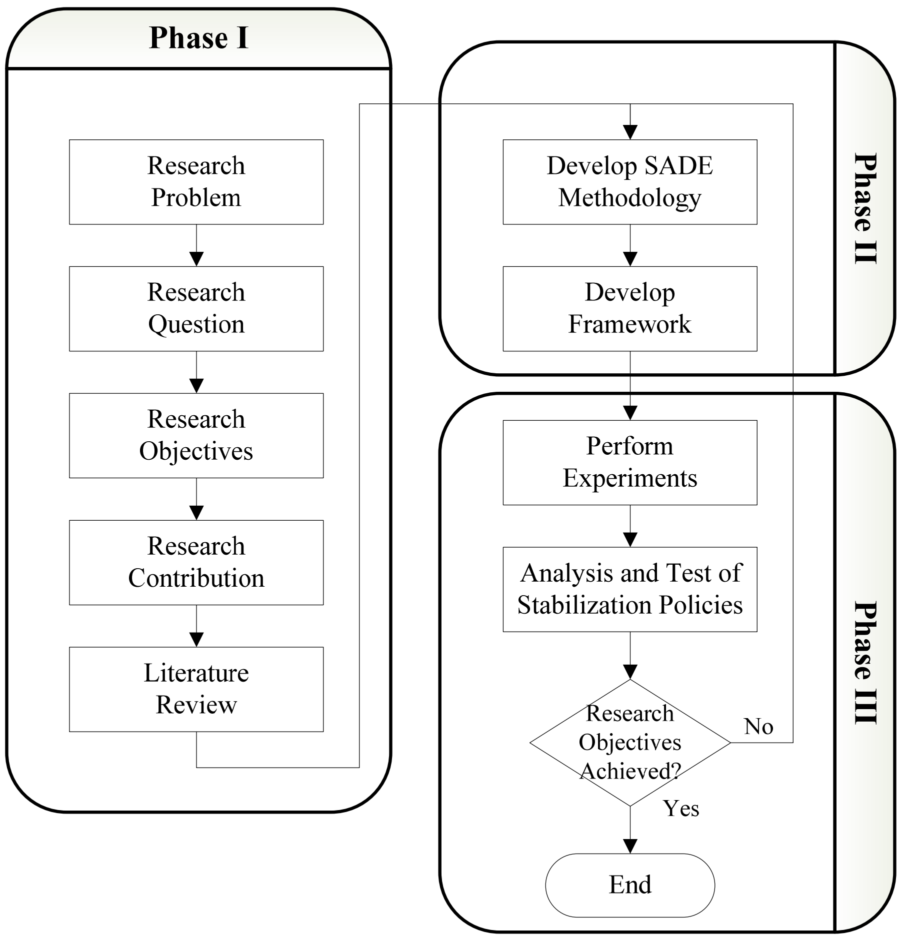

Figure 3.1. High level research methodology

Phase I covers the identification of the research problem, question, objectives and contribution that justify the development of this dissertation. These issues were explained in chapter one. Moreover, this phase identifies the research gaps, related to the research question, that were found after reviewing the relevant literature. This information was presented in chapter two.

Phase II covers the development of the theoretical aspects of the SADE methodology and the implementation of these concepts in a computerized framework. Section 3.1 provides a description of the functioning of the SADE methodology which considers the SD model, the optimization problem and the PSO solution algorithm. Justification of the stabilization properties of the objective function used in the optimization problem is discussed in chapter four. This justification involves the presentation of several theorems and their proofs. Section 3.2 presents the optimization framework.

In Phase III the framework is used to perform the stability analysis of several case studies. Section 3.3 explains how to validate the SADE methodology based on the results obtained from the experiments performed in those case studies.

If the research objectives are not achieved then Phase II has to be evaluated in order to see if it can be reformulated to meet the research goals.

3.1. Description of the SADE Methodology

The purpose of this dissertation is to develop a methodology that captures the dynamics of the supply chain and indicates potentials for modifications in the SC settings in order to avoid (or mitigate) the undesirable behaviors and performances. Figure 3.2 shows the different stages of the methodology and the general functioning is explained in the following lines.

The supply chain environment represents the actual participants, structure, strategies, policies, objectives, variables, constraints and parameters that configure different scenarios of the supply chain over time. All configurations require making different decisions that when implemented will produce changes in the behavior of the supply chain. Behavior in the supply chain is referred as observed patterns in the state variables (e.g. oscillatory behavior).

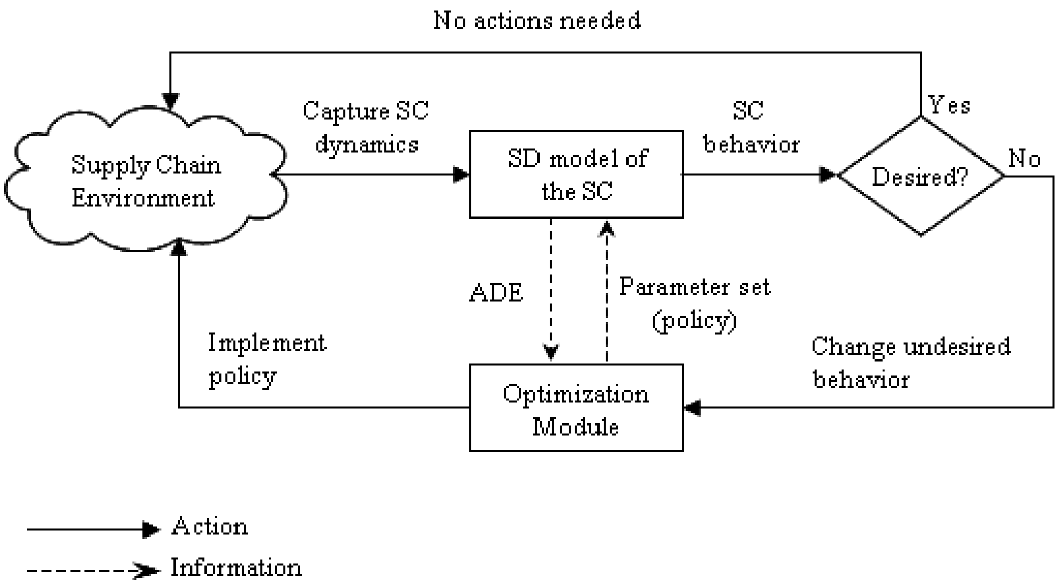

Figure 3.2. General procedure of the SADE methodology

The first step of the methodology uses a SD model to replicate the dynamic behavior of the supply chain. A SD model is chosen because it can capture the complex relationships, feedback processes, and multiple time delays necessary to track accurately the evolution of important endogenous variables. Section 3.1.1 provides a detailed description of the different type of variables, feedback structures and model equations used to represent a model in SD.

If the current behavior of the SC does not show instability patterns, such as ripple effects, then no actions are needed to be carried out over the supply chain, otherwise a new management policy must be found to remove the instability or minimize its impact.

The second step of the methodology uses a simulation optimization technique to find such a policy. This technique uses a hybrid algorithm that combines PSO and PHC methods to modify the set of parameters that constitute the current policy in order to minimize the ADE and thus achieve stability. In every iteration of the algorithm, the parameter set is sent to the SD model in order to calculate, through simulation, the value of the ADE (objective function). Simulation is used due to the difficulty of solving the complex dynamic equations by analytical methods. The optimization problem and the hybrid algorithm are described in sections 3.1.2 and 3.1.3.

Once the best setting of parameters (stabilization policy) is obtained, then it is implemented in the actual supply chain to ensure it is kept stable and robust.

3.1.1. System Dynamic Model of the Supply Chain

The dynamic relationships of the supply chain are represented by using a SD model that consists in feedback structures linked with stock and flow structures.

The basic building block in the feedback structure is the feedback loop. The feedback loop is a path coupling decision, action, stock (or state) of the system, and information, with the path returning to the decision point (Forrester 1990) as shown in Figure 3.3. Causal relationships of the SC that tend to move the behavior toward a goal are modeled as *negative feedback loops*. In contrast, causal relationships that amplify disturbances in the system to create even higher variations in behavior are modeled as *positive feedback loops*.

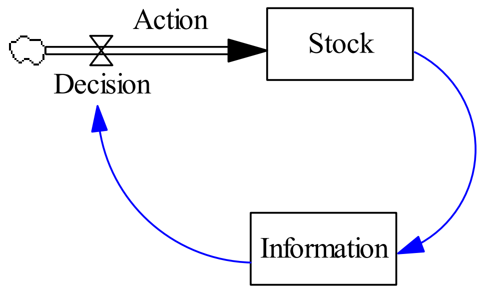

Figure 3.3. Feedback loop

Stocks and flows compose a substructure within feedback loops. The stock and flow structure consists in two types of variable elements: the stocks and the rates. The stock (or state) variables describe the condition of the system at any particular time. They accumulate the results of action within the system. On the other hand, the rate (action) variables tell how fast the levels are changing. They are the policy statements that describe action in the system.

The *feedback* and *stock and flow* structures of the SD model are converted into mathematical equations. These equations can be classified into four categories: level, rate, auxiliary and initial-value equations. A description of each category is provided as follows:

- Stock equations. They are equations of the form:

$$\text{Stock}(t) = \int_{t_0}^{t} [\text{Inflow}(s) - \text{Outflow}(s)]ds + \text{Stock}(t_0)$$

These equations calculate the value of the stock variables as the accumulation over time of the difference between the inflows to a process and its outflows (Figure 3.4). For example, the inventory can be expressed as the integral of the difference between Production and Shipment rates as follows:

$$\text{Inventory}(t) = \int_{0}^{t} [\text{Production Rate}(s) - \text{Shipment Rate}(s)]ds + \text{Inventory}(0)$$

Figure 3.4. Stock and flow diagram

- Rate equations. They state how the flows within a system are controlled. Unlike, stock equations, rate equations have not standard form. Each rate equation represents an understanding of some process of change in a particular system. For example, the Production rate can be expressed as the sum of two terms:

$$\text{Production Rate} = \text{Desired Production} + \frac{(\text{Desired Inventory - Inventory(t)})}{\text{Inventory Adjustment Time}}$$

The first term aligns the current production rate with its desired value. The second term modifies the production rate to keep the inventory in line with the desired inventory level.

- Auxiliary equations. The auxiliary equations are merely algebraic subdivisions of the rates used with the purpose of providing more clarity and meaning to the rate equations. For instance, the auxiliary variable "Adjustment for Inventory" can be used to represent the second term in the *Production Rate* equation of the previous example. Thus, now there is a more simple and meaningful rate equation:

$$\text{Production Rate} = \text{Desired Production} + \text{Adjustment for Inventory}$$

Along with the auxiliary equation:

$$\text{Adjustment for Inventory} = \frac{(\text{Desired Inventory - Inventory(t)})}{\text{Inventory Adjustment Time}}$$

- Initial-value equations. They are used to define initial values of all levels and initially to compute values of some constants from other constants. For example, the expression: $\text{Inventory}(0) = 100$, sets the initial value of the inventory variable to 100 units.

3.1.2. Optimization Problem

The SD model can be described by an equation of the form $\dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}(t), \mathbf{p})$, where $\dot{\mathbf{x}}(t) = \partial \mathbf{x}(t)/\partial t$, $\mathbf{x}(t)$ is the vector of state variables (dimension *n*) and $\mathbf{p}$ is a vector of adjustable parameters (dimension n_p) with lower and upper bounds $\mathbf{p}^L$ and $\mathbf{p}^U$ respectively.

Using the results of Theorem 5 (see chapter 4) an optimization problem can be formulated to find the parameter vector $\mathbf{p}^*$ that causes the state variable x_s to become asymptotically stable around the equilibrium point $x_s^{eq}(\mathbf{p}^*)$. This optimal parameter vector can be found by minimizing the ADE for predetermined time horizon T and making use of Theorem 5. That is, the optimization problem will find the vector that makes ADE converge[16]. The optimization problem is then stated as

$$\underset{\mathbf{p}}{\text{Minimize}}\ J(\mathbf{p}) = \sum_{s=1}^{m} \left\{ w_s \int_0^T \left| x_s(t) - x_s^{eq} \right| dt \right\}, \text{ where } \sum_{s=1}^{m} w_s = 1 \tag{3.1}$$

Subject to

$$\dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}(t), \mathbf{p}) \tag{3.2}$$

$$\mathbf{x}(0) = \mathbf{x}_0 \tag{3.3}$$

$$\mathbf{p}^L \le \mathbf{p} \le \mathbf{p}^U \tag{3.4}$$

$$\mathbf{x}(t) \in R^n, \mathbf{p} \in R^{n_p},\ \mathbf{p}^L \in R^{n_p}, \mathbf{p}^U \in R^{n_p}, t \in [0, T]$$

The objective function J(**p**) is defined as the weighted average value of the ADE. The term $\int_0^T \left| x_s(t) - x_s^{eq} \right| dt$ is the mathematical expression of the ADE for the state variable x_s, where the symbol |*c*| represents the absolute value of *c*. If necessary the time horizon T should be increased to obtain similar effects of convergence that when time goes to infinity, as stated in Theorem 5.

The use of weights, w_s, means that J(**p**) will support the simultaneous stabilization of any subset of *m* state variables $(m \leq n)$. Positive weights can be assigned to these variables in any way, provided the normalization constraint ($\sum_{s=1}^{m} w_s = 1$) is met. This allows higher weights to be assigned to the variables that are considered more important.

If the equilibrium point x_s^{eq} is not known in advance, J(**p**) can be modified to include it as a variable (a_s) and change to optimization of the problem[17] as follows:

$$\underset{\mathbf{p}}{\text{Minimize}}\ J(\mathbf{p}) = \sum_{s=1}^{m} \left\{ w_s \int_0^T \left| x_s(t) - a_s \right| dt \right\}, \text{ where } \sum_{s=1}^{m} w_s = 1 \qquad (3.5)$$

This amounts to including a_s (s=1,..,m) as part of the solution vector **p**. Theorem 6 (see chapter 4) guarantees that the values of a_s obtained from the optimization will, in fact, coincide with the equilibrium points x_s^{eq} (s=1,..,m).

---

[16] One way to check the convergence of ADE is by adding a new state variable to the model, called "ADE" (see Figure 3.5), and graphically verify that its graph becomes a flat line when time goes to T.

[17] For example, for an inventory variable, the interval of variation of its EP in the optimization problem would be based on the minimum and maximum levels of inventory determined by the production plan.

The objective function defined in (3.5) can be incorporated very easily into any SD formulation by adding a “stock and flow” piece to the model that is linked to the state variables of interest as illustrated in Figure 3.5. Then the variables DE(t) and ADE are defined as follows:

$$DE(t) = \sum_{s=1}^{m} \left\{ w_s \left| x_s(t) - a_s \right| \right\}$$

$$ADE = \int_0^T DE(t)\, dt$$

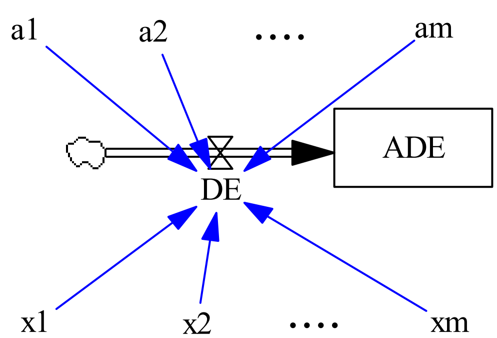

Figure 3.5. Stock and flow diagram for the objective function

3.1.3. Optimization Algorithm

The method used to solve the optimization problem is a hybrid algorithm that combines the advantage of PSO optimization to determine good regions of the search space with the advantage of local optimization to find quickly the optimal point within those regions. In other words, the local search is an improvement procedure over the solution obtained from the PSO algorithm that assures a fast convergence of the ADE.

The local search technique selected was the Powell hill-climbing algorithm. This method was chosen because: (1) it can be applied to solve multi-dimensional optimization problems, (2) it is a relatively simple heuristic that does not require the calculation of derivatives.

The general structure of the method is illustrated in Figure 3.6. This figure indicates that the solution to the optimization problem obtained by the PSO algorithm becomes the initial point to perform a local search using the PHC algorithm. Finally, if the ADE has converged then the solution provided by the PHC method is the stabilization policy; otherwise the parameter settings of the PSO algorithm have to be changed in order to improve the search that makes ADE to converge.

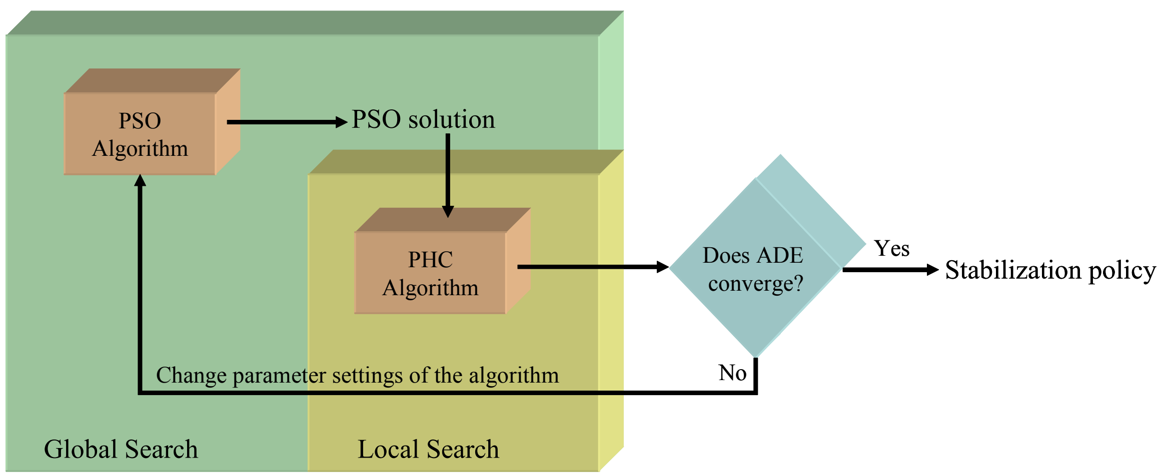

Figure 3.6. Optimization algorithm

The details of the functioning of each algorithm are explained in the following lines.

3.1.3.1. <u>Global Search: PSO Algorithm</u>

The algorithm used is called "local best PSO" (Engelbrecht 2005) and is based on a social network composed of neighborhoods related to each particle. The algorithm maintains a swarm of particles, where each particle represents a candidate solution to the optimization problem. These particles move across the search space communicating good positions to each other within the neighborhood and adjusting their own position and velocity based on these good positions. For this purpose, each particle keeps a memory of its own best position found so far and the neighborhood best position among all the neighbor particles. The goodness of a position is determined by using a fitness function. The stopping condition of the algorithm is when the maximum number of iterations has been exceeded. The basic elements of the algorithm are defined as follows:

- Particle. A particle *i* is represented by a n_p-dimensional real-valued vector $\mathbf{p}_i$. This vector is composed of particle positions p_{ij}, i.e., $\mathbf{p}_i = [p_{i1}, p_{i2}, .., p_{in_p}]$. Each particle position corresponds to one of the parameters of the parameter vector defined in the optimization problem 3.1.2.
- Swarm size. It is the number of particles in the swarm, and it is denoted by N.
- Fitness function. It is a mathematical function used to quantify how good the solution represented by a particle is. For a particle *i* the fitness function is the objective function $J(\mathbf{p}_i)$ as defined in 3.1.2.
- Personal best position. As a particle moves through the search space, it compares its fitness value at the current position to the fitness value it has ever attained so far, which is called the personal best position. For each particle *i* the personal best position can be expressed as the real-valued vector $\mathbf{y}_i = [y_{i1}, y_{i2}, .., y_{in_p}]$, and it is determined so that $J(\mathbf{y}_i) \leq J(\mathbf{p}_i),\ i = 1, .., N$.
- Neighborhood size. Defines the extent of the social iteration within the swarm (Engelbrecht 2005) and it is denoted by H. Selection of neighbors was done based on particle indexes. Each particle *i* has a neighborhood associated to, where B_i defines the set of indexes for the

particles neighbors. The neighborhood associated to particle *i* is composed by $B_i = \{i, i+1, i+2,\ldots, i+H-1\}$. It can be noted that neighborhoods overlap.

- Neighborhood best position. It is the best position among all the personal best positions in the neighborhood. It is denoted by the real-valued vector $\hat{\mathbf{y}}_i = [\hat{y}_{i1}, \hat{y}_{i2},.., \hat{y}_{in_p}]$ and it is determined so that $J(\hat{\mathbf{y}}_i) \le J(\mathbf{y}_j),\ j \in B_i$.
- Global best position. It is the best position among all the personal best positions achieved so far. It is denoted by the real-valued vector $\mathbf{g} = [g_1, g_2,.., g_{n_p}]$ and it is determined so that $J(\mathbf{g}) \le J(\mathbf{y}_i),\ i = 1,.., N$.
- Particle velocity. It is the velocity of the moving particle *i* represented by the real-valued vector $\mathbf{v}_i = [v_{i1}, v_{i2},.., v_{in_p}]$. This vector reflects both the experiential knowledge of the particle and socially exchanged information from the particle's neighborhood (Engelbrecht 2005). The experiential knowledge of a particle is generally referred as the *cognitive component*, which quantifies the performance of particle *i* relative to past performances. It is represented by the term $c_1\mathbf{r}_1(\mathbf{y}_i - \mathbf{p}_i)$. The socially exchanged information is referred as the *social component* of the velocity equation. It is represented by the term $c_2\mathbf{r}_2(\hat{\mathbf{y}}_i - \mathbf{p}_i)$.
- Acceleration coefficients. The acceleration coefficients, c_1 and c_2, together with the random vectors $\mathbf{r}_1$ and $\mathbf{r}_2$, control the stochastic influence of the cognitive and social components on the overall velocity of a particle (Engelbrecht 2005). The constants c_1 and c_2 are also referred as trust parameters, where c_1 expresses how much confidence a particle has in itself, while c_2 expresses how much a particle has in its neighbors. The random vectors are defined as

$\mathbf{r}_1 = [r_{11}, r_{12},.., r_{1n_p}]$ and $\mathbf{r}_2 = [r_{21}, r_{22},.., r_{2n_p}]$, where r_{1j} and r_{2j} are uniformly distributed random numbers in [0,1].

- Inertia weight. It is a parameter "w" that is used to control the influence in the new velocity of a particle by its previous velocity (flight direction). Thus, it influences the tradeoff between the global and local exploration abilities of the particles (Shi and Eberhart 1998). For initial stages of the search process where global exploration is required, it is recommended to set a large inertia weight, while for the last stages, the inertia weight should be reduced for better local exploration. A decrement function for decreasing the inertia weight at the iteration k can be given by $w(k) = \alpha w(k')$, where α=0.98, and k' is the last iteration when the fitness function was improved. A parameter "iteration_lag" is defined to set the number of iterations that are required to change the inertia weight if the fitness function has not been improved.

The following empirical rules are recommended to guide the choice of selecting the initial values for the parameters of the PSO algorithm.

Table 3.1. Empirical rules for selecting the PSO parameters

Parameter	**Empirical rule of choice**
Swarm size	From 20 to 40 (Clerc 2006)
Inertia weight	In]0,1[(Shi and Eberhart 1998)
Cognitive coefficient	Suggestion 1.43 (Clerc 2006)
Social coefficient	Suggestion 1.43 (Clerc 2006)

The steps of the algorithm are described in the following lines.

Step 1) Initialization:

- Set iteration k=0

- Generate N particles $\mathbf{p}_i(0) = [p_{i1}(0), p_{i2}(0),.., p_{in_p}(0)]$, i=1,..,N; where $p_{ij}(0)$ is randomly selected according to a uniform distribution in the interval $[p_j^L, p_j^U]$, j=1,..,n_p.
- Generate velocities $\mathbf{v}_i(0) = [0, 0,..,0]$, i=1,..,N.
- Evaluate the fitness of each particle using J($\mathbf{p}_i$(0)) , i=1,..,N.
- Set the initial value of the personal best position vector as $\mathbf{y}_i(0) = \mathbf{p}_i(0)$, i=1,..,N.
- Determine the neighborhood best position vector $\hat{\mathbf{y}}_i(0)$ using the formula $J(\hat{\mathbf{y}}_i(0)) = \min\{J(\mathbf{y}_j(0))\}, j \in B_i$.
- Determine the global best position $\mathbf{g}(0)$ using the formula $J(\mathbf{g}(0)) = \min\{J(\mathbf{y}_i(0))\},\ i = 1,.., N$.
- Set the initial value of the inertia weight $w(0)$. Set k'=0.

Step 2) Iteration updating: Set k=k+1.

Step 3) Weight updating: If k-1-k'≥ iteration_lag then update the inertia weight using:

$w(k) = \alpha w(k')$.

Step 4) Velocity updating: Calculate the velocity of particle *i* by using:

$$\mathbf{v}_i(k) = w(k)\mathbf{v}_i(k-1) + c_1\mathbf{r}_1(k)[\mathbf{y}_i(k) - \mathbf{p}_i(k)] + c_2\mathbf{r}_2(k)[\hat{\mathbf{y}}_i(k) - \mathbf{p}_i(k)]$$

Step 5) Position updating: Based on the updated velocities, each particle changes its position according to the following equation:

$$\mathbf{p}_i(k) = \mathbf{v}_i(k) + \mathbf{p}_i(k-1)$$

Step 6) Personal best updating: Determine the personal best position visited so far by each particle:

- Evaluate the fitness of each particle using $J(\mathbf{p}_i(k))$, i=1,..,N.
- Set $\mathbf{y}_i(k) = \begin{cases} \mathbf{y}_i(k-1) & \text{if } J(\mathbf{p}_i(k+1)) \geq J(\mathbf{y}_i(k-1)) \\ \mathbf{p}_i(k) & \text{if } J(\mathbf{p}_i(k)) < J(\mathbf{y}_i(k-1)) \end{cases}$

Step 7) Neighborhood best updating: Determine the neighborhood best position $\hat{\mathbf{y}}_i(k)$ visited so far by the whole swarm by using the formula

$J(\hat{\mathbf{y}}_i(k)) = \min\{J(\mathbf{y}_j(k))\}, j \in B_i$

Step 8) Global best updating: Determine the global best position $\mathbf{g}(k)$ visited so far by the whole swarm by using the formula

$J(\mathbf{g}(k)) = \min\{J(\mathbf{y}_i(k))\}$, i=1,..,N.

If $J(\mathbf{g}(k)) < J(\mathbf{g}(k-1))$ then set k'=k

Step 9) Stopping criteria: If the maximum number of iterations is achieved then stop, $\mathbf{g}^* = \mathbf{g}(k)$ is the optimal solution; otherwise go to step 2.

3.1.3.2. Local Search: Powell Hill-Climbing Algorithm

PHC method basically uses one-dimensional minimization algorithms to solve multi-dimensional optimization problems. The procedure searches into a region by constructing a set of linearly independent, mutually "non-interfering" or conjugate search directions and applies linear minimization to move into each direction (Press *et al*. 1992). The number of conjugate directions coincides with the dimension of the search space and their linear independence guarantees the whole search space can be covered. The use of conjugate directions has the advantage that minimization in one direction is not interfered by subsequent minimization along another direction, avoiding endless cycling through the set of directions.

The steps of the algorithm are described in the following lines:

Step 1) Initialization:

- Set iteration k=0
- Set the initial search point $\mathbf{Z}_0 = [z_1, z_2,..,z_{n_p}]$ as the optimal solution of the PSO algorithm, i.e. $\mathbf{Z}_0 = \mathbf{g}*$
- Initialize directions $\mathbf{u}_d$ to the basis vectors, i.e. $\mathbf{u}_d=\mathbf{e}_d$, d=1,..,$n_p$, where $\mathbf{e}_1 = [1, 0,..,0], \mathbf{e}_2 = [0, 1,..,0],..., \mathbf{e}_{n_p} = [0, 0,..,1]$

Step 2) Define the iteration start point: Set $\mathbf{S}_0 = \mathbf{Z}_k$

Step 3) Minimize objective function along direction $\mathbf{u}_d$

For every direction d=1,..,n_p

- Find the value γ_d that minimizes $J(\mathbf{S}_{d-1} + \gamma_d \mathbf{u}_d)$
- Set $\mathbf{S}_d = \mathbf{S}_{d-1} + \gamma_d \mathbf{u}_d$

Step 4) Update directions

- Set $\mathbf{u}_d = \mathbf{u}_{d+1}$, d=1,..,$n_p$-1
- $\mathbf{u}_{n_p} = \mathbf{S}_{n_p} - \mathbf{S}_0$

Step 5) Iteration updating: Set k=k+1.

Step 6) Minimize objective function along direction $\mathbf{u}_{n_p}$

- Find the value γ that minimizes $J(\mathbf{S}_0 + \gamma \mathbf{u}_{n_p})$
- Set $\mathbf{Z}_k = \mathbf{S}_0 + \gamma \mathbf{u}_{n_p}$

Step 7) Stopping criteria: If $J(\mathbf{Z}_k) > J(\mathbf{Z}_{k-1})$ then stop, $\mathbf{Z}_k^*$ is the optimal solution; otherwise go to step 2.

3.2. Optimization Framework

The structure of the optimization framework is divided into the framework architecture and the framework interface.

3.2.1. Framework Architecture

The framework architecture consists of three components: the simulation, optimization and report modules. Figure 3.7 shows the interactions between these components.

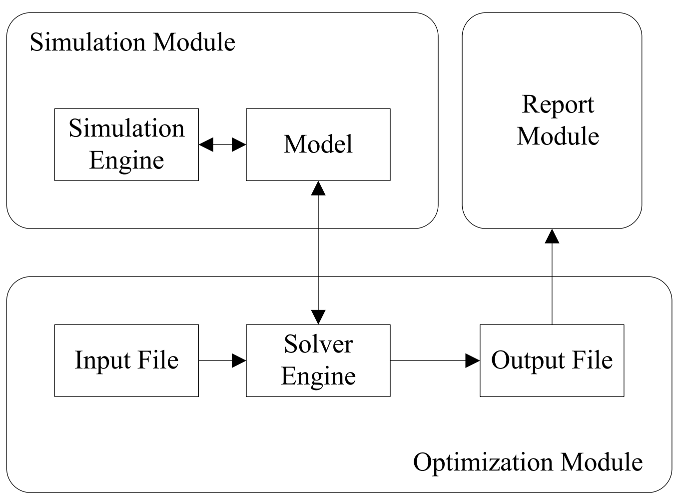

Figure 3.7. Framework architecture

The description of these components is explained in the following lines:

- Simulation module. This module is composed by the model equations file and the simulation engine. Vensim DSS simulation package will be used to build the SD model and to run the simulations. The model and the simulation engine can be accessed from the optimization module by using the following functions incorporated in the Vensim Dynamic Link Library.
  - Vensim_command. This function is used to load the SD model, to pass values to the parameters selected in the optimization problem, to run the simulation and the PHC algorithm. However, before calling this function, the model file must be saved in binary format as a *.vmf* file.
  - Vensim_be_quiet. This function is used to turn off the work in progress dialog that Vensim displays during the simulation, and to prevent the appearance of "*yes* or *no*" dialogs.
  - Vensim_get_data. This function is used to retrieve the value of ADE from the simulation run.
- Optimization module. This module is composed by the solver engine, the input and output files. The input file is a text file that contains the settings of the PSO algorithm, such as the inertia weight, social and cognitive coefficients, maximum number of iterations, etc. along with the parameter vector **p**. The output file is also a text file that displays the optimal value of **p** and the value of the best fitness. The solver engine follows the steps of the PSO and PHC algorithms defined in section 3.1.3. The code is built using the programming language Visual Basic from the Microsoft Visual Studio environment. The solver engine consists in a module file: *Main.bas* and five class files: *SwarmType.cls*, *NeighborhoodType.cls*,

ParticleType.cls, *VensimCalculations.cls* and *RandomNumberType.cls*.

- Main.bas: This module controls the reading and writing of the *input* and *output* files, and runs the optimization algorithm. First the subroutine *ReadOptData* is called to read the values of the input file (previously loaded in the interface). Then the subroutine *Algorithm* performs the steps of the PSO algorithm by using the *SwarmType* class and the PCH algorithm by using the *VensimCalculations* class. The final solution is written in the *output* file which is immediately opened with the Notepad application.
- SwarmType.cls: This class enables manipulation of the entire swarm by calling the methods and properties of the individual particles defined in the *ParticleType* class. Calculations for the neighborhoods of particles are also done here by using the *NeighborhoodType* class.
- NeighborhoodType.cls: This class basically keeps the information of the best position for each neighborhood and the particle index associated to it.
- ParticleType.cls: The methods used to calculate the position and velocity of each particle are defined in this class.
- VesimCalculations.cls: This class contains the logic to load the SD model, pass the values of **p** to the model, simulate the model with these values, and retrieve the ADE value. Moreover, this class calls a function in the Vensim Dynamic Link Library to compute the PHC algorithm. The results are then passed to the Main.bas module.
- RandomNumberType.cls: This class generates random numbers between a lower and upper limit.

- Report module. This module uses the class *Report* to build the solution and detailed reports. The solution report presents the values for the parameters of the stabilization policy (Figure 3.8). Both the PSO and PHC solutions are provided. The detailed report presents information of each particle for every iteration of the PSO algorithm (Figure 3.9). Because this report presents a lot of information, it can be loaded in Excel and with the help of a macro extract only pertinent information such as the value of the ADE at each iteration of the algorithm.

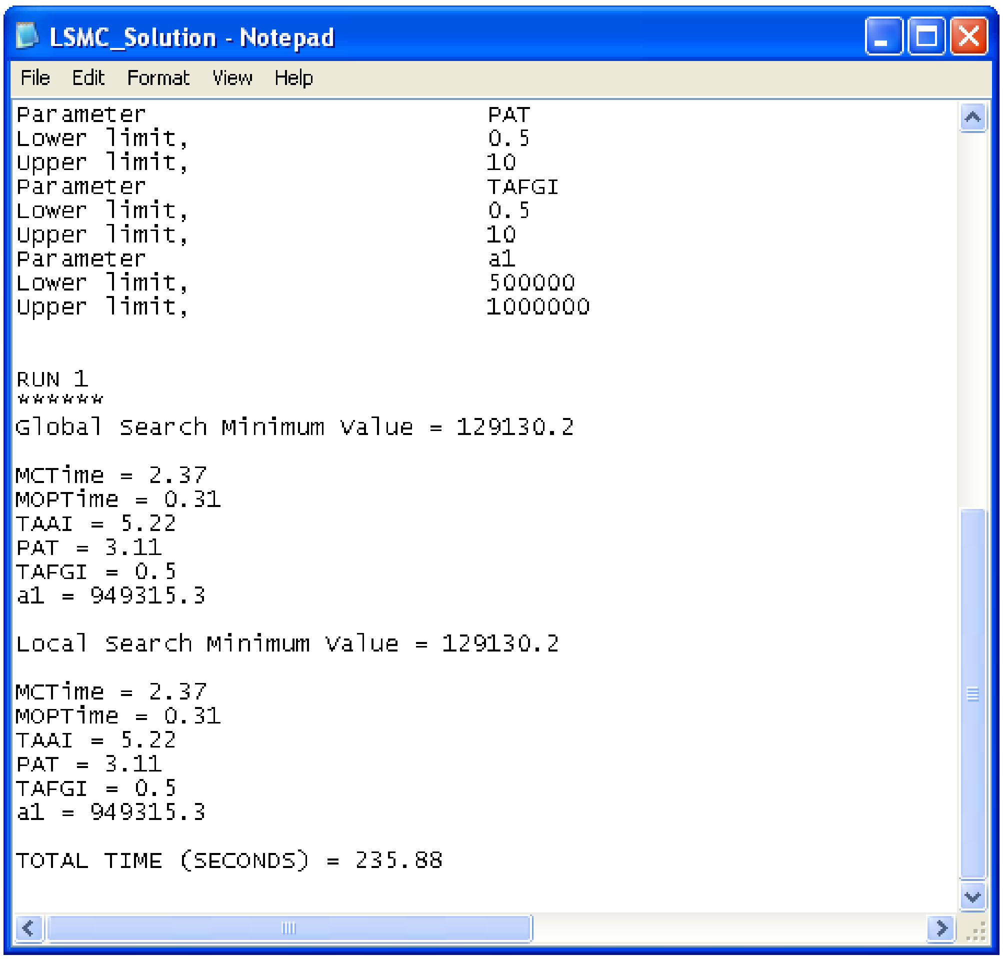

Figure 3.8. Solution report

	A	B	C	D	E	F	G	H	I	J	K	L	M	N
1	Number of Runs	Number of Iterations	Number of Neighborhoods	Number of Particles	Number of Parameters									
2	3	3	6	5	13									
3	Run	Iteration	gBest Value	gBest Particle	gBest Iteration	Neighborhood Index	nBest Value	Particle	Fitness Value	pBest Value	Vector Index	Velocity Vector Value	Position Vector Value	pBest Vector Value
4	1	0	2.46E+07	5	0	1	2.46E+07	1	1.67E+10	1.67E+10	1	0	35.57183	35.57183
5	.	.	.	.	.	.	.	.	.	.	2	0	27.13778	27.13778
6	.	.	.	.	.	.	.	.	.	.	3	0	107.9519	107.9519
7	.	.	.	.	.	.	.	.	.	.	4	0	15.18856	15.18856
8	.	.	.	.	.	.	.	.	.	.	5	0	15.79545	15.79545
9	.	.	.	.	.	.	.	.	.	.	6	0	38.96227	38.96227
10	.	.	.	.	.	.	.	.	.	.	7	0	1.686864	1.686864
11	.	.	.	.	.	.	.	.	.	.	8	0	38.27546	38.27546
12	.	.	.	.	.	.	.	.	.	.	9	0	40.91001	40.91001
13	.	.	.	.	.	.	.	.	.	.	10	0	82090.38	82090.38
14	.	.	.	.	.	.	.	.	.	.	11	0	35453.53	35453.53
15	.	.	.	.	.	.	.	.	.	.	12	0	79.14033	79.14033
16	.	.	.	.	.	.	.	.	.	.	13	0	1072.524	1072.524
17	.	.	.	.	.	.	.	2	1.63E+08	1.63E+08	1	0	39.73352	39.73352
18	.	.	.	.	.	.	.	.	.	.	2	0	19.30327	19.30327
19	.	.	.	.	.	.	.	.	.	.	3	0	146.1953	146.1953
20	.	.	.	.	.	.	.	.	.	.	4	0	43.70085	43.70085
21	.	.	.	.	.	.	.	.	.	.	5	0	3.755606	3.755606
22	.	.	.	.	.	.	.	.	.	.	6	0	47.52827	47.52827
23	.	.	.	.	.	.	.	.	.	.	7	0	18.83692	18.83692
24	.	.	.	.	.	.	.	.	.	.	8	0	26.71855	26.71855
25	.	.	.	.	.	.	.	.	.	.	9	0	38.58847	38.58847
26	.	.	.	.	.	.	.	.	.	.	10	0	75535.05	75535.05
27	.	.	.	.	.	.	.	.	.	.	11	0	40924.58	40924.58
28	.	.	.	.	.	.	.	.	.	.	12	0	79.687	79.687
29	.	.	.	.	.	.	.	.	.	.	13	0	959.6331	959.6331
30	.	.	.	.	.	.	.	3	7.12E+07	7.12E+07	1	0	31.51214	31.51214
31	.	.	.	.	.	.	.	.	.	.	2	0	32.74324	32.74324
32	.	.	.	.	.	.	.	.	.	.	3	0	76.3793	76.3793

Figure 3.9. Detailed report

3.2.2. Framework Interface

The interface consists of a window with a File menu, a Run option and three tabs: Model Parameters, PSO Settings and Vensim Settings.

- File menu. This menu contains the commands to create, open and save a file with the settings and parameters used in the optimization algorithm.
- Run option. This command is used to run the optimization algorithm using the settings defined in the three tabs.
- Model Parameters tab. This tab is used to enter the list of model parameters to be searched over in the optimization. For each parameter it is required to enter the parameter name and its lower and upper bound (see Figure 3.10).

- PSO Settings tab. This tab is used to enter the parameter settings required by the PSO algorithm. The names of the solution report and detailed report can also be entered here (see Figure 3.11).
- Vensim Settings tab. This tab is used to enter the parameter settings, the SD model and the data file required by Vensim DSS application to simulate the model. Moreover, the payoff file and the optimization parameter file used by the PHC algorithm are defined here (see Figure 3.12). The payoff file includes the variable that has to be optimized. The Vensim optimizer is designed to maximize the payoff, therefore to minimize the objective function the variable has to be entered as a negative expression in the payoff file. The optimization parameter file is built using the information entered in the Model Parameters tab.

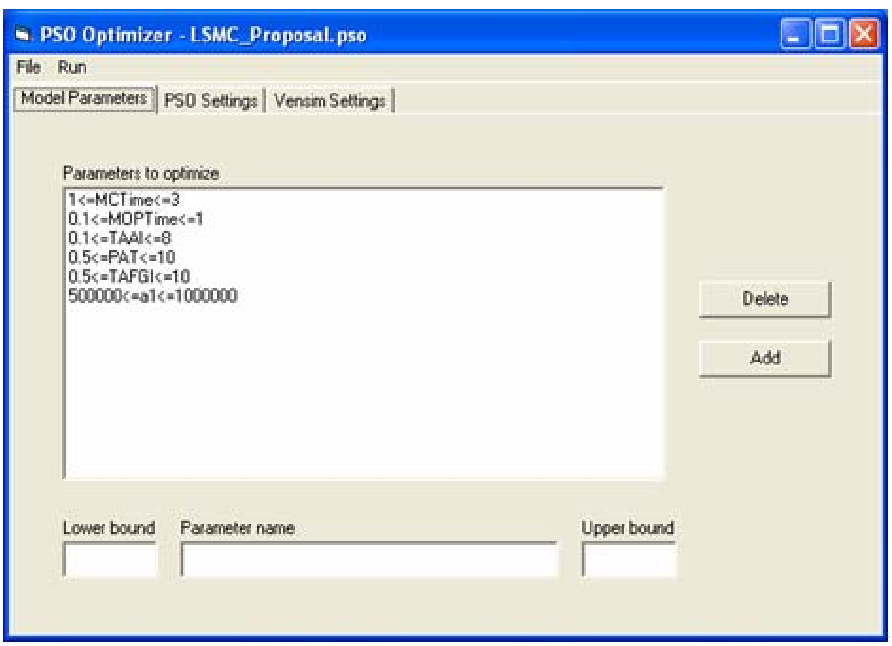

Figure 3.10. Model Parameters Tab

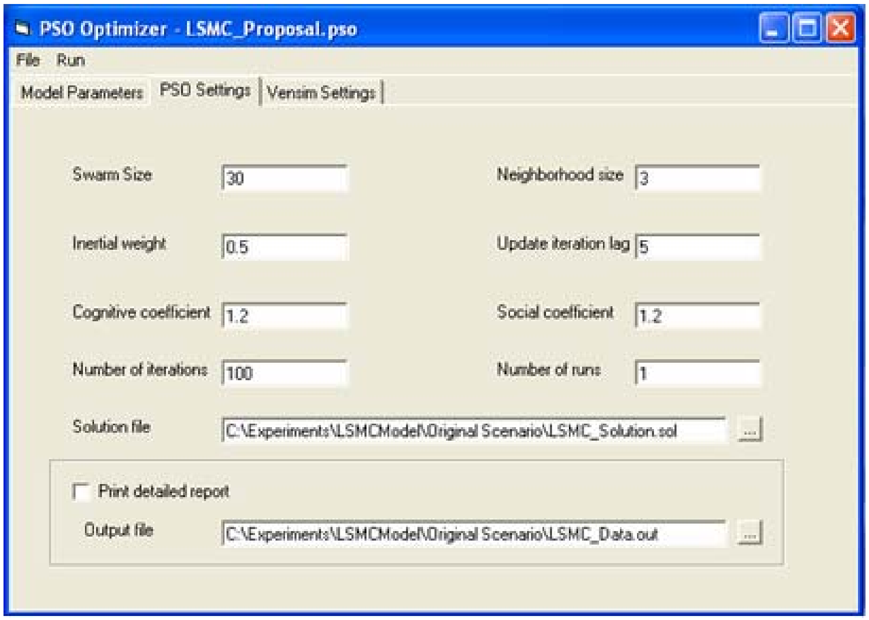

Figure 3.11. PSO Settings Tab

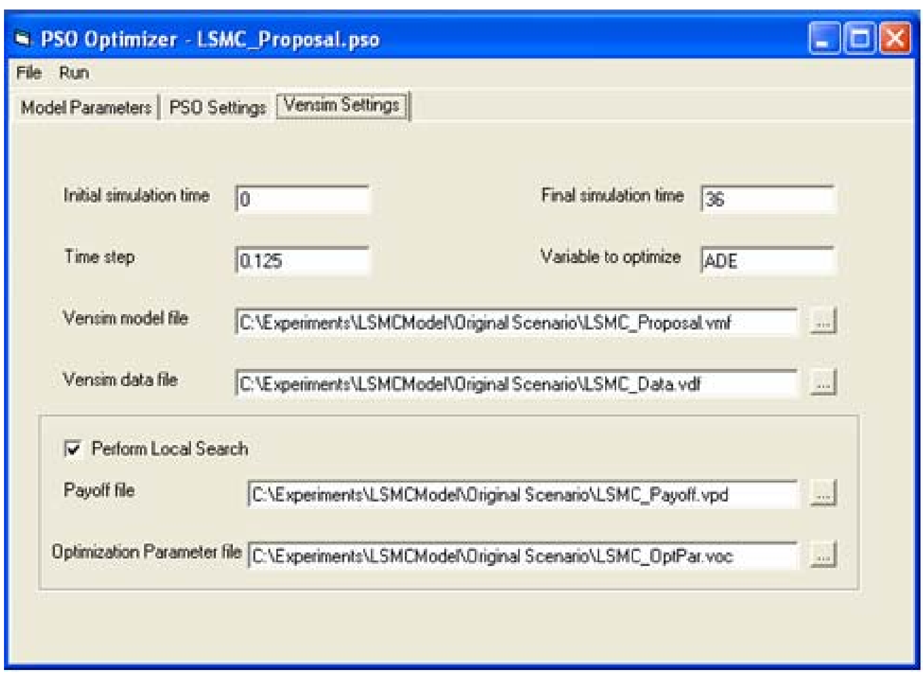

Figure 3.12. Vensim Settings Tab

3.3. Validation of the Methodology

The stabilization methodology will be validated by performing several experiments on the framework using the SC models of four case studies.

