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On the range of the Attenuated Radon Transform in strictly convex sets.

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ON THE RANGE OF THE ATTENUATED RADON TRANSFORM IN STRICTLY CONVEX SETS

by

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ABSTRACT

In the present dissertation, we characterize the range of the attenuated Radon transform of zero, one, and two tensor fields, supported in strictly convex set. The approach is based on a Hilbert transform associated with A-analytic functions of A. Bukhgeim.

We first present new necessary and sufficient conditions for a function on $\partial\Omega \times S^1$, to be in the range of the attenuated Radon transform of a sufficiently smooth function supported in the convex set $\overline{\Omega} \subset \mathbb{R}^2$. The approach is based on an explicit Hilbert transform associated with traces of the boundary of A-analytic functions in the sense of A. Bukhgeim [18]. We then uses the range characterization of the Radon transform of functions to characterize the range of the attenuated Radon transform of vector fields as they appear in the medical diagnostic techniques of Doppler tomography. As an application we determine necessary and sufficient conditions for the Doppler and X-ray data to be mistaken for each other. We also characterize the range of real symmetric second order tensor field using the range characterization of the Radon transform of zero tensor field.

To my parents.

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CHAPTER 1: INTRODUCTION

Inverse Problems originate from practical situations such as medical imaging or exploration geophysics. In a typical Inverse Problem the mathematical model is assumed to be known, but the media in which it takes place it is not. By sending some signals (electromagnetic waves) through the object one measures the effect of the signals at the boundary of the object. The goal in an inverse boundary value problem is to recover the interior structure of the object from the measured data at the boundary of the object.

In general, even though the forward model is linear and well-posed in the sense of Hadamard [29], the Inverse Problem is ill posed and often non-linear. For example in a coefficient identification problem the boundary data depend on the products of the coefficients (to be determined) with solutions of the equations (which in turn depend on the coefficients) also unknown. Solution of an Inverse Problem requires understanding from different areas of mathematics, including Partial Differential Equations, Harmonic Analysis, Complex Analysis, Functional Analysis and Differential Geometry.

One of the inverse boundary value problem is X-ray tomography, where the structure of a two-dimensional object is to be determined by its integrals over lines. In particularly, an object is exposed with a beam of X-rays with known intensity from a source. On the other side of the source a detector is placed to measure the intensity of these X-rays due to attenuating effects of the object. If assume that the medium of the object is non-refractive (X-ray beams traveling along straight line from the source when entered the object follow the same straight line) and X-rays are monochromatic (single energy or wavelength), then the intensity of the X-ray beam, I satisfies Beer's law $\dfrac{dI}{ds} = -f(x)I$, where f is the linear attenuation coefficient of the object [24]. If the initial intensity is I_0 and after traveling the line L the intensity at the detector is I_1, then integrating

Beer's law, we obtain $\log \frac{I_0}{I_1} = \int_L f(x)dx =: Rf$, where Rf is the Radon transform of f over the line L.

$$Rf(s,w) = \int_{\infty}^{\infty} f(sw + tw^{\perp})dt, \quad s \in \mathbb{R}, w \in \mathbf{S}^1.$$

The properties of this transform have been well studied [31]. Radon transforms were developed at the beginning of the twentieth century by P. Funk, G. Lorenz, and J. Radon [52]. In the 1970s, Allan Cormack and Godfrey Hounsfield recognized their work and apply these transforms in the field of medical imaging and was awarded a Nobel Prize in Medicine in 1979.

Further developments led to many different medical imaging methods in common use today, such as X-ray Computerized Tomography (CT), Single Photon Emission Computerized Tomography (SPECT), Positron Emission Tomography (PET), Electrical impedance Tomography (EIT), Ultrasound Tomography and Magnetic Resonance Imaging (MRI), see e.g. [44], [32], [45], [24], [40].

Due to the importance of Radon transform used in some of these medical imaging methods, range characterization of Radon transform and inversion formula of it are of particular interest. Inversion methods of the attenuated Radon transform in the plane appeared first in [10], and [47], and various developments can be found in [46], [16], [26], [12]. Necessary and sufficient constraints on range of the non-attenuated (classical) Radon Transform in the Euclidean space have been known since the works of Gelfand-Graev [28], Helgason [30], and Ludwig [37]. These constraints, known as the Cavalieri or the moment conditions, are in terms of the angular variable: They state that the angular average of the p-moment $\int_{\mathbb{R}} s^p g(s,w)ds$ of the data $g(s,w)$ are homogeneous polynomial of degree p in w. For function in the Schwartz class, they are essentially unique due to a Paley-Wiener type theorem. Moreover, the Helgason support theorem extends the conditions to smooth functions of compact support [31]. However, in the case of functions of compact support, it is possible to obtain essentially different range conditions since more than one operator can annihilate functions of compact support in the range of the Radon transform. The results here in chapter 4, constitute one such example.

For the attenuating media in the Euclidean space analogous range characterization based on the inversion procedure in [47] have been given by Novikov [48]. These constraints are also in terms of the angular variable.

The new range characterization presented here in chapter 4, is in terms of a Hilbert transform associated with the A-analytic maps à la Bukhgeim [18], and represents constraints in the spatial variable; see Theorems 4.1.3, and 4.2.1. These results offer a completely different alternative to Gelfand-Graev [28], Helgason [30], and Ludwig [37], because the constraints are in the spatial variable. Range conditions for the exponential Radon transform [61] has been discussed in [2]. Range characterizations in terms of a Hilbert type transform were first introduced by L.Pestov and G.Uhlmann [51] in the non-attenuated case for smooth functions on two dimensional compact simple Riemmanian manifolds. Extensions to the attenuated case and to tensor tomography has been recently obtained by G.Paternain, M.Salo, and G.Uhlmann [49]: these results are in terms of the angular variable.

Range characterization for higher order tensors are much more recent. In the non-attenuated case, range characterizations of the Doppler transform of one tensors on compact simple manifolds were first introduced in [51] in the more general geometric setting of simple Riemannian surfaces with boundary. Their characterization is in terms of the scattering relation (an involution on the boundary of the unit tangent bundle). Extensions to tensors of higher order as in [35] for the Euclidean case or [50] for the Riemannian case do not address range characterization.

We consider also the problem of the characterization of the attenuated Radon transform of vector fields in the Euclidean plane in chapter 5, as they appear in the medical diagnostic technique of Doppler tomography, e.g [17]. The approach is based on the theory of A-analytic functions developed by A. Bukhgeim [18], and the Hilbert transform associated with A-analytic maps discussed in chapter 4. As an application of zero and one tensor characterization, we address the question of when can the Radon and the Doppler data be mistaken for each other, see chapter 5, section 5.3. We carry similar range characterization for symmetric two tensor in chapter 6.

CHAPTER 2: TRANSPORT EQUATION AND RADON TRANSFORM

In this chapter, we establish a close connection between an Inverse source problem for the transport equation and the attenuated Radon transform. Boundary value problems for the transport equation have been considered in [1], [14], [19], [20], [21], [22], [23], [63]. The identification coefficient problem for the transport equation have been considered in [3], [4], [5], [6], [7], [8], [9], [11], [15], [25], [36], [38], [39], [47], [43], [46], [53], [55].

Let $\Omega \subset \mathbb{R}^2$ be a convex bounded domain with C^2-smooth boundary $\varGamma$ with strictly positive curvature bound. By $\mathbf{S}^1$, we denote the unit circle.

Let $a, f \in C(\overline{\Omega})$ be extended by zero outside.

Definition 2.0.1. *The* divergence beam transform *of a is defined as*

$$Da(x,\theta) := \int_0^\infty a(x + t\theta)dt, \tag{2.1}$$

for each $x \in \Omega$ and $\theta = (\cos\varphi, \sin\varphi) \in S^1$.

The integration in (2.1) is with respect to arc length.
If ∇ denote the gradient in x then the directional derivative of Da, is

$$\theta \cdot \nabla \left(Da(x,\theta)\right) = -a(x,\theta),$$

for each $x \in \Omega$ and $\theta = (\cos\varphi, \sin\varphi) \in S^1$.

Consider the stationary linear transport equation

$$\theta \cdot \nabla u(x,\theta) + a(x)u(x,\theta) = f(x), \quad (x,\theta) \in \Omega \times S^1, \tag{2.2}$$

where $u(x,\theta)$ is the density of particles at x moving in the direction θ, $f(x)$ is the density of radiating particles per unit path-length, and $a(x)$ is the medium capability of absorption per unit path-length at x.

The equation (2.2) can be multiplied by the integrating factor e^{-Da}, where Da as in (2.1) and can be rewritten in the advection form as

$$\theta \cdot \nabla \left(e^{-Da(x,\theta)}u(x,\theta)\right) = f(x)e^{-Da(x,\theta)}, \tag{2.3}$$

for every $(x,\theta) \in \Omega \times \mathbf{S}^1$.

Definition 2.0.2. *The* attenuated Radon transform *of f (with attenuation a) is defined as*

$$\int_{-\infty}^{\infty} f(x+t\theta)e^{-Da(x+t\theta,\theta)}dt. \tag{2.4}$$

for each $x \in \Omega$ and $\theta = (\cos\varphi, \sin\varphi) \in S^1$.

The integral in (2.4) is constant in x in the direction of θ, and this defines a function on the cotangent bundle of the circle $\mathbf{S}^1$.

For any $(x,\theta) \in \overline{\Omega} \times S^1$, let $\tau_\pm(x,\theta)$ denote the distance from x in the $\pm\theta$ direction to the boundary, and distinguish the endpoints $x_\theta^\pm \in \varGamma$ of the chord in the direction of θ passing through x by

$$x_\theta^\pm := x \pm \tau_\pm(x,\theta)\theta. \tag{2.5}$$

Note that

$$\tau(x,\theta) = \tau_+(x,\theta) + \tau_-(x,\theta) \tag{2.6}$$

is the length of the chord, see Figure (2.1).

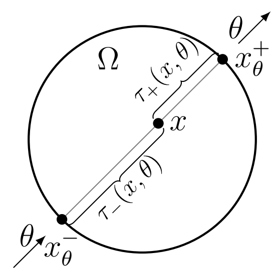

Figure 2.1: Definition of $\tau_\pm(x,\theta)$

Definition 2.0.3. *The function g on $\Gamma \times \mathbf{S}^1$ is an* attenuated Radon transform *of f with attenuation a, if*

$$g(x_\theta^+,\theta) - \left[e^{-Da}g\right](x_\theta^-,\theta) = \int_{\tau_-(x,\theta)}^{\tau_+(x,\theta)} f(x+t\theta)e^{-Da(x+t\theta,\theta)}dt, \tag{2.7}$$

for every $(x,\theta) \in \overline{\Omega} \times \mathbf{S}^1$.

The function g in definition 2.0.3 is not unique, since we can add to g any function h on $\Gamma \times S^1$ such that

$$h(x_\theta^+,\theta) = \left[e^{-Da}h\right](x_\theta^-,\theta). \tag{2.8}$$

If g is an attenuated Radon transform in the sense above, we use the notation $g \in R_a f$. In the case $a \equiv 0$, we use the notation $g \in Rf$.

Note that the only way non-uniqueness occurs is as in (2.8), and that, for functions defined in the whole plane with Radon data at infinity, such an ambiguity cannot occur.

Returning to transport equation, the equation (2.3) can be integrated along lines in direction θ to obtain

$$
\begin{aligned}
&e^{-Da(x+t\theta,\theta)}u(x+t\theta,\theta)\Big|_{\tau_-(x,\theta)}^{\tau_+(x,\theta)} = \int_{\tau_-(x,\theta)}^{\tau_+(x,\theta)} f(x+t\theta)e^{-Da(x+t\theta,\theta)}dt, \text{ which implies} \\
&e^{-Da(x_\theta^+,\theta)}u(x_\theta^+,\theta) - e^{-Da(x_\theta^-,\theta)}u(x_\theta^-,\theta) = \int_{\tau_-(x,\theta)}^{\tau_+(x,\theta)} f(x+t\theta)e^{-Da(x+t\theta,\theta)}dt, \text{ which implies} \\
&u(x_\theta^+,\theta) - e^{-Da(x_\theta^-,\theta)}u(x_\theta^-,\theta) = \int_{\tau_-(x,\theta)}^{\tau_+(x,\theta)} f(x+t\theta)e^{-Da(x+t\theta,\theta)}dt, \quad \because Da(x_\theta^+,\theta) = 0, \quad (2.9)
\end{aligned}
$$

where the notation $x_\theta^\pm$ as in (2.5).

Note that if the function g is the trace on $\Gamma \times \mathbf{S}^1$ of solutions u to the transport equation (2.2), then (2.9) becomes

$$
g(x_\theta^+,\theta) - \left[e^{-Da}g\right](x_\theta^-,\theta) = \int_{\tau_-(x,\theta)}^{\tau_+(x,\theta)} f(x+t\theta)e^{-Da(x+t\theta,\theta)}dt.
$$

This shows that g is an attenuated Radon transform of f with attenuation a, i.e. $g \in R_a f$ and so the function g in our definition (2.7) is precisely the trace on $\Gamma \times \mathbf{S}^1$ of solutions u to the transport equation (2.2).

As the transport equation (2.2) is a parametrized family of ordinary differential equations so to find a solution u, boundary values of u should be known. If the incoming flux is known, $u(x,\theta) = g(x,\theta)$ for $x \in \Gamma$ and $\theta \cdot n(x) < 0$, with $n(x)$ denoting the outer unit normal at x, then the transport equation (2.2) will have a unique solution. The theory of boundary value problems (2.2), and $u|_\Gamma = g$, started with the work of Vladimirov [63] in connection with neutron transport theory, see also [19], [1] , [14], [20], [21], [22], [23].

Thus if $g \in R_a f$, then g in (2.7) completes the specification of the boundary values of u.

$$\begin{aligned} \theta \cdot \nabla u + au = f, \\ u|_{\Gamma} = g, \end{aligned} \quad \Longleftrightarrow g \in R_a f. \tag{2.10}$$

2.1 Special Integrating factor

In this section, we will introduce a special integrating factor for (2.2), the significance of which will be apparent later, see chapter 4, section 4.1.

Let h be defined in $\Omega \times \mathbf{S}^1$ by

$$h(z,\theta) := Da(z,\theta) - \frac{1}{2}\left(I - iH\right) Ra(z\cdot\theta^{\perp},\theta), \tag{2.11}$$

where $Ra(s,\theta) = \displaystyle\int_{-\infty}^{\infty} a\left(s\theta^{\perp} + t\theta\right) dt$ is the Radon transform of the attenuation, and the classical Hilbert transform $Hh(s) = \dfrac{1}{\pi}\displaystyle\int_{-\infty}^{\infty} \frac{h(t)}{s-t} dt$ is taken in the first variable and evaluated at $s = z\cdot\theta^{\perp}$.

It is important to note that the second term in (2.11) varies only in the spatial variable and orthogonal to θ. Hence

$$\theta \cdot \nabla \left((I - iH) Ra(z\cdot\theta^{\perp},\theta)\right) = 0,$$

yielding e^{-h} is an integrating factor for (2.2).

The integrating factor in (2.11) was first considered in the work of Natterrer [44]; see also [16],

[26], and [57] for elegant arguments that show

$$\int_0^{2\pi} h(z, \langle\cos\varphi, \sin\varphi\rangle) e^{in\varphi} = 0, \quad \forall n \leq 0.$$

One of the main result in chapter 4 , gives necessary and sufficient conditions for $g \in R_a f$. These conditions characterize the traces $u|_{\Gamma\times\mathbf{S}^1}$ of solutions of (2.2), as traces on Γ of solutions of A-analytic functions.

CHAPTER 3: A- ANALYTIC FUNCTIONS

We recall some preliminary notions and results from the theory of A-analytic sequence valued maps developed by A. Bukhgeim [18], singular integral and harmonic analysis.

Let $\Omega \subset \mathbb{R}^2$ be a convex bounded domain with C^2-smooth boundary $\varGamma$ with strictly positive curvature bound.

Let l_∞ be the space of bounded sequences and let $\mathcal{L}$ be the left shift operator

$$\mathcal{L}\mathbf{v} = \langle v_{-1}, v_{-2}, v_{-3}, ...\rangle,$$

with $\mathcal{L}^k$ the k-composition $\underbrace{\mathcal{L} \circ \cdots \circ \mathcal{L}}_{k}$.

Definition 3.0.1. *The sequence valued map* $z \mapsto \mathbf{v}(z) := \langle v_0(z), v_{-1}(z), v_{-2}(z), ...\rangle$ *is called* A*-analytic if* $\mathbf{v} \in C(\overline{\Omega}; l_\infty) \cap C^1(\Omega; l_\infty)$ *and*

$$\overline{\partial}\mathbf{v}(z) + \mathcal{L}^2 \partial \mathbf{v}(z) = 0, \tag{3.1}$$

where

$$z = x_1 + ix_2, \quad \overline{\partial} = (\partial_{x_1} + i\partial_{x_2})/2, \quad \partial = (\partial_{x_1} - i\partial_{x_2})/2.$$

For a compact set $K \subset \mathbb{R}^2$, such as $\Gamma, \overline{\Omega}$, $\mathbf{S}^1$, or $\overline{\Omega} \times \mathbf{S}^1$, by $C^\alpha(K)$ we denote the Banach space of uniform α- Hölder continuous functions endowed with the norm

$$\|f\|_{C^\alpha(K)} := \sup_{z\in K} |f(z)| + \sup_{z,w\in K,\, z\neq w} \frac{|f(z) - f(w)|}{|z-w|^\alpha}.$$

By $C^\alpha(\Omega)$ we denote the space of locally uniform α- Hölder continuous functions.

We note the general fact that, for a sequence of nonnegative numbers.

Lemma 3.0.1. *Let $\{c_n\}$ be a sequence of nonnegative numbers. Then*

$$(i) \quad \sum_{k=1}^{\infty}\sum_{n=0}^{\infty} k\ c_{k+n} = \sum_{j=1}^{\infty} \frac{j(j+1)}{2}\ c_j,$$
$$(ii) \quad \sum_{k=1}^{\infty}\sum_{n=0}^{\infty} c_{k+n} = \sum_{j=1}^{\infty} j\ c_j,$$

whenever one of the sides in (i) and (ii) is finite.

Proof. (i) Indeed, if we introduce the change of index $j = k+n$, for $k \geq 1$, ($j - n \geq 1$, and $n \leq j-1$) we get

$$\sum_{k=1}^{\infty}\sum_{n=0}^{\infty} kc_{k+n} = \sum_{j=1}^{\infty}\sum_{n=0}^{j-1}(j-n)\ c_j = \sum_{j=1}^{\infty} c_j \sum_{n=0}^{j-1}(j-n) = \sum_{j=1}^{\infty}\frac{j(j+1)}{2}c_j.$$

(ii) Indeed the change of index $j = k+n$, for $k \geq 1$, yields

$$\sum_{k=1}^{\infty}\sum_{n=0}^{\infty} c_{k+n} = \sum_{j=1}^{\infty}\sum_{n=0}^{j-1} c_j = \sum_{j=1}^{\infty} c_j \sum_{n=0}^{j-1} 1 = \sum_{j=1}^{\infty} jc_j.$$

□

In several of the arguments we make use of the following Bernstein's lemma below (see, e.g., [33]).

Lemma 3.0.2. *Let $f \in C^{k,\alpha}(\mathbf{S}^1)$, $\alpha > 1/2$, and $\{\hat{f}_n\}$ be the sequence of its Fourier coefficients. Then* $\sum_{n=-\infty}^{\infty} |n|^k|\hat{f}_n| < \infty$.

To characterize traces of A-analytic functions we need to control the decay in the Fourier terms. We work in the following Banach spaces

$$l^{1,1}_{\infty}(\Gamma) := \left\{ \mathbf{v} = \langle v_0, v_{-1}, ... \rangle : \sup_{w \in \varGamma} \sum_{j=1}^{\infty} j |v_{-j}(w)| < \infty \right\}, \tag{3.2}$$

and

$$C^{\epsilon}(\Gamma; l_1) := \left\{ \mathbf{v} = \langle v_0, v_{-1}, ... \rangle : \sup_{\xi \in \varGamma} \|\mathbf{v}(\xi)\|_{l_1} + \sup_{\substack{\xi, \eta \in \varGamma \\ \xi \neq \eta}} \frac{\|\mathbf{v}(\xi) - \mathbf{v}(\eta)\|_{l_1}}{|\xi - \eta|^{\epsilon}} < \infty \right\}, \tag{3.3}$$

where l_1 is the space of sumable sequences. By replacing Γ with $\overline{\Omega}$ and l_1 with l_∞ in (3.3) we similarly define $C^{\epsilon}(\overline{\Omega}; l_1)$, respectively, $C^{\epsilon}(\overline{\Omega}; l_\infty)$, where l_∞ denotes the space of bounded sequences.

We describe next the two operators which define the Hilbert transform associated with A-analytic maps. For $\mathbf{v} \in C^{\epsilon}(\Gamma, l_1)$, we consider the Cauchy integral operators defined componentwise by

$$(C\mathbf{v})_n(\xi) := (Cv_n)(\xi) = \frac{1}{2\pi i} \int_{\varGamma} \frac{v_n(w)}{w - \xi} dw, \quad \xi \in \Omega, \tag{3.4}$$

and

$$(S\mathbf{v})_n(\xi) := (Sv_n)(\xi) = \frac{1}{\pi i} \int_{\varGamma} \frac{v_n(w)}{w - \xi} dw, \quad \xi \in \Gamma, \quad n = 0, -1, -2, ... \tag{3.5}$$

The later integral is understood in the Cauchy principal value sense.

The following result is a componentwise extension of Sokhotski-Plemelj formula (e.g., [42]) to sequence valued maps.

Proposition 3.0.1. *Let* $\mathbf{v} \in C^{\epsilon}(\varGamma; l_1)$ *as in* (3.3). *Then, for every* $\xi_0 \in \Gamma$*, the limit*

$$\lim_{\Omega \ni \xi \to \xi_0} \left\| (C\mathbf{v})(\xi) - \frac{1}{2}\mathbf{v}(\xi_0) - \frac{1}{2} S\mathbf{v}(\xi_0) \right\|_{l_1} = 0, \tag{3.6}$$

defines an extension of $C\mathbf{v}$ *from* Ω *to* $\overline{\Omega}$ *as a* $Holder$ *continuous map with values in* l_1*, i.e,*

$$C : C^{\epsilon}(\mathit{\Gamma}; l_1) \longrightarrow C^{\epsilon}\left(\overline{\Omega}; l_1\right) \cap C^1\left(\Omega; l_1\right).$$

The fact that $C\mathbf{v} \in C^1(\Omega; l_1)$ follows directly from the local character of differentiability and from the fact that $\sum_{n=1}^{\infty} \int_{\mathit{\Gamma}} |v_{-n}(w)dw| < \infty$.

Next we introduce the second operator which appears in the definition of the Hilbert transform. It is defined componentwise for each index $n \leq 0$, $\xi \in \overline{\Omega}$, $w \in \mathit{\Gamma}$, and $\mathbf{v} \in l^{1,1}_{\infty}(\Gamma)$ by

$$(G\mathbf{v})_n(\xi) = \frac{1}{\pi i} \int_{\mathit{\Gamma}} \left\{ \frac{dw}{w - \xi} - \frac{d\overline{w}}{\overline{w - \xi}} \right\} \sum_{j=1}^{\infty} v_{n-2j}(w) \left(\frac{\overline{w - \xi}}{w - \xi} \right)^j. \tag{3.7}$$

We will use the following mapping property of G.

