Tropical curves corresponding to certain singular curves on toric surfaces

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博　士　論　文

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(トーリック曲面上のある特異曲線に対応するトロピカル曲線について)

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Abstract

A degeneration of a singular curve on a toric surface, called a tropicalization, was constructed by E. Shustin. He classified the degeneration of 1-cuspidal curves using polyhedral complexes called tropical curves. In this thesis, we define a tropical version of a 1-tacnodal curve, that is, a curve having exactly one singular point whose topological type is A_3, and classify tropical curves which correspond to 1-tacnodal curves by applying the tropicalization method.

Contents

Introduction

Tropical geometry is a field of combinatorial algebraic geometry developing in recent years. The objects treated in this geometry are polyhedral complexes, which can be obtained as the non-linear locus of a polynomial over the tropical algebra. Here, the tropical algebra is an algebraic system $(\mathbb{R}; \oplus, \odot)$ where the addition $\oplus$ is max and the multiplication $\odot$ is $+$. Tropical geometry was introduced in a paper of G. Mikhalkin [13] and used to describe pair-of-pants decompositions of smooth complex algebraic hypersurfaces.

Among known results on tropical geometry, the most famous result is an application to the enumeration problem on toric surfaces by Mikhalkin [14]. He focused on nodal curves on toric surfaces and solved the following enumeration problem of nodal curves, which was proposed by physicists in the study of mirror symmetry:

> *For a lattice polygon $\Delta \subset \mathbb{R}^2$ and $\delta \in \mathbb{Z}_{\geq 0}$, how many δ-nodal curves on the toric surface $X(\Delta)$ which are contained in the complete linear system $|D(\Delta)|$ and pass through $r(\delta, \Delta)$-points lying in general position do there exist?*

Here, $(X(\Delta), D(\Delta))$ is the polarized toric surface associated with Δ and $r(\delta, \Delta) = \sharp(\Delta \cap \mathbb{Z}^2) - 1 - \delta$. For the projective plane $\mathbb{CP}^2$, the enumeration problem for rational curves was studied by M. Kontsevich [11] using the theory of quantum cohomology. In case of any geometric genus, L. Caporaso and J. Harris [2] proved a recursive formula on the enumerative number using intersection theory on Severi variety. R. Vakil [21] also proved that similar results hold on several rational surfaces.

Mikhalkin [14] studied a tropical analogy of the above problem and proved the tropical enumeration problem is equal to the classical one by giving appropriate multiplicities for enumerated tropical curves. T. Nishinou and B. Siebert [16] also showed that the enumeration problem on toric varieties equals the enumeration of a certain type of tropical curves.

It is natural to extend the above enumeration problem to that of general singular curves on toric surfaces. To formulate the problem for singular curves, we introduce some terminology and notations. We denote by $\mathfrak{S}$ a finite collection of topological types of plane curve singularities. A plane curve C is called an $\mathfrak{S}$-curve if there exists a bijection $\sigma : \mathrm{Sing}(C) \to \mathfrak{S}$ which maps a singular point (C, p) to an element of topological type in $\mathfrak{S}$.

Then the above enumeration problem is extended as follows:

> *For a lattice polygon $\Delta \subset \mathbb{R}^2$, how many $\mathfrak{S}$-curves on $X(\Delta)$ which are contained in $|D(\Delta)|$ and pass through $r(\Delta, \mathfrak{S})$-points lying in general position do there exist?*

Here, $r(\Delta, \mathfrak{S})$ is the dimension of the space of $\mathfrak{S}$-curves on $X(\Delta)$. As noted before, Mikhalkin [14] proved in the case of $\mathfrak{S} = \{\delta A_1\}$. The following cases had been studied by using the theory of characteristic classes:

- the case of $\mathfrak{S} = \{\text{a triple point}\}$ on any smooth surface (S. Kleiman and R. Piene [10]),
- the case of $|\mathfrak{S}| = 1$ on $\mathbb{CP}^2$ (D. Kerner [9]),
- the case of some $\mathfrak{S}$ on any smooth surface (M. Kazarian [8]).

Tropical approach to the enumeration problem of general singular curves on toric surfaces began with E. Shustin [19]. He introduced a degeneration of a curve, called a *tropicalization*, and showed that the tropicalization of a curve which has only one singular point whose topological type is A_2 (he called such a curve a *1-cuspidal curve* for simplicity) is related to a certain tropical curve, called a tropical 1-cuspidal curve. Furthermore, using the theory of patchworking, he showed that the enumeration of 1-cuspidal curves reduced into that of the tropical 1-cuspidal curves.

In this thesis, we apply the tropicalization method to 1-tacnodal curves, that is, curves which have exactly one singular point whose topological type is A_3, on a toric surface, and classify them using tropical curves.

To state our result, we prepare some terminology. Let F be a polynomial in two variables over the field of convergent Puiseux series over $\mathbb{C}$, denoted by $K := \mathbb{C}\{\{t\}\}$. Then we can define a valuation $\mathrm{val} : K^* := K \setminus \{0\} \to \mathbb{R}$ as follows. For a given element $b(t) \in K^*$, take the minimal exponent q of $b(t)$ in t, then define $\mathrm{val}(b(t)) := q$. We set

$$\mathrm{Trop} : (K^*)^2 \to \mathbb{R}^2; (z, w) \mapsto (-\mathrm{val}(z), -\mathrm{val}(w)).$$

We call the closure

$$T_F := \mathrm{Closure}(\mathrm{Trop}(\{F = 0\} \cap (K^*)^2)) \subset \mathbb{R}^2$$

of the curve defined by F in $(K^*)^2$ the *tropical amoeba* defined by F. More generally, a *tropical curve* is defined as the non-linear locus of a tropical polynomial function.

The tropical amoeba defined by F is related to a degeneration, called *tropicalization*, of the curve defined by F in $X(N_F)$. We will explain the details on the tropicalization in Subsection 1.6 of this thesis.

It is known that any tropical curve has the structure of 1-dimensional polyhedral complex, whose 1-dimensional polyhedron has a rational slope and a positive integral weight (See Theorem 1.11). We call a 1-simplex an *edge* and a 0-simplex a *vertex*.

Each tropical curve T has a positive integer $\mathrm{rk}(T)$ called a *rank*, which, roughly speaking, is the dimension of the space of tropical curves which are combinatorially same as T. The formal definition of the rank will be given in Section 1.5.

Definition. For a tropical curve T and a vertex $V \in T$, V is a *smooth vertex* of T if the following two conditions are satisfied:

- V is trivalent.
- Let $v_1, v_2, v_3 \in \mathbb{Z}^2$ and $w_1, w_2, w_3 \in \mathbb{Z}_{\geq 1}$ be the slopes and weights of edges adjacent to V, respectively. Then the *multiplicity* of V

$$m(V) := w_1 w_2 |v_1 \times v_2| = w_2 w_3 |v_2 \times v_3| = w_3 w_1 |v_3 \times v_1|$$

  is 1, where $|u \times v|$ means the area of the parallelogram spanned by u and v.

In other words, any smooth vertex is a vertex which is dual to the standard 2-dimensional simplex $\mathrm{Conv}\{(0,0), (1,0), (0,1)\}$.

Definition (Shustin [19, Section 4.1])**.** A *tropical 1-cuspidal curve* is a tropical curve having exactly one of the parts (i), ..., (v) in Figure A, up to the $\mathbb{R}^2 \rtimes \mathrm{GL}(\mathbb{Z}^2)$-equivalence, and the rest of the vertices are smooth, where two tropical curves T_1 and T_2 are $\mathbb{R}^2 \rtimes \mathrm{GL}(\mathbb{Z}^2)$-*equivalent* if there exists an element $\Psi \in \mathbb{R}^2 \rtimes \mathrm{GL}(\mathbb{Z}^2)$ such that $\Psi(T_1) = T_2$.

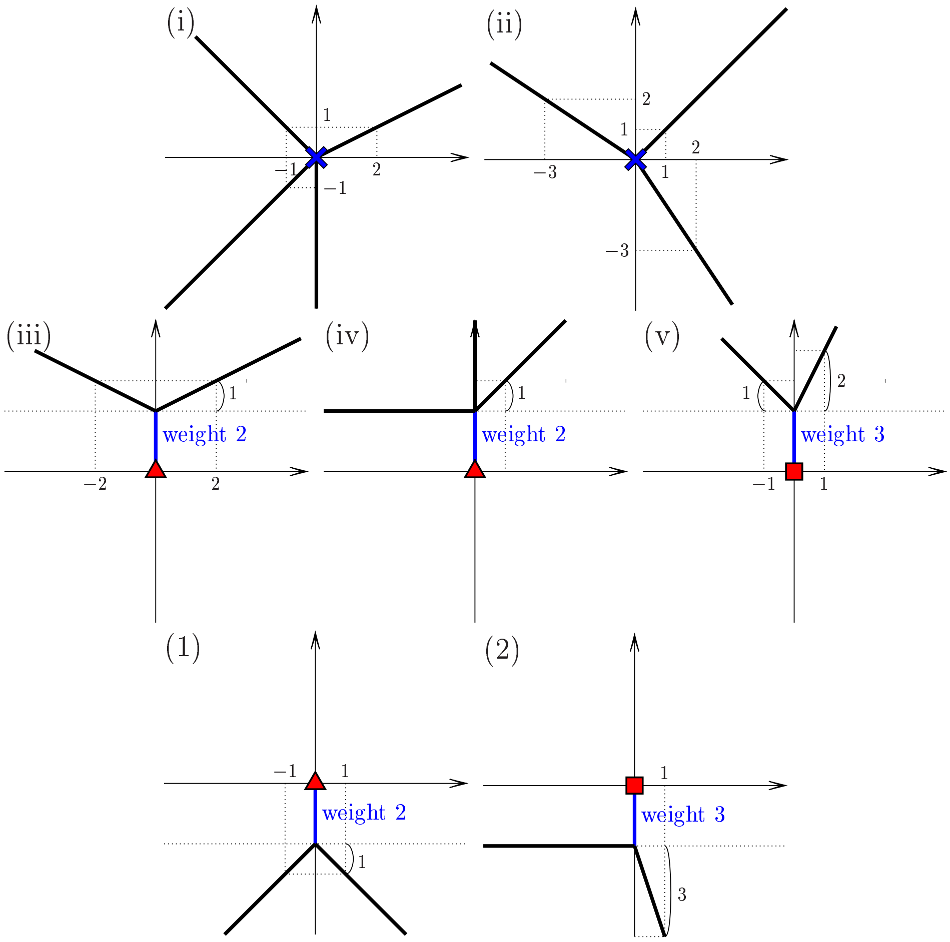

Figure A: Parts of tropical 1-cuspidal curves [19], where the ends $\blacktriangle$ of (iii) and (iv) are connected to $\blacktriangle$ of (1), and the end $\blacksquare$ of (v) is connected to $\blacksquare$ of (2).

In [19], for each tropical 1-cuspidal curve T, he defined a *multiplicity* $m(T)$ of T and the notion of general position of points in tropical geometry appropriately. We omit the details in this thesis.

For a compact lattice polygon $\Delta \subset \mathbb{R}^2$, let $N^{\mathrm{trop}}(\Delta, A_2, \mathcal{Q})$ be the number of tropical 1-cuspidal amoebas in $\mathbb{R}^2$ whose Newton polygons are Δ and which pass through $r(\Delta, A_2)$ tropical generic points $\mathcal{Q}$ counted with the multiplicities $m(T)$. Let $N(\Delta, A_2, \mathcal{P})$ be the number of 1-cuspidal curves on $X(\Delta)$ which are contained in the complete linear system $|D(\Delta)|$ and pass through $r(\Delta, A_2)$ generic points $\mathcal{P}$.

Using the above notations, Shustin [19] proved the following equality. This is a 1-cuspidal version of Mikhalkin's result [14, Theorem 1].

Theorem (Shustin [19, Theorem 4])**.** *For any tropically $r(A_2,\Delta)$-generic points $\mathcal{Q} \subset \mathbb{R}^2$, there exist $r(A_2,\Delta)$-generic points $\mathcal{P} \subset \mathrm{Trop}^{-1}(\mathcal{Q})$ such that the following holds:*

$$N(\Delta, A_2, \mathcal{P}) = N^{\mathrm{trop}}(\Delta, A_2, \mathcal{Q}).$$

This theorem is proved by combining the following two claims:

Claim 1. *Let $F \in K[z,w]$ be a polynomial which defines a 1-cuspidal curve. If the rank of the tropical amoeba T_F defined by F is more than or equal to the number of the lattice points of the Newton polygon of F minus <u>three</u>, then T_F is a tropical 1-cuspidal curve.*

Claim 2. *For a given tropical 1-cuspidal amoeba T with the Newton polygon Δ which passes through tropically $r(A_2,\Delta)$-generic points, there exist $r(A_2,\Delta)$-generic points in $X(\Delta)$ and $m(T)$ 1-cuspidal curves which pass through the points.*

Notice that, the condition of the rank of Claim 1 corresponds to the assumption that "*passes through the tropically $r(A_2,\Delta)$-general points*", and the number $\sharp(\Delta \cap \mathbb{Z}^2) - 3$ is equal to the dimension $r(A_2,\Delta)$ of the space of the 1-cuspidal curves.