3.3.1. Case Studies

The validation process will be based on the effective application of the methodology to two academic case studies (see case studies A and B) and two case studies of real companies referred as LSMC and PMOC (see case studies C and D). A brief description of the case studies is presented in the following lines:

- Case study A: the Inventory-Workforce model.

  This is a short case about a manufacturing supply chain that includes labor as an explicit factor of production. Saleh *et al*. (2007) developed a linear SD model for this supply chain by modifying Sterman's original model (2000). The interactions between inventory management policies and the labor adjustment policies are the main cause for the oscillatory behavior of the supply chain.

- Case study B: the Mass model

  Mass (Mass 1975) developed a nonlinear SD model of a manufacturing supply chain to explore the economic processes underlying business-cycle behavior. Business cycles are recurring fluctuations that affect total production, prices, employment, inventories and capital investment. This case is designed to show how production, hiring and investment policies within a SC can interact to create fluctuations in inventory and employment that are characteristic of a business cycle. This model contains a production sector plus two factors of production: labor sector and capital sector.

- Case study C: the LSMC model.

  This case, which is based on the work of Lertpattarapong (2002), describes the operations of an actual electronics manufacturing company called LSMC. LSMC supplies products for personal computers to original equipment manufacturers such as Dell, Gateway, and Hewlett-Packard. Since 1998, many original equipment manufacturers have changed their strategies by adopting built-to-order and just-in-time processes. These changes in personal computers in addition to their short life cycles have amplified the coordination problems in the company's supply chain, which in turn has caused excess inventories and sometimes difficulties to keep up with demand. LSMC was facing a problem of persistent oscillations in its finished goods inventory and desired capacity.

- Case study D: the PMOC model.

  This case, which is based on the work of Helal (2008), describes the operations of a real industrial company (referred as PMOC) that produces various optical products. The case study focuses on the lenses production process which constitutes 65% of the company's production. The SD model covers the production process of the enterprise system which is composed by the following sub-systems: internal supply chain, suppliers and labor. The goal of management is to find a policy that maintains the stocks at equilibrium through the setting of various parameters in the model.

3.3.2. Experimental Analysis

The dynamic behavior of the supply chain can be studied through experiments by applying various types of inputs to the model and modifying its parameters to respond to those changes while keeping the supply chain stable (policy optimization).

First, the model needs to be placed in balanced equilibrium when possible[18]. At the equilibrium, the model does not generate any dynamic behavior, i.e. nothing changes over time, and it lies at the equilibrium unless otherwise disturbed. This action facilitates the process of experimentation because the system is disturbed only by the inputs the tester chooses to impose avoiding confusion with the transient behavior induced by initial disequilibrium (Sterman 2000).

Second, from equilibrium, the SC will be disturbed by various types of inputs affecting some variables of the model and generating different scenarios:

- Scenario 1 (step input): This is a sudden, permanent increase/decrease in the input from one rate to another. It can serve to "excite" any mode of response that may be inherent in the system model. If the system has oscillatory behavior, the step input gives an immediate indication of the natural period of oscillation and the rapidity of damping or of growth of the oscillation (Forrester 1961).

  This input will be implemented by using the STEP function. For example, in the following equation the variable *Sales* returns the value 100 units/week until week 20, and then it changes to 110 units/week.

  Sales=100+STEP(10,20)

- Scenario 2 (linear growth or decline input): These inputs contain underlying growth and decline trends on which the other variations are superimposed. Instead of being a one-shot

[18] Many times it is difficult to calculate the equilibrium point of a system due to the complexity of the system equations.

change like the one obtained from the STEP function, this input represents continual changes of a fixed magnitude.

This input will be implemented by using the RAMP function. For example, in the following equation the variable *Demand* increases linearly with rate of 20 units/week, beginning at week 2.

Demand=RAMP(20,2)

- Scenario 3 (combination of different inputs): This scenario is built by adding the effects of several inputs.

Third, using the framework, several experiments will be performed with the result of stabilization policies created as a response to the inputs mentioned in the previous step. Robustness of the solutions then will be investigated after varying several parameters of the model. For small perturbations of the system (small variations in parameters of the model – typically they are chosen to be exogenous variables) the asymptotic stability of the SC must be kept.

The effectiveness of the stabilization methodology will be demonstrated by the comparison of the stabilization policies obtained in the experimentation step against the base policy. The base policy is the one that has been disturbed after applying some of the inputs mentioned before. Although, it is possible to verify graphically if the application of a policy has made the system to achieve asymptotic stability, it is necessary other indicators to measure the characteristics of a stabilization policy. Two useful quantitative indicators are shown next.

- Amplification of a variable. It is the maximum value in a variable of interest due to the change in a parameter. When testing robustness, a policy with lower amplification value may indicate a more robust solution.

- Response time. It is the time the system takes to achieve asymptotic stability. A policy with longer response times may indicate trouble in adjusting to growth or decline in business.

CHAPTER FOUR: DEVELOPMENT OF STABILITY CONDITIONS

The direction of this chapter points toward the demonstration of the properties of the objective function used in the optimization problem presented in chapter 3, which is used to achieve stability of nonlinear dynamic systems. As it was explained in the previous chapters, this optimization problem relies on the concept that the minimization of the ADE will make the trajectory of the state variables to converge to the equilibrium point. Throughout this chapter it will be proved several lemmas and theorems that will facilitate the calculation of the state vector equation, and therefore the calculation of the ADE in order to lead to the conditions of convergence of a dynamic system. The challenge is to demonstrate the general applicability of the objective function to reach stability of any linear or nonlinear dynamic system formulated as a system of first-order differential equations.

Two objectives are set for this purpose. The first is to define the concepts for stability of linear dynamic systems. The second is to extend these ideas to the nonlinear dynamic case.

For the linear case, Theorems 1 through 3 provide different forms of expressing the state vector equation. From an equation stated in terms of the matrix exponential (see Definition 4) to a more simple form to operate, expressed in terms of eigenvalues and eigenvectors. This last form, which is more suitable for integration, is applied to find a bound for the ADE that is used to set the conditions for its convergence. Theorem 4 will guarantee that ADE convergence implies the restriction and convergence of the state variable trajectory, and thus asymptotic stability.

For the nonlinear case, the system equations are approximated by an infinite number of linear system equations defined at very small intervals of time. The ADE is calculated as the summation of the accumulated deviations of each of these linear systems. Expressing ADE in this form facilitates the use of the theory of convergence of an infinite series to establish the condition in which Theorem 5 is based on to prove asymptotic stability. Theorem 6 is presented to cover the cases when the EP is unknown, letting the optimization problem to decide which EP leads to fewer oscillations and faster stability. Finally, if the ADE convergence is close, but not achieved completely, it can be useful to amplify the deviations from the equilibrium point (DE) to accelerate the asymptotic stability of the variables of interest. Theorem 7 provides a mechanism to do that.

4.1. Definition of the Concept of Stability

The intuition for stability of a dynamic system captures the idea that if the system is started at a particular set of initial values near an equilibrium point (as stated in Definition 1), it will stay near that equilibrium point for all future time.

Definition 1 (Khalil 1996) The point $\mathbf{x}^{eq} \in R^n$ is said to be an equilibrium point of the differential equation $\dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}(t))$ if it has the property that once the corresponding system reaches $\mathbf{x}^{eq}$ at time t_{eq} it will remain at $\mathbf{x}^{eq}$ for all future time; in other words, $\mathbf{f}(\mathbf{x}(t)) = \mathbf{0}$ for all t $\geq t_{eq}$.

A more rigorous mathematical description for stability is given in the following definition.

Definition 2 Consider the system defined by $\dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}(t))$; $\mathbf{x}(0) = \mathbf{x}_0$; where $\mathbf{x}(t) \in R^n$;

$\mathbf{f}: R^n \rightarrow R^n; \mathbf{x}(t) = [x_1(t), x_2(t),.., x_n(t)]^T = [x_s(t)], s = 1,.., n$. The state variable x_s is defined to be stable (around the EP x_s^{eq}) if it is bounded, that is, there is a finite number M_s such that $|x_s(t) - x_s^{eq}| \leq M_s$. If this condition holds for all state variables then the system is said to be stable.

Because the notion of supply chain stability should consider also the reduction or minimization of oscillatory behavior, then the concept of *asymptotic stability* is preferred over the mere stability. A system is to be said asymptotically stable, if the system trajectory converges to the equilibrium point as time increases indefinitely. Next, the concept of asymptotic stability will be restated in a formal mathematical context.

Definition 3 Consider the system defined by $\dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}(t))$; $\mathbf{x}(0) = \mathbf{x}_0$; where $\mathbf{x}(t) \in R^n$; $\mathbf{f}: R^n \rightarrow R^n; \mathbf{x}(t) = [x_s(t)], s = 1,.., n$. The state variable x_s is defined to be *asymptotically stable* (around the EP x_s^{eq}) if it is both stable (satisfies Definition 2), and additionally, it satisfies $\lim_{t \rightarrow \infty}(x_s(t) - x_s^{eq}) = 0$. If these two conditions hold for all state variables then the system is said to be *asymptotically stable*.

Definitions 2 and 3 were adapted from the formal definitions of stability and asymptotic stability used in control theory (Khalil 1996). The stability conditions used in this research work are defined in terms of "one state variable" and not in terms of "all state variables" like traditional control theory that uses the norm of the state vector. This facilitates performing the stability analysis of specific state variables of the system (e.g. finished goods inventory). If necessary, stability can be extended to the whole system by using a weighted average function that includes all state variables.

4.2. Stability of Linear Systems

Stability analysis of linear systems is more tractable than the nonlinear case due to the well know structure of the system represented as a set of first-order linear differential equations. In section 4.2.1 it is found an equation for the solution of the linear system in terms of eigenvalues and eigenvectors that clear the path for justifying the notion of stability based on the convergence of ADE. This last step is presented in section 4.2.2.

4.2.1. Solution of Linear Systems

The structure of a linear dynamic system can be represented compactly as:

$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{b}$; $\mathbf{x}(0) = \mathbf{x}_0$; where $\mathbf{x} \in R^n, \mathbf{A} \in R^{n\,x\,n}, \mathbf{b} \in R^{n\,x\,1}$.

The matrix $\mathbf{A} = \partial\dot{\mathbf{x}}(t)/\partial\mathbf{x}(t)$ is commonly known as the *Jacobian* of the system.

The solution of this system can be expressed in terms of the matrix exponential (Definition 4) as it is shown in Theorem 1.

Definition 4 (Edwards and Penney 2001) For each matrix $\mathbf{A} \in R^{n\,x\,n}$, define the matrix exponential of ***A**t* to be the matrix:

$$e^{\mathbf{A}t} = \sum_{k=0}^{\infty} \frac{(\mathbf{A}t)^k}{k!} = \mathbf{I} + \mathbf{A}t + \frac{(\mathbf{A}t)^2}{2!} + \frac{(\mathbf{A}t)^3}{3!} + \cdots;$$

where $t \in R$ and $\mathbf{I} \in R^{n\,x\,n}$ is the identity matrix.

Theorem 1 (Umez-Eronini 1999) Consider the system defined by $\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{b}$;

$\mathbf{x}(0) = \mathbf{x}_0$; where $\mathbf{x}(t) \in R^n$, $\mathbf{A} \in R^{n \times n}$, $\mathbf{b} \in R^{n \times 1}$. The solution to this system is given by

$$\mathbf{x}(t) = e^{\mathbf{A}t}\mathbf{x}_0 + \int_0^t e^{\mathbf{A}(t-\tau)}\mathbf{b}d\tau$$

Proof: The solution to the system can be written as the sum of two terms: $\mathbf{x}(t) = \mathbf{x}_h(t) + \mathbf{x}_p(t)$,

where the subscripts *h* and *p* denote *homogeneous* and *particular*.

The homogeneous solution is the solution to the equation:

$$\dot{\mathbf{x}}_h(t) = \mathbf{A}\mathbf{x}_h(t);\ \mathbf{x}_h(0) = \mathbf{x}_0 \tag{4.1}$$

The particular solution is the solution to the equation:

$$\dot{\mathbf{x}}_{\mathbf{p}}(t) = \mathbf{A}\mathbf{x}_p(t) + \mathbf{b};\ \mathbf{x}_p(0) = \mathbf{0} \tag{4.2}$$

First, the homogeneous solution will be calculated. Equation (4.1) can be expressed as:

$$\frac{d\mathbf{x}_h(t)}{dt} = \mathbf{A}\mathbf{x}_h(t) \text{ or } \frac{d\mathbf{x}_h(t)}{\mathbf{x}_h(t)} = \mathbf{A}dt$$

Integrating: $\int \frac{d\mathbf{x}_h(t)}{\mathbf{x}_h(t)} = \int \mathbf{A}dt$

It follows that $\ln(\mathbf{x}_h(t)) = \mathbf{A}t + K$, where K is a constant.

Hence $\mathbf{x}_h(t) = e^{\mathbf{A}t}e^K$. Substituting the initial condition $\mathbf{x}_h(0) = \mathbf{x}_0$, the homogeneous solution is

given by $\mathbf{x}_h(t) = e^{\mathbf{A}t}\mathbf{x}_0$ (4.3)

Second, the particular solution (4.2) is calculated. Doing the following transformation:

$$\mathbf{z}(t) = e^{-\mathbf{A}t}\mathbf{x}_p(t) \tag{4.4}$$

After taking the derivatives to both terms:

$$\dot{\mathbf{z}}(t)=\frac{d}{dt}\left(e^{-\mathbf{A}t}\right)\mathbf{x}_p(t)+e^{-\mathbf{A}t}\dot{\mathbf{x}}_p(t)=-\mathbf{A}e^{-\mathbf{A}t}\mathbf{x}_p(t)+e^{-\mathbf{A}t}\dot{\mathbf{x}}_p(t) \tag{4.5}$$

Substituting (4.2) into (4.5)

$$\dot{\mathbf{z}}(t)=-\mathbf{A}e^{-\mathbf{A}t}\mathbf{x}_p(t)+e^{-\mathbf{A}t}\left[\mathbf{A}\mathbf{x}_p(t)+\mathbf{b}\right]$$

Simplifying: $\dot{\mathbf{z}}(t)=e^{-\mathbf{A}t}\mathbf{b}$

Integrating both sides from 0 to "t" gives

$$\mathbf{z}(t)-\mathbf{z}(0)=\int_0^t e^{-\mathbf{A}\tau}\mathbf{b}\,d\tau$$

But $\mathbf{z}(0)=e^{-\mathbf{A}0}\mathbf{x}_p(0)=\mathbf{0}$, then

$$\mathbf{z}(t)=\int_0^t e^{-\mathbf{A}\tau}\mathbf{b}\,d\tau \tag{4.6}$$

Substituting the transformation (4.4) into (4.6)

$$e^{-\mathbf{A}t}\mathbf{x}_p(t)=\int_0^t e^{-\mathbf{A}\tau}\mathbf{b}\,d\tau$$

Solving for $\mathbf{x}_p(t)$ in the above equation

$$\mathbf{x}_p(t)=e^{\mathbf{A}t}\int_0^t e^{-\mathbf{A}\tau}\mathbf{b}\,d\tau$$

$$=\int_0^t e^{\mathbf{A}(t-\tau)}\mathbf{b}\,d\tau \tag{4.7}$$

Combining the homogeneous and particular solutions (4.3) and (4.7) into a total solution

$\mathbf{x}(t)=\mathbf{x}_h(t)+\mathbf{x}_p(t)$, it is proved that

$$\mathbf{x}(t) = e^{\mathbf{A}t}\mathbf{x}_0 + \int_0^t e^{\mathbf{A}(t-\tau)}\mathbf{b}\,d\tau \ \blacksquare$$

Example 1 Consider the manufacturing supply chain shown in Figure 4.1. The SD model is composed of three state variables: *Inventory*, *Work in Process* (*WIP) Inventory* and *Expected Demand*. The current level for these variables is 50, 70 and 0 units respectively. Inventory integrates the difference between production and shipments. Production starts to replenish inventory to its desired level and satisfy the expected demand. The expected demand is a smooth function of actual demand. Shipment rate depends on the current inventory level and the shipment delay. In this model it is assumed that demand is greater or equal to the shipment rate and all orders not immediately filled are lost as customers seek alternate suppliers. Obtain the trajectory equations for the state variables of this system knowing that the model equations are the following:

$$\frac{\partial \text{Expected Demand(t)}}{\partial t} = \frac{(\text{Demand} - \text{Expected Demand(t)})}{\text{Time to Average Demand}}$$

$$= \frac{-\text{Expected Demand(t)}}{\text{Time to Average Demand}} + \frac{\text{Demand}}{\text{Time to Average Demand}}$$

$$\frac{\partial \text{Inventory(t)}}{\partial t} = \frac{\text{WIP Inventory(t)}}{\text{Production Delay}} - \frac{\text{Inventory(t)}}{\text{Shipment Delay}}$$

$$\frac{\partial \text{WIP Inventory(t)}}{\partial t} = \text{Expected Demand(t)} + \frac{(\text{Desired Inventory} - \text{Inventory(t)})}{\text{Production Adjustment Time}}$$

$$- \frac{\text{WIP Inventory(t)}}{\text{Production Delay}}$$

$$= \text{Expected Demand(t)} - \frac{\text{Inventory(t)}}{\text{Production Adjustment Time}}$$

$$-\frac{\text{WIP Inventory(t)}}{\text{Production Delay}}+\frac{\text{Desired Inventory}}{\text{Production Adjustment Time}}$$

Table 4.1 shows the values for the demand, which is an exogenous variable, and the set of parameters that define the inventory management policy for this supply chain.

Table 4.1. Parameter values for the supply chain of Example 1

Parameter	**Value**	**Unit**
Demand	2000	Units
Production Delay	1	Weeks
Shipment Delay	1/3	Weeks
Desired Inventory	100	Units
Production Adjustment Time	1	Weeks
Time to Average Demand	10	Weeks

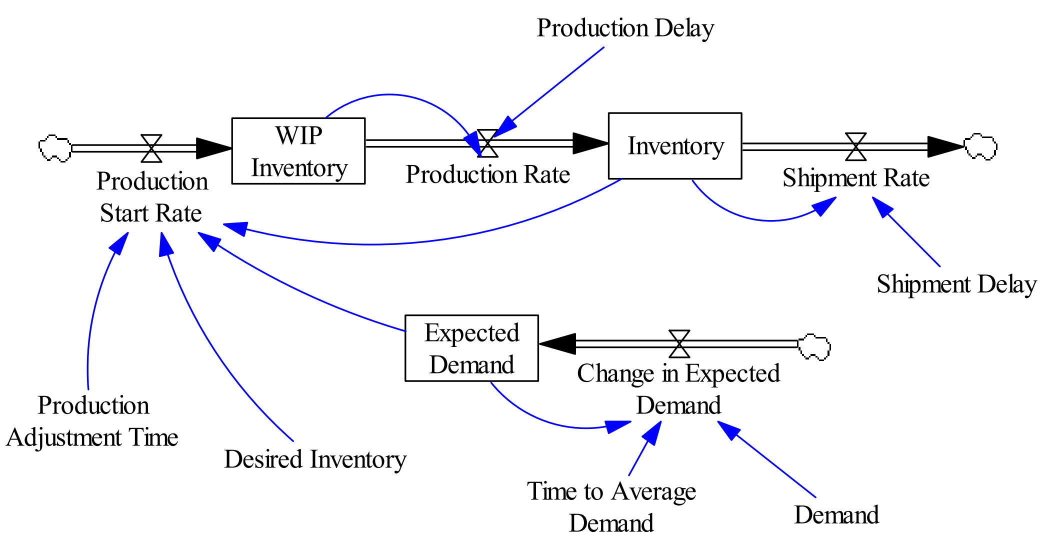

Figure 4.1. Manufacturing supply chain

Solution: The model equations of this supply chain can be expressed as the linear system $\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{b};\ \mathbf{x}(0) = \mathbf{x}_0$, where

$$\mathbf{x}(t) = \begin{bmatrix} \text{Expected Demand}(t) \\ \text{Inventory}(t) \\ \text{WIP Inventory}(t) \end{bmatrix}, \mathbf{A} = \begin{bmatrix} -10 & 0 & 0 \\ 0 & -3 & 1 \\ 1 & -1 & -1 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} 200 \\ 0 \\ 100 \end{bmatrix}, \mathbf{x}_0 = \begin{bmatrix} 0 \\ 50 \\ 70 \end{bmatrix}$$

To obtain the solution of this model, first the matrix exponential has to be determined.

For the given matrix **A**, the following terms are calculated

$$\mathbf{A}^2 = \begin{bmatrix} 100 & 0 & 0 \\ 1 & 8 & -4 \\ -11 & 4 & 0 \end{bmatrix}, \mathbf{A}^3 = \begin{bmatrix} -1000 & 0 & 0 \\ -14 & -20 & 12 \\ 110 & -12 & 4 \end{bmatrix}, \mathbf{A}^4 = \begin{bmatrix} 10000 & 0 & 0 \\ 152 & 48 & -32 \\ 1096 & 32 & -16 \end{bmatrix}, \text{etc.} \tag{4.8}$$

By Definition 4,

$$e^{\mathbf{A}t} = \sum_{k=0}^{\infty} \frac{(\mathbf{A}t)^k}{k!} = \mathbf{I} + \mathbf{A}t + \frac{(\mathbf{A}t)^2}{2!} + \frac{(\mathbf{A}t)^3}{3!} + \frac{(\mathbf{A}t)^4}{4!} + \cdots$$

Substituting the values of (4.8) into the expression above it is obtained

$$e^{\mathbf{A}t} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \begin{bmatrix} -10t & 0 & 0 \\ 0 & -3t & t \\ t & -t & -t \end{bmatrix} + \begin{bmatrix} 100t^2 & 0 & 0 \\ t^2 & 8t^2 & -4t^2 \\ -11t^2 & 4t^2 & 0 \end{bmatrix} \frac{1}{2!} + \begin{bmatrix} -1000t^3 & 0 & 0 \\ -14t^3 & -20t^3 & 12t^3 \\ 110t^3 & -12t^3 & 4t^3 \end{bmatrix} \frac{1}{3!} + \cdots \tag{4.9}$$

Let $e^{\mathbf{A}t} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$

Then by (4.9) and the theory of convergence of infinite series results that

$$a_{11} = 1 - 10t + \frac{100t^2}{2!} - \frac{1000t^3}{3!} + \cdots = e^{-10t}$$

$$a_{12} = 0 + 0 + 0 + 0 + \cdots = 0$$

$$a_{13} = 0 + 0 + 0 + 0 + \cdots = 0$$

$$a_{21} = 0 + 0 + t^2 - 14t^3 + \cdots = \frac{e^{-10t}}{64} - \frac{e^{-2t}}{64} + \frac{e^{-2t}t}{8}$$

$$a_{22} = 1 - 3t + \frac{8t^2}{2!} - \frac{20t^3}{3!} + \cdots = e^{-2t} - e^{-2t}t$$

$$a_{23} = 0 + t - \frac{4t^2}{2!} + \frac{12t^3}{3!} + \cdots = e^{-2t}t$$

$$a_{31} = 0 + t - \frac{11t^2}{2!} + \frac{110t^3}{3!} + \cdots = -\frac{7e^{-10t}}{64} + \frac{7e^{-2t}}{64} + \frac{e^{-2t}t}{8}$$

$$a_{32} = 0 - t + \frac{4t^2}{2!} - \frac{12t^3}{3!} + \cdots = -e^{-2t}t$$

$$a_{33} = 1 - t + 0 - \frac{4t^3}{3!} + \cdots = e^{-2t} + e^{-2t}t$$

Thus, the matrix exponential is given by

$$e^{\mathbf{A}t} = \begin{bmatrix} e^{-10t} & 0 & 0 \\ \frac{e^{-10t}}{64} - \frac{e^{-2t}}{64} + \frac{e^{-2t}t}{8} & e^{-2t} - e^{-2t}t & e^{-2t}t \\ -\frac{7e^{-10t}}{64} + \frac{7e^{-2t}}{64} + \frac{e^{-2t}t}{8} & -e^{-2t}t & e^{-2t} + e^{-2t}t \end{bmatrix} \tag{4.10}$$

It is know from Theorem 1 that,

$$\mathbf{x}(t) = e^{\mathbf{A}t}\mathbf{x}_0 + \int_0^t e^{\mathbf{A}(t-\tau)}\mathbf{b}d\tau$$

$$= e^{\mathbf{A}t}\mathbf{x}_0 + e^{\mathbf{A}t}\int_0^t e^{-\mathbf{A}\tau}\mathbf{b}d\tau \tag{4.11}$$

Substituting the matrix exponential obtained in (4.10) into (4.11)

$$
\mathbf{x}(t) = \begin{bmatrix} e^{-10t} & 0 & 0 \\ \frac{e^{-10t}}{64} - \frac{e^{-2t}}{64} + \frac{e^{-2t}t}{8} & e^{-2t} - e^{-2t}t & e^{-2t}t \\ -\frac{7e^{-10t}}{64} + \frac{7e^{-2t}}{64} + \frac{e^{-2t}t}{8} & -e^{-2t}t & e^{-2t} + e^{-2t}t \end{bmatrix} \begin{bmatrix} 0 \\ 50 \\ 70 \end{bmatrix} +
$$

$$
\int_0^t \begin{bmatrix} e^{-10(\tau-t)} & 0 & 0 \\ \frac{e^{-10(\tau-t)}}{64} - \frac{e^{-2(\tau-t)}}{64} + \frac{e^{-2(\tau-t)}(\tau-t)}{8} & e^{-2(\tau-t)} - e^{-2(\tau-t)}(\tau-t) & e^{-2(\tau-t)}(\tau-t) \\ -\frac{7e^{-10(\tau-t)}}{64} + \frac{7e^{-2(\tau-t)}}{64} + \frac{e^{-2(\tau-t)}(\tau-t)}{8} & -e^{-2(\tau-t)}(\tau-t) & e^{-2(\tau-t)} + e^{-2(\tau-t)}(\tau-t) \end{bmatrix} \begin{bmatrix} 200 \\ 0 \\ 100 \end{bmatrix} d\tau
$$

(4.12)

After applying matrix and integral operations to the expression in (4.12) and simplifying, the solution of the system is obtained.

$$
\mathbf{x}(t) = \begin{bmatrix} \text{Expected Demand}(t) \\ \text{Inventory}(t) \\ \text{WIP Inventory}(t) \end{bmatrix} = \begin{bmatrix} 20 - 20e^{-10t} \\ 30 - \frac{5e^{-10t}}{16} + \frac{325e^{-2t}}{16} - \frac{85e^{-2t}t}{2} \\ 90 + \frac{35e^{-10t}}{16} - \frac{355e^{-2t}}{16} - \frac{85e^{-2t}t}{2} \end{bmatrix} \blacksquare
$$

It follows from Example 1 that the calculation of the matrix exponential involves the computation of several infinite series. However, this calculation can be simplified using the eigenvalue-eigenvector method. The essential idea in this method is to transform matrix **A** into a Jordan canonical form **J** (Definition 7) by using a transformation matrix **T** composed by generalized eigenvectors (Definition 6). The transformation $\mathbf{A} = \mathbf{T}\mathbf{J}\mathbf{T}^{-1}$ then leads to the equation $e^{\mathbf{A}t} = \mathbf{T}e^{\mathbf{J}t}\mathbf{T}^{-1}$, which is much simpler to compute than the formula given in Definition 4 and can be used for any matrix **A** as it is stated in Lemma 1. First, it is necessary to provide the following definitions required to understand the proof of this lemma.

Definition 5 (Edwards and Penney 2001) For a matrix $\mathbf{A} \in R^{n \times n}$, if the number $\lambda \in C$ and nonzero vector $\mathbf{r} \in C^{n}$ satisfy: $\mathbf{Ar} = \lambda\mathbf{r}$, then λ is called an *eigenvalue* of **A** and **r** its corresponding *eigenvector*.

Definition 6 (Edwards and Penney 2001) If $\lambda \in C$ is an eigenvalue of matrix $\mathbf{A} \in R^{n \times n}$, then a generalized eigenvector $\mathbf{u} \in C^{n}$ with multiplicity *k* associated with λ satisfies: $(\mathbf{A} - \lambda\mathbf{I})^{k}\mathbf{u} = \mathbf{0}$. Ordinary eigenvectors as on Definition 5 are obtained for *k*=1.

Definition 7 (Edwards and Penney 2001) A square matrix is in *Jordan canonical form* if it has a block decomposition in which all diagonal blocks are *Jordan blocks* $\mathbf{J}_{i}$ (i=1,..,m) and all other blocks are zero:

$$\mathbf{J} = \begin{bmatrix} \mathbf{J}_1 & & & \mathbf{0} \\ & \mathbf{J}_2 & & \\ & & \ddots & \\ \mathbf{0} & & & \mathbf{J}_m \end{bmatrix} \tag{4.13}$$

where a *Jordan block* $\mathbf{J}_{i}$ is a square matrix in which all diagonal entries are equal to a single eigenvalue λ_i, all entries immediately above the diagonal are one, and all other entries are zero:

$$\mathbf{J}_i = \begin{bmatrix} \lambda_i & 1 & & & 0 \\ & \lambda_i & 1 & & \\ & & \lambda_i & \ddots & \\ & & & \ddots & 1 \\ 0 & & & & \lambda_i \end{bmatrix} \tag{4.14}$$

If the dimension of $\mathbf{J}_i$ is n_i (i=1,..,m) then the dimension of **J** is $n_1+n_2+\ldots+n_m$

Definition 8 (Edwards and Penney 2001) A set of *k*-vectors $\{\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_k\}$ is called *linearly independent* when the equation $\alpha_1\mathbf{u}_1 + \alpha_2\mathbf{u}_2 + \cdots + \alpha_k\mathbf{u}_k = \mathbf{0}$ is satisfied only by the trivial choice of scalars $\alpha_1 = \alpha_2 = \cdots = \alpha_k = 0$.

Lemma 1 (Khalil 1996) For any matrix $\mathbf{A} \in R^{n \times n}$ there is an invertible matrix **T** such that it is possible to find the following transformations:

i) $\mathbf{A} = \mathbf{TJT}^{-1}$

ii) $e^{\mathbf{A}t} = \mathbf{T}e^{\mathbf{J}t}\mathbf{T}^{-1}$

where $\mathbf{J} \in C^{n \times n}$ is a matrix in Jordan canonical form and $\mathbf{T} \in C^{n \times n}$ is a matrix composed by generalized eigenvectors.

Proof: **Part i**. Refer to Gel'fand (1977) to see the proof of this part. There, it is also proved that the generalized eigenvectors that compose **T** are linearly independent and they span R^n. Moreover, matrix **T** has the form

$$\mathbf{T} = [\mathbf{u}_1^J, \mathbf{u}_2^J, \cdots, \mathbf{u}_m^J] \tag{4.15}$$

where

$$\mathbf{u}_i^J = [\mathbf{u}_{i1} \quad \mathbf{u}_{i2} \quad \cdots \quad \mathbf{u}_{in_i}] \tag{4.16}$$

$$\mathbf{u}_{ij} = \begin{bmatrix} u_{ij1} \\ u_{ij2} \\ \vdots \\ u_{ijn_i} \end{bmatrix} \tag{4.17}$$

$\mathbf{u}_i^J$ denotes the group of generalized eigenvectors associated to the i^{th} Jordan block.

$\mathbf{u}_{ij}$ denotes the j^{th} generalized eigenvector of the i^{th} Jordan block.

u_{ijs} denotes the element of the s^{th} row of the j^{th} generalized eigenvector of the i^{th} Jordan block.

The generalized eigenvectors satisfy the following equations:

$$\mathbf{A}\mathbf{u}_{i1} = \lambda_i \mathbf{u}_{i1} \tag{4.18}$$

$$\mathbf{A}\mathbf{u}_{ij} = \lambda_i \mathbf{u}_{ij} + \mathbf{u}_{i,j\text{-}1}\text{, if j>1} \tag{4.19}$$

Part ii. Using Definition 4 the expression for $e^{\mathbf{A}t}$ is given by

$$e^{\mathbf{A}t} = \sum_{k=0}^{\infty} \frac{(\mathbf{A}t)^k}{k!} \tag{4.20}$$

Solving for $\mathbf{J}$ in equation $\mathbf{A} = \mathbf{TJT}^{-1}$

$$\mathbf{J} = \mathbf{T}^{-1}\mathbf{AT}$$

Thus, now $\mathbf{J}^2$ can be calculated as

$$\mathbf{J}^2 = (\mathbf{T}^{-1}\mathbf{AT})(\mathbf{T}^{-1}\mathbf{AT}) = (\mathbf{T}^{-1}\mathbf{A})(\mathbf{TT}^{-1})(\mathbf{AT}) = (\mathbf{T}^{-1}\mathbf{A})(\mathbf{I})(\mathbf{AT}) = \mathbf{T}^{-1}\mathbf{A}^2\mathbf{T}$$

In a similar fashion, $\mathbf{J}^k$ is obtained as

$$\mathbf{J}^k = \mathbf{T}^{-1}\mathbf{A}^k\mathbf{T} \text{ (for k=0 the expression } \mathbf{J}^0 \text{ gives } \mathbf{A}^0 = \mathbf{I})$$

Therefore

$$(\mathbf{J}t)^k = \mathbf{J}^k t^k = \mathbf{T}^{-1}\mathbf{A}^k t^k \mathbf{T} = \mathbf{T}^{-1}(\mathbf{A}t)^k \mathbf{T} \tag{4.21}$$

Using Definition 4 the expression for $e^{\mathbf{J}t}$ can be written as

$$e^{\mathbf{J}t} = \sum_{k=0}^{\infty} \frac{(\mathbf{J}t)^k}{k!} \tag{4.22}$$

Substituting (4.21) into (4.22) yields

$$e^{\mathbf{J}t} = \sum_{k=0}^{\infty} \mathbf{T}^{-1} \frac{(\mathbf{A}t)^k}{k!} \mathbf{T} = \mathbf{T}^{-1}\left(\sum_{k=0}^{\infty} \frac{(\mathbf{A}t)^k}{k!} \right)\mathbf{T} \tag{4.23}$$

Using (4.23) the expression $\mathbf{T}e^{\mathbf{J}t}\mathbf{T}^{-1}$ can be expanded as

$$\mathbf{T}e^{\mathbf{J}t}\mathbf{T}^{-1} = \mathbf{T}\left[\mathbf{T}^{-1}\left(\sum_{k=0}^{\infty} \frac{(\mathbf{A}t)^k}{k!} \right)\mathbf{T} \right]\mathbf{T}^{-1} = \sum_{k=0}^{\infty} \frac{(\mathbf{A}t)^k}{k!} \tag{4.24}$$

From (4.20) and (4.24) it is proved that $e^{\mathbf{A}t} = \mathbf{T}e^{\mathbf{J}t}\mathbf{T}^{-1}$ ■

Now it is possible to express $\mathbf{x}(t)$ in terms of the deviations from the equilibrium point of the system. To obtain the trajectory equation $\mathbf{x}(t)$ two cases have to be considered. The first case is when at least one of the eigenvalues of the Jacobian matrix of the system has real part equal to zero. Trajectory equations of this type cannot achieve asymptotic stability (Khalil 1996). The second case is when the Jacobian matrix has no eigenvalues with zero real part. These trajectories can be shaped to attain asymptotic behavior, and therefore the equations for $\mathbf{x}(t)$ will be derived based on this second case. The equations will be obtained by applying the results obtained in Lemma 1 to the solution provided in Theorem 1, where the DE can be written as a linear combination of the generalized eigenvectors associated with the eigenvalues of the Jacobian matrix, as shown in the following theorem.

Theorem 2 Consider the system defined by $\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{b};\ \mathbf{x}(0) = \mathbf{x}_0$; where $\mathbf{x}(t) \in R^n$, $\mathbf{A} \in R^{n\,x\,n}$, $\mathbf{b} \in R^{n\,x\,1}$, then the solution to this system can be expressed as

$\mathbf{x}(t) = [x_s(t)],\ s = 1,..,n$

$$x_s(t) = x_s^{eq} + \sum_{i=1}^{m}\sum_{j=1}^{n_i}\sum_{k=j}^{n_i}\left\{ \left(\alpha_{ik} + \sum_{h=1}^{n_i-k+1} \frac{(-1)^{h-1}\beta_{i,k+h-1}}{(\lambda_i)^h} \right) u_{i,k-j+1,s} \frac{t^{j-1}}{(j-1)!} e^{\lambda_i t} \right\} \tag{4.25}$$

where

u_{ijs} is the element of the s^{th} row of the generalized eigenvector $\mathbf{u}_{ij}$ associated to the nonzero eigenvalue of $\mathbf{A}$ $\lambda_i \in R$, and $\alpha_{ij} \in R$, $\beta_{ij} \in R$ are constants, i=1,..,m; j=1,..,n_i.