Proposition 3.0.2.

$$G : C^{\epsilon}(\mathit{\Gamma}; l_1) \cap l^{1,1}_{\infty}(\Gamma) \longrightarrow C^{\epsilon}\left(\overline{\Omega}; l_{\infty}\right) \cap C^1\left(\Omega; l_{\infty}\right).$$

Proof. Let ξ, $\xi_0 \in \Omega$. Since $\mathbf{v} \in l^{1,1}_{\infty}(\Gamma)$, it follows from (3.7) that each component $(G\mathbf{v})_n(\xi)$ is well-defined for $n \leq 0$.

Now let $w(\varphi) = \xi + l_{\xi}(\varphi)e^{i\varphi}$ be a parametrization of Γ, where $l_{\xi}(\varphi) = |\xi - w(\varphi)|$. Since the boundary Γ is at least C^1, we have that $\xi \mapsto l_{\xi}$ is Lipschitz in $\overline{\Omega}$ uniformly in $\varphi \in [0, 2\pi]$, i.e.,

$$|l_{\xi}(\varphi) - l_{\xi_0}(\varphi)| \leq L|\xi - \xi_0|, \tag{3.8}$$

for some constant $L > 0$. Moreover,

$$\frac{dw}{w-\xi} = \left[\frac{l'_\xi}{l_\xi} + i\right] d\varphi, \quad \frac{d\overline{w}}{\overline{w-\xi}} = \left[\frac{l'_\xi}{l_\xi} - i\right] d\varphi, \quad \left(\frac{\overline{w-\xi}}{w-\xi}\right) = e^{-2i\varphi},$$

and note that the measure $\dfrac{dw}{w-\xi} - \dfrac{d\overline{w}}{\overline{w-\xi}} = 2id\varphi$, in (3.7), is nonsingular.

For each integer $n \leq 0$, the equation (3.7) rewrites

$$(G\,\mathbf{v})_n(\xi) = \frac{2}{\pi}\int_0^{2\pi} \sum_{j=1}^{\infty} g_{n-2j}(\xi + l_\xi(\varphi)\, e^{i\varphi})\; e^{-2ij\varphi} d\varphi,\;\; \xi \in \overline{\Omega}.$$

Since $\mathbf{v} \in C^\epsilon(\Gamma; l_1)$, we have

$$\kappa := \sup_{\substack{w_1, w_2 \in \Gamma \\ w_1 \neq w_2}} \sum_{n=0}^{\infty} \frac{|g_{-n}(w_1) - g_{-n}(w_2)|}{|w_1 - w_2|^\epsilon} < \infty.$$

We estimate for each n,

$$\begin{aligned}
&|(G\mathbf{v})_n(\xi) - (G\mathbf{v})_n(\xi_0)| \\
&\qquad \leq \frac{2}{\pi}\sum_{j=1}^{\infty}\int_0^{2\pi} \left| v_{n-2j}(\xi + l_\xi(\varphi)\, e^{i\varphi}) - v_{n-2j}(\xi_0 + l_{\xi_0}(\varphi)\, e^{i\varphi}) \right| d\varphi \\
&\qquad \leq \frac{2\kappa}{\pi}\int_0^{2\pi} \left|(\xi - \xi_0) + [l_\xi(\varphi) - l_{\xi_0}(\varphi)]\; e^{i\varphi}\right|^\epsilon d\varphi, \\
&\qquad \leq \frac{2\kappa}{\pi}\int_0^{2\pi} \left(2\,|\xi - \xi_0|^\epsilon + |l_\xi(\varphi) - l_{\xi_0}(\varphi)|^\epsilon\right) d\varphi, \\
&\qquad \leq (8\kappa + 4\kappa L^\epsilon)\,|\xi - \xi_0|^\epsilon .
\end{aligned}$$

In the third inequality above we used $|a+b|^\epsilon \leq 2|a|^\epsilon + |b|^\epsilon$, and the fourth inequality uses (3.8).

Next we show that $G\mathbf{v} \in C^1(\Omega; l_\infty)$. Suffices to carry the estimates in the neighborhood

$\overline{B(\xi_0, r_0)} \subset \Omega$ of an arbitrary point $\xi_0 \in \Omega$, where $r_0 = \text{dist}(\xi_0, \Gamma)/2 > 0$.

For $\xi \in B(\xi_0, r_0)$ arbitrary, we have

$$\left| \nabla_\xi \left\{ \frac{dw}{w - \xi} - \frac{d\overline{w}}{\overline{w - \xi}} \right\} \right| = \left| 2\, \mathbb{I}\text{m} \left(\frac{dw}{(w - \xi)^2} \right) \right| \leq c|dw|, \tag{3.9}$$

where $c = 2/r_0^2$.

For each $n \leq 0$, we have

$$\nabla_\xi (G\,\mathbf{v})_n(\xi) = \frac{1}{\pi i} \int_\Gamma \nabla_\xi \left\{ \frac{dw}{w - \xi} - \frac{d\overline{w}}{\overline{w - \xi}} \right\} \sum_{j=1}^{\infty} v_{n-2j}(w) \left(\frac{\overline{w - \xi}}{w - \xi} \right)^j$$
$$+ \frac{1}{\pi i} \int_\Gamma \left\{ \frac{dw}{w - \xi} - \frac{d\overline{w}}{\overline{w - \xi}} \right\} \sum_{j=1}^{\infty} v_{n-2j}(w) \nabla_\xi \left(\frac{\overline{w - \xi}}{w - \xi} \right)^j .$$

For $\mathbf{v} \in C^\epsilon(\Gamma; l_1) \cap l_\infty^{1,1}(\Gamma)$, and $\xi \in \overline{B(\xi_0, r_0)}$, the right hand side above is bounded uniformly in n, since

$$|\nabla_\xi (G\,\mathbf{v})_n(\xi)| \leq \frac{c}{\pi} \int_\Gamma \sum_{j=1}^{\infty} |v_{n-2j}(w)|\, dw$$
$$+ \frac{c}{\pi} \int_\Gamma \sum_{j=1}^{\infty} j\ |v_{n-2j}(w)|\, dw < \infty.$$

Therefore $G\mathbf{v} \in C^1(\Omega; l_\infty)$. $\square$

CHAPTER 4: RANGE CHARACTERIZATION OF ZERO TENSOR

In this chapter, we first consider the zero tensor non-attenuated case, and gives necessary and sufficient conditions for some function g to be in the range of attenuated Radon transform of some f, i.e. $g \in Rf$ as in Definition 2.0.3. These conditions characterize the traces $u|_{\Gamma \times \mathbf{S}^1}$ of solutions of (2.2), as traces on Γ of solutions of A-analytic functions. We then consider the zero tensor attenuated case and see that it will reduces to non-attenuated case. We use these results to characterize the range of one and two tensor.

4.1 Non Attenuated Case

In the non-attenuated case ($a \equiv 0$) , the transport equation simplifies to

$$\theta \cdot \nabla v(x, \theta) = f(x), \quad (x, \theta) \in \Omega \times S^1. \tag{4.1}$$

With the complex notations

$$z = x_1 + ix_2, \quad \overline{\partial} = (\partial_{x_1} + i\partial_{x_2})/2, \quad \partial = (\partial_{x_1} - i\partial_{x_2})/2,$$

the advection operator becomes

$$\theta \cdot \nabla = e^{-i\varphi}\overline{\partial} + e^{i\varphi}\partial,$$

where $\varphi = \arg(\theta)$ denotes an angular variable .

Let $v(z, \theta) = \sum_{-\infty}^{\infty} v_n(z)e^{in\varphi}$, be the (formal) Fourier expansion of v in the angular variable. Provided appropriate convergence of the series as specified in the theorems, we see that v solves (4.1) if and only if its Fourier coefficients solve

$$\overline{\partial} v_{-1}(z) + \partial v_1(z) = f(z), \tag{4.2}$$

and, for $n \neq 1$,

$$\overline{\partial} v_n(z) + \partial v_{n-2}(z) = 0.$$

Since v is real-valued, its Fourier coefficients appear in complex-conjugate pairs, $\overline{v_n} = v_{-n}$, so that it suffices to work with the sequence of non-positive indexes (this choice preserves the original notation in [18]).

We characterize the range of the non-attenuated Radon transform by introducing the Hilbert transform $\mathcal{H}_0$ associated with the traces on Γ of A-analytic maps in Ω.

Recall the operator S and G as defined in (3.5), and (3.7).

Definition 4.1.1. *The Hilbert transform $\mathcal{H}_0$ for $\mathbf{g} = \langle g_0, g_{-1}, ... \rangle \in l^{1,1}_{\infty}(\Gamma) \cap C^{\epsilon}(\Gamma; l_1)$ is defined by*

$$\mathcal{H}_0 \mathbf{g} := i[S + G]\mathbf{g}, \tag{4.3}$$

and written componentwise, for $n = 0, -1, -2, ...$, as

$$\begin{aligned}(\mathcal{H}_0 \mathbf{g})_n(\xi) =& \frac{1}{\pi} \int_{\Gamma} \frac{g_n(w)}{w - \xi} dw \\ &+ \frac{1}{\pi} \int_{\Gamma} \left\{ \frac{dw}{w - \xi} - \frac{d\overline{w}}{\overline{w - \xi}} \right\} \sum_{j=1}^{\infty} g_{n-2j}(w) \left(\frac{\overline{w - \xi}}{w - \xi} \right)^j, \quad \xi \in \Gamma.\end{aligned}$$

The mapping properties of S, and G in Propositions 3.0.1, and 3.0.2, together with the continuous embedding of $l_1 \subset l_\infty$, yields

Proposition 4.1.1.

$$\mathcal{H}_0 : C^\epsilon(\mathit{\Gamma}; l_1) \cap l^{1,1}_\infty(\Gamma) \longrightarrow C^\epsilon\left(\Gamma; l_\infty\right), \tag{4.4}$$

is a continuous map.

The name of this transform will be motivated in the next section, where we show that traces on Γ of A-analytic maps lie in the kernel of $[I + i\mathcal{H}_0]$ in analogy with the classical Hilbert transform for analytic functions.

At the heart of the theory of A-analytic maps lies a Cauchy integral formula. A class of such Cauchy integral formulae were first introduced by Bukhgeim in [18]. The explicit form (4.5) below is due to Finch [26]; see also [58, 59, 60] where one works with square summable sequences.

Theorem 4.1.1 (Bukhgeim-Cauchy Integral Formula)**.** *Let* $\mathbf{g} = \langle g_0, g_{-1}, ...\rangle \in l^{1,1}_\infty(\Gamma) \cap C^\epsilon(\Gamma; l_1)$ *be a sequence valued map defined at the boundary* $\mathit{\Gamma}$*. For* $\xi \in \Omega$*, and each index* $n \leq 0$*, we consider the Bukhgeim-Cauchy operator* $(B\mathbf{g})_n(\xi)$ *defined by*

$$(B\mathbf{g})_n(\xi) := \frac{1}{2}(G\mathbf{g})_n(\xi) + (C\mathbf{g})_n(\xi). \tag{4.5}$$

Then $\mathbf{v} := \langle (B\mathbf{g})_0, (B\mathbf{g})_{-1}, (B\mathbf{g})_{-2}, ...\rangle \in C^{1,\epsilon}(\Omega; l_\infty)$*, and for each* $n = 0, -1, ...,$

$$\overline{\partial}(B\mathbf{g})_n(\xi) + \partial(B\mathbf{g})_{n-2}(\xi) = 0, \quad \xi \in \Omega.$$

Moreover, for each $n = 0, -1, ...,$

$$(B\mathbf{g})_n \in C^\infty(\Omega). \tag{4.6}$$

Furthermore, for each $n = 0, -1, -2, ...$, the component $(B\mathbf{g})_n$ extends continuously to $\overline{\Omega}$ with limiting values

$$(B\mathbf{g})_n^+(\xi_0) := \lim_{\Omega \ni \xi \to \xi_0 \in \Gamma} (B\mathbf{g})_n(\xi), \tag{4.7}$$

where

$$(B\mathbf{g})_n^+(\xi_0) = \frac{1}{2}(G\mathbf{g})_n(\xi_0) + \frac{1}{2}(S + I)g_n(\xi_0). \tag{4.8}$$

Proof. Let $\xi \in \Omega$ and $n \leq 0$ arbitrarily fixed. Since $\mathbf{g} \in l_\infty^{1,1}(\Gamma) \cap C^\epsilon(\Gamma, l_1)$, both $(G\mathbf{g})_n(\xi)$ and $(C\mathbf{g})_n)(\xi)$ are well-defined. Moreover, from Propositions 3.0.1, and 3.0.2, we have that $\mathbf{v} \in C(\overline{\Omega}; l_\infty) \cap C^1(\Omega; l_\infty)$.

For each $n \leq 0$, by its definition in (4.5), we have

$$2\pi i (B\mathbf{g})_n(\xi) = \sum_{j=0}^{\infty} \int_\Gamma \frac{g_{n-2j}(w)\overline{(w-\xi)}^j}{(w-\xi)^{j+1}} dw - \sum_{j=1}^{\infty} \int_\Gamma \frac{g_{n-2j}(w)\overline{(w-\xi)}^{j-1}}{(w-\xi)^j} d\overline{w}.$$

From where

$$2\pi i \partial (B\mathbf{g})_{n-2}(\xi) = \sum_{j=1}^{\infty} \int_\Gamma \frac{j g_{n-2j}(w)\overline{(w-\xi)}^{j-1}}{(w-\xi)^{j+1}} dw - \sum_{j=2}^{\infty} \int_\Gamma \frac{(j-1) g_{n-2j}(w)\overline{(w-\xi)}^{j-2}}{(w-\xi)^j} d\overline{w}, \tag{4.9}$$

and

$$2\pi i\overline{\partial}(B\mathbf{g})_n(\xi) = -\sum_{j=1}^{\infty}\int_{\Gamma}\frac{jg_{n-2j}(w)\overline{(w-\xi)}^{j-1}}{(w-\xi)^{j+1}}dw + \sum_{j=2}^{\infty}\int_{\Gamma}\frac{(j-1)g_{n-2j}(w)\overline{(w-\xi)}^{j-2}}{(w-\xi)^{j}}d\overline{w}. \tag{4.10}$$

By summing (4.9) and (4.10) we obtain $\overline{\partial}(B\mathbf{g})_n + \partial(B\mathbf{g})_{n-2} = 0$ for each $n = 0, -1, -2, ...$

The regularity $(B\mathbf{g})_n \in C^{\infty}(\Omega)$ follows from the explicit formula (4.9), and the fact that $\xi \mapsto \frac{\overline{(w-\xi)}^{j}}{(w-\xi)^{j+k}}$, for $k \geq 2$, are locally uniform ϵ-Hölder continuous.

The continuity to the boundary are consequences of Propositions 3.0.1, and 3.0.2. In the limits below $\xi \in \Omega$, and $\xi_0 \in \Gamma$:

$$\begin{aligned}
\lim_{\xi\to\xi_0}(B\mathbf{g})_n(\xi) &= \lim_{\xi\to\xi_0}\frac{1}{2}(G\mathbf{v})_n(\xi) \; + \; \lim_{\xi\to\xi_0}(C\mathbf{g})_n)(\xi),\\
&= \frac{1}{2}(G\mathbf{v})_n(\xi_0) \; + \; \frac{1}{2}g_n(\xi_0) + \frac{1}{2}\sum_{n=0}^{\infty}(S\mathbf{g})_n(\xi_0),\\
&= \frac{1}{2}(G\mathbf{v})_n(\xi_0) + \frac{1}{2}(S+I)g_n(\xi_0).
\end{aligned}$$

□

The following theorem presents necessary and sufficient conditions for sufficiently regular sequence valued map to be the trace at the boundary of an A-analytic function.

Theorem 4.1.2. *Let* $\mathbf{g} = \langle g_0, g_{-1}, g_{-2}, \cdots \rangle \in l^{1,1}_{\infty}(\Gamma) \cap C^{\epsilon}(\Gamma, l^1)$. *For* $\mathbf{g}$ *to be boundary value of an* A*-analytic function it is necessary and sufficient that*

$$(I + i\mathcal{H}_0)\mathbf{g} = 0. \tag{4.11}$$

Proof. For the necessity, let $\mathbf{v} = \langle v_0, v_{-1}, v_{-2}, \cdots \rangle$ be A-analytic as in (3.1) whose trace $\mathbf{v}|_\Gamma = \mathbf{g}$, in the sense that

$$\lim_{\Omega \ni \xi \to \xi_0 \in \varGamma} v_n(\xi) = g_n(\xi_0), \quad n \leq 0.$$

By (4.8), we obtain

$$g_n(\xi_0) = \frac{1}{2}(G\mathbf{v})_n(\xi_0) + \frac{1}{2}Sg_n(\xi_0) + \frac{1}{2}g_n(\xi_0),$$

or,

$$[(I - S - G)\mathbf{g}]_n = 0, \quad n \leq 0. \tag{4.12}$$

Since $\mathcal{H}_0 = i[S + G]$, (4.12) is a componentwise representation of (4.11).

Next we prove sufficiency. Let $\mathbf{g} \in l^{1,1}_{\infty}(\Gamma) \cap C^{\epsilon}(\Gamma, l_1)$ satisfy (4.11), and define

$$\mathbf{v} := \langle (B\mathbf{g})_0, (B\mathbf{g})_{-1}, ... \rangle,$$

where B is the Bukhgeim-Cauchy operator as defined in (4.5). From Propositions 3.0.1, and 3.0.2, we have that $\mathbf{v} \in C^1(\Omega; l_\infty) \cap C(\overline{\Omega}; l_\infty)$, and from Theorem 4.1.1, we see that $\overline{\partial}(B\mathbf{g})_n + \partial(B\mathbf{g})_{n-2} = 0$, for each $n \leq 0$. Therefore $\mathbf{v}$ is A-analytic. Moreover,

$$\begin{aligned}
\lim_{\Omega \ni \xi \to \xi_0 \in \varGamma} (B\mathbf{g})_n(\xi) &= \frac{1}{2}(G\mathbf{v})_n(\xi_0) + \frac{1}{2}(S + I)g_n(\xi_0), \\
&= \frac{1}{2}(I - S)g_n(\xi_0) + \frac{1}{2}(S + I)g_n(\xi_0), \\
&= g_n(\xi_0),
\end{aligned}$$

where the first equality uses (4.8), whereas the second equality uses (4.11). $\square$

4.1.1 Range characterization of the non-attenuated Radon transform of zero tensors

This section concerns our main result in the non-attenuated case ($a \equiv 0$). The results require a stronger topology. For $\epsilon > 0$, we consider the space $Y_\epsilon = C^\epsilon(\Gamma; l^{1,1}(\mathbf{S}^1)) \cap C^0(\Gamma; l^{1,2}(\mathbf{S}^1))$ i.e

$$Y_\epsilon = \left\{ \mathbf{g} \in l^{1,2}_\infty(\Gamma) : \sup_{\substack{\xi,\mu \in \Gamma \\ \xi \neq \mu}} \sum_{j=1}^{\infty} j \frac{|g_{-j}(\xi) - g_{-j}(\mu)|}{|\xi - \mu|^\epsilon} < \infty \right\}, \tag{4.13}$$

where

$$l^{1,2}_\infty(\Gamma) := \left\{ \mathbf{g} = \langle g_0, g_{-1}, g_{-2}, \cdots \rangle \ : \sup_{w \in \Gamma} \sum_{j=1}^{\infty} j^2 \, |g_{-j}(w)| < \infty \right\}. \tag{4.14}$$

For the sake of clarity in the statement of the main result we introduce the following projections.

Definition 4.1.2. *Given a function $g \in C(\overline{\Omega}; L^1(\mathbf{S}^1))$, we consider the projections*

$$\mathcal{P}^-(g) := \langle g_0, g_{-1}, g_{-2}, ... \rangle, \qquad \mathcal{P}^+(g) := \langle g_0, g_1, g_2, ... \rangle, \tag{4.15}$$

where $g_n(z) = \frac{1}{2\pi} \int_0^{2\pi} g(z, \theta) e^{-in\varphi} d\varphi$, for $z \in \overline{\Omega}$, is the n-th Fourier coefficients for $n \in \mathbb{Z}$. Conversely, given $\mathbf{g}(z) = \langle g_0(z), g_{-1}(z), g_{-2}(z), ... \rangle \in C(\overline{\Omega}; l^1)$, we define a corresponding real valued function g on $\overline{\Omega} \times \mathbf{S}^1$ by

$$\mathcal{P}^*(\mathbf{g}) := g_0(z) + 2\,\mathbb{R}\mathrm{e} \left(\sum_{n=1}^{\infty} g_{-n}(z) e^{-in\varphi} \right). \tag{4.16}$$

The properties below are immediate:

If g is a function on $\mathit{\Gamma} \times \mathbf{S}^1$ then

$$(i)\ \mathcal{P}^- \mathcal{P}^* \mathcal{P}^-(g) = \mathcal{P}^-(g), \tag{4.17}$$

$$(ii)\ \mathcal{P}^-(e^{\pm h} g) = (\mathcal{P}^+ e^{\pm h}) *_n (\mathcal{P}^-(g)), \tag{4.18}$$

where $*_n$ is the convolution operator on sequences and h is a function on $\mathit{\Gamma} \times \mathbf{S}^1$ with only non negative Fourier modes.

The following result gives some of the properties of the $P^{\pm}$ and $\mathcal{P}^*$ operators. Recall the definition of the space Y_ϵ in (4.13).

Proposition 4.1.2. *Let $\alpha > 1/2$, and $\epsilon > 0$ be arbitrarily small. Then*

(i) $\mathcal{P}^- : C^\epsilon\left(\mathit{\Gamma}; C^{1,\alpha}(\mathbf{S}^1)\right) \to l^{1,1}_\infty(\Gamma) \cap C^\epsilon(\Gamma; l^1)$,

(ii) $\mathcal{P}^- : C^\epsilon\left(\mathit{\Gamma}; C^{1,\alpha}(\mathbf{S}^1)\right) \cap C^0\left(\mathit{\Gamma}; C^{2,\alpha}(\mathbf{S}^1)\right) \to Y_\epsilon$,

(iii) $\mathcal{P}^* : C^{1,\alpha}(\Omega; l^1) \cap C^\alpha(\overline{\Omega}; l^1) \to C^{1,\alpha}(\Omega \times \mathbf{S}^1) \cap C^\alpha(\overline{\Omega} \times \mathbf{S}^1)$.

Proof. Let $g \in C^\epsilon\left(\mathit{\Gamma}; C^{1,\alpha}(\mathbf{S}^1)\right)$. Then

$$\sup_{\xi \in \mathit{\Gamma}} \|g(\xi, \cdot)\|_{C^{1,\alpha}} + \sup_{\substack{\xi, \mu \in \mathit{\Gamma} \\ \xi \neq \mu}} \frac{\|g(\xi, \cdot) - g(\mu, \cdot)\|_{C^{1,\alpha}}}{|\xi - \mu|^\epsilon} < \infty. \tag{4.19}$$

From

$$\sup_{\xi \in \mathit{\Gamma}} \sum_{j=1}^{\infty} j\, |g_{-j}(\xi)| \leq \sup_{\xi \in \mathit{\Gamma}} \|g(\xi, \cdot)\|_{C^{1,\alpha}} < \infty, \tag{4.20}$$

and by Lemma 3.0.2, $\mathcal{P}^-(g) \in l^{1,1}_\infty(\Gamma)$.