The aim of this thesis is to study a 1-tacnodal version of the above arguments. As the main theorem, we prove the 1-tacnodal version of Claim 1. Furthermore, in this case, we see that the criterion for using patchworking of singular curves, which is necessary in the proof of Claim 2, does not work.

First, we define a tropical version of 1-tacnodal curves as follows.

Definition (Definition 4.1 of this thesis)**.** A *tropical 1-tacnodal curve* is a tropical curve having exactly one of the parts (I), ..., (IX), (E) in Figure B, up to the $\mathbb{R}^2 \rtimes \mathrm{GL}(\mathbb{Z}^2)$-equivalence, and the rest of the vertices are smooth.

The following statement is the main result in this thesis, which corresponds to Claim 1.

Main Theorem. *Let $F \in K[z,w]$ be a polynomial which defines an irreducible 1-tacnodal curve. If the rank of the tropical amoeba T_F defined by F is more than or equal to the number of the lattice points of the Newton polytope of F minus <u>four</u> and the tropicalization of the curve defined by F in $X(N_F)$ has only isolated singularities, then T_F is a tropical 1-tacnodal curve.*

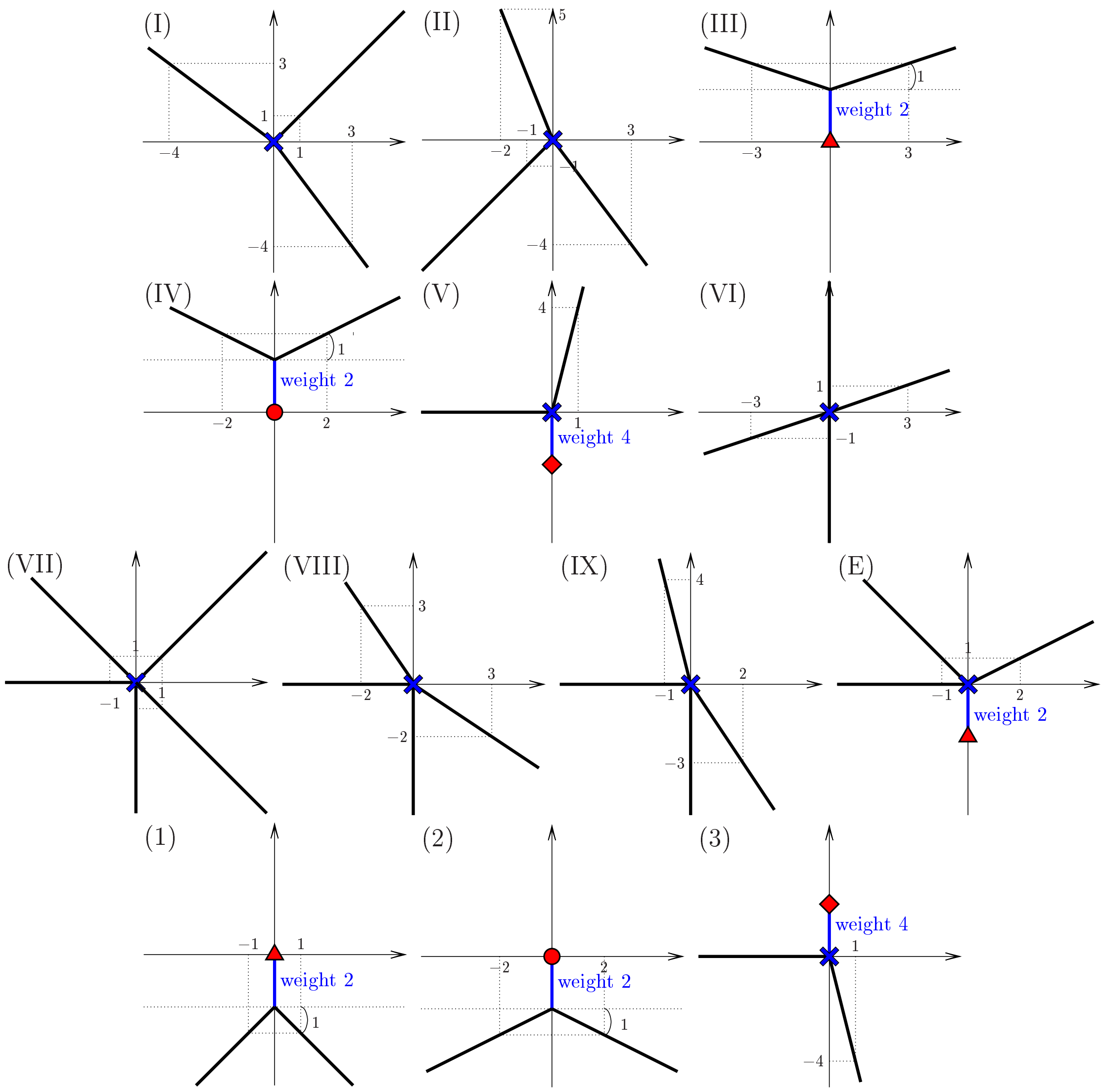

Figure B: Parts of tropical 1-tacnodal curves, where the ends ▲ of (III) and (E) are connected to ▲ of (1), the end ● of (IV) is connected to ● of (2), and the end ◆ of (V) is connected to ◆ of (3).

Note that, the number $\sharp(\Delta \cap \mathbb{Z}^2) - 4$, which is in the above theorem, is equal to the lower bound of the dimension of the space of 1-tacnodal curves in $X(\Delta)$ (see Corollary 2.6 of this thesis). Note also that there is a possibility there exists a tropicalization which has non-isolated singularities, see Remark 4.20.

In [19], Claim 2 is proved by using patchworking method. The key of the proof is the criterion which makes sure that we can use the patchworking method [18, 19]. In the 1-cuspidal case, the criterion works and we can apply it to the proof of the existence of the nodal and 1-cuspidal curves. On the other hand, it does not work in the 1-tacnodal case. We will discuss this in Remark 4.21 of Chapter 4 in details.

We organize this thesis as follows. In Chapter 1, we introduce some basic terminology and properties of tropical hypersurfaces such as the duality theorem, the structure theorem, tropical amoeba and the rank of a tropical curve. In Chapter 2, we prepare some notions on plane curve singularities, such as invariants and Newton diagrams of singularities. We also consider a necessary and sufficient condition for a complex curve to have a tacnode, and estimate the dimension of the space of 1-tacnodal curves on a toric surface. In Chapter 3, we give some attempts to use tropical geometry to the theory of plane curve singularities. In Sections 4.1 and 4.2 of Chapter 4, before the proof of Main Theorem, we prepare some definition and lemmata on relation between singular curves and their Newton polytopes. The proof of Main Theorem is carried out in Section 4.3.

Conventions and Notions

We here introduce some definitions and facts from convex geometry which will be used in this thesis.

(I) Some notions on polyhedra.

A set in $\mathbb{R}^N$ is a *(lattice) polyhedron* if it is the intersection of a finite number of half-spaces in $\mathbb{R}^N$ whose vertices are contained in the lattice $\mathbb{Z}^N \subset \mathbb{R}^N$. A polyhedron is *$d$-dimensional* if its *affine span*, which is the smallest affine space containing the polyhedron, has dimension d. In this thesis, we assume that any polyhedron always has the maximal dimension. A set is a *polytope* in $\mathbb{R}^d$ if it is a compact polyhedron, that is, the convex hull of a finite number of lattice points. Here, for a subset $A \subset \mathbb{R}^n$, the *convex hull* $\mathrm{Conv}(A)$ of A is the smallest convex set containing A. Note that we will use the symbols Δ or P as a polytope. A subset in a d-dimensional polyhedron is a *sub-polyhedron* if it is a polyhedron as a subset on $\mathbb{R}^d$. In particular, if a sub-polyhedron is a polytope then the sub-polyhedron is called a *sub-polytope*.

A *face* of a d-dimensional polyhedron $\Delta \subset \mathbb{R}^d$ is the set

$$\{x \in \Delta; f(x) \geq f(y),\ \forall y \in \Delta\}$$

for a linear function f. Particularly, a face of codimension 1 is called a *facet*. The *boundary* $\partial\Delta$ is the union of all facets of Δ. The *interior* $\mathrm{Int}\Delta$ is defined by $\Delta \setminus \partial\Delta$.

Let $\Delta \subset \mathbb{R}^d$ be a polytope. We denote the interior lattice points of Δ, $\mathrm{Int}\Delta$ and $\partial\Delta$ as $\Delta_{\mathbb{Z}}$, $\mathrm{Int}\Delta_{\mathbb{Z}}$ and $\partial\Delta_{\mathbb{Z}}$, respectively. That is,

$$\Delta_{\mathbb{Z}} := \Delta \cap \mathbb{Z}^d, \quad \mathrm{Int}\Delta_{\mathbb{Z}} := \mathrm{Int}\Delta \cap \mathbb{Z}^d, \quad \partial\Delta_{\mathbb{Z}} := \partial\Delta \cap \mathbb{Z}^d.$$

For any ring R with the zero element 0_R and a polynomial denoted by

$$f = \sum_{(i_1,\dots,i_n)} c_{(i_1,\dots,i_n)} x_1^{i_1} \dots x_n^{i_n} \in R[x_1, \dots, x_n],$$

the *Newton polytope* N_f of f is defined by

$$N_f := \mathrm{Conv}\big(\{(i_1, \dots, i_n) \in \mathbb{Z}^n; c_{(i_1,\dots,i_n)} \neq 0_R\}\big) \subset \mathbb{R}^n.$$

In this thesis, a *polygon* means a 2-dimensional polytope. We call a facet of a polygon an *edge*. Similarly, we call the 0-dimensional face obtained as a corner of a polygon a *vertex*. We denote the set of the vertices as $V(\Delta)$ for a polygon Δ. A polygon is said to be *parallel* if the opposite edges have the same directional vector (up to orientation) and the same lattice length. A polygon is called an *m-gon* if the number of its edges is m.

In two dimensional case, the following classical facts are well-known as the *Pick's formula*:

Theorem (Pick). *For a polygon $\Delta \subset \mathbb{R}^2$, the following formula holds:*

$$\mathrm{Area}(\Delta) = \sharp \mathrm{Int}\Delta_{\mathbb{Z}} + \frac{\sharp \partial \Delta_{\mathbb{Z}}}{2} - 1,$$

where the notation $\mathrm{Area}(\Delta)$ *means the area of* Δ.

It is known that, for a polygon $\Delta \subset \mathbb{R}^2$, we can construct a *polarized toric surface* associated with Δ over $\mathbb{C}$, denoted by $(X(\Delta), D(\Delta))$, where $D(\Delta)$ is the polarization on $X(\Delta)$ associated with Δ (see [4] for details).

For a topological space X and a subset $A \subset X$, the notation $\mathrm{Closure}(A)$ stands for the *closure* of A in X.

(II) SUBDIVISION OF POLYTOPES.

Secondly, we introduce some notions on subdivisions of polytopes. Let $\Delta \subset \mathbb{R}^d$ be a d-dimensional polytope. A collection $S := \{\Delta_1, \dots, \Delta_N\}$ of sub-polytopes of Δ is a *(lattice) subdivision* of Δ if S satisfies the following three conditions:

(1) For each $i = 1, \dots, N$, the dimension of Δ_i is d.

(2) The polytope Δ is the union of $\Delta_1, \dots, \Delta_N$.

(3) If $i \neq j$, the intersection $\Delta_i \cap \Delta_j$ is a common proper face of Δ_i and Δ_j or the empty set.

A subdivision S of Δ is *regular* if there exists a continuous convex PL-function $\nu : \Delta \to \mathbb{R}$ such that S is obtained as the collection of the linearity domains of ν, where a *linearity domain* of ν means a maximal sub-polytope R contained in the domain Δ such that the restriction $\nu|_R$ is an affine linear function.

(III) POLYHEDRAL COMPLEX.

Thirdly, we introduce the notion of polyhedral complex. A finite collection $\mathcal{PC}$ of polyhedrons in $\mathbb{R}^d$ is a *polyhedral complex* if the collection $\mathcal{PC}$ satisfies the following conditions:

(1) $\mathcal{PC}$ contains the empty set $\emptyset$.

