Hyperlogarithms, Bernoulli polynomials, and related multiple zeta values

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博 士 論 文

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(Hyperlogarithms, Bernoulli多項式, および関連する多重ゼータ値)

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Contents

Multiple zeta(-star) values are real numbers first studied by Leonhard Euler. These numbers have been appeared in various contexts in number theory, geometry, knot theory, mathematical physics and related areas. In 1994, Don Zagier conjectured the dimensions of the vector spaces spanned by the multiple zeta values over $\mathbb{Q}$. This conjecture was partially solved by Tomohide Terasoma, Alexander Goncharov and Pierre Deligne in 2000's. According to this result, there are many relations over $\mathbb{Q}$ among the multiple zeta values. One of the main problems in the theory of multiple zeta values is to clarify all relations among multiple zeta values. Masanobu Kaneko and Shuji Yamamoto recently introduced Integral-Series identity for multiple zeta values and conjectured that the identity deduces all relations among multiple zeta values in [8].

This doctoral thesis has six chapters and we mainly consider the multiple zeta values and the hyperlogarithms. In Chapter 2, we give the Integral-Series identity and the regularization theorem for hyperlogarithms. We introduce our alternative and simpler proofs of three identities for special values of Arakawa-Kaneko and Kaneko-Tsumura multiple zeta functions in Chapter 3. A generalization of Akiyama-Tanigawa's algorithm for Bernoulli numbers to the multi-poly-Bernoulli polynomials is given in Chapter 4. We prove the cyclic sum formulas for finite multiple zeta and zeta-star values in Chapter 5. Finally, we discuss the problem of deducing the duality formula from the extended double shuffle relation in Chapter 6. The details are as follows.

In Chapter 2, we consider the hyperlogarithms and its Integral-Series identity. The Integral-Series identity is expressed by Yamamoto's integral representation ([15]) which is associated with 2-labeled partially ordered set. Hyperlogarithm is a function of one variable which takes multiple zeta value as its special value. We approach linear relations among multiple zeta values through a study of linear relations among the hyperlogarithms. In this chapter, first, we introduce a generalization of Yamamoto's integral representation and present Integral-Series identity for hyperlogarithms. The new identity includes the Integral-Series identity for both multiple zeta values and a kind of Euler sums as its specializations. Next, we give the regularization theorem for hyperlogarithms by using its equivalence to the Integral-Series identity under double shuffle relation. We refer the shuffle and the harmonic products introduced by Minoru Hirose and Nobuo Sato [4].

In Chapter 3, we consider the special values of Arakawa-Kaneko multiple zeta functions via Yamamoto's integral representation ([15]). Tsuneo Arakawa and Kaneko introduced a function which is called Arakawa-Kaneko multiple zeta function as a generalization of the Riemann zeta function by using multiple polylogarithm functions in [2]. This zeta function is related to multiple zeta values and poly-Bernoulli numbers. Kaneko and Hirofumi Tsumura introduced a function which is called Kaneko-Tsumura multiple zeta function, a twin sibling of the Arakawa-Kaneko multiple zeta function. Kaneko and Tsumura gave an interesting expression of special values of these two zeta functions at positive integral points in terms of multiple zeta and zeta-star values, respectively. In this chapter, we give an alternative method to obtain the expression by using Yamamoto's integral representation. We also give an alternative proof of a particular case of Yamamoto's duality formula for Kaneko-Tsumura multiple zeta functions [16]. Both of our proofs are combinatorial and simpler than the original ones. This chapter is based on a joint work with Yasuo Ohno (Tohoku Univ.) [11].

In Chapter 4, we discuss the computational method of various generalizations of Bernoulli numbers and polynomials. Akiyama-Tanigawa's algorithm ([1]) and Chen's algorithm ([3]) are known as simple and easy calculating ways for two types of poly-Bernoulli numbers. The multi-poly-Bernoulli polynomials contain various generalizations of Bernoulli numbers and polynomials including Bernoulli polynomials, two types of poly-Bernoulli numbers, and so on. These numbers and polynomials are related to the special values at negative integral points of both Arakawa-Kaneko and Kaneko-Tsumura multiple zeta functions. In this chapter, we introduce an algorithm for the multi-poly-Bernoulli polynomials. It is a joint work with Yasuo Ohno (Tohoku Univ.) [10].

We introduce the cyclic sum formulas for finite multiple zeta and zeta-star values in Chapter 5. Zagier conjectured the dimensions of the vector spaces spanned by finite multiple zeta values. According to this conjecture, there are many relations over $\mathbb{Q}$ among the finite multiple zeta values more than those among the multiple zeta values. One of the main problems in the theory of finite multiple zeta values is to construct analogous relations of classical relations among multiple zeta values. In this chapter, we prove the cyclic sum formula for finite multiple zeta(-star) values. The cyclic sum formula is a clean-cut decomposition of the sum formula, which is one of the most important and famous relations for multiple zeta(-star) values ([5, 14]). This chapter is based on a joint work with Kojiro Oyama [12]. Moreover, we give the cyclic sum formula for $\widehat{\mathcal{A}}$-finite multiple zeta(-star) values or equivalently the cyclic sum formula for $\mathcal{A}_n$-finite multiple zeta(-star) values.

In the final chapter, Chapter 6, we consider the problem of deducing the duality formula from the extended double shuffle relation. This famous problem comes from the conjecture which claims that the extended double shuffle relations give all relations among the multiple zeta values. Jun Kajikawa showed that the duality formula for the sum of multiple zeta values with fixed weight, depth, and height is deduced from the extended double shuffle relation. In this chapter, we show that the duality formula for two cases are deduced from the extended double shuffle relation. The first case is for each double zeta value, thus it is strong enough result for double zeta case. The second case is for the sum of certain type of multiple zeta values. This chapter is a reconstruction and extended version of the half part of the author's master's thesis (Kyoto Sangyo Univ. 2016), and it is a joint work with Tatsushi Tanaka [13].

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