Another application of Lemma 3.0.2, together with (4.19) imply

$$\sup_{\substack{\xi,\mu\in\varGamma\\ \xi\neq\mu}} \sum_{j=1}^{\infty} \frac{j|g_{-j}(\xi)-g_{-j}(\mu)|}{|\xi-\mu|^{\epsilon}} \leq \sup_{\substack{\xi,\mu\in\varGamma\\ \xi\neq\mu}} \frac{\|g(\xi,\cdot\,)-g(\mu,\cdot\,)\|_{C^{1,\alpha}}}{|\xi-\mu|^{\epsilon}} < \infty. \tag{4.21}$$

By combining the estimates (4.20) and (4.21) we showed that $\mathcal{P}^{-}(g) \in C^{\epsilon}(\varGamma; l_1)$. This proves part (i).

Now let $g \in C^{\epsilon}\left(\varGamma; C^{1,\alpha}(\mathbf{S}^1)\right) \cap C^{0}\left(\varGamma; C^{2,\alpha}(\mathbf{S}^1)\right)$. Since $g \in C^{0}\left(\varGamma; C^{2,\alpha}(\mathbf{S}^1)\right)$, then

$$\sup_{\xi\in\varGamma}\|g(\xi,\cdot\,)\|_{C^{2,\alpha}} < \infty.$$

Lemma 3.0.2, applied to $g(\xi,\cdot\,) \in C^{2,\alpha}$ for $\xi \in \Gamma$, yields

$$\sup_{w\in\varGamma} \sum_{j=1}^{\infty} j^{2}\,|g_{-j}(w)| \leq \|g(\xi,\cdot\,)\|_{C^{2,\alpha}}. \tag{4.22}$$

This shows that $\mathcal{P}^{-}(g) \in l^{1,2}_{\infty}(\Gamma)$. Now (4.21) yields $\mathcal{P}^{-}(g) \in Y_{\epsilon}$.

By triangle inequality in (4.16), we have $\mathbf{g} \in C^{1,\alpha}(\Omega; l^1) \cap C^{\alpha}(\overline{\Omega}; l^1)$, yields

$$\sup_{\xi\in\overline{\Omega}} \|\mathbf{g}(\xi)\|_{l^1} + \sup_{\substack{\xi,\mu\in\overline{\Omega}\\ \xi\neq\mu}} \frac{\|\mathbf{g}(\xi)-\mathbf{g}(\mu)\|_{l^1}}{|\xi-\mu|^{\alpha}} < \infty.$$

For $\xi \in \Omega$, and $r > 0$ with $\overline{B(\xi;r)} \subset \Omega$, there is an $M_{\xi,r} > 0$, with

$$\sup_{\xi\in\overline{B(\xi;r)}} \|\nabla\mathbf{g}(\xi)\|_{l^1} + \sup_{\substack{\mu\in\overline{B(\xi;r)}\\ \xi\neq\mu}} \frac{\|\nabla\mathbf{g}(\xi)-\nabla\mathbf{g}(\mu)\|_{l^1}}{|\xi-\mu|^{\alpha}} \leq M_{\xi,r}.$$

These proves part(iii).

□

The following result refines the mapping properties of the operator G in (3.7), when restricted to the subspace Y_ϵ.

Proposition 4.1.3. *Let* Y_ϵ *be the space defined in* (4.13). *Then*

$$(i)\ G : Y_\epsilon \longrightarrow C^\epsilon\left(\overline{\Omega}; l^1\right) \cap C^1\left(\Omega; l^1\right),$$
$$(ii)\ \mathcal{H}_0 : Y_\epsilon \longrightarrow C^\epsilon\left(\Gamma; l^1\right).$$

Proof. (i) Let $\xi, \xi_0 \in \overline{\Omega}$ and $\mathbf{g} \in Y_\epsilon$. Using the parametrization $w(\varphi) = \xi + l_\xi(\varphi)e^{i\varphi}$, where $l_\xi(\varphi) = |\xi - w(\varphi)|$, we obtain as in the proof in Proposition 3.0.2, that

$$(G\mathbf{g})_{-n}(\xi) = \frac{2}{\pi}\int_0^{2\pi}\sum_{j=1}^{\infty} g_{-n-2j}(\xi + l_\xi(\varphi)\,e^{i\varphi})\,e^{-2ij\varphi}d\varphi,$$

is well defined for $\xi \in \overline{\Omega}$.

Since $\mathbf{g} \in Y_\epsilon$, we have

$$\kappa := \sup_{\substack{\xi,\mu\in\Gamma\\ \xi\neq\mu}}\sum_{j=1}^{\infty} j\frac{|g_{-j}(\xi) - g_{-j}(\mu)|}{|\xi - \mu|^\epsilon} < \infty.$$

We estimate

$$
\begin{aligned}
\sum_{n=0}^{\infty} & |(G\,\mathbf{g})_{-n}(\xi) - (G\,\mathbf{g})_{-n}(\xi_0)| \\
&\leq \frac{2}{\pi} \sum_{n=0}^{\infty} \sum_{j=1}^{\infty} \int_0^{2\pi} \left| g_{-n-2j}(\xi + l_\xi(\varphi)\, e^{i\varphi}) - g_{-n-2j}(\xi_0 + l_{\xi_0}(\varphi)\, e^{i\varphi}) \right| d\varphi, \\
&\leq \frac{2}{\pi} \sum_{j=1}^{\infty} \int_0^{2\pi} j \left| g_{-j}(\xi + l_\xi(\varphi)\, e^{i\varphi}) - g_{-j}(\xi_0 + l_{\xi_0}(\varphi)\, e^{i\varphi}) \right| d\varphi, \\
&\leq \frac{2\kappa}{\pi} \int_0^{2\pi} \left| (\xi - \xi_0) + [l_\xi(\varphi) - l_{\xi_0}(\varphi)]\; e^{i\varphi} \right|^\epsilon d\varphi, \\
&\leq \frac{2\kappa}{\pi} \int_0^{2\pi} \left(2\, |\xi - \xi_0|^\epsilon + |l_\xi(\varphi) - l_{\xi_0}(\varphi)|^\epsilon\right) d\varphi, \\
&\leq (8\kappa + 4\kappa L^\epsilon)\, |\xi - \xi_0|^\epsilon .
\end{aligned}
$$

In the second inequality, we used Lemma 3.0.1 part (ii), in the third inequality we used $|a+b|^\epsilon \leq 2|a|^\epsilon + |b|^\epsilon$, and in the fourth inequality we used (3.8). This shows $G\mathbf{v} \in C^\epsilon(\overline{\Omega}; l^1)$.

We will show next that $G\mathbf{g} \in C^1(\Omega; l^1)$. Suffices to carry the estimates in the neighborhood $\overline{B(\xi_0, r_0)} \subset \Omega$ of an arbitrary point $\xi_0 \in \Omega$, where $r_0 = \text{dist}(\xi_0, \Gamma)/2 > 0$. Recall the estimate (3.9) where $c = 2/r_0^2$.

For each $n \leq 0$, we have

$$
\begin{aligned}
\nabla_\xi (G\,\mathbf{g})_n(\xi) = &\frac{1}{\pi i} \int_\Gamma \nabla_\xi \left\{ \frac{dw}{w - \xi} - \frac{d\overline{w}}{\overline{w - \xi}} \right\} \sum_{j=1}^{\infty} g_{n-2j}(w) \left(\frac{\overline{w - \xi}}{w - \xi} \right)^j \\
&+ \frac{1}{\pi i} \int_\Gamma \left\{ \frac{dw}{w - \xi} - \frac{d\overline{w}}{\overline{w - \xi}} \right\} \sum_{j=1}^{\infty} g_{n-2j}(w) \nabla_\xi \left(\frac{\overline{w - \xi}}{w - \xi} \right)^j ,
\end{aligned}
$$

which estimate using (3.9) by

$$\sum_{n=0}^{\infty} |\nabla_\xi (G\,\mathbf{g})_{-n}(\xi)| \leq \frac{c}{\pi} \sum_{n=0}^{\infty} \int_\Gamma \sum_{j=1}^{\infty} |g_{-n-2j}(w)|\, dw + \frac{c}{\pi} \sum_{n=0}^{\infty} \int_\Gamma \sum_{j=1}^{\infty} j\, |g_{-n-2j}(w)|\, dw.$$

By Lebesgue Dominated Convergence Theorem, Lemma 3.0.1, and $\mathbf{g} \in l^{1,2}_{\infty}(\Gamma)$, the right hand side above is finite.

To prove part (ii), we note that $Y_\epsilon \subset C^\epsilon(\Gamma; l^1)$, so that the Sokhotzki-Plemelj limit in (3.6) holds. The result follows from Definition 4.1.1 of $\mathcal{H}_0$ and part(i) above. □

Corollary 4.1.1. *Let Y_ϵ be the space defined in* (4.13)*, and $\mathbf{g} \in Y_\epsilon$ satisfying*

$$(I + i\mathcal{H}_0)\mathbf{g} = 0. \tag{4.23}$$

Define $\mathbf{v} := \langle (B\mathbf{g})_0, (B\mathbf{g})_{-1}, ... \rangle$, where B is the Bukhgeim-Cauchy operator as defined in (4.5)*. Then $\mathbf{v} \in C^{1,\epsilon}(\Omega; l^1)$ extends continuously to a map in $C^\epsilon(\overline{\Omega}; l^1)$. Moreover, $\mathbf{v}$ is A-analytic and $\mathbf{v}|_\Gamma = \mathbf{g}$, in the sense*

$$\lim_{\Omega \ni z \to z_0 \in \Gamma} \|\mathbf{v}(z) - \mathbf{g}(z_0)\|_{l_1} = 0,$$

and, for $\mathcal{P}^*$ in (4.16)*, we have*

$$\lim_{\Omega \ni z \to z_0 \in \Gamma} \mathcal{P}^*(\mathbf{v})(z) = \mathcal{P}^*(\mathbf{g})(z_0).$$

Proof. Since $\mathbf{g} \in Y_\epsilon$, by Proposition 4.1.3, we have $\mathbf{v} \in C^\epsilon(\overline{\Omega}; l^1) \cap C^1(\Omega; l^1)$. By summing (4.9), and (4.10), we obtain

$$\overline{\partial}(B\mathbf{g})_n + \partial(B\mathbf{g})_{n-2} = 0, \quad n = 0, -1, -2, ...,$$

and so $\mathbf{v}$ is A-analytic. Next we will show that $\mathbf{v}|_{\varGamma}= \mathbf{g}$. Let $z \in \Omega$ and $z_0 \in \varGamma$. Then

$$\begin{aligned}\|\mathbf{v}(z)-\mathbf{g}(z_0)\|_{l_1} &= \left\|\frac{1}{2}(G\mathbf{g})(z)+(C\mathbf{g})(z)-\mathbf{g}(z_0)\right\|_{l_1}\\ &= \left\|\frac{1}{2}\left((G\mathbf{g})(z)-(G\mathbf{g})(z_0)\right)+\left((C\mathbf{g})(z)-\frac{1}{2}\mathbf{g}(z_0)-\frac{1}{2}S\mathbf{g}(z_0)\right)\right\|_{l_1}\end{aligned}$$

In the equality above we use the fact that g satisfy (4.23). From Proposition 3.0.1, and Proposition 4.1.3 part (i), we have

$$\lim_{\Omega\ni z\to z_0\in\varGamma}\left\|(C\mathbf{g})(z)-\frac{1}{2}\mathbf{g}(z_0)-\frac{1}{2}S\mathbf{g}(z_0)\right\|_{l_1}=0,$$

$$\lim_{\Omega\ni z\to z_0\in\varGamma}\|(G\mathbf{g})(z)-(G\mathbf{g})(z_0)\|_{l_1}=0,$$

and so

$$\lim_{\Omega\ni z\to z_0\in\varGamma}\|\mathbf{v}(z)-\mathbf{g}(z_0)\|_{l_1}=0, \tag{4.24}$$

$$\text{i.e.}\mathbf{v}|_{\varGamma}=\mathbf{g}.$$

Since $\mathbf{v}\in C^{\epsilon}(\overline{\Omega};l^1)\cap C^1(\Omega;l^1)$, it follows from Proposition 4.1.2 part (iii), that $\mathcal{P}^*v(z)\in C^{1,\epsilon}(\Omega\times \mathbf{S}^1)\cap C^{\epsilon}(\overline{\Omega}\times\mathbf{S}^1)$. The triangle inequality yields

$$|[\mathcal{P}^*\mathbf{v}](z,\theta)-[\mathcal{P}^*\mathbf{g}](z_0,\theta)|\leq\|\mathbf{v}(z)-\mathbf{g}(z_0)\|_{l_1},$$

and the result follows from (4.24). □

Lemma 4.1.1. *Let Ω be a (convex) domain with C^2-boundary Γ with a strictly positive curvature lower bound $\delta>0$. Let $\tau(z,\theta)$ be as in* (2.6) *for $(z,\theta)\in\overline{\Omega}\times\mathbf{S}^1$, then the angular derivative*

$\partial_\varphi \tau(z,\theta)$ has a jump discontinuity across the variety Z as defined by

$$Z := \{(z,\theta) \in \Gamma \times \mathbf{S}^1 : \ \mathbf{n}(z)\cdot\theta = 0\}. \tag{4.25}$$

Proof. Let $z_0 \in \mathit{\Gamma}$ be fixed and let $\theta_0 := \mathbf{n}(z_0)^\perp$ with $\varphi_0 = \arg(\theta_0)$.

Let $\tilde{\tau}(z_0,\theta)$ be the length of the chord corresponding to the osculating circle at z_0 of radius R_0 and let $\varphi = \arg(\theta)$. Let $\tau(z_0,\theta)$ be the length of the chord from z_0 to the boundary in the θ direction as defined in (2.6).

Consider a local parametrization $t \mapsto (t, y(t))$ of the boundary near $z_0 = (0, y(0)) \in \Gamma$, with $y(0) = y'(0) = 0$. Then the curvature of the boundary at z_0 is $k(0) = y''(0)$, and, by the Taylor series expansion,

$$y(t) = \frac{\kappa(0)t^2}{2} + r(t)t^2,$$

for some $r(t)$ with $\lim_{t\to 0} r(t) = 0$.

The equation of the line passing through z_0 and making an angle $\varphi - \varphi_0$ with the positive t axis is $(t, \tan(\varphi-\varphi_0)t)$. The point of intersection of this line with $\mathit{\Gamma}$ gives $t = \dfrac{2\tan(\varphi-\varphi_0)}{\kappa(0)+2r(t)}$. Thus,

$$\begin{aligned}
\tau(z_0,\theta) &= t\sec(\varphi-\varphi_0),\\
&= \frac{\sin(\varphi-\varphi_0)}{\cos^2(\varphi-\varphi_0)}\frac{2}{\kappa(0)+2r(t)},\\
&\le \frac{2c_1}{\kappa(0)}\frac{\sin(\varphi-\varphi_0)}{\cos^2(\varphi-\varphi_0)},\\
&\le \frac{2c_1R_0|\sin(\varphi-\varphi_0)|}{\cos^2(\varphi-\varphi_0)}.
\end{aligned}$$

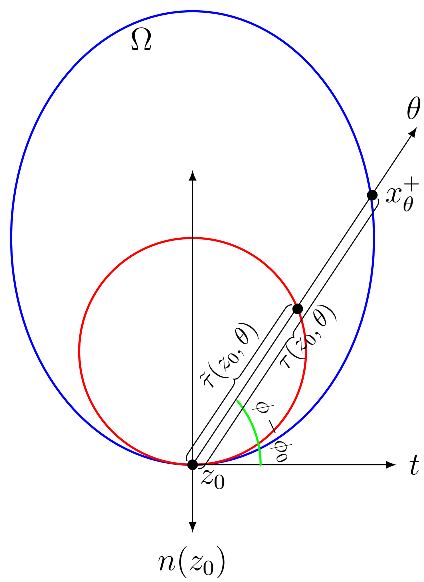

Figure 4.1: Geometry of the osculating circle

From the geometry of the osculating circle (see Figure 4.1), we have

$$\tilde{\tau}(z_0,\theta) = 2R_0|\sin(\varphi-\varphi_0)| \leq \frac{2}{\delta}|\varphi-\varphi_0|, \tag{4.26}$$

and so there is a constant $C>0$, such that, for all $(z_0,\theta)\in\Gamma\times\mathbf{S}^1$,

$$\tau(z_0,\theta) \leq C\tilde{\tau}(z_0,\theta). \tag{4.27}$$

A derivative in φ at φ_0 in the equality in (4.26) also yields the jump value of $4R_0$, as the direction θ crosses the tangent direction from outgoing to incoming. □

In order for the integral in (2.4) to inherit the regularity of f it is then necessary for f to vanish at the boundary. The following proposition makes this statement precise.

Corollary 4.1.2. *Let Ω be a (convex) domain with C^2-boundary Γ with a strictly positive curvature* *lower bound* $\delta > 0$. *If* $f \in C_0^{1,\alpha}(\overline{\Omega})$, *then* $Rf \cap C^{1,\alpha}(\mathit{\Gamma} \times \mathbf{S}^1) \neq \emptyset$.

Proof. For every $(z, \theta) \in \Gamma \times \mathbf{S}^1$, let us define

$$g(z,\theta) = \begin{cases} \int_{-\tau(z,\theta)}^{0} f(z+t\theta)dt, & \mathbf{n}(z)\cdot\theta > 0, \\ 0, & \mathbf{n}(z)\cdot\theta \le 0, \end{cases} \tag{4.28}$$

where $\mathbf{n}(z)$ is the unit outer normal at $z \in \mathit{\Gamma}$. Since

$$g(z_\theta^+,\theta) = \begin{cases} \int_{-\tau(z_\theta^+,\theta)}^{0} f(z_\theta^+ + t\theta)dt, & \mathbf{n}(z_\theta^+)\cdot\theta > 0, \\ 0, & \mathbf{n}(z_\theta^+)\cdot\theta \le 0, \end{cases}$$

and $g(z_\theta^-, \theta) = 0$, condition (2.7) is satisfied with $a \equiv 0$ to show that $g \in Rf$. We will show next that $g \in C^{1,\alpha}(\mathit{\Gamma} \times \mathbf{S}^1)$. Let ∂ be the partial derivative with respect to one of the spacial or angular variable. At points $(z_0, \theta_0) \in (\Gamma \times \mathbf{S}^1) \setminus Z$, differentiation in (4.28) together with $f|_\Gamma = 0$ yield

$$\partial g(z_0,\theta_0) = \begin{cases} \int_{-\tau(z_0,\theta_0)}^{0} \partial f(z_0+t\theta_0)dt, & \mathbf{n}(z_0)\cdot\theta_0 > 0, \\ 0, & \mathbf{n}(z_0)\cdot\theta_0 < 0, \end{cases} \tag{4.29}$$

Since $\partial f \in C^\alpha(\overline{\Omega})$, it remains to show that ∂f extends C^α across the variety Z. We first consider the case for a fixed $z_0 \in \Gamma$ and study the dependence of ∂g in θ near the tangential direction $\theta_0 := \mathbf{n}(z_0)^\perp$. The other case, studying the dependence of ∂g as $z \in \mathit{\Gamma}$ approach z_1 along $\mathit{\Gamma}$ for a fixed $\theta_1 \in \mathbf{S}^1$ with $(z_1, \theta_1) \in \mathit{\Gamma} \times \mathbf{S}^1$ reduces to the first case.

For this we first analyze the speed of convergence of $\tau(z_0, \theta) \to 0$ as $\theta \to \theta_0$. Let $\tilde{\tau}(z_0, \theta)$ be the length of the chord corresponding to the osculating circle at z_0 of radius R_0. From Lemma 4.1.1, we have that there is a constant $C > 0$, such that, for all $(z_0, \theta) \in \Gamma \times \mathbf{S}^1$,

$$\tau(z_0, \theta) \le C\tilde{\tau}(z_0, \theta). \tag{4.30}$$

From the geometry of the osculating circle (see Figure 4.1), we have

$$\tilde{\tau}(z_0, \theta) = 2R_0|\sin(\varphi - \varphi_0)| \le \frac{2}{\delta}|\varphi - \varphi_0|, \tag{4.31}$$

where $\varphi = \arg(\theta)$ and $\varphi_0 = \arg(\theta_0)$. A derivative in φ at φ_0 in the equality in (4.31) also yields the jump value of $4R_0$, as the direction θ crosses the tangent direction from outgoing to incoming direction. Since $\lim_{\theta\to\theta_0}\tau(z_0, \theta) = 0$, the formula (4.28) shows that $g \in C^1(\Gamma \times \mathbf{S}^1)$. To prove that ∂g is α-Hölder continuous, we estimate using (4.26)

$$|\partial g(z_0, \theta)| \le \|\nabla f\|_\infty \tau(z_0, \theta) \le \|\nabla f\|_\infty C\tilde{\tau}(z_0, \theta)| \le \tilde{C}|\varphi - \varphi_0|, \tag{4.32}$$

for some constant dependent on the sup-norm of the $|\nabla f|$ and the minimum curvature δ.

$$\begin{aligned}
&|g(z_0, \theta) - g(z_0, \theta_0)| \\
&= \left| \int_{-\tau(z_0,\theta_0)}^{0} (\partial f(z_0 + t\theta_0) - \partial f(z_0 + t\theta_0))dt + \int_{-\tau(z_0,\theta_0)}^{-\tau(z_0,\theta)} \partial f(z_0 + t\theta_0)dt \right|, \\
&\le C_1|\varphi - \varphi_0|^\alpha \tau(z_0, \theta) + \|\nabla f\|_\infty |\tau(z_0, \theta) - \tau(z_0, \theta_0)|, \\
&\le \tilde{C}|\varphi - \varphi_0|^\alpha + \|\nabla f\|_\infty C_2|\varphi - \varphi_0|, \\
&\le C|\varphi - \varphi_0|^\alpha.
\end{aligned}$$

Therefore, $g \in Rf \cap C^{1,\alpha}(\mathit{\Gamma} \times \mathbf{S}^1)$. □

One of our main results establishes necessary and sufficient conditions for a sufficiently smooth function on $\varGamma \times \mathbf{S}^1$ to be the Radon data of some sufficiently smooth source as follows.

Theorem 4.1.3 (Range characterization for Radon transform)**.** *Let $\Omega \subset \mathbb{R}^2$ be a domain with C^2 boundary Γ of strictly positive curvature, and $\alpha > 1/2$.*

(i) Let $f \in C_0^{1,\alpha}(\overline{\Omega})$ be real valued, and $g \in Rf \cap C^{\alpha}(\varGamma; C^{1,\alpha}(\mathbf{S}^1))$. Then $\mathcal{P}^-(g)$ as defined in (4.15)*, solves*

$$[I + i\mathcal{H}_0]\mathcal{P}^-(g) = 0, \tag{4.33}$$

where $\mathcal{H}_0$ is the Hilbert transform in (4.3)*.*

(ii) Let $g \in C^{\alpha}\left(\varGamma; C^{1,\alpha}(\mathbf{S}^1)\right) \cap C^0(\varGamma; C^{2,\alpha}(\mathbf{S}^1))$ be real valued and such that $\mathcal{P}^-(g)$ satisfies (4.33)*. Then there exists a real valued $f \in C^{\alpha}(\Omega) \cap L^1(\Omega)$, and such that $g \in Rf$.*

Proof. (i) By Corollary 4.1.2, we note first that $Rf \cap C^{\alpha}\left(\varGamma; C^{1,\alpha}(\mathbf{S}^1)\right) \supset Rf \cap C^{1,\alpha}(\varGamma \times \mathbf{S}^1) \neq \emptyset$. Since $g \in C^{\alpha}\left(\varGamma; C^{1,\alpha}(\mathbf{S}^1)\right)$, by Proposition 4.1.2 part (i), we have that $\mathcal{P}^-(g) \in l^{1,1}_{\infty}(\Gamma) \cap C^{\alpha}(\Gamma; l^1)$. Now the necessity in Theorem 4.1.2, yields $(I + i\mathcal{H}_0)\mathcal{P}^-(g) = 0$.

