Axiomatization of the away goals rule in football

Takumi Kongo *

Abstract

In this study, we axiomatize the away goals rule extensively used in football (soccer) competitions held in a two-legged tie format. We formalize the rule as a binary relation defined on the two-dimensional natural number vector space. Then, we introduce three axioms: scoreless draw independence, antisymmetry, and away goals weak superiority and demonstrate that the away goals rule uniquely satisfies the three axioms.

1 Introduction

Football (soccer) is one of the most popular sports in the world. Most of the domestic and international football tournaments are held in a round-robin, a knockout, or their concomitantly-used formats. In a knockout tournament, we usually decide on a winner between each pair of football teams in each round, and the winner will advance to the next round. It is rather unfair to determine the winner by only one match that is organized on the home ground of either of the two teams. This is because, in many sports including football, there is well-documented tendency for the home team

* Faculty of Economics, Fukuoka University, Fukuoka, Japan

to win (also known as a home advantage, see e.g., Courneya and Carron 1992, and Nevill and Holder 1999). Bearing in mind that there exists a home advantage, one possible way to fairly decide on a winner between two teams is to organize a match in a neutral location. Another way is that each of the two teams organizes one match on each of their home ground (each match is often called as first leg and second leg, corresponding to their sequence), and a winner between the two teams is determined by the results of first and second legs.

In this two-legged format in football, an away goals rule is often applied to determine a winner. In this rule, first, we compare the numbers of the aggregate goals of each team in two matches, and the team scoring the maximum number of aggregate goals wins the round. When both teams score the same number of aggregate goals, then we compare the numbers of away goals, i.e., goals scored in the match organized on opponent's home ground. In this case, the team scoring the maximum number of away goals is the winner. The away goals rule is extensively used in many football tournaments throughout the world, including the high-visibility ones such as the UEFA Champions League and the qualifying rounds for the FIFA World Cup for some continental zones. Then, a natural question arises: what are the advantages of the away goals rule over other rules for two-legged ties in football?

In this paper, we answer the question using an axiomatic approach. We formalize rules mathematically as binary relations on the space of two-dimensional natural number vectors, introduce some properties (axioms) on rules, and associate the away goals rule with a set of axioms for characterization. More precisely, we demonstrate that the away goals rule is the only rule that satisfies scoreless-draw independence, antisymmetry, and away goals weak superiority. Scoreless-draw independence requires that a match ending in a scoreless draw is irrelevant to the determination of the winner in two-legged ties. Antisymmetry avoids the draw in the two-legged ties as much as possible. Away goals weak superiority treats both teams fairly, even in a match organized on the home ground of either team by giving the away team a slight advantage. Rules other than away goals rule lack a minimum of one of the three axioms. Therefore, we clarify the essence of the away goals rule from a theoretical viewpoint.

The reminder of the paper is organized as follows. Our model is in Section 2. Three axioms are introduced and discussed in Section 3. A characterization of the away goals rule are provided in Section 4. Some remarks are given in Section 5.

2 Model

Consider two football teams playing two matches against each other. For each team, one of the two matches is its home match and the other is its away match (the opponent's home match). A team, which scores $a \in \mathbb{N}$ goals in its home match and $b \in \mathbb{N}$ goals in its away match, is represented by a two-dimensional natural number vector $(a, b) \in \mathbb{N}^2$. Similarly, the opponent team, which scores $c \in \mathbb{N}$ goals at its home match and $d \in \mathbb{N}$ goals at its away match, is represented by $(c, d) \in \mathbb{N}^2$. At this time, two matches between the two teams result in a vs. d (the first team's home/second team's away match) and b vs. c (the first team's away/second team's home match).