Proof: From Lemma 1 it is known that:

$$e^{\mathbf{A}t} = \mathbf{T}e^{\mathbf{J}t}\mathbf{T}^{-1} \tag{4.26}$$

From Theorem 1 the solution to the linear system is given by

$$\mathbf{x}(t) = e^{\mathbf{A}t}\mathbf{x}_0 + \int_0^t e^{\mathbf{A}(t-\tau)}\mathbf{b}d\tau \tag{4.27}$$

Substituting (4.26) into (4.27)

$$\mathbf{x}(t) = \mathbf{T}e^{\mathbf{J}t}\mathbf{T}^{-1}\mathbf{x}_0 + \int_0^t \mathbf{T}e^{\mathbf{J}(t-\tau)}\mathbf{T}^{-1}\mathbf{b}d\tau$$

$$= \mathbf{T}e^{\mathbf{J}t}\mathbf{T}^{-1}\mathbf{x}_0 + \mathbf{T}e^{\mathbf{J}t}\left(\int_0^t e^{-\mathbf{J}\tau}d\tau\right)\mathbf{T}^{-1}\mathbf{b} \tag{4.28}$$

Because the number zero is not an eigenvalue of $\mathbf{A}$ (from the hypothesis) then $\mathbf{J}$ is invertible and from Lemma A.1 (see Appendix A):

$$\mathbf{J}^{-1} = \begin{bmatrix} \mathbf{J}_1^{-1} & & & \mathbf{0} \\ & \mathbf{J}_2^{-1} & & \\ & & \ddots & \\ \mathbf{0} & & & \mathbf{J}_m^{-1} \end{bmatrix} \tag{4.29}$$

$$\text{where } \mathbf{J}_i^{-1} = \begin{bmatrix} 1/\lambda_i & -1/\lambda_i^2 & 1/\lambda_i^3 & \cdots & (-1)^{n_i-1}/\lambda_i^{n_i} \\ & 1/\lambda_i & -1/\lambda_i^2 & \cdots & (-1)^{n_i-2}/\lambda_i^{n_i-1} \\ & & \ddots & & \vdots \\ & & & \ddots & \vdots \\ 0 & & & & 1/\lambda_i \end{bmatrix}, \text{ i=1,..,m} \tag{4.30}$$

Moreover, from Lemma A.2 the matrix exponential is given by

$$e^{\mathbf{J}t} = \begin{bmatrix} e^{\mathbf{J}_1 t} & & & \mathbf{0} \\ & e^{\mathbf{J}_2 t} & & \\ & & \ddots & \\ \mathbf{0} & & & e^{\mathbf{J}_m t} \end{bmatrix} \tag{4.31}$$

$$\text{where } e^{\mathbf{J}_i t} = \begin{bmatrix} e^{\lambda_i t} & te^{\lambda_i t} & \frac{t^2 e^{\lambda_i t}}{2} & \cdots & \frac{t^{n_i-1}e^{\lambda_i t}}{(n_i-1)!} \\ & e^{\lambda_i t} & te^{\lambda_i t} & \cdots & \frac{t^{n_i-2}e^{\lambda_i t}}{(n_i-2)!} \\ & & \ddots & & \vdots \\ & & & \ddots & \vdots \\ 0 & & & & e^{\lambda_i t} \end{bmatrix}, \text{ i=1,..,m} \tag{4.32}$$

Calculating the integral in (4.28) and simplifying

$$\begin{aligned} \mathbf{x}(t) &= \mathbf{T}e^{\mathbf{J}t}\mathbf{T}^{-1}\mathbf{x}_0 + \mathbf{T}e^{\mathbf{J}t}\left[-(\mathbf{J}^{-1})(e^{-\mathbf{J}t} - \mathbf{I})\right]\mathbf{T}^{-1}\mathbf{b} \\ &= \mathbf{T}e^{\mathbf{J}t}\mathbf{T}^{-1}\mathbf{x}_0 + \mathbf{T}\mathbf{J}^{-1}(e^{\mathbf{J}t} - \mathbf{I})\mathbf{T}^{-1}\mathbf{b} \\ &= \mathbf{T}e^{\mathbf{J}t}\mathbf{T}^{-1}\mathbf{x}_0 + \mathbf{T}\mathbf{J}^{-1}e^{\mathbf{J}t}\mathbf{T}^{-1}\mathbf{b} - \mathbf{T}\mathbf{J}^{-1}\mathbf{T}^{-1}\mathbf{b} \\ &= \mathbf{T}e^{\mathbf{J}t}\mathbf{T}^{-1}\mathbf{x}_0 + \mathbf{T}e^{\mathbf{J}t}\mathbf{J}^{-1}\mathbf{T}^{-1}\mathbf{b} - \mathbf{T}\mathbf{J}^{-1}\mathbf{T}^{-1}\mathbf{b} \\ &= \mathbf{T}e^{\mathbf{J}t}\left(\mathbf{T}^{-1}\mathbf{x}_0 + \mathbf{J}^{-1}\mathbf{T}^{-1}\mathbf{b}\right) - \mathbf{T}\mathbf{J}^{-1}\mathbf{T}^{-1}\mathbf{b} \end{aligned} \tag{4.33}$$

Making

$$\mathbf{T}^{-1}\mathbf{x}_0 = \begin{bmatrix} \boldsymbol{\alpha}_1^J \\ \boldsymbol{\alpha}_2^J \\ \vdots \\ \boldsymbol{\alpha}_m^J \end{bmatrix} \text{ and } \mathbf{T}^{-1}\mathbf{b} = \begin{bmatrix} \boldsymbol{\beta}_1^J \\ \boldsymbol{\beta}_2^J \\ \vdots \\ \boldsymbol{\beta}_m^J \end{bmatrix} \tag{4.34}$$

where $\boldsymbol{\alpha}_i^J = \begin{bmatrix} \alpha_{i1} \\ \alpha_{i2} \\ \vdots \\ \alpha_{in_i} \end{bmatrix}$ and $\boldsymbol{\beta}_i^J = \begin{bmatrix} \beta_{i1} \\ \beta_{i2} \\ \vdots \\ \beta_{in_i} \end{bmatrix}$ are column vectors of constants associated to the i^{th} Jordan block. (4.35)

It therefore follows from (4.29) and (4.34) that

$$\mathbf{T}^{-1}\mathbf{x}_0 + \mathbf{J}^{-1}\mathbf{T}^{-1}\mathbf{b} = \begin{bmatrix} \boldsymbol{\alpha}_1^J \\ \boldsymbol{\alpha}_2^J \\ \vdots \\ \boldsymbol{\alpha}_m^J \end{bmatrix} + \begin{bmatrix} \mathbf{J}_1^{-1} & & & \mathbf{0} \\ & \mathbf{J}_2^{-1} & & \\ & & \ddots & \\ \mathbf{0} & & & \mathbf{J}_m^{-1} \end{bmatrix} \begin{bmatrix} \boldsymbol{\beta}_1^J \\ \boldsymbol{\beta}_2^J \\ \vdots \\ \boldsymbol{\beta}_m^J \end{bmatrix} = \begin{bmatrix} \boldsymbol{\alpha}_1^J + \mathbf{J}_1^{-1}\boldsymbol{\beta}_1^J \\ \boldsymbol{\alpha}_2^J + \mathbf{J}_2^{-1}\boldsymbol{\beta}_2^J \\ \vdots \\ \boldsymbol{\alpha}_m^J + \mathbf{J}_m^{-1}\boldsymbol{\beta}_m^J \end{bmatrix} \tag{4.36}$$

Using (4.15), (4.32) and (4.36) the term $\mathbf{T}e^{\mathbf{J}t}\left(\mathbf{T}^{-1}\mathbf{x}_0 + \mathbf{J}^{-1}\mathbf{T}^{-1}\mathbf{b}\right)$ can be written as

$$[\mathbf{u}_1^J, \mathbf{u}_2^J, \cdots, \mathbf{u}_m^J] \begin{bmatrix} e^{\mathbf{J}_1 t} & & & \mathbf{0} \\ & e^{\mathbf{J}_2 t} & & \\ & & \ddots & \\ \mathbf{0} & & & e^{\mathbf{J}_m t} \end{bmatrix} \begin{bmatrix} \boldsymbol{\alpha}_1^J + \mathbf{J}_1^{-1}\boldsymbol{\beta}_1^J \\ \boldsymbol{\alpha}_2^J + \mathbf{J}_2^{-1}\boldsymbol{\beta}_2^J \\ \vdots \\ \boldsymbol{\alpha}_m^J + \mathbf{J}_m^{-1}\boldsymbol{\beta}_m^J \end{bmatrix} = \sum_{i=1}^{m} \mathbf{u}_i^J e^{\mathbf{J}_i t}\left(\boldsymbol{\alpha}_i^J + \mathbf{J}_i^{-1}\boldsymbol{\beta}_i^J\right) \tag{4.37}$$

Expanding the terms of the summation by using (4.17), (4.30), (4.32), and (4.35)

$$\sum_{i=1}^{m} \mathbf{u}_i^J e^{\mathbf{J}_i t}\left(\boldsymbol{\alpha}_i^J + \mathbf{J}_i^{-1}\boldsymbol{\beta}_i^J\right) = \sum_{i=1}^{m}\sum_{j=1}^{n_i}\sum_{k=j}^{n_i}\left\{\left(\alpha_{ik} + \sum_{h=1}^{n_i-k+1}\frac{(-1)^{h-1}\beta_{i,k+h-1}}{(\lambda_i)^h}\right)\frac{t^{k-j}}{(k-j)!}e^{\lambda_i t}\mathbf{u}_{i,j}\right\} \tag{4.38}$$

After rearranging the terms of (4.38) around the common factor $\dfrac{t^{j-1}}{(j-1)!}$ then

$$\sum_{i=1}^{m}\mathbf{u}_i^J e^{\mathbf{J}_i t}\left(\boldsymbol{\alpha}_i^J+\mathbf{J}_i^{-1}\boldsymbol{\beta}_i^J\right)=\sum_{i=1}^{m}\sum_{j=1}^{n_i}\sum_{k=j}^{n_i}\left\{\left(\alpha_{ik}+\sum_{h=1}^{n_i-k+1}\frac{(-1)^{h-1}\beta_{i,k+h-1}}{(\lambda_i)^h}\right)\mathbf{u}_{i,k-j+1}\frac{t^{j-1}}{(j-1)!}e^{\lambda_i t}\right\} \quad (4.39)$$

In view of Lemma 1, it is known that $\mathbf{A}=\mathbf{TJT}^{-1}$. This leads to the expression

$$\mathbf{A}^{-1}=\mathbf{TJ}^{-1}\mathbf{T}^{-1}$$

Multiplying both terms of the expression above by vector $\mathbf{b}$ yields

$$\mathbf{A}^{-1}\mathbf{b}=\mathbf{TJ}^{-1}\mathbf{T}^{-1}\mathbf{b} \quad (4.40)$$

It is also known that the EP is determined by making $\dot{\mathbf{x}}(t)=\mathbf{Ax}(t)+\mathbf{b}=\mathbf{0}$, therefore

$$\mathbf{x}^{eq}=-\mathbf{A}^{-1}\mathbf{b} \quad (4.41)$$

Using (4.40) and (4.41) gives

$$\mathbf{x}^{eq}=-\mathbf{TJ}^{-1}\mathbf{T}^{-1}\mathbf{b} \quad (4.42)$$

Substituting (4.15), (4.29), and (4.34) into (4.42) gives

$$\mathbf{x}^{eq}=-[\mathbf{u}_1^J,\mathbf{u}_2^J,\cdots,\mathbf{u}_m^J]\begin{bmatrix}\mathbf{J}_1^{-1} & & & \mathbf{0}\\ & \mathbf{J}_2^{-1} & & \\ & & \ddots & \\ \mathbf{0} & & & \mathbf{J}_m^{-1}\end{bmatrix}\begin{bmatrix}\boldsymbol{\beta}_1^J\\ \boldsymbol{\beta}_2^J\\ \vdots\\ \boldsymbol{\beta}_m^J\end{bmatrix}=\sum_{i=1}^{m}\mathbf{u}_i^J\mathbf{J}_i^{-1}\boldsymbol{\beta}_i^J$$

Expanding the terms of the summation by using (4.17), (4.30), and (4.35)

$$\mathbf{x}^{eq}=-\sum_{i=1}^{m}\mathbf{u}_i^J\mathbf{J}_i^{-1}\boldsymbol{\beta}_i^J=-\sum_{i=1}^{m}\sum_{j=1}^{n_i}\sum_{k=1}^{j}\left\{\frac{(-1)^{j-k}\beta_{i,j}}{(\lambda_i)^{j-k+1}}\mathbf{u}_{i,k}\right\} \quad (4.43)$$

From (4.37), (4.39), (4.42) and (4.33), it is clear that

$$\mathbf{x}(t)=\mathbf{x}^{eq}+\sum_{i=1}^{m}\sum_{j=1}^{n_i}\sum_{k=j}^{n_i}\left\{\left(\alpha_{ik}+\sum_{h=1}^{n_i-k+1}\frac{(-1)^{h-1}\beta_{i,k+h-1}}{(\lambda_i)^h}\right)\mathbf{u}_{i,k-j+1}\frac{t^{j-1}}{(j-1)!}e^{\lambda_i t}\right\} \quad (4.44)$$

By expressing this equation in terms of each state variable $x_s(t)$ of vector $\mathbf{x}(t)$ it is proved that

$$x_s(t) = x_s^{eq} + \sum_{i=1}^{m}\sum_{j=1}^{n_i}\sum_{k=j}^{n_i}\left\{\left(\alpha_{ik} + \sum_{h=1}^{n_i-k+1}\frac{(-1)^{h-1}\beta_{i,k+h-1}}{(\lambda_i)^h}\right)u_{i,k-j+1,s}\frac{t^{j-1}}{(j-1)!}e^{\lambda_i t}\right\} \quad \blacksquare$$

Example 2 Consider the manufacturing supply chain presented in Example 1. Assuming the same initial conditions and parameters shown in Table 4.1, obtain the trajectory equations for the state variables of this system using the results of Theorem 2.

Solution: The model equations of this supply chain can be expressed as the linear system $\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{b};\ \mathbf{x}(0) = \mathbf{x}_0$, where

$$\mathbf{x}(t) = \begin{bmatrix} \text{Expected Demand}(t) \\ \text{Inventory}(t) \\ \text{WIP Inventory}(t) \end{bmatrix}, \mathbf{A} = \begin{bmatrix} -10 & 0 & 0 \\ 0 & -3 & 1 \\ 1 & -1 & -1 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} 200 \\ 0 \\ 100 \end{bmatrix}, \mathbf{x}_0 = \begin{bmatrix} 0 \\ 50 \\ 70 \end{bmatrix}$$

The real eigenvalues of **A** are: $\lambda_1 = -10, \lambda_2 = -2, \lambda_3 = -2$; thus there are two Jordan blocks (m=2); the first one of dimension n_1=1 associated to eigenvalue λ_1 and the second one of dimension n_2=2 associated to eigenvalues λ_2 and λ_3. The Jordan canonical form is given by

$$\mathbf{J} = \begin{bmatrix} -10 & 0 & 0 \\ 0 & -2 & 1 \\ 0 & 0 & -2 \end{bmatrix}.$$

The matrix of generalized eigenvectors is: $\mathbf{T} = \begin{bmatrix} \mathbf{u}_{11} & \mathbf{u}_{21} & \mathbf{u}_{22} \end{bmatrix} = \begin{bmatrix} -64 & 0 & 0 \\ -1 & 1 & -1 \\ 7 & 1 & 0 \end{bmatrix}$.

The constants are found as follows:

$$\begin{bmatrix} \alpha_{11} \\ \alpha_{21} \\ \alpha_{22} \end{bmatrix} = \mathbf{T}^{-1}\mathbf{x}_0 = \begin{bmatrix} 0 \\ 70 \\ 20 \end{bmatrix}$$

$$\begin{bmatrix}\beta_{11}\\ \beta_{21}\\ \beta_{22}\end{bmatrix} = \mathbf{T}^{-1}\mathbf{b} = \begin{bmatrix}-25/8\\ 975/8\\ 125\end{bmatrix}$$

Let $\mathbf{x}(t) = \begin{bmatrix}x_1(t)\\ x_2(t)\\ x_3(t)\end{bmatrix} = \begin{bmatrix}\text{Expected Demand}(t)\\ \text{Inventory}(t)\\ \text{WIP Inventory}(t)\end{bmatrix}$

The equilibrium point is calculated using (4.43), for m=2, $n_1=1$, $n_2=2$

$$\mathbf{x}^{eq} = -\sum_{i=1}^{m}\sum_{j=1}^{n_i}\sum_{k=1}^{j}\left\{\frac{(-1)^{j-k}\beta_{i,j}}{(\lambda_i)^{j-k+1}}\mathbf{u}_{i,k}\right\} = -\frac{\beta_{1,1}}{\lambda_1}\mathbf{u}_{1,1} + \frac{\beta_{2,2}}{(\lambda_2)^2}\mathbf{u}_{2,1} - \frac{\beta_{2,1}}{\lambda_2}\mathbf{u}_{2,1} - \frac{\beta_{2,2}}{\lambda_2}\mathbf{u}_{2,2} \tag{4.45}$$

Replacing the values of the eigenvalues and constants in (4.45) gives

$$\mathbf{x}^{eq} = \begin{bmatrix}x_1^{eq}\\ x_2^{eq}\\ x_3^{eq}\end{bmatrix} = \begin{bmatrix}20\\ 30\\ 90\end{bmatrix}$$

The variables Expected Demand(t), Inventory(t) and WIP Inventory(t) are calculated replacing the corresponding values in (4.25), for m=2, $n_1=1$, $n_2=2$, s=1,..,3

$$\text{Expected Demand}(t) = x_1^{eq} + \sum_{i=1}^{m}\sum_{j=1}^{n_i}\sum_{k=j}^{n_i}\left\{\left(\alpha_{ik} + \sum_{h=1}^{n_i-k+1}\frac{(-1)^{h-1}\beta_{i,k+h-1}}{(\lambda_i)^h}\right)u_{i,k-j+1,1}\frac{t^{j-1}}{(j-1)!}e^{\lambda_i t}\right\}$$

$$\begin{aligned}\text{Expected Demand}(t) = 20 &+ \left(\alpha_{11} + \frac{\beta_{11}}{\lambda_1}\right)u_{111}e^{\lambda_1 t} + \left(\alpha_{21} - \frac{\beta_{22}}{(\lambda_2)^2} + \frac{\beta_{21}}{\lambda_2}\right)u_{211}e^{\lambda_2 t}\\ &+ \left(\alpha_{22} + \frac{\beta_{22}}{\lambda_2}\right)u_{211}te^{\lambda_2 t} + \left(\alpha_{22} - \frac{\beta_{22}}{\lambda_2}\right)u_{221}e^{\lambda_2 t}\end{aligned}$$

$$\text{Expected Demand}(t) = 20 - 20e^{-10t} \;\blacksquare$$

$$\text{Inventory}(t) = x_2^{eq} + \sum_{i=1}^{m}\sum_{j=1}^{n_i}\sum_{k=j}^{n_i}\left\{\left(\alpha_{ik} + \sum_{h=1}^{n_i-k+1}\frac{(-1)^{h-1}\beta_{i,k+h-1}}{(\lambda_i)^h}\right)u_{i,k-j+1,2}\frac{t^{j-1}}{(j-1)!}e^{\lambda_i t}\right\}$$

$$\text{Inventory(t)} = 30 + \left(\alpha_{11} + \frac{\beta_{11}}{\lambda_1}\right)u_{112}e^{\lambda_1 t} + \left(\alpha_{21} - \frac{\beta_{22}}{(\lambda_2)^2} + \frac{\beta_{21}}{\lambda_2}\right)u_{212}e^{\lambda_2 t}$$

$$+ \left(\alpha_{22} + \frac{\beta_{22}}{\lambda_2}\right)u_{212}te^{\lambda_2 t} + \left(\alpha_{22} - \frac{\beta_{22}}{\lambda_2}\right)u_{222}e^{\lambda_2 t}$$

$$\text{Inventory(t)} = 30 - \frac{5e^{-10t}}{16} + \frac{325e^{-2t}}{16} - \frac{85e^{-2t}t}{2} \ \blacksquare$$

$$\text{WIP Inventory(t)} = x_3^{eq} + \sum_{i=1}^{m}\sum_{j=1}^{n_i}\sum_{k=j}^{n_i}\left\{\left(\alpha_{ik} + \sum_{h=1}^{n_i-k+1}\frac{(-1)^{h-1}\beta_{i,k+h-1}}{(\lambda_i)^h}\right)u_{i,k-j+1,3}\frac{t^{j-1}}{(j-1)!}e^{\lambda_i t}\right\}$$

$$\text{WIP Inventory(t)} = 90 + \left(\alpha_{11} + \frac{\beta_{11}}{\lambda_1}\right)u_{113}e^{\lambda_1 t} + \left(\alpha_{21} - \frac{\beta_{22}}{(\lambda_2)^2} + \frac{\beta_{21}}{\lambda_2}\right)u_{213}e^{\lambda_2 t}$$

$$+ \left(\alpha_{22} + \frac{\beta_{22}}{\lambda_2}\right)u_{213}te^{\lambda_2 t} + \left(\alpha_{22} - \frac{\beta_{22}}{\lambda_2}\right)u_{223}e^{\lambda_2 t}$$

$$\text{WIP Inventory(t)} = 90 + \frac{35e^{-10t}}{16} - \frac{355e^{-2t}}{16} - \frac{85e^{-2t}t}{2} \ \blacksquare$$

The restriction of Theorem 2 is that requires the eigenvalues to be real numbers in order to obtain a vector $\mathbf{x}(t)$ in the real space. However, knowing that the eigenvalues are complex numbers (real numbers are a subset of the complex numbers with a zero imaginary part), it is necessary to develop a state trajectory equation that considers all type of eigenvalues. In order to do that, first it has to be proved that the eigenvalues, eigenvectors and constants of expression $\mathbf{x}(t)$ occur in conjugate pairs when they are complex numbers. This is done in Lemma 2. This facilitates the conversion of the state trajectory into an expression of pure real numbers as shown in Theorem 3. This new expression obtained in Theorem 3 decomposes $\mathbf{x}(t)$ into several modes of behavior (exponential growth, exponential decay, expanding oscillations, etc.) each characterized by an eigenvalue.

Lemma 2 Consider the system defined by $\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{b};\ \mathbf{x}(0) = \mathbf{x}_0$; where $\mathbf{x}(t) \in R^n$, $\mathbf{A} \in R^{n\,x\,n}$, $\mathbf{b} \in R^{n\,x\,1}$. If the complex eigenvalues of **A**, λ_z and $\lambda_{z'}$ are a conjugate pair corresponding to the Jordan blocks $\mathbf{J}_z$ and $\mathbf{J}_{z'}$ respectively, then the following are conjugate pairs:

i) $\mathbf{u}_{z,j}$ and $\mathbf{u}_{z',j}$

ii) $\alpha_{z,j}$ and $\alpha_{z',j}$

$\beta_{z,j}$ and $\beta_{z',j}$

iii) $\left(\alpha_{z,k} + \sum_{h=1}^{n_z-k+1} \frac{(-1)^{h-1}\beta_{z,k+h-1}}{(\lambda_z)^h}\right)\mathbf{u}_{z,k-j+1}$ and $\left(\alpha_{z',k} + \sum_{h=1}^{n_{z'}-k+1} \frac{(-1)^{h-1}\beta_{z',k+h-1}}{(\lambda_{z'})^h}\right)\mathbf{u}_{z',k-j+1}$

where $\alpha_{z,j} \in C$, $\beta_{z,j} \in C$ are constants defined as in Theorem 2, and $\mathbf{u}_{z,j}$, $\mathbf{u}_{z',j}$ are the corresponding generalized eigenvectors of λ_z and $\lambda_{z'}$.

Proof: Part i. Let $\lambda_z = c_z + d_z i,\quad \lambda_{z'} = c_z - d_z i,\quad \mathbf{u}_{z,j} = \mathbf{p}_{z,j} + \mathbf{q}_{z,j} i$ (4.46)

It has to be shown that $\mathbf{u}_{z',j} = \mathbf{p}_{z,j} - \mathbf{q}_{z,j} i$, j=1,..,$n_z$

<u>Case for j=1</u>

By equation (4.18)

$$\mathbf{A}\mathbf{u}_{z,1} = \lambda_z \mathbf{u}_{z,1} \tag{4.47}$$

Use of (4.46) for j=1 into (4.47) yields

$$\mathbf{A}(\mathbf{p}_{z,1} + \mathbf{q}_{z,1} i) = (c_z + d_z i)(\mathbf{p}_{z,1} + \mathbf{q}_{z,1} i)$$

$$\mathbf{A}\mathbf{p}_{z,1} + \mathbf{A}\mathbf{q}_{z,1} i = (c_z + d_z i)(\mathbf{p}_{z,1} + \mathbf{q}_{z,1} i)$$

$$= c_z \mathbf{p}_{z,1} + c_z \mathbf{q}_{z,1} i + d_z \mathbf{p}_{z,1} i - d_z \mathbf{q}_{z,1}$$

$$= (c_z \mathbf{p}_{z,1} - d_z \mathbf{q}_{z,1}) + (c_z \mathbf{q}_{z,1} + d_z \mathbf{p}_{z,1}) i$$

Then

$$\mathbf{A}\mathbf{p}_{z,1} = (c_z \mathbf{p}_{z,1} - d_z \mathbf{q}_{z,1}) \tag{4.48}$$

$$\mathbf{A}\mathbf{q}_{z,1} i = (c_z \mathbf{q}_{z,1} + d_z \mathbf{p}_{z,1}) i \tag{4.49}$$

Subtracting (4.48) minus (4.49)

$$\mathbf{A}\mathbf{p}_{z,1} - \mathbf{A}\mathbf{q}_{z,1} i = c_z \mathbf{p}_{z,1} - d_z \mathbf{q}_{z,1} - c_z \mathbf{q}_{z,1} i - d_z \mathbf{p}_{z,1} i$$

$$\mathbf{A}(\mathbf{p}_{z,1} - \mathbf{q}_{z,1} i) = (c_z - d_z i)\mathbf{p}_{z,1} - (d_z + c_z i)\mathbf{q}_{z,1}$$

$$= (c_{z,1} - d_{z,1} i)\mathbf{p}_z - (-d_z i^2 + c_z i)\mathbf{q}_z$$

$$= (c_z - d_z i)\mathbf{p}_z - (c_z - d_z i)\mathbf{q}_z i$$

$$= (c_z - d_z i)(\mathbf{p}_{z,1} - \mathbf{q}_{z,1} i)$$

It follows from (4.46) that $\lambda_{z'} = c_z - d_z i$, thus

$$\mathbf{A}(\mathbf{p}_{z,1} - \mathbf{q}_{z,1} i) = \lambda_{z'}(\mathbf{p}_{z,1} - \mathbf{q}_{z,1} i)$$

However, by comparing with equation (4.18) the term $\mathbf{p}_{z,1} - \mathbf{q}_{z,1} i$ is the eigenvector $\mathbf{u}_{z',1}$ associated to the eigenvalue $\lambda_{z'}$, therefore

$$\mathbf{u}_{z',1} = \mathbf{p}_{z,1} - \mathbf{q}_{z,1} i \ \blacksquare$$

<u>Case for j>1</u>

By equation (4.19)

$$\mathbf{A}\mathbf{u}_{z,j} = \lambda_i \mathbf{u}_{z,j} + \mathbf{u}_{z,j-1} \tag{4.50}$$

From (4.46) and (4.50) yields

$$\mathbf{A}(\mathbf{p}_{z,j} + \mathbf{q}_{z,j}i) = (c_z + d_z i)(\mathbf{p}_{z,j} + \mathbf{q}_{z,j}i) + \mathbf{p}_{z,j-1} + \mathbf{q}_{z,j-1}i$$

After some operations then

$$\mathbf{A}\mathbf{p}_{z,j} = (c_z\mathbf{p}_{z,j} - d_z\mathbf{q}_{z,j}) + \mathbf{p}_{z,j-1} \tag{4.51}$$

$$\mathbf{A}\mathbf{q}_{z,j}i = (c_z\mathbf{q}_{z,j} + d_z\mathbf{p}_{z,j})i + \mathbf{q}_{z,j-1}i \tag{4.52}$$

Subtracting (4.51) minus (4.52)

$$\mathbf{A}\mathbf{p}_{z,j} - \mathbf{A}\mathbf{q}_{z,j}i = c_z\mathbf{p}_{z,j} - d_z\mathbf{q}_{z,j} - c_z\mathbf{q}_{z,j}i - d_z\mathbf{p}_{z,j}i + \mathbf{p}_{z,j-1} - \mathbf{q}_{z,j-1}i$$

After simplifying it is obtained

$$\mathbf{A}(\mathbf{p}_{z,j} - \mathbf{q}_{z,j}i) = (c_z - d_z i)(\mathbf{p}_{z,j} - \mathbf{q}_{z,j}i) + \mathbf{p}_{z,j-1} - \mathbf{q}_{z,j-1}i$$

It follows from (4.46) that $\lambda_{z'} = c_z - d_z i$, thus

$$\mathbf{A}(\mathbf{p}_{z,j} - \mathbf{q}_{z,j}i) = \lambda_{z'}(\mathbf{p}_{z,j} - \mathbf{q}_{z,j}i) + (\mathbf{p}_{z,j-1} - \mathbf{q}_{z,j-1}i)$$

However, by comparing with equation (4.19) the term $\mathbf{p}_{z,j} - \mathbf{q}_{z,j}i$ is the generalized eigenvector $\mathbf{u}_{z',j}$ associated to the eigenvalue $\lambda_{z'}$, therefore

$$\mathbf{u}_{z',j} = \mathbf{p}_{z,j} - \mathbf{q}_{z,j}i \ \blacksquare$$

Part ii. First, it has to be shown that if $\mathbf{T}^{-1} = \begin{bmatrix} \mathbf{w}_1^J \\ \mathbf{w}_2^J \\ \vdots \\ \mathbf{w}_m^J \end{bmatrix}$ (4.53)

where $\mathbf{w}_z^J = \begin{bmatrix} \mathbf{w}_{z,1} \\ \mathbf{w}_{z,2} \\ \vdots \\ \mathbf{w}_{z,n_z} \end{bmatrix}$, $\mathbf{w}_{z,j} \in C^n$ are row vectors; then (4.54)

$$\mathbf{w}_{z,n_z}\mathbf{A} = \lambda_z \mathbf{w}_{z,n_z}$$

$$\mathbf{w}_{z,j}\mathbf{A} = \lambda_z \mathbf{w}_{z,j} + \mathbf{w}_{z,j+1}\text{, if } j<n_z$$

From Lemma 1-*part i*, it follows

$$\mathbf{A} = \mathbf{TJT}^{-1}$$

Rearranging terms

$$\mathbf{T}^{-1}\mathbf{A} = \mathbf{JT}^{-1}$$

$$\begin{bmatrix} \mathbf{w}_1^J \\ \mathbf{w}_2^J \\ \vdots \\ \mathbf{w}_m^J \end{bmatrix}\mathbf{A} = \begin{bmatrix} \mathbf{J}_1 & & & 0 \\ & \mathbf{J}_2 & & \\ & & \ddots & \\ 0 & & & \mathbf{J}_m \end{bmatrix}\begin{bmatrix} \mathbf{w}_1^J \\ \mathbf{w}_2^J \\ \vdots \\ \mathbf{w}_m^J \end{bmatrix}$$

$$\begin{bmatrix} \mathbf{w}_1^J\mathbf{A} \\ \mathbf{w}_2^J\mathbf{A} \\ \vdots \\ \mathbf{w}_m^J\mathbf{A} \end{bmatrix} = \begin{bmatrix} \mathbf{J}_1\mathbf{w}_1^J \\ \mathbf{J}_2\mathbf{w}_2^J \\ \vdots \\ \mathbf{J}_n\mathbf{w}_m^J \end{bmatrix} \text{ then } \mathbf{w}_z^J\mathbf{A} = \mathbf{J}_z\mathbf{w}_z^J\text{, z=1,..,m} \tag{4.55}$$

Substituting (4.14) and (4.54) into (4.55) gives

$$\begin{bmatrix} \mathbf{w}_{z,1} \\ \mathbf{w}_{z,2} \\ \vdots \\ \mathbf{w}_{z,n_z} \end{bmatrix}\mathbf{A} = \begin{bmatrix} \lambda_i & 1 & & & 0 \\ & \lambda_i & 1 & & \\ & & \lambda_i & \ddots & \\ & & & \ddots & 1 \\ 0 & & & & \lambda_i \end{bmatrix}\begin{bmatrix} \mathbf{w}_{z,1} \\ \mathbf{w}_{z,2} \\ \vdots \\ \mathbf{w}_{z,n_z} \end{bmatrix}$$

which implies that

$$\mathbf{w}_{z,n_z}\mathbf{A} = \lambda_z \mathbf{w}_{z,n_z} \tag{4.56}$$

$$\mathbf{w}_{z,j}\mathbf{A} = \lambda_z \mathbf{w}_{z,j} + \mathbf{w}_{z,j+1}\text{, if } j<n_z \tag{4.57}$$

Similar to *part i* of this lemma, using (4.56) and (4.57), it can be shown that $\mathbf{w}_{z,j}$ and $\mathbf{w}_{z',j}$ are a conjugate pair, if λ_z and $\lambda_{z'}$ are a conjugate pair of eigenvalues of $\mathbf{A}$ (4.58)

Theorem 2 shows that $\begin{bmatrix} \boldsymbol{\alpha}_1^J \\ \boldsymbol{\alpha}_2^J \\ \vdots \\ \boldsymbol{\alpha}_m^J \end{bmatrix} = \mathbf{T}^{-1}\mathbf{x}_0$ (4.59)

where $\boldsymbol{\alpha}_z^J = \begin{bmatrix} \alpha_{z,1} \\ \alpha_{z,2} \\ \vdots \\ \alpha_{z,n_z} \end{bmatrix}$ is a column vector of constants associated to $\mathbf{J}_z$ (4.60)

Substituting (4.53) into (4.59) yields

$$\begin{bmatrix} \boldsymbol{\alpha}_1^J \\ \boldsymbol{\alpha}_2^J \\ \vdots \\ \boldsymbol{\alpha}_m^J \end{bmatrix} = \begin{bmatrix} \mathbf{w}_1^J \\ \mathbf{w}_2^J \\ \vdots \\ \mathbf{w}_m^J \end{bmatrix} \mathbf{x}_0 = \begin{bmatrix} \mathbf{w}_1^J\mathbf{x}_0 \\ \mathbf{w}_2^J\mathbf{x}_0 \\ \vdots \\ \mathbf{w}_m^J\mathbf{x}_0 \end{bmatrix}$$

Thus

$$\boldsymbol{\alpha}_z^J = \mathbf{w}_z^J\mathbf{x}_0,\ z \in \{1,..,m\} \ \text{ and } \ \boldsymbol{\alpha}_{z'}^J = \mathbf{w}_{z'}^J\mathbf{x}_0,\ z' \in \{1,..,m\} - \{z\} \qquad (4.61)$$

From (4.54), (4.60) and (4.61), it can be obtained

$$\alpha_{z,j} = \mathbf{w}_{z,j}\mathbf{x}_0 \ \text{ and } \ \alpha_{z',j} = \mathbf{w}_{z',j}\mathbf{x}_0 \qquad (4.62)$$

From (4.58) it is known that $\mathbf{w}_{z,j}$ and $\mathbf{w}_{z',j}$ are a conjugate pair, then they can be expressed as:

$$\mathbf{w}_{z,j} = \mathbf{g}_{z,j} + \mathbf{v}_{z,j}i,\ \mathbf{w}_{z',j} = \mathbf{g}_{z,j} - \mathbf{v}_{z,j}i \qquad (4.63)$$

Using (4.62) and (4.63) leads to

$$\alpha_{z,j} = \mathbf{w}_{z,j}\mathbf{x}_0 = \mathbf{g}_{z,j}\mathbf{x}_0 + \mathbf{v}_{z,j}\mathbf{x}_0 i$$

$$\alpha_{z',j} = \mathbf{w}_{z',j}\mathbf{x}_0 = \mathbf{g}_{z,j}\mathbf{x}_0 - \mathbf{v}_{z,j}\mathbf{x}_0 i$$

This means that $\alpha_{z,j}$ and $\alpha_{z',j}$ are a conjugate pair too ■

Similarly,

It follows from Theorem 2 that $\begin{bmatrix} \boldsymbol{\beta}_1^J \\ \boldsymbol{\beta}_2^J \\ \vdots \\ \boldsymbol{\beta}_m^J \end{bmatrix} = \mathbf{T}^{-1}\mathbf{x}_0$ (4.64)

where $\beta_z^J = \begin{bmatrix} \beta_{z,1} \\ \beta_{z,2} \\ \vdots \\ \beta_{z,n_z} \end{bmatrix}$ is a column vector of constants associated to $\mathbf{J}_z$ (4.65)

Substituting (4.53) into (4.64) gives

$$\begin{bmatrix} \boldsymbol{\beta}_1^J \\ \boldsymbol{\beta}_2^J \\ \vdots \\ \boldsymbol{\beta}_m^J \end{bmatrix} = \begin{bmatrix} \mathbf{w}_1^J \\ \mathbf{w}_2^J \\ \vdots \\ \mathbf{w}_m^J \end{bmatrix} \mathbf{x}_0 = \begin{bmatrix} \mathbf{w}_1^J\mathbf{x}_0 \\ \mathbf{w}_2^J\mathbf{x}_0 \\ \vdots \\ \mathbf{w}_m^J\mathbf{x}_0 \end{bmatrix}$$

Thus

$$\boldsymbol{\beta}_z^J = \mathbf{w}_z^J\mathbf{x}_0,\ z \in \{1,..,m\} \text{ and } \boldsymbol{\beta}_{z'}^J = \mathbf{w}_{z'}^J\mathbf{x}_0,\ z' \in \{1,..,m\} - \{z\} \quad (4.66)$$

Substituting (4.54) and (4.65) into (4.66) yields

$$\beta_{z,j} = \mathbf{w}_{z,j}\mathbf{x}_0 \text{ and } \beta_{z',j} = \mathbf{w}_{z',j}\mathbf{x}_0$$

In a similar fashion that was proved that $\alpha_{z,j}$ and $\alpha_{z',j}$ are a conjugate pair it is possible to conclude that $\beta_{z,j}$ and $\beta_{z',j}$ are a conjugate pair too ■

Part iii. Let denote $\overline{y} \in C$ the conjugate of $y \in C$

$$\left(\alpha_{z,k} + \sum_{h=1}^{n_z-k+1} \frac{(-1)^{h-1}\beta_{z,k+h-1}}{(\lambda_z)^h}\right)\mathbf{u}_{z,k-j+1}$$

Because $\lambda_z, \lambda_{z'}$ are conjugate pairs and by the property of complex numbers: $\overline{(y)^h} = (\bar{y})^h$ then

$$(\lambda_z)^h \text{ and } (\lambda_{z'})^h \text{ are a conjugate pair} \tag{4.67}$$

From (4.67) and because $\beta_{z,k+h-1}, \beta_{z',k+h-1}$ are a conjugate pair (*part ii* of this lemma) and by the property of complex numbers: $\overline{\left(y_1/y_2\right)} = \overline{y_1}/\overline{y_2}$, $y_2 \neq 0$ then

$$\frac{\beta_{z,k+h-1}}{(\lambda_z)^h} \text{ and } \frac{\beta_{z',k+h-1}}{(\lambda_z)^h} \text{ are a conjugate pair} \tag{4.68}$$

From (4.68) and by the property of complex numbers: $\overline{cy} = c\bar{y}$, c=constant then

$$\frac{(-1)^{h-1}\beta_{z,k+h-1}}{(\lambda_z)^h} \text{ and } \frac{(-1)^{h-1}\beta_{z',k+h-1}}{(\lambda_{z'})^h} \text{ are a conjugate pair} \tag{4.69}$$

From (4.69) and because $\alpha_{z,k}$ and $\alpha_{z',k}$ are a conjugate pair, and by the property of complex numbers: $\overline{\left(\sum y\right)} = \sum \bar{y}$ then

$$\left(\alpha_{z,k} + \sum_{h=1}^{n_z-k+1} \frac{(-1)^{h-1}\beta_{z,k+h-1}}{(\lambda_z)^h}\right) \text{and} \left(\alpha_{z',k} + \sum_{h=1}^{n_{z'}-k+1} \frac{(-1)^{h-1}\beta_{z',k+h-1}}{(\lambda_{z'})^h}\right) \text{ are a conjugate pair} \tag{4.70}$$

From (4.70) and because $\mathbf{u}_{z,k-j+1}, \mathbf{u}_{z',k-j+1}$ are a conjugate pair (*part i* of this lemma), and by the property of complex numbers: $\overline{(y_1 y_2)} = \overline{y_1}\,\overline{y_2}$ then it can be deduced that

$\left(\alpha_{z,k} + \sum_{h=1}^{n_z-k+1} \frac{(-1)^{h-1}\beta_{z,k+h-1}}{(\lambda_z)^h}\right)\mathbf{u}_{z,k-j+1}$ and $\left(\alpha_{z',k} + \sum_{h=1}^{n_{z'}-k+1} \frac{(-1)^{h-1}\beta_{z',k+h-1}}{(\lambda_{z'})^h}\right)\mathbf{u}_{z',k-j+1}$ are a conjugate pair ■

Theorem 3 Consider the system defined by $\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{b};\ \mathbf{x}(0) = \mathbf{x}_0$; where $\mathbf{x}(t) \in R^n$, $\mathbf{A} \in R^{n\,x\,n}, \mathbf{b} \in R^{n\,x\,1}$. If matrix $\mathbf{A}$ has nonzero eigenvalues, then the solution to this system can be expressed as

$\mathbf{x}(t) = [x_s(t)], s = 1,..,n$

$$x_s(t) = x_s^{eq} + \sum_{i\in H_1}\sum_{j=1}^{n_i}\sum_{k=j}^{n_i}\left\{w_{ijks} t^{j-1} e^{\mathrm{Re}(\lambda_i)t}\right\} + \sum_{q\in H_2}\sum_{j=1}^{n_q}\sum_{k=j}^{n_q}\left\{v_{qjks}\sin\left(\mathrm{Im}(\lambda_q)t + \theta_{qjks}\right) t^{j-1} e^{\mathrm{Re}(\lambda_q)t}\right\}$$

where

$$w_{ijks} = \mathrm{Re}\left(\left(\alpha_{ik} + \sum_{h=1}^{n_i-k+1}\frac{(-1)^{h-1}\beta_{i,k+h-1}}{(\lambda_i)^h}\right)\frac{u_{i,k-j+1,s}}{(j-1)!}\right)$$

$$v_{qjks} = 2\left\|\left(\alpha_{qk} + \sum_{h=1}^{n_q-k+1}\frac{(-1)^{h-1}\beta_{q,k+h-1}}{(\lambda_q)^h}\right)\frac{u_{q,k-j+1,s}}{(j-1)!}\right\|$$, being $\|z\|$ the modulus[19] of $z \in C$

$x_s(t)$ denotes the state variable *s* of vector $\mathbf{x}(t)$, s=1,..,n

λ_p and $\mathbf{u}_{pj}$ are the corresponding eigenvalues and generalized eigenvectors of the p^{th} Jordan block, p=1,..,m; j=1,..,n_p

Re(z) and Im(z) mean the real and imaginary parts of $z \in C$

[19] Given z=a+bi then $\|z\| = \sqrt{a^2 + b^2}$

$$\theta_{qjks} = \arctan\left(\frac{\mathrm{Re}\left(\left(\alpha_{qk} + \sum_{h=1}^{n_q-k+1}\frac{(-1)^{h-1}\beta_{q,k+h-1}}{(\lambda_q)^h}\right)u_{q,k-j+1,s}\right)}{-\mathrm{Im}\left(\left(\alpha_{qk} + \sum_{h=1}^{n_q-k+1}\frac{(-1)^{h-1}\beta_{q,k+h-1}}{(\lambda_q)^h}\right)u_{q,k-j+1,s}\right)}\right), \text{ expressed in radians}$$

H_1 is a set of Jordan blocks J_i such that $\mathrm{Im}(\lambda_i) = 0$

H_2 is a set of pair of Jordan blocks $\{J_q, J_{q'}\}$ such that $\mathrm{Im}(\lambda_q) \neq 0$, where *q* denotes the conjugate pair of eigenvalues λ_q and $\lambda_{q'}$, i.e. one index *q* represents two eigenvalues. Therefore, $\mathrm{Re}(\lambda_q) = \mathrm{Re}(\lambda_{q'})$ and $\mathrm{Im}(\lambda_q) = \mathrm{Im}(\lambda_{q'})$

Proof: Dividing equation (4.25) in two parts associated to sets H_1 and $H\text{-}H_1$:

$$x_s(t) = x_s^{eq} + \sum_{i\in H_1}\sum_{j=1}^{n_i}\sum_{k=j}^{n_i}\left\{\left(\alpha_{ik} + \sum_{h=1}^{n_i-k+1}\frac{(-1)^{h-1}\beta_{i,k+h-1}}{(\lambda_i)^h}\right)u_{i,k-j+1,s}\frac{t^{j-1}}{(j-1)!}e^{\lambda_i t}\right\} +$$

$$\sum_{i\in\{H-H_1\}}\sum_{j=1}^{n_i}\sum_{k=j}^{n_i}\left\{\left(\alpha_{ik} + \sum_{h=1}^{n_i-k+1}\frac{(-1)^{h-1}\beta_{i,k+h-1}}{(\lambda_i)^h}\right)u_{i,k-j+1,s}\frac{t^{j-1}}{(j-1)!}e^{\lambda_i t}\right\} \tag{4.71}$$

where H_1 is defined as in the text of this theorem and H is the set of indexes that represent all the Jordan blocks of matrix **J**; thus H={1,..,m}.

All the terms in the first summation of equation (4.71) are real numbers (which are a subset of the complex numbers); thus, summation

$$\sum_{i\in H_1}\sum_{j=1}^{n_i}\sum_{k=j}^{n_i}\left\{\left(\alpha_{ik} + \sum_{h=1}^{n_i-k+1}\frac{(-1)^{h-1}\beta_{i,k+h-1}}{(\lambda_i)^h}\right)u_{i,k-j+1,s}\frac{t^{j-1}}{(j-1)!}e^{\lambda_i t}\right\} \text{ is equivalent to}$$

$$\sum_{i\in H_1}\sum_{j=1}^{n_i}\sum_{k=j}^{n_i}\left\{\mathrm{Re}\left(\left(\alpha_{ik} + \sum_{h=1}^{n_i-k+1}\frac{(-1)^{h-1}\beta_{i,k+h-1}}{(\lambda_i)^h}\right)\frac{u_{i,k-j+1,s}}{(j-1)!}\right)t^{j-1}e^{\mathrm{Re}(\lambda_i)t}\right\} \tag{4.72}$$

Now it is necessary to obtain an expression in terms of pure real numbers for the complex numbers in the second summation of equation (4.71) (whose imaginary part is different from zero).

Sets $\{H\text{-}H_1\}$ and H_2 (as defined in the text of this theorem) point the same Jordan blocks; thus, the cardinality of $\{H\text{-}H_1\}$ is twice the cardinality of H_2.