Next we prove the sufficiency of (4.33) in part (ii).

Since $g \in C^{\alpha}\left(\varGamma; C^{2,\alpha}(\mathbf{S}^1)\right) \cap C^0\left(\varGamma; C^{2,\alpha}(\mathbf{S}^1)\right)$, it follows from the Proposition 4.1.2 part (ii), that $\mathbf{g} := \mathcal{P}^-(g) \in Y_{\epsilon}$. For each $z \in \Omega$, construct the vector valued function $\mathbf{v} = \langle v_0, v_{-1}, v_{-2}, ...\rangle$ by

$$v_n(z) = (B\mathbf{g})(z), n = 0, -1, -2...$$

where B is the Bukhgeim-Cauchy operator as defined in (4.5). By Corollary 4.1.1, $\mathbf{v} \in C^{1,\epsilon}(\Omega; l^1) \cap C^{\epsilon}(\overline{\Omega}; l^1)$ is A-analytic, in particular for each $n = 0, -1, -2, ...$, we have

$$\overline{\partial} v_n + \partial v_{n-2} = 0.$$

Using $v_{-1} \in C^{1,\alpha}(\Omega)$, we define the Hölder continuous function $f \in C^{\alpha}(\Omega)$ by

$$f(z) := 2\,\mathbb{R}\mathrm{e}\left(\partial v_{-1}(z)\right), \quad z \in \Omega, \tag{4.34}$$

and show that f integrates along any line and that $g \in Rf$.

Since $\mathbf{v} \in C^{1,\epsilon}(\Omega; l^1) \cap C^{\epsilon}(\overline{\Omega}; l^1)$, it follows from the Proposition 4.1.2 part (iii), that

$$v(z,\theta) := \mathcal{P}^*(\mathbf{v}(z)) \in C^{1,\alpha}(\Omega \times \mathbf{S}^1) \cap C^{\alpha}(\overline{\Omega} \times \mathbf{S}^1).$$

Also from Corollary 4.1.1, $\mathbf{v}|_{\Gamma} = \mathbf{g}$ and $\lim_{\Omega \ni z \to z_0 \in \Gamma} v(z,\theta) = \mathcal{P}^*\mathbf{g}(z_0)$. Now using the fact that g is real valued yields

$$\lim_{\Omega \ni z \to z_0 \in \Gamma} v(z,\theta) = g(z_0,\theta),$$

$$\text{i.e. } v|_{\Gamma \times \mathbf{S}^1} = g.$$

Using $\theta \cdot \nabla v = e^{-i\varphi}\overline{\partial v} + e^{i\varphi}(\partial v)$, we obtain

$$\begin{aligned} \theta \cdot \nabla v(z,\theta) &= 2\,\mathbb{R}\mathrm{e}\left(\partial v_{-1}(z)\right) + 2\,\mathbb{R}\mathrm{e}\left(\sum_{n=0}^{\infty}(\overline{\partial} v_{-n}(z) + \partial v_{-n-2}(z))e^{-in\varphi}\right), \\ &= 2\,\mathbb{R}\mathrm{e}\left(\partial v_{-1}(z)\right), \\ &= f(z). \end{aligned}$$

By integrating $f(z) = \theta \cdot \nabla v(z,\theta)$, we obtain

$$\int_{\tau_-(z,\theta)}^{\tau_+(z,\theta)} f(z+s\theta)ds = \lim_{\substack{t_1 \to -\tau_-(z,\theta) \\ t_2 \to \tau_+(z,\theta)}} \int_{t_1}^{t_2} f(z+s\theta)dsm$$

$$= \lim_{\substack{t_1 \to -\tau_-(z,\theta) \\ t_2 \to \tau_+(z,\theta)}} \left[v(z+t_2\theta,\theta) - v(z+t_1\theta,\theta) \right],$$

$$= g\left(z \,+\, \tau_+(z,\theta)\,\theta\,,\ \theta\right) - g\left(z \,-\, \tau_-(z,\theta)\,\theta\,,\ \theta\right).$$

This shows that f integrates along any arbitrary line, in particular $f \in L^1(\Omega)$, and that $g \in Rf$. $\square$

4.2 Attenuated Case

In this section, we consider the attenuated case, where $a \not\equiv 0$ is a real valued map. The method of proof is based on the reduction to the non-attenuated case. Since e^{-Da} where Da is as in (2.1), and the equation (2.2) can be rewritten as

$$\theta \cdot \nabla \left(e^{-Da(z,\theta)} u(z,\theta) \right) = f(z) e^{-Da(z,\theta)}.$$

However, the right hand side is now angularly dependent with nonzero positive and negative modes, and one cannot use the A-analytic equations (3.1) directly.

The key idea in the reduction of the attenuated to the non-attenuated case is to choose a special integrating factor in such a way that all the negative Fourier modes vanish.

Recall the special integrating factor e^{-h} where h is as in (2.11), extends from $\mathbf{S}^1$ inside the disk as an analytic map, see [44], [26], and [16]. Since $e^{\pm h}$ are also extension of analytic functions in the disk they still have vanishing negative modes.

Now the equation (2.2) can be rewritten as

$$\theta\cdot\nabla\left(e^{-h(z,\theta)}u(z,\theta)\right)=f(z)e^{-h(z,\theta)}.$$

Proposition 4.2.1. *Let* $a\in C_0^{1,\alpha}(\overline{\Omega}),\alpha>1/2$*, and* h *be defined in* (2.11). *Then* $h\in C^{1,\alpha}(\overline{\Omega}\times\mathbf{S}^1)$.

Proof. Since $a\in C_0^{1,\alpha}(\overline{\Omega})$, we use the proof of Corollary 4.1.2 applied to a to conclude $Da\in C^{1,\alpha}(\overline{\Omega}\times\mathbf{S}^1)$ and also $Ra\in C^{1,\alpha}(\mathbb{R}\times\mathbf{S}^1)$. The Hilbert Transform in the linear variable preserve the smoothness class to yeild $HRa\in C^{1,\alpha}(\overline{\Omega}\times\mathbf{S}^1)$ and thus $h\in C^{1,\alpha}(\overline{\Omega}\times\mathbf{S}^1)$. □

Consider the Fourier expansions of $e^{-h(z,\theta)}$ and $e^{h(z,\theta)}$

$$e^{-h(z,\theta)}=\sum_{k=0}^{\infty}\alpha_k(z)e^{ik\varphi},\quad e^{h(z,\theta)}=\sum_{k=0}^{\infty}\beta_k(z)e^{ik\varphi},\quad (z,\theta)\in\overline{\Omega}\times\mathbf{S}^1, \tag{4.35}$$

where $h\in C^{1,\alpha}(\Gamma\times\mathbf{S}^1)$ is as defined in (2.11). Since $e^{-h}e^h=1$ the Fourier modes $\alpha_k,\beta_k,k\geq 0$ satisfy

$$\alpha_0\beta_0=1,\qquad \sum_{m=0}^{k}\alpha_m\beta_{k-m}=0,\ k\geq 1. \tag{4.36}$$

The following mapping property is used in defining Hilbert Transform associated with attenuated Radon Transform. Recall the operator $\mathcal{P}^+$ in (4.15), e^h be as in (4.35), and Y_α in (4.13) with $\epsilon=\alpha$.

Proposition 4.2.2. *Let* $a\in C_0^{1,\alpha}(\overline{\Omega})$ *with* $\alpha>1/2$. *Then* $\mathcal{P}^+(e^{\pm h})\in C^\alpha(\overline{\Omega};l^1)$. *Moreover*

$(i)\ \mathcal{P}^+(e^h)*_n(\cdot):C^\alpha(\overline{\Omega};l_\infty)\to C^\alpha(\overline{\Omega};l_\infty);$

$(ii)\ \mathcal{P}^+(e^h)*_n(\cdot):C^\alpha(\overline{\Omega};l_1)\to C^\alpha(\overline{\Omega};l_1);$

$(iii)\ \mathcal{P}^+(e^h)*_n(\cdot):Y_\alpha\to Y_\alpha,$

where $*_n$ denotes the convolution operator on sequences.

Proof. Since $a \in C_0^{1,\alpha}(\overline{\Omega})$, it follows from Proposition 4.2.1 that

$$e^{\pm h} \in C^{1,\alpha}(\overline{\Omega} \times \mathbf{S}^1) \subset C^{\alpha}(\overline{\Omega}; C^{\alpha}(\mathbf{S}^1)).$$

Then

$$\sup_{z \in \overline{\Omega}} \|e^{h(\xi,\cdot)}\|_{C^{\alpha}(\mathbf{S}^1)} + \sup_{\substack{\xi,\mu \in \overline{\Omega} \\ \xi \neq \mu}} \frac{\|e^{h(\xi,\cdot)} - e^{h(\mu,\cdot)}\|_{C^{\alpha}(\mathbf{S}^1)}}{|\xi - \mu|^{\alpha}} < \infty. \tag{4.37}$$

Let $\mathcal{P}^+(e^h) := \langle \beta_0, \beta_1, \beta_2, \cdots \rangle$. Then

$$\sup_{\xi \in \overline{\Omega}} \sum_{k=1}^{\infty} |\beta_k(\xi)| \leq \sup_{\xi \in \overline{\Omega}} \|e^{h(\xi,\cdot)}\|_{C^{\alpha}(\mathbf{S}^1)} < \infty. \tag{4.38}$$

Another application of Lemma 3.0.2 together with (4.37) imply

$$\sup_{\substack{\xi,\mu \in \overline{\Omega} \\ \xi \neq \mu}} \sum_{k=1}^{\infty} \frac{|\beta_k(\xi) - \beta_k(\mu)|}{|\xi - \mu|^{\alpha}} \leq \sup_{\substack{\xi,\mu \in \overline{\Omega} \\ \xi \neq \mu}} \frac{\|e^{h(\xi,\cdot)} - e^{h(\mu,\cdot)}\|_{C^{\alpha}}}{|\xi - \mu|^{\alpha}} < \infty. \tag{4.39}$$

By combining the estimates (4.38) and (4.39) we showed that $\mathcal{P}^+(e^h) \in C^{\alpha}(\overline{\Omega}; l_1)$. A similar estimate shows $\mathcal{P}^+(e^{-h}) \in C^{\alpha}(\overline{\Omega}; l_1)$.

Next we prove part (i). Let $\mathbf{g} \in C^{\alpha}(\overline{\Omega}; l_{\infty})$, and $\mathbf{v} := \mathcal{P}^+(e^h) *_n \mathbf{g}$, given by

$$v_n = \sum_{k=0}^{\infty} \beta_k g_{n-k}, \quad n \leq 0,$$

where β_k are the Fourier coefficients of e^h, as in (4.35). Since $\mathbf{g} \in C^{\alpha}(\overline{\Omega}; l_{\infty})$ and $\mathcal{P}^+(e^h) \in$

$C^\alpha(\overline{\Omega}; l^1)$, we have

$$c_1 := \sup_{n\le 0}\sup_{\xi\in\overline{\Omega}} |g_n(\xi)| < \infty,\ \ \kappa_1 := \sup_{n\le 0}\sup_{\substack{\xi,\eta\in\overline{\Omega}\\ \xi\ne\eta}} \frac{|g_n(\xi)-g_n(\eta)|}{|\xi-\eta|^\alpha} < \infty, \tag{4.40}$$

and

$$c_2 := \sup_{\xi\in\overline{\Omega}} \sum_{k=0}^{\infty} |\beta_k(\xi)| < \infty,\ \ \kappa_2 := \sup_{\substack{\xi,\eta\in\overline{\Omega}\\ \xi\ne\eta}} \sum_{k=0}^{\infty} \frac{|\beta_k(\xi)-\beta_k(\eta)|}{|\xi-\eta|^\alpha} < \infty. \tag{4.41}$$

By taking the supremum in $\xi \in \overline{\Omega}$, for each $n \le 0$, in

$$|v_n(\xi)| \le \sum_{k=0}^{\infty} |\beta_k(\xi) g_{n-k}(\xi)| \le c_1 \sum_{k=0}^{\infty} |\beta_k(\xi)| \le c_1 c_2,$$

we obtain

$$\sup_{n\le 0}\sup_{\xi\in\overline{\Omega}} |v_{-n}(\xi)| < \infty. \tag{4.42}$$

From (4.42), and by taking the supremum in $\xi, \eta \in \overline{\Omega}$ with $\xi \ne \eta$, for each $n \le 0$, in

$$\begin{aligned}
\frac{|\mathbf{v}_n(\xi)-\mathbf{v}_n(\eta)|}{|\xi-\eta|^\alpha} \le \sum_{k=0}^{\infty} & \frac{|\beta_k(\xi)-\beta_k(\eta)|}{|\xi-\eta|^\alpha} |g_{n-k}(\xi)| \\
&+ \sum_{k=0}^{\infty} |\beta_k(\eta)| \frac{|g_{n-k}(\xi)-g_{n-k}(\eta)|}{|\xi-\eta|^\alpha}, \\
&\le c_1 \sum_{k=0}^{\infty} \frac{|\beta_k(\xi)-\beta_k(\eta)|}{|\xi-\eta|^\alpha} + \kappa_1 \sup_{\eta\in\overline{\Omega}} \sum_{k=0}^{\infty} |\beta_k(\eta)| \\
&\le c_1\kappa_2 + c_2\kappa_1,
\end{aligned}$$

we obtain that $\mathbf{v} \in C^\alpha(\overline{\Omega}; l_\infty)$.

Next we prove part (ii). Let $\mathbf{g} \in C^{\alpha}(\overline{\Omega}; l^1)$, and let $\mathbf{v} = \mathcal{P}^+(e^h) *_n \mathbf{g}$ be as before. Since $\mathbf{g}, \mathcal{P}^+(e^h) \in C^{\alpha}(\overline{\Omega}; l^1)$, we have

$$c_3 := \sup_{\xi \in \overline{\Omega}} \sum_{n=0}^{\infty} |g_{-n}(\xi)| < \infty, \kappa_3 := \sup_{\substack{\xi, \eta \in \overline{\Omega} \\ \xi \neq \eta}} \sum_{n=0}^{\infty} \frac{|g_{-n}(\xi) - g_{-n}(\eta)|}{|\xi - \eta|^{\alpha}} < \infty. \tag{4.43}$$

By taking the supremum in $\xi \in \overline{\Omega}$ in

$$\begin{aligned}
\sum_{n=0}^{\infty} |v_{-n}(\xi)| \leq \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} |\beta_k(\xi)||g_{n-k}(\xi)| &\leq \sum_{k=0}^{\infty} |\beta_k(\xi)| \sum_{n=0}^{\infty} |g_{-n-k}(\xi)| \\
&\leq c_3 \sum_{k=0}^{\infty} |\beta_k(\xi)| \leq c_2 c_3,
\end{aligned}$$

we obtain

$$\sup_{\xi \in \overline{\Omega}} \sum_{n=0}^{\infty} |v_{-n}(\xi)| < \infty. \tag{4.44}$$

From (4.44), and by taking the supremum in $\xi, \eta \in \overline{\Omega}$ with $\xi \neq \eta$ in

$$\begin{aligned}
\frac{\|\mathbf{v}(\xi) - \mathbf{v}(\eta)\|_{l_1}}{|\xi - \eta|^{\alpha}} &\leq \sum_{k=0}^{\infty} \frac{|\beta_k(\xi) - \beta_k(\eta)|}{|\xi - \eta|^{\alpha}} \sum_{n=0}^{\infty} |g_{-n-k}(\xi)| \\
&\quad + \sum_{k=0}^{\infty} |\beta_k(\eta)| \sum_{n=0}^{\infty} \frac{|g_{-n-k}(\xi) - g_{-n-k}(\eta)|}{|\xi - \eta|^{\alpha}} \\
&\leq c_3 \kappa_2 + c_2 \kappa_3,
\end{aligned}$$

we obtain that $\mathbf{v} \in C^{\alpha}(\overline{\Omega}; l^1)$.

Last we prove part (iii).

Since $a \in C_0^{1,\alpha}(\overline{\Omega})$, it follows from Proposition 4.2.1, that $e^h \in C^{1,\alpha}(\varGamma \times \mathbf{S}^1) \subset C^{\alpha}(\varGamma; C^{\alpha}(\mathbf{S}^1))$, and from (4.38) and (4.39), we have

$$c_4 := \sup_{\xi\in\Gamma}\sum_{k=0}^{\infty}|\beta_k(\xi)| < \infty,\ \ \kappa_4 := \sup_{\substack{\xi,\eta\in\Gamma\\ \xi\neq\eta}}\sum_{k=0}^{\infty}\frac{|\beta_k(\xi)-\beta_k(\eta)|}{|\xi-\eta|^{\alpha}} < \infty.$$

Let $\mathbf{g}\in Y_\alpha$, and let $\mathbf{v}=\mathcal{P}^{+}(e^h)*_n\mathbf{g}$, be as before.

Since $\mathbf{g}\in Y_\alpha$, we have

$$\begin{aligned} c_5 &:= \sup_{\xi\in\Gamma}\sum_{j=1}^{\infty} j^2|g_{-j}(w)| < \infty,\\ \kappa_5 &:= \sup_{\substack{\xi,\mu\in\Gamma\\ \xi\neq\mu}}\sum_{j=1}^{\infty} j\frac{|g_{-j}(\xi)-g_{-j}(\mu)|}{|\xi-\mu|^{\alpha}} < \infty. \end{aligned}$$

By taking the supremum in $w\in\Gamma$ in

$$\begin{aligned} \sum_{j=1}^{\infty} j^2|v_{-j}(w)| &\leq \sum_{j=1}^{\infty} j^2\sum_{k=0}^{\infty}|\beta_k(w)||g_{-j-k}(w)|,\\ &\leq \sum_{k=0}^{\infty}|\beta_k(w)|\sum_{j=1}^{\infty} j^2|g_{-j-k}(w)|,\\ &\leq \sum_{k=0}^{\infty}|\beta_k(w)|\sum_{j=1}^{\infty} j^2|g_{-j}(w)|,\\ &\leq c_4c_5, \end{aligned}$$

we obtain that $\mathbf{v}\in l^{1,2}_{\infty}(\Gamma)$.

Finallym we show that $\mathbf{v}$ obeys the estimate in (4.13). By taking the supremum in $\xi,\mu\in\Gamma$ with

$\xi \neq \mu$ in

$$\begin{aligned}
&\sum_{j=1}^{\infty} \frac{j|v_{-j}(\xi) - v_{-j}(\mu)|}{|\xi - \mu|^{\alpha}} \\
&\quad \leq \sum_{j=1}^{\infty} \frac{j}{|\xi - \mu|^{\alpha}} \sum_{k=0}^{\infty} |\beta_k(\xi)\; g_{-j-k}(\xi) - \beta_k(\mu)\; g_{-j-k}(\mu)|, \\
&\quad \leq \sum_{j=1}^{\infty} j \sum_{k=0}^{\infty} \frac{|\beta_k(\xi) - \beta_k(\mu)|}{|\xi - \mu|^{\alpha}} |g_{-j-k}(\xi)| \\
&\qquad + \sum_{j=1}^{\infty} j \sum_{k=0}^{\infty} \frac{|g_{-j-k}(\xi) - g_{-j-k}(\mu)|}{|\xi - \mu|^{\alpha}} |\beta_k(\mu)|, \\
&\quad \leq \sum_{k=0}^{\infty} \frac{|\beta_k(\xi) - \beta_k(\mu)|}{|\xi - \mu|^{\alpha}} \sum_{j=1}^{\infty} j|g_{-j}(\xi)| \\
&\qquad + \sum_{k=0}^{\infty} |\beta_k(\mu)| \sum_{j=1}^{\infty} \frac{j|g_{-j}(\xi) - g_{-j}(\mu)|}{|\xi - \mu|^{\alpha}}, \\
&\quad \leq \kappa_4 c_5 + c_4\; \kappa_5,
\end{aligned}$$

we obtain that $\mathbf{v} \in Y_{\alpha}$. □

4.2.1 Range characterization of the attenuated Radon transform of zero tensor

Recall the Hilbert transform $\mathcal{H}_0$ in Definition 4.1.1, $\mathcal{P}_{\pm}$ in (4.15), and $e^{\pm h}$ in (4.35).

Definition 4.2.1. *The Hilbert transform associated with the attenuated Radon transform for $g \in C^{1,\alpha}(\mathit{\Gamma} \times \mathbf{S}^1)$ is given by*

$$\mathcal{H}_a(\mathcal{P}^-(g)) := \mathcal{P}^+(e^h) *_n \mathcal{H}_0 \left(\mathcal{P}^+(e^{-h}) *_n \mathcal{P}^-(g)\right), \tag{4.45}$$

where $*_n$ is the convolution operator on sequences.

Using the Fourier coefficients of $e^{\pm h}$, we can also write for $\mathbf{u} := \langle u_0, u_{-1}, u_{-2}, ...\rangle$, the Hilbert transform as

$$\mathcal{H}_a \mathbf{u} := \sum_{m=0}^{\infty} \beta_m \mathcal{L}^m \left(\mathcal{H}_0 \left(\sum_{k=0}^{\infty} \alpha_k \mathcal{L}^k \right) \right) \mathbf{u},$$

where $\mathcal{L}$ is the left translation operator and α_k, β_k are the Fourier coefficients of $e^{-h(x,\theta)}$, respectively, $e^{h(x,\theta)}$ as in (4.35).

The following result describes the mapping properties of the Hilbert transform $\mathcal{H}_a$ needed later.

Proposition 4.2.3. *Let $l^{1,1}_{\infty}(\mathit{\Gamma})$ and $C^{\epsilon}(\Gamma; l_1)$ be the spaces in* (3.2) *and* (3.3) *respectively. Assume $a \in C^{1,\alpha}_0(\overline{\Omega})$ with $\alpha > 1/2$, and $\epsilon > 0$ be arbitrarily small. Then*

$$\mathcal{H}_a : C^{\epsilon}(\mathit{\Gamma}; l_1) \cap l^{1,1}_{\infty}(\mathit{\Gamma}) \longrightarrow C^{\epsilon}\left(\Gamma; l_{\infty}\right). \tag{4.46}$$

Proof. Let $g \in C^{1,\alpha}(\mathit{\Gamma} \times \mathbf{S}^1) \subset C^{\epsilon}(\mathit{\Gamma}; C^{1,\alpha}(\mathbf{S}^1))$, then by Proposition 4.1.2 (i), $\mathcal{P}^- g \in l^{1,1}_{\infty}(\mathit{\Gamma}) \cap C^{\epsilon}(\Gamma; l^1)$. Since $a \in C^{1,\alpha}_0(\overline{\Omega})$, it follows from Proposition 4.2.1, that $e^{\pm h} \in C^{\epsilon}(\mathit{\Gamma}; C^{1,\alpha}(\mathbf{S}^1))$.

Since $e^{-h} g \in C^{\epsilon}(\mathit{\Gamma}; C^{1,\alpha}(\mathbf{S}^1))$, it follows from Proposition 4.1.2 (i), that $\mathcal{P}^-(e^{-h} g) \in l^{1,1}_{\infty}(\mathit{\Gamma}) \cap C^{\epsilon}(\Gamma; l^1)$. By (4.18), $\mathcal{P}^-(e^{-h} g) = (\mathcal{P}^+ e^{-h}) *_n (\mathcal{P}^-(g))$ and so by Proposition 4.1.1,
$\mathcal{H}_0 \left(\mathcal{P}^+(e^h) *_n \mathcal{P}^-(g) \right) \in C^{\epsilon}(\mathit{\Gamma}; l_{\infty})$. Finally by Proposition 4.2.2 (ii),

$$\mathcal{P}^+(e^h) *_n \mathcal{H}_0 \left(\mathcal{P}^+(e^h) *_n \mathcal{P}^-(g) \right) \in C^{\epsilon}(\mathit{\Gamma}; l_{\infty}).$$

□

Now we are able to state and prove our main result.