(2) If $P \in \mathcal{PC}$, all faces of P are contained in $\mathcal{PC}$.

(3) The intersection of polyhedrons $P, Q \in \mathcal{PC}$ is a common face of P and Q.

A polyhedral complex $\mathcal{PC}$ is *rational* if, for each polyhedron in $\mathcal{PC}$, the affine span of the polyhedron is parallel to some subspace of $\mathbb{R}^n$ defined over $\mathbb{Q}$.

The *dimension* of a polyhedral complex $\mathcal{PC}$ is defined by the maximum of the dimensions of polyhedrons contained in $\mathcal{PC}$.

Let $\mathcal{PC}^{[k]}$ be the set of k-dimensional polyhedrons in $\mathcal{PC}$. An n-dimensional polyhedral complex $\mathcal{PC}$ is *weighted* if the $\mathcal{PC}$ is assigned a function $w : \mathcal{PC}^{[n]} \to \mathbb{Z}_{>0}$.

Chapter 1

Basics of tropical geometry

In this chapter, we discuss some elementally facts on tropical geometry.

1.1 Tropical algebra and tropical polynomial

Definition 1.1 (Semi-ring)**.** Let S be a set and $\oplus, \odot : S \times S \to S$ be binary operations. A triple $(S; \oplus, \odot)$ is a *semi-ring* if the triple satisfies the following conditions:

(1) The double $(S; \oplus)$ is an abelian semigroup, i.e.,

  (1-1) $(a \oplus b) \oplus c = a \oplus (b \oplus c)$,

  (1-2) $a \oplus b = b \oplus a$.

(2) The double $(S; \odot)$ is an abelian group with unit element 1_S.

(3) The $\odot$ is distributive on $\oplus$, i.e.,

  (3-1) $a \odot (b \oplus c) = (a \odot b) \oplus (a \odot c)$,

  (3-2) $(a \oplus b) \odot c = (a \odot c) \oplus (b \odot c)$.

In some literature, instead of (1), it may be requested that $(S; \oplus)$ is a monoid. I.e., each semi-ring does not necessarily have the unit element with respect to the addition (we call such an element the *zero element* for simplicity).

Next we introduce a semi-ring which we work on. We define the triple $\mathbb{T} := (\mathbb{R}; \oplus, \odot)$ as

$$a \oplus b := \max(a, b), \quad a \odot b := a + b, \quad a, b \in \mathbb{R},$$

where the notation $+$ is the usual addition on $\mathbb{R}$. The following statement holds for $\mathbb{T}$:

Proposition 1.2. *The triple* $\mathbb{T} = (\mathbb{R}; \oplus, \odot)$ *has the structure of semi-ring with unit element* $0 \in \mathbb{R}$.

The proof of this statement is done by easily computations. We call $\mathbb{T}$ the *tropical algebra*. It is also known as the *max-plus algebra*. The tropical algebra is an example of idempotent algebra. I.e., for any $a \in \mathbb{T}$,

$$a \oplus a = \max(a, a) = a.$$

We also remark that, $-\infty$ plays a role of zero element of the tropical algebra.

A polynomial over the tropical algebra is naturally defined as follows:

$$\begin{aligned}\tau(X_1, \dots, X_n) &:= \bigoplus_{(i_1,\dots,i_n)\in A} a_{(i_1,\dots,i_n)} \odot X_1^{\odot i_1} \odot \cdots \odot X_n^{\odot i_n} \\ &= \max\{a_{(i_1,\dots,i_n)} + i_1X_1 + \cdots + i_nX_n; (i_1, \dots, i_n) \in A\},\end{aligned}$$

where $a_{(i_1,\dots,i_n)} \in \mathbb{R}$ is a real number and $A \subset \mathbb{Z}^n$ is a finite set. We call such a polynomial as a *tropical polynomial* in n variables and denote the set of tropical polynomials as

$$\mathbb{T}[X_1, \dots, X_n].$$

We also use multi-index for describing tropical polynomials, that is, we denote a tropical polynomial as

$$\tau(\mathbf{X}) = \bigoplus_{\mathbf{i}\in A} a_{\mathbf{i}} \odot \mathbf{X}^{\odot \mathbf{i}} = \max_{\mathbf{i}\in A}\{a_{\mathbf{i}} + \langle \mathbf{i}, \mathbf{X}\rangle\},$$

for simplicity, where $\mathbf{i} = (i_1, \dots, i_n)$, $\mathbf{X} = (X_1, \dots, X_n)$ and $\langle \mathbf{i}, \mathbf{X}\rangle := i_1X_1 + \cdots + i_nX_n$. Each tropical polynomial in n variables is a continuous concave PL-function from $\mathbb{R}^n$ to $\mathbb{R}$. We always treat each tropical polynomial as a function.

Furthermore, each tropical polynomial τ is obtained as the discrete Legendre transform of the function

$$A \to \mathbb{R}; \quad (i_1, \dots, i_n) \mapsto -a_{(i_1,\dots,i_n)},$$

where, for a function κ, the *discrete Legendre transform* κ^* of κ is defined as

$$\kappa^*(\mathbf{X}) := \max_{\mathbf{i}\in A}\{-\kappa(\mathbf{i}) + \langle \mathbf{i}, \mathbf{X}\rangle\}.$$

For each tropical polynomial τ described as above, the polytope

$$N_\tau := \mathrm{Conv}(A) \subset \mathbb{R}^n$$

is called the *Newton polytope* of τ.

Next, we define an analogy of a hypersurface defined by a tropical polynomial.

Definition 1.3 (Tropical Hypersurface)**.** Let $\tau \in \mathbb{T}[X_1, \dots, X_n]$ be a tropical polynomial.

The *tropical hypersurface* $V_\tau \subset \mathbb{R}^n$ defined by τ is the set of non-differentiable points of $\tau : \mathbb{R}^n \to \mathbb{R}$.

This definition can be paraphrased as follows:

Proposition 1.4 (Mikhalkin [14, Proposition 3.3]). *Let $\tau \in \mathbb{T}[X_1, \ldots, X_n]$ be a tropical polynomial. The tropical hypersurface V_τ defined by τ is the set of points such that two or more terms of τ attain their maximum.*

Proof. If exactly one term of τ attains the maximum at a point $\mathbf{X} \in \mathbb{R}^n$, then, since τ matches the term at $\mathbf{X}$ locally, τ is a linear function on a sufficiently small neighborhood of $\mathbf{X}$ and therefore smooth at $\mathbf{X}$. $\square$

By this proof, the set V_τ is equal to the projection of the intersection of two or more hyperplanes in $\mathbb{R}^n \times \mathbb{R}$ obtained as the graph of some terms of τ. In other words, V_τ is the projection of the "corner" points of the graph of $\tau : \mathbb{R}^n \to \mathbb{R}$ in $\mathbb{R}^n \times \mathbb{R}$. From this fact, V_τ is also called the *corner locus* of τ.

By paying attention that $-\infty$ has a role as a zero element of $\mathbb{T}$, we can describe the relationship between the definition of a classical hypersurface and that of the tropical hypersurface as follows.

Lemma 1.5 (Mikhalkin [14, Proposition 3.5]). *Let $\tau \in \mathbb{T}[X_1, \ldots, X_n]$ be a tropical polynomial and $\mathrm{Gr}(\tau) \subset \mathbb{R}^n \times \mathbb{R}$ be the graph of τ. The set*

$$\overline{\mathrm{Gr}}(\tau) := \mathrm{Gr}(\tau) \cup \{(\mathbf{X}, Y) \in \mathbb{R}^n \times \mathbb{R};\ \mathbf{X} \in V_\tau,\ Y \leq \tau(\mathbf{X})\}$$

is the tropical hypersurface defined by $\tau(\mathbf{X}) \oplus Y$.

Proof. If $(\mathbf{X}, Y) \in \mathrm{Gr}(\tau)$, then Y and some terms of τ attain the maximum of $\tau(\mathbf{X}) \oplus Y$. On the other hand, if $\mathbf{X} \in V_\tau$ and $Y < \tau(\mathbf{X})$, some two terms of τ attain the maximum of $\tau(\mathbf{X}) \oplus Y$. $\square$

For sufficiently small $t \in \mathbb{R}$, we obtain $V_\tau \simeq \overline{\mathrm{Gr}}(\tau) \cap \{Y = t\}$. Therefore we can call V_τ the "zero set" of τ.

1.2 Non-Archimedean field and tropical amoeba

A field K is *non-Archimedean* if there exists a function $\mathrm{val} : K^* := K \setminus \{0_K\} \to \mathbb{R}$, called a *valuation* on K, i.e., a function such that, for $a, b \in K^*$,

(1) $\mathrm{val}(ab) = \mathrm{val}(a) + \mathrm{val}(b)$,

(2) $\mathrm{val}(a + b) \leq \max\{\mathrm{val}(a), \mathrm{val}(b)\}$.

The pair (K, val) is called a *non-Archimedean field.* For the zero element 0_K of K, we define $\mathrm{val}(0_K) = -\infty$ formally.

Example 1.6. We here give an example of a non-Archimedean field. Let $\mathbb{C}\{\{t\}\}$ be the set of convergent Puiseux series, which is a convergent series described as

$$\sum_{k=k_0}^{\infty} c_k t^{\frac{k}{M}}, \quad c_k \in \mathbb{C},$$

where $k_0 \in \mathbb{Z}$ and $c_{k_0} \neq 0$. It is well-known that the set $\mathbb{C}\{\{t\}\}$ has the structure of an algebraic closed field of characteristic zero. We define the function $\mathrm{val} : \mathbb{C}\{\{t\}\}^* \to \mathbb{R}$ on the field as

$$c(t) = \sum_{k=k_0}^{\infty} c_k t^{\frac{k}{M}} \mapsto \mathrm{val}(c(t)) := \frac{k}{M}.$$

This function is a non-Archimedean valuation. Therefore, $(\mathbb{C}\{\{t\}\}, \mathrm{val})$ is a non-Archimedean field. The field is called the *field of convergent Puiseux series over* $\mathbb{C}$.

A relationship between hypersurfaces over a non-Archimedean field and tropical hypersurfaces is described as follows. Let

$$F(x_1, \dots, x_n) = \sum_{(i_1,\dots,i_n)} c_{(i_1,\dots,i_n)} x_1^{i_1} \dots x_n^{i_n} \in K[x_1, \dots, x_n]$$

be a polynomial over K in n variables and $V_F^K \subset (K^*)^n$ be the hypersurface defined by F in $(K^*)^n$.

Definition 1.7 (Tropical Amoeba)**.** Set

$$\mathrm{Trop} : (K^*)^n \to \mathbb{R}^n; \quad (x_1, \dots, x_n) \mapsto (-\mathrm{val}(x_1), \dots, -\mathrm{val}(x_n)).$$

The set

$$T_F := \mathrm{Closure}\big(\mathrm{Trop}(V_F^K)\big) \subset \mathbb{R}^n$$

is called the *tropical amoeba* or *non-Archimedean amoeba* defined by F.

The *tropical polynomial* $\mathrm{trop}F$ *defined by* F is defined by

$$\mathrm{trop}F(\mathbf{X}) := \bigoplus_{\mathbf{i}=(i_1,\dots,i_n)} -\mathrm{val}(c_{\mathbf{i}}) \odot \mathbf{X}^{\odot \mathbf{i}} \in \mathbb{T}[X_1, \dots, X_n].$$

Then the following fact, which is well-known as *Kapranov's Theorem*, holds:

Theorem 1.8 (Kapranov [3, Theorem 2.2.5])**.** *Let $F \in K[x_1, \dots, x_n]$ be a polynomial over the non-Archimedean field K. The tropical amoeba T_F defined by F is equal to the tropical hypersurface defined by the tropical polynomial* $\mathrm{trop}F$*.*

1.3 Duality theorem of tropical hypersurfaces

Let
$$\tau(\mathbf{X}) = \bigoplus_{\mathbf{i}\in A} a_{\mathbf{i}} \odot \mathbf{X}^{\odot \mathbf{i}} \in \mathbb{T}[X_1, \ldots, X_n]$$
be a tropical polynomial whose Newton polytope is $\Delta \subset \mathbb{R}^n$. We define the $(n+1)$-dimensional polyhedron $\hat{\Delta}(\tau)$ as
$$\hat{\Delta}(\tau) := \mathrm{Conv}\big(\{(\mathbf{i}, t) \in \mathbb{R}^n \times \mathbb{R};\ \mathbf{i} \in A,\ t \leq a_{\mathbf{i}}\}\big) \subset \mathbb{R}^n \times \mathbb{R}.$$

The projection $\mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ induces a homeomorphism from the union of compact faces of $\hat{\Delta}(\tau)$ to Δ. In particular, by the projection $\mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$, each compact face of $\hat{\Delta}(\tau)$ is mapped into some sub-polytope of Δ. Therefore, we obtain a subdivision S_τ of Δ from τ.