For any index $q \in H_2$, the conjugate eigenvalues represented by q as can be expresses as

$$\lambda_q = c_q + d_q \hat{\imath}$$

$$\lambda_{q'} = c_q - d_q \hat{\imath}$$

Also, as the terms $\left(\alpha_{qk} + \sum_{h=1}^{n_q-k+1} \frac{(-1)^{h-1}\beta_{q,k+h-1}}{(\lambda_q)^h} \right)$ and $\left(\alpha_{q'k} + \sum_{h=1}^{n_{q'}-k+1} \frac{(-1)^{h-1}\beta_{q',k+h-1}}{(\lambda_{q'})^h} \right)$ are conjugates (by Lemma 2), the following terms can be written them as

$$\left(\alpha_{qk} + \sum_{h=1}^{n_q-k+1} \frac{(-1)^{h-1}\beta_{q,k+h-1}}{(\lambda_q)^h} \right) \frac{u_{q,k-j+1,s}}{(j-1)!} = f_{qjks} + g_{qjks}\hat{\imath}$$

$$\left(\alpha_{q'k} + \sum_{h=1}^{n_{q'}-k+1} \frac{(-1)^{h-1}\beta_{q',k+h-1}}{(\lambda_{q'})^h} \right) \frac{u_{q',k-j+1,s}}{(j-1)!} = f_{qjks} - g_{qjks}\hat{\imath}$$

Using the conjugate terms defined above, the summation

$$\sum_{i\in\{H-H_1\}} \sum_{j=1}^{n_i} \sum_{k=j}^{n_i} \left\{ \left(\alpha_{ik} + \sum_{h=1}^{n_i-k+1} \frac{(-1)^{h-1}\beta_{i,k+h-1}}{(\lambda_i)^h} \right) u_{i,k-j+1,s} \frac{t^{j-1}}{(j-1)!} e^{\lambda_i t} \right\}$$ can be expressed as

$$\sum_{q\in H_2} \sum_{j=1}^{n_q} \sum_{k=j}^{n_q} \left\{ (f_{qjks} + g_{qjks}\hat{\imath}) e^{(c_q + d_q \hat{\imath})t} t^{j-1} + (f_{qjks} - g_{qjks}\hat{\imath}) e^{(c_q - d_q \hat{\imath})t} t^{j-1} \right\}$$

$$= \sum_{q\in H_2} \sum_{j=1}^{n_q} \sum_{k=j}^{n_q} \left\{ \left(f_{qjks} + g_{qjks}\hat{\imath}\right) e^{c_q t} e^{d_q \hat{\imath} t} t^{j-1} + \left(f_{qjks} - g_{qjks}\hat{\imath}\right) e^{c_q t} e^{-d_q \hat{\imath} t} t^{j-1} \right\}$$

By Euler's formula (Spivak 1967) it is known that $e^{\gamma\hat{\imath}} = \cos(\gamma) + \sin(\gamma)\hat{\imath}$; thus replacing this formula in the expression above:

$$= \sum_{q \in H_2} \sum_{j=1}^{n_q} \sum_{k=j}^{n_q} \left\{ (f_{qjks} + g_{qjks}\hat{\imath}) e^{c_q t} t^{j-1} \left(\cos(d_q t) + \sin(d_q t)\hat{\imath} \right) + (f_{qjks} - g_{qjks}\hat{\imath}) e^{c_q t} t^{j-1} \left(\cos(-d_q t) + \sin(-d_q t)\hat{\imath} \right) \right\}$$

$$= \sum_{q \in H_2} \sum_{j=1}^{n_q} \sum_{k=j}^{n_q} \left\{ (f_{qjks} + g_{qjks}\hat{\imath}) e^{c_q t} t^{j-1} \left(\cos(d_q t) + \sin(d_q t)\hat{\imath} \right) + (f_{qjks} - g_{qjks}\hat{\imath}) e^{c_q t} t^{j-1} \left(\cos(d_q t) - \sin(d_q t)\hat{\imath} \right) \right\}$$

Simplifying:

$$= \sum_{q \in H_2} \sum_{j=1}^{n_q} \sum_{k=j}^{n_q} \left\{ e^{c_q t} t^{j-1} \left(2f_{qjks} \cos(d_q t) - 2g_{qjks} \sin(d_q t) \right) \right\}$$

$$= \sum_{q \in H_2} \sum_{j-1}^{n_q} \sum_{k=j}^{n_q} \left\{ 2e^{c_q t} t^{j-1} \left(f_{qjks} \cos(d_q t) - g_{qjks} \sin(d_q t) \right) \right\} \tag{4.73}$$

Making $\theta_{qjks} = \arctan\left(\dfrac{f_{qjks}}{-g_{qjks}} \right)$ then

$$\sin(\theta_{qjks}) = \left(\frac{f_{qjks}}{\sqrt{(f_{qjks})^2 + (g_{qjks})^2}} \right) \text{ and } \cos(\theta_{qjks}) = \left(\frac{-g_{qjks}}{\sqrt{(f_{qjks})^2 + (g_{qjks})^2}} \right) \tag{4.74}$$

Multiplying and dividing (4.73) by $\sqrt{(f_{qjks})^2 + (g_{qjks})^2}$ and replacing the terms of (4.74)

$$= \sum_{q \in H_2} \sum_{j=1}^{n_q} \sum_{k=j}^{n_q} \left\{ 2\sqrt{(f_{qjks})^2 + (g_{qjks})^2}\ e^{c_q t} t^{j-1} \left(\sin(\theta_{qjks}) \cos(d_q t) + \cos(\theta_{qjks}) \sin(d_q t) \right) \right\}$$

$$= \sum_{q \in H_2} \sum_{j=1}^{n_q} \sum_{k=j}^{n_q} \left\{ 2\sqrt{(f_{qjks})^2 + (g_{qjks})^2}\ e^{c_q t} t^{j-1} \sin(d_q t + \theta_{qjks}) \right\} \tag{4.75}$$

It is known that

$$\sqrt{(f_{qjks})^2+(g_{qjks})^2}=\left\|f_{qjks}+g_{qjks}\hat{\imath}\right\|=\left\|\left(\alpha_{qk}+\sum_{h=1}^{n_q-k+1}\frac{(-1)^{h-1}\beta_{q,k+h-1}}{(\lambda_q)^h}\right)\frac{u_{q,k-j+1,s}}{(j-1)!}\right\|$$

Also, it is clear that

$$c_q=\mathrm{Re}(\lambda_q),\, d_q=\mathrm{Im}(\lambda_q)\,,\, f_{qjks}=\mathrm{Re}\left(\left(\alpha_{qk}+\sum_{h=1}^{n_q-k+1}\frac{(-1)^{h-1}\beta_{q,k+h-1}}{(\lambda_q)^h}\right)\frac{u_{q,k-j+1,s}}{(j-1)!}\right) \text{ and}$$

$$g_{qjks}=\mathrm{Im}\left(\left(\alpha_{qk}+\sum_{h=1}^{n_q-k+1}\frac{(-1)^{h-1}\beta_{q,k+h-1}}{(\lambda_q)^h}\right)\frac{u_{q,k-j+1,s}}{(j-1)!}\right) \tag{4.76}$$

Thus, $\theta_{qjks}=\arctan\left(\dfrac{f_{qjks}}{-g_{qjks}}\right)=\arctan\left(\dfrac{\mathrm{Re}\left(\left(\alpha_{qk}+\sum_{h=1}^{n_q-k+1}\frac{(-1)^{h-1}\beta_{q,k+h-1}}{(\lambda_q)^h}\right)u_{q,k-j+1,s}\right)}{-\mathrm{Im}\left(\left(\alpha_{qk}+\sum_{h=1}^{n_q-k+1}\frac{(-1)^{h-1}\beta_{q,k+h-1}}{(\lambda_q)^h}\right)u_{q,k-j+1,s}\right)}\right)$

Making $2\left\|\left(\alpha_{qk}+\sum_{h=1}^{n_q-k+1}\frac{(-1)^{h-1}\beta_{q,k+h-1}}{(\lambda_q)^h}\right)\frac{u_{q,k-j+1,s}}{(j-1)!}\right\|=v_{qjks}$ and replacing (4.76) in (4.75) it follows

that:

$$\sum_{i\in\{H-H_1\}}\sum_{j=1}^{n_i}\sum_{k=j}^{n_i}\left\{\left(\alpha_{ik}+\sum_{h=1}^{n_i-k+1}\frac{(-1)^{h-1}\beta_{i,k+h-1}}{(\lambda_i)^h}\right)u_{i,k-j+1,s}\frac{t^{j-1}}{(j-1)!}e^{\lambda_i t}\right\}=$$

$$\sum_{q\in H_2}\sum_{j=1}^{n_q}\sum_{k=j}^{n_q}\left\{v_{qjks}\sin\left(\mathrm{Im}(\lambda_q)t+\theta_{qjks}\right)t^{j-1}e^{\mathrm{Re}(\lambda_q)t}\right\} \tag{4.77}$$

From (4.72) making $w_{ijks}=\mathrm{Re}\left(\left(\alpha_{ik}+\sum_{h=1}^{n_i-k+1}\frac{(-1)^{h-1}\beta_{i,k+h-1}}{(\lambda_i)^h}\right)\frac{u_{i,k-j+1,s}}{(j-1)!}\right)$ and substituting this term

and (4.77) into (4.71) it is demonstrated that

$$x_s(t) = x_s^{eq} + \sum_{i\in H_1}\sum_{j=1}^{n_i}\sum_{k=j}^{n_i}\left\{w_{ijks}t^{j-1}e^{Re(\lambda_i)t}\right\} + \sum_{q\in H_2}\sum_{j=1}^{n_q}\sum_{k=j}^{n_q}\left\{v_{qjks}\sin\left(Im(\lambda_q)t+\theta_{qjks}\right)t^{j-1}e^{Re(\lambda_q)t}\right\} \blacksquare$$

Example 3 Consider the manufacturing supply chain presented in Example 1. Considering the same initial conditions and the parameters shown in Table 4.2, obtain the trajectory equations for the state variables of this system using the results of Theorem 3.

Table 4.2. Parameter values for the supply chain of Example 3

Parameter	Value	Unit
Demand	200	Units
Production Delay	1	Weeks
Shipment Delay	1	Weeks
Desired Inventory	100	Units
Production Adjustment Time	1	Weeks
Time to Average Demand	1	Weeks

Solution: The model equations of this supply chain can be expressed as the linear system $\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{b};\ \mathbf{x}(0) = \mathbf{x}_0$, where

$$\mathbf{x}(t) = \begin{bmatrix} \text{Expected Demand}(t) \\ \text{Inventory}(t) \\ \text{WIP Inventory}(t) \end{bmatrix}, \mathbf{A} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 1 \\ 1 & -1 & -1 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} 200 \\ 0 \\ 100 \end{bmatrix}, \mathbf{x}_0 = \begin{bmatrix} 0 \\ 50 \\ 70 \end{bmatrix}$$

The eigenvalues of **A** are: $\lambda_1 = -1, \lambda_2 = -1 - \hat{\imath}, \lambda_3 = -1 + \hat{\imath}$ (4.78)

Thus, there are three Jordan blocks (m=3) of dimension n_i=1 (i=1,..,3) associated to each of the eigenvalues. The Jordan canonical form is given by $\mathbf{J} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1-\hat{\imath} & 0 \\ 0 & 0 & -1+\hat{\imath} \end{bmatrix}$.

The matrix of generalized eigenvectors is:

$$\mathbf{T} = [\mathbf{u}_{11} \quad \mathbf{u}_{21} \quad \mathbf{u}_{31}] = \begin{bmatrix} 1 & 0 & 0 \\ 1 & \hat{\imath} & -\hat{\imath} \\ 0 & 1 & 1 \end{bmatrix}$$

The constants are found as follows:

$$\begin{bmatrix} \alpha_{11} \\ \alpha_{21} \\ \alpha_{31} \end{bmatrix} = \mathbf{T}^{-1}\mathbf{x}_0 = \begin{bmatrix} 0 \\ 35 - 25\hat{\imath} \\ 35 + 25\hat{\imath} \end{bmatrix}$$

$$\begin{bmatrix} \beta_{11} \\ \beta_{21} \\ \beta_{31} \end{bmatrix} = \mathbf{T}^{-1}\mathbf{b} = \begin{bmatrix} 200 \\ 50 + 100\hat{\imath} \\ 50 - 100\hat{\imath} \end{bmatrix}$$

The following terms are obtained as follows:

$$\left(\alpha_{11} + \frac{\beta_{11}}{\lambda_1}\right)\mathbf{u}_{11} = \begin{bmatrix} -200 \\ -200 \\ 0 \end{bmatrix}, \left(\alpha_{21} + \frac{\beta_{21}}{\lambda_2}\right)\mathbf{u}_{21} = \begin{bmatrix} 0 \\ 50 - 40\hat{\imath} \\ -40 - 50\hat{\imath} \end{bmatrix}, \left(\alpha_{31} + \frac{\beta_{31}}{\lambda_3}\right)\mathbf{u}_{31} = \begin{bmatrix} 0 \\ 50 + 40\hat{\imath} \\ -40 + 50\hat{\imath} \end{bmatrix} \tag{4.79}$$

$$\frac{\beta_{11}}{\lambda_1}\mathbf{u}_{11} = \begin{bmatrix} -200 \\ -200 \\ 0 \end{bmatrix}, \ \frac{\beta_{21}}{\lambda_2}\mathbf{u}_{21} = \begin{bmatrix} 0 \\ 25 - 75\hat{\imath} \\ -75 - 25\hat{\imath} \end{bmatrix}, \ \frac{\beta_{31}}{\lambda_3}\mathbf{u}_{31} = \begin{bmatrix} 0 \\ 25 + 75\hat{\imath} \\ -75 + 25\hat{\imath} \end{bmatrix} \tag{4.80}$$

$$\text{Let } \mathbf{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} = \begin{bmatrix} \text{Expected Demand}(t) \\ \text{Inventory}(t) \\ \text{WIP Inventory}(t) \end{bmatrix}$$

The equilibrium point is calculated using (4.43) and (4.80), for m=3, n_i=1, i=1,..,3.

$$\mathbf{x}^{eq} = -\sum_{i=1}^{m}\sum_{j=1}^{n_i}\sum_{k=1}^{j}\left\{\frac{(-1)^{j-k}\beta_{ij}}{(\lambda_i)^{j-k+1}}\mathbf{u}_{ik}\right\} = -\frac{\beta_{11}}{\lambda_1}\mathbf{u}_{11} - \frac{\beta_{21}}{\lambda_2}\mathbf{u}_{21} - \frac{\beta_{31}}{\lambda_3}\mathbf{u}_{31} \tag{4.81}$$

Substituting the values above in (4.81) it is obtained

$$\mathbf{x}^{eq} = \begin{bmatrix} x_1^{eq} \\ x_2^{eq} \\ x_3^{eq} \end{bmatrix} = \begin{bmatrix} 200 \\ 150 \\ 150 \end{bmatrix}$$

From (4.78) it is known know that $H_1 = \{1\}, H_2 = \{2\}$. Variables Expected Demand(t), Inventory(t) and WIP Inventory(t) are calculated substituting the values above in the equation of Theorem 3, for n_1=1, n_3=1, s=1,..,3.

Calculating Expected Demand(t)

$$\text{Expected Demand}(t) = x_1^{eq} + \sum_{i \in H_1} \sum_{j=1}^{n_i} \sum_{k=j}^{n_i} \left\{ w_{ijk1} t^{j-1} e^{\mathrm{Re}(\lambda_i)t} \right\}$$
$$+ \sum_{q \in H_2} \sum_{j=1}^{n_q} \sum_{k=j}^{n_q} \left\{ v_{qjk1} \sin\left(\mathrm{Im}(\lambda_q)t + \theta_{qjk1}\right) t^{j-1} e^{\mathrm{Re}(\lambda_q)t} \right\}$$

$$\text{Expected Demand}(t) = -200 + w_{1111} e^{\mathrm{Re}(\lambda_1)t} + v_{2111} \sin\left(\mathrm{Im}(\lambda_2)t + \theta_{2111}\right) e^{\mathrm{Re}(\lambda_2)t}$$

where

$$w_{1111} = \mathrm{Re}\left(\alpha_{11} u_{111} + \frac{\beta_{11}}{\lambda_1} u_{111} \right) = -200$$

$$f_{2111} = 0,\, g_{2111} = 0 \Rightarrow \theta_{2111} = \arctan\left(\frac{f_{2111}}{-g_{2111}} \right) = \arctan(0) = 0$$

$$v_{2111} = 2\sqrt{f_{2111}^{\ 2} + g_{2111}^{\ 2}} = 0$$

Thus,

$$\text{Expected Demand}(t) = 200 - 200e^{-t} \quad \blacksquare$$

It can be verified that

$$\text{Expected Demand}(0) = 200 - 200 = 0$$

Calculating Inventory(t)

$$\text{Inventory(t)} = x_2^{eq} + \sum_{i\in H_1}\sum_{j=1}^{n_i}\sum_{k=j}^{n_i}\left\{w_{ijk2}t^{j-1}e^{\mathrm{Re}(\lambda_i)t}\right\}$$
$$+\sum_{q\in H_2}\sum_{j=1}^{n_q}\sum_{k=j}^{n_q}\left\{v_{qjk2}\sin\left(\mathrm{Im}(\lambda_q)t+\theta_{qjk2}\right)t^{j-1}e^{\mathrm{Re}(\lambda_q)t}\right\}$$

$$\text{Inventory(t)} = 150 + w_{1112}e^{\mathrm{Re}(\lambda_1)t} + v_{2112}\sin\left(\mathrm{Im}(\lambda_2)t+\theta_{2112}\right)e^{\mathrm{Re}(\lambda_2)t}$$

where

$$w_{1112} = \mathrm{Re}\left(\alpha_{11}u_{112} + \frac{\beta_{11}}{\lambda_1}u_{112}\right) = -200$$

$$f_{2112} = 50, g_{2112} = -40 \Rightarrow \theta_{2112} = \arctan\left(\frac{f_{2112}}{-g_{2112}}\right) = \arctan\left(\frac{5}{4}\right) = 0.8961$$

$$v_{2112} = 2\sqrt{{f_{2112}}^2 + {g_{2112}}^2} = 128.0625$$

Thus,

$$\text{Inventory(t)} = 150 - 200e^{-t} + 128.0625\, e^{-t}\sin\left(-t + 0.8961\right) \blacksquare$$

It can be verified that

$$\text{Inventory(0)} = 150 - 200 + 128.0625\sin\left(0.8961\right) = 50$$

Calculating WIP Inventory(t)

$$\text{WIP Inventory(t)} = x_3^{eq} + \sum_{i\in H_1}\sum_{j=1}^{n_i}\sum_{k=j}^{n_i}\left\{w_{ijk3}t^{j-1}e^{\mathrm{Re}(\lambda_i)t}\right\}$$
$$+\sum_{q\in H_2}\sum_{j=1}^{n_q}\sum_{k=j}^{n_q}\left\{v_{qjk3}\sin\left(\mathrm{Im}(\lambda_q)t+\theta_{qjk3}\right)t^{j-1}e^{\mathrm{Re}(\lambda_q)t}\right\}$$

$$\text{WIP Inventory(t)} = 150 + w_{1113}e^{\mathrm{Re}(\lambda_1)t} + v_{2113}\sin\left(\mathrm{Im}(\lambda_2)t+\theta_{2113}\right)e^{\mathrm{Re}(\lambda_2)t}$$

where

$$w_{1113} = \mathrm{Re}\left(\alpha_{11} u_{113} + \frac{\beta_{11}}{\lambda_1} u_{213} \right) = 0$$

$$f_{2113} = -40, g_{2113} = -50 \Rightarrow \theta_{2113} = \arctan\left(\frac{f_{2113}}{-g_{2113}} \right) = \arctan\left(\frac{-4}{5} \right) = -0.6747$$

$$v_{2113} = 2\sqrt{f_{2113}{}^2 + g_{2113}{}^2} = 128.0625$$

Thus,

$$\text{WIP Inventory(t)} = 150 + 128.0625\, e^{-t} \sin(-t - 0.6747) \ \blacksquare$$

It can be verified that

$$\text{WIP Inventory(0)} = 150 + 128.0625 \sin(-0.6747) = 70$$

From the state variables equations the solution of the system is obtained

$$\mathbf{x}(t) = \begin{bmatrix} \text{Expected Demand(t)} \\ \text{Inventory(t)} \\ \text{WIP Inventory(t)} \end{bmatrix} = \begin{bmatrix} (200 - 200e^{-t}) \\ (150 - 200e^{-t} + 128.0625\, e^{-t} \sin(-t + 0.8961)) \\ (150 + 128.0625\, e^{-t} \sin(-t - 0.6747)) \end{bmatrix} \ \blacksquare$$

4.2.2. Conditions for linear stability

In this section the concept of ADE is applied in order to achieve stability of linear systems. It is demonstrated that if ADE converge then the system is asymptotically stable. In order to do that, first, it is proved in Theorem 4 that the convergence of ADE assures the convergence of the state variable trajectory to the equilibrium point.

Theorem 4 Consider the system defined by $\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{b}$; $\mathbf{x}(0) = \mathbf{x}_0$; where

$\mathbf{x}(t) \in R^n$, $\mathbf{x}(t) = [x_s(t)]$, $s = 1,..,n$; $\mathbf{A} \in R^{n\,x\,n}$, $\mathbf{b} \in R^{n\,x\,1}$. The state variable x_s is asymptotically stable (around the EP x_s^{eq}), if $\int_0^\infty \left|x_s(t) - x_s^{eq}\right| dt$ converges.

Proof: As initial step, it will be proved by contradiction that if $\int_0^\infty \left|x_s(t) - x_s^{eq}\right| dt$ converges then the real part of all the eigenvalues of **A** has to be negative. (4.82)

Thus, assume to contrary that there is at least one eigenvalue of **A** with a real part greater than or equal to zero.

Recalling the following equation for $x_s(t)$ from Theorem 3

$$x_s(t) = x_s^{eq} + \sum_{i \in H_1} \sum_{j=1}^{n_i} \sum_{k=j}^{n_i} \left\{ w_{ijks} t^{j-1} e^{\mathrm{Re}(\lambda_i)t} \right\} + \sum_{q \in H_2} \sum_{j=1}^{n_q} \sum_{k=j}^{n_q} \left\{ v_{qjks} \sin\left(\mathrm{Im}(\lambda_q)t + \theta_{qjks}\right) t^{j-1} e^{\mathrm{Re}(\lambda_q)t} \right\}$$

After rearranging terms

$$x_s(t) - x_s^{eq} = \sum_{i \in H_1} \sum_{j=1}^{n_i} \sum_{k=j}^{n_i} \left\{ w_{ijks} t^{j-1} e^{\mathrm{Re}(\lambda_i)t} \right\} + \sum_{q \in H_2} \sum_{j=1}^{n_q} \sum_{k=j}^{n_q} \left\{ v_{qjks} \sin\left(\mathrm{Im}(\lambda_q)t + \theta_{qjks}\right) t^{j-1} e^{\mathrm{Re}(\lambda_q)t} \right\} \tag{4.83}$$

Taking absolute value in both sides and rearranging terms again

$$\left| \sum_{i \in H_1} \sum_{j=1}^{n_i} \sum_{k=j}^{n_i} \left\{ w_{ijks} t^{j-1} e^{\mathrm{Re}(\lambda_i)t} \right\} + \sum_{q \in H_2} \sum_{j=1}^{n_q} \sum_{k=j}^{n_q} \left\{ v_{qjks} \sin\left(\mathrm{Im}(\lambda_q)t + \theta_{qjks}\right) t^{j-1} e^{\mathrm{Re}(\lambda_q)t} \right\} \right| = \left| x_s(t) - x_s^{eq} \right|$$

By the property of absolute value: $\left\| a \right| - \left| b \right\| \le \left| a + b \right|$

$$\left\| \sum_{i \in H_1} \sum_{j=1}^{n_i} \sum_{k=j}^{n_i} \left\{ w_{ijks} t^{j-1} e^{\mathrm{Re}(\lambda_i)t} \right\} \right| - \left| \sum_{q \in H_2} \sum_{j=1}^{n_q} \sum_{k=j}^{n_q} \left\{ v_{qjks} \sin\left(\mathrm{Im}(\lambda_q)t + \theta_{qjks}\right) t^{j-1} e^{\mathrm{Re}(\lambda_q)t} \right\} \right\| \le \left| x_s(t) - x_s^{eq} \right|$$

Applying the inequality property ($-\sum |a_i| \le -\left|\sum a_i\right|$) to the second summation

$$\left|\left|\sum_{i\in H_1}\sum_{j=1}^{n_i}\sum_{k=j}^{n_i}\left\{w_{ijks}t^{j-1}e^{\mathrm{Re}(\lambda_i)t}\right\}\right|-\sum_{q\in H_2}\sum_{j=1}^{n_q}\sum_{k=j}^{n_q}\left|v_{qjks}\sin\left(\mathrm{Im}(\lambda_q)t+\theta_{qjks}\right)t^{j-1}e^{\mathrm{Re}(\lambda_q)t}\right|\right|\le\left|x_s(t)-x_s^{eq}\right|$$

(4.84)

By applying the property: $|\sin(t)|\le 1, \forall t\in R$ to the second summation

$$\sum_{q\in H_2}\sum_{j=1}^{n_q}\sum_{k=j}^{n_q}\left|v_{qjks}\sin\left(\mathrm{Im}(\lambda_q)t+\theta_{qjks}\right)t^{j-1}e^{\mathrm{Re}(\lambda_q)t}\right|\le\sum_{q\in H_2}\sum_{j=1}^{n_q}\sum_{k=j}^{n_q}\left|v_{qjks}t^{j-1}e^{\mathrm{Re}(\lambda_q)t}\right|$$

Because $v_{qjks}t^{j-1}e^{\mathrm{Re}(\lambda_q)t}\ge 0, \forall t\ge 0$ then the absolute vale can be removed from the second summation

$$\sum_{q\in H_2}\sum_{j=1}^{n_q}\sum_{k=j}^{n_q}\left|v_{qjks}\sin\left(\mathrm{Im}(\lambda_q)t+\theta_{qjks}\right)t^{j-1}e^{\mathrm{Re}(\lambda_q)t}\right|\le\sum_{q\in H_2}\sum_{j=1}^{n_q}\sum_{k=j}^{n_q}\left\{v_{qjks}t^{j-1}e^{\mathrm{Re}(\lambda_q)t}\right\}$$

Multiplying by (-1) both terms

$$-\sum_{q\in H_2}\sum_{j=1}^{n_q}\sum_{k=j}^{n_q}\left\{v_{qjks}t^{j-1}e^{\mathrm{Re}(\lambda_q)t}\right\}\le-\sum_{q\in H_2}\sum_{j=1}^{n_q}\sum_{k=j}^{n_q}\left|v_{qjks}\sin\left(\mathrm{Im}(\lambda_q)t+\theta_{qjks}\right)t^{j-1}e^{\mathrm{Re}(\lambda_q)t}\right| \qquad (4.85)$$

From inequalities (4.84) and (4.85) it follows

$$\left|\left|\sum_{i\in H_1}\sum_{j=1}^{n_i}\sum_{k=j}^{n_i}\left\{w_{ijks}t^{j-1}e^{\mathrm{Re}(\lambda_i)t}\right\}\right|-\sum_{q\in H_2}\sum_{j=1}^{n_q}\sum_{k=j}^{n_q}\left\{v_{qjks}t^{j-1}e^{\mathrm{Re}(\lambda_q)t}\right\}\right|\le\left|x_s(t)-x_s^{eq}\right| \qquad (4.86)$$

Making $h(t)=\left|x_s(t)-x_s^{eq}\right|$ and $g(t)=\left|\left|\sum_{i\in H_1}\sum_{j=1}^{n_i}\sum_{k=j}^{n_i}\left\{w_{ijks}t^{j-1}e^{\mathrm{Re}(\lambda_i)t}\right\}\right|-\sum_{q\in H_2}\sum_{j=1}^{n_q}\sum_{k=j}^{n_q}\left\{v_{qjks}t^{j-1}e^{\mathrm{Re}(\lambda_q)t}\right\}\right|$

satisfies the hypothesis of Lemma A.3, i.e. g(t) and h(t) are continuous functions on [0,∞), and $0\le g(t)\le h(t)$.

Integrating g(t) from zero to infinity

$$\int_0^\infty g(t)\,dt = \int_0^\infty \left\| \sum_{i\in H_1}\sum_{j=1}^{n_i}\sum_{k=j}^{n_i}\left\{w_{ijks} t^{j-1} e^{\mathrm{Re}(\lambda_i)t}\right\} \right| - \sum_{q\in H_2}\sum_{j=1}^{n_q}\sum_{k=j}^{n_q}\left\{v_{qjks} t^{j-1} e^{\mathrm{Re}(\lambda_q)t}\right\} \Bigg| dt$$

By the property of absolute value: $\int |w(t)\,dt| \geq \left|\int w(t)\,dt\right|$

$$\int_0^\infty g(t)\,dt \geq \left| \int_0^\infty \left| \sum_{i\in H_1}\sum_{j=1}^{n_i}\sum_{k=j}^{n_i}\left\{w_{ijks} t^{j-1} e^{\mathrm{Re}(\lambda_i)t}\right\} \right| - \sum_{q\in H_2}\sum_{j=1}^{n_q}\sum_{k=j}^{n_q}\left\{v_{qjks} t^{j-1} e^{\mathrm{Re}(\lambda_q)t}\right\} dt \right|$$

By the property of integrals: $\int (a-b)\,dt = \int a\,dt - \int b\,dt$

$$\int_0^\infty g(t)\,dt \geq \left| \int_0^\infty \left| \sum_{i\in H_1}\sum_{j=1}^{n_i}\sum_{k=j}^{n_i}\left\{w_{ijks} t^{j-1} e^{\mathrm{Re}(\lambda_i)t}\right\} \right| dt - \int_0^\infty \sum_{q\in H_2}\sum_{j=1}^{n_q}\sum_{k=j}^{n_q}\left\{v_{qjks} t^{j-1} e^{\mathrm{Re}(\lambda_q)t}\right\} dt \right|$$

Applying the property: $\int |w(t)\,dt| \geq \left|\int w(t)\,dt\right|$ to the first summation and by the property of integrals: $\int\left(\sum a_i\right) = \sum\left(\int a_i\right)$

$$\int_0^\infty g(t)\,dt \geq \left\| \sum_{i\in H_1}\sum_{j=1}^{n_i}\sum_{k=j}^{n_i}\left\{\int_0^\infty w_{ijks} t^{j-1} e^{\mathrm{Re}(\lambda_i)t}\,dt\right\} \right| - \sum_{q\in H_2}\sum_{j=1}^{n_q}\sum_{k=j}^{n_q}\left\{\int_0^\infty v_{qjks} t^{j-1} e^{\mathrm{Re}(\lambda_q)t}\,dt\right\} \Bigg|$$

Taking the constant terms out of the integrals

$$\int_0^\infty g(t)\,dt \geq \left\| \sum_{i\in H_1}\sum_{j=1}^{n_i}\sum_{k=j}^{n_i}\left\{w_{ijks}\int_0^\infty t^{j-1} e^{\mathrm{Re}(\lambda_i)t}\,dt\right\} \right| - \sum_{q\in H_2}\sum_{j=1}^{n_q}\sum_{k=j}^{n_q}\left\{v_{qjks}\int_0^\infty t^{j-1} e^{\mathrm{Re}(\lambda_q)t}\,dt\right\} \Bigg| \tag{4.87}$$

By Lemma A.5 and the assumption that there is at least one eigenvalue of **A** with a real part greater than or equal to zero then there is at least one integral in (4.87) that diverges. This implies that $\int_0^\infty g(t)\,dt \geq \infty$.

Buy using *part ii* of Lemma A.3, it is concluded that $\int_0^\infty h(t)\,dt = \int_0^\infty \left| x_s(t) - x_s^{eq} \right| dt$ diverges because

$\int_0^\infty g(t)\,dt$ diverges. But this is a contradiction to the hypothesis that says that $\int_0^\infty \left| x_s(t) - x_s^{eq} \right| dt$

converges. Therefore, the assumption that the real part of at least one eigenvalue of **A** has to be greater or equal to zero is false.

Now it will be shown that if the real part of all the eigenvalues of **A** is negative then

$$\lim_{t\to\infty}\left(x_s(t) - x_s^{eq}\right) = 0 \tag{4.88}$$

Taking limits to both sides of equation (4.83) gives

$$\begin{aligned}\lim_{t\to\infty}\left(x_s(t) - x_s^{eq}\right) = {} & \lim_{t\to\infty} \sum_{i\in H_1}\sum_{j=1}^{n_i}\sum_{k=j}^{n_i}\left\{w_{ijks}\, t^{j-1} e^{\mathrm{Re}(\lambda_i)t}\right\} + \\ & \lim_{t\to\infty} \sum_{q\in H_2}\sum_{j=1}^{n_q}\sum_{k=j}^{n_q}\left\{v_{qjks}\sin\left(\mathrm{Im}(\lambda_q)t + \theta_{qjks}\right) t^{j-1} e^{\mathrm{Re}(\lambda_q)t}\right\}\end{aligned} \tag{4.89}$$

By property of limits the first limit can be expressed as $\sum_{i\in H_1}\sum_{j=1}^{n_i}\sum_{k=j}^{n_i}\lim_{t\to\infty}\left\{w_{ijks}\, t^{j-1} e^{\mathrm{Re}(\lambda_i)t}\right\}$

By using the L'Hospital rule and considering that $\mathrm{Re}(\lambda_i) < 0, \forall i \in H_1$ the first limit in (4.89) is calculated as follows

$$\sum_{i\in H_1}\sum_{j=1}^{n_i}\sum_{k=j}^{n_i}\lim_{t\to\infty}\left\{w_{ijks}\, t^{j-1} e^{\mathrm{Re}(\lambda_i)t}\right\} = \sum_{i\in H_1}\sum_{j=1}^{n_i}\sum_{k=j}^{n_i}\lim_{t\to\infty}\frac{w_{ijks}\, t^{j-1}}{e^{-\mathrm{Re}(\lambda_i)t}}$$

Taking the first derivative to the expression inside the limit

$$= \sum_{i\in H_1}\sum_{j=1}^{n_i}\sum_{k=j}^{n_i}\lim_{t\to\infty}\frac{w_{ijks}(j-1)t^{j-2}}{-\mathrm{Re}(\lambda_i)\, e^{-\mathrm{Re}(\lambda_i)t}}$$

Taking the second derivative

$$= \sum_{i \in H_1} \sum_{j=1}^{n_i} \sum_{k=j}^{n_i} \lim_{t \to \infty} \frac{w_{ijks}(j-1)(j-2)t^{j-3}}{\left[-\mathrm{Re}(\lambda_i)\right]^2 e^{-\mathrm{Re}(\lambda_i)t}}$$

Continuing in this fashion after taking (j-1) derivatives yields

$$\sum_{i \in H_1} \sum_{j=1}^{n_i} \sum_{k=j}^{n_i} \lim_{t \to \infty} \left\{ w_{ijks} t^{j-1} e^{\mathrm{Re}(\lambda_i)t} \right\} = \sum_{i \in H_1} \sum_{j=1}^{n_i} \sum_{k=j}^{n_i} \lim_{t \to \infty} \frac{w_{ijks}(j-1)!}{\left[-\mathrm{Re}(\lambda_i)\right]^{(j-1)} e^{-\mathrm{Re}(\lambda_i)t}} = 0$$

Then $\lim_{t \to \infty} \sum_{i \in H_1} \sum_{j=1}^{n_i} \sum_{k=j}^{n_i} \left\{ w_{ijks} t^{j-1} e^{\mathrm{Re}(\lambda_i)t} \right\} = 0$ (4.90)

The calculation of the second limit in (4.89) requires the use of the sandwich theorem. The function of the second limit can be bounded as follows by using (4.85):

$$-\sum_{q \in H_2} \sum_{j=1}^{n_q} \sum_{k=j}^{n_q} \left\{ v_{qjks} t^{j-1} e^{\mathrm{Re}(\lambda_q)t} \right\} \le \sum_{q \in H_2} \sum_{j=1}^{n_q} \sum_{k=j}^{n_q} \left\{ v_{qjks} \sin\left(\mathrm{Im}(\lambda_q)t + \theta_{qjks}\right) t^{j-1} e^{\mathrm{Re}(\lambda_q)t} \right\} \le$$

$$\sum_{q \in H_2} \sum_{j=1}^{n_q} \sum_{k=j}^{n_q} \left\{ v_{qjks} t^{j-1} e^{\mathrm{Re}(\lambda_q)t} \right\}$$

Similar to the results of (4.90) and considering that $\mathrm{Re}(\lambda_q) < 0, \forall q \in H_2$ it is known that

$$\lim_{t \to \infty} \left(-\sum_{q \in H_2} \sum_{j=1}^{n_q} \sum_{k=j}^{n_q} \left\{ v_{qjks} t^{j-1} e^{\mathrm{Re}(\lambda_q)t} \right\} \right) = \lim_{t \to \infty} \left(\sum_{q \in H_2} \sum_{j=1}^{n_q} \sum_{k=j}^{n_q} \left\{ v_{qjks} t^{j-1} e^{\mathrm{Re}(\lambda_q)t} \right\} \right) = 0$$

Then by the sandwich theorem

$$\lim_{t \to \infty} \sum_{q \in H_2} \sum_{j=1}^{n_q} \sum_{k=j}^{n_q} \left\{ v_{qjks} \sin\left(\mathrm{Im}(\lambda_q)t + \theta_{qjks}\right) t^{j-1} e^{\mathrm{Re}(\lambda_q)t} \right\} = 0 \qquad (4.91)$$

Substituting the results of (4.90) and (4.91) into (4.89) gives $\lim_{t \to \infty}\left(x_s(t) - x_s^{eq}\right) = 0$

From (4.82) and (4.88) it is clear that if $\int_0^\infty \left|x_s(t) - x_s^{eq}\right| dt$ converges then $\operatorname*{Lim}_{t\to\infty}\left(x_s(t) - x_s^{eq}\right) = 0$

(4.92)

Finally, Definition 2 will be used to prove that if $\int_0^\infty \left|x_s(t) - x_s^{eq}\right| dt$ converges then the state variable

x_s is stable. (4.93)

In view of (4.82), if $\int_0^\infty \left|x_s(t) - x_s^{eq}\right| dt$ converges then all the eigenvalues of **A** have a negative real

part.

By applying properties of absolute value in (4.83)

$$\left|x_s(t) - x_s^{eq}\right| \le \sum_{i\in H_1}\sum_{j=1}^{n_i}\sum_{k=j}^{n_i}\left|w_{ijks}t^{j-1}e^{Re(\lambda_i)t}\right| + \sum_{q\in H_2}\sum_{j=1}^{n_q}\sum_{k=j}^{n_q}\left|v_{qjks}\sin\left(Im(\lambda_q)t + \theta_{qjks}\right)t^{j-1}e^{Re(\lambda_q)t}\right|$$

By applying the property: $\left|\sin(t)\right| \le 1, \forall t \in R$ to the second summation

$$\left|x_s(t) - x_s^{eq}\right| \le \sum_{i\in H_1}\sum_{j=1}^{n_i}\sum_{k=j}^{n_i}\left|w_{ijks}t^{j-1}e^{Re(\lambda_i)t}\right| + \sum_{q\in H_2}\sum_{j=1}^{n_q}\sum_{k=j}^{n_q}\left|v_{qjks}t^{j-1}e^{Re(\lambda_q)t}\right| \quad (4.94)$$

By using calculus it is derived that the maximum value of the function $f(t) = t^{j-1}e^{Re(\lambda_i)t}$ is in

$$t_M = \frac{-(j-1)}{Re(\lambda_i)}, Re(\lambda_i) < 0 \text{. Thus, } f(t) = t^{j-1}e^{Re(\lambda_i)t} \le f(t_M) = \left[\frac{-(j-1)}{Re(\lambda_i)}\right]^{j-1} e^{-(j-1)} \quad (4.95)$$

From (4.94) and (4.95), it follows

$$\left|x_s(t) - x_s^{eq}\right| \le \sum_{i\in H_1}\sum_{j=1}^{n_i}\sum_{k=j}^{n_i}\left|w_{ijks}\left[\frac{-(j-1)}{Re(\lambda_i)}\right]^{j-1} e^{-(j-1)}\right| + \sum_{q\in H_2}\sum_{j=1}^{n_q}\sum_{k=j}^{n_q}\left|v_{qjks}\left[\frac{-(j-1)}{Re(\lambda_q)}\right]^{j-1} e^{-(j-1)}\right|$$

Making $M_s = \sum_{i \in H_1} \sum_{j=1}^{n_i} \sum_{k=j}^{n_i} \left| w_{ijks} \left[\frac{-(j-1)}{\mathrm{Re}(\lambda_i)} \right]^{j-1} e^{-(j-1)} \right| + \sum_{q \in H_2} \sum_{j=1}^{n_q} \sum_{k=j}^{n_q} \left| v_{qjks} \left[\frac{-(j-1)}{\mathrm{Re}(\lambda_q)} \right]^{j-1} e^{-(j-1)} \right|$

then $\left| x_s(t) - x_s^{eq} \right| \le M_s$. This means that $\left| x_s(t) - x_s^{eq} \right|$ is bounded, implying that x_s is stable.

From (4.92), (4.93) and Definition 3, it is proved that the state variable x_s is asymptotically stable around the EP x_s^{eq} if $\int_0^\infty \left| x_s(t) - x_s^{eq} \right| dt$ converges ■

4.3. Stability of Nonlinear Systems

The most difficult task for studying the stability of nonlinear systems is not having a well defined structure of the system like in the linear case. This problem can be overcome by the linearization of the system at infinite number of operating points. Using this approach it is obtained a linearized model, as presented in Definition 10, which makes easier to apply the conditions for stability derived for the linear system.

4.3.1. Linearization of a Nonlinear System

The linearization of the nonlinear system equations at an operating point can be accomplished using the Taylor series expansion, as it is shown in Definition 9.

Definition 9 (Khalil 1996) Consider the nonlinear system defined by equation $\dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}(t))$; $\mathbf{x}(0) = \mathbf{x}_0$; where $\mathbf{x}(t) \in R^n; \mathbf{x}(t) = [x_s(t)], s = 1,..,n$; and function $\mathbf{f}$ is defined by $\mathbf{f}: R^n \to R^n; \mathbf{f}(\mathbf{x}(t)) = [f_s(\mathbf{x}(t))], s = 1,..,n$. The linear approximation $\dot{z}_s(t)$ for the s^{th} component of vector $\dot{\mathbf{x}}(t)$ around the operating point $\mathbf{x}_\theta = [x_{\theta 1}, x_{\theta 2}, \cdots, x_{\theta n}]$ is given by

$$\dot{z}_s(t) = f_s(\mathbf{x}_\theta) + \sum_{i=1}^{n} \left\{ \left.\frac{\partial f_s}{\partial x_i}\right|_{\mathbf{x}=\mathbf{x}_\theta} (z_i - x_{\theta i}) \right\} \tag{4.96}$$

A nonlinear system can be approximated by linearizing the model around several operating points as shown in the following definition.

Definition 10 Consider the nonlinear system defined by $\dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}(t));\ \mathbf{x}(0) = \mathbf{x}_0;$ where $\mathbf{x}(t) \in R^n; \mathbf{f} : R^n \to R^n$. The linearized model $\dot{\mathbf{z}}(t)$ of system $\dot{\mathbf{x}}(t)$ around *m* operating points $\{\mathbf{z}(t_{p-1})$, p=1,..,m; $t_0<t_1<...<t_m\}$ is represented by the following equations

$$\dot{\mathbf{z}}(t) = \begin{cases} \mathbf{A}_1\mathbf{z}(t) + \mathbf{b}_1, t_0 \le t < t_1; & \text{Initial condition}: \mathbf{z}(t_0) = \mathbf{x}(t_0) \\ \mathbf{A}_2\mathbf{z}(t) + \mathbf{b}_2, t_1 \le t < t_2; & \text{Initial condition}: \mathbf{z}(t_1) \\ \vdots & \vdots \\ \mathbf{A}_m\mathbf{z}(t) + \mathbf{b}_m, t_{m-1} \le t < t_m; & \text{Initial condition}: \mathbf{z}(t_m) \end{cases} \tag{4.97}$$

where $\mathbf{z}(t_p) = \lim_{t \to t_p^-} \mathbf{z}(t_p), p = 1,..,m-1$ (4.98)

This definition implies that trajectory $\mathbf{x}(t)$ is been approximated by trajectories $\mathbf{z}(t)$ of *p* linear systems. Note that $\mathbf{z}(t)$ is a continuous piecewise function. This is because $\mathbf{z}(t)$ is differentiable and therefore continuous in $[t_{p-1}, t_p), p = 1,..,m$, and condition (4.98).