Theorem 4.2.1 (Range characterization for the attenuated Radon transform)**.** *Let $\Omega \subset R^2$ be a* *domain with C^2 boundary Γ of strictly positive curvature, and $a \in C_0^{1,\alpha}(\overline{\Omega})$, $\alpha > 1/2$ be real valued.*

(i) Let $f \in C_0^{1,\alpha}(\overline{\Omega})$ be real valued. Then $R_a f \cap C^\alpha(\mathit{\Gamma}; C^{1,\alpha}(\mathbf{S}^1)) \neq \emptyset$, and if $g \in R_a f \cap C^\alpha(\mathit{\Gamma}; C^{1,\alpha}(\mathbf{S}^1))$, its projection $\mathcal{P}^-(g)$ must solve

$$[I + i\mathcal{H}_a]\mathcal{P}^-(g) = 0, \tag{4.47}$$

with the Hilbert transform $\mathcal{H}_a$ defined in (4.45).

(ii) Let $g \in C^\alpha(\mathit{\Gamma}; C^{1,\alpha}(\mathbf{S}^1)) \cap C^0(\mathit{\Gamma}; C^{2,\alpha}(\mathbf{S}^1))$ be real valued with the projection $\mathcal{P}^-(g)$ satisfying (4.47). *Then there exists a real valued $f \in C^\alpha(\Omega) \cap L^1(\Omega)$ for which $g \in R_a f$.*

Proof. (i) For $z \in \Omega$ and $\theta \in S^1$, let $u(z,\theta)$ be the solution of

$$\begin{aligned} \theta \cdot \nabla u(z,\theta) + a(z)u(z,\theta) = f(z), \quad (z,\theta) \in \Omega \times S^1, \\ u(z,\theta) = 0, \quad (z,\theta) \in \mathit{\Gamma}_-, \end{aligned} \tag{4.48}$$

namely $u(z + t\theta, \theta) = \displaystyle\int_0^t f(z + s\theta)\, e^{-Da(z+s\theta,\theta)} ds$, for $(z,\theta) \in \mathit{\Gamma}_-$ and $0 \leq t \leq \tau_+(z,\theta)$, where $\mathit{\Gamma}_\pm = \{(z,\theta) \in \mathit{\Gamma} \times \mathbf{S}^1 : \pm\mathbf{n}(z) \cdot \theta > 0\}$ denote the incoming $(-)$, respectively, outgoing $(+)$ boundary and $n(z)$ denotes the outer normal at some boundary point z.

Let $g(z,\theta) := u(z,\theta)|_{\mathit{\Gamma} \times \mathbf{S}^1}$. Note that $\mathit{\Gamma} \times \mathbf{S}^1 = \mathit{\Gamma}_- \cup \mathit{\Gamma}_+ \cup Z$, where Z is the variety in (4.25). Since $g(z,\theta) = 0$ for $(z,\theta) \in \mathit{\Gamma}_- \cup Z$ and $g(z,\theta) = \displaystyle\int_0^{\tau_+(z,\theta)} f(z + s\theta)\, e^{-Da(z+s\theta,\theta)} ds$, for $(z,\theta) \in \mathit{\Gamma}_+$, it follows that g satisfies (2.7) and thus $g \in R_a f$.

Since $a \in C_0^{1,\alpha}(\overline{\Omega})$, it follows from Proposition 4.2.1, that $e^{-Da} \in C^{1,\alpha}(\overline{\Omega} \times \mathbf{S}^1)$ and so $fe^{-Da} \in C_0^{1,\alpha}(\overline{\Omega} \times \mathbf{S}^1) \subset C^\alpha(\overline{\Omega}; C^{1,\alpha}(\mathbf{S}^1))$. The proof of Corollary 4.1.2, applied to fe^{-Da} shows that

$g \in C^{1,\alpha}(\varGamma \times \mathbf{S}^1)$ and therefore $g \in R_a f \cap C^{\alpha}(\varGamma; C^{1,\alpha}(\mathbf{S}^1))$.

For $z \in \overline{\Omega}$ and $\theta \in S^1$, if we let

$$v(z,\theta) := e^{-h(z,\theta)} u(z,\theta), \tag{4.49}$$

where $u(z,\theta)$ solves (4.48) with $u(z,\theta)|_{\varGamma \times \mathbf{S}^1} = g(z,\theta)$, and $e^{-h(z,\theta)}$ as in (4.35) then $v(z,\theta)$ solves

$$\begin{aligned} \theta \cdot \nabla v(z,\theta) &= f(z) e^{-h(z,\theta)}, \quad (z,\theta) \in \Omega \times \mathbf{S}^1, \\ v|_{\varGamma \times \mathbf{S}^1} &= g\, e^{-h}|_{\varGamma \times \mathbf{S}^1}, \end{aligned} \tag{4.50}$$

If $\mathbf{v} := \langle v_0, v_{-1}, v_{-2}, ... \rangle$ is the projection on the non-positive Fourier coeficients of $\displaystyle\sum_{n=-\infty}^{\infty} v_n(x) e^{in\varphi}$ then the equation (4.50) yields for each $n = 0, -1, -2, ...$

$$\overline{\partial} v_n(z) + \partial v_{n-2}(z) = 0, \quad z \in \Omega.$$

This makes $\mathbf{v} := \langle v_0, v_{-1}, v_{-2}, ... \rangle$ be A-analytic.

The convolution applied to (4.49) rewrites $\mathbf{v}$ as

$$\mathbf{v}(z) = \mathcal{P}^{+}(e^{-h(z,\theta)}) *_n \mathcal{P}^{-}(u(z,\theta)), \quad (z,\theta) \in \Omega \times \mathbf{S}^1. \tag{4.51}$$

Since $a \in C_0^{1,\alpha}(\overline{\Omega})$ and $g \in C^{\alpha}(\varGamma; C^{1,\alpha}(\mathbf{S}^1))$, we have from Proposition 4.2.1,

$$e^{-h} g \in C^{\alpha}(\varGamma; C^{1,\alpha}(\mathbf{S}^1)).$$

Hence, by Proposition 4.1.2 (i), $\mathcal{P}^{-}(e^{-h} g) \in l^{1,1}_{\infty}(\varGamma) \cap C^{\alpha}(\Gamma, l^1)$.

Since $\mathcal{P}^-(e^{-h}g)$ is the boundary value of the A-analytic function $\mathbf{v}$, we can apply necessity part in Theorem 4.1.2, to conclude that

$$(I + i\mathcal{H}_0)\mathcal{P}^-(g\, e^{-h}|_{\Gamma\times\mathbf{S}^1}) = 0. \tag{4.52}$$

The convolution of (4.52) by $\mathcal{P}^+(e^h)$ yields

$$\begin{aligned} 0 &= \mathcal{P}^+(e^h) *_n (I + i\mathcal{H}_0)\mathcal{P}^-(e^{-h}g),\\ &= \mathcal{P}^+(e^h) *_n \mathcal{P}^-(e^{-h}g) + i\mathcal{P}^+(e^h) *_n \mathcal{H}_0\mathcal{P}^-(e^{-h}g),\\ &= \mathcal{P}^-(g) + i\mathcal{H}_a\mathcal{P}^-(g),\\ &= [I + i\mathcal{H}_a]\mathcal{P}^-(g). \end{aligned}$$

In the third equality above we use (4.18) to simplify

$$\mathcal{P}^+(e^h) *_n \mathcal{P}^-(e^{-h}g) = \mathcal{P}^-(e^h e^{-h}g) = \mathcal{P}^-(g),$$

and definition 4.2.1 of $\mathcal{H}_a$ to obtain

$$\mathcal{P}^+(e^h) *_n \mathcal{H}_0\left(\mathcal{P}^+(e^h) *_n \mathcal{P}^-(g)\right) = \mathcal{H}_a\mathcal{P}^-(g).$$

Conversely, let $g \in C^\alpha\left(\Gamma; C^{1,\alpha}(\mathbf{S}^1)\right) \cap C^0(\Gamma; C^{2,\alpha}(\mathbf{S}^1))$ be real valued and such that $\mathcal{P}^-(g)$ satisfies (4.47). Then by Proposition 4.1.2 (ii), we have $\mathcal{P}^-(g) \in Y_\alpha$. Since $a \in C_0^{1,\alpha}(\overline{\Omega})$, it follows from Propositions 4.2.1, and 4.1.2(i), that $\mathcal{P}^+(e^h) \in l^{1,1}_\infty(\Gamma) \cap C^\alpha(\Gamma; l^1)$. Finally we apply Proposition 4.2.2(iv), to yield $\mathcal{P}^-(e^{-h}g) \in Y_\alpha$. From $\mathcal{P}^-(g)$ satisfying (4.47), we have

$$0 = [I + i\mathcal{H}_a]\mathcal{P}^-(g) = \mathcal{P}^-(g) + i\mathcal{H}_a\mathcal{P}^-(g). \tag{4.53}$$

The convolution of (4.53) by $\mathcal{P}^+(e^{-h})$ yields

$$
\begin{aligned}
0 &= \mathcal{P}^+(e^{-h}) *_n \left(\mathcal{P}^-(g) + i\mathcal{H}_a\mathcal{P}^-(g)\right), \\
&= \mathcal{P}^+(e^{-h}) *_n \mathcal{P}^-(g) + i\mathcal{P}^+(e^{-h}) *_n \mathcal{H}_a\mathcal{P}^-(g), \\
&= \mathcal{P}^-(e^{-h}g) + i\mathcal{P}^+(e^{-h}) *_n \mathcal{P}^+(e^{h}) *_n \mathcal{H}_0\mathcal{P}^-(e^{-h}g), \\
&= \mathcal{P}^-(e^{-h}g) + i\mathcal{P}^+(1) *_n \mathcal{H}_0\mathcal{P}^-(e^{-h}g), \\
&= \mathcal{P}^-(e^{-h}g) + i\mathcal{H}_0\mathcal{P}^-(e^{-h}g), \\
&= [I + i\mathcal{H}_0]\mathcal{P}^-(e^{-h}g).
\end{aligned}
$$

In the third equality above we use the Proposition 4.2.2 part(iii), to simplify $\mathcal{P}^+(e^{-h}) *_n \mathcal{P}^-(g) = \mathcal{P}^-(e^{-h}g)$, and Definition 4.2.1 of $\mathcal{H}_a$. In the fourth equality above, we use $\mathcal{P}^+(e^{-h}) *_n \mathcal{P}^+(e^{h}) = \mathcal{P}^+(1) := \langle 1, 0, 0, \cdots \rangle$, and the fact that $\mathcal{P}^+(1)$ is the identity element for convolution in sequences to conclude $\mathcal{P}^+(1) *_n \mathcal{H}_0\mathcal{P}^-(e^{-h}g) = \mathcal{H}_0\mathcal{P}^-(e^{-h}g)$.

For each $z \in \Omega$, construct the vector valued function $\mathbf{v} = \langle v_0, v_{-1}, v_{-2}, ...\rangle$ by

$$
v_n(z) = (B\mathbf{g})(z), n = 0, -1, -2...
$$

where $\mathbf{g} := \mathcal{P}^-(e^{-h}g)$ and B is the Bukhgeim-Cauchy operator as defined in (4.5). By the Corollary 4.1.1, $\mathbf{v} \in C^{1,\epsilon}(\Omega; l_1) \cap C^{\epsilon}(\overline{\Omega}; l_1)$ is A-analytic and $\mathbf{v}|_{\varGamma} = \mathbf{g}$.

Construct the vector valued function $\mathbf{u} := \langle u_0, u_{-1}, u_{-2}, ...\rangle$ from $\mathbf{v}$ by the convolution formula $\mathbf{u}(z) = \mathcal{P}^+(e^{h(z,\cdot)}) *_n \mathbf{v}(z)$ for $(z, \cdot) \in \Omega \times \mathbf{S}^1$. By the Proposition 4.2.2(ii), we have $\mathbf{u} \in C^{\alpha}(\overline{\Omega}; l_1)$ and by Proposition 4.1.2(iii), we have $u(z, \theta) := \mathcal{P}^*(\mathbf{u}(z)) \in C^{1,\alpha}(\Omega \times \mathbf{S}^1) \cap C^{\alpha}(\overline{\Omega} \times \mathbf{S}^1)$. Note that

$$\begin{aligned}\mathcal{P}^-(u|_{\Gamma\times\mathbf{S}^1}) &= \mathcal{P}^+(e^h|_{\Gamma\times\mathbf{S}^1}) *_n \mathbf{v}|_\Gamma, \\ &= \mathcal{P}^+(e^h|_{\Gamma\times\mathbf{S}^1}) *_n \mathcal{P}^-(e^{-h}|_{\Gamma\times\mathbf{S}^1}\, g), \\ &= \mathcal{P}^-(g).\end{aligned}$$

Taking $\mathcal{P}^*$ on both sides of the above equation and using the fact that u and g are real valued yields $u|_{\Gamma\times\mathbf{S}^1}= g$.

We define the Hölder continuous function $f \in C^\alpha(\Omega)$ by

$$f(z) := \theta\cdot\nabla u(z,\theta) + a(z)u(z,\theta), \quad (z,\theta)\in\Omega\times\mathbf{S}^1, \tag{4.54}$$

and show that f integrates along any line and that $g \in R_a f$.

Since e^{-Da} in (2.1) is an integrating factor, the equation (4.54) can be rewritten in the advection form as

$$f(z)e^{-Da(z,\theta)} = \theta\cdot\nabla\left(e^{-Da(z,\theta)}u(z,\theta)\right).$$

and integrated along lines in direction θ to obtain

$$\begin{aligned}\int_{\tau_-(x,\theta)}^{\tau_+(x,\theta)} f(x+t\theta)e^{-Da(x+t\theta,\theta)}dt &= e^{-Da(z+t\theta,\theta)}u(z+t\theta,\theta)\Big|_{\tau_-(z,\theta)}^{\tau_+(z,\theta)}, \\ &= e^{-Da(z_\theta^+,\theta)}u(z_\theta^+,\theta) - e^{-Da(z_\theta^-,\theta)}u(z_\theta^-,\theta), \\ &= g(x_\theta^+,\theta) - \left[e^{-Da}g\right](x_\theta^-,\theta),\end{aligned}$$

where the notation $z_\theta^\pm = z\pm\tau_\pm(z,\theta)\theta$ as in (2.5). This shows that f integrates along any arbitrary line, in particular $f \in L^1(\Omega)$, and that $g \in R_a f$. $\square$

CHAPTER 5: RANGE CHARACTERIZATION OF ONE TENSOR

We consider here the problem of the characterization of the attenuated Radon transform of vector fields in the Euclidean plane as they appear in the medical diagnostic technique of Doppler tomography, e.g [17].

We consider both the case of attenuating and non-attenuating media. Formulas are derived on data collected on a circular domain (as in practice), while the approach applies to strictly convex domains. For the non-attenuated case our result can be understood as an intrinsic characterization of the scattering relation in the Euclidean case.

To simplify the exhibition we represent the case in which $\Omega = \mathbb{D}$ and this is infact what occurs in practice where detectors moves on a circular path around the body. Let $\mathbb{D} \subset \mathbb{R}^2$ denote the unit disc in the plane and $\varGamma = \mathbf{S}^1$ be its boundary. We will consider two real functions $f, a \in C_0^2(\overline{\mathbb{D}})$ and a real vector field $\mathbf{F} = (F_1, F_2) \in C_0^2(\overline{\mathbb{D}}; \mathbb{R}^2)$.

Definition 5.0.2. *For each $x \in \mathbb{D}$, and $\theta = (\cos\varphi, \sin\varphi) \in S^1$, the* attenuated Doppler Transform *of* $\mathbf{F}$ *is defined as*

$$\int_{-\infty}^{\infty} (\theta \cdot \mathbf{F})(x + t\theta) e^{-Da(x+t\theta,\theta)} dt. \tag{5.1}$$

Definition 5.0.3. *For $\mathbf{F} \in L^1(\mathbb{D}; \mathbb{R}^2)$, we say that g on $\varGamma \times \mathbf{S}^1$ is an* attenuated Doppler transform *of* $\mathbf{F}$ *with attenuation a, if*

$$g(x_\theta^+, \theta) - \left[e^{-Da} g\right](x_\theta^-, \theta) = \int_{\tau_-(x,\theta)}^{\tau_+(x,\theta)} (\theta \cdot \mathbf{F})(x + t\theta) e^{-Da(x+t\theta,\theta)} dt, \tag{5.2}$$

for $(x, \theta) \in \overline{\mathbb{D}} \times \mathbf{S}^1$.

For the Doppler case we have that $g(a,0,\mathbf{F})$ is the trace on $\Gamma \times \mathbf{S}^1$ of solution to the transport equation

$$\begin{aligned} \theta \cdot \nabla u(x,\theta) + a(x)u(x,\theta) &= \theta \cdot \mathbf{F}(x), \quad (x,\theta) \in \mathbb{D} \times \mathbf{S}^1, \\ u|_{\Gamma} &= g(a,0,\mathbf{F}). \end{aligned} \tag{5.3}$$

In the non-attenuated case ($a \equiv 0$), it is easy to see that superposition of the gradient of a compactly supported function does not change the data. This non-uniqueness is made explicit in Theorem 5.1.1, where a class of vector fields of all which yield the same data is constructed. In contrast, in the strictly attenuated case ($a > 0$) the attenuated Doppler transform is uniquely invertible as observed first in [35].

Let u be solution of (5.3). Since it is real it suffices to work with non-positive Fourier modes of the Fourier expansion

$$u(z,\theta) = \sum_{-\infty}^{\infty} u_n(z) e^{in\varphi}.$$

Provided appropriate convergence of the series we see that u solves (5.3) if and only if its Fourier coefficients solve the system

$$\overline{\partial} u_1(z) + \partial u_{-1}(z) + a(z)u_0(z) = 0, \tag{5.4}$$

$$\overline{\partial} u_0(z) + \partial u_{-2}(z) + a(z)u_{-1}(z) = f_1(z), \tag{5.5}$$

$$\overline{\partial} u_n(z) + \partial u_{n-2}(z) + a(z)u_{n-1}(z) = 0, \quad n \leq -1, \tag{5.6}$$

where $f_1 = (F_1 + iF_2)/2$, and v solves (2.2) if and only if its Fourier coefficients solve the system

$$\overline{\partial} v_1(z) + \partial v_{-1}(z) + a(z)v_0(z) = f(z), \tag{5.7}$$

$$\overline{\partial} v_0(z) + \partial v_{-2}(z) + a(z)v_{-1}(z) = 0, \tag{5.8}$$

$$\overline{\partial} v_n(z) + \partial v_{n-2}(z) + a(z)v_{n-1}(z) = 0, \quad n \leq -1. \tag{5.9}$$

Note that u_0 and f are real valued, while f_1 is complex valued.

The operators ∂, $\overline{\partial}$ can be rewritten in terms the angular derivative ∂_η and radial derivative ∂_r,

$$\partial_r = \nabla \cdot \theta = \langle \partial_x, \partial_y \rangle \cdot \langle \cos\eta, \sin\eta \rangle = \cos\eta \partial_x + \sin\eta \partial_y,$$
$$\partial_\eta = \nabla \cdot \theta^\perp = \langle \partial_x, \partial_y \rangle \cdot \langle -\sin\eta, \cos\eta \rangle = -\sin\eta \partial_x + \cos\eta \partial_y,$$

as

$$\partial = \frac{\partial_x - i\partial_y}{2} = \frac{e^{-i\eta}}{2}(\partial_r - i\partial_\eta),$$
$$\overline{\partial} = \frac{\partial_x + i\partial_y}{2} = \frac{e^{i\eta}}{2}(\partial_r + i\partial_\eta).$$

When treating the attenuated case it is useful to introduce the following operator T as in the proposition below

Proposition 5.0.4. *Let* $\alpha > 1/2$, $g \in C^\alpha(\mathit{\Gamma}; C^{1,\alpha}(\mathbf{S}^1)) \cap C^0(\mathit{\Gamma}; C^{2,\alpha}(\mathbf{S}^1))$, $a \in C_0^2(\overline{\mathbb{D}})$ *and* h *be as in* (2.11). *Let us define*

$$Tg := \mathcal{P}^+ e^h *_n B\mathcal{P}^-(e^{-h}g), \tag{5.10}$$

where $*_n$ is the convolution operator on sequences, $\mathcal{P}^{\pm}$ as in (4.15) *and B as in* (4.5). *Then*

$$T : C^{\alpha}\left(\varGamma; C^{1,\alpha}(\mathbf{S}^1)\right) \cap C^0(\varGamma; C^{2,\alpha}(\mathbf{S}^1)) \to C^{1,\alpha}(\mathbb{D}; l_1) \cap C^{\alpha}(\overline{\mathbb{D}}; l_1).$$

Moreover, the components $(Tg)_n$, for $n \leq -1$, solve

$$\overline{\partial}(Tg)_n + \partial(Tg)_{n-2} + a(Tg)_{n-1} = 0. \tag{5.11}$$

Furthermore, for each $n = 0, -1, ...,$

$$(B\mathbf{g})_n \in C^{\infty}(\Omega). \tag{5.12}$$

Proof. Let $g \in C^{\alpha}\left(\varGamma; C^{1,\alpha}(\mathbf{S}^1)\right) \cap C^0(\varGamma; C^{2,\alpha}(\mathbf{S}^1))$ with $\alpha > 1/2$. By Proposition 4.2.2(iii), it follows that $\mathcal{P}^-(e^{-h}g) \in Y_{\alpha}$. By Corollary 4.1.1, we get $\mathcal{L}B\mathcal{P}^-(e^{-h}g) \in C^{1,\alpha}(\mathbb{D}; l_1) \cap C^{\alpha}(\overline{\mathbb{D}}; l_1)$ is A-analytic. Finally, by Proposition 4.2.2(ii), $Tg \in C^{1,\alpha}(\mathbb{D}; l_1) \cap C^{\alpha}(\overline{\mathbb{D}}; l_1)$ and the components $(Tg)_n$, for $n \leq -1$, solve (5.11). The regularity $(Tg)_n \in C^{\infty}(\Omega)$ follows from (4.6), in Theorem 4.1.1. □

5.1 Range Characterization of the non-attenuated Doppler Transform

In this section, we established necessary and sufficient conditions for a sufficiently smooth function g on $\varGamma \times \mathbf{S}^1$ to be the Doppler data of some sufficiently smooth vector field $\mathbf{F}$. The first variable describes the boundary of the domain and we refer to it as the *spatial* variable. The second variable describes a direction and we refer to it as the *angular* variable.

To address the non-uniqueness (up to a gradient field) in the characterization of the non-attenuated Doppler transform we introduce the class of functions Π_g with prescribed trace and gradient on the boundary Γ as

$$\Pi_g := \left\{\psi \in C^1(\overline{\mathbb{D}};\mathbb{R}) : \psi|_\Gamma = g_0,\, \partial_r\psi|_\Gamma = -\,\mathbb{R}\mathrm{e}\, e^{-2i\eta}(\partial_r - i\partial_\eta)(B\,\mathcal{P}^- g)_{-2}\big|_\Gamma\right\},$$

where the operator B is defined in (4.5), and $\mathcal{P}^-$ as defined in (4.15).