The following theorem is the so-called *Duality Theorem of tropical hypersurfaces.*

Theorem 1.9 (Mikhalkin [14, Theorem 3.11])**.** *Let $\tau \in \mathbb{T}[X_1, \ldots, X_n]$ be a tropical polynomial whose Newton polytope is $\Delta \subset \mathbb{R}^n$. The subdivision S_τ is combinatorially dual to the tropical hypersurface V_τ defined by τ, that is, for any k-dimensional polytope $P \in S_\tau$, there exists a polyhedron $V_\tau^P \subset V_\tau$ uniquely such that*

(1) the polyhedron V_τ^P is contained in an $(n-k)$-dimensional affine subspace $L^P \subset \mathbb{R}^n$ and V_τ^P is orthogonal to P,

(2) the relative interior U^P of V_τ^P in L^P is non-empty,

(3) $V_\tau = \bigcup_{P\in S_\tau} U^P$,

(4) if $P_1 \neq P_2$ then $U^{P_1} \cap U^{P_2} = \emptyset$,

(5) non-compactness of V_τ^P is equivalent to $P \subset \partial\Delta$.

Clearly, for a given tropical hypersurface T which is defined by a tropical polynomial τ, the subdivision S_τ is determined uniquely. Therefore we use the notation S_T in place of S_τ, and call it the *dual subdivision* of T.

1.4 Structure theorem of tropical hypersurfaces

Let $\mathcal{PC}$ be an n-dimensional weighted rational polyhedral complex and $w : \mathcal{PC}^{[n]} \to \mathbb{Z}_{>0}$ be the weight. By definition, for an n-dimensional polyhedron $P \in \mathcal{PC}^{[n]}$, there exists a unique $\mathbb{Z}$-linear map $c_P : \mathbb{Z}^n \to \mathbb{Z}$ up to sign such that

(1) the kernel $\mathrm{Ker}(c_P)$ is parallel to P, and

(2) $c_P/w(P)$ is a primitive $\mathbb{Z}$-linear map.

Note that, if an orientation of P is fixed, then the sign of c_P is determined.

Definition 1.10 (Balancing Condition [13, Definition 3]). The polyhedral complex $\mathcal{PC}$ satisfies the *balancing condition* if the following condition is satisfied: For each $(n-1)$-dimensional polyhedron $Q \in \mathcal{PC}^{[n-1]}$, let $P_1, \dots, P_k \in \mathcal{PC}^{[n]}$ be n-dimensional polyhedrons adjacent to Q. A choice of a rational direction about Q defines a coherent co-orientation on these n-dimensional polyhedrons. Then

$$\sum_{j=1}^{k} c_{P_j} = 0$$

holds.

Theorem 1.11 (Mikhalkin [14, Theorem 3.15]). *Let $\mathcal{PC}$ be an n-dimensional weighted rational polyhedral complex and $w : \mathcal{PC}^{[n]} \to \mathbb{Z}_{>0}$ be the weight. The polyhedral complex $\mathcal{PC}$ is a tropical hypersurface if and only if the polyhedral complex $\mathcal{PC}$ satisfies the balancing condition.*

1.5 The space of tropical curves and the rank of tropical curves

For a fixed polytope $\Delta \subset \mathbb{R}^d$, the set of tropical hypersurfaces having the fixed Newton polytope Δ is denoted by $\mathfrak{T}(\Delta)$, that is,

$$\mathfrak{T}(\Delta) := \big\{\text{tropical hypersurface in } \mathbb{R}^d \text{ whose Newton polytope is } \Delta\big\}.$$

Proposition 1.12 (Mikhalkin [14, Proposition 3.8]). *The set $\mathfrak{T}(\Delta)$ has the structure of closed convex polyhedral cone in $\mathbb{R}^{\sharp\Delta_{\mathbb{Z}}-1}$. The cone $\mathfrak{T}(\Delta)$ is well-defined up to natural isomorphism of $SL_{\sharp\Delta_{\mathbb{Z}}-1}(\mathbb{Z})$.*

Let S be a regular subdivision of $\Delta \subset \mathbb{R}^d$. Set

$$\mathfrak{T}(\Delta; S) := \big\{T \in \mathfrak{T}(\Delta); S_T = S\big\},$$

where S_T means the subdivision dual to T in Theorem 1.9.

Lemma 1.13 (Mikhalkin [14, Lemma 3.14]). *The set $\mathfrak{T}(\Delta; S) \subset \mathfrak{T}(\Delta)$ has the structure of convex polyhedral domain, which is open in its affine span.*

Next, we discuss the dimension of the space of tropical curves. Let S be the dual subdivision of $T \in \mathfrak{T}(\Delta)$ and define the *rank* of the tropical curve T (or of S) as

$$\mathrm{rk}(T) := \mathrm{rk}(S) := \dim \mathfrak{T}(\Delta; S).$$

Let $\Delta_1, \dots, \Delta_N$ be the polygons of S. According to [19], we define the *expected rank* of the tropical curve T (or of S) as

$$\mathrm{rk}_{\mathrm{exp}}(T) := \mathrm{rk}_{\mathrm{exp}}(S) := \sharp V(S) - 1 - \sum_{k=1}^{N} (\sharp V(\Delta_k) - 3),$$

where $V(S)$ is the set of vertices of S and $V(\Delta_k)$ is the set of vertices of Δ_k.

Definition 1.14 (TP-subdivision)**.** A lattice subdivision of a polygon is a *TP-subdivision* if the subdivision consists of only triangles and parallelograms.

We remark that, this definition is same as the definition of the *nodal subdivision* in [19, Subsection 3.1] except the condition on the boundary $\partial\Delta$.

For any subdivision S, we denote the number of ℓ-gons and the number of parallel $(2m)$-gons contained in S as N_ℓ and N'_{2m}, respectively.

Lemma 1.15 (Shustin [19, Lemma 2.2])**.** *For a tropical curve T, the difference*

$$d(T) := \mathrm{rk}(T) - \mathrm{rk}_{\mathrm{exp}}(T)$$

satisfies $d(T) \geq 0$. Moreover, for the dual subdivision S of T, the difference $d(T)$ satisfies

- *$d(T) = 0$ if S is a TP-subdivision and*
- *$0 \leq 2d(T) \leq \mathcal{N}_S$ otherwise,*

where

$$\begin{aligned}\mathcal{N}_S &:= \sum_{m\geq 2} ((2m-3)N_{2m} - N'_{2m}) + \sum_{m \geq 2} ((2m-2)N_{2m+1}) - 1 \\ &= \sum_{\ell \geq 3} (\ell - 3) N_\ell - \sum_{m\geq 2} N'_{2m} - 1.\end{aligned}$$

1.6 Tropicalization of classical curves and its refinement

We briefly introduce the *tropicalization* of a curve and its refinement (see [19, Section 3] for more details). In this section, let K be the field of convergent Puiseux series $\mathbb{C}\{\{t\}\}$.

Let $F \in K[z, w]$ be a reduced polynomial which defines a curve $C \subset X(N_F)$. Set $\Delta = N_F$ and let T_F be the tropical amoeba defined by F introduced in Section 1.2 and S_F

be the dual subdivision of T_F. We consider the 3-dimensional unbounded polyhedron

$$\check{\Delta}_F := \mathrm{Conv}\{(i,j,t) \in \mathbb{R}^2 \times \mathbb{R}; t \geq -\nu_F(i,j)\} \subset \mathbb{R}^3.$$

We remark that a compact facet $\check{\Delta}_i$ of $\check{\Delta}_F$ corresponds to a polygon Δ_i in S_F by the projection $\check{\Delta}_F \subset \mathbb{R}^2 \times \mathbb{R} \to \mathbb{R}^2$.

We then obtain a toric flat morphism $X(\check{\Delta}_F) = \mathfrak{X} \to \mathbb{C}$ from the toric 3-fold associated with $\check{\Delta}_F$ to the complex line, which is called a *toric degeneration*. A generic fiber $\mathfrak{X}_t$ is isomorphic to $X(\Delta)$, and its central fiber $\mathfrak{X}_0$ is isomorphic to $\bigcup_{i=1,\ldots,N} X(\Delta_i)$ (see [16, Section 3] for more details). Let $D \subset \mathbb{C}$ be a small disk centered at the origin. We regard the indeterminate t of K as the variable in $D^* := D \setminus \{0\}$. Then we can get an analytic function $F(t; z, w)$ in three variables. From this analytic function, we obtain an equisingular family on the toric surface $X(\Delta)$

$$\{C^{(t)} := \mathrm{Closure}(\{F(t; z, w) = 0\})\}_{t \in D^*}.$$

The limit $C^{(0)}$ of this family is constructed as follows: For each $i = 1, \ldots, N$, a complex polynomial $f_i \in \mathbb{C}[z, w]$ whose Newton polygon is $\Delta_i \in S_F$ is induced from the face function of F on $\check{\Delta}_i$ by the transformation induced by the projection from $\check{\Delta}_i$ to Δ_i. The union of these curves is the limit $C^{(0)}$, which is a curve on the central fiber $\mathfrak{X}_0$ of the toric degeneration. The limit $C^{(0)}$ is called a *tropicalization* of C.

For each singular point z of C, there exists a continuous family of singular points $\{z_t\}$ for $t \in D^*$, where $z_t \in C^{(t)}$, and this family defines a section $s : D^* \to X(\check{\Delta}_F)$. If the limit $s(0) = \lim_{t\to 0} s(t)$ does not belong to the intersection lines $\bigcup_{i\neq j} X(\Delta_i \cap \Delta_j)$ and bears just one singular point of $C^{(t)}$, the point $s(0)$ is called a *regular singular point*. Otherwise it is called an *irregular singular point*. Note that if $s(0)$ is a regular singular point then it is topologically equivalent to the original singularity.

If the singular point $s(0)$ is irregular, additional information can be obtained by the *refinement* of the tropicalization, see Figure C. In the rest of this section, we explain this method briefly. See [19, Subsection 3.5] for the details of the refinement.

Hereafter, we assume that F defines a 1-tacnodal curve in $X(\Delta)$. Let $\Delta_1 \in S_F$ and $\Delta_2 \in S_F$ be polygons which have a common edge σ of length $m \geq 2$ and we observe the case where an irregular singularity degenerates into the subvariety $X(\sigma)$ of $\mathfrak{X}_0$. For each $i = 1, 2$, let f_i be a polynomial whose Newton polygon is Δ_i such that the union of curves $C_1 \cup C_2 \subset C^{(0)}$ defined by $f_1 = f_2 = 0$ intersects $X(\sigma)$ at $z \in \mathfrak{X}_0$. In this thesis, by later discussion, we can assume that, for each $i = 1, 2$, the polynomial f_i has an isolated singularity at $z \in X(\sigma)$ and their Newton boundary intersects the x- and y-axes at $(m_i, 0)$ and $(0, m)$, respectively, where the y-axis corresponds to $X(\sigma)$.

Find an automorphism $M_\sigma \in \mathrm{Aff}(\mathbb{Z}^2)$ such that $M_\sigma(\Delta)$ is contained in the right halfplane of $\mathbb{R}^2$ and $M_\sigma(\sigma) =: \sigma'$ is a horizontal segment, see Figure C. The automorphism M_σ induces a transformation $(x, y) \mapsto (x', y')$, by which we obtain a new polynomial $F'(x', y')$ from F. We can assume that $F' \in K[x', y']$ by multiplying a monomial. We remark that the point z corresponds to a root $\xi \neq 0$ of the truncation polynomial $F'^{\sigma'}(x', y')$ of F' on σ'. Here the truncation polynomial F^σ of a polynomial F on a facet σ of N_F is the sum of the terms of F corresponding to the lattice points on σ.

Then we choose an element $\tau \in K$ such that the coefficient of $\tilde{x}^{m-1}$ in $\tilde{F}(\tilde{x}, \tilde{y}) = F'(\tilde{x} + \xi + \tau, \tilde{y})$ is zero. Moreover, the dual subdivision of the tropical amoeba defined by $\tilde{F}$ contains a subdivision of the triangle $\Delta_z := \mathrm{Conv}\{(m, 0), (0, m_1), (0, -m_2)\}$. In this thesis, we call the polygon Δ_z the *exceptional polygon* for the irregular singularity $z \in \mathfrak{X}_0$. We remark that, the exceptional polygon is the union of the complements of the Newton diagrams of the polynomials f_1 and f_2 at $z \in X(\sigma)$ in the first quadrant of $\mathbb{R}^2$. Making the exceptional polygon Δ_z by the translation is an operation similar to a blowing-up of the 3-fold $\mathfrak{X}$. We can restore the topological type of the irregular singularity z in $X(\Delta_z)$ by this operation.