Example 4 Consider the system defined by $\dot{x}_1 = \frac{x_1}{15}(-0.1x_2 + 1.1) - \frac{x_1}{30}e^{(x_2-11)};$ $\dot{x}_2 = 0.01x_1;$ $\mathbf{x_0} = (12,2)$. Obtain the linearized model $\dot{\mathbf{z}}(t)$ for the operating points: $\mathbf{z}(t=0), \mathbf{z}(t=1)$.

Solution: To use equation (4.96) it is necessary to determine the initial values and first derivatives for function **f**. The following calculations are required to linearize the model around an operating point.

For the component $f_1(x_1, x_2) = \dot{x}_1 = \frac{x_1}{15}(-0.1x_2 + 1.1) - \frac{x_1}{30} e^{(x_2 - 11)}$, it results:

$$\frac{\partial f_1}{\partial x_1} = \frac{(-0.1x_2 + 1.1)}{15} - \frac{e^{(x_2 - 11)}}{30}$$

$$\frac{\partial f_1}{\partial x_2} = \frac{-0.1x_1}{15} - \frac{x_1}{30} e^{(x_2 - 11)}$$

Using equation (4.96) to calculate component $\dot{z}_1(t)$ yields

$$\dot{z}_1(t) = f_1(12,2) + \left.\frac{\partial f_1}{\partial x_1}\right|_{\mathbf{x}=(12,2)} (z_1 - 12) + \left.\frac{\partial f_1}{\partial x_2}\right|_{\mathbf{x}=(12,2)} (z_2 - 2)$$

Replacing values and simplifying:

$$\dot{z}_1(t) = 0.72 + 0.06(z_1 - 12) + (-0.08)(z_2 - 2)$$

$$\dot{z}_1(t) = 0.06z_1 - 0.08z_2 + 0.16 \qquad (4.99)$$

For the component $f_2(x_1, x_2) = \dot{x}_2 = 0.01x_1$, it results:

$$\frac{\partial f_2}{\partial x_1} = 0.01$$

$$\frac{\partial f_2}{\partial x_2} = 0$$

Using equation (4.96) to calculate component $\dot{z}_2(t)$

$$\dot{z}_2(t) = f_2(12,2) + \left.\frac{\partial f_2}{\partial x_1}\right|_{\mathbf{x}=(12,2)} (z_1 - 12) + \left.\frac{\partial f_2}{\partial x_2}\right|_{\mathbf{x}=(12,2)} (z_2 - 2)$$

Replacing values and simplifying:

$$\dot{z}_2(t) = 0.12 + 0.01(z_1 - 12) + 0(z_2 - 2)$$

$$\dot{z}_2(t) = 0.01z_1 \tag{4.100}$$

From (4.99) and (4.100), $\dot{\mathbf{z}}(t)$ can be expressed as the linear system

$$\dot{\mathbf{z}}(t) = \mathbf{A}_1\mathbf{z}(t) + \mathbf{b}_1, 0 \le t < 1 \tag{4.101}$$

where

$$\mathbf{A}_1 = \begin{bmatrix} 0.06 & -0.08 \\ 0.01 & 0 \end{bmatrix}, \mathbf{b}_1 = \begin{bmatrix} 0.16 \\ 0 \end{bmatrix}$$

In order to find the second operating point, it is required to solve (4.101) and obtain $\mathbf{z}(t=1)$. The solution trajectory is given by equation (4.44). The terms of this equation are the following:

The eigenvalues of $\mathbf{A}_1$ are: $\lambda_1 = 0.04, \lambda_2 = 0.02$, which are different and implies a Jordan canonical form decomposition of two Jordan blocks.

The matrix of generalized eigenvectors is: $\mathbf{T} = [\mathbf{u}_{11}\ \mathbf{u}_{21}] = \begin{bmatrix} 0.9701 & 0.8947 \\ 0.2427 & 0.4467 \end{bmatrix}$

The constants are:

$$\begin{bmatrix} \alpha_{11} \\ \alpha_{21} \end{bmatrix} = \begin{bmatrix} 16.5188 \\ -4.4986 \end{bmatrix}$$

$$\begin{bmatrix} \beta_{11} \\ \beta_{21} \end{bmatrix} = \begin{bmatrix} 0.3308 \\ -0.1798 \end{bmatrix}$$

For m=2, $n_1=n_2=1$ equation (4.44) is simplified as

$$\mathbf{z}(t) = \mathbf{z}^{eq} + \sum_{i=1}^{m} \left(\alpha_{i1} + \frac{\beta_{i1}}{\lambda_i} \right) \mathbf{u}_{i1} \tag{4.102}$$

where $\mathbf{z}^{eq} = -\frac{\beta_{i1}}{\lambda_i}\mathbf{u}_{i1}$

Substituting the terms above in (4.102) yields:

$$\mathbf{z}(t) = \begin{bmatrix} 24.0546\, e^{0.04t} - 12.0546\, e^{0.02t} \\ 6.0186\, e^{0.04t} - 6.0186\, e^{0.02t} \end{bmatrix}$$

For t=1, $\mathbf{z}(t=1) = (12.74, 2.12)$. The following calculations are required to linearize the model around the operating point $(12.74, 2.12)$.

By using equation (4.96) component $\dot{z}_1(t)$ is calculated as

$$\dot{z}_1(t) = f_1(12.74,2.12) + \left.\frac{\partial f_1}{\partial x_1}\right|_{\mathbf{x}=(12.74,2.12)} (z_1 - 12.74) + \left.\frac{\partial f_1}{\partial x_2}\right|_{\mathbf{x}=(12.74,2.12)} (z_2 - 2.12)$$

Replacing values and simplifying:

$$\dot{z}_1(t) = 0.75 + 0.059(z_1 - 12.74) + (-0.085)(z_2 - 2.12)$$

$$\dot{z}_1(t) = 0.059 z_1 - 0.085 z_2 + 0.18 \qquad (4.103)$$

By using equation (4.96) component $\dot{z}_2(t)$ is calculated as

$$\dot{z}_2(t) = f_2(12.74,2.12) + \left.\frac{\partial f_2}{\partial x_1}\right|_{\mathbf{x}=(12.74,2.12)} (z_1 - 12.74) + \left.\frac{\partial f_2}{\partial x_2}\right|_{\mathbf{x}=(12,2.12)} (z_2 - 2.12)$$

Replacing values and simplifying:

$$\dot{z}_2(t) = 0.1274 + 0.01(z_1 - 12.74) + 0(z_2 - 2.12)$$

$$\dot{z}_2(t) = 0.01 z_1 \qquad (4.104)$$

From (4.103) and (4.104) $\dot{\mathbf{z}}(t)$ can be expressed as the linear system

$$\dot{\mathbf{z}}(t) = \mathbf{A}_2 \mathbf{z}(t) + \mathbf{b}_2, 1 \le t \qquad (4.105)$$

where

$$\mathbf{A}_2 = \begin{bmatrix} 0.059 & -0.085 \\ 0.01 & 0 \end{bmatrix},\ \mathbf{b}_1 = \begin{bmatrix} 0.18 \\ 0 \end{bmatrix}$$

After combining (4.101) and (4.105) into one equation it is obtained the linearized model

$$\dot{\mathbf{z}}(t) = \begin{cases} \begin{bmatrix} 0.06 & -0.08 \\ 0.01 & 0 \end{bmatrix}\mathbf{z}(t) + \begin{bmatrix} 0.16 \\ 0 \end{bmatrix}, 0 \le t < 1 \\ \begin{bmatrix} 0.059 & -0.085 \\ 0.01 & 0 \end{bmatrix}\mathbf{z}(t) + \begin{bmatrix} 0.18 \\ 0 \end{bmatrix}, 1 \le t \end{cases} \quad \blacksquare$$

4.3.2. Conditions for Nonlinear Stability

In this section, it is extended the stabilization concept of ADE applied for the linear system to cover the nonlinear stability as well. Theorem 5 shows that the condition for the ADE convergence of the nonlinear system (approximated by a linearized model) can be derived from the convergence of an infinite series of linear systems. This condition states that all eigenvalues of the m^{th} linear system have to be negative when m goes to infinity, which assures asymptotic stability of the linearized model and therefore asymptotic stability of the nonlinear system.

Theorem 5 Consider the system defined by $\dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}(t))$; $\mathbf{x}(0) = \mathbf{x}_0$; where $\mathbf{x}(t) \in R^n$; $\mathbf{f}: R^n \to R^n$; $\mathbf{x}(t) = [x_s(t)], s = 1,..,n$. The state variable x_s is asymptotically stable (around the EP x_s^{eq}), if $\int_0^\infty |x_s(t) - x_s^{eq}|\,dt$ converges.

Proof: First, it will be proved by contradiction that if $\int_0^\infty |x_s(t) - x_s^{eq}|\,dt$ converges the state variable

x_s is stable around the EP x_s^{eq}. (4.106)

Thus, it is assumed to contrary that state variable x_s is not stable, which by Definition 2 means that $\left|x_s(t)-x_s^{eq}\right|$ is not bounded, i.e.

$$\forall M_s, \exists t_M \text{ such that} \left|x_s(t)-x_s^{eq}\right| > M_s, \forall s$$

Making $y_s(t)=x_s(t)-x_s^{eq}$, thus

$$\forall M_s, \exists t_M \text{ such that} \left|y_s(t_M)\right| > M_s \qquad (4.107)$$

Expressing $\int_0^\infty \left|y_s(t)\right| dt$ as Riemann sums (Yuen and Yuan 2000)

$$\int_0^\infty \left|y_s(t)\right| dt = \sum_{i=1}^\infty \left|y_s(c_i)\Delta t_i\right| \qquad (4.108)$$

where $\Delta t_i = t_i - t_{i-1}$, and $c_i \in [t_i, t_{i-1}]$

By hypothesis it is known that the integral $\int_0^\infty \left|x_s(t)-x_s^{eq}\right| dt$ converges (i.e. it is bounded), and thus

there is a number W_s such that $\int_0^\infty \left|x_s(t)-x_s^{eq}\right| dt \le W_s, \forall t \ge 0, \forall s$

Expressing the above statement in terms of $y_s(t)$: there is a number W_s such that

$$\int_0^\infty \left|y_s(t)\right| dt \le W_s, \forall t \ge 0, \forall s \qquad (4.109)$$

From (4.108) and (4.109) gives

$$\sum_{i=1}^\infty \left|y_s(c_i)\Delta t_i\right| \le W_s \Rightarrow \left|y_s(c_i)\Delta t_i\right| \le W_s, \forall i \Rightarrow \left|y_s(c_i)\right| \le \frac{W_s}{\left|\Delta t_i\right|}, \forall i \qquad (4.110)$$

Because "t" is a continuous variable from 0 to infinity, then there is an index i=b such that $c_b = t_M$. Moreover, condition (4.110) holds for every c_i and particularly for $c_b = t_M$, therefore

$$|y_s(t_M)| \leq \frac{W_s}{|\Delta t_b|} \tag{4.111}$$

Condition (4.107) holds for every M_s and particularly for $M_s = \frac{W_s}{|\Delta t_b|}$, thus

$$|y_s(t_M)| > \frac{W_s}{|\Delta t_b|}$$

But this is a contradiction to the statement in (4.111). Therefore, the assumption that the equilibrium point x_s^{eq} is not stable is false.

Second, it will be proved that if $\int_0^\infty |x_s(t) - x_s^{eq}|\,dt$ converges then $\lim_{t\to\infty}\left(x_s(t) - x_s^{eq}\right) = 0$

(4.112)

In order to do that, the nonlinear system has to be linearized around *m* operating points. It is important to note that the equilibrium points of these linear systems do not have to coincide with the equilibrium point of the nonlinear system. However, it will be shown that when the system is asymptotically stable the equilibrium points of the linear systems tend to converge to the equilibrium point of the nonlinear system when t goes to infinity.

Making the transformation $y_s(t) = x_s(t) - x_s^{eq}$. The equilibrium point for the new nonlinear system will be the origin, i.e. $x_s^{eq} = 0$, and therefore

$\int_0^\infty |x_s(t) - x_s^{eq}|\,dt = \int_0^\infty |y_s(t)|\,dt$, $\forall s$ implying that

$$\text{if } \int_0^\infty |x_s(t) - x_s^{eq}|\,dt \text{ converges then } \int_0^\infty |y_s(t)|\,dt \text{ converges} \tag{4.113}$$

Applying Definition 10, it is possible to approximate $\dot{\mathbf{y}}(t)$ by $\dot{\mathbf{z}}(t)$ after linearizing the system

$\dot{\mathbf{y}}(t)$ around *m* operating points $\{z(t_{p-1}), p=1,..,m; t_0<t_1<...<t_m\}$ as follows:

$$\dot{\mathbf{z}}(t) = \begin{cases} \mathbf{A}_1\mathbf{z}(t) + \mathbf{b}_1, t_0 \le t < t_1 \\ \mathbf{A}_2\mathbf{z}(t) + \mathbf{b}_2, t_1 \le t < t_2 \\ \vdots \\ \mathbf{A}_m\mathbf{z}(t) + \mathbf{b}_m, t_{m-1} \le t < t_m \end{cases}$$

After considering $\Delta t_p = t_p - t_{p-1} = \text{constant} = h > 0, \quad p = 1,..,m$, the interval of validity for each linear system is $[t_{p-1}, t_{p-1} + h)$.

Now the integral $\int_0^\infty |y_s(t)|\,dt$ can be calculated as the sum of the integrals of *m* linear systems when *m* goes to infinity as follows

$$\int_0^\infty |y_s(t)|\,dt = \lim_{m\to\infty} \sum_{i=1}^{m} \int_{t_{i-1}}^{t_{i-1}+h} |z_s(t)|\,dt \tag{4.114}$$

Making $\Psi_i = \int_{t_{i-1}}^{t_{i-1}+h} |z_s(t)|\,dt$ and $S_m = \sum_{i=1}^{m} \Psi_i$ (4.115)

Replacing S_m in (4.114) results

$$\int_0^\infty |y_s(t)|\,dt = \lim_{m\to\infty} S_m = \sum_{i=1}^{\infty} \Psi_i \tag{4.116}$$

From (4.113), it follows that $\int_0^\infty |y_s(t)|\,dt$ converges and therefore from (4.116) it is obtained that $\sum_{i=1}^{\infty} \Psi_i$ converges also. Therefore, $\lim_{m\to\infty} \Psi_m = 0$ (from Lemma A.4)

Using (4.115) yields:

$$\lim_{m\to\infty}\int_{t_{m-1}}^{t_{m-1}+h}\left|z_s(t)\right|dt = 0 \tag{4.117}$$

From (4.86) it is known that for the system $\dot{\mathbf{z}} = \mathbf{A}_m\mathbf{z}(t) + \mathbf{b}_m, t_{m-1} \le t < t_m$, the following inequality holds:

$$\left|\left|\sum_{i\in H_1}\sum_{j=1}^{n_i}\sum_{k=j}^{n_i}\left\{w_{ijksm}t^{j-1}e^{\mathrm{Re}(\lambda_{im})t}\right\}\right| - \sum_{q\in H_2}\sum_{j=1}^{n_q}\sum_{k=j}^{n_q}\left\{v_{qjksm}t^{j-1}e^{\mathrm{Re}(\lambda_{qm})t}\right\}\right| \le \left|z_s(t) - z_{sm}^{eq}\right| \tag{4.118}$$

Note that all the parameters on the left-hand side of the inequality have also a subindex *m*, denoting that they are dependant of the m^{th} linear model. In other words, each linear model *p* (p=1,..,m) has its own parameters (constants, eigenvalues and eigenvectors).

By the property of absolute value $|a-b| \le |a| + |b|$ and by (4.118)

$$\left|\left|\sum_{i\in H_1}\sum_{j=1}^{n_i}\sum_{k=j}^{n_i}\left\{w_{ijksm}t^{j-1}e^{\mathrm{Re}(\lambda_{im})t}\right\}\right| - \sum_{q\in H_2}\sum_{j=1}^{n_q}\sum_{k=j}^{n_q}\left\{v_{qjksm}t^{j-1}e^{\mathrm{Re}(\lambda_{qm})t}\right\}\right| \le \left|z_s(t)\right| + \left|z_{sm}^{eq}\right|$$

Rearranging terms

$$\left|\left|\sum_{i\in H_1}\sum_{j=1}^{n_i}\sum_{k=j}^{n_i}\left\{w_{ijksm}t^{j-1}e^{\mathrm{Re}(\lambda_{im})t}\right\}\right| - \sum_{q\in H_2}\sum_{j=1}^{n_q}\sum_{k=j}^{n_q}\left\{v_{qjksm}t^{j-1}e^{\mathrm{Re}(\lambda_{qm})t}\right\}\right| - \left|z_{sm}^{eq}\right| \le \left|z_s(t)\right|$$

Integrating both terms of the inequality from t_{m-1} to $t_{m-1}+h$

$$\int_{t_{m-1}}^{t_{m-1}+h}\left|\left|\sum_{i\in H_1}\sum_{j=1}^{n_i}\sum_{k=j}^{n_i}\left\{w_{ijksm}t^{j-1}e^{\mathrm{Re}(\lambda_{im})t}\right\}\right| - \sum_{q\in H_2}\sum_{j=1}^{n_q}\sum_{k=j}^{n_q}\left\{v_{qjksm}t^{j-1}e^{\mathrm{Re}(\lambda_{qm})t}\right\}\right|dt - \int_{t_{m-1}}^{t_{m-1}+h}\left|z_{sm}^{eq}\right|dt \le \int_{t_{m-1}}^{t_{m-1}+h}\left|z_s(t)\right|dt$$

Applying different properties of the absolute value and the integral results

$$\left|\left|\sum_{i\in H_1}\sum_{j=1}^{n_i}\sum_{k=j}^{n_i}\left\{w_{ijksm}\int_{t_{m-1}}^{t_{m-1}+h}t^{j-1}e^{\mathrm{Re}(\lambda_{im})t}dt\right\}\right| - \sum_{q\in H_2}\sum_{j=1}^{n_q}\sum_{k=j}^{n_q}\left\{v_{qjksm}\int_{t_{m-1}}^{t_{m-1}+h}t^{j-1}e^{\mathrm{Re}(\lambda_{qm})t}dt\right\}\right| - \int_{t_{m-1}}^{t_{m-1}+h}\left|z_{sm}^{eq}\right|dt$$

$$\leq \int_{t_{m-1}}^{t_{m-1}+h} \left| z_s(t) \right| dt \tag{4.119}$$

Integrating by parts the integral $\int_{t_{m-1}}^{t_{m-1}+h} t^{j-1} e^{\mathrm{Re}(\lambda_{im})t} dt$ is calculated as

$$\int_{t_{m-1}}^{t_{m-1}+h} t^{j-1} e^{\mathrm{Re}(\lambda_{im})t} dt = e^{\mathrm{Re}(\lambda_{im})t_{m-1}} \sum_{c=0}^{j-1} \left\{ \frac{n!(-1)^{j-1-c}[(t_{m-1}+h)^c e^{\mathrm{Re}(\lambda_{im})h} - (t_{m-1})^c]}{c!\mathrm{Re}(\lambda_{im})^{(j-1)-c+1}} \right\}$$

Let define

$$F(t_{m-1}) = e^{\mathrm{Re}(\lambda_{im})t_{m-1}} \sum_{c=0}^{j-1} \left\{ \frac{n!(-1)^{j-1-c}[(t_{m-1}+h)^c e^{\mathrm{Re}(\lambda_{im})h} - (t_{m-1})^c]}{c!\mathrm{Re}(\lambda_{im})^{(j-1)-c+1}} \right\} \tag{4.120}$$

Similarly

$$\int_{t_{m-1}}^{t_{m-1}+h} t^{j-1} e^{\mathrm{Re}(\lambda_{qm})t} dt = e^{\mathrm{Re}(\lambda_{qm})t_{m-1}} \sum_{c=0}^{j-1} \left\{ \frac{n!(-1)^{j-1-c}[(t_{m-1}+h)^c e^{\mathrm{Re}(\lambda_{qm})h} - (t_{m-1})^c]}{c!\mathrm{Re}(\lambda_{qm})^{(j-1)-c+1}} \right\}$$

$$G(t_{m-1}) = e^{\mathrm{Re}(\lambda_{qm})t_{m-1}} \sum_{c=0}^{j-1} \left\{ \frac{n!(-1)^{j-1-c}[(t_{m-1}+h)^c e^{\mathrm{Re}(\lambda_{qm})h} - (t_{m-1})^c]}{c!\mathrm{Re}(\lambda_{qm})^{(j-1)-c+1}} \right\} \tag{4.121}$$

Evaluating the integral $\int_{t_{m-1}}^{t_{m-1}+h} \left| z_{sm}^{eq} \right| dt$

$$\int_{t_{m-1}}^{t_{m-1}+h} \left| z_{sm}^{eq} \right| dt = \left| z_{sm}^{eq} \right| h \tag{4.122}$$

Substituting (4.120-4.122) into (4.119)

$$\left\| \left| \sum_{i \in H_1} \sum_{j=1}^{n_i} \sum_{k=j}^{n_i} \left\{ w_{ijksm} F(t_{m-1}) \right\} \right| - \sum_{q \in H_2} \sum_{j=1}^{n_q} \sum_{k=j}^{n_q} \left\{ v_{qjksm} G(t_{m-1}) \right\} \right| - \left| z_{sm}^{eq} \right| h$$

$$\leq \int_{t_{m-1}}^{t_{m-1}+h} \left| z_s(t) \right| dt \tag{4.123}$$

Taking the limit when *m* goes to infinity, and knowing that

$(m \to \infty) \Rightarrow (t_{m-1} \to \infty) \Rightarrow (t \to \infty)$ results (4.124)

$$\lim_{t_{m-1}\to\infty}\left\|\sum_{i\in H_1}\sum_{j=1}^{n_i}\sum_{k=j}^{n_i}\left\{w_{ijksm}F(t_{m-1})\right\}\right| - \sum_{q\in H_2}\sum_{j=1}^{n_q}\sum_{k=j}^{n_q}\left\{v_{qjksm}G(t_{m-1})\right\}\Bigg| - \lim_{m\to\infty}\left|z_{sm}^{eq}\right| h\, dt$$

$$\le \lim_{m\to\infty}\int_{t_{m-1}}^{t_{m-1}+h}\left|z_s(t)\right| dt \tag{4.125}$$

However, by (4.117) it is known that $\lim_{m\to\infty}\int_{t_{m-1}}^{t_{m-1}+h}\left|z_s(t)\right| dt = 0$, and therefore the only way to satisfy this condition is if the terms on the left-hand side of inequality (4.125) are zero.

The first term, $\lim_{t_{m-1}\to\infty}\left\|\sum_{i\in H_1}\sum_{j=1}^{n_i}\sum_{k=j}^{n_i}\left\{w_{ijksm}F(t_{m-1})\right\}\right| - \sum_{q\in H_2}\sum_{j=1}^{n_q}\sum_{k=j}^{n_q}\left\{v_{qjksm}G(t_{m-1})\right\}\Bigg|$, can take two values when t_{m-1} goes to infinity: zero or infinity. The requirement for this term to be zero is that the real part of all the eigenvalues of $\mathbf{A}_m$ has to be negative.

The second term, $\lim_{m\to\infty}\left|z_{sm}^{eq}\right| h$, will be zero only if $\left|z_{sm}^{eq}\right|$ is zero (because h>0). Therefore, $z_{sm}^{eq} = 0$ when *m* goes to infinity, which coincides with the equilibrium point of the nonlinear system $\mathbf{y}(t)$ that is also zero.

Following similar steps to (4.88-4.91) it can be shown that $\lim_{t\to\infty}\left(z_s(t) - z_{sm}^{eq}\right) = 0$.

Considering that $z_{sm}^{eq} = 0$ when $m \to \infty$ and (4.124) the previous expression can be written as

$$\lim_{t\to\infty}\left(z_s(t)\right) = 0 \tag{4.126}$$

Because $\mathbf{z}(t)$ is as an approximation of $\mathbf{y}(t)$ and from (4.126) it is concluded that

$$\lim_{t\to\infty}\left(y_s(t)\right)=0 \tag{4.127}$$

But it is known that $y_s(t)=x_s(t)-x_s^{eq}$. Thus, taking limits to both sides when t goes to infinity and from (4.127) gives

$$\lim_{t\to\infty}\left(y_s(t)\right)=\lim_{t\to\infty}\left(x_s(t)-x_s^{eq}\right)=0 \tag{4.128}$$

From (4.112-4.128) it follows that

$$\text{if } \int_0^\infty \left|x_s(t)-x_s^{eq}\right| dt \text{ converges then } \lim_{t\to\infty}\left(x_s(t)-x_s^{eq}\right)=0 \tag{4.129}$$

Using (4.106), (4.129) and Definition 3 it is proved that the state variable x_s is asymptotically stable around the EP x_s^{eq} if $\int_0^\infty \left|x_s(t)-x_s^{eq}\right| dt$ converges ■

In complex models where the EP is difficult to estimate, it can be easily added as one more variable to calculate in the optimization problem. The following theorem guarantees that if the ADE of a state variable converge to a variable a_s then the value of a_s is the EP of the state variable.

Theorem 6 Consider the system defined by $\dot{\mathbf{x}}(t)=\mathbf{f}(\mathbf{x}(t))$; $\mathbf{x}(0)=\mathbf{x}_0$; where $\mathbf{x}(t)\in R^n$; $\mathbf{f}: R^n \to R^n$; $\mathbf{x}(t)=\left[x_s(t)\right], s=1,..,n$. If $\int_0^\infty \left|x_s(t)-a_s\right| dt$ converges then $a_s = x_s^{eq}$.

Proof: It will be proved by contradiction that if $\int_0^\infty \left|x_s(t)-a_s\right| dt$ converges then $a_s = x_s^{eq}$.

Thus, it is assumed to contrary that $a_s \neq x_s^{eq}$. (4.130)

Making $y_s(t)=x_s(t)-a_s$, thus (4.131)

$\int_0^\infty |x_s(t) - a_s|\, dt = \int_0^\infty |y_s(t)|\, dt$, $\forall s$ implying that

$$\text{if } \int_0^\infty |x_s(t) - a_s|\, dt \text{ converges then } \int_0^\infty |y_s(t)|\, dt \text{ converges}$$

From (4.130) and (4.131) it is derived that $y_s^{eq} = x_s^{eq} - a_s \neq 0$ (4.132)

In view of Theorem 5, if $\int_0^\infty |y_s(t)|\, dt$ converges then $\operatorname*{Lim}_{t\to\infty}(y_s(t)) = 0$, and this statement is satisfied independently of the initial conditions of the system $\dot{\mathbf{y}}(t) = \mathbf{f}(\mathbf{y}(t))$ (4.133)

By Definition 1, if the system $\dot{\mathbf{y}}(t) = \mathbf{f}(\mathbf{y}(t))$ starts at the equilibrium point y_s^{eq} then

$y_s(t) = y_s^{eq}, \forall t$, and thus

$$\operatorname*{Lim}_{t\to\infty}(y_s(t)) = \operatorname*{Lim}_{t\to\infty}(y_s^{eq}) = y_s^{eq} \tag{4.134}$$

From (4.132) and (4.134) yields

$$\operatorname*{Lim}_{t\to\infty}(y_s(t)) \neq 0$$

But this is a contradiction to the statement in (4.133). Therefore, the assumption that $a_s \neq x_s^{eq}$ is false ■

There are situations where achieving the convergence of the ADE is close but it is not totally obtained. This can happen when near the end of the time horizon the DE are small but in an increasing rate. To help accelerate the convergence of the objective function, which initially is expressed only in terms of the ADE (see section 3.1.2), these small DE have to be amplified. This is done by raising them to the exponential power. By summing these values (associated to a state variable) it is obtained a new term called accumulated exponential deviations from equilibrium (AEDE). The mathematical expression of this term is provided in Definition 11.

Definition 11 Consider the system defined by $\dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}(t))$; $\mathbf{x}(0) = \mathbf{x}_0$; where $\mathbf{x}(t) \in R^n$; $\mathbf{f} : R^n \to R^n$; $\mathbf{x}(t) = \left[x_s(t)\right], s = 1,..,n$. For the state variable x_s the accumulated exponential deviations from its EP x_s^{eq} is defined as $\int_0^\infty e^{\left|x_s(t)-x_s^{eq}\right|}\, dt$.

Similar to the ADE case, the following theorem states that the AEDE convergence of a state variable also guarantees the asymptotic stability of that variable.

Theorem 7 Consider the system defined by $\dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}(t))$; $\mathbf{x}(0) = \mathbf{x}_0$; where $\mathbf{x}(t) \in R^n$; $\mathbf{f} : R^n \to R^n$; $\mathbf{x}(t) = \left[x_s(t)\right], s = 1,..,n$. The state variable x_s is asymptotically stable (around the EP x_s^{eq}), if $\int_0^\infty e^{\left|x_s(t)-x_s^{eq}\right|}\, dt$ converges.

Proof: First, it will be proved that $e^{\left|x_s(t)-x_s^{eq}\right|}$ is greater than or equal to the term $\left|x_s(t) - x_s^{eq}\right|$, $\forall t \geq 0$.

By the property of the exponential function: $e^{y(t)} \geq y(t), \forall y(t) \geq 0$

Making $y(t) = \left|x_s(t) - x_s^{eq}\right| \geq 0, \forall t \geq 0$ it follows

$$e^{\left|x_s(t)-x_s^{eq}\right|} \geq \left|x_s(t) - x_s^{eq}\right|, \forall t \geq 0 \tag{4.135}$$

Making $h(t) = e^{\left|x_s(t)-x_s^{eq}\right|}$ and $g(t) = \left|x_s(t) - x_s^{eq}\right|$ satisfies the hypothesis of Lemma A.3, i.e. g(t) and h(t) are continuous functions on $[0,\infty)$, and $0 \leq g(t) \leq h(t)$.

From the hypothesis, it is known that $\int_0^\infty h(t)\, dt = \int_0^\infty e^{\left|x_s(t)-x_s^{eq}\right|}\, dt$ converges (4.136)

Thus, by using *part i* of Lemma A.3. and (4.136) then

$$\int_0^\infty g(t)\,dt = \int_0^\infty \left| x_s(t) - x_s^{eq} \right| dt \ \text{ converges} \tag{4.137}$$

From Theorem 5 and (4.137) then the state variable x_s is asymptotically stable (4.138)

Finally, from (4.136) and (4.138), it is proved that the state variable x_s is asymptotically stable around the EP x_s^{eq} if $\int_0^\infty e^{\left|x_s(t)-x_s^{eq}\right|}\,dt$ converges ■

CHAPTER FIVE: CASE STUDIES ANALYSIS AND RESULTS

This chapter applies the SADE methodology to several case studies. For each case study it is provided a SD model that represents the structure of the supply chain, a description of the problem, and a description of the business if the case study was created based on a real manufacturing company (LSMC and PMOC models).

A general optimization problem, following the guidelines of section 3.1.2., is formulated to test different scenarios and develop alternative stabilization policies. The analysis and results of these experiments are presented to demonstrate the quality and robustness of the policies obtained.

5.1. Case Study A: The Inventory-Workforce Model

5.1.1. Description

The Inventory-Workforce (I-W) model is the case of a manufacturing supply chain that includes labor as an explicit factor of production. The purpose of this case study is to illustrate how production scheduling and hiring policies can interact to generate instability in the SC. Moreover, it is intended to illustrate how instability can feed back undermining trust among partners in a SC and leading to behavior that worsens the instability. The goal of management is to find a policy that maintains the finished goods inventory and labor at equilibrium.

5.1.2. SD Model

Saleh *et al*. (2007) developed a linear SD model for this supply chain by modifying

Sterman's original model (2000). This linear model is divided in two sectors: (1) the inventory management sector and (2) the labor sector. These sectors are described and depicted below.

The **inventory management sector** (Figure 5.1) is represented by two state variables: *Inventory* and *Work in Process Inventory*. The variable *Work in Process Inventory* represents all the stages of the production process where intermediate inventory is created. The variable *Inventory* represents the finished goods inventory. This model assumes that orders are filled as they arrive and the ones that cannot be filled immediately are lost as customers seek other sources of supply.

The **labor sector** (Figure 5.2) is represented by two state variables: *Vacancies* and *Labor*. The stock of vacancies is the supply line or order of workers that have been placed but not yet filled. This states that workers cannot be instantly hired. Hiring takes time: positions must be authorized and vacancies must be created. The labor force is a stock of people, which is increased by the *Hiring Rate* and decreased by the *Quit Rate*. This last rate includes voluntary quits and retirements, excluding the possibility of layoffs.

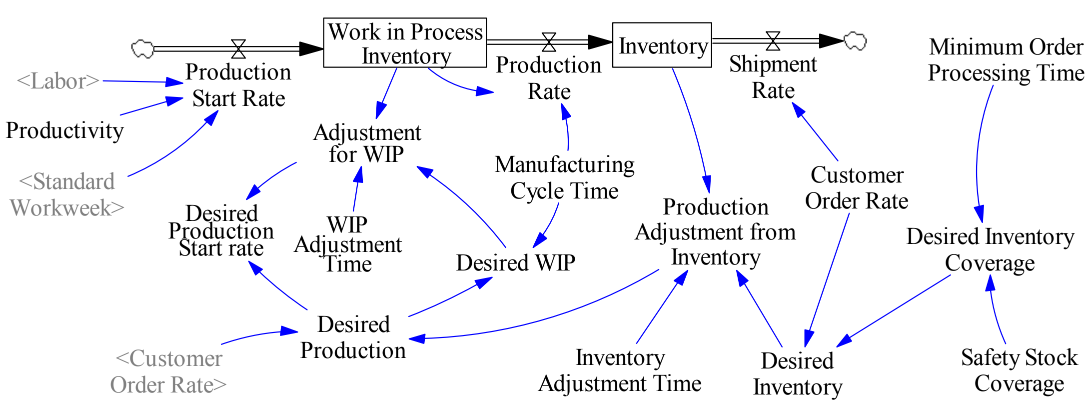

Figure 5.1. I-W model: Structure of inventory management sector

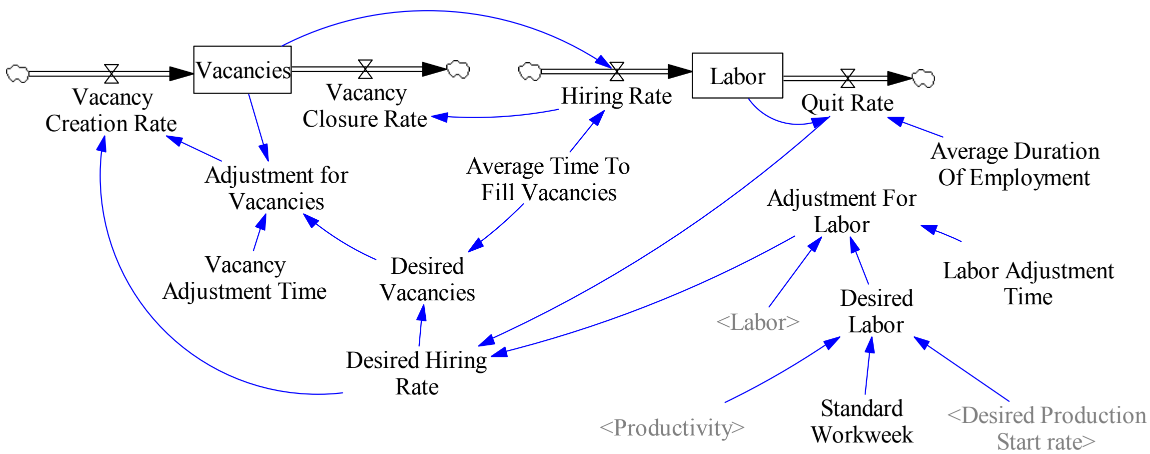

Figure 5.2. I-W model: Structure of labor sector

5.1.3. Current Policy and SC Instability

The set of parameters in Table 5.1 defines the current policy for this supply chain.

Table 5.1. I-W model: Parameter values for the current policy

Parameter	Value	Unit
Manufacturing Cycle Time	8	Weeks
Inventory Adjustment Time	12	Weeks
Average Duration of Employment	100	Weeks
Average Time to Fill Vacancies	8	Weeks
Labor Adjustment Time	19	Weeks
Vacancy Adjustment Time	4	Weeks
WIP Adjustment Time	6	Weeks
Minimum Order Processing Time	2	Weeks
Safety Stock Coverage	2	Widgets

At time 0, the system starts at the equilibrium points: 40,000, 1,000, 80 and 8,0000 for the variables Inventory, Labor, Vacancies and WIP Inventory respectively. Customer orders are arriving at the rate of 10,000 widgets/week. After the system remains in equilibrium for the first five weeks, customer orders experienced a linear increment for the next twenty five weeks until reaching 20 percent of their original value, where they remain constant. As a result, Figure 5.3 shows sharp increases in the variables of interest: *Inventory* and *Labor*, follow by several oscillatory fluctuations.

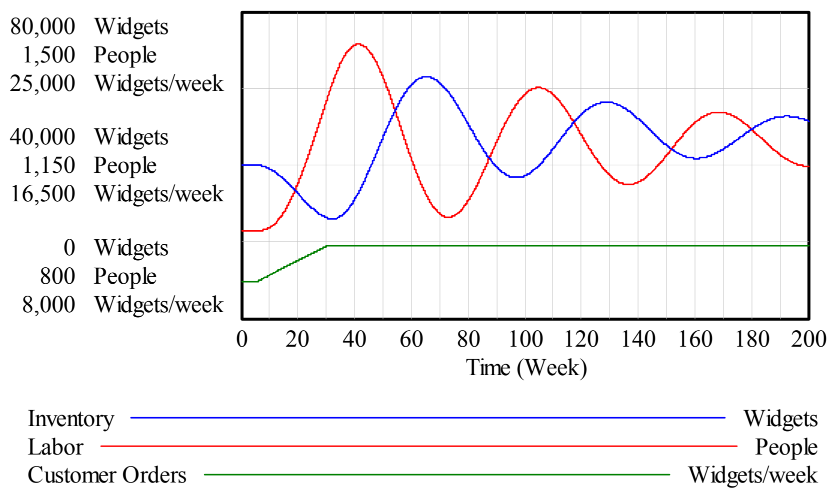

Figure 5.3. I-W model: Behavior of variables of interest for the current policy

In the next section an optimization problem is formulated to determine a new policy that can damp these fluctuations.

5.1.4. Optimization Problem

This optimization problem considers the simultaneous stabilization of the *Inventory* and *Labor* state variables according to the equations described in section 3.1.2. Equal weights (w_s=0.5, s=1,2) were assigned to these two variables. The time horizon (T) considered was 200 weeks.

Let x_1= Inventory, x_2=Labor

Let a_i=the new equilibrium point associated to the i^{th} state variable (i=1,2)

$$\underset{\mathbf{p}}{\text{Minimize}}\ J(\mathbf{p}) = \sum_{s=1}^{2}\left\{0.5\int_{0}^{200}\left|x_s(t)-a_s\right|dt\right\}$$

Subject to

$\dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}(t), \mathbf{p})$ (This notation represents the SD model equations)

$\mathbf{x}_0^{\ T} = [40000 \quad 1000 \quad 80000 \quad 80]$

$1 \leq$ Manufacturing Cycle Time ≤ 8

$1 \leq$ Inventory Adjustment Time ≤ 50

$50 \leq$ Average Duration of Employment ≤ 150

$1 \leq$ Average Time to Fill Vacancies ≤ 50

$1 \leq$ Labor Adjustment Time ≤ 50

$1 \leq$ Vacancy Adjustment Time ≤ 50

$1 \leq$ WIP Adjustment Time ≤ 50

$1 \leq$ Minimum Order Processing Time ≤ 50

$1 \leq$ Safety Stock Coverage ≤ 50

$10{,}000 \leq a_1 \leq 150{,}000$

$10 \leq a_2 \leq 1{,}000$

5.1.5. Stabilization Policy

The stabilization policy is obtained after solving the optimization problem presented in the previous section. The optimization algorithm was run at the fifth week using the following settings: swarm size = 30 particles, neighborhood size = 3 particles, initial inertia weight = 0.5, iteration lag = 5, cognitive coefficient = 1.2, social coefficient = 1.2. These settings were obtained after performing some initial experiments using the empirical rules defined in Table 3.1. They will be used as the initial settings in the other case studies. The time to obtain the optimal policy (after 150 PSO iterations and 1,393 PHC iterations) was 206 seconds.

The solution yielded the parameter values shown in Table 5.2. This table also includes parameters a_1, a_2 which are the new equilibrium points for the state variables of interest.

Table 5.2. I-W model: Parameter values for the stabilization policy

Parameter	Value	Unit
Manufacturing Cycle Time	5.02	Weeks
Inventory Adjustment Time	2.53	Weeks
Average Duration of Employment	50.14	Weeks
Average Time to Fill Vacancies	1	Weeks
Labor Adjustment Time	1	Weeks
Vacancy Adjustment Time	1	Weeks
WIP Adjustment Time	10.96	Weeks
Minimum Order Processing Time	1	Weeks
Safety Stock Coverage	3.54	Widgets
a_1 (EP for Inventory)	54,482.22	Widgets
a_2 (EP for Labor)	1,200.2	People

Figure 5.4 shows the behavior of the state variables when this revised policy is applied at the fifth week. While there are, indeed, changes to these variables, their fluctuations have all but disappeared approximately in 30 weeks (response time) since the system was disturbed. This figure also shows that the convergence of ADE has caused the asymptotic stability of the two state variables of interest.

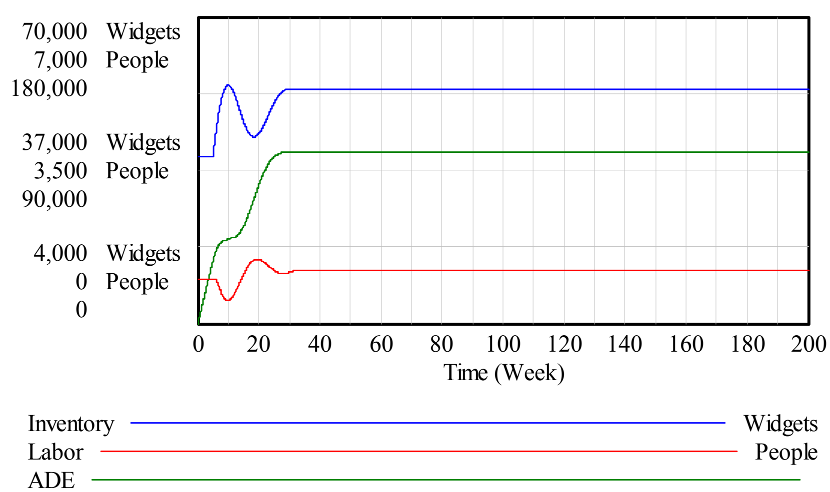

Figure 5.4. I-W model: Behavior of variables of interest for the stabilization policy

An interpretation of the improved policy points out that to keep up with the increased customer orders is necessary to increase the production rate reducing the manufacturing cycle time and the time to adjust inventory. Because production and labor are directly proportional, decreasing the time to adjust labor and vacancies will help production to track the desired production rates more closely.