If $a \equiv 0$, then u solves (5.3) if and only if its Fourier coefficients solve the system

$$\overline{\partial} u_1(z) + \partial u_{-1}(z) = 0, \tag{5.13}$$

$$\overline{\partial} u_0(z) + \partial u_{-2}(z) = f_1(z), \tag{5.14}$$

$$\overline{\partial} u_n(z) + \partial u_{n-2}(z) = 0, \quad n \le -1, \tag{5.15}$$

where $f_1 = (F_1 + iF_2)/2$.

The basic idea in the characterization below is that the negative modes $\langle g_{-1}, g_{-2}, \cdots\rangle$ of the data determines the negative modes of the solution to the transport equation (5.3) via the Hilbert transform $\mathcal{H}_a$, see (5.16). This accounts for all but one differential equation which is investigated separately.

Theorem 5.1.1 (Range characterization of non-attenuated Doppler Transform)**.** *Let $\alpha > 1/2$, and $a \equiv 0$. (i) For $\mathbf{F} \in C_0^{1,\alpha}(\overline{\mathbb{D}};\mathbb{R}^2)$, let $g = g(0,0,\mathbf{F})$ be the Doppler data of $\mathbf{F}$ as in* (5.3)*. Then $g \in C^\alpha(\Gamma; C^{1,\alpha}(\mathbf{S}^1))$, satisfies*

$$[I + i\mathcal{H}_0]\mathcal{L}\mathcal{P}^- g = 0, \tag{5.16}$$

$$\mathbb{R}\mathrm{e}\left\{\partial\left(B\,\mathcal{P}^- g\right)_{-1}\right\} = 0, \quad \textit{in } \mathbb{D}, \tag{5.17}$$

$$\partial_\eta(\mathcal{P}^- g)_0 = -\,\mathbb{I}\mathrm{m}\left\{e^{-2i\eta}(\partial_r - i\partial_\eta)(B\,\mathcal{P}^- g)_{-2}|_\Gamma\right\}, \tag{5.18}$$

where the operator B is defined in (4.5)*, and $\mathcal{H}_0$ is the Hilbert transform in* (4.3)*.*

(ii) Let $g \in C^{\alpha}(\Gamma; C^{1,\alpha}(\mathbf{S}^1)) \cap C^0(\Gamma; C^{2,\alpha}(\mathbf{S}^1))$*, be real valued satisfying* (5.16)*,* (5.17)*, and* (5.18)*. Then, for each* $\psi \in \Pi_g$*, there exist a unique real valued vector field* $\mathbf{F}_\psi \in C_0(\overline{\mathbb{D}}; \mathbb{R}^2)$*, such that* $g = g(0, 0, \mathbf{F})$ *is the Doppler data of* $\mathbf{F}_\psi$*.*

Proof. (i) For simplicity we use the notation $\mathbf{g} = \mathcal{P}^- g := \langle g_0, g_{-1}, \cdots \rangle$. Recall that g is the trace of the solution u of (5.3) and by the equivalence with the system (5.13), (5.14), (5.15), its negative Fourier modes u_n satisfy

$$u_n = (B\mathbf{g})_n, \quad n \leq -1,$$

where B as in (4.5).

The equation (5.15) implies that the sequence $\mathcal{L}(B\mathbf{g})) = \langle u_{-1}, u_{-2}, \cdots \rangle$ is A-analytic. By the necessity part in Theorem 4.1.2, $[I + i\mathcal{H}_0]\mathcal{L}\mathbf{g} = 0$.

The equation (5.13) implies the condition (5.17).

The restriction of (5.14) to the boundary yields

$$\begin{aligned}
&\overline{\partial} u_0|_\Gamma + \partial u_{-2}|_\Gamma = f_1|_\Gamma, \\
&\frac{e^{i\eta}}{2}(\partial_r + i\partial_\eta) u_0|_\Gamma + \frac{e^{-i\eta}}{2}(\partial_r - i\partial_\eta) u_{-2}|_\Gamma = 0, \because f_1 \in C_0^1(\overline{\mathbb{D}}), \\
&(\partial_r + i\partial_\eta) u_0|_\Gamma = -e^{-2i\eta}(\partial_r - i\partial_\eta) u_{-2}|_\Gamma, \\
&\partial_r u_0|_\Gamma + i\partial_\eta g_0 = -e^{-2i\eta}(\partial_r - i\partial_\eta)(B\mathbf{g})_{-2}|_\Gamma,
\end{aligned} \tag{5.19}$$

From the above equation (5.19), we get condition (5.18),

$$\partial_\eta g_0 = -\operatorname{\mathbb{I}m}\left\{e^{-2i\eta}(\partial_r - i\partial_\eta)(B\mathbf{g})_{-2}|_\Gamma\right\}.$$

This proves the necessity part of the theorem.

Conversely, assume that we have $g \in C^{\alpha}(\Gamma; C^{1,\alpha}(\mathbf{S}^1)) \cap C^0(\Gamma; C^{2,\alpha}(\mathbf{S}^1))$, satisfying (5.16), (5.17), and (5.18).

Let $\mathbf{g} = \mathcal{P}^- g := \langle g_0, g_{-1}, \cdots \rangle$. and define $u_n = (B\mathbf{g})_n$ for $n \le -1$, where B is as in (4.5). By Theorem 4.1.2, (5.16) implies $\overline{\partial} u_n + \partial u_{n-2} = 0$ for $n \le -1$, and (5.17) implies $\overline{\partial} u_1 + \partial u_{-1} = 0$.

Let $\psi \in \Pi_g$. Define $f_1 := \overline{\partial}\psi + \partial (B\mathbf{g})_{-2}$. By Theorem 4.1.1, $f_1 \in C(\mathbb{D})$, and its trace satisfies

$$
\begin{aligned}
f_1|_\Gamma &= \overline{\partial}\psi\big|_\Gamma + \partial (B\mathbf{g})_{-2}\big|_\Gamma \\
f_1|_\Gamma &= \frac{e^{i\eta}}{2}(\partial_r + i\partial_\eta)\,\psi|_\Gamma + \frac{e^{-i\eta}}{2}(\partial_r - i\partial_\eta)\,(B\mathbf{g})_{-2}\Big|_\Gamma \\
2e^{-i\eta}\, f_1|_\Gamma &= \partial_r\psi|_\Gamma + i\partial_\eta g_0 + e^{-2i\eta}(\partial_r - i\partial_\eta)(B\mathbf{g})_{-2}\big|_\Gamma \\
2e^{-i\eta}\, f_1|_\Gamma &= 0, \quad \text{from (5.18)}.
\end{aligned}
$$

From the above equation $f_1|_\Gamma = 0$, implying $f_1 \in C_0(\overline{\mathbb{D}})$.

Define the real valued vector field $\mathbf{F}_\psi = \langle 2\,\mathbb{R}\mathrm{e}\, f_1, 2\,\mathbb{I}\mathrm{m}\, f_1 \rangle$ and let

$$
u(z,\theta) := \psi(z) + 2\,\mathbb{R}\mathrm{e}\left\{\sum_{n=1}^{\infty} u_{-n}(z) e^{-in\varphi}\right\}.
$$

By the one to one correspondence between (5.13), (5.14), (5.15) and the boundary value problem (5.3), we have that u solves

$$
\theta \cdot \nabla u = \theta \cdot \mathbf{F}_\psi,
$$

$$
u|_\Gamma = g,
$$

i.e., g is the Doppler data of $\mathbf{F}_\psi$.

$\square$

5.2 Range Characterization of the attenuated Doppler Transform

We now consider the attenuated case.

Theorem 5.2.1 (Range characterization of attenuated Doppler Transform)**.** *Let $a \in C_0^2(\overline{\mathbb{D}})$, $\alpha > 1/2$, with $a > 0$ in $\mathbb{D}$. (i) For $\mathbf{F} \in C_0^{1,\alpha}(\overline{\mathbb{D}}; \mathbb{R}^2)$, let $g = g(a, 0, \mathbf{F})$ be the Doppler data of $\mathbf{F}$ as in* (5.3)*. Then $g \in C^\alpha(\Gamma; C^{1,\alpha}(\mathbf{S}^1))$ satisfies*

$$[I + i\mathcal{H}_a]\mathcal{L}\mathcal{P}^-(g) = 0, \tag{5.20}$$

$$(\mathcal{P}^- g)_0(z_0) = \lim_{\mathbb{D} \ni z \to z_0 \in \varGamma} \frac{-2\,\mathbb{R}\mathrm{e}\left\{\partial (Tg)_{-1}(z)\right\}}{a(z)}, \tag{5.21}$$

$$\partial_\eta (\mathcal{P}^- g)_0 = -\,\mathbb{I}\mathrm{m}\left\{e^{-2i\eta}(\partial_r - i\partial_\eta)(Tg)_{-2}\right\}, \tag{5.22}$$

$$\partial_r \left(\frac{\mathbb{R}\mathrm{e}\left(\partial (Tg)_{-1}\right)}{a} \right)\Bigg|_{\varGamma} = \frac{1}{2}\,\mathbb{R}\mathrm{e}\left\{e^{-2i\eta}(\partial_r - i\partial_\eta)(Tg)_{-2}\right\}, \tag{5.23}$$

where the operator T is defined in (5.10) *and the Hilbert transform $\mathcal{H}_a$ defined in* (4.45).

(ii) Let $g \in C^\alpha\left(\varGamma; C^{1,\alpha}(\mathbf{S}^1)\right) \cap C^0(\varGamma; C^{2,\alpha}(\mathbf{S}^1))$, be real valued satisfying (5.20)*,* (5.21)*,* (5.22) *and* (5.23)*. Then there exists a unique real valued vector field $\mathbf{F} \in C_0(\overline{\mathbb{D}}; \mathbb{R}^2)$ such that $g = g(a, 0, \mathbf{F})$ is the Doppler data of* $\mathbf{F}$.

Proof. (i) Let g be the Doppler data of $\mathbf{F} \in C_0^{1,\alpha}(\overline{\mathbb{D}}; \mathbb{R}^2)$. Recall that g is the trace of the solution u of the Transport equation (5.3), and by the equivalence with the system (5.4), (5.5), (5.6), its negative Fourier modes u_n satisfy

$$u_n = (Tg)_n, \quad n \leq -1,$$

where T as in (5.10). The Proposition 5.0.4, implies $\overline{\partial} u_n + \partial u_{n-2} + a u_{n-1} = 0$, for $n \leq -1$ and the necessity part in Theorem 4.2.1, yields the condition (5.20).

From (5.4), for $z \in \mathbb{D}$ we have

$$u_0(z) = \frac{-2\,\mathbb{R}\mathrm{e}\,\{\partial(u_{-1})(z)\}}{a(z)}, \tag{5.24}$$

and the restriction of (5.4) to the boundary yields

$$\begin{aligned}\lim_{z\to z_0\in\varGamma} u_0(z) &= \lim_{z\to z_0\in\varGamma} \frac{-2\,\mathbb{R}\mathrm{e}\,\{\partial u_{-1}(z)\}}{a(z)},\\ g_0(z_0) &= \lim_{z\to z_0\in\varGamma} \frac{-2\,\mathbb{R}\mathrm{e}\,\{\partial u_{-1}(z)\}}{a(z)},\end{aligned}$$

thus (5.21) holds.

The restriction of (5.5) to the boundary yields

$$\begin{aligned}&\overline{\partial} u_0|_\varGamma + \partial u_{-2}|_\varGamma + a u_{-1}|_\varGamma = f_1|_\varGamma,\\ &\overline{\partial} u_0|_\varGamma + \partial u_{-2}|_\varGamma = 0, \ \because a \in C_0^2(\overline{\mathbb{D}}),\ f_1 \in C_0^1(\overline{\mathbb{D}}),\\ &\frac{e^{i\eta}}{2}(\partial_r + i\partial_\eta) u_0|_\varGamma + \frac{e^{-i\eta}}{2}(\partial_r - i\partial_\eta) u_{-2}|_\varGamma = 0,\\ &(\partial_r + i\partial_\eta) u_0|_\varGamma = -e^{-2i\eta}(\partial_r - i\partial_\eta)(Tg)_{-2}|_\varGamma,\\ &-2\,\partial_r \left(\frac{\mathbb{R}\mathrm{e}(\partial u_{-1})}{a}\right)\bigg|_\varGamma + i\partial_\eta g_0 = -e^{-2i\eta}(\partial_r - i\partial_\eta)(Tg)_{-2}|_\varGamma, \text{ from (5.24) and } u_0|_\varGamma = g_0.\end{aligned} \tag{5.25}$$

From the equation (5.25) above, we get conditions (5.22), and (5.23),

$$\partial_\eta g_0 = -\,\mathbb{I}\mathrm{m}\left\{e^{-2i\eta}(\partial_r - i\partial_\eta)(Tg)_{-2}|_\Gamma\right\},$$
$$\partial_r\left(\frac{\mathbb{R}\mathrm{e}(\partial u_{-1})}{a}\right)\Bigg|_\Gamma = \frac{1}{2}\,\mathbb{R}\mathrm{e}\left\{e^{-2i\eta}(\partial_r - i\partial_\eta)(Tg)_{-2}|_\Gamma\right\}.$$

This proves the necessity part of the theorem.

Conversely, assume $g \in C^\alpha\left(\Gamma; C^{1,\alpha}(\mathbf{S}^1)\right) \cap C^0(\Gamma; C^{2,\alpha}(\mathbf{S}^1))$, satisfying (5.20), (5.21), (5.22) and (5.23). Define $u_n = (Tg)_n$ for $n \leq -1$, where T is as in (5.10). By Proposition 5.0.4, (5.20) implies $\overline{\partial} u_n + \partial u_{n-2} + a u_{n-1} = 0$ for $n \leq -1$.

Now define u_0 in $\mathbb{D}$ by

$$u_0 := -\frac{2\,\mathbb{R}\mathrm{e}\,\partial u_{-1}}{a}, \tag{5.26}$$

in particular $\overline{\partial} u_1 + \partial u_{-1} + a u_0 = 0$ holds. By Proposition 5.0.4, $u_0 \in C^1(\mathbb{D})$, and (5.21) yields $u_0|_\Gamma = g_0$.

Next we define $f_1 := \overline{\partial} u_0 + \partial u_{-2} + a u_{-1}$. By Proposition 5.0.4, $f_1 \in C(\mathbb{D})$ and the trace on the boundary satisfies

$$f_1|_\Gamma = \overline{\partial} u_0|_\Gamma + \partial u_{-2}|_\Gamma + a u_{-1}|_\Gamma,$$
$$f_1|_\Gamma = \frac{e^{i\eta}}{2}(\partial_r + i\partial_\eta)u_0|_\Gamma + \frac{e^{-i\eta}}{2}(\partial_r - i\partial_\eta)u_{-2}|_\Gamma, \ \because a \in C_0^2(\overline{\mathbb{D}}),$$
$$2e^{-i\eta} f_1|_\Gamma = -2\,\partial_r\left(\frac{\mathbb{R}\mathrm{e}(\partial u_{-1})}{a}\right)\Bigg|_\Gamma + i\partial_\eta g_0 + e^{-2i\eta}(\partial_r - i\partial_\eta)(Tg)_{-2}|_\Gamma, \text{ from (5.26), and } u_0|_\Gamma = g_0,$$
$$2e^{-i\eta} f_1|_\Gamma = 0, \qquad \text{from (5.22), and (5.23).}$$

From the above equation $f_1|_\Gamma = 0$, implying $f_1 \in C_0(\overline{\mathbb{D}})$.

Finally, we define the real valued vector field $\mathbf{F} = \langle 2\,\mathbb{R}\mathrm{e}\, f_1, 2\,\mathbb{I}\mathrm{m}\, f_1\rangle$, and let

$$u(z,\theta) := u_0(z) + 2\,\mathbb{R}\mathrm{e}\left\{\sum_{n=1}^{\infty} u_{-n}(z)e^{-in\varphi}\right\}.$$

By the one to one correspondence between (5.4), (5.5), (5.6) and the boundary value problem (5.3), we have that

$$\theta\cdot\nabla u + au = \theta\cdot\mathbf{F},$$
$$u|_{\Gamma} = g,$$

i.e., g is the Doppler data of $\mathbf{F}$.

□

5.3 When can the X-ray and Doppler data be mistaken for each other ?

To distinguish between the Radon and Doppler cases we used the notation $g(a,f,\mathbf{0})$ for the attenuated Radon transform of f, respectively, $g(a,0,\mathbf{F})$ for the attenuated Doppler transform of $\mathbf{F}$. Recall that the function $g(a,f,\mathbf{0})$ is precisely the trace on $\Gamma\times\mathbf{S}^1$ of solutions v to the transport equation

$$\theta\cdot\nabla v(x,\theta) + a(x)v(x,\theta) = f(x), \quad (x,\theta)\in\mathbb{D}\times S^1,$$
$$v|_{\Gamma} = g(a,f,\mathbf{0}),$$

and $g(a, 0, \mathbf{F})$ is the trace of an $\Gamma \times \mathbf{S}^1$ of solutions u to the transport equation

$$\theta \cdot \nabla u(x, \theta) + a(x)u(x, \theta) = \theta \cdot \mathbf{F}(x), \quad (x, \theta) \in \mathbb{D} \times \mathbf{S}^1,$$
$$u|_\Gamma = g(a, 0, \mathbf{F}).$$

In the non-attenuated case ($a \equiv 0$), the Radon and the Doppler data cannot be mistaken for each other unless they are both zero. This is due to the new constraints (5.17) and (5.18) in the Doppler case. However, in the attenuated case the following can hold.

Theorem 5.3.1. *Let $a > 0$ in $\mathbb{D}$, with $a \in C^2(\mathbb{D}) \cap C_0^1(\overline{\mathbb{D}})$.*
If $f \in C^2(\mathbb{D}) \cap C_0^1(\overline{\mathbb{D}})$ with $f/a \in C^1(\mathbb{D}) \cap C_0(\overline{\mathbb{D}})$, then $-\nabla \left(\frac{f}{a}\right)$ is a vector field whose Doppler data $g\left(a, 0, -\nabla\left(\frac{f}{a}\right)\right)$ is the same as the attenuated Radon data $g(a, f, \mathbf{0})$ of f.

Conversely, if $\mathbf{F} \in C^2(\mathbb{D}) \cap C_0^1(\overline{\mathbb{D}})$ has the attenuated Doppler data equal to the attenuated Radon data of some $f \in C^2(\mathbb{D}) \cap C_0^1(\overline{\mathbb{D}})$ then $\mathbf{F}$ must be a gradient field and $\mathbf{F} = -\nabla\left(\frac{f}{a}\right)$.

Proof. Assume $g = g(a, f, \mathbf{0})$ is the Radon data of f. The function $g(a, f, \mathbf{0})$ is the trace on $\Gamma \times \mathbf{S}^1$ of solutions v to the transport equation

$$\theta \cdot \nabla v + av = f,$$
$$v|_\Gamma = g(a, f, \mathbf{0}).$$

Let $u = v - \frac{f}{a}$ and $\mathbf{F} = -\nabla\left(\frac{f}{a}\right)$. Then for the Doppler case the transport equation

$$\theta \cdot \nabla u + au = \theta \cdot \mathbf{F},$$

becomes

$$\theta \cdot \nabla \left(v - \frac{f}{a} \right) + av - f = -\theta \cdot \nabla \left(\frac{f}{a} \right),$$
$$\theta \cdot \nabla v + av = f,$$

and

$$v|_{\Gamma} = u|_{\Gamma} + \left. \frac{f}{a} \right|_{\Gamma},$$
$$v|_{\Gamma} = u|_{\Gamma} \quad \because f/a \in C_0(\overline{\mathbb{D}}),$$
$$g(a, f, \mathbf{0}) = g(a, 0, \mathbf{F}).$$

Next, we prove that this is the only case where the two measurements can be confounded. Let $\mathbf{F}$ be a vector field as in the hypotheses for which the attenuated Doppler data $g(a, 0, \mathbf{F})$ matches the attenuated Radon data $g(a, f, \mathbf{0})$ of some f.

Let v be the solution of the transport equation

$$\theta \cdot \nabla v + av = f,$$
$$v|_{\Gamma} = g(a, f, \mathbf{0}),$$

with trace of the solution v be the Radon data $g(a, f, \mathbf{0})$. The non-positive Fourier modes v_n of v solves the system (5.7), (5.8), (5.9).

Let u be the solution of the transport equation

$$\theta \cdot \nabla u + au = \theta \cdot \mathbf{F},$$
$$u|_{\Gamma} = g(a, 0, \mathbf{F}),$$

with trace of the solution u be the Doppler data $g(a, 0, \mathbf{F})$. The non-positive Fourier modes u_n of u solves the system (5.4), (5.5), (5.6).

By Theorem 5.2.1 and Theorem 4.2.1, both Doppler data $g(a, 0, \mathbf{F})$ and the Radon data $g(a, f, \mathbf{0})$ satisfy

$$[I + i\mathcal{H}_a]\mathcal{L}\mathcal{P}^{-}(g) = 0,$$

where $\mathcal{P}^{-}$ defined in (4.15) and $\mathcal{H}_a$ defined in (4.45). It follows that $v_n = u_n$, for all $n \leq -1$. Now (5.7) and (5.4) yield that $f = a(v_0 - u_0)$.

By equation (5.5),

$$\begin{aligned} f_1 &= \overline{\partial} u_0 + \partial u_{-2} + a u_{-1}, \\ &= \overline{\partial} v_0 - \overline{\partial}\left(\frac{f}{a}\right) + \partial v_{-2} + a v_{-1}, \quad \because u_{-1} = v_{-1},\ u_{-2} = v_{-2}, \\ &= -\overline{\partial}\left(\frac{f}{a}\right), \quad \text{from (5.8)}, \end{aligned}$$

which implies $\mathbf{F} = -\nabla\left(\frac{f}{a}\right) \in C_0(\overline{\mathbb{D}})$. $\square$

CHAPTER 6: RANGE CHARACTERIZATION OF SYMMETRIC SECOND ORDER TENSOR

Any sufficiently smooth symmetric m-tensor field f on a compact oriented two dimensional Riemannian manifold M with smooth boundary can be decomposed in a potential and solenoidal part [55]:

$$f = f^s + dg, \quad div(f^s) = 0, \; g|_{\partial M} = 0,$$

where g is a smooth symmetric $(m-1)$-tensor field on M.

The fundamental theorem of Calculus shows that the geodesic ray transform of the potential part of the tensor is zero, which means that one can at most be able to recover the solenoidal part of the tensor field from its ray transform.

In the case of 2-tensor, the geodesic ray transform arises in the linearization of the boundary rigidity problem [55], which reads as can one recover the Riemannian metric of a compact manifold with boundary from the distances function between boundary points. This problem arose in travel time tomography in geophysics where one attempt to determine the inner structure of the Earth by measuring the travel times of seismic waves.

Let $\mathbb{D} \subset \mathbb{R}^2$ denote the unit disc in the plane and $\varGamma = \mathbf{S}^1$ be its boundary. We will consider real function $a \in C_0^2(\overline{\mathbb{D}})$ and a real valued symmetric second order tensor field

$$\mathbf{F}(x) = \begin{pmatrix} f_{11}(x) & f_{12}(x) \\ f_{12}(x) & f_{22}(x) \end{pmatrix}.$$

Definition 6.0.1. *For each* $x \in \mathbb{D}$ *and* $\theta = (\cos\varphi, \sin\varphi) \in S^1$, *the* attenuated Radon transform *of* $\mathbf{F}$ *(with attenuation* a*) is defined as*

$$\int_{-\infty}^{\infty} \theta^T \, \mathbf{F}(x + t\theta) \, \theta \, e^{-Da(x+t\theta,\theta)} dt. \tag{6.1}$$

Definition 6.0.2. *For* $\mathbf{F} \in L^1(\mathbb{D}; \mathbb{R}^{2\times 2})$*, we say that* g *on* $\Gamma \times \mathbf{S}^1$ *is an* attenuated Radon transform *of* $\mathbf{F}$ *with attenuation* a*, if*

$$g(x_\theta^+, \theta) - \left[e^{-Da} g\right](x_\theta^-, \theta) = \int_{\tau_-(x,\theta)}^{\tau_+(x,\theta)} \theta^T \, \mathbf{F}(x + t\theta) \, \theta \, e^{-Da(x+t\theta,\theta)} dt. \tag{6.2}$$

for $(x, \theta) \in \overline{\mathbb{D}} \times \mathbf{S}^1$.