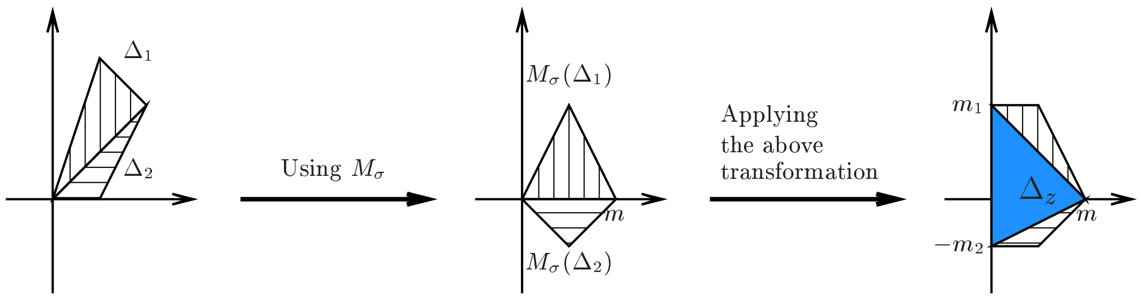

Figure C: A refinement of a tropicalization

Definition 1.16. For each $i = 1, 2$, let f_i be a polynomial which defines C_i such that $f_1^\sigma = f_2^\sigma$, and ϕ_i denote the composition of f_i and the translation which maps z to the origin of $\mathbb{C}^2$. Set

$$\hat{\sigma}_i := \Delta_z \cap N_{\phi_i} \subset \Delta_z,$$

where N_{ϕ_i} is the Newton polygon of ϕ_i. We assume that $\hat{\sigma}_i$ is an edge of Δ_z. We call a polynomial ϕ whose Newton polygon is Δ_z and that satisfies

(a) the coefficient of x^{m-1} is zero and

(b) the truncation polynomial $\phi^{\hat{\sigma}_i}$ is equal to ϕ_i for each edge $\hat{\sigma}_i$ of Δ_z

a *deformation pattern compatible with given data* (f_1, f_2, z).

We use this deformation pattern in the proof of Main Theorem about 1-tacnodal curves. In that case, by the same reason as in [19, Subsection 3.5], except case (E) in Figure B, if the curve defined by F has only one singular point which is an irregular singularity and there does not exist a deformation pattern compatible with the irregular singularity which defines a 1-tacnodal curve, then F does not define a 1-tacnodal curve. We will discuss what happen in case (E) in Subsection 4.2.2.

Chapter 2

Preliminaries on singularity theory

2.1 Basics of plane curve singularity

Let $\mathbb{C}\{x, y\}$ be the ring of convergent series in two variables over the field of complex numbers and $\mathfrak{m}$ be the maximal ideal given by $\{f \in \mathbb{C}\{x, y\}; f(0) = 0\}$.

The *Jacobi ideal* $J(f)$ associated by a series $f \in \mathbb{C}\{x, y\}$ is defined by

$$J(f) := \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right\rangle \subset \mathbb{C}\{x, y\}.$$

For $f \in \mathfrak{m}$, the number (possibly infinity) defined by

$$\mu(f) := \dim_{\mathbb{C}} \mathbb{C}\{x, y\}/J(f)$$

is called the *Milnor number* of f.

Let

$$\mathcal{O}_f := \mathbb{C}\{x, y\}/\langle f \rangle \hookrightarrow \mathcal{O}\{t_1\} \oplus \mathcal{O}\{t_2\} \oplus \cdots \oplus \mathcal{O}\{t_{r(f)}\} =: \overline{\mathcal{O}}_f$$

be the normalization of $\mathcal{O}_f$. Then the *delta number* of f is defined as

$$\delta(f) := \dim_C \overline{\mathcal{O}}_f/\mathcal{O}_f.$$

A relationship between the Milnor number and the delta number can be described as follows:

Theorem 2.1 (Milnor [15, Theorem 10.5]). *For a reduced holomorphic germ $f \in \mathfrak{m}$, the equality*

$$2\delta(f) = \mu(f) + r(f) - 1$$

holds, where $r(f)$ is the number of irreducible factors of f.

Let X be a projective surface with only isolated singularity over $\mathbb{C}$ and $C \subset X$ be an algebraic curve in X such that $C \cap \mathrm{Sing}(X) = \emptyset$. For any germ of curve (C, p), we can take

$f \in \mathfrak{m}$ which defines the curve C locally at p.

A point $p \in C$ is a *singular point* (or a *singularity of* C) if the germ (C, p) defines an $f \in \mathfrak{m}$ which satisfies $df(0) = 0$.

The *Milnor number* $\mu(C, p)$ of (C, p) is defined by

$$\mu(C, p) = \mu(f),$$

where $f \in \mathfrak{m}$ defines (C, p) locally. Similarly, the *delta number* $\delta(C, p)$ of (C, p) is defined by

$$\delta(C, p) = \delta(f).$$

Note that $p \in C$ is a singular point if and only if $\mu(C, p) > 0$.

2.2 Newton diagram of plane curve singularity

Let $f \in \mathfrak{m}$ be a convergent series described as

$$f(x, y) := \sum_{(i,j) \in (\mathbb{Z}_{\geq 0})^2} c_{(i,j)} x^i y^j.$$

Definition 2.2 (Newton diagram/boundary)**.** We define the polyhedron $\Gamma_+(f)$ as the convex hull

$$\operatorname{Conv}\left(\bigcup_{(i,j)} \left\{ (i, j) + (\mathbb{R}_{\geq 0})^2 ; c_{(i,j)} \neq 0 \right\} \right)$$

and call it the *Newton diagram* of f at the origin. We denote the union of all compact facet of the Newton diagram $\Gamma_+(f)$ by $\Gamma(f)$ and call it the *Newton boundary* of f.

We auxiliary define $\Gamma_-(f)$ as the cone over the Newton boundary $\Gamma(f)$ whose vertex is the origin of $\mathbb{R}^2$.

For a face $P \subset \Gamma_+(f)$, the *truncation function* (or the *face function*) of f on P is defined by

$$f_P(x, y) := \sum_{(i,j) \in P \cap \mathbb{Z}^2} c_{(i,j)} x^i y^j.$$

Definition 2.3 (Newton non-degenerate/convenient)**.** A function $f \in \mathfrak{m}$ is *Newton non-degenerate* if, for any compact facet $P \subset \Gamma_+(f)$, the equation

$$\frac{\partial f_P}{\partial x} = \frac{\partial f_P}{\partial y} = 0$$

has no solution in $(\mathbb{C}^*)^2$.

A function $f \in \mathfrak{m}$ is *convenient* if the Newton boundary $\Gamma(f)$ intersects both of x- and y-axes.

If the Milnor number of f is finite, we can always assume that f is convenient by the fact that, for a sufficiently large integer $n \in \mathbb{Z}_{\geq 0}$, the topological types of the singularities of f and $f + x^n$ at the origin are the same.

We denote by V_k the sum of the volumes of k-dimensional faces of $\Gamma_-(f)$ containing the origin of $\mathbb{R}^2$. We define the *Newton number* $\nu(f)$ of f as the alternating sum of $k!V_k$, i.e.,

$$\nu(f) := 2!V_2 - 1!V_1 + V_0.$$

Theorem 2.4 (Kouchnirenko [12, Théorème I])**.** *For any convenient $f \in \mathbb{C}\{x, y\}$, the inequality $\mu(f) \geq \nu(f)$ holds. Moreover, $\mu(f) = \nu(f)$ if and only if f is Newton non-degenerate.*

2.3 Some remarks on 1-tacnodal curves

In this thesis, a curve on a projective surface is called a *1-tacnodal curve* if the curve has exactly one singular point at a smooth point of the surface whose topological type is A_3. The term "tacnode" means an A_3-singularity. In this subsection we prepare a lemma concerning a 1-tacnodal curve.

For a polynomial f and $p \in \mathbb{C}^2$, we use the notations $f_x(p) = \frac{\partial f}{\partial x}(p), f_y(p) = \frac{\partial f}{\partial y}(p)$ and so on. We set $\mathrm{Hess}(f)(p) = f_{xx}(p)f_{yy}(p) - f_{xy}(p)^2$ and

$$\begin{aligned} K(f)(p) := -f_{xy}(p)^3 f_{xxx}(p) &+ 3f_{xx}(p)f_{xy}(p)^2 f_{xxy}(p) \\ &- 3f_{xx}(p)^2 f_{xy}(p)f_{xyy}(p) + f_{xx}(p)^3 f_{yyy}(p). \end{aligned}$$

Lemma 2.5. *Suppose that a polynomial $f \in \mathbb{C}[x, y]$ satisfies $f_{xx}(p) \neq 0$. Then the curve $\{f = 0\} \subset \mathbb{C}^2$ has a tacnode at p if and only if f satisfies*

(1) $f(p) = f_x(p) = f_y(p) = 0,$

(2) $\mathrm{Hess}(f)(p) = 0,$

(3) $K(f)(p) = 0,$

(4) $a_{12}(p)^2 - 4f_{xx}(p)a_{04}(p) \neq 0,$

where

$$\begin{aligned} a_{12}(p) :=& f_{xy}(p)^2 f_{xxx}(p) - 2f_{xx}(p)f_{xy}(p)f_{xxy}(p) + f_{xx}(p)^2 f_{xyy}(p), \\ a_{04}(p) :=& f_{xy}(p)^4 f_{xxxx}(p) - 4f_{xx}(p)f_{xy}(p)^3 f_{xxxy}(p) \\ &+ 6f_{xx}(p)^2 f_{xy}(p)^2 f_{xxyy}(p) - 4f_{xx}(p)^3 f_{xy}(p)f_{xyyy}(p) + f_{xx}(p)^4 f_{yyyy}(p). \end{aligned}$$

Proof. For simplicity, we assume that p is the origin $(0,0)$ of $\mathbb{C}^2$. First, if the origin is a singular point then we can represent f as

$$f = Ax^2 + Bxy + Cy^2 + (\text{higher terms}),$$

where $(A, B, C) = (f_{xx}(0,0)/2, f_{xy}(0,0), f_{yy}(0,0)/2)$. If $\mathrm{Hess}(f)(0,0) \neq 0$, then the origin is an A_1-singularity of $\{f = 0\}$. Therefore $\mathrm{Hess}(f)(0,0) = 0$ for the origin to be an A_3-singularity. Then we can rewrite f as

$$f = \frac{1}{4A}(2Ax + By)^2 + (\text{higher terms}).$$

The tangent line of $\{f = 0\}$ at the origin is defined by

$$f_{xx}(0,0)x + f_{xy}(0,0)y = 0.$$

Now we define new coordinates (u, v) as

$$\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} f_{xx}(0,0) & f_{xy}(0,0) \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

and set

$$\hat{f}(u, v) := f(x(u, v), y(u, v)).$$

Note that the condition $f(0,0) = f_x(0,0) = f_y(0,0) = \mathrm{Hess}(f)(0,0) = 0$ is equivalent to $\hat{f}(0,0) = \hat{f}_u(0,0) = \hat{f}_v(0,0) = \mathrm{Hess}(\hat{f})(0,0) = 0$.

By direct computation, we obtain the equalities:

$$\begin{aligned}
\hat{f}_{uu}(0,0) &= \frac{1}{f_{xx}(0,0)}, \\
\hat{f}_{uv}(0,0) &= 0, \\
\hat{f}_{vv}(0,0) &= \frac{1}{f_{xx}(0,0)} \mathrm{Hess}(f)(0,0), \\
\hat{f}_{uvv}(0,0) &= \frac{1}{f_{xx}(0,0)^3} a_{12}(0,0), \\
\hat{f}_{vvv}(0,0) &= \frac{1}{f_{xx}(0,0)^3} K(f)(0,0), \\
\hat{f}_{vvvv}(0,0) &= \frac{1}{f_{xx}(0,0)^4} a_{04}(0,0).
\end{aligned} \tag{*}$$

By Kouchnirenko's Theorem 2.4, the condition that the singularity at the origin is A_3 can be rewritten as

$$\hat{f}_{uv}(0,0) = \hat{f}_{vv}(0,0) = \hat{f}_{vvv}(0,0) = 0, \;\; \hat{f}_{uu}(0,0) \neq 0$$

and

$$\hat{f}_{uvv}(0,0)^2 - 4\hat{f}_{uu}(0,0)\hat{f}_{vvvv}(0,0) \neq 0$$

on the new coordinate system. By (*), these conditions coincide with the conditions in the assertion. ☐

For $\mu = 1, 3$, let $U(\Delta, A_\mu)$ denote a locally closed subvariety in the complete linear system $|D(\Delta)|$ of $D(\Delta)$ which parametrizes the set of curves having exactly one singular point whose topological type is A_μ. Let $V(\Delta, A_\mu)$ be the closure of $U(\Delta, A_\mu)$ in $|D(\Delta)|$.