Based on (a, b) and (c, d), a **rule** determines the result of the matches from the three outcomes as follows: (i) the first team wins (i.e., the second team loses); (ii) both teams result in a draw; and (iii) the second team wins (i.e., the first team loses). A rule is represented by a binary relation $\succsim$ defined on $\mathbb{N}^2$, where $\mathbb{N}$ denotes the set of natural numbers. We assume that $\succsim$ satisfy completeness, reflexivity, and transitivity. Given $\succsim$, the outcome that (i) the first team wins corresponds to $(a, b) \succ (c, d)$ (i.e., $(a, b) \succsim (c, d)$ and $(c, d) \not\succsim (a, b)$); (ii) both teams result in a draw corresponds to $(a, b) \sim (c, d)$ (i.e., $(a, b) \succsim (c, d)$ and $(c, d) \succsim (a, b)$); and (iii) the second team wins corresponds to $(c, d) \succ (a, b)$ (i.e., $(c, d) \succsim (a, b)$ and $(a, b) \not\succsim (c, d)$).

An **away goals rule** $\succsim^A$ is a rule that, for any $(a, b), (c, d) \in \mathbb{N}^2$, (i) $(a, b) \succ^A (c, d)$ if $a+b > c+d$, or if $a+b = c+d$ and $b > d$, (ii) $(a, b) \sim^A (c, d)$ if $a + b = c + d$ and $b = d$, and (iii) $(c, d) \succ^A (a, b)$ otherwise. In the away goals rule, the team that scores the maximum number of aggregate goals in the two matches wins. If both teams score the same number of aggregate goals in the two matches, then the team scoring the maximum number of goals in its away match wins. Otherwise, both teams result in a draw and go into extra time or a penalty shootout depending on the detailed regulations of each tournament.

3 Axioms

We consider the three axioms on rules, each of which is well represented in the context of football.

Scoreless draw independence: For any $a, b \in \mathbb{N}$, $a > b$ implies $(0, a) \succ (b, 0)$ and $(a, 0) \succ (0, b)$.

As for the winner in one match, the rulebook of football describes as follows: “The team scoring the greater number of goals during a match is the winner” (p.35 of FIFA 2015). Because a two-legged tie consists of two matches, it is natural to consider that a team wins as a whole if it scores the greater numbers of goals during both matches (i.e., the team wins both matches), or the team that scores the greater number of goals in a match and scores the same number of goals as the opponent does in the other match (i.e., the team wins a match and the other match ends in a draw.) The above axiom requires a milder condition of the latter case, implying that it only considers the case in which the draw is scoreless. In other words, a match ending in a scoreless draw is insignificant in terms of determining the winner of the two-legged tie. The axiom requires nothing for the other cases, and thus, it is quite a weak axiom that is satisfied by many rules.

Antisymmetry: For any $(a, b), (c, d) \in \mathbb{N}^2$, $(a, b) \sim (c, d)$ implies $(a, b) = (c, d)$.

By the reflexivity of rules, we cannot decide a winner between two teams scoring the same (i.e., $(a, b) = (c, d)$). However, if the two teams score in different ways even in the slightest terms (i.e., $(a, b) \neq (c, d)$), we should be able to decide a winner, which is the requirement of antisymmetry. By this requirement, we can avoid rematches, extra time, and penalty shootouts when possible. This helps to decrease fatigue and the risk of injury to football players, and thus, it benefits to maintain the quality of matches across a tight schedule.

Away goals weak superiority: For any $a, b \in \mathbb{N}$, $(a, b + 1) \succsim (\mathrm{a} + 1, b)$.

This implies that adding an away goal is at least as valuable as adding a home goal. As mentioned in the Introduction, the home advantage is undeniable. Thus, we maintain fairness by organizing matches at both home and away locations. However, the home advantage that each team enjoys is not always the same.[1] If possible, it is better to maintain fairness in each match, and one possible way is to provide the away team with an advantage. Note that the above property is formalized as a very weak form: not $(a, b + 1) \succ (a + 1, b)$ but $(a, b + 1) \succsim (a + 1, b)$. Thus, many rules satisfy it.

4 Results

Our main results is that the three axioms discussed in the previous section uniquely characterize the away goals rule.

Theorem 1. *The away goals rule is a unique rule that satises scoreless draw independence, antisymmetry, and away goals weak superiority.*

[1] For example, home advantages are affected by the existence, size, and composition of spectators (Garicano et al. 2005 and Pettersson-Lidbom and Priks 2010.)