One of the open problems in the field of tensor tomography discussed in [50] consists of finding inversion formulas of the solenoidal part of 2-tensor in the Euclidean setting. The problem can be cast as inverse source problem for the linear transport model

$$\theta \cdot \nabla u(x, \theta) + a(x) u(x, \theta) = \theta^T \mathbf{F}(x) \theta, \tag{6.3}$$

where $\mathbf{F}$ is a symmetric second order 2D tensor field, $u(x, \theta)$ is the density of particles at $x \in \mathbb{D} \subset \mathbb{R}^2$ moving in the direction $\theta = (\cos\varphi, \sin\varphi) \in \mathbf{S}^1$, and $a(x)$ is the medium capability of absorption per unit path-length at x. Calculation shows that

$$
\begin{aligned}
\theta^T \mathbf{F}(x)\theta &= (\cos\varphi, \sin\varphi) \begin{pmatrix} f_{11}(x) & f_{12}(x) \\ f_{12}(x) & f_{22}(x) \end{pmatrix} \begin{pmatrix} \cos\varphi \\ \sin\varphi \end{pmatrix}, \\
&= (\cos\varphi, \sin\varphi) \begin{pmatrix} f_{11}(x)\cos\varphi + f_{12}(x)\sin\varphi \\ f_{12}(x)\cos\varphi + f_{22}(x)\sin\varphi \end{pmatrix}, \\
&= f_{11}(x)\cos^2\varphi + f_{12}(x)\sin 2\varphi + f_{22}(x)\sin^2\varphi, \\
&= \frac{f_{11}(x)+f_{22}(x)}{2} + \frac{f_{11}(x)-f_{22}(x)}{2}\cos 2\varphi + f_{12}(x)\sin 2\varphi, \\
&= \frac{f_{11}(x)+f_{22}(x)}{2} + \left(\frac{f_{11}(x)}{4} - i\frac{f_{12}(x)}{2}\right)e^{2i\varphi} + \left(\frac{f_{11}(x)}{4} + i\frac{f_{12}(x)}{2}\right)e^{-2i\varphi}, \\
&= f_0 + \bar{f}_2 e^{2i\varphi} + f_2 e^{-2i\varphi}.
\end{aligned}
$$

where

$$f_0 = \frac{f_{11}(x)+f_{22}(x)}{2}, \text{ and} \tag{6.4}$$

$$f_2 = \left(\frac{f_{11}(x)}{4} + i\frac{f_{12}(x)}{2}\right)e^{-2i\varphi}. \tag{6.5}$$

So (6.3), becomes

$$\theta \cdot \nabla u(x,\theta) + a(x)u(x,\theta) = \theta^T \mathbf{F}(x)\theta = f_0 + \bar{f}_2 e^{2i\varphi} + f_2 e^{-2i\varphi}, \tag{6.6}$$

$$u|_\Gamma = g(a, f_0, f_2). \tag{6.7}$$

Let $u(z,\theta) = \sum_{-\infty}^{\infty} u_n(z)e^{in\varphi}$, be the (formal) Fourier expansions of u in the angular variable. Provided appropriate convergence of the series as specified in the theorems, identifying the like

Fourier modes we see that u solves (6.6) if and only if its Fourier coefficients solve the system

$$\overline{\partial} u_1(z) + \partial u_{-1}(z) + a(z)u_0(z) = f_0(z), \tag{6.8}$$

$$\overline{\partial} u_0(z) + \partial u_{-2}(z) + a(z)u_{-1}(z) = 0, \tag{6.9}$$

$$\overline{\partial} u_{-1}(z) + \partial u_{-3}(z) + a(z)u_{-2}(z) = f_2(z), \tag{6.10}$$

$$\overline{\partial} u_n(z) + \partial u_{n-2}(z) + a(z)u_{n-1}(z) = 0, \quad n \leq -2. \tag{6.11}$$

Note that u_0 and f_0 are real valued, while f_2 is complex valued.

Since u is real-valued, its Fourier coefficients appear in complex-conjugate pairs, $\overline{u_n} = u_{-n}$, so that it suffices to work with the sequence of non-positive indexes.

Note that there are six unknown u_0, f_0, u_{-1}, f_2 and five equations yielding underdetermined system. For 2-tensors there will be nonuniqueness even if $a > 0$.

6.1 Range Characterization of the non-attenuated Radon Transform of real valued symmetric second order tensor

In this section, we established necessary and sufficient conditions for a sufficiently smooth function g on $\Gamma \times \mathbf{S}^1$ to be the Radon data of some sufficiently smooth real valued symmetric second order tensor $\mathbf{F}$.

To address the non-uniqueness in the characterization of the non-attenuated Radon transform of real valued symmetric second order tensor we introduce the class of functions Ψ_g with prescribed trace and gradient on the boundary $\varGamma$ as

$$\Psi_g := \Big\{ \psi \in C^1(\overline{\mathbb{D}};\mathbb{C}) : \psi|_\varGamma = (\mathcal{P}^- g)_{-1},$$
$$\partial_r\left(\mathbb{R}\mathrm{e}\,\psi|_\varGamma\right) = -\,\mathbb{R}\mathrm{e}\left(e^{-2i\eta}(\partial_r - i\partial_\eta)(B\mathcal{P}^- g)_{-3}|_\varGamma\right) + \partial_\eta\,\mathbb{I}\mathrm{m}(\mathcal{P}^- g)_{-1},$$
$$\partial_r\left(\mathbb{I}\mathrm{m}\,\psi|_\varGamma\right) = -\,\mathbb{I}\mathrm{m}\left(e^{-2i\eta}(\partial_r - i\partial_\eta)(B\mathcal{P}^- g)_{-3}|_\varGamma\right) - \partial_\eta\,\mathbb{R}\mathrm{e}(\mathcal{P}^- g)_{-1}\Big\},$$

where the operator B is defined in (4.5), and $\mathcal{P}^-$ as defined in (4.15).

If $a \equiv 0$, then u solves (6.6) if and only if its Fourier coefficients solve the system

$$\overline{\partial} u_1(z) + \partial u_{-1}(z) = f_0(z), \tag{6.12}$$

$$\overline{\partial} u_0(z) + \partial u_{-2}(z) = 0, \tag{6.13}$$

$$\overline{\partial} u_{-1}(z) + \partial u_{-3}(z) = f_2(z), \tag{6.14}$$

$$\overline{\partial} u_n(z) + \partial u_{n-2}(z) = 0, \quad n \le -2. \tag{6.15}$$

Recall the Cauchy integral formula for antiholmorphic functions [62],

$$\overline{\partial}\left\{-\frac{1}{\pi}\iint_{\mathbb{D}} \frac{g(z)}{z-\xi}dxdy\right\} = g(\xi), \quad \xi \in \mathbb{D},$$

and we define one of the right inverse $(\overline{\partial})^{-1}$ (unique up to an analytic function), by

$$(\overline{\partial})^{-1} g(\xi) = -\frac{1}{\pi}\iint_{\mathbb{D}} \frac{g(z)}{z-\xi}dxdy, \quad \xi \in \mathbb{D}. \tag{6.16}$$

Theorem 6.1.1 (Range characterization for the 2-tensor (Non-Attenuated))**.** *Let* $\alpha > 1/2$*, and* $a \equiv 0$*. (i) For* $\mathbf{F} \in C_0^{1,\alpha}(\overline{\mathbb{D}};\mathbb{R}^{2\times 2})$*, let* g *be the Radon data of* $\mathbf{F}$*. Then* $g \in C^\alpha(\varGamma; C^{1,\alpha}(\mathbf{S}^1))$*,* *satisfies*

$$[I + i\mathcal{H}_0]\mathcal{L}^2\mathcal{P}^- g = 0, \tag{6.17}$$

$$(I + i\mathcal{H}_c)\left[(\mathcal{P}^- g)_0 + (\overline{\partial})^{-1}\partial\left(B\mathcal{P}^- g\right)_{-2}|_\Gamma\right] = 0, \tag{6.18}$$

$$\mathbb{I}\mathrm{m}\left\{\partial^2\left(Tg\right)_{-2}\right\} = 0, \quad \textit{in } \mathbb{D}, \tag{6.19}$$

$$\begin{aligned}&\cos\eta\,\partial_\eta\left\{\mathbb{I}\mathrm{m}(\mathcal{P}^- g)_{-1}\right\} - \sin\eta\,\partial_\eta\left\{\mathbb{R}\mathrm{e}(\mathcal{P}^- g)_{-1}\right\} = \\ &\quad \mathbb{R}\mathrm{e}\left\{e^{-2i\eta}(\partial_r - i\partial_\eta)(B\mathcal{P}^- g)_{-3}|_\Gamma\right\}\frac{\cos\eta}{2} + \mathbb{I}\mathrm{m}\left\{e^{-2i\eta}(\partial_r - i\partial_\eta)(B\mathcal{P}^- g)_{-3}|_\Gamma\right\}\frac{\sin\eta}{2},\end{aligned} \tag{6.20}$$

where the operator B is defined in (4.5)*, $\mathcal{H}_0$ is defined in* (4.3)*, $\mathcal{H}_c$ is the Hilbert transform on the circle, $(\overline{\partial})^{-1}$ is defined in* (6.16) *and $\mathcal{P}^-$ as defined in* (4.15)*.*

(ii) Let $g \in C^\alpha\left(\Gamma; C^{1,\alpha}(\mathbf{S}^1)\right) \cap C^0(\Gamma; C^{2,\alpha}(\mathbf{S}^1))$, be real valued satisfying (6.17)*,* (6.18)*,* (6.19) *and* (6.20)*. Then, for each $\psi \in \Psi_g$, there exist a unique real valued symmetric second order tensor field $\mathbf{F}_\psi$ such that g is the Radon data of $\mathbf{F}_\psi$.*

Proof. (i) For simplicity we use the notation $\mathbf{g} = \mathcal{P}^- g := \langle g_0, g_{-1}, \cdots\rangle$. Recall that g is the trace of the solution u of (6.6) and by the equivalence with the system (6.12), (6.13), (6.14), (6.15), its negative Fourier modes u_n satisfy

$$u_n = (B\mathbf{g})_n, \quad n \leq -2,$$

where B as in (4.5).

The equation (6.15) implies that the sequence $\mathcal{L}^2(B\mathbf{g})) = \langle u_{-2}, u_{-3}, \cdots\rangle$ is A-analytic. By the necessity part in Theorem 4.1.2, $[I + i\mathcal{H}_0]\mathcal{L}^2\mathbf{g} = 0$, satisfying (6.17).

From (6.13), we get $\overline{\partial} u_0(z) = -\partial u_{-2}(z),\ z \in \mathbb{D}$ and using $(\overline{\partial})^{-1}$ as defined in (6.16), yields

$$u_0(z) = -(\overline{\partial})^{-1}\partial \left(B\mathbf{g}\right)_{-2}(z) + \mathcal{A}(z),$$

where $\mathcal{A}(z)$ is an analytic function. The restriction to the boundary yields that $g_0 + (\overline{\partial})^{-1}\partial \left(B\mathbf{g}\right)_{-2}|_{\Gamma}$ is the trace of an analytic function and thus lies in the kernel $(I + i\mathcal{H}_c)\left[g_0 + (\overline{\partial})^{-1}\partial \left(B\mathbf{g}\right)_{-2}|_{\Gamma}\right] = 0$, satisfying (6.18).

Taking Laplacian of u_0, and using $\triangle \equiv 4\partial\overline{\partial}$, we get

$$\triangle u_0 = -4\partial^2 u_{-2}.$$

Since u_0 is real valued, it follows that $\triangle u_0$ is also real valued, implying $\mathbb{Im}\left\{\partial^2(B\mathbf{g})_{-2}\right\} = 0$ in $\mathbb{D}$, hence satisfying (6.19).

For simplicity we use the notations

$$\begin{aligned} \alpha &= \mathbb{Re}\, u_{-1}, \\ \beta &= \mathbb{Im}\, u_{-1}, \end{aligned} \tag{6.21}$$

$$\begin{aligned} U &= -\,\mathbb{Re}\left\{e^{-2i\eta}(\partial_r - i\partial_\eta)(B\mathbf{g})_{-3}|_{\Gamma}\right\}, \\ V &= -\,\mathbb{Im}\left\{e^{-2i\eta}(\partial_r - i\partial_\eta)(B\mathbf{g})_{-3}|_{\Gamma}\right\}, \end{aligned} \tag{6.22}$$

along with $u_{-1}|_{\Gamma} = g_{-1}$ to prove the condition (6.20).

The restriction of (6.14) to the boundary yields

$$\begin{aligned}
&\overline{\partial} u_{-1}|_{\Gamma}+\partial u_{-3}|_{\Gamma}= f_2|_{\Gamma},\\
&\frac{e^{i\eta}}{2}(\partial_r + i\partial_\eta)(\alpha+i\beta)|_{\Gamma}+\frac{e^{-i\eta}}{2}(\partial_r - i\partial_\eta)u_{-3}|_{\Gamma}= 0, \ \because f_2 \in C_0^1(\overline{\mathbb{D}}),\\
&(\partial_r + i\partial_\eta)(\alpha+i\beta)|_{\Gamma}= -e^{-2i\eta}(\partial_r - i\partial_\eta)(B\mathbf{g})_{-3}|_{\Gamma},\\
&(\partial_r\alpha - \partial_\eta\beta)|_{\Gamma}+i(\partial_\eta\alpha + \partial_r\beta)|_{\Gamma}= U+iV.
\end{aligned} \tag{6.23}$$

From the above equation (6.23), we get

$$\begin{aligned}
\partial_r\alpha|_{\Gamma} &= U + \partial_\eta\beta|_{\Gamma},\\
\partial_r\beta|_{\Gamma} &= V - \partial_\eta\alpha|_{\Gamma},
\end{aligned} \tag{6.24}$$

where α, β defined in (6.21) and U, V defined in (6.22).

The restriction of (6.12) to the boundary yields

$$\begin{aligned}
&2\,\mathbb{R}\mathrm{e}(\partial u_{-1})|_{\Gamma}= f_0|_{\Gamma},\\
&2\,\mathbb{R}\mathrm{e}\left\{\frac{e^{-i\eta}}{2}(\partial_r - i\partial_\eta)(\alpha+i\beta)|_{\Gamma}\right\} = 0, \ \because f_0 \in C_0^1(\overline{\mathbb{D}}),\\
&\mathbb{R}\mathrm{e}\left\{(\cos\eta - i\sin\eta)(\partial_r - i\partial_\eta)(\alpha+i\beta)|_{\Gamma}\right\} = 0,\\
&\mathbb{R}\mathrm{e}\left\{(\cos\eta - i\sin\eta)\left((\partial_r\alpha+\partial_\eta\beta)|_{\Gamma}+i(\partial_r\beta-\partial_\eta\alpha)\right)|_{\Gamma}\right\} = 0,\\
&\cos\eta(\partial_r\alpha+\partial_\eta\beta)|_{\Gamma}+\sin\eta(\partial_r\beta-\partial_\eta\alpha)|_{\Gamma}= 0.
\end{aligned} \tag{6.25}$$

From the above equation (6.25), we get

$$\cos\eta\,\partial_r\alpha|_{\Gamma}+\sin\eta\,\partial_r\beta|_{\Gamma} = \sin\eta\partial_\eta\alpha|_{\Gamma}-\cos\eta\partial_\eta\beta|_{\Gamma}. \tag{6.26}$$

Plugging $\partial_r \alpha$ and $\partial_r \beta$ from (6.24) into (6.26) yields

$$\begin{aligned}
&\cos\eta(U + \partial_\eta \beta|_\Gamma) + \sin\eta(V - \partial_\eta \alpha|_\Gamma) = \sin\eta\partial_\eta\alpha|_\Gamma - \cos\eta\partial_\eta\beta|_\Gamma,\\
&U\cos\eta + \cos\eta\partial_\eta\beta|_\Gamma + V\sin\eta - \sin\eta\partial_\eta\alpha|_\Gamma = \sin\eta\partial_\eta\alpha|_\Gamma - \cos\eta\partial_\eta\beta|_\Gamma,\\
&\cos\eta\partial_\eta\beta|_\Gamma - \sin\eta\partial_\eta\alpha|_\Gamma = -\frac{\cos\eta}{2}U - \frac{\sin\eta}{2}V,\\
&\cos\eta\partial_\eta \operatorname{Im}(g_{-1}) - \sin\eta\partial_\eta \operatorname{Re}(g_{-1}) = -\frac{\cos\eta}{2}U - \frac{\sin\eta}{2}V.
\end{aligned}$$

Plugging the expressions (6.22) for U and V yields the condition (6.20)

$$\begin{aligned}
&\cos\eta\partial_\eta \operatorname{Im}(\mathcal{P}^- g)_{-1} - \sin\eta\partial_\eta \operatorname{Re}(\mathcal{P}^- g)_{-1} =\\
&\frac{\cos(\eta)}{2}\operatorname{Re}\left\{e^{-2i\eta}(\partial_r - i\partial_\eta)(B\mathbf{g})_{-3}|_\Gamma\right\} + \frac{\sin(\eta)}{2}\operatorname{Im}\left\{e^{-2i\eta}(\partial_r - i\partial_\eta)(B\mathbf{g})_{-3}|_\Gamma\right\}.
\end{aligned}$$

This proves the necessity part of the theorem.

Conversely, assume that we have $g \in C^\alpha(\Gamma; C^{1,\alpha}(\mathbf{S}^1)) \cap C^0(\Gamma; C^{2,\alpha}(\mathbf{S}^1))$, satisfying (6.17), (6.18), (6.19), and (6.20).

Let $\mathbf{g} = \mathcal{P}^- g := \langle g_0, g_{-1}, \cdots \rangle$. and define $u_n = (B\mathbf{g})_n$, for $n \leq -2$, where B is as in (4.5). By Theorem 4.1.2, (6.17) implies $\overline{\partial} u_n + \partial u_{n-2} = 0$, for $n \leq -2$.

The condition (6.18) implies that $g_0 + (\overline{\partial})^{-1}\partial\,(B\mathbf{g})_{-2}|_\Gamma$ is the trace of an analytic function, where $(\overline{\partial})^{-1}$ is defined in (6.16).

For any analytic function $\mathcal{A}$, define

$$u_0 := -(\overline{\partial})^{-1}\partial\,(B\mathbf{g})_{-2} + \mathcal{A},$$

which taking $\overline{\partial}$ will satisfy $\overline{\partial} u_0 + \partial u_{-2} = 0$.

Claim: Among these u_0's (indexed) by $\mathcal{A}$ there exits a unique u_0 such that $u_0|_\Gamma = g_0$.

Taking $4\bar{\partial}\partial \equiv \triangle$ of u_0, yields

$$\triangle u_0 = -4\partial^2(B\mathbf{g})_{-2},$$
$$u_0|_\Gamma = g_0.$$

This boundary value Poisson equation has a unique solution, see [27]. Furthermore the condition (6.19), implies u_0 is real valued.

Let $\psi \in \Psi_g$, then $\psi|_\Gamma = g_{-1}$ and

$$\partial_r\left(\mathbb{R}\mathrm{e}\,\psi|_\Gamma\right) = -\,\mathbb{R}\mathrm{e}\left(e^{-2i\eta}(\partial_r - i\partial_\eta)(B\mathbf{g})_{-3}|_\Gamma\right) + \partial_\eta\,\mathbb{I}\mathrm{m}\,g_{-1}, \tag{6.27}$$
$$\partial_r\left(\mathbb{I}\mathrm{m}\,\psi|_\Gamma\right) = -\,\mathbb{I}\mathrm{m}\left(e^{-2i\eta}(\partial_r - i\partial_\eta)(B\mathbf{g})_{-3}|_\Gamma\right) - \partial_\eta\,\mathbb{R}\mathrm{e}\,g_{-1}. \tag{6.28}$$

For simplicity we use the notations

$$\alpha = \mathbb{R}\mathrm{e}\,\psi, \qquad \beta = \mathbb{I}\mathrm{m}\,\psi, \tag{6.29}$$

$$U = -\,\mathbb{R}\mathrm{e}\left\{e^{-2i\eta}(\partial_r - i\partial_\eta)(B\mathbf{g})_{-3}|_\Gamma\right\}, \qquad V = -\,\mathbb{I}\mathrm{m}\left\{e^{-2i\eta}(\partial_r - i\partial_\eta)(B\mathbf{g})_{-3}|_\Gamma\right\}. \tag{6.30}$$

Using the above notations the equations (6.27) and (6.28) can be rewritten as

$$\partial_r\alpha|_\Gamma = U + \partial_\eta\beta|_\Gamma, \qquad \partial_r\beta|_\Gamma = V - \partial_\eta\alpha|_\Gamma, \tag{6.31}$$

and (6.19) as

$$2\cos\eta\partial_\eta\beta|_\Gamma - 2\sin\eta\partial_\eta\alpha|_\Gamma + U\cos\eta + V\sin\eta = 0. \tag{6.32}$$

where α, β defined in (6.29) and U, V defined in (6.30).

Define $f_0 = 2\,\mathbb{R}\mathrm{e}(\partial\psi)$. As $\psi \in \Psi_g$, $f_0 \in C(\overline{\mathbb{D}})$, and its trace satisfies

$$\begin{aligned}
f_0|_\Gamma &= 2\,\mathbb{R}\mathrm{e}(\partial\psi)|_\Gamma,\\
&= 2\,\mathbb{R}\mathrm{e}\left\{\frac{e^{-i\eta}}{2}(\partial_r - i\partial_\eta)(\mathbb{R}\mathrm{e}\,\psi + i\,\mathbb{I}\mathrm{m}\,\psi)|_\Gamma\right\},\\
&= \mathbb{R}\mathrm{e}\left\{(\cos\eta - i\sin\eta)(\partial_r - i\partial_\eta)(\alpha + i\beta)|_\Gamma\right\},\\
&= \mathbb{R}\mathrm{e}\left\{(\cos\eta - i\sin\eta)\left((\partial_r\alpha + \partial_\eta\beta)|_\Gamma + i(\partial_r\beta - \partial_\eta\alpha)\right)|_\Gamma\right\},\\
&= \cos\eta(\partial_r\alpha + \partial_\eta\beta)|_\Gamma + \sin\eta(\partial_r\beta - \partial_\eta\alpha)|_\Gamma,\\
&= \cos\eta(U + 2\partial_\eta\beta|_\Gamma) + \sin\eta(V - 2\partial_\eta\alpha|_\Gamma),\\
&= 0, \qquad \text{from (6.32).}
\end{aligned}$$

From the above equation $f_0|_\Gamma = 0$, implying $f_0 \in C_0(\overline{\mathbb{D}})$.