Corollary 2.6. *If $V(\Delta, A_3)$ is non-empty then* $\dim V(\Delta, A_3) \geq \sharp\Delta_{\mathbb{Z}} - 4$.

Proof. For $\mu = 1, 3$, we set

$$\Sigma(\Delta, A_\mu) := \{(C, p); p \text{ is a singular point of } C\} \subset U(\Delta, A_\mu) \times X(\Delta) \subset |D(\Delta)| \times X(\Delta).$$

For a curve $C \in U(\Delta, A_\mu) \subset V(\Delta, A_\mu)$, we choose a local coordinate system (x, y) of $X(\Delta)$ around the singular point $p = (x_0, y_0) \in C$. By Lemma 2.5, $\Sigma(\Delta, A_3)$ is defined in a neighborhood of C in $\Sigma(\Delta, A_3)$ by

$$f(x_0, y_0) = f_x(x_0, y_0) = f_y(x_0, y_0) = \mathrm{Hess}(f)(x_0, y_0) = K(f)(x_0, y_0) = 0. \qquad (**)$$

where f is a polynomial whose Newton polygon is Δ. Note that, by [7, Theorem (1.49)], the dimension of the Severi variety $V(\Delta, A_1)$ satisfies

$$\dim V(\Delta, A_1) = \dim \Sigma(\Delta, A_1) = \sharp\Delta_{\mathbb{Z}} - 1 - 1$$

and $\Sigma(\Delta, A_1)$ is defined by the first three equations of (**). Therefore, we obtain

$$\dim V(\Delta, A_3) \geq \dim \Sigma(\Delta, A_3) \geq \sharp\Delta_{\mathbb{Z}} - 1 - 3 = \sharp\Delta_{\mathbb{Z}} - 4.$$

☐

Chapter 3

Construction of certain singular curves via tropical geometry

The main claim in this chapter can be found in [20].

3.1 Statement of a result

We first prepare several notations and notions. Let $\Delta \subset \mathbb{R}^2$ be a polygon and $T \in \mathfrak{T}(\Delta)$ be a tropical curve whose Newton polygon is Δ. We use the notation S as the dual subdivision of T.

To state the claim we consider a union of polygons corresponding to a part of S. Let Δ' be a sub-polygon of Δ. A subset T' of T is called the *tropical sub-curve with respect to* Δ' if Δ' is a union of sub-polygons of S and T' is dual to the subdivision of Δ' induced by S. We denote it by $T|_{\Delta'}$. Note that if $\Delta' = \Delta$ then $T|_{\Delta'} = T$. The exact definition of this notion will be given in Definition 3.2 in Section 3.2. For a tropical sub-curve $T|_{\Delta'}$, let $v(T|_{\Delta'})$ denote the number of 4-valent vertices of $T|_{\Delta'}$ and $r(T|_{\Delta'})$ denote the number of bounded components of $\mathbb{R}^2 \setminus T|_{\Delta'}$.

The following claim asserts that we can get the Milnor number of an isolated plane curve singularity $(f, 0)$ from the tropical curve associated with a subdivision of $\Gamma_-(f)$.

Theorem 3.1 ([20]). *For any Newton non-degenerate and convenient isolated singularity $(f, 0)$, there is a polynomial $F := F_f \in \mathbb{C}\{\{t\}\}[x, y]$ such that $N_F = \mathrm{Conv}(\Gamma_-(f)) \subset \mathbb{R}^2$ and $T_F|_{\Gamma_-(f)}$ gives the Milnor number by*

$$\mu(f, 0) = v(T_F|_{\Gamma_-(f)}) + r(T_F|_{\Gamma_-(f)}).$$

For plane curve singularities, the real morsification due to A'Campo [1] and Gusein-Zade [6] gives an explicit way to understand the configuration of vanishing cycles. Our hope is that we can perform the same observation for tropical curves realized in $\Gamma_-(f)$.

Theorem 3.1 is an analogy of the following equality valid for a real morsification:

$$\mu(f) = \delta(f_s) + r(f_s),$$

where f_s is a real morsification of f, $\delta(f_s)$ is the number of double points of $\{f_s = 0\}|_{\mathbb{R}^2 \cap U}$, where U is a small neighborhood of the origin fixed before the deformation, and $r(f_s)$ is the number of regions in U bounded by $\{f_s = 0\}|_{\mathbb{R}^2}$. As a corollary of Theorem 3.1 we have the equality $\delta(f) = v(T_F|_{\Gamma_-(f)})$, where $\delta(f)$ is the δ-invariant of $(f, 0)$, see Corollary 3.6.

3.2 Proof of Theorem 3.1

In this section, let K denote the field of convergent Puiseux series $\mathbb{C}\{\{t\}\}$.

Let $F \in K[x, y]$ be a polynomial over K and

$$S_F : \Delta_1, \dots, \Delta_N$$

be the dual subdivision of the tropical amoeba T_F defined by F. Due to the structure theorem in tropical geometry, a tropical hypersurface has the structure of a polyhedral complex. In particular, a plane tropical curve is an embedded plane graph in $\mathbb{R}^2$.

Let $[u, v]$ denote the edge of the tropical curve T_F whose endpoints are 0-cells u and v. Set $[u, v) = [u, v] \setminus \{v\}$. We allow that one of the endpoints is at ∞. In this case, the other endpoint is contained in the 0-cells of the curve.

Let

$$S'_F : \Delta_{k_1}, \dots, \Delta_{k_m}$$

be a subset of S_F such that $\bigcup S'_F \setminus S_F^{[0]}$ is connected, where $S_F^{[0]}$ is the set of vertices of S_F. Let Δ' be a sub-polyhedron of N_F given as the union of S'_F.

Let $V = \{v_1, \dots, v_N\}$ and $E = \{[u, v]\ ;\ u, v \in V\}$ be the set of vertices and edges of T_F respectively.

Definition 3.2 (Tropical sub-curve)**.** A subset of T_F is called the *tropical sub-curve with respect to* Δ' if it has the structure of a metric (open) sub-graph (V', E') of the tropical curve T_F which satisfies the following conditions:
(1) the set of vertices $V' \subset V$ is given by $\{v_{k_1}, \dots, v_{k_m}\}$,
(2) the set of edges E' is given by the following manners: for each $[u, v] \in E$,

(i) if $u, v \in V'$ then $[u, v] \in E'$,

(ii) if $v = \infty$ and $u \in V'$ then $[u, v] \in E'$,

(iii) if $u \in V'$ and $v \in V \setminus V'$ then $[u, v') \in E'$, where v' is taken as the middle point of $[u, v]$.

We denote the tropical sub-curve of T_F with respect to Δ' by $T_F|_{\Delta'}$.

Example 3.3. (1) Let F be a polynomial over K given by

$$\begin{aligned} F =& 1 + tz + tw + t^3z^2 + t^2zw + t^3w^2 + t^6z^3 + t^4z^2w + t^4zw^2 + t^6w^3 \\ &+ t^{10}z^4 + t^7z^3w + t^{12}z^2w^2 + t^7zw^3 + t^{10}w^4 + t^{15}z^5 + t^{15}w^5. \end{aligned}$$

The Newton polygon N_F of F is $\mathrm{Conv}\{(0,0),(0,5),(5,0)\}$. See on the left in Figure D. This polynomial F is F_f in Theorem 3.1 for the singularity of $x^5 + x^2y^2 + y^5$ at the origin. The polyhedron Δ' in the figure is $\Gamma_-(f)$ for the singularity $(f,0)$. The tropical sub-curve $T_F|_{\Gamma_-(f)}$ with respect to $\Gamma_-(f)$ is as shown on the right. Since $\mu(f) = 11, v(T_F|_{\Gamma_-(f)}) = 6$ and $r(T_F|_{\Gamma_-(f)}) = 5$, the equality $\mu(f) = v(T_F|_{\Gamma_-(f)}) + r(T_F|_{\Gamma_-(f)})$ in Theorem 3.1 is verified.

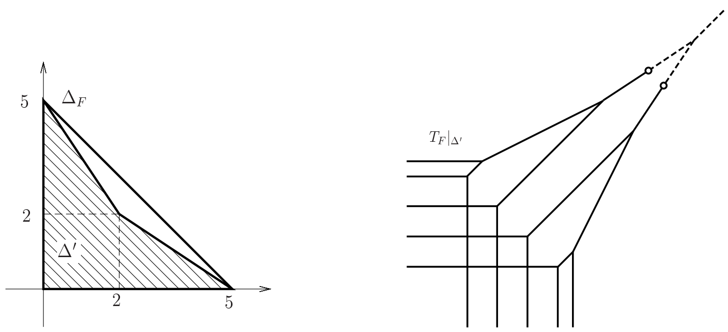

Figure D: N_F, Δ' and $T_F|_{\Delta'}$ in Example 3.3 (1).

(2) Let F be a polynomial over K given by

$$F = 1 + tz + tw + t^3z^2 + t^2zw + t^3w^2 + t^6w^3.$$

The Newton polygon $N_F = \Delta'$ of F is $\mathrm{Conv}\{(0,0),(2,0),(0,3)\}$. See on the left in Figure E. This polynomial F is F_f in Theorem 3.1 for the singularity of $x^2 + y^3$ at the origin. The polyhedron N_F in the figure is $\Gamma_-(f)$ for the singularity $(f,0)$. The tropical sub-curve $T_F|_{\Gamma_-(f)}$ with respect to $\Gamma_-(f)$ is as shown on the right. Since $\mu(f) = 2, v(T_F|_{\Gamma_-(f)}) = 1$ and $r(T_F|_{\Gamma_-(f)}) = 1$, the equality in Theorem 3.1 holds.

Suppose that f is convenient. For the lattice points $(i,j) \in \Gamma_-(f) \cap \mathbb{Z}^2$, we define a map $\nu_f|_{\Gamma_-(f)\cap\mathbb{Z}^2} : \Gamma_-(f) \cap \mathbb{Z}^2 \to \mathbb{R}$ by

$$\nu_f(i,j) = a_0 + a_1 + \cdots + a_i + b_0 + b_1 + \cdots + b_j,$$

where $\{a_k\}_{k\in\mathbb{N}}, \{b_k\}_{k\in\mathbb{N}}$ are non-negative strictly increasing sequences of integers. We then extend it to the whole domain $\Gamma_-(f)$ as a continuous piecewise linear function and obtain a map $\nu_f : \Gamma_-(f) \to \mathbb{R}$. Taking sufficiently large values for ν_f at the lattice points of the

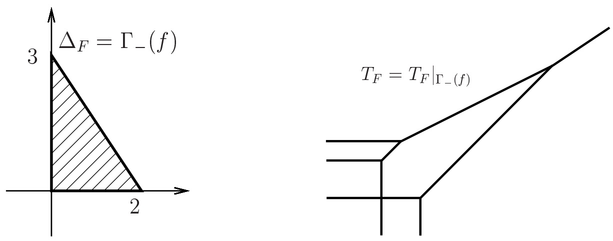

Figure E: $N_F(= \Delta' = \Gamma_-(f))$ and T_F in Example 3.3 (2).

Newton boundary of f, we may assume that each sub-polygon which is not a square is a triangle of area $1/2$.

Definition 3.4. We call the subdivision of $\Gamma_-(f)$ defined as above the *special subdivision of* $\Gamma_-(f)$ and each square in this subdivision the *special square*.

For example, the special subdivisions of the polyhedrons Δ' in Example 3.3 are as in the following figures. Here, Δ' in (1), (2) of Example 3.3 are represented by Δ_1, Δ_2, respectively.

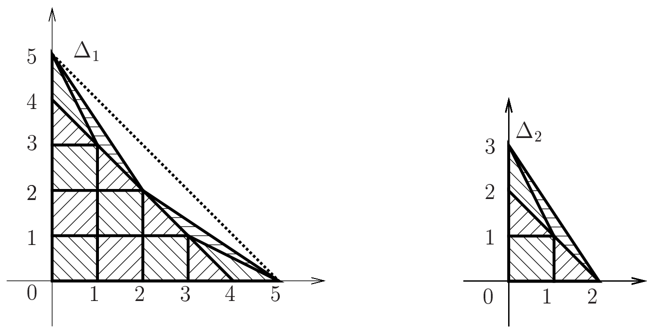

Figure F: The special subdivisions of Δ_1 and Δ_2.