Proof. It is straightforward that $\succsim^A$ satisfies three axioms. Let $\succsim$ be a complete, reflexive, and transitive binary relation on $\mathbb{N}^2$ that satisfies the three axioms.

First we show the fact that for any $(a, b), (c, d) \in \mathbb{N}^2$, it holds that $(a, b) \succ (c, d)$ if $a + b > c + d$. We prove the fact with a contradiction. Suppose that there exists $(a, b), (c, d) \in \mathbb{N}^2$ such that $a + b > c + d$ and $(c, d) \succsim (a, b)$. Because $a + b > c + d$, antisymmetry implies that $(c, d) \nsim (a, b)$. Thus, it holds that $(c, d) \succ (a, b)$. We consider two cases. We have (i) $b = c = 0$: by supposition, $a > d$ and $(0, d) \succ (a, 0)$. However, this immediately contradicts scoreless draw independence. Moreover, we have (ii) $b \neq 0$ or $c \neq 0$: by away goals weak superiority and transitivity, at least either $(0, c + d) \succ (c, d)$ or $(a, b) \succ (a + b, 0)$ holds. Together with supposition and transitivity, it holds that $(0, c + d) \succ (a + b, 0)$. However, because $a + b > c + d$, this contradicts scoreless draw independence.

Next, we show the fact that $(a, b) \succ (c, d)$ if $a + b = c + d$ and $b > d$. Now, it holds that $c > a$. Let $e = b - d = c - a > 0$, then $(a, b) = (c - e, d + e)$. By away goals weak superiority, it holds that $(a, b) = (c - e, d + e) \succsim (c - (e - 1), \mathrm{d} + (e - 1)) \succsim \cdots \succsim (c, d)$. By transitivity, the fact that $(a, b) \neq (c, d)$, and antisymmetry, we obtain $(a, b) \succ (c, d)$.

If $a + b = c + d$ and $b = d$, reflexivity implies that $(a, b) \sim (c, d)$. We obtain the desired results for the rest of the cases ($a + b < c + d$, and $a + b = c + d$ and $b < d$) similarly as we did above. Therefore, $\succsim = \succsim^A$. □

In the following, we mention that all the three axioms are not redundant in Theorem 1.

A **home goals rule** $\succsim^H$ is a rule that, for any $(a, b), (c, d) \in \mathbb{N}^2$, (i) $(a, b) \succ^H (c, d)$ if $a + b > c + d$ or if $a + b = c + d$ and $a > c$, (ii) $(a, b) \sim^H (c, d)$ if $a + b = c + d$ and $a = c$, (iii) $(c, d) \succ^H (a, b)$ otherwise. This rule satisfies scoreless draw independence and antisymmetry but not away goals weak superiority.

An **aggregate goals rule** $\succsim^{AG}$ is a rule that, for any $(a, b), (c, d) \in \mathbb{N}^2$, (i) (a, b)

$\succ^{AG}$ (c, d) if $a + b > c + d$, (ii) $(a, b) \sim^{AG} (c, d)$ if $a + b = c + d$, (iii) (c, d) $\succ^{AG}$ (a, b) otherwise. This rule satisfies scoreless draw independence and away goals weak superiority but not antisymmetry.

An **away lexicographic rule** $\succsim^{AL}$ is a rule that, for any $(a, b), (c, d) \in \mathbb{N}^2$, (i) $(a, b) \succ^{AL} (c, d)$ if $b > d$ or if $b = d$ and $a > c$, (ii) $(a, b) \sim^{AL} (c, d)$ if $b = d$ and $a = c$, (iii) $(c, d) \succ^{AL} (a, b)$ otherwise. This rule satisfies antisymmetry and away goals weak superiority, but not scoreless draw independence.

5 Concluding remarks

In this paper we characterize the away goals rule by three axioms. The away goals rule is one of the "permitted procedures for determining the winning team" (p.35 of FIFA 2015). Our characterization can rationalize this description theoretically.

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