5.1.6. Testing for Policy Robustness

The stabilization policy is tested by generating a sudden change in week 40 in the customer orders and showing the system's response to this change. The customer order rate is increased or decreased to new levels calculated as a percentage of its initial value (Figure 5.5). Figures 5.6 and 5.7 depict the robust behavior of the *Inventory* and *Labor* variables to the changes. These variables show a sharp increase or decrease in their levels and few oscillations before reaching new equilibrium points (see Table 5.3). Stability returns approximately 60 weeks after the system was perturbed (response time). This represents 37.5% of the remaining time horizon since the system was perturbed.

For each variable of interest, the new EP levels have moved from their previous value (Table 5.2) almost in the same percentage that the corresponding change in customer orders. For instance, for a 10% increase in customer orders the new EP of 59,927 reached by the *Inventory* variable represents a 9.99% increment of its prior value of 54,482.22.

Table 5.3. I-W model: New equilibrium points for the variables of interest

Percentage change in customer orders	New EP for Inventory (Widgets)	New EP for Labor (People)
-10%	49,031	1,080
-5%	51,756	1,139

Percentage change in customer orders	New EP for Inventory (Widgets)	New EP for Labor (People)
+5%	57,203	1,260
+10%	59,927	1,320

From Table 5.4 it is possible to conclude that the adaptation to the changes in customer orders shows diverse types of amplifications for the two variables of interest. For the *Inventory* variable the amplification effect (between 3% and 7%) is in a lower magnitude that the one affected to the *Labor* variable (between 18% and 43%). This indicates that *Labor* is more sensible to a sudden change in customer orders before reaching equilibrium again, with amplifications 3 or 4 times the effect of the change.

Table 5.4. I-W model: Amplification over/under the new equilibrium points

Percentage change in customer orders	Amplification over/under the new Inventory EP (Percentage)	Amplification over/under the new Labor EP (Percentage)
-10%	-6.95%	-42.52%
-5%	-3.30%	-20.17%
+5%	+2.97%	+18.15%
+10%	+5.68%	+34.70%

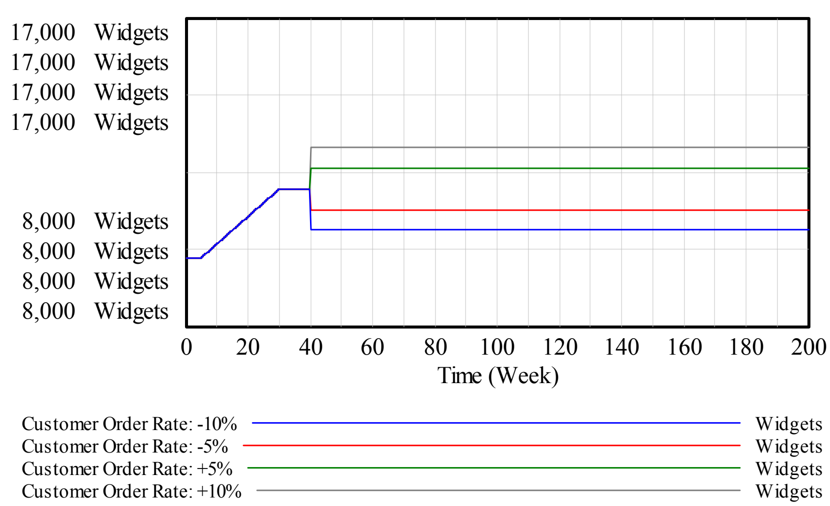

Figure 5.5. I-W model: Changes in the customer order rate to test policy robustness

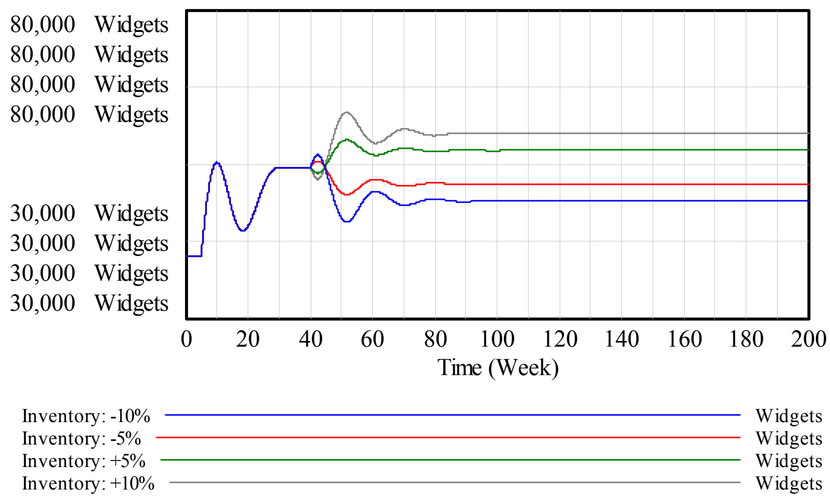

Figure 5.6. I-W model: Behavior of Inventory due to changes in customer orders

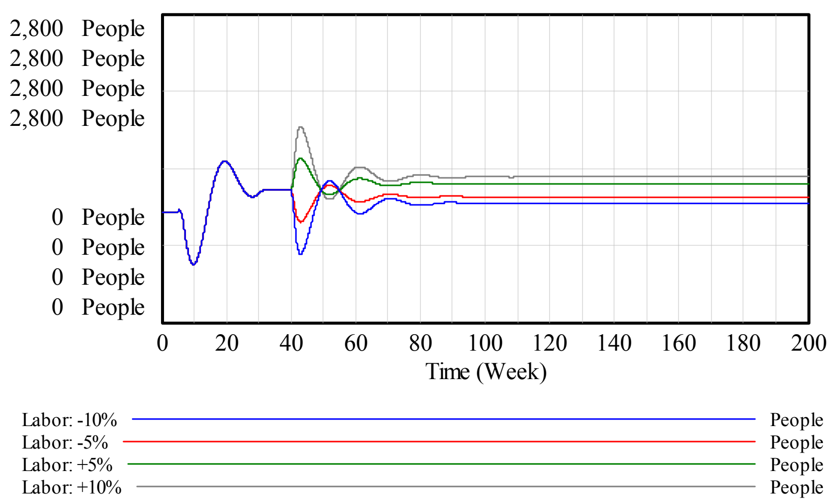

Figure 5.7. I-W model: Behavior of Labor due to changes in customer orders

5.2. Case Study B: The Mass Model

5.2.1. Description

Mass (Mass 1975) developed a nonlinear SD model to explore the economic processes underlying business-cycle behavior. Business cycles are recurring fluctuations in the macro-economy that affect total production, prices, employment, inventories and capital investment. A better understanding of the causes of cyclic behavior is critical to the formulation of effective stabilization policies by decision makers. The case study is a simplified version of the model developed by Mass and was designed by Kampmann and Oliva (2006). It allows the analysis of the role of labor-adjustment (hiring and termination) policies and capital-investment policies in generating business cycles in a supply chain. The objective of this case study is to use the SADE methodology to propose a stabilization policy for the three main state variables of the model: *Capital*, *Inventory* and *Labor*.

5.2.2. SD Model

The model interrelates inventories, backlogs, employment and investment decisions to provide a deeper understanding of the factors underlying intermediate-run (fifteen- to twenty-year) economic cycles. It contains (1) a production sector plus two factors of production: (2) a labor sector and (3) a capital sector. These sectors are described and depicted below.

Business cycles are characterized by amplification of demand in successive stages of production. To represent these amplification effects on a SC, a model would need to represent these sectors: consumer, retail, wholesale and production. In order to study the response to incoming orders emanating from the consumer and retail sectors, these two sectors have been aggregated into one sector and the wholesale and production sectors into another sector. This simplified system has been called the **production management sector** (Figure 5.8). Within this sector a *Desired Production Rate* is calculated on the bases of an *Average Production Rate* and *Inventory* and *Backlog* conditions. The sector can maintain a given *Production Rate* using different combinations of capital and labor.

The **labor sector** (Figure 5.9) introduces the influence of labor availability on production rate. Labor is a production resource whose lead time is affected by the tightness of labor markets and by the length of any training delays. Variations in labor and over- or undertime change the utilization of company's capital equipment. The state variables in this model are *Labor*, *Vacancies* and *Average New Vacancy Creation*. This last variable is defined as an exponentially averaged value of *New Vacancy Creation*. The dependence of new hiring decisions on *Average New Vacancy Creation* reflects the position that arises from reluctance to restrain recruitment activities during temporary business slowdowns and from other factors.

The **capital sector** (Figure 5.10) allows incorporating the decisions played by capacity expansion policies to determine how much to invest in production capacity. These decisions are critical to match demand in long lead time resources such as capital equipment, balancing the costs of shortfall against the costs of excess. The state variables in this model are *Capital*, *Capital on Order* (which corresponds to an unfilled order backlog for capital goods), and *Average Orders for Capital* that represents an exponential average of *Orders for Capital*.

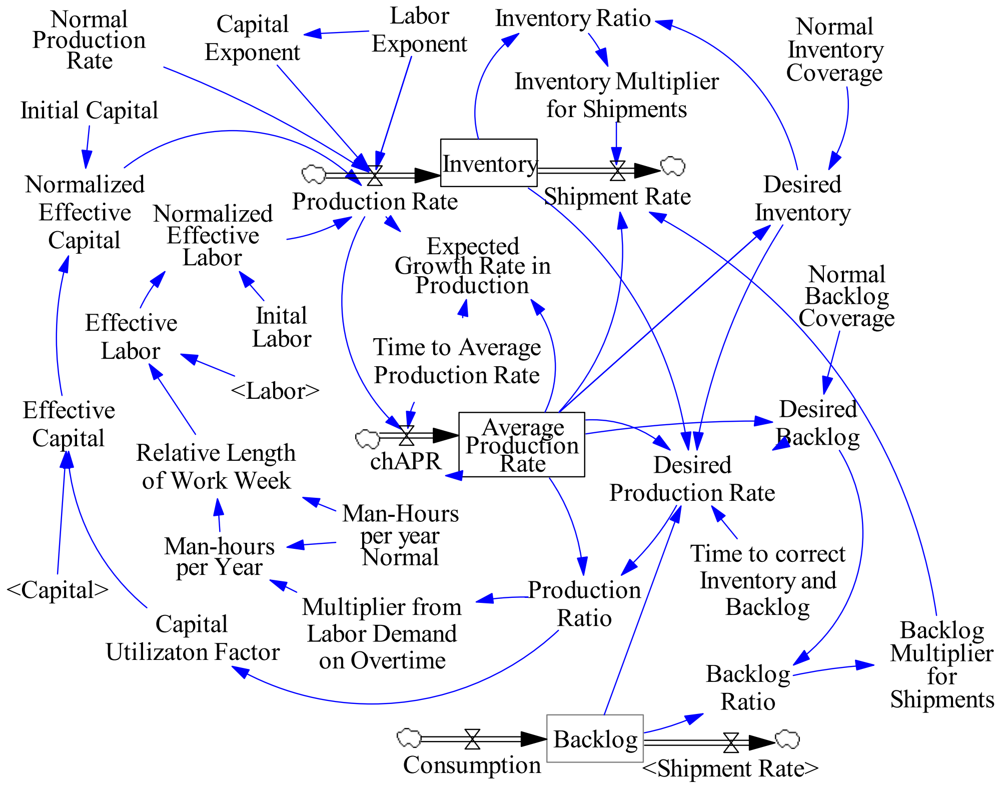

Figure 5.8. Mass model: Structure of production management sector

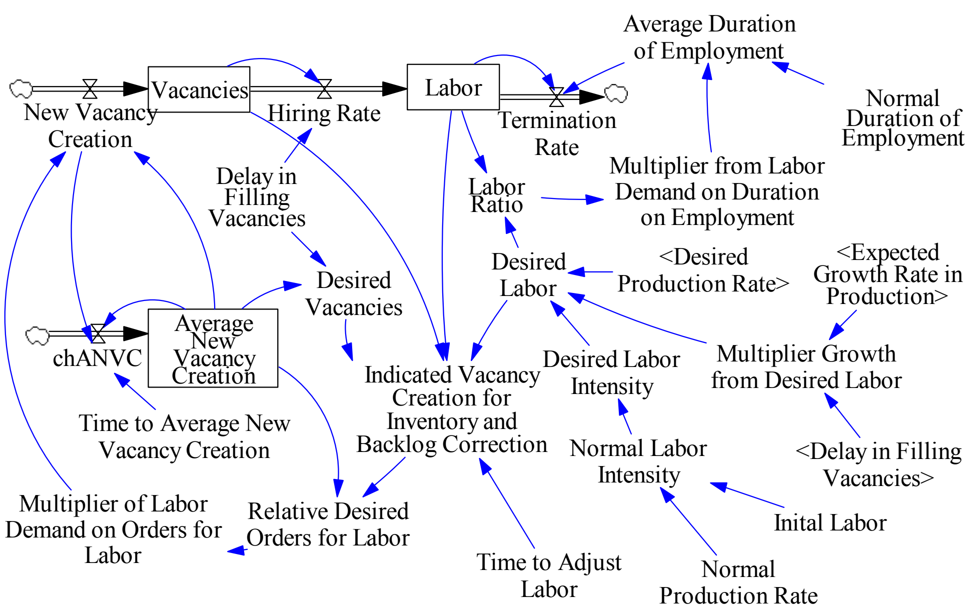

Figure 5.9. Mass model: Structure of labor sector

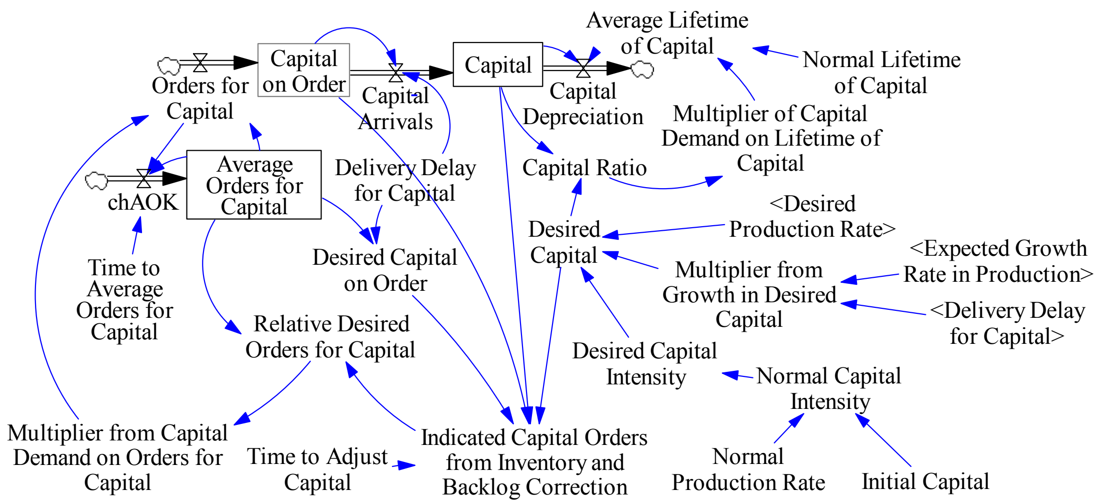

Figure 5.10. Mass model: Structure of capital sector

5.2.3. Current Policy and SC Instability

The set of parameters in Table 5.5 defines the current policy for this model.

Table 5.5. Mass model: Parameter values for the current policy

Parameter	**Value**	**Unit**
Normal Production Rate	3E06	Units/year
Initial Capital	7.5E06	Capital Units
Initial Labor	1500	People
Time to Average Production Rate	1	Years
Normal Inventory Coverage	0.5	Years
Time to Correct Inventory and Backlog	0.8	Years
Normal Backlog Coverage	0.2	Years
Delay in Filling Vacancies	0.25	Years
Time to Average New Vacancy Creation	0.5	Years
Time to Adjust Labor	0.5	Years
Normal Duration of Employment	2	Years
Time to Average Orders for Capital	4	Years
Delivery Delay for Capital	2	Years
Time to Adjust Capital	4	Years
Normal Life of Capital	15	Years

For a consumption rate of 1,400,000 units/year the system starts out of equilibrium. The behavior of the three variables of interest is depicted in Figure 5.11. Variables *Inventory* and *Labor* have several oscillatory fluctuations before they start to settle down. Capital shows a decreasing rate for a long period and then a small increment before starting to settle down.

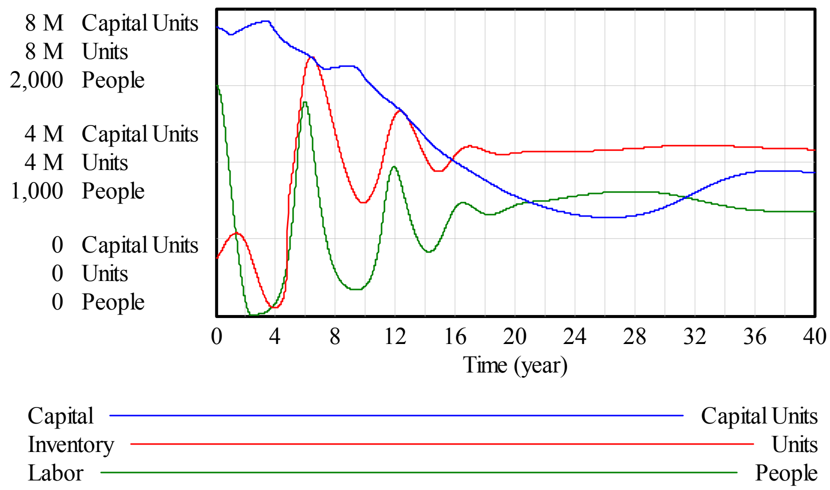

Figure 5.11. Mass model: Behavior of variables of interest for the current policy

The new policy to eliminate these fluctuations is obtained from the optimization problem formulated in the next section.

5.2.4. Optimization Problem

This optimization problem considers the simultaneous stabilization of the following state variables: *Capital*, *Inventory* and *Labor* according to the equations described in section 3.1.2. Equal weights (w_s=0.33, s=1,..,3) were assigned to these variables. The time horizon (T) considered was 40 years.

Let x_1=Capital, x_2=Inventory, x_3=Labor

Let a_i=the new equilibrium point associated to the i^{th} state variable (i=1,..,3)

$$\underset{\mathbf{p}}{\text{Minimize}}\ J(\mathbf{p}) = \sum_{s=1}^{3}\left\{0.33\int_{0}^{40}\left|x_s(t) - a_s\right|dt\right\}$$

Subject to

$$\dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}(t), \mathbf{p}) \quad \text{(This notation represents the SD model equations)}$$
$$\mathbf{x}_0^{\ T} = [7.5E06 \quad 1.5E06 \quad 1500]$$
1E06 ≤ Normal Production Rate ≤ 1E07
1E06 ≤ Initial Capital ≤ 1E07
1E02 ≤ Initial Labor ≤ 1E04
0.1 ≤ Time to Average Product Rate ≤ 5
0.1 ≤ Normal Inventory Coverage ≤ 5
0.1 ≤ Time to Correct Inventory and Backlog ≤ 5
0.1 ≤ Normal Backlog Coverage ≤ 5
0.1 ≤ Delay in Filling Vacancies ≤ 5
0.1 ≤ Time to Average New Vacancy Creation ≤ 5
0.1 ≤ Time to Adjust Labor ≤ 5
0.1 ≤ Normal Duration of Employment ≤ 5
1 ≤ Time to Average Order for Capital ≤ 10
1 ≤ Delivery Delay Capital ≤ 10
1 ≤ Time to Adjust Capital ≤ 10
1 ≤ Normal Lifetime of Capital ≤ 20
$1E06 \leq a_1 \leq 1E07$
$5E05 \leq a_2 \leq 5E06$
$1E02 \leq a_3 \leq 1E04$

5.2.5. Stabilization Policy

The stabilization policy is obtained after solving the optimization problem presented in the previous section. The optimization algorithm was run at time 0 using the following settings: swarm size = 30 particles, neighborhood size = 3 particles, initial inertia weight = 0.5, iteration lag = 5, cognitive coefficient = 1.2, social coefficient = 1.2. It can be seen from Figure 5.12 that the ADE does not converge, i.e., it is not showed as a horizontal line. Although the slope of the

ADE curve is smooth, due to fact of no convergence there is not guarantee that the stabilization policy obtained will be robust.

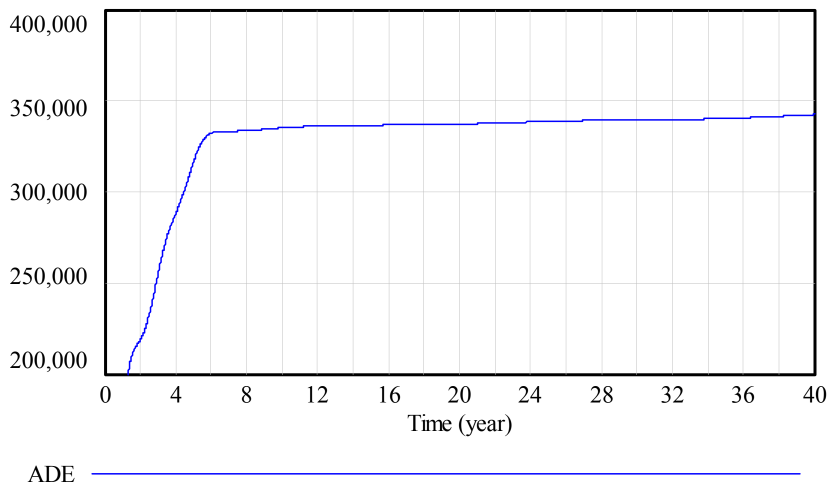

Figure 5.12. Mass model: ADE curve

To obtain a new solution that satisfies the robustness condition, the optimization algorithm will be run again using the following new settings: initial inertia weight = 0.1, cognitive coefficient = 1, social coefficient = 1, neighborhood size = 5 particles. The other settings remain the same. Moreover, to speed up the convergence it will be used the AEDE (see Theorem 7 in chapter 4) in the objective function for small DE. This can be done by using a tolerance factor L_s (s=1,..,3). When DE are above this factor, the objective function is calculated normally using the ADE; otherwise the value of the AEDE is used in the objective function. The optimization problem will change as follows:

$$\underset{\mathbf{p}}{\text{Minimize}}\ J(\mathbf{p}) = \sum_{s=1}^{3} 0.33\, J_s(\mathbf{p})$$

$$J_s(\mathbf{p}) = \begin{cases} \int_0^{40} \left| x_s(t) - a_s \right| dt, & \text{if } \left| x_s(t) - a_s \right| > L_s \\ \int_0^{40} e^{\left| x_s(t) - a_s \right|} dt, & \text{if } \left| x_s(t) - a_s \right| \le L_s \end{cases}$$

where L_1=10, L_2=5, L_3=1 are set for the variables *Capital*, *Inventory* and *Labor* respectively.

The time to obtain the optimal policy (after 150 PSO iterations and 3,171 PHC iterations) was 306 seconds.

The solution yielded the parameter values shown in Table 5.6. This table also includes parameters a_1, a_2, a_3 which are the new equilibrium points for the state variables of interest.

Table 5.6. Mass model: Parameter values for the stabilization policy

Parameter	**Value**	**Unit**
Normal Production Rate	1,160,299	Units/year
Initial Capital	5,102,877	Capital Units
Initial Labor	7,630.05	People
Time to Average Production Rate	0.86	Years
Normal Inventory Coverage	0.42	Years
Time to Correct Inventory and Backlog	0.49	Years
Normal Backlog Coverage	0.14	Years
Delay in Filling Vacancies	0.1	Years
Time to Average New Vacancy Creation	0.61	Years
Time to Adjust Labor	0.15	Years
Normal Duration of Employment	0.54	Years
Time to Average Orders for Capital	3	Years
Delivery Delay for Capital	1.17	Years
Time to Adjust Capital	1	Years
Normal Life of Capital	9.37	Years
a_1 (EP for Capital)	6,159,479	Capital Units
a_2 (EP for Inventory)	587,767.5	Units
a_3 (EP for Labor)	9,223.15	People

Figure 5.13 shows that after applying the revised policy the system has reached equilibrium in 14 years (response time). This figure also shows that the convergence of ADE has caused the asymptotic stability of the three state variables of interest. This was achieved by increasing the parameter values *Initial Labor* and *Time to Average New Vacancy Creation* and decreasing several other parameter values including *Normal Production Rate*, *Time to Correct Inventory and Backlog*, *Time to Adjust Labor*, and *Time to Adjust Capital*.

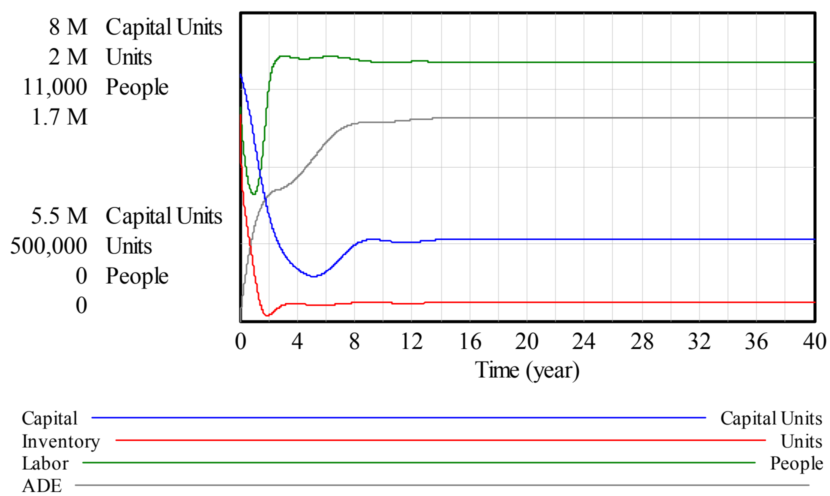

Figure 5.13. Mass model: Behavior of variables of interest for the stabilization policy

5.2.6. Testing for Policy Robustness

The stabilization policy is tested by generating a linear change in the consumption rate from year 10 to year 20. The consumption rate is increased or decreased to new levels calculated as a percentage of its initial value (Figure 5.14). Figures 5.15, 5.16 and 5.17 depict the robust behavior of the *Capital*, *Inventory* and *Labor* variables to the changes. The adaptation to the changes is smooth with amplifications less than 2% over/under the new EPs for the three variables of interest. The values for the new EPs are shown in Table 5.7. Stability returns approximately 15 years after applying the stabilization policy (response time). This represents 50% of the time horizon since the system was disturbed.

For each variable of interest, the new EP levels have moved from their previous value (Table 5.6) almost in the same percentage that the corresponding change in the consumption rate. For instance, for a 5% increase in the consumption rate the new EP of 9,685 reached by the variable *Labor* represents a 5.01% increment of its prior value of 9,223.15.

Table 5.7. Mass model: New equilibrium points for the variables of interest

Percentage change in consumption rate	New EP for Capital (Capital Units)	New EP for Inventory (Units)	New EP for Labor (People)
-10%	5,543,000	528,235	8,301
-5%	5,851,000	557,571	8,762
+5%	6,467,000	616,269	9,685
+10%	6,775,000	645,618	10,146

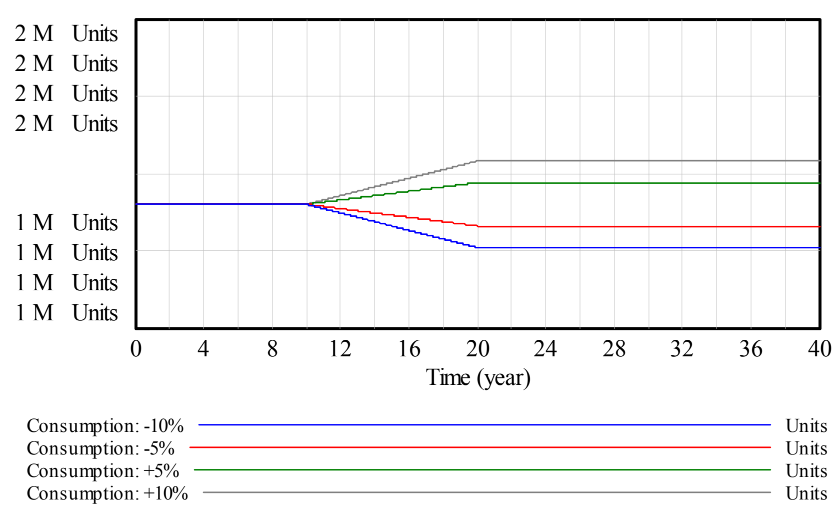

Figure 5.14. Mass model: Changes in the consumption rate to test policy robustness

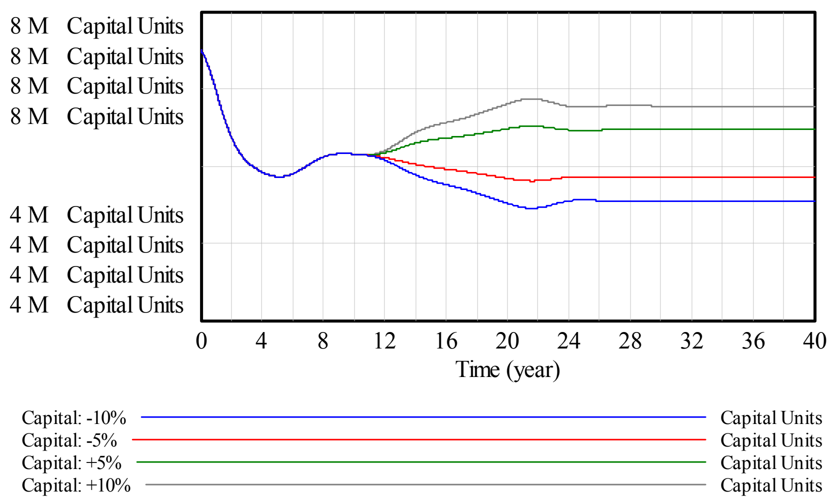

Figure 5.15. Mass model: Behavior of Capital due to changes in consumption

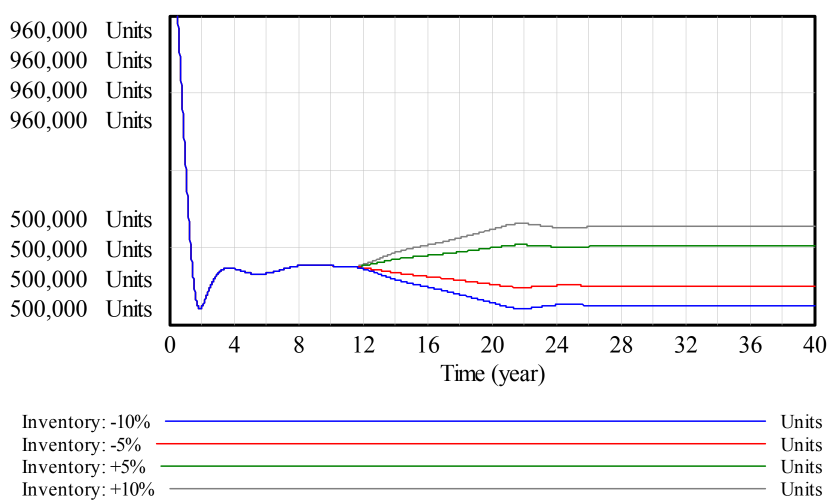

Figure 5.16. Mass model: Behavior of Inventory due to changes in consumption

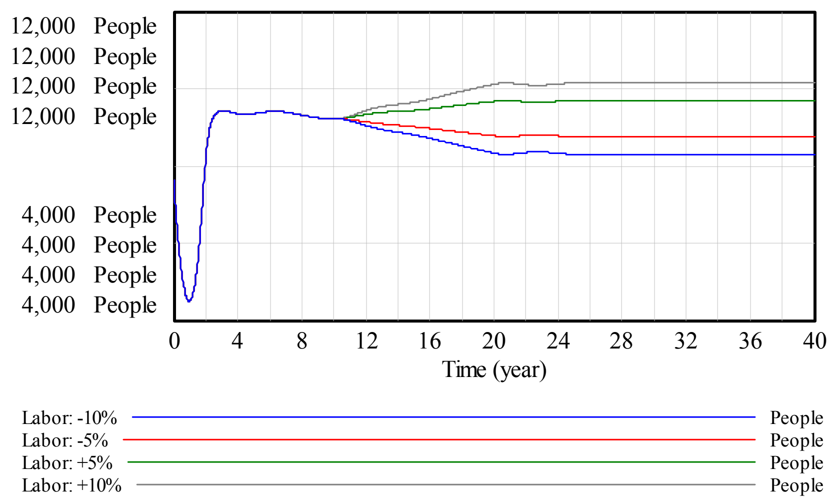

Figure 5.17. Mass model: Behavior of Labor due to changes in consumption

5.3. Case Study C: The LSMC Model

5.3.1. Description

This is the case study involving a real electronics manufacturer, which is designated as LSMC (Lertpattarapong 2002) to respect confidentiality. LSMC products are technological gadgets and personal computer complementary products. LSMC is the major supplier of companies like Compaq or Dell. The increasing competition in the market of personal computers has caused fluctuations in the demand that resulted in oscillatory behavior of LSMC finished goods inventory and capacity.

Since 1998, led by Dell, many original equipment manufacturers have changed their strategies by aggressively eliminating slack in their inventories through a build-to-order manufacturing and just-in-time processes. Further, because of fast dynamic changes in the market of personal computers, the short lifecycle associated with them and other complementary products has also amplified coordination problems, which in turn have often caused excess inventory and sometimes difficulties to keep up with demand. Moreover, the competition has forced the company to introduce more product varieties at lower prices into the market to protect its existing and potential market share. Production capacity is another factor that adds to supply chain complexity because its long delays, huge investments, and new products with more complex manufacturing processes than previous generations. In addition, these complementary products are at the upstream of the supply chain for personal computers and their resulting fluctuations are higher.

Given the complex and dynamic nature of the supply chain at LSMC, it is difficult for LSMC to see how its policy decision might impact its performance or cause unexpected and undesirable consequences. The objective in this case study is to use the SADE methodology to propose a policy to eliminate instabilities in the finished goods inventory.

5.3.2. SD Model

This nonlinear SD model is based on the original work described on Lertpattarapong (2002). It comprises three connected stock and flow submodels: (1) the market share and shipment submodel, (2) the demand forecast and capacity submodel and (3) the production submodel. These submodels are described and depicted below.

The **market share and shipment submodel** (Figure 5.18) comprises two parts: market-share and inventory-backlog-shipping. The first part represents the links between orders filled, market share, and demand. The second part represents the links between inventories and customer orders, which are filled from the finished goods inventory and shipped to customers.

The state variables in this model are *Finished Goods Inventory*, the *Channel Order Backlog*, and *Perceived Fraction Orders Filled*. This last variable represents an "information delay" that captures the customer's perception regarding his/her order. This delay was forecasted using a third order smoothing function.

The **demand forecast and capacity submodel** (Figure 5.19) represents the link between demand and production capacity. The part of the model related to the demand forecast consists of two state variables: *Historical Demand* and *Perceived Present Demand*, which are smooth functions with the time horizon constant. The variable *Unit Forecast Demand* is then calculated from the *Historical Demand* and *Perceived Present Demand*. An important state variable of the capacity part of the model is *Available Capacity* which is a function of *Capacity Acquisition* (an estimate of how fast LSMC can build a pre-assembly facility) and *Capacity Obsolescence* (an estimate of an average life expectancy of a pre-assembly facility).

The **production submodel** (Figure 5.20) implements a push-pull strategy. The 'push' is from the pre-assembly processes to the assembly process. The 'pull' is from the assembly process to packaging and shipping. Inventories represent the principal variables in this model. Three types of inventory were modeled and represented by the state variables: *Pre-assembly Inventory*, *Assembly Inventory* and *Finished Goods Inventory*. The variable *Expected Channel Demand for LSMC* is a smooth function of *Channel Demand for LSMC Products*.

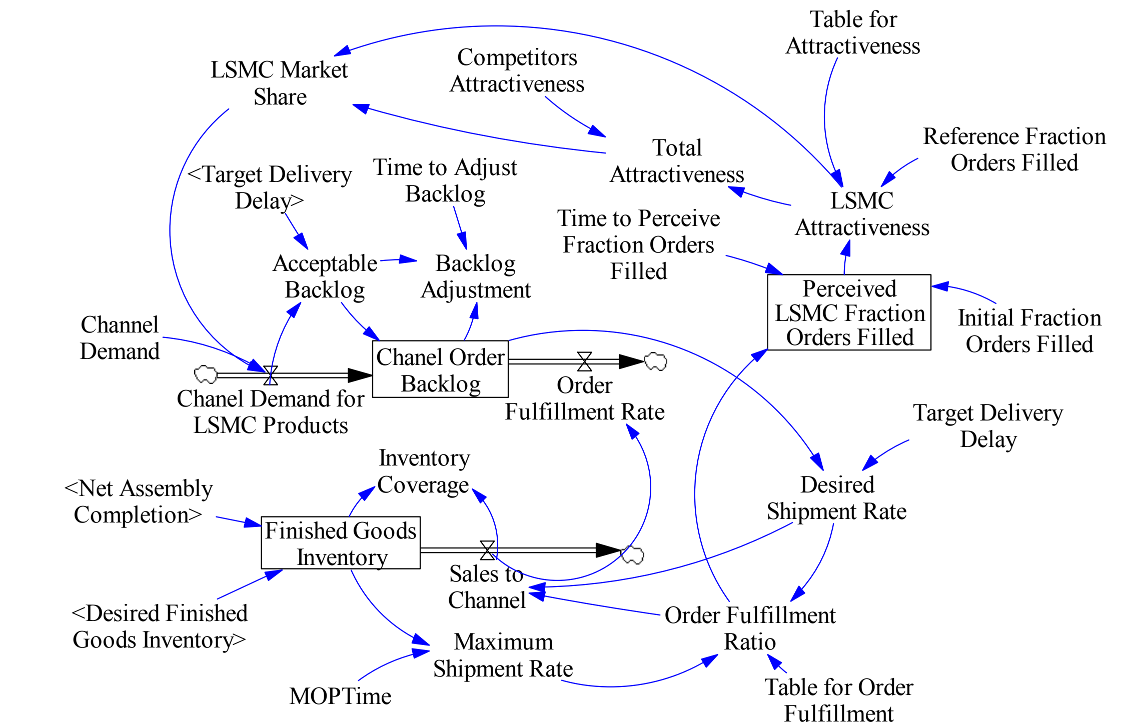

Figure 5.18. LSMC model: Market share and shipment submodel

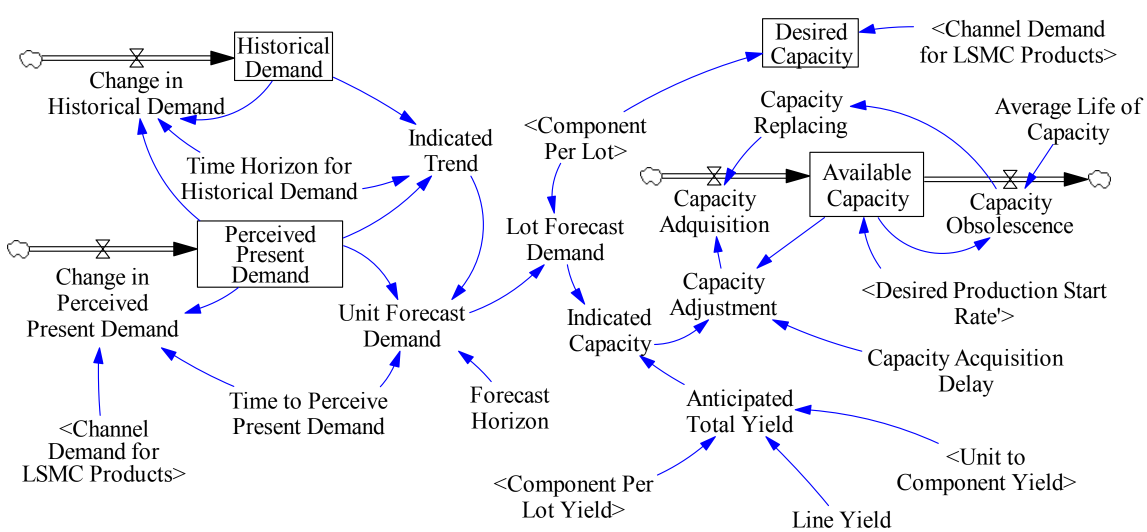

Figure 5.19. LSMC model: Demand forecast and capacity submodel

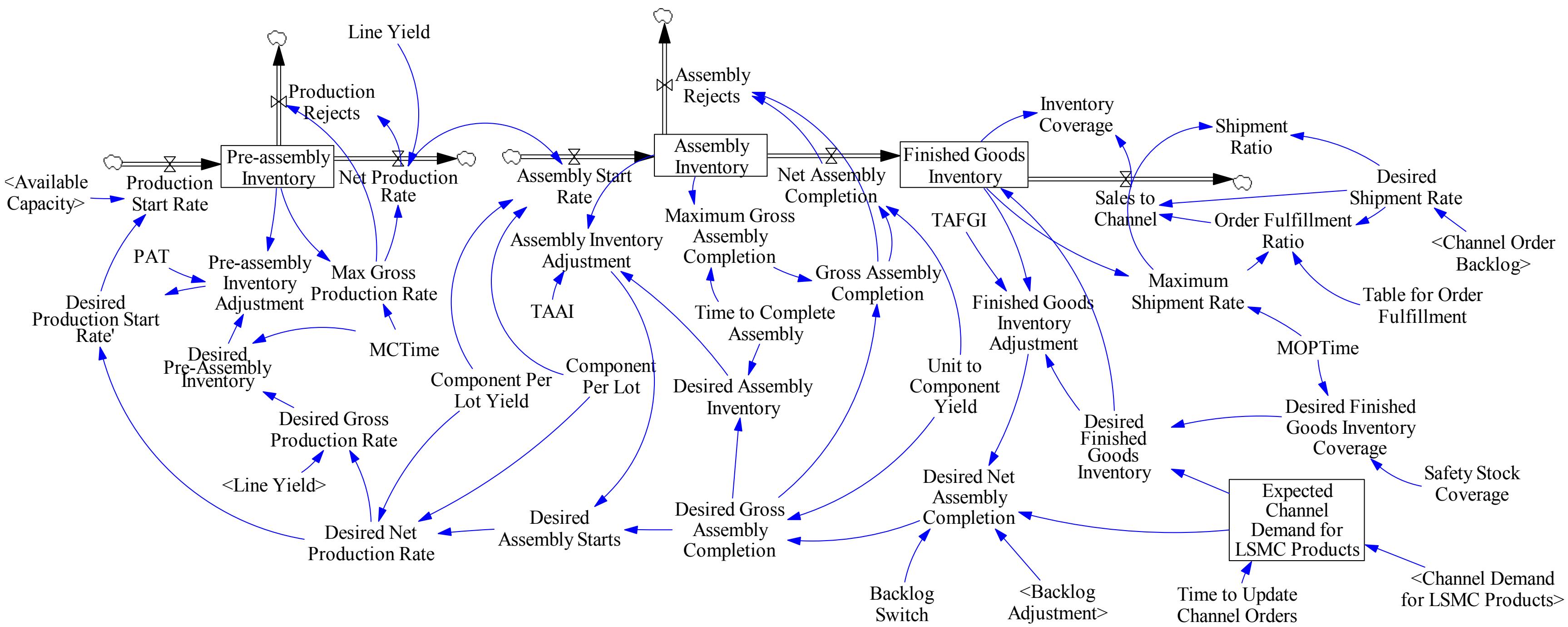

Figure 5.20. LSMC model: Production submodel

5.3.3. Current Policy and SC Instability

The current inventory policy is defined by five main parameters that are in control of the supply chain managers. These parameters are shown in Table 5.8.