Lemma 3.5. *Let $p, q \in \mathbb{N}$ be coprime integers. The number of special squares in the special subdivision of* $\Delta_{(p,q)} = \mathrm{Conv}\{(0,0),(p,0),(0,q)\} \subset \mathbb{R}^2$ *is* $(p-1)(q-1)/2$.

Proof. Let $\hat{\Delta}$ be the rectangle given by

$$\hat{\Delta} = \mathrm{Conv}\{(0,0),(p,0),(0,q),(p,q)\} \subset \mathbb{R}^2.$$

We consider the special subdivision of $\hat{\Delta}$. We decompose it into p vertical rectangles

$$\hat{\Delta}_i = ([i, i+1] \times \mathbb{R}) \cap \hat{\Delta} \subset \hat{\Delta}, \quad i = 0, \dots, p-1.$$

The special subdivision of $\hat{\Delta}$ induces a special subdivision of each $\hat{\Delta}_i$. Let ℓ be the segment connecting $(p,0)$ and $(0,q)$. We denote by I the number of special squares in $\hat{\Delta}$ which intersect ℓ. Similarly, we denote by I_i the number of special squares in $\hat{\Delta}_i$ which intersect ℓ. Obviously $I = \sum_{i=0}^{p-1} I_i$.

Let λ be the number of special squares of $\Delta_{(p,q)}$. Notice that $\lambda = \frac{1}{2}(pq - I)$ since $pq = 2\lambda + I$. Thus it is enough to show $I = p + q - 1$. Without loss of generality, we may assume $p < q$. Let k, l be integers such that $q = pk + l$ and $0 < l < p$. The segment ℓ can be denoted as

$$\Big(x, -\frac{q}{p}x + q\Big) = \Big(x, (pk + l - kx) - \frac{l}{p}x\Big), \quad x \in [0, p].$$

Set $\xi(x) = \frac{l}{p}x$. Then, I_i is calculated as

$$\begin{aligned} I_i &= pk + l - nk - \lfloor \xi(i) \rfloor - \Big\{pk + l - (i+1)k - \lfloor \xi(i+1) \rfloor - 1\Big\} \\ &= k + 1 - \lfloor \xi(i) \rfloor + \lfloor \xi(i+1) \rfloor \end{aligned}$$

for $i = 1, \ldots, p-2$ and

$$I_0 = I_{p-1} = k + 1,$$

where $\lfloor \alpha \rfloor$ means the largest integer not greater than $\alpha \in \mathbb{R}$. Thus, we obtain

$$\begin{aligned} I = \sum_{i=0}^{p-1} I_i &= I_0 + (p-2)(k+1) + \lfloor \xi(p-1) \rfloor - \lfloor \xi(1) \rfloor + I_{p-1} \\ &= pk + p + \lfloor \frac{l}{p}(p-1) \rfloor = p + q - 1. \end{aligned}$$

□

Proof of Theorem 3.1. Choose a polynomial F such that the Newton polygon N_F coincides with $\mathrm{Conv}(\Gamma_-(f))$. To determine coefficients of F, we take the convex function $\nu : N_F \cap \mathbb{Z}^2 \to \mathbb{R}$ as a linear extension of ν_f used in the definition of the special subdivision of $\Gamma_-(f)$, and define F as the *patchworking polynomial* defined by ν, that is,

$$F(x, y) = \sum_{(i,j) \in N_F \cap \mathbb{Z}^2} t^{-\nu(i,j)} x^i y^j.$$

In the rest of the proof, we check that $T_F|_{\Gamma_-(f)}$ satisfies the equality in the assertion. To calculate the number of special squares, we decompose $\Gamma_-(f)$ into two sub-polyhedrons as follows. First, set the coordinates of the intersection points of the Newton boundary $\Gamma(f)$ of f and the lattice as

$$\Gamma(f) \cap \mathbb{Z}^2 = \{(0, q), (P_1, Q_1), \ldots, (P_{n-1}, Q_{n-1}), (p, 0)\},$$

where $0 < P_1 < \cdots < P_{n-1} < p$. We set $P_0 = 0, P_n = p, Q_0 = q, Q_n = 0$ and define

$$p_i = |P_i - P_{i-1}|,\ q_i = |Q_i - Q_{i-1}|, \quad i = 1, \ldots, n.$$

Notice that $p = p_1 + \cdots + p_n$ and $q = q_1 + \cdots + q_n$. For $i = 1, \ldots, n$, we define the subset Δ_i of $\Gamma_-(f)$ as

$$\Delta_i = \mathrm{Conv}\{(P_{i-1}, Q_{i-1}), (P_{i-1}, Q_i), (P_i, Q_i)\} = \Delta_{(p_i, q_i)}$$

and

$$\Xi_1 := \bigcup_{i=1}^{n} \Delta_i \subset \Gamma_-(f), \quad \Xi_2 := \mathrm{Closure}\big(\Gamma_-(f) \setminus \Xi_1\big) \subset \Gamma_-(f).$$

Then, $\Gamma_-(f)$ decomposes as $\Gamma_-(f) = \Xi_1 \cup \Xi_2$. For $i = 1, 2$, we denote by $|\Xi_i|$ the number of special squares contained in the special subdivision of Ξ_i induced by that of $\Gamma_-(f)$. Then, using Lemma 3.5, we have

$$\begin{aligned}
|\Xi_1| &= \sum_{i=1}^{n} \frac{1}{2}(p_i - 1)(q_i - 1),\\
|\Xi_2| &= \sum_{i=1}^{n-1} p_i \cdot (q_i + \cdots + q_n)\\
&= \sum_{i=1}^{n-1} p_i \cdot \{q - (q_1 + \cdots + q_{i-1})\} = \mathrm{Vol}(\Xi_2).
\end{aligned}$$

Next we will show the following equalities:

$$|\Xi_1| + |\Xi_2| = v(T_F|_{\Gamma_-(f)}), \tag{3.1}$$

$$|\Xi_1| + |\Xi_2| - (n-1) = r(T_F|_{\Gamma_-(f)}). \tag{3.2}$$

The correspondence between subdivisions and tropical curves, introduced in Theorem 1.9, gives a 1-to-1 correspondence between parallelograms and 4-valent vertices. In our case, any 4-valent vertex corresponds to a special square. Thus, the number of special squares, $|\Xi_1| + |\Xi_2|$, coincides with the number of 4-valent vertices of $T_F|_{\Gamma_-(f)}$. Hence equality (3.1) holds.

We prove the other equality. There is a 1-to-1 correspondence between

$$\{\text{special square } \boxplus \text{ contained in special subdivision of } \Gamma_-(f)\ ;\ V(\boxplus) \cap \Gamma(f) = \emptyset\}$$

and

$$\mathrm{int}(\Gamma_-(f)) \cap \mathbb{Z}^2,$$

where $V(\boxplus)$ is the set of vertices of a special square $\boxplus$ in $\Gamma_-(f)$. Moreover, by Theorem 1.9, we have a 1-to-1 correspondence between the bounded regions contained in the complement of $T_F|_{\Gamma_-(f)} \subset \mathbb{R}^2$ and the interior lattice points $\mathrm{int}(\Gamma_-(f) \cap \mathbb{Z}^2)$ in $\Gamma_-(f)$. Since

$$\begin{aligned}&\sharp\{\text{special square } \boxplus \text{ contained in special subdivision of } \Gamma_-(f); V(\boxplus) \cap \Gamma(f) \neq \emptyset\}\\ &= n-1,\end{aligned}$$

we get

$$\begin{aligned} r(T_F|_{\Gamma_-(f)}) &= \sharp(\mathrm{int}(\Gamma_-(f)) \cap \mathbb{Z}^2) \\ &= |\Xi_1| + |\Xi_2| - (n-1). \end{aligned}$$

Thus equality (3.2) holds.

Set $L = \sum_{i=1}^n p_i q_i$. From equality (3.1), we get $L = 2|\Xi_1| + (p+q) - n$ as

$$\begin{aligned} |\Xi_1| &= \sum_{i=1}^n \frac{1}{2}(p_i - 1)(q_i - 1) = \frac{1}{2}\sum_{i=1}^n (p_i q_i - p_i - q_i + 1) \\ &= \frac{1}{2}\{L - (p+q) + n\}. \end{aligned}$$

For the Milnor number $\mu(f)$, we use Kouchnirenko's Theorem 2.4:

$$\mu(f) = 2V_2 - V_1 + 1,$$

where V_2 is the area of $\Gamma_-(f)$ and V_1 is the sum of lengths of the segments obtained as the intersection of $\Gamma_-(f)$ and the x- and y-axes. In our case, they are given by

$$V_2 = \frac{L}{2} + |\Xi_2|, \ V_1 = p + q.$$

Thus, we get

$$\begin{aligned} \mu(f) &= 2V_2 - V_1 + 1 \\ &= L + 2|\Xi_2| - (p+q) + 1 \\ &= \left\{2|\Xi_1| + (p+q) - n\right\} + 2|\Xi_2| - (p+q) + 1 \\ &= v(T_F|_{\Gamma_-(f)}) + r(T_F|_{\Gamma_-(f)}). \end{aligned}$$

□

Corollary 3.6. *Let $F := F_f$ be a polynomial obtained in Theorem 3.1. Then the δ-invariant $\delta(f)$ of $(f, 0)$ coincides with $v(T_F|_{\Gamma_-(f)})$.*

Proof. In [17, Theorem 22(4)], we have $r(f) = \sharp(\mathbb{Z}^2 \cap \Gamma(f)) - 1$, where $r(f)$ is the number of local irreducible components of $\{f = 0\}$ at 0. We also have

$$\begin{aligned}\mu(f) &= 2(|\Xi_1| + |\Xi_2|) - \left\{\sharp(\mathbb{Z}^2 \cap \Gamma(f)) - 2\right\} \\ &= 2v(T_F|_{\Gamma_-(f)}) - \left\{\sharp(\mathbb{Z}^2 \cap \Gamma(f)) - 2\right\}\end{aligned}$$

from the argument in the proof of Theorem 3.1. By Theorem 2.1, we obtain

$$\begin{aligned}2\delta(f) &= \mu(f) + r(f) - 1 \\ &= 2v(T_F|_{\Gamma_-(f)}) - \left\{\sharp(\mathbb{Z}^2 \cap \Gamma(f)) - 2\right\} + \sharp(\mathbb{Z}^2 \cap \Gamma(f)) - 1 - 1 \\ &= 2v(T_F|_{\Gamma_-(f)}).\end{aligned}$$

□

As in [1, 6], we can obtain the intersection form of vanishing cycles from the immersed curve of a real morsification. To study the intersection form in our tropical curve we need to fix "framings" on the edges of the curve in $\Gamma_-(f)$, though we do not have any good way to see these "framings".

Chapter 4

Tropicalization of 1-tacnodal curves

4.1 Tropical 1-tacnodal curves

4.1.1 Definition of tropical 1-tacnodal curves

In this subsection, we define a tropical 1-tacnodal curve. We can think of it as a tropical version of a 1-tacnodal curve, which is the main theorem in this thesis.

Set

$$\begin{aligned}
&\Delta_{\mathrm{I}} := \mathrm{Conv}\{(0,7),(1,0),(2,0)\},\ \Delta_{\mathrm{II}} := \mathrm{Conv}\{(0,7),(2,0),(3,0)\},\\
&\Delta_{\mathrm{III}} := \mathrm{Conv}\{(0,0),(2,0),(1,3)\},\ \Delta_{\mathrm{IV}} := \mathrm{Conv}\{(0,0),(2,0),(1,2)\}\\
&\Delta_{\mathrm{V}} := \mathrm{Conv}\{(0,0),(4,0),(0,1)\},\ \Delta_{\mathrm{VI}} := \mathrm{Conv}\{(1,0),(2,0),(0,3),(1,3)\},\\
&\Delta_{\mathrm{VII}} := \mathrm{Conv}\{(0,0),(1,0),(2,1),(0,1),(1,2)\},\\
&\Delta_{\mathrm{VIII}} := \mathrm{Conv}\{(0,0),(1,0),(0,1),(3,3)\},\ \Delta_{\mathrm{IX}} := \mathrm{Conv}\{(0,0),(1,0),(0,1),(4,2)\},\\
&\Delta_{\mathrm{E}} := \mathrm{Conv}\{(0,0),(2,0),(0,1),(1,2)\},
\end{aligned}$$

see Figure G.

We say that a polygon $P \subset \mathbb{R}^2$ is *equivalent* to $P' \subset \mathbb{R}^2$ if there exists an affine isomorphism $A \in \mathrm{Aff}(\mathbb{Z}^2)$ such that $A(P) = P'$, and denote it as $P \simeq P'$.