Define $f_2 = \overline{\partial}\psi + \partial(B\mathbf{g})_{-3}$. By the definition of the class Ψ_g, and the Theorem 4.1.1, $f_2 \in C(\mathbb{D})$, and its trace satisfies

$$\begin{aligned}
f_2|_{\varGamma} &= \overline{\partial}\psi|_{\varGamma}+\partial(B\mathbf{g})_{-3}|_{\varGamma},\\
f_2|_{\varGamma} &= \frac{e^{i\eta}}{2}(\partial_r + i\partial_\eta)(\alpha+i\beta)|_{\varGamma}+\frac{e^{-i\eta}}{2}(\partial_r - i\partial_\eta)(B\mathbf{g})_{-3}|_{\varGamma},\\
2e^{-i\eta}f_2|_{\varGamma} &= (\partial_r + i\partial_\eta)(\alpha+i\beta)|_{\varGamma}+e^{-2i\eta}(\partial_r - i\partial_\eta)(B\mathbf{g})_{-3}|_{\varGamma},\\
2e^{-i\eta}f_2|_{\varGamma} &= (\partial_r + i\partial_\eta)(\alpha+i\beta)|_{\varGamma}-(U+iV),\\
2e^{-i\eta}f_2|_{\varGamma} &= ((\partial_r\alpha - \partial_\eta\beta)|_{\varGamma}+i(\partial_\eta\alpha+\partial_r\beta)|_{\varGamma})-(U+iV),\\
2e^{-i\eta}f_2|_{\varGamma} &= (\partial_r\alpha - \partial_\eta\beta)|_{\varGamma}-U+i((\partial_\eta\alpha+\partial_r\beta)|_{\varGamma}-V),\\
2e^{-i\eta}f_2|_{\varGamma} &= 0, \qquad \text{from (6.31).}
\end{aligned}$$

From the above equation $f_2|_{\varGamma}=0$, implying $f_2 \in C_0(\overline{\mathbb{D}})$.

Define the real valued symmetric second order 2D tensor field

$$\mathbf{F} = \begin{pmatrix} 4\,\mathbb{R}\mathrm{e}\{f_2\} & 2\,\mathbb{I}\mathrm{m}\{f_2\} \\ 2\,\mathbb{I}\mathrm{m}\{f_2\} & -4\,\mathbb{R}\mathrm{e}\{f_2\}+2f_0 \end{pmatrix},$$

or

$$\mathbf{F} = \begin{pmatrix} 4\,\mathbb{R}\mathrm{e}\left(\overline{\partial}\psi+\partial(B\mathbf{g})_{-3}\right) & 2\,\mathbb{I}\mathrm{m}\left(\overline{\partial}\psi+\partial(B\mathbf{g})_{-3}\right) \\ 2\,\mathbb{I}\mathrm{m}\left(\overline{\partial}\psi+\partial(B\mathbf{g})_{-3}\right) & 4\,\mathbb{R}\mathrm{e}\left(\partial\psi-\overline{\partial}\psi-\partial(B\mathbf{g})_{-3}\right) \end{pmatrix},$$

and let

$$u(z,\theta) := u_0(z) + 2\,\mathbb{R}\mathrm{e}\left\{\psi(z)e^{-i\varphi}\right\} + 2\,\mathbb{R}\mathrm{e}\left\{\sum_{n=2}^{\infty} u_{-n}(z)e^{-in\varphi}\right\}.$$

By the one to one correspondence between (6.12), (6.13), (6.14), (6.15) and the boundary value problem (6.6), we have that u solves

$$\theta \cdot \nabla u = \theta^T \mathbf{F}_\psi \theta,$$

$$u|_\Gamma = g,$$

i.e., g is the Radon data of $\mathbf{F}_\psi$. □

6.2 Range Characterization of the attenuated Radon Transform of real valued symmetric second order tensor

We now consider the attenuated case.

Let $a > 0$, $a \in C_0^2(\overline{\mathbb{D}})$ be real valued, $\alpha > 1/2$.

To address the non-uniqueness in the characterization of the attenuated Radon transform of real valued symmetric second order tensor we introduce the class of functions Ψ_g^a with prescribed trace and gradient on the boundary Γ as

$$\Psi_g^a := \Big\{ \psi \in C^1(\overline{\mathbb{D}}; \mathbb{C}) : \psi|_\Gamma = (\mathcal{P}^- g)_{-1}, \mathbb{Im}\left(\partial(a\psi) + \partial^2 (Tg)_{-2} \right) = 0,$$
$$\partial_r \left(\mathbb{Re}\, \psi|_\Gamma \right) = -\mathbb{Re}\left(e^{-2i\eta} (\partial_r - i\partial_\eta)(Tg)_{-3}|_\Gamma \right) + \partial_\eta \mathbb{Im}(\mathcal{P}^- g)_{-1},$$
$$\partial_r \left(\mathbb{Im}\, \psi|_\Gamma \right) = -\mathbb{Im}\left(e^{-2i\eta} (\partial_r - i\partial_\eta)(Tg)_{-3}|_\Gamma \right) - \partial_\eta \mathbb{Re}(\mathcal{P}^- g)_{-1} \Big\},$$

where the operator T is defined in (5.10) and $\mathcal{P}^-$ as defined in (4.15).

Theorem 6.2.1 (Range characterization for the 2-tensor (Attenuated))**.** *Let $a > 0$, $a \in C_0^2(\overline{\mathbb{D}})$ be real valued, $\alpha > 1/2$. (i) For $\mathbf{F} \in C_0^{1,\alpha}(\overline{\mathbb{D}}; \mathbb{R}^{2\times 2})$, let g be the attenuated Radon data of $\mathbf{F}$. Then*

$g \in C^{\alpha}(\Gamma; C^{1,\alpha}(\mathbf{S}^1))$ *satisfies*

$$[I + i\mathcal{H}_a]\mathcal{L}^2\mathcal{P}^- g = 0, \tag{6.33}$$

$$(I + i\mathcal{H}_c)\left[(\mathcal{P}^- g)_0 + (\overline{\partial})^{-1}\partial\,(Tg)_{-2}\,|_{\varGamma}\right] = 0, \tag{6.34}$$

$$\begin{aligned} &\cos\eta\,\partial_\eta\,\mathbb{I}\mathrm{m}(\mathcal{P}^- g)_{-1} - \sin\eta\,\partial_\eta\,\mathbb{R}\mathrm{e}(\mathcal{P}^- g)_{-1} = \\ &\quad \mathbb{R}\mathrm{e}\left\{e^{-2i\eta}(\partial_r - i\partial_\eta)(Tg)_{-3}|_{\varGamma}\right\}\frac{\cos\eta}{2} + \mathbb{I}\mathrm{m}\left\{e^{-2i\eta}(\partial_r - i\partial_\eta)(Tg)_{-3}|_{\varGamma}\right\}\frac{\sin\eta}{2}, \end{aligned} \tag{6.35}$$

where the operator T is defined in (5.10)*, $\mathcal{H}_a$ is as defined in* (4.2.1)*, $\mathcal{H}_c$ is the Hilbert transform on the circle, $(\overline{\partial})^{-1}$ is defined in* (6.16) *and $\mathcal{P}^-$ as defined in* (4.15)*.*

(ii) Let $g \in C^{\alpha}\left(\varGamma; C^{1,\alpha}(\mathbf{S}^1)\right) \cap C^0(\varGamma; C^{2,\alpha}(\mathbf{S}^1))$, be real valued satisfying (6.33)*,* (6.34)*, and* (6.35)*. Then, for each $\psi \in \Psi_g^a$, there exist a unique real valued symmetric second order tensor field $\mathbf{F}_\psi$ such that g is the attenuated Radon data of $\mathbf{F}_\psi$.*

Proof. (i) Let g be the attenuated Radon data of $\mathbf{F} \in C_0^{1,\alpha}(\overline{\mathbb{D}}; \mathbb{R}^{2\times 2})$. Recall that g is the trace of the solution u of the Transport equation (6.6) and by the equivalence with the system (6.8), (6.9), (6.10), (6.11), its negative Fourier modes u_n satisfy

$$u_n = (Tg)_n, \quad n \le -2,$$

where T as in (5.10). The Proposition 5.0.4, implies $\overline{\partial}u_n + \partial u_{n-2} + au_{n-1} = 0$, for $n \le -2$ and the necessity part in Theorem 4.2.1, yields the condition (6.33).

From (6.9), we get $\overline{\partial}u_0(z) = -\partial u_{-2}(z) - au_{-1}(z),\ z \in \mathbb{D}$ and using $(\overline{\partial})^{-1}$ as defined in (6.16), yields

$$u_0(z) = -(\overline{\partial})^{-1}\partial\,(Tg)_{-2}\,(z) - (\overline{\partial})^{-1}\left(a(z)u_{-1}(z)\right) + \mathcal{A}(z),$$

where $\mathcal{A}(z)$ is an analytic function. The restriction to the boundary yields that $g_0+(\overline{\partial})^{-1}\partial\,(Tg)_{-2}\,|_\Gamma$ is the trace of an analytic function and thus lies in the kernel $(I+i\mathcal{H}_c)\left[g_0+(\overline{\partial})^{-1}\partial\,(Tg)_{-2}\,|_\Gamma\right]=0$, satisfying (6.34).

For simplicity we use the notations

$$\begin{aligned}\alpha &= \mathbb{R}\mathrm{e}\,u_{-1},\\ \beta &= \mathbb{I}\mathrm{m}\,u_{-1},\end{aligned} \tag{6.36}$$

$$\begin{aligned}U &= -\,\mathbb{R}\mathrm{e}\left\{e^{-2i\eta}(\partial_r - i\partial_\eta)(Tg)_{-3}|_\Gamma\right\},\\ V &= -\,\mathbb{I}\mathrm{m}\left\{e^{-2i\eta}(\partial_r - i\partial_\eta)(Tg)_{-3}|_\Gamma\right\},\end{aligned} \tag{6.37}$$

along with $u_{-1}|_\Gamma = g_{-1}$ to prove the condition (6.35).

The restriction of (6.10) to the boundary yields

$$\begin{aligned}&\overline{\partial}u_{-1}|_\Gamma + \partial u_{-3}|_\Gamma + a u_{-2}|_\Gamma = f_2|_\Gamma, \quad \because a \in C_0^2(\overline{\mathbb{D}})\\ &\frac{e^{i\eta}}{2}(\partial_r + i\partial_\eta)(\alpha + i\beta)|_\Gamma + \frac{e^{-i\eta}}{2}(\partial_r - i\partial_\eta)u_{-3}|_\Gamma = 0, \ \because f_2 \in C_0^1(\overline{\mathbb{D}})\\ &(\partial_r + i\partial_\eta)(\alpha + i\beta)|_\Gamma = -e^{-2i\eta}(\partial_r - i\partial_\eta)(Tg)_{-3}|_\Gamma,\\ &(\partial_r\alpha - \partial_\eta\beta)|_\Gamma + i(\partial_\eta\alpha + \partial_r\beta)|_\Gamma = U + iV.\end{aligned} \tag{6.38}$$

From the above equation (6.38), we get

$$\begin{aligned}\partial_r\alpha|_\Gamma &= U + \partial_\eta\beta|_\Gamma,\\ \partial_r\beta|_\Gamma &= V - \partial_\eta\alpha|_\Gamma,\end{aligned} \tag{6.39}$$

where α, β defined in (6.36) and U, V defined in (6.37). The restriction of (6.8) to the boundary yields

$$
\begin{aligned}
&2\,\mathbb{R}\mathrm{e}(\partial u_{-1})|_{\Gamma} + a u_0|_{\Gamma} = f_0|_{\Gamma}, \quad \because a \in C_0^2(\overline{\mathbb{D}}),\\
&\mathbb{R}\mathrm{e}(\partial u_{-1})|_{\Gamma} = 0, \ \because f_0 \in C_0^1(\overline{\mathbb{D}}),\\
&\mathbb{R}\mathrm{e}\left\{\frac{e^{-i\eta}}{2}(\partial_r - i\partial_\eta)(\alpha + i\beta)|_{\Gamma}\right\} = 0,\\
&\mathbb{R}\mathrm{e}\left\{(\cos\eta - i\sin\eta)(\partial_r - i\partial_\eta)(\alpha + i\beta)|_{\Gamma}\right\} = 0,\\
&\mathbb{R}\mathrm{e}\left\{(\cos\eta - i\sin\eta)\left((\partial_r\alpha + \partial_\eta\beta)|_{\Gamma} + i(\partial_r\beta - \partial_\eta\alpha)\right)|_{\Gamma}\right\} = 0,\\
&\cos\eta(\partial_r\alpha + \partial_\eta\beta)|_{\Gamma} + \sin\eta(\partial_r\beta - \partial_\eta\alpha)|_{\Gamma} = 0. \qquad (6.40)
\end{aligned}
$$

From the above equation (6.40), we get

$$
\cos\eta\,\partial_r\alpha|_{\Gamma} + \sin\eta\,\partial_r\beta|_{\Gamma} = \sin\eta\partial_\eta\alpha|_{\Gamma} - \cos\eta\partial_\eta\beta|_{\Gamma}. \qquad (6.41)
$$

Plugging $\partial_r\alpha$ and $\partial_r\beta$ from (6.39) into (6.41) yields

$$
\begin{aligned}
&\cos\eta(U + \partial_\eta\beta|_{\Gamma}) + \sin\eta(V - \partial_\eta\alpha|_{\Gamma}) = \sin\eta\partial_\eta\alpha|_{\Gamma} - \cos\eta\partial_\eta\beta|_{\Gamma},\\
&U\cos\eta + \cos\eta\partial_\eta\beta|_{\Gamma} + V\sin\eta - \sin\eta\partial_\eta\alpha|_{\Gamma} = \sin\eta\partial_\eta\alpha|_{\Gamma} - \cos\eta\partial_\eta\beta|_{\Gamma},\\
&\cos\eta\partial_\eta\beta|_{\Gamma} - \sin\eta\partial_\eta\alpha|_{\Gamma} = -\frac{\cos\eta}{2}U - \frac{\sin\eta}{2}V,\\
&\cos\eta\partial_\eta\,\mathbb{I}\mathrm{m}(\mathcal{P}^- g)_{-1} - \sin\eta\partial_\eta\,\mathbb{R}\mathrm{e}(\mathcal{P}^- g)_{-1} = -\frac{\cos\eta}{2}U - \frac{\sin\eta}{2}V.
\end{aligned}
$$

Plugging the expressions (6.37) for U and V yields the condition (6.35)

$$
\begin{aligned}
&\cos\eta\,\partial_\eta\,\mathbb{I}\mathrm{m}(\mathcal{P}^- g)_{-1} - \sin\eta\,\partial_\eta\,\mathbb{R}\mathrm{e}(\mathcal{P}^- g)_{-1} =\\
&\quad \mathbb{R}\mathrm{e}\left\{e^{-2i\eta}(\partial_r - i\partial_\eta)(Tg)_{-3}|_{\Gamma}\right\}\frac{\cos\eta}{2} + \mathbb{I}\mathrm{m}\left\{e^{-2i\eta}(\partial_r - i\partial_\eta)(Tg)_{-3}|_{\Gamma}\right\}\frac{\sin\eta}{2}.
\end{aligned}
$$

This proves the necessity part of the theorem.

Conversely, assume $g \in C^{\alpha}\left(\varGamma; C^{1,\alpha}(\mathbf{S}^1)\right) \cap C^0(\varGamma; C^{2,\alpha}(\mathbf{S}^1))$, satisfying (6.33), (6.34), and (6.35). Define $u_n = (Tg)_n$ for $n \leq -2$, where T is as in (5.10). By Proposition 5.0.4, (6.33) implies $\overline{\partial} u_n + \partial u_{n-2} + a u_{n-1} = 0$, for $n \leq -2$.

Let $\psi \in \Psi_g^a$, then $\psi|_{\varGamma} = (\mathcal{P}^- g)_{-1}$, $\partial(a\psi) + \partial^2 (Tg)_{-2}$ is real valued, and

$$\partial_r \left(\mathbb{R}\mathrm{e}\, \psi|_{\varGamma}\right) = -\,\mathbb{R}\mathrm{e}\left(e^{-2i\eta}(\partial_r - i\partial_\eta)(Tg)_{-3}|_{\varGamma}\right) + \partial_\eta\, \mathbb{I}\mathrm{m}(\mathcal{P}^- g)_{-1}, \tag{6.42}$$

$$\partial_r \left(\mathbb{I}\mathrm{m}\, \psi|_{\varGamma}\right) = -\,\mathbb{I}\mathrm{m}\left(e^{-2i\eta}(\partial_r - i\partial_\eta)(Tg)_{-3}|_{\varGamma}\right) - \partial_\eta\, \mathbb{R}\mathrm{e}(\mathcal{P}^- g)_{-1}. \tag{6.43}$$

The condition (6.34) implies that $g_0 + (\overline{\partial})^{-1} \partial \left(Tg\right)_{-2}|_{\varGamma}$ is the trace of an analytic function, where $(\overline{\partial})^{-1}$ is defined in (6.16).

For any analytic function $\mathcal{A}$, define

$$u_0 := -(\overline{\partial})^{-1} \partial (Tg)_{-2} - (\overline{\partial})^{-1}(a\psi) + \mathcal{A},$$

which taking $\overline{\partial}$ will satisfy (6.9).

Claim: Among these u_0's (indexed) by $\mathcal{A}$ there exits a unique u_0 such that $u_0|_{\varGamma} = g_0$.

Taking $\triangle \equiv 4\overline{\partial}\partial$ of u_0, yields

$$\triangle u_0 = -4\partial(a\psi) - 4\partial^2 (Tg)_{-2},$$

$$u_0|_{\varGamma} = g_0.$$

This boundary value Poisson equation has a unique solution, see [27]. Furthermore the condition $\mathbb{I}\mathrm{m}\left(\partial(a\psi) + \partial^2 (Tg)_{-2}\right) = 0$, implies u_0 is real valued.

For simplicity we use the notations

$$\begin{aligned} \alpha &= \mathbb{R}\mathrm{e}\,\psi, \\ \beta &= \mathbb{I}\mathrm{m}\,\psi, \end{aligned} \tag{6.44}$$

$$\begin{aligned} U &= -\,\mathbb{R}\mathrm{e}\left\{e^{-2i\eta}(\partial_r - i\partial_\eta)(Tg)_{-3}|_\Gamma\right\}, \\ V &= -\,\mathbb{I}\mathrm{m}\left\{e^{-2i\eta}(\partial_r - i\partial_\eta)(Tg)_{-3}|_\Gamma\right\}. \end{aligned} \tag{6.45}$$

Using the above notations the equations (6.42) and (6.43) can be rewritten as

$$\begin{aligned} \partial_r\alpha|_\Gamma &= U + \partial_\eta\beta|_\Gamma, \\ \partial_r\beta|_\Gamma &= V - \partial_\eta\alpha|_\Gamma, \end{aligned} \tag{6.46}$$

and (6.35) as

$$2\cos\eta\partial_\eta\beta|_\Gamma - 2\sin\eta\partial_\eta\alpha|_\Gamma + U\cos\eta + V\sin\eta = 0, \tag{6.47}$$

where α, β defined in (6.44) and U, V defined in (6.45).

Define $f_0 = 2\,\mathbb{R}\mathrm{e}(\partial\psi) + au_0$.

By Proposition 5.0.4, and $\psi \in C^1(\overline{\mathbb{D}};\mathbb{C})$, it follows that $f_0 \in C(\mathbb{D})$, and its trace satisfies

$$
\begin{aligned}
f_0|_\Gamma &= 2\,\mathbb{R}\mathrm{e}(\partial\psi)|_\Gamma, \quad \because a \in C_0^2(\overline{\mathbb{D}}),\\
&= 2\,\mathbb{R}\mathrm{e}\left\{\frac{e^{-i\eta}}{2}(\partial_r - i\partial_\eta)(\alpha+i\beta)|_\Gamma\right\},\\
&= \mathbb{R}\mathrm{e}\left\{(\cos\eta - i\sin\eta)(\partial_r - i\partial_\eta)(\alpha+i\beta)|_\Gamma\right\},\\
&= \mathbb{R}\mathrm{e}\left\{(\cos\eta - i\sin\eta)\left((\alpha_r+\beta_\eta)|_\Gamma + i(\beta_r-\alpha_\eta)\right)|_\Gamma\right\},\\
&= \cos\eta(\partial_r\alpha+\partial_\eta\beta)|_\Gamma + \sin\eta(\partial_r\beta-\partial_\eta\alpha)|_\Gamma, \qquad \text{from (6.46)},\\
&= \cos\eta(U+2\partial_\eta\beta|_\Gamma) + \sin\eta(V-2\partial_\eta\alpha|_\Gamma),\\
&= 0, \qquad \text{from (6.47)}.
\end{aligned}
$$

From the above equation $f_0|_\Gamma = 0$, implying $f_0 \in C_0(\overline{\mathbb{D}})$.

Define $f_2 = \overline{\partial}\psi + a(Tg)_{-2} + \partial(Tg)_{-3}$. By the Proposition 5.0.4, $f_2 \in C(\mathbb{D})$, and its trace satisfies

$$
\begin{aligned}
f_2|_\Gamma &= \overline{\partial}\psi|_\Gamma + \partial(Tg)_{-3}|_\Gamma, \quad \because a \in C_0^2(\overline{\mathbb{D}}),\\
f_2|_\Gamma &= \frac{e^{i\eta}}{2}(\partial_r + i\partial_\eta)(\alpha+i\beta)|_\Gamma + \frac{e^{-i\eta}}{2}(\partial_r - i\partial_\eta)(Tg)_{-3}|_\Gamma,\\
2e^{-i\eta}f_2|_\Gamma &= (\partial_r + i\partial_\eta)(\alpha+i\beta)|_\Gamma + e^{-2i\eta}(\partial_r - i\partial_\eta)(Tg)_{-3}|_\Gamma,\\
2e^{-i\eta}f_2|_\Gamma &= (\partial_r + i\partial_\eta)(\alpha+i\beta)|_\Gamma - (U+iV),\\
2e^{-i\eta}f_2|_\Gamma &= ((\partial_r\alpha - \partial_\eta\beta)|_\Gamma + i(\partial_\eta\alpha + \partial_r\beta)|_\Gamma) - (U+iV),\\
2e^{-i\eta}f_2|_\Gamma &= (\partial_r\alpha - \partial_\eta\beta)|_\Gamma - U + i((\partial_\eta\alpha + \partial_r\beta)|_\Gamma - V),\\
2e^{-i\eta}f_2|_\Gamma &= 0, \qquad \text{from (6.46)}.
\end{aligned}
$$

From the above equation $f_2|_\Gamma = 0$, implying $f_2 \in C_0(\overline{\mathbb{D}})$.

Define the real valued symmetric second order 2D tensor field

$$\mathbf{F} = \begin{pmatrix} 4\,\mathbb{R}\mathrm{e}\,\{f_2\} & 2\,\mathbb{I}\mathrm{m}\,\{f_2\} \\ 2\,\mathbb{I}\mathrm{m}\,\{f_2\} & -4\,\mathbb{R}\mathrm{e}\,\{f_2\} + 2f_0 \end{pmatrix},$$

and let

$$u(z,\theta) := u_0(z) + 2\,\mathbb{R}\mathrm{e}\left\{\psi(z)e^{-i\varphi}\right\} + 2\,\mathbb{R}\mathrm{e}\left\{\sum_{n=2}^{\infty} u_{-n}(z)e^{-in\varphi}\right\}.$$

By the one to one correspondence between (6.8), (6.9), (6.10) and (6.11) and the boundary value problem (6.6), we have that u solves

$$\theta \cdot \nabla u + au = \theta^T \mathbf{F}_\psi \theta,$$

$$u|_\Gamma = g,$$

i.e., g is the attenuated Radon data of $\mathbf{F}_\psi$. □

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