Table 5.8. LSMC model: Parameter values for the current policy

Parameter	**Value**	**Unit**
Manufacturing Cycle Time (MCTime)	2	Months
Minimum Order Processing Time (MOPTime)	0.25	Months
Time to Adjust Assembly Inventory (TAAI)	0.5	Months
Pre Assembly Adjustment Time (PAT)	2	Months
Time to Adjust Finished Goods Inventory (TAFGI)	2	Months

The system starts and remains at equilibrium for the following eight months. Then the demand, which has a rate of five million units per month, is reduced by 20 percent.

The response of the supply chain to this increment in demand is a persistent ripple effect on the *Finished Goods Inventory* variable. Figure 5.21 shows this oscillatory behavior.

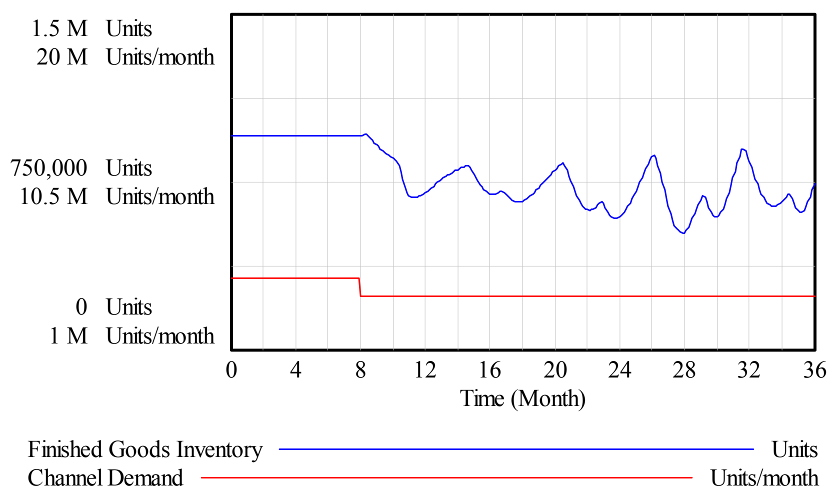

Figure 5.21. LSMC model: Behavior of the variable of interest for the current policy

A new policy to minimize these oscillations will be determined by solving the optimization problem presented in the next section.

5.3.4. Optimization Problem

This optimization problem considers the stabilization of the *Finished Goods Inventory* state variable according to the equations described in section 3.1.2. The time horizon (T) considered was 36 months.

Let x_1= Finished Goods Inventory

Let a_1=the new equilibrium point associated to the state variable x_1

$$\underset{\mathbf{p}}{\text{Minimize}}\ J(\mathbf{p}) = \int_0^{36} \left| x_1(t) - a_1 \right| dt$$

Subject to

$\dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}(t), \mathbf{p})$ (This notation represents the SD model equations)

$\mathbf{x}_0^{T}$ (Vector with initial values of all state variables)

$1 \leq$ Manufacturing Cycle Time ≤ 3

$0.1 \leq$ Minimum Order Processing Time ≤ 1

$0.1 \leq$ Time to Adjust Assembly Inventory ≤ 8

$0.5 \leq$ Pre Assembly Adjustment Time ≤ 10

$0.5 \leq$ Time to Adjust Finished Goods Inventory ≤ 10

$5E05 \leq a_1 \leq 1E06$

5.3.5. Stabilization Policy

After solving the optimization problem presented in the previous section, the stabilization policy shown in Table 5.9 is obtained. It is important to note that the new equilibrium point for the *Finished Goods Inventory* has decreased not significantly from its original value of 956,971 units.

The optimization algorithm used the following settings: swarm size = 30 particles, neighborhood size = 3 particles, initial inertia weight = 0.5, iteration lag = 5, cognitive coefficient = 1.2, social coefficient = 1.2. The time to obtain the optimal policy (after 100 PSO iterations and 79 PHC iterations) was 236 seconds.

Table 5.9. LSMC model: Parameter values for the stabilization policy

Parameter	Value	Unit
Manufacturing Cycle Time (MCTime)	2.37	Months
Minimum Order Processing Time (MOPTime)	0.31	Months
Time to Adjust Assembly Inventory (TAAI)	5.22	Months
Pre Assembly Adjustment Time (PAT)	3.11	Months
Time to Adjust Finished Goods Inventory (TAFGI)	0.5	Months
a_1 (EP for Finished Goods Inventory)	949,315	Units

Figure 5.22 shows the behavior of the *Finished Goods Inventory* when this improved policy is applied at the eighth month. This variable reaches a stable level in the 10th month caused by the convergence of ADE. This represents a response time of two months.

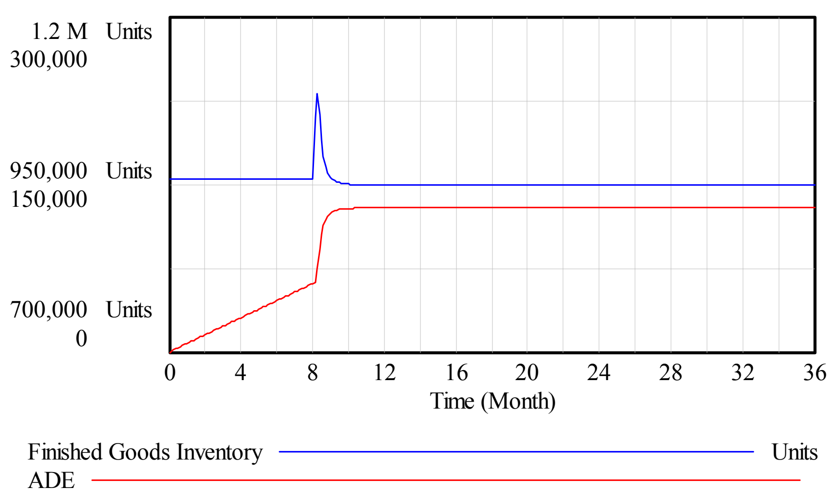

Figure 5.22. LSMC model: Behavior of the variable of interest for the stabilization policy

An interpretation of the improved policy points out that to stabilize the system close to the initial *Finished Goods Inventory* equilibrium point, it is necessary to increase the time to adjust assembly and pre assembly inventory. This means fewer inventory corrections will be needed in response to the customer orders change. On the other hand, by decreasing the time to adjust the finished goods inventory reduces the likelihood that this inventory will fall to unacceptable levels.

5.3.6. Testing for Policy Robustness

To test the stabilization policy a sudden change in demand is generated in month 20. The demand is increased or decreased to new levels calculated as a percentage of its initial value (Figure 5.23). The system response is depicted in Figure 5.24. There, it is shown that in all cases new EPs are reached to the variable *Finished Goods Inventory*. The values for these EPs are presented in Table 5.10.

The new EP levels have moved from their previous value (Table 5.9) almost in the same percentage that the corresponding change in the demand. For instance, for a 5% decrease in demand the new EP of 901,849 reached by the variable *Finished Goods Inventory* represents a 5% decrease of its prior value of 949,315.

The adaptation to the equilibrium state is smooth and fast in the case where demand is decreased. There are not amplifications under de EP. Stability returns approximately two weeks after the system was disturbed (response time) which represents 12.5% of the remaining time since the alteration. On the contrary, in the case where demand is increased it takes more time to reach stability, about 6 or 8 months to reach the new equilibrium points. Amplifications are on the order of 2% and 6% over the EPs for +5% and +10% increments in demand respectively.

Table 5.10. LSMC model: New equilibrium points for the variable of interest

Percentage change in demand	New EP for Finished Goods Inventory (Widgets)
-10%	854,383
-5%	901,849
+5%	996,771
+10%	1,044,216

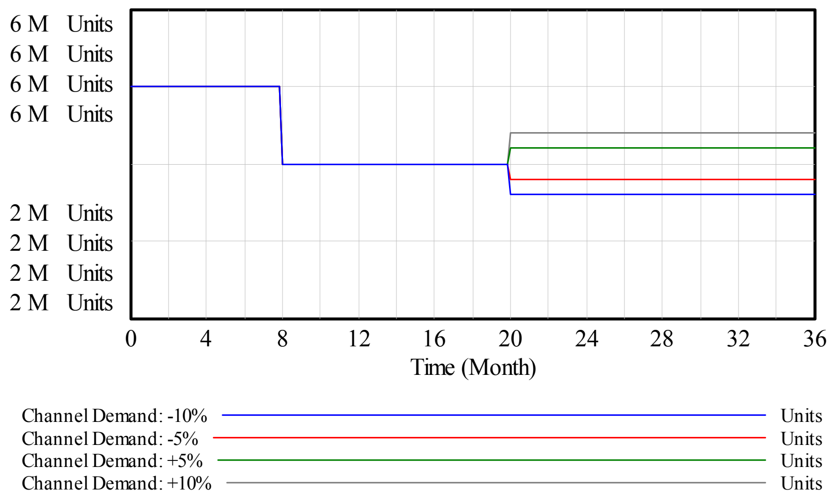

Figure 5.23. LSMC model: Changes in demand to test policy robustness

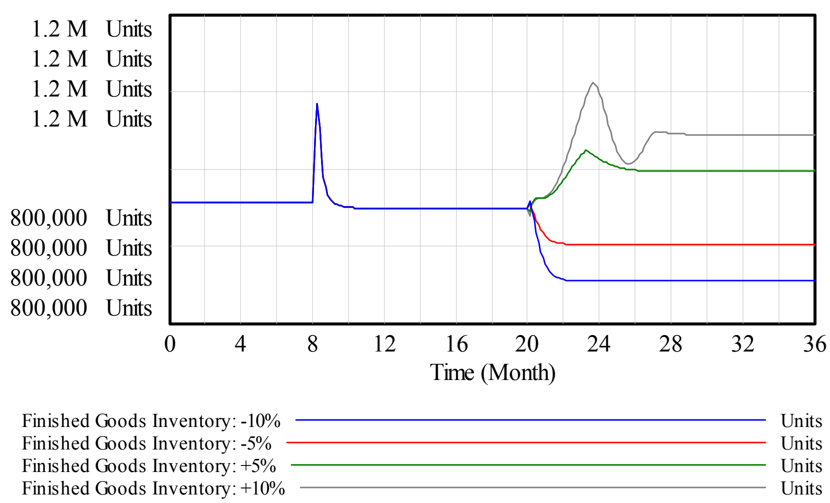

Figure 5.24. LSMC model: Behavior of Finished Goods Inventory due to changes in demand

5.3.7. Policy Comparison with another Method

This section compares the stabilization policies obtained by using two methods: (1) the SADE method described in this dissertation and (2) the eigenvalue and elasticity analysis as explained in Lertpattarapong (2002). A new scenario is presented to compare the policies. From equilibrium, the LSMC model is disturbed by a 10% step increase in Channel Demand at the sixth month. This causes an oscillatory behavior in the final inventory which is shown in Figure 5.25.

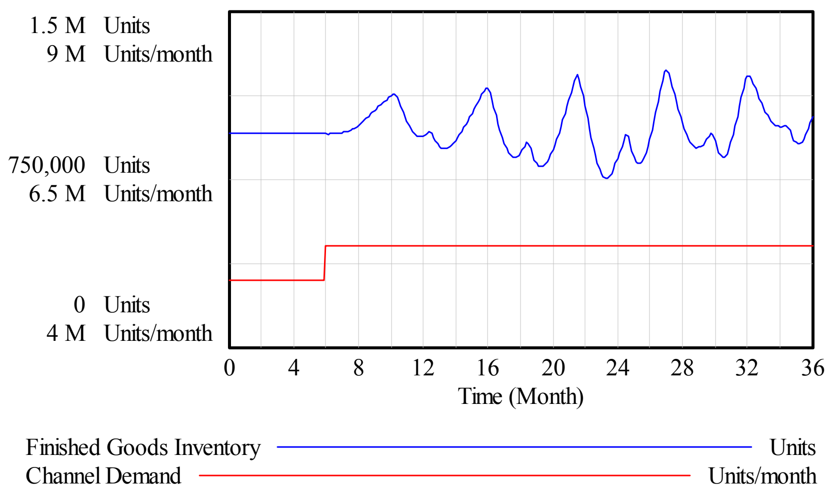

Figure 5.25. LSMC model: Oscillatory behavior of the Finished Goods Inventory

The stabilization policies to minimize this instability are presented in the next lines.

5.3.7.1. Stabilization Policy by using the SADE method

Using the same settings defined in 5.3.5., the optimization algorithm found the optimal policy in 392 seconds (after 100 PSO iterations and 73 PHC iterations). The stabilization policy is shown in the next table. The *Finished Goods Inventory* reaches the equilibrium level approximately in the 16th month (see Figure 5.26).

Table 5.11. LSMC model: SADE stabilization policy

Parameter	Value	Unit
Manufacturing Cycle Time (MCTime)	1	Months
Minimum Order Processing Time (MOPTime)	0.23	Months
Time to Adjust Assembly Inventory (TAAI)	6.69	Months
Pre Assembly Adjustment Time (PAT)	9.21	Months
Time to Adjust Finished Goods Inventory (TAFGI)	0.5	Months
a_1 (EP for Finished Goods Inventory)	968,448	Units

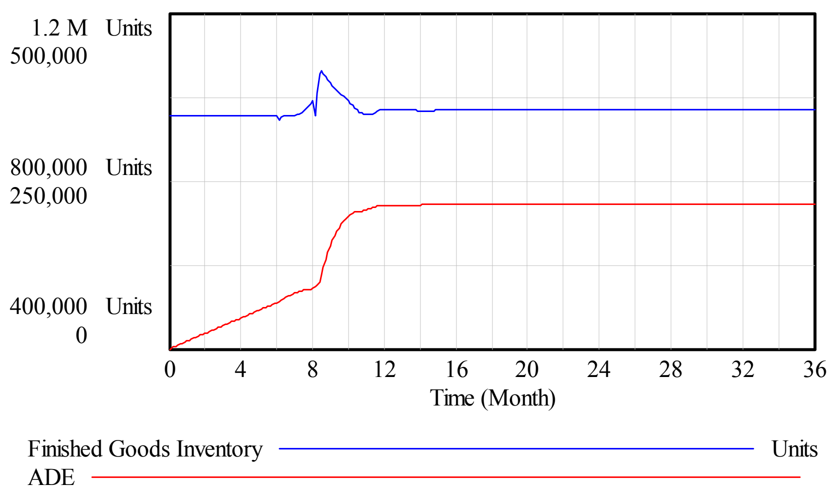

Figure 5.26. SADE method: Stable behavior of the Finished Goods Inventory

5.3.7.2. Stabilization Policy by using the EEA method

Before applying the EEA method, Lertpattarapong (2002) first linearized the nonlinear system at any point in time. Then, the eigenvalues and eigenvalue elasticity were calculated. This information was used to identify which eigenvalues contributed to the oscillations and then investigating the elasticities to determine which links and loops affected this oscillatory behavior. This analysis revealed that Loop L1, composed by the variables Channel Order

Backlog, Pre-assembly, Assembly and Finished Goods Inventory, caused the oscillation in the supply chain model. The interpretation is that LSMC is building up backlog due to the increment in demand. When the backlog occurs the order fulfillment ratio drops as well. This leads to a reduction in the demand. However, the order to increase the production was already sent, building up the inventory. When the inventory exceeds the backlog, LSMC will cut its productions. However, with the decrease in production, the backlog will occur again.

Thus, policies for lessening or stopping the oscillations should involve Loop L1. In his analysis, Lertpattarapong proposes to build up a safety stock to reduce backlog. He suggests building up a 1-week or 0.25 month for Safety Stock Coverage. This stabilization policy makes the *Finished Goods Inventory* to reach equilibrium around the 22nd month. This is depicted in the next figure.

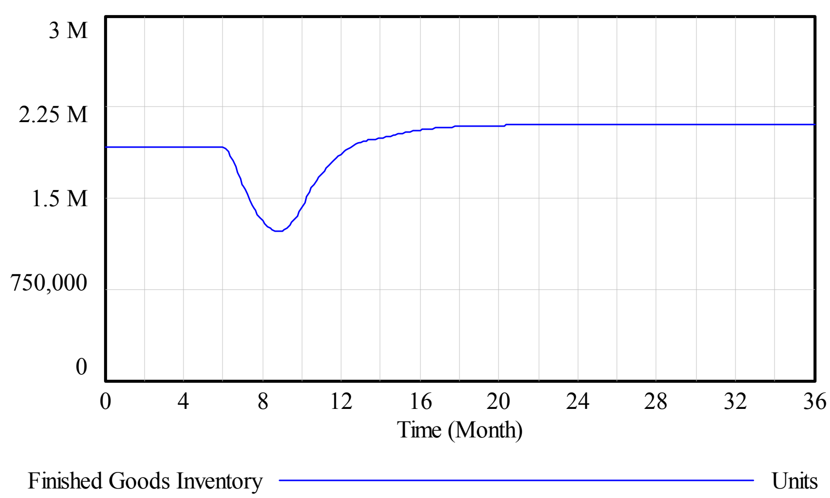

Figure 5.27. EEA method: Stable behavior of the Finished Goods Inventory

5.3.7.3. Comparison of Stabilization Policies

Figure 5.28 shows clearly that the stabilization policy obtained by SADE has a better response time and less amplification that the one obtained by EEA. This is due to the fact that SADE policy is considering important parameters to adjust the inventory levels, while EEA policy relies only in building up a safety stock to reduce the oscillations. Moreover, in terms of costs the SADE policy is also more economical than its counterpart because it requires fewer inventories.

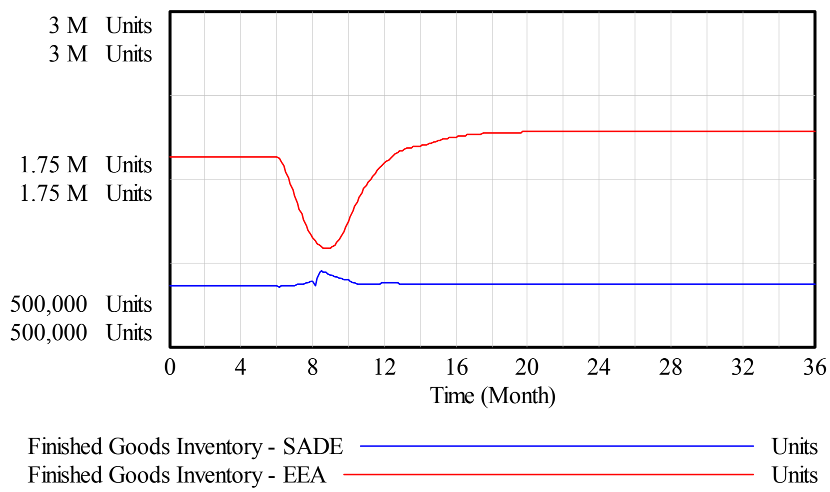

Figure 5.28. Inventory behaviors after using SADE and EEA methods

To perform the robustness analysis it is generated a sudden change in demand in month 22. The demand is again increased by 10 percent. The system response is depicted in Figure 5.29. The SADE policy makes the *Finished Goods Inventory* to reach equilibrium around the 28th month. The EEA policy starts stabilizing the system around the 35th month. The robustness analysis also shows that the EEA policy generates more amplification than the SADE policy before reaching the equilibrium state. Having lower levels of amplification keeps the inventory level closer to its new equilibrium point.

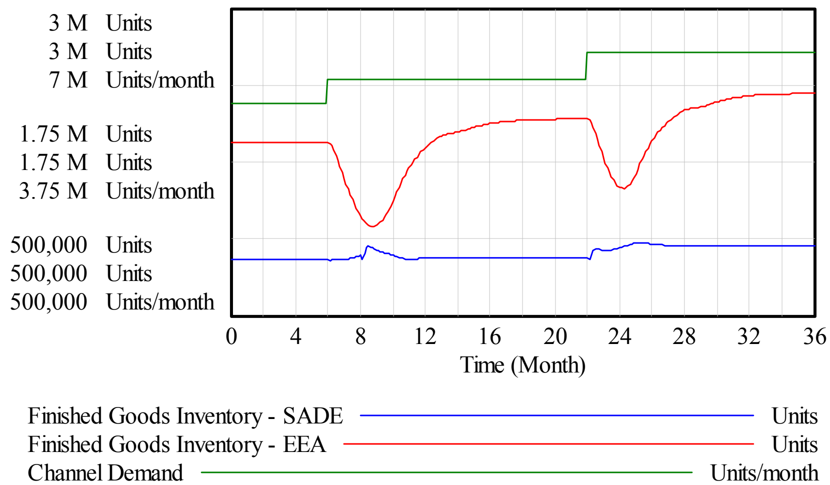

Figure 5.29. Policy robustness for SADE and EEA methods

The following table shows the comparison of the stabilization policies generated by the SADE and EEA methods.

Table 5.12. Results of the analysis for the SADE and EEA methods

	SADE	**EEA**
Stability analysis		
Response time as a percentage of the remaining time since the stabilization policy was applied	33.3%	53.3%
Amplification as a percentage over/under the equilibrium point	9.6% (over EP)	58.3% (under EP)
Robustness analysis		
Response time as a percentage of the remaining time since the stabilization policy was applied	42.9%	92.9%
Amplification as a percentage over/under the equilibrium point	2.7% (over EP)	66.7% (under EP)

It can be concluded that the policy obtained by the SADE method is more efficient (faster and smoother) and more economical to implement than the policy proposed by the EEA method.

5.4. Case Study D: The PMOC Model

5.4.1. Description

PMOC Technologies Inc. is a manufacturer of optical solutions for medical, industrial, communications, defense, test, and measurement applications. PMOC Technologies Inc. is an integrator of families of precision molded aspheric optics, glass products, and high performance fiber-optic collimators and isolators. The precision molded optics (PMO) process produces lenses for industrial laser and other optical applications and is the focus of the simulation model.

PMOC Inc. has built its reputation on providing customized products to long-term customers who have designed their equipment to use PMOC lenses. Lenses make up to 65% of the company's operations. It has a stable customer base of around 1,700 customers. With special requirements in lenses in addition to high quality level of service and support, customers are willing to pay relatively higher than traditional market prices. This has helped PMOC Inc. maintain a stable market share over the past few years despite using an old manufacturing technology with limited capacity.

Manufacturing equipment is utilized such that a maximum of 40% overtime is allowed. And due to relatively long term plan to move the lenses operations to Asia, the company desires to continue serving its customer base using existing workers and overtime; without hiring or training more workers. Workers will be moved to new productions lines and trained.

The company depends for the remaining periods on its stable base of customers who continue to rely on PMOC specially designed lenses until they upgrade to new technologies. The company however, should minimize expenses in the form of scrape and maintain stable operations. The goal of management is to find a policy that avoids large oscillations in the inventory if expected increase of customer orders on regular types of lenses occurs.

5.4.2. SD Model

The nonlinear SD model used in this case study is a subsystem of the enterprise system developed by Helal (2008). It is focused on the production process of PMOC and is composed by the following submodels: (1) supplier submodel, (2) labor management submodel and (3) internal supply chain submodel. These submodels are described and depicted below.

The **supplier submodel** (Figure 5.30) represents how the capacity of the supplier affects the rate at which the company orders raw materials (*Parts Order Rate*). To simplify the model it is assumed that only one supplier provides raw materials to PMOC. The state variables of this model are *Supplier Production Capacity* and *Supplier Order Backlog*.

The **labor management submodel** (Figure 5.31) estimates the required capacity level (including overtime when necessary) based on the production rate obtained from the production planning. The opening positions for recruiting new workers are represented in the state variable *Labor Being Recruited*. Labor being recruited moves to become *Labor* (get hired) after some hiring delay, according to the *Labor Hiring Rate*. Similarly, *Labor* can be fired o leave voluntarily the company at the *Labor Firing Rate*.

The **internal supply chain submodel** (Figure 5.32) consists of two overlapping constructs. The first construct is the materials ordering and inventory. The state variables for this part of the model are *Parts on Order*, and *Parts Inventory*. The usage rate of parts (raw material) being taken from *Parts Inventory*, to be converted into semi finished products (WIP inventory) is given by the *Production Start Rate*. The second construct is the production planning. This part of the model regulates the WIP inventory at the Preforms and Presses departments to ensure smooth production rate and the availability of the final products for shipping. The state variables of this part of the model are *Preforms WIP* and *Presses WIP* and *Finished Goods Inventory*.

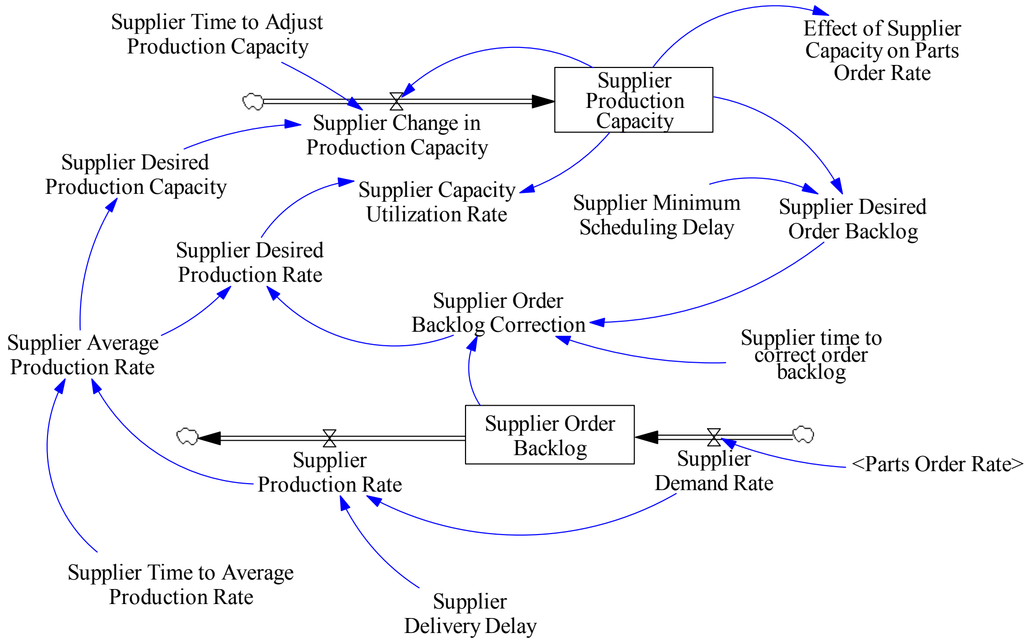

Figure 5.30. PMOC model: Supplier submodel

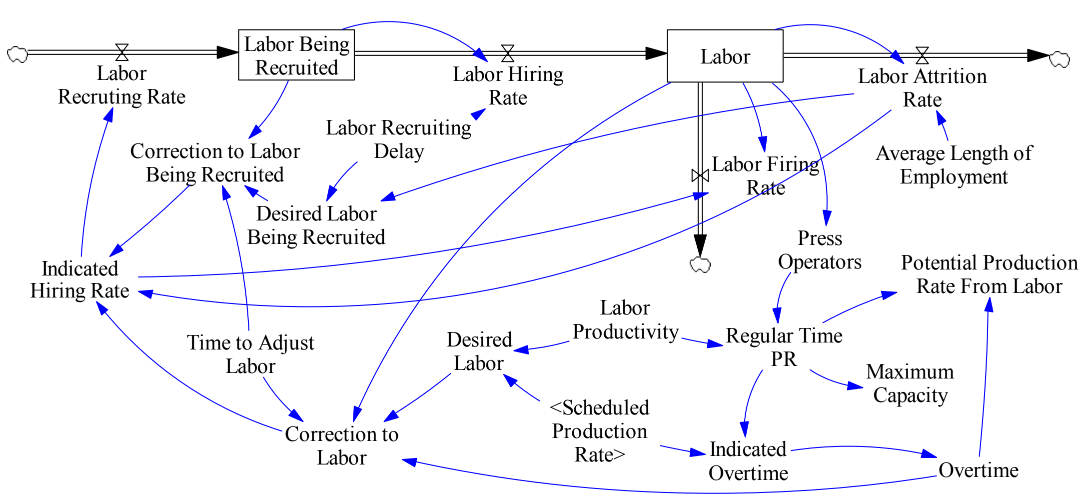

Figure 5.31. PMOC model: Labor management submodel

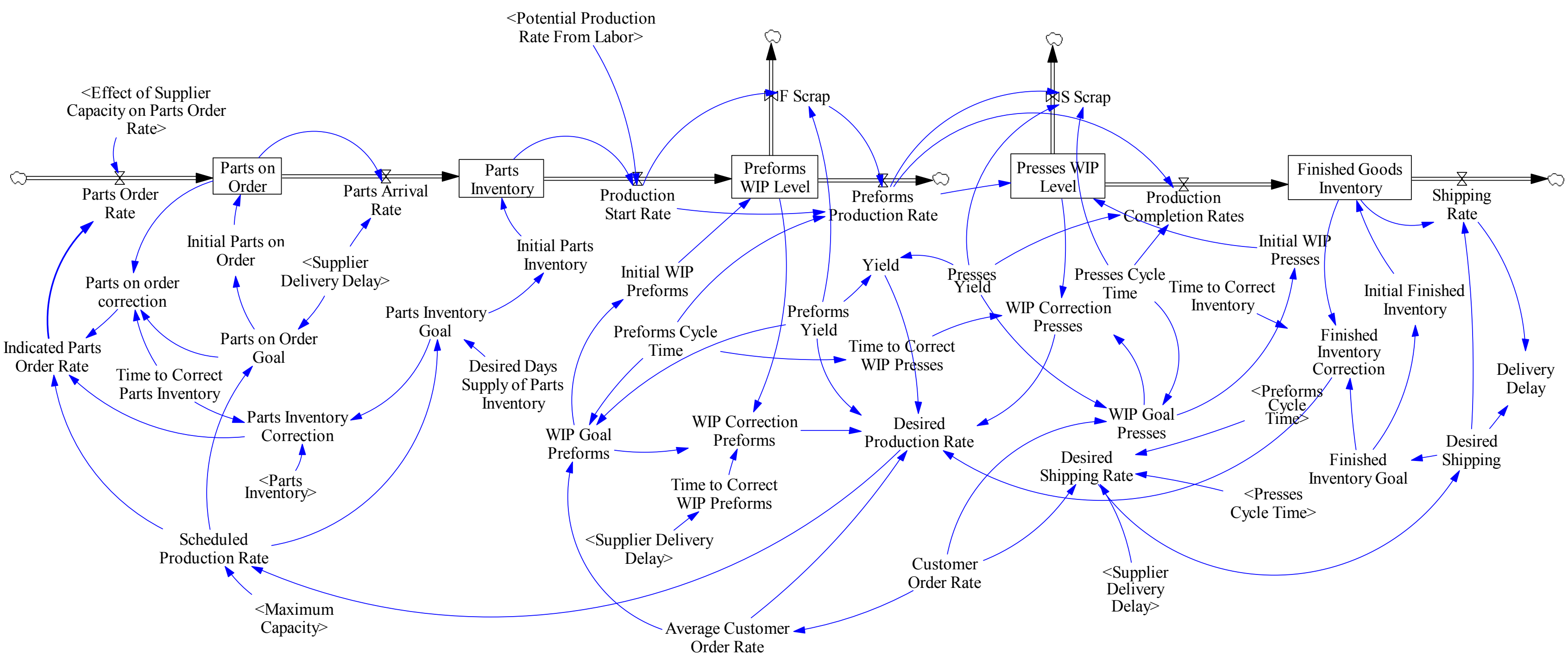

Figure 5.32. PMOC model: Internal supply chain submodel

5.4.3. Current Policy and SC Instability

The set of parameters in Table 5.13 defines the current policy for this supply chain.

Table 5.13. PMOC model: Parameter values for the current policy

Parameter	Value	Unit
Desired Days Supply of Parts Inventory	2	Weeks
Time to Correct Parts Inventory	1	Weeks
Preforms Cycle Time	3	Weeks
Presses Cycle Time	3	Weeks
Time to Correct Inventory	1	Weeks
Supplier Delivery Delay	2	Weeks
Time to Adjust Labor	1	Weeks
Labor Recruiting Delay	5	Weeks

For a customer order rate of 5,000 units/week the system starts out of equilibrium. The behavior of the four variables of interest is depicted in Figure 5.33. Variables *Preforms WIP Level*, *Presses WIP Level* and *Labor* have several oscillatory fluctuations. Variable *Finished Goods Inventory* is starting to settle down, although it has not reach equilibrium yet.

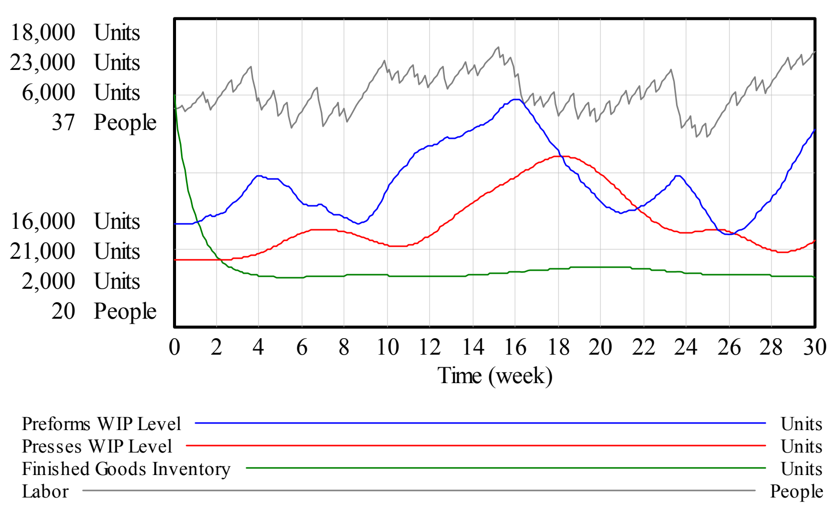

Figure 5.33. PMOC model: Behavior of variables of interest for the current policy

A new policy to minimize these oscillations will be determined by solving the optimization problem presented in the next section.

5.4.4. Optimization Problem

This optimization problem considers the simultaneous stabilization of the following state variables: *Preforms WIP Level*, *Presses WIP Level*, *Finished Goods Inventory* and *Labor* according to the equations described in section 3.1.2.

Let x_1=Preforms WIP Level, x_2= Presses WIP Level, x_3= Finished Goods Inventory, x_4=Labor

Let a_i=the new equilibrium point associated to the i^{th} state variable (i=1,..,4)

The following weights were assigned: w_1=0.4, w_2=0.4, w_3=0.1, w_4=0.1 to represent the concern of management in the inventory and considering that variables x_1 and x_2 exhibit higher oscillations. The time horizon (T) considered was 30 weeks.

$$\underset{\mathbf{p}}{\text{Minimize}}\ J(\mathbf{p}) = \sum_{s=1}^{2}\left\{0.4\int_0^{30}\left|x_s(t) - a_s\right| dt\right\} + \sum_{s=3}^{4}\left\{0.1\int_0^{30}\left|x_s(t) - a_s\right| dt\right\}$$

Subject to

$\dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}(t), \mathbf{p})$ (This notation represents the SD model equations)

$\mathbf{x}_0^{\,T}$ (Vector with initial values of all state variables)

$0.5 \leq$ Desired Days Supply of Parts Inventory ≤ 5

$0.5 \leq$ Time to Correct Parts Inventory ≤ 5

$0.5 \leq$ Preforms Cycle Time ≤ 3

$0.5 \leq$ Presses Cycle Time ≤ 3

$0.5 \leq$ Time to Correct Inventory ≤ 5

$0.5 \leq$ Supplier Delivery Delay ≤ 5

$0.5 \leq$ Time to Adjust Labor ≤ 5

$0.5 \leq$ Labor Recruiting Delay ≤ 5

$5000 \leq a_1 \leq 50000$

$5000 \leq a_2 \leq 50000$

$1000 \leq a_3 \leq 50000$

$10 \leq a_4 \leq 100$

5.4.5. Stabilization Policy

The stabilization policy is obtained after solving the optimization problem presented in the previous section. The optimization algorithm was run at time 0 using the following settings: swarm size = 30 particles, neighborhood size = 3 particles, initial inertia weight = 0.5, iteration lag = 5, cognitive coefficient = 1.2, social coefficient = 1.2. The time to obtain the optimal policy (after 150 PSO iterations and 1,243 PHC iterations) was 89 seconds.

The solution yielded the results shown in Table 5.14. This table also includes parameters a_1, a_2, a_3, a_4 which are the new equilibrium points for the state variables of interest.

Table 5.14. PMOC model: Parameter values for the stabilization policy

Parameter	Value	Unit
Desired Days Supply of Parts Inventory	3.46	Weeks
Time to Correct Parts Inventory	2.79	Weeks
Preforms Cycle Time	1.36	Weeks
Presses Cycle Time	1.70	Weeks
Time to Correct Inventory	1.47	Weeks
Supplier Delivery Delay	2.93	Weeks
Time to Adjust Labor	1.24	Weeks
Labor Recruiting Delay	0.5	Weeks
a_1 (EP for Preforms WIP Level)	8828	Units
a_2 (EP for Presses WIP Level)	13739	Units
a_3 (EP for Finished Goods Inventory)	3275	Units
a_4 (EP for Labor)	44	People

Figure 5.34 shows the behavior of the state variables when this revised policy is applied. The system has reached equilibrium approximately in 9 weeks (response time). This figure also shows that the convergence of ADE has caused the asymptotic stability of the four variables of interest. This was achieved mainly by increasing the parameter values *Desired Days Supply of Parts Inventory*, *Time to Correct Parts Inventory* and *Supplier Delivery Delay* and decreasing

several other parameter values including *Labor Recruiting Delay*, *Preforms Cycle Time*, and *Presses Cycle Time*. This stabilization policy has been reached using the maximum production capacity of 5,600 units/week as shown in Figure 5.35. This is due to the constraint in manpower in the lenses manufacturing department.

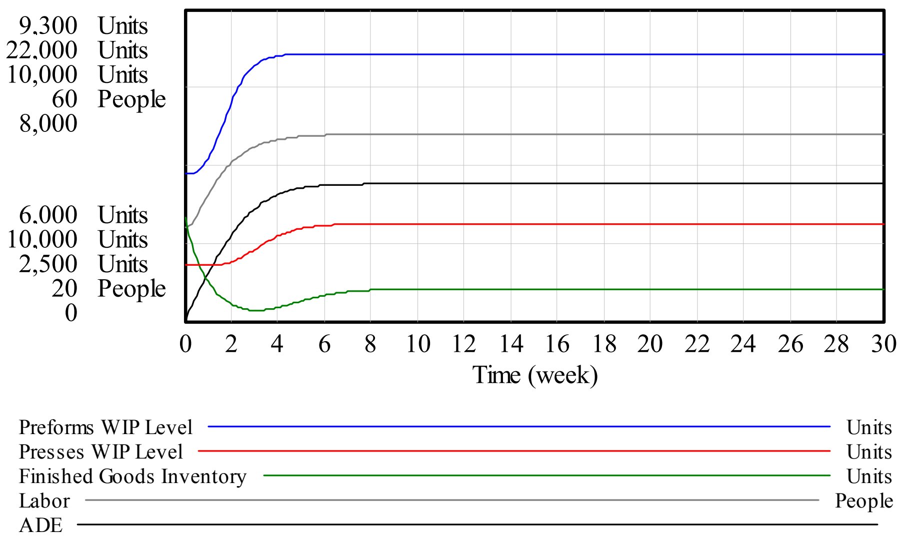

Figure 5.34. PMOC model: Behavior of variables of interest for the stabilization policy

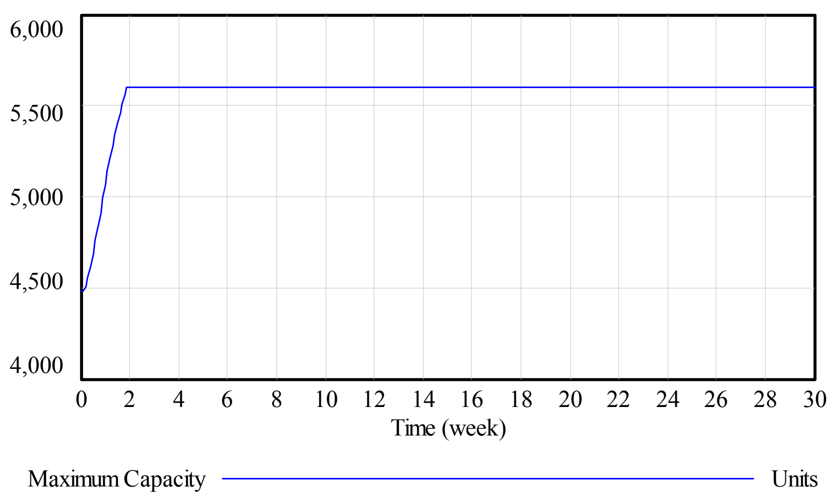

Figure 5.35. PMOC model: Maximum capacity of lenses manufacturing department

5.4.6. Testing for Policy Robustness

To test the stabilization policy it is generated a sudden change in the customer order rate in week 10. The customer order rate is increased or decreased to new levels calculated as a percentage of its initial value. This is displayed in Figure 5.36. Moreover, Figures 5.37, 5.38 and 5.39 depict the robust behavior of the *Preforms WIP Level*, *Presses WIP Level*, and *Finished Goods Inventory* variables to the changes in customer orders. The values for the new EPs are shown in Table 5.15.

The EP levels of the three inventory variables remain the same for a 10% increment in customer orders. The reason is simple; the stabilization policy was reached by using the maximum production capacity and orders over the original customer order rate are considered backlog and therefore they do not affect the production rates and the stability. Similarly, for a 5% decrease in customer orders, production is working close to maximum capacity and the EPs remain the same. In the case where customer orders are decreased by 10% and 15% the new EPs are reduced too but in a lower percentage that the change in customer orders.

Stability returns approximately 10 weeks and 16 weeks after the system was disturbed (response time) for -10% and -15% decrease in customer orders respectively. Amplifications are on the order of 1% under the EPs for both -10% and -15% decrease in customer orders.

Table 5.15. PMOC model: Parameter values for the stabilization policy

Percentage change in customer order rate	New EP for Preforms WIP Level (Units)	New EP for Presses WIP Level (Units)	New EP for Finished Goods Inventory (Units)
-15%	8377	13178	3045
-10%	8789	13691	3256
-5%	8828	13739	3275
+10%	8828	13739	3275

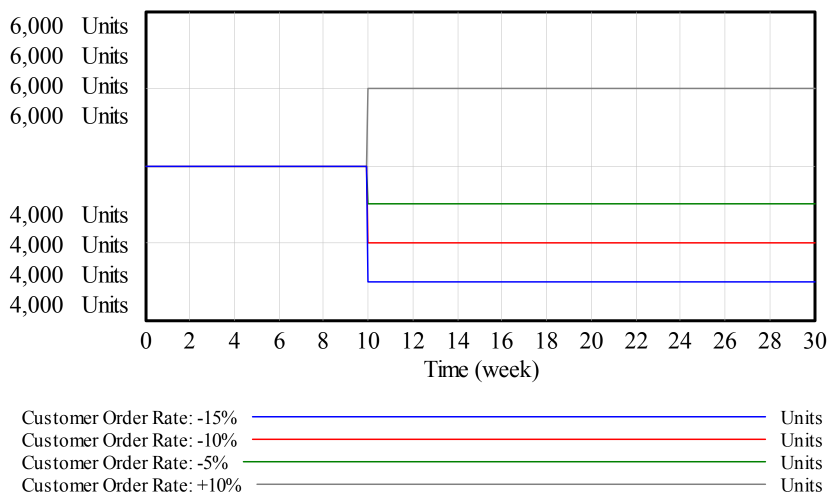

Figure 5.36. PMOC model: Changes in customer orders to test policy robustness

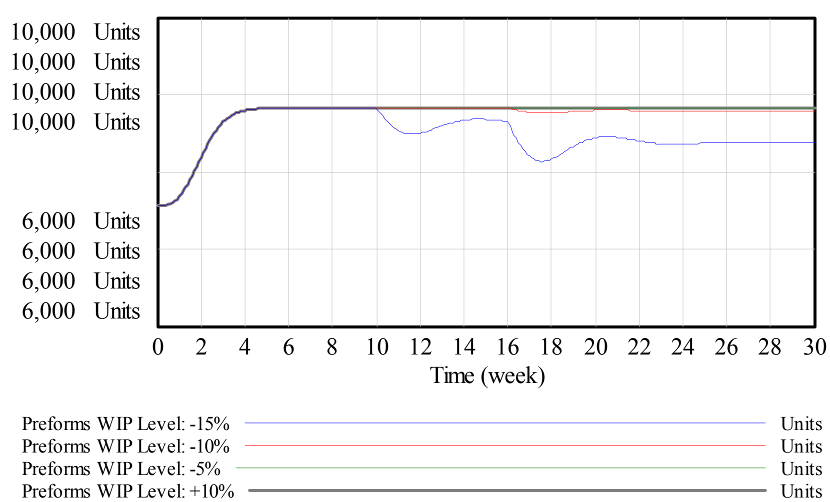

Figure 5.37. PMOC model: Behavior of Preforms WIP Level due to changes in customer orders

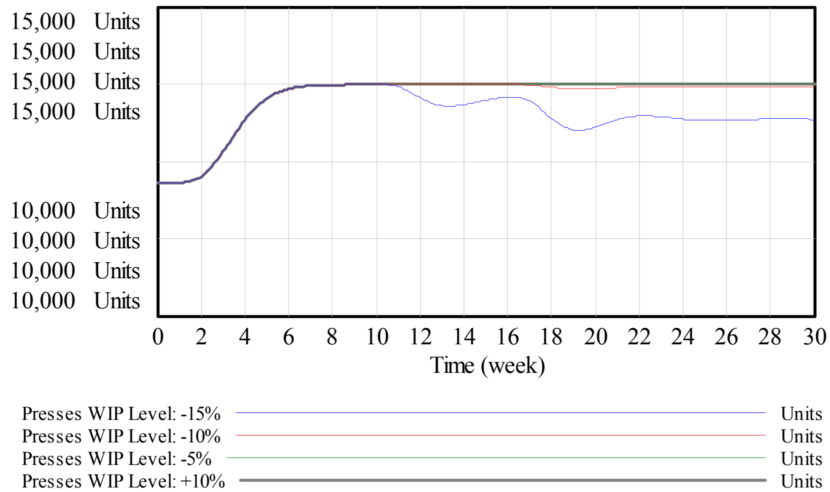

Figure 5.38. PMOC model: Behavior of Presses WIP Level due to changes in customer orders

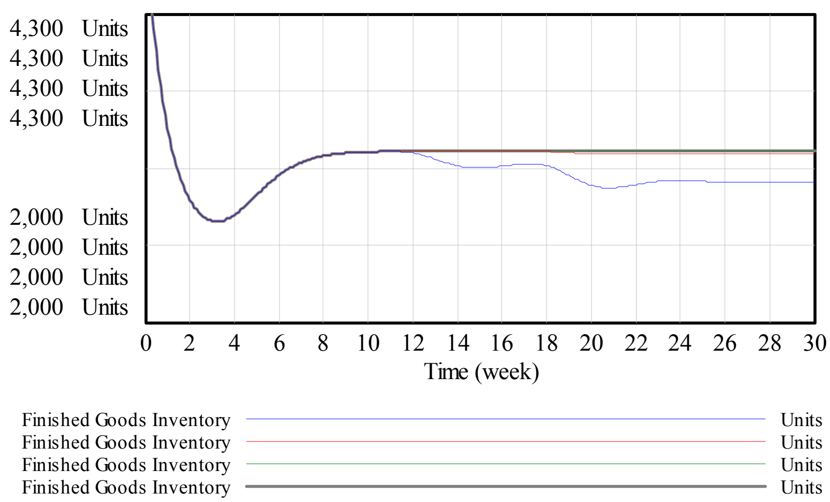

Figure 5.39. PMOC model: Behavior of Finished Goods Inv. due to changes in customer orders

5.5. Summary of the Experimental Analysis

The following table presents a summary of the results for the stability and robustness analysis as well as the values that show the performance of the optimization algorithm.

Table 5.16. Summary of the stability and robustness analysis

	Supply Chain Model			
	I-W	**Mass**	**LSMC**	**PMOC**
Model size				
Number of variables of interest	2	3	2	4
Number of parameters (including EPs)	11	18	6	12
Stability analysis				
Does the system start in equilibrium?	Yes	No	Yes	No
Type of Perturbation	Gradual and linear change	Sudden change	Sudden change	Sudden change
Does system reach stability?	Yes	Yes	Yes	Yes
Concept used in objective function	ADE	ADE+AEDE	ADE	ADE
Response time as a percentage of the remaining time since the stabilization policy was applied	15.4%	35%	7.1%	30%
Optimization algorithm				
Solution time (seconds)	206	306	236	89
PSO iterations	150	150	100	150
PHC iterations	1,393	3,171	79	1243
Robustness analysis				
Type of Perturbation	Sudden change	Gradual and linear change	Sudden change	Sudden change
Was the policy robust?	Yes	Yes	Yes	Yes
Are EPs changes proportional to the disturbance change?	Yes	Yes	Yes	No

CHAPTER SIX: CONCLUSIONS AND FUTURE RESEARCH

This research proposes the SADE methodology to reduce or eliminate instability in supply chains. The method provides an effective tool for managers to react quickly to unexpected events generating new policies and plans to cope with these changes in the business dynamics. This chapter summarizes the conclusions and highlights the directions for future research.

6.1. Summary of Research and Conclusions

We propose the SADE methodology that uses the concept of asymptotic stability to minimize oscillatory behaviors of specific (state) variables of interest of the supply chain model. If necessary stability can be extended to the whole SC system by using a weighted average function that includes all state variables. This also allows higher weights to be assigned to those variables considered more important. This approach does not require direct knowledge of the internal structure of the model. It also does not require linearization of the system or eigenvalue calculations. We argue that the simplicity of our approach makes it a powerful tool that can be applied very easily for practitioners – especially when dealing with systems that exhibit highly nonlinear oscillatory behavior.

We develop stability conditions based on the ADE. These conditions can be used as a general procedure to stabilize supply chains represented by linear or nonlinear dynamic models. We prove several theorems that show that ADE convergence of a state variable will make its trajectory approach asymptotic stability. Achieving ADE convergence requires the solution of a policy optimization problem. Moreover, we introduce the concept of AEDE to be used in problems where obtaining ADE convergence is not easy. AEDE is most beneficial for amplifying small DE. Thus, expressing the objective function as a combination of ADE and AEDE can improve the asymptotic stability of the state variables under study.

We propose a hybrid algorithm to obtain a quick convergence of the ADE. This algorithm is based on a search engine that combines the advantage of PSO optimization to determine the most promising regions of the search space and the properties of PHC algorithm to accelerate locating the optimum that makes the ADE to convergence. Although it is not required to find the global optimum to obtain a satisfactory reduction in instability, our hybrid algorithm provides solutions that escape local convergence and lead to stabilization polices with few oscillations and fast stability. This broader search to find more effective stabilization policies is also possible due to the fact that we incorporate a theorem that allows finding the best equilibrium levels that minimize the ADE.

We perform the experimental analysis over four case studies. These cases consist on one linear model (I-W) and three nonlinear models (Mass, LSMC, PMOC) of the supply chain. The methodology is applied to stabilize some variables of interest that show several oscillatory fluctuations. The optimization algorithm generated stabilization policies in a few minutes. The results show that our method makes the trajectory of these variables to achieve asymptotic stability. For the I-W and LSMC models stability is reached in a very short time (less than one fifth of the remaining time since the stabilization policy was applied). For the Mass and PMOC models stability took some time longer, approximately one third of the time horizon. The Mass model, which is highly nonlinear, required a combination of ADE and AEDE to obtain the convergence of the objective function. For the LSMC model, we compared the stabilization policies obtained using our method (SADE) and the EEA method. Results show that the policy generated with the SADE method is faster and smoother to reach the equilibrium state. Moreover, the lower inventory level required by the SADE policy to stabilize the system makes it more economical to implement than the policy proposed by the EEA method.

We conclude that the convergence of the ADE generates stabilization policies that are robust. To test robustness on these policies we produced a perturbation in the stable system by changing the value of an exogenous variable. The results show that the variables of interest reach new equilibrium points after a period of adaptation to the alteration of the system. Moreover, perturbations generated by sudden changes produce amplifications before reaching new EPs. The experiments also show that in most cases the change of level in the EPs is proportional to the change of the exogenous variable.

6.2. Research Contributions

This research contributes to the industrial engineering science by developing a novel stabilization method that can be broadly applied to supply chains modeled as dynamic systems, independently of their nature: linear or nonlinear. The stabilization policies obtained by the method help to identify the impact of important parameters of the model in the behavior of the system. This will also permit to advance the understanding on how the dynamic and complex interactions of the supply chain components affect the behavior of the whole supply chain at the strategic and tactical levels, creating conditions of constant disequilibrium and change.

The stabilization method presented in this research work is a more general and simpler approach than the methods based on linearized models like eigenvalue optimization. Moreover, it is not limited to particular characteristics of the system like many of the methods used by nonlinear control theory like Lyapunov functions. Due to the simplicity of the method that only requires solving a policy optimization problem in order to obtain a stabilization policy, it can be accessed not only by academics but also by practitioners. This is an important contribution because managers often reject using complex approaches that they do not understand. They prefer basic approaches that are simple to comprehend and easy to communicate with other people.

This research advances the field of system dynamics with the development of stability conditions based on the ADE that can be applied to several state variables of the system simultaneously. We propose and prove new theorems that determine the conditions for (1) the convergence of a state variable around its equilibrium point and (2) finding the best equilibrium point that minimize instability. These theorems are incorporated into an optimization problem to achieve stability. We show a simple way to add the objective function of the optimization problem into any SD formulation.

This research presents and implements a framework to plan and design robust supply chains. To facilitate the modeling activity, this framework was designed in such a way that can simulate SD models created with Vensim, one of the leading producers of SD software. A PSO solver was developed and incorporated into the framework to be used with the PHC optimizer that comes with Vensim. The PSO solver is a tool that will allow SD users to solve policy optimization problems associated with dynamic systems in general, expanding its use beyond the supply chain cases.

6.3. Directions for Future Research

There are several additional aspects that must be addressed and investigated for enhancing this methodology. The proposed future research directions are outlined in the following sections.

6.3.1 Controlling the Characteristics of Stability

Currently the concept to achieve stability relies on minimizing the deviations of controlled variables from the equilibrium state. The stabilization policies obtained by solving the optimization problem have the characteristic of asymptotic stability, which make them robust. However, to improve our methodology it will be necessary to have control over the following factors before reaching stability: (1) possible fluctuations appearance, (2) amplifications magnitude, and (3) value of the response time.

The ideal stabilization policy will have a very short response time, no fluctuations, and no amplifications before reaching stability. However, policies with shorter response times generally show greater amplifications or fluctuations, making it difficult to obtain the ideal stability. One idea that can help to minimize the fluctuation behavior of a policy is introducing in the objective function a penalty every time the curve crosses the equilibrium point. It has to be demonstrated that the convergence of this new objective function will still achieve asymptotic stability. In addition, to control the amplification and response time factors, we should be able to add in the optimization problem new constraints that represent the maximum and minimum tolerances for these factors. This addition will help to adapt the resulting policy into a more desired shape. Moreover, a modification in the solution algorithm will have to be made to check that the curve of the variable of interest is inside these tolerances.

6.3.2. Multi-level Stabilization Policy of the Supply Chain

Supply chains exhibit complex dynamics consisting of a hierarchical nesting of both continuous and discrete dynamics. The discrete dynamics would represent activities at the operational level where the status of individual items is traced (e.g. shop floor activities) while continuous dynamics would represent aggregate flows and decisions at the tactical and strategic level (e.g. aggregate production planning or new product market dynamics).

We plan to extend the stabilization policies obtained from the strategic and tactical levels (higher levels) to the operational level (lower level). Thus, we will use SD and discrete event simulation techniques to capture the different dynamics of the SC forming an integrated and hybrid two-level simulation model. This hybrid simulation model will be used to develop a top-down hierarchical stabilization methodology that will search for new supply chain configurations to avoid instability. The top level does aggregate planning across the entire supply chain. The aggregate level activities, which take place at the manufacturer, include planning and dispatching decisions. These decisions are evaluated using system dynamics simulation. The stabilization policies generated at this level will be based on the convergence of the ADE as stated in this research work. The detailed bottom level activities, which take place at the manufacturer, transporter, and retailer, include scheduling decisions and production activities. These are evaluated using discrete-event simulation.

6.3.3. Detecting Instabilities in the Supply Chain

Having the capability to detect instability (ripple effects) at a very early stage provides companies enough time to design and implement stabilization policies. This capability should go far beyond current monitoring systems, such as the popular dashboard, which can provide alerts, but cannot predict the impact of those alerts.

Some preliminary work has been done to detect structural changes in the supply chain by using neural networks (NNs) and system dynamics (Shah 2001). The method described in Shah's work uses pattern recognition analysis to map a set of inputs to the most likely future behavior of the supply chain. Then it classifies possible behaviors of state variables of the SD model into categories of similar graphs by using fuzzy art NNs. After that it uses backpropagation NNs to predict the behavior of a variable of interest. Although this method has demonstrated to be efficient capturing the behavior of a complex supply chain, still there is potential for extension of this work. As the next step in this line of research, we propose to (1) investigate other classification techniques to categorize the behavior of state variables, (2) determine which NN topologies are the most appropriate to produce less training and testing errors, and (3) encapsulate the detection capabilities into a monitoring agent.

When the monitoring system predicts the future occurrence of instability, a new management strategy must be found. Therefore, the detection capability (behavior monitor module) can be incorporated with the SADE methodology in order to predict ripple effects in the supply chain at an early stage and then remove the instability or minimize its impact (see Figure 6.1). Finally, these two methodologies should be integrated in a framework for detecting and modifying the behavior of SC models.

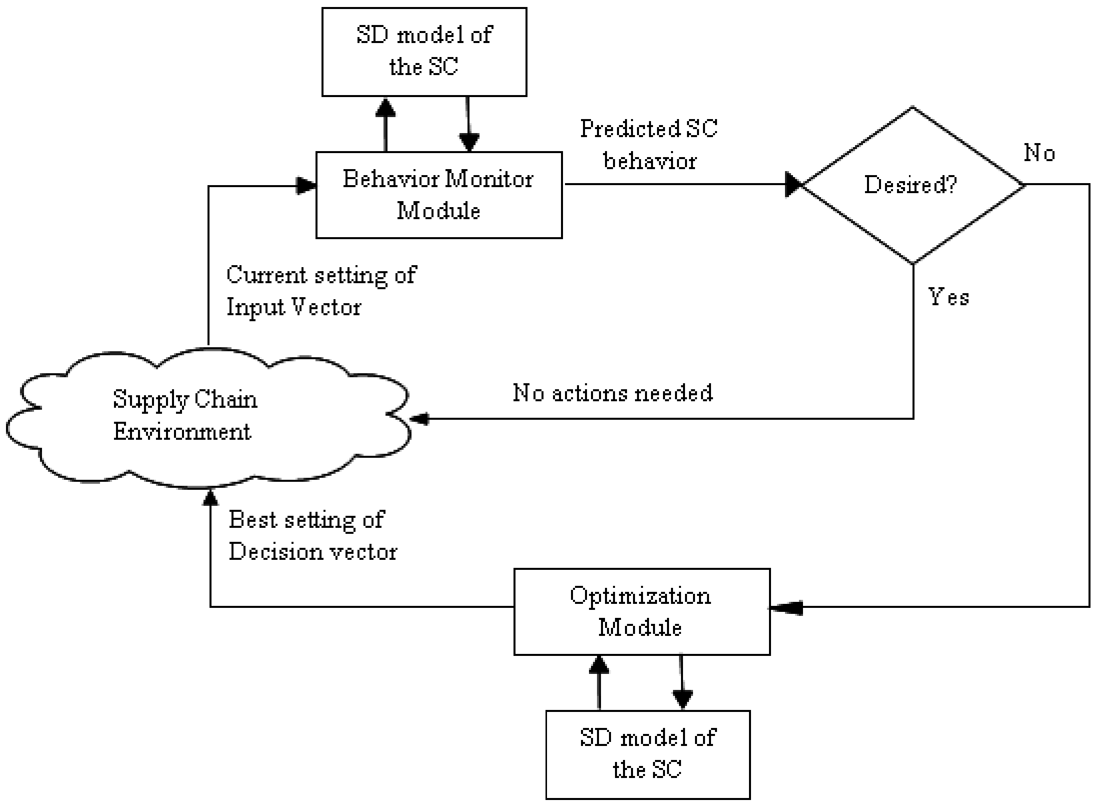

Figure 6.1. SADE methodology with detection capability

APPENDIX A: ADDITIONAL LEMMAS

Lemma A.1 The inverse matrix of the Jacobian $\mathbf{J}$ is given by

$$\mathbf{J}^{-1} = \begin{bmatrix} \mathbf{J}_1^{-1} & & & \mathbf{0} \\ & \mathbf{J}_2^{-1} & & \\ & & \ddots & \\ \mathbf{0} & & & \mathbf{J}_m^{-1} \end{bmatrix}$$

where $\mathbf{J}_i^{-1} = \begin{bmatrix} 1/\lambda_i & -1/\lambda_i^2 & 1/\lambda_i^3 & \cdots & (-1)^{n_i-1}/\lambda_i^{n_i} \\ & 1/\lambda_i & -1/\lambda_i^2 & \cdots & (-1)^{n_i-2}/\lambda_i^{n_i-1} \\ & & \ddots & & \vdots \\ & & & \ddots & \vdots \\ 0 & & & & 1/\lambda_i \end{bmatrix}$, i=1,..,m

Proof: First, it will be proved by contradiction that matrix $\mathbf{J}^{-1} = \begin{bmatrix} \mathbf{J}_1^{-1} & & & \mathbf{0} \\ & \mathbf{J}_2^{-1} & & \\ & & \ddots & \\ \mathbf{0} & & & \mathbf{J}_m^{-1} \end{bmatrix}$ is not the inverse of the Jacobian $\mathbf{J}$.

If $\mathbf{J}^{-1} = \begin{bmatrix} \mathbf{J}_1^{-1} & & & \mathbf{0} \\ & \mathbf{J}_2^{-1} & & \\ & & \ddots & \\ \mathbf{0} & & & \mathbf{J}_m^{-1} \end{bmatrix}$ is not the inverse matrix of the Jacobian $\mathbf{J}$ then by the definition of inverse matrix it follows that $\mathbf{J}\mathbf{J}^{-1} \neq \mathbf{I}$. (A.1)

Multiplying matrices $\mathbf{J}$ and $\mathbf{J}^{-1}$ yields

$$\mathbf{J}\mathbf{J}^{-1} = \begin{bmatrix} \mathbf{I}_1 & & & \mathbf{0} \\ & \mathbf{I}_2 & & \\ & & \ddots & \\ \mathbf{0} & & & \mathbf{I}_m \end{bmatrix}$$

where $\mathbf{I}_i = \mathbf{J}_i\mathbf{J}_i^{-1} = \begin{bmatrix} 1 & & & 0 \\ & 1 & & \\ & & \ddots & \\ 0 & & & 1 \end{bmatrix}$ is the identity matrix of dimension n_i (i=1,..,m)

Thus, $\mathbf{J}\mathbf{J}^{-1} = \begin{bmatrix} \mathbf{I}_1 & & & \mathbf{0} \\ & \mathbf{I}_2 & & \\ & & \ddots & \\ \mathbf{0} & & & \mathbf{I}_m \end{bmatrix}$ is the identity matrix $\mathbf{I}$ of dimension $n_1+n_2+\ldots+n_m$

But this is a contradiction to the statement in (A.1). Therefore, the assumption that

$\mathbf{J}^{-1} = \begin{bmatrix} \mathbf{J}_1^{-1} & & & \mathbf{0} \\ & \mathbf{J}_2^{-1} & & \\ & & \ddots & \\ \mathbf{0} & & & \mathbf{J}_m^{-1} \end{bmatrix}$ is not the inverse of the Jacobian $\mathbf{J}$ is false ■

Second, it will be proved using the Gauss-Jordan elimination method that

$$\mathbf{J}_i^{-1} = \begin{bmatrix} 1/\lambda_i & -1/\lambda_i^2 & 1/\lambda_i^3 & \cdots & (-1)^{n_i-1}/\lambda_i^{n_i} \\ & 1/\lambda_i & -1/\lambda_i^2 & \cdots & (-1)^{n_i-2}/\lambda_i^{n_i-1} \\ & & \ddots & & \vdots \\ & & & \ddots & \vdots \\ 0 & & & & 1/\lambda_i \end{bmatrix}, \text{ i=1,..,m.}$$

The method requires augmenting the original matrix $\mathbf{J}_i$ by the identity, and then the form $[\mathbf{J}_i \mid \mathbf{I}]$ is obtained

$$\left[\begin{array}{ccccc|ccccc} \lambda_i & 1 & & & 0 & 1 & & & & 0 \\ & \lambda_i & 1 & & & & 1 & & & \\ & & \lambda_i & \ddots & & & & 1 & & \\ & & & \ddots & 1 & & & & \ddots & \\ 0 & & & & \lambda_i & 0 & & & & 1 \end{array}\right]$$

The following operations are performed to transform to the form $[\mathbf{I} \mid \mathbf{J}_i^{-1}]$

$$\left[\begin{array}{ccccc|ccccc} 1 & 1/\lambda_i & & & 0 & 1/\lambda_i & & & & 0 \\ & \lambda_i & 1 & & & & 1 & & & \\ & & \lambda_i & \ddots & & & & 1 & & \\ & & & \ddots & 1 & & & & \ddots & \\ 0 & & & & \lambda_i & 0 & & & & 1 \end{array}\right]$$

Dividing the first row by λ_i

$$\left[\begin{array}{ccccc|ccccc} 1 & 0 & -1/\lambda_i^2 & \cdots & 0 & 1/\lambda_i & -1/\lambda_i^2 & & \cdots & 0 \\ & \lambda_i & 1 & & & & 1 & & & \\ & & \lambda_i & \ddots & & & & 1 & & \\ & & & \ddots & 1 & & & & \ddots & \\ 0 & & & & \lambda_i & 0 & & & & 1 \end{array}\right]$$

Multiplying the second row by $-1/\lambda_i^2$ and adding it to the first row

$$\left[\begin{array}{ccccc|ccccc} 1 & 0 & -1/\lambda_i^2 & \cdots & 0 & 1/\lambda_i & -1/\lambda_i^2 & & \cdots & 0 \\ & 1 & 1/\lambda_i & & & & 1/\lambda_i & & & \\ & & \lambda_i & \ddots & & & & 1 & & \\ & & & \ddots & 1 & & & & \ddots & \\ 0 & & & & \lambda_i & 0 & & & & 1 \end{array}\right]$$

Dividing the second row by λ_i

$$\left[\begin{array}{ccccc|ccccc} 1 & 0 & 0 & \cdots & 0 & 1/\lambda_i & -1/\lambda_i^2 & 1/\lambda_i^3 & \cdots & 0 \\ & 1 & 1/\lambda_i & & & & 1/\lambda_i & & & \\ & & \lambda_i & \ddots & & & & 1 & & \\ & & & \ddots & 1 & & & & \ddots & \\ 0 & & & & \lambda_i & 0 & & & & 1 \end{array}\right]$$

Multiplying the third row by $1/\lambda_i^3$ and adding it to the first row

$$\left[\begin{array}{ccccc|ccccc} 1 & 0 & 0 & \cdots & 0 & 1/\lambda_i & -1/\lambda_i^2 & 1/\lambda_i^3 & \cdots & 0 \\ & 1 & 0 & & & & 1/\lambda_i & -1/\lambda_i^2 & & \\ & & \lambda_i & \ddots & & & & 1 & & \\ & & & \ddots & 1 & & & & \ddots & \\ 0 & & & & \lambda_i & 0 & & & & 1 \end{array}\right]$$

Multiplying the third row by $-1/\lambda_i^2$ and adding it to the second row

After continuing with these operations until the final row is reached then the form $\left[\mathrm{I} \mid \mathbf{J}_i^{-1}\right]$ is obtained.

$$\left[\begin{array}{ccccc|ccccc} 1 & 0 & 0 & \cdots & 0 & 1/\lambda_i & -1/\lambda_i^2 & 1/\lambda_i^3 & \cdots & (-1)^{n_i-1}/\lambda_i^{n_i} \\ & 1 & 0 & & & & 1/\lambda_i & -1/\lambda_i^2 & \cdots & (-1)^{n_i-2}/\lambda_i^{n_i-1} \\ & & \ddots & \ddots & & & & \ddots_i & & \vdots \\ & & & \ddots & 0 & & & & \ddots & \vdots \\ 0 & & & & 1 & 0 & & & & 1/\lambda_i \end{array}\right]$$

Thus, from the form $\left[\mathrm{I} \mid \mathbf{J}_i^{-1}\right]$ it is clear that

$$\mathbf{J}_i^{-1} = \begin{bmatrix} 1/\lambda_i & -1/\lambda_i^2 & 1/\lambda_i^3 & \cdots & (-1)^{n_i-1}/\lambda_i^{n_i} \\ & 1/\lambda_i & -1/\lambda_i^2 & \cdots & (-1)^{n_i-2}/\lambda_i^{n_i-1} \\ & & \ddots & & \vdots \\ & & & \ddots & \vdots \\ 0 & & & & 1/\lambda_i \end{bmatrix}, \text{ i=1,..,m } \blacksquare$$

Lemma A.2 (DeCarlo 1989) The matrix exponential of the Jacobian **J** is given by

$$e^{\mathbf{J}t} = \begin{bmatrix} e^{\mathbf{J}_1 t} & & & \mathbf{0} \\ & e^{\mathbf{J}_2 t} & & \\ & & \ddots & \\ \mathbf{0} & & & e^{\mathbf{J}_m t} \end{bmatrix}$$

where $$e^{\mathbf{J}_i t} = \begin{bmatrix} e^{\lambda_i t} & te^{\lambda_i t} & \dfrac{t^2 e^{\lambda_i t}}{2} & \cdots & \dfrac{t^{n_i-1} e^{\lambda_i t}}{(n_i-1)!} \\ & e^{\lambda_i t} & te^{\lambda_i t} & \cdots & \dfrac{t^{n_i-2} e^{\lambda_i t}}{(n_i-2)!} \\ & & \ddots & & \vdots \\ & & & \ddots & \vdots \\ 0 & & & & e^{\lambda_i t} \end{bmatrix}, \text{ i=1,..,m}$$

Proof: By applying the property of the exponentiation of a diagonal matrix, the exponentiation of $\mathbf{J} = \begin{bmatrix} \mathbf{J}_1 & & & \mathbf{0} \\ & \mathbf{J}_2 & & \\ & & \ddots & \\ \mathbf{0} & & & \mathbf{J}_m \end{bmatrix}$ is calculated simply by exponentiating each of the diagonal elements;

therefore

$$e^{\mathbf{J}t} = \begin{bmatrix} e^{\mathbf{J}_1 t} & & & \mathbf{0} \\ & e^{\mathbf{J}_2 t} & & \\ & & \ddots & \\ \mathbf{0} & & & e^{\mathbf{J}_m t} \end{bmatrix}$$

Thus, now the matrix exponential of the Jordan block $\mathbf{J}_i$ has to be calculated. It is known that

$$\mathbf{J}_i = \begin{bmatrix} \lambda_i & 1 & & & 0 \\ & \lambda_i & 1 & & \\ & & \lambda_i & \ddots & \\ & & & \ddots & 1 \\ 0 & & & & \lambda_i \end{bmatrix} = \begin{bmatrix} \lambda_i & & & & 0 \\ & \lambda_i & & & \\ & & \lambda_i & & \\ & & & \ddots & \\ 0 & & & & \lambda_i \end{bmatrix} + \begin{bmatrix} 0 & 1 & & & 0 \\ & 0 & 1 & & \\ & & 0 & \ddots & \\ & & & \ddots & 1 \\ 0 & & & & 0 \end{bmatrix}$$

Thus, making $\mathbf{D}_i = \begin{bmatrix} 0 & 1 & & & 0 \\ & 0 & 1 & & \\ & & 0 & \ddots & \\ & & & \ddots & 1 \\ 0 & & & & 0 \end{bmatrix}$ then

$$\mathbf{J}_i = \lambda_i \mathbf{I} + \mathbf{D}_i \text{, for i=1,..,m}$$

After multiplying by “t” and exponentiating both sides

$$e^{\mathbf{J}_i t} = e^{\lambda_i t \mathbf{I}} e^{\mathbf{D}_i t} \tag{A.2}$$

Applying Definition 4 to the first factor on the right-hand side of (A.2)

$$e^{\lambda_i t \mathbf{I}} = \sum_{k=0}^{\infty} \frac{(t)^k}{k!} = \mathbf{I} + (\lambda_i t \mathbf{I}) + \frac{(\lambda_i t \mathbf{I})^2}{2!} + \frac{(\lambda_i t \mathbf{I})^3}{3!} + \cdots$$

$$= \mathbf{I} + (\lambda_i t)\mathbf{I} + \frac{(\lambda_i t)^2 \mathbf{I}}{2!} + \frac{(\lambda_i t)^3 \mathbf{I}}{3!} + \cdots$$

$$= \left(1 + (\lambda_i t) + \frac{(\lambda_i t)^2}{2!} + \frac{(\lambda_i t)^3}{3!} + \cdots \right) \mathbf{I}$$

$$= e^{\lambda_i t}\mathbf{I} \tag{A.3}$$

Applying Definition 4 to the second factor on the right-hand side of (A.2)

$$e^{\mathbf{D}_i t} = \sum_{k=0}^{\infty}\frac{(\mathbf{D}_i t)^k}{k!} = \mathbf{I} + (\mathbf{D}_i t) + \frac{(\mathbf{D}_i t)^2}{2!} + \frac{(\mathbf{D}_i t)^3}{3!} + \cdots \tag{A.4}$$

The powers of matrix $\mathbf{D}_i$ of dimension n_i (i=1,..,m) are computed as follows

$$\mathbf{D}_i = \begin{bmatrix} 0 & 1 & & & 0 \\ & 0 & 1 & & \\ & & 0 & \ddots & \\ & & & \ddots & 1 \\ 0 & & & & 0 \end{bmatrix},\ \mathbf{D}_i^2 = \begin{bmatrix} 0 & 0 & 1 & & 0 \\ & 0 & 0 & \ddots & \\ & & 0 & \ddots & 1 \\ & & & \ddots & 0 \\ 0 & & & & 0 \end{bmatrix}, \ldots,\ \mathbf{D}_i^{n_i-1} = \begin{bmatrix} 0 & 0 & 0 & & 1 \\ & 0 & 0 & \ddots & \\ & & 0 & \ddots & 0 \\ & & & \ddots & 0 \\ 0 & & & & 0 \end{bmatrix},$$

$$\mathbf{D}_i^p = 0,\ p \ge n_i$$

Replacing the powers of $\mathbf{D}_i$ in (A.4) gives

$$e^{\mathbf{D}_i t} = \begin{bmatrix} 1 & t & \frac{t^2}{2} & \cdots & \frac{t^{n_i-1}}{(n_i-1)!} \\ & 1 & t & \cdots & \frac{t^{n_i-2}}{(n_i-2)!} \\ & & \ddots & & \vdots \\ & & & \ddots & \vdots \\ 0 & & & & 1 \end{bmatrix},\ \text{i=1,..,m} \tag{A.5}$$

Substituting (A.3), (A.5) into (A.2) it is proved that

$$e^{\mathbf{J}_i t} = e^{\lambda_i t} e^{\mathbf{D}_i t}\mathbf{I} = \begin{bmatrix} e^{\lambda_i t} & te^{\lambda_i t} & \frac{t^2 e^{\lambda_i t}}{2} & \cdots & \frac{t^{n_i-1}e^{\lambda_i t}}{(n_i-1)!} \\ & e^{\lambda_i t} & te^{\lambda_i t} & \cdots & \frac{t^{n_i-2}e^{\lambda_i t}}{(n_i-2)!} \\ & & \ddots & & \vdots \\ & & & \ddots & \vdots \\ 0 & & & & e^{\lambda_i t} \end{bmatrix},\ \text{i=1,..,m} \ \blacksquare$$

Lemma A.3 (Fong and Wang 2000) If *g* and *h* are continuous functions on $[a,\infty)$ and if $0 \le g(t) \le h(t)$ for all $t \in [a,\infty)$, then

i) $\int_a^\infty g(t)\,dt$ converges if $\int_a^\infty h(t)\,dt$ converges

ii) $\int_a^\infty h(t)\,dt$ diverges if $\int_a^\infty g(t)\,dt$ diverges

Proof: Refer to Fridy (2000) to see the proof of this lemma.

Lemma A.4 (Fong and Wang 2000) (Convergence of Infinite Series) If the series $\sum_{i=1}^{\infty} \Psi_i$ converges, then $\lim_{m\to\infty} \Psi_m = 0$.

Proof: Let $\{S_m\}$ be the sequence defined by

$$S_1 = \Psi_1$$

$$S_2 = \Psi_1 + \Psi_2$$

$$S_3 = \Psi_1 + \Psi_2 + \Psi_3$$

$$\vdots$$

$$S_m = \sum_{i=1}^{m} \Psi_i$$

It is known by the hypothesis that the sequence S_m converges when $m\to\infty$, then

$$\sum_{i=1}^{\infty} \Psi_i = \lim_{m\to\infty} S_m = L$$

Note that if $\lim_{m\to\infty} S_m = L$ then $\lim_{m\to\infty} S_{m-1} = L$ (A.6)

The m^{th} term of the series can be expressed as: $\Psi_m = S_m - S_{m-1}$ (A.7)

Thus, from (A.6) and (A.7) results

$$\lim_{m\to\infty} \Psi_m = \lim_{m\to\infty}(S_m - S_{m-1})$$

$$= \lim_{m\to\infty} S_m - \lim_{m\to\infty} S_{m-1} = L - L = 0 \blacksquare$$

Lemma A.5 The integral $\int_0^\infty t^{j-1} e^{\mathrm{Re}(\lambda)}\, dt$ converges only if $\mathrm{Re}(\lambda) < 0$, for $j \geq 1$.

Proof: Integrating $\int_0^\infty t^{j-1} e^{\mathrm{Re}(\lambda)}\, dt$ by parts

$$\int_0^\infty t^{j-1} e^{\mathrm{Re}(\lambda)}\, dt = t^{j-1} \left.\frac{e^{\mathrm{Re}(\lambda)t}}{\mathrm{Re}(\lambda)}\right|_0^\infty - \frac{(j-1)}{\mathrm{Re}(\lambda)} \int_0^\infty t^{j-2} e^{\mathrm{Re}(\lambda)}\, dt \tag{A.8}$$

The term $t^{j-1} \left.\frac{e^{\mathrm{Re}(\lambda)t}}{\mathrm{Re}(\lambda)}\right|_0^\infty$ converges to zero only if $\mathrm{Re}(\lambda) < 0$. (A.9)

Otherwise the term and the whole integral go to infinity.

From (A.8) and (A.9) gives

$$\int_0^\infty t^{j-1} e^{\mathrm{Re}(\lambda)}\, dt = \frac{(j-1)}{-\mathrm{Re}(\lambda)} \int_0^\infty t^{j-2} e^{\mathrm{Re}(\lambda)}\, dt \tag{A.10}$$

Integrating $\int_0^\infty t^{j-2} e^{\mathrm{Re}(\lambda)}\, dt$ by parts

$$\int_0^\infty t^{j-2} e^{\mathrm{Re}(\lambda)}\, dt = t^{j-2} \left.\frac{e^{\mathrm{Re}(\lambda)t}}{\mathrm{Re}(\lambda)}\right|_0^\infty - \frac{(j-1)}{\mathrm{Re}(\lambda)} \int_0^\infty t^{j-3} e^{\mathrm{Re}(\lambda)}\, dt$$

Again, the term $t^{j-2} \left.\frac{e^{\mathrm{Re}(\lambda)t}}{\mathrm{Re}(\lambda)}\right|_0^\infty$ converges to zero only if $\mathrm{Re}(\lambda) < 0$ and

$$\int_0^\infty t^{j-2} e^{\mathrm{Re}(\lambda)}\, dt = \frac{(j-2)}{-\mathrm{Re}(\lambda)} \int_0^\infty t^{j-3} e^{\mathrm{Re}(\lambda)}\, dt \tag{A.11}$$

Substituting (A.11) into (A.10) yields

$$\int_0^\infty t^{j-1} e^{\mathrm{Re}(\lambda)}\, dt = \frac{(j-1)(j-2)}{(-\mathrm{Re}(\lambda))^2} \int_0^\infty t^{j-3} e^{\mathrm{Re}(\lambda)}\, dt$$

After integrating by parts (j-1) times it follows that the integral $\int_0^\infty t^{j-1} e^{\mathrm{Re}(\lambda)}\, dt$ converges to

$\frac{(j-1)!}{(-\mathrm{Re}(\lambda))^j}$ if $\mathrm{Re}(\lambda) < 0$ ■

APPENDIX B: STABILIZATION WITH A LOCAL SEARCH ALGORITHM

The SADE methodology does not require finding the global optimum to obtain satisfactory reduction in instability. A local search algorithm can obtain a quick convergence of the ADE in just few seconds. Although the time to find the optimal solution is an important factor in selecting a search algorithm, the quality of such solution in terms of oscillation reduction has to be analyzed. For that reason, in this appendix are compared the results obtained by solving the optimization problem using the hybrid algorithm (PSO+PHC) with the one obtained by using the local search algorithm (PHC). The case study to do the comparison is the Mass model described in section 5.2.

Due to its highly nonlinear equations the Mass model complicates the task of finding a good starting point for the local search algorithm. A simple way to choose the starting point will use the parameter values of the current policy and consider the lower limits for the equilibrium points, i.e., a_1=1,000,000, a_2=500,000 and a_3=100. This is shown in the next table.

Table B.1. Mass model: initial point for the local search

Parameter	Value	Unit
Initial Labor	1500	People
Time to Average Production Rate	1	Years
Normal Inventory Coverage	0.5	Years
Normal Backlog Coverage	0.2	Years
Delay in Filling Vacancies	0.25	Years
Time to Average New Vacancy Creation	0.5	Years
Normal Duration of Employment	2	Years
Time to Average Orders for Capital	4	Years
Delivery Delay for Capital	2	Years
Time to Adjust Capital	4	Years
Normal Life of Capital	15	Years
a_1 (EP for Capital)	1,000,000	Capital Units
a_2 (EP for Inventory)	500,000	Units
a_3 (EP for Labor)	100	People
Normal Production Rate	3E06	Units/year

Parameter	Value	Unit
Initial Capital	7.5E06	Capital Units
Time to Correct Inventory and Backlog	0.8	Years
Time to Adjust Labor	0.5	Years

Figures B.1, B.2 and B.3 show that although labor and inventory levels are similar in both policies, the result obtained with the PHC algorithm requires much more capital to stabilize the system. Moreover, the stabilization with the hybrid algorithm generates fewer fluctuations before reaching the equilibrium level. The explanation relies on the characteristics of the PSO method to perform a more expanded and deeper search of the space to find a better starting point for the PHC algorithm.

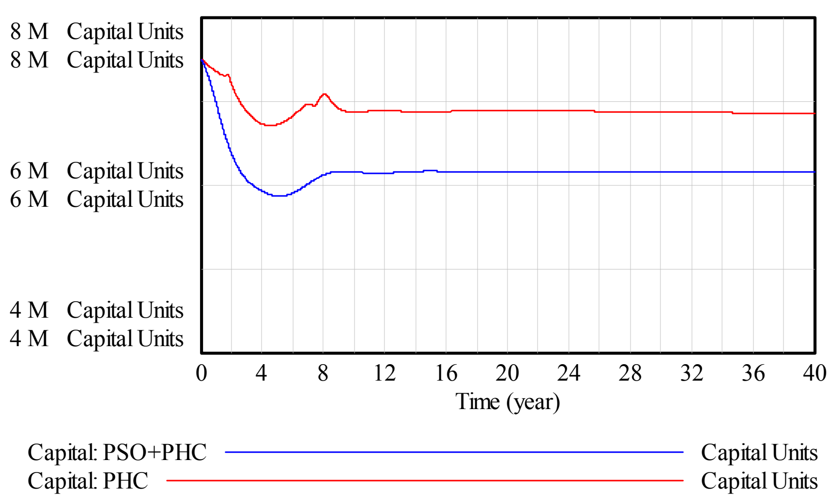

Figure B.1. Capital behaviors using the hybrid and local search algorithms

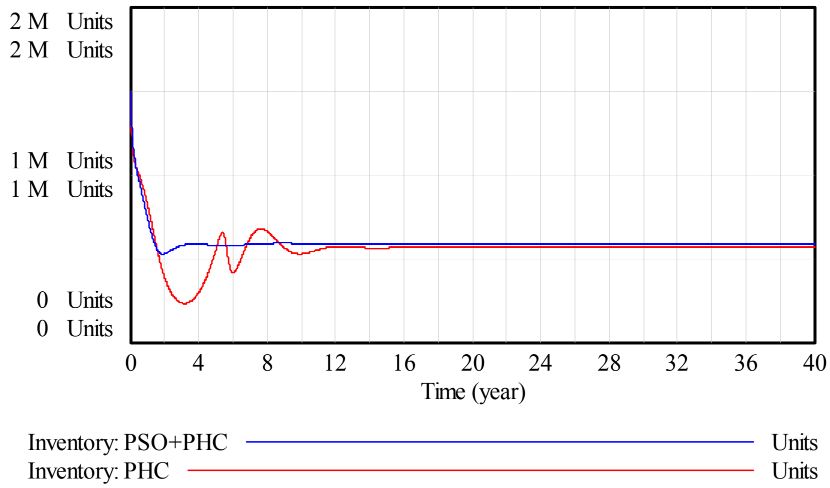

Figure B.2. Inventory behaviors using the hybrid and local search algorithms

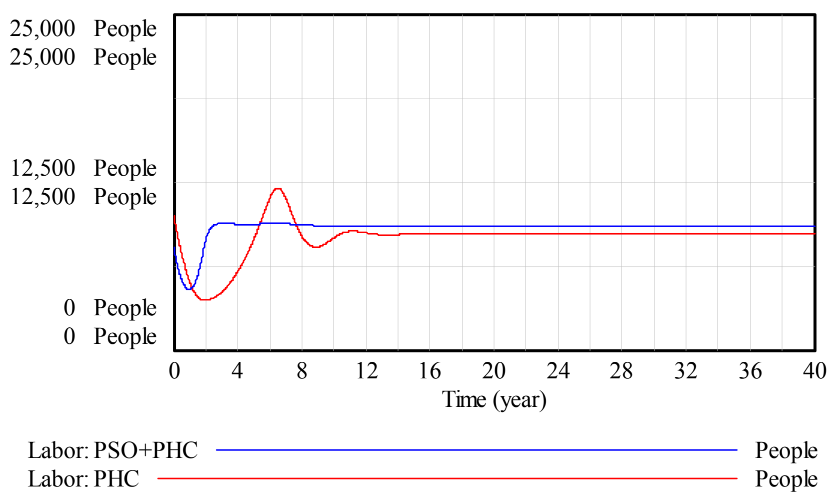

Figure B.3. Labor behaviors using the hybrid and local search algorithms

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