Definition 4.1. A tropical curve T is said to be *tropical* 1*-tacnodal* if the dual subdivision S of T contains one of the following polygons or unions of polygons:

(I) a triangle equivalent to Δ_{I},

(II) a triangle equivalent to Δ_{II},

(III) the union of a triangle equivalent to Δ_{III} and a triangle with edges of lattice length 1, 1 and 2 and without interior lattice point glued in such a way that they share the edge of lattice length 2,

(IV) the union of two triangles equivalent to Δ_{IV} which share the edge of lattice length 2,

(V) the union of two triangles equivalent to Δ_{V} which share the edge of lattice length 4,

(VI) a parallelogram equivalent to Δ_{VI},

(VII) a pentagon equivalent to Δ_{VII},

(VIII) a quadrangle equivalent to Δ_{VIII},

(IX) a quadrangle equivalent to Δ_{IX},

(E) the union of a quadrangle equivalent to Δ_{E} and a triangle with edges of lattice length 1, 1 and 2 and without interior lattice point which share the edge of lattice length 2,

and the rest of S consists of triangles of area 1/2.

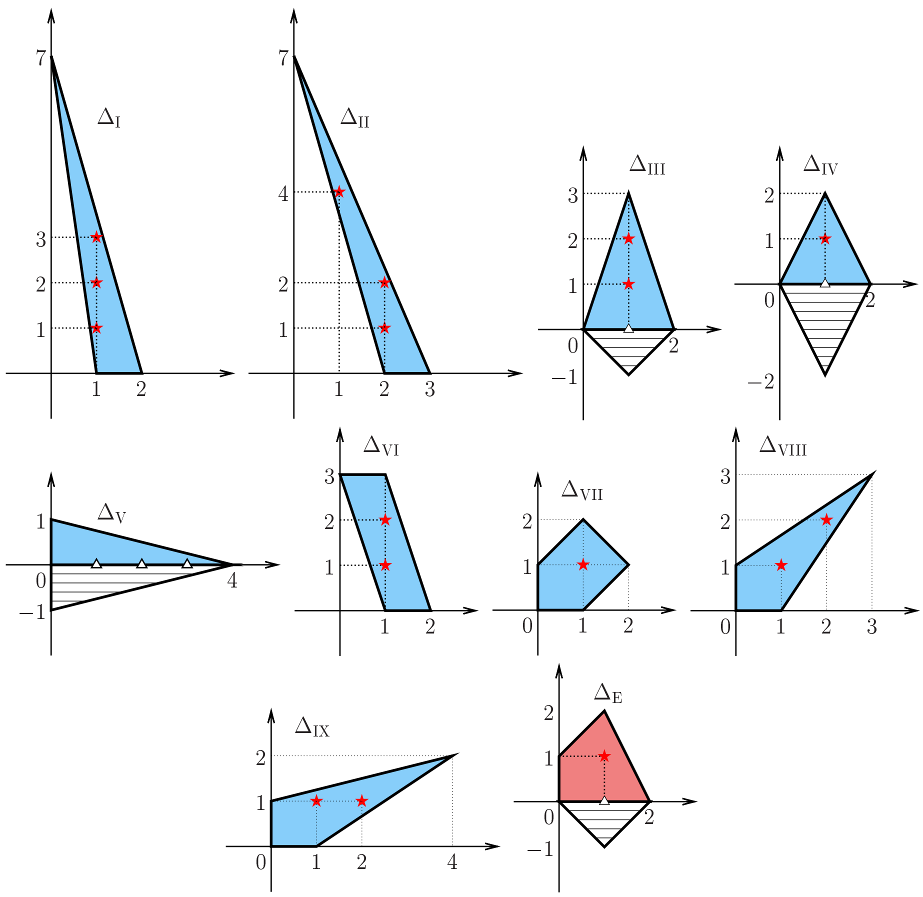

Figure G: Polygons in Definition 4.1. The notation $\triangle$ means a lattice point on the boundary which is not a vertex and the notation $\star$ means an interior lattice point.

4.1.2 Polygons corresponding to tropical 1-tacnodal curves

In this subsection, we mention some remark on polygons appearing in Definition 4.1.

We denote an m-gon which has edges of lattice lengths $\ell_1, \dots, \ell_m$ and I interior lattice points by

$$\Delta_m(I; \ell_1, \dots, \ell_m).$$

Similarly, we denote a parallel $2m$-gon which has m pairs of antipodal parallel edges of lattice length $\ell_1, \dots, \ell_m$ by

$$\Delta_{2m}^{\mathrm{par}}(I; \ell_1, \dots, \ell_m).$$

When we consider polygons of the same type $(I; \ell_1, \dots, \ell_m)$ simultaneously, we denote one as $\Delta_m(I; \ell_1, \dots, \ell_m)$ and the others as $\Delta'_m(I; \ell_1, \dots, \ell_m)$, $\Delta''_m(I; \ell_1, \dots, \ell_m)$ and so on.

Lemma 4.2. *The following holds up to the equivalence:*
(1) A triangle $\Delta_3(3; 1, 1, 1)$ is either Δ_{I} or Δ_{II}.
(2) A triangle $\Delta_3(2; 2, 1, 1)$ is Δ_{III}.
(3) A triangle $\Delta_3(1; 2, 1, 1)$ is Δ_{IV}.
(4) A triangle $\Delta_3(0; 4, 1, 1)$ is Δ_{V}.
(5) A parallelogram $\Delta_4^{\mathrm{par}}(2; 1, 1)$ is Δ_{VI}.
(6) A pentagon $\Delta_5(1; 1, 1, 1, 1, 1)$ is Δ_{VII}.
(7) A non-parallel quadrangle $\Delta_4(2; 1, 1, 1, 1)$ is equivalent to one of the following polygons:

$$\Delta_{\mathrm{VIII}}, \quad \Delta_{\mathrm{IX}}, \quad \mathrm{Conv}\{(1,0), (0,1), (2,1), (1,3)\}.$$

Proof. (1) We can take $A \in \mathrm{Aff}(\mathbb{Z}^2)$ which maps $\Delta_3(3; 1, 1, 1)$ to

$$\hat{\Delta}_n := \mathrm{Conv}\{(0, q), (n, 0), (n+1, 0)\}$$

for some q, n. By Pick's formula, we obtain $q = 7$. We remark that, $\hat{\Delta}_n$ and $\hat{\Delta}_{n+7}$ are equivalent by

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}. \tag{***}$$

Moreover we do not have to discuss the cases $n = 0$ and $n = 6$ since they have an edge of lattice length more than 1.

We get the isomorphisms

$$\hat{\Delta}_1 \simeq \hat{\Delta}_5, \quad \hat{\Delta}_2 \simeq \hat{\Delta}_4$$

by the reflection, and $\hat{\Delta}_1 \simeq \hat{\Delta}_3$ by

$$\begin{pmatrix} 3 & 1 \\ -7 & -2 \end{pmatrix}.$$

Because of the configuration of interior lattice points, we can show that $\hat{\Delta}_1 = \Delta_{\mathrm{I}}$ and

$\hat{\Delta}_2 = \Delta_{\mathrm{II}}$ are not isomorphic.

(2) For any $\Delta_3(2; 2, 1, 1)$, there exists $A \in \mathrm{Aff}(\mathbb{Z}^2)$ such that $\Delta_3(2; 2, 1, 1)$ maps to

$$\mathrm{Conv}\{(p, 0), (p + 2, 0), (0, q)\}$$

for some $p, q \in \mathbb{N}$. Then we have $q = 3$ by Pick's formula, and we may assume $p = 0, 1, 2$ by the isomorphism (***). But the cases $p = 0, 1$ do not satisfy the conditions of lattice length. Hence we get $p = 2$. This triangle is equivalent to Δ_{III}.

The claims (3), (4), (5) and (6) can be proved by the same method.

(7) We can split $P := \Delta_4(2; 1, 1, 1, 1)$ into two triangles which satisfy one of the following:

- $\Delta_3(1; 2, 1, 1)$ and $\Delta_3(0; 2, 1, 1)$ such that their intersection is a segment of length 2,
- $\Delta_3(2; 1, 1, 1)$ and $\Delta_3(0; 1, 1, 1)$ such that their intersection is a segment of length 1,
- $\Delta_3(0; 3, 1, 1)$ and $\Delta'_3(0; 3, 1, 1)$ such that their intersection is a segment of length 3,
- $\Delta_3(1; 1, 1, 1)$ and $\Delta'_3(1; 1, 1, 1)$ such that their intersection is a segment of length 1.

In the first case, $\Delta_3(1; 2, 1, 1)$ is uniquely determined as $\mathrm{Conv}\{(0, 0), (2, 0), (1, 2)\}$, so P has two descriptions

$$\hat{P}_1 := \mathrm{Conv}\{(0, 0), (2, 0), (1, 2), (0, -1)\}, \quad \hat{P}_2 := \mathrm{Conv}\{(0, 0), (2, 0), (1, 2), (1, -1)\}.$$

In the second case, by [19, Lemma 4.1], any triangle $\Delta_3(2; 1, 1, 1)$ is isomorphic to

$$Q := \mathrm{Conv}\{(0, 0), (3, 2), (2, 3)\}.$$

We denote the other triangle, which is $\Delta_3(0; 1, 1, 1)$, by R. We can easily check that Q is equivalent to

$$Q_1 := \mathrm{Conv}\{(0, 1), (0, 2), (0, 5)\}, \quad Q_2 := \mathrm{Conv}\{(0, 2), (0, 3), (0, 5)\}.$$

If the intersection of Q with R is $\mathrm{Conv}\{(0, 0), (3, 2)\} \subset Q$ or $\mathrm{Conv}\{(0, 0), (2, 3)\} \subset Q$, then we can assume that the intersection is the bottom edge of Q_1. Similarly, if the intersection is $\mathrm{Conv}\{(2, 3), (3, 2)\} \subset Q$, then we can assume that R shares the bottom edge of Q_2. Thus, the polygon P is equivalent to either

$$\hat{P}_3 := \mathrm{Conv}\{(1, 0), (2, 0), (0, 5), (2, -1)\} \text{ or } \hat{P}_4 := \mathrm{Conv}\{(2, 0), (3, 0), (0, 5), (3, -1)\}.$$

In the third and fourth cases, we obtain the following polygons in the same way as above:

$$\hat{P}_5 := \mathrm{Conv}\{(0, 0), (0, 1), (1, -1), (3, 0)\}, \quad \hat{P}_6 := \mathrm{Conv}\{(0, 0), (0, 1), (2, -1), (3, 0)\}.$$

Between the polygons $\hat{P}_1, \dots, \hat{P}_6$, we have the following isomorphisms:

$$\hat{P}_1 \simeq \hat{P}_3 \text{ by } \begin{pmatrix} -1 & 0 \\ 3 & -1 \end{pmatrix}, \quad \hat{P}_5 \simeq \hat{P}_4 \text{ by } \begin{pmatrix} -1 & -1 \\ 2 & 1 \end{pmatrix}, \quad \hat{P}_6 \simeq \hat{P}_2 \text{ by } \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}.$$

Notice that, the polygon $\hat{P}_2$ is the translation of $\mathrm{Conv}\{(1,0),(0,1),(2,1),(1,3)\}$. Also, the polygons $\hat{P}_3$ and $\hat{P}_4$ are equivalent to Δ_{IX} and Δ_{VIII} by

$$\begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix} : \mathbb{Z}^2 \to \mathbb{Z}^2,$$

respectively.

Furthermore, by the configuration of interior lattice points and vertices, we obtain $\Delta_{\mathrm{VIII}} \not\simeq \Delta_{\mathrm{IX}}$, $\Delta_{\mathrm{IX}} \not\simeq \mathrm{Conv}\{(1,0),(0,1),(2,1),(1,3)\}$, and $\mathrm{Conv}\{(1,0),(0,1),(2,1),(1,3)\} \not\simeq \Delta_{\mathrm{VIII}}$. □

Lemma 4.3. *A quadrangle* $\Delta_4(1;2,1,1,1)$ *is* Δ_{E}.

Proof. We can split $P = \Delta_4(1;2,1,1,1)$ into two polygons Q, R which are either

(3-1) $Q = \Delta_3(0;1,1,1)$, $R = \Delta_4(1;1,1,1,1)$ and these polygons share an edge of length 1,

(3-2) $Q = \Delta_3(0;2,1,1)$, $R = \Delta_4(0;2,1,1,1)$ and these polygons share the edge of length 2,

(3-3) $Q = \Delta_3(1;1,1,1)$, $R = \Delta_4(0;1,1,1,1)$ and these polygons share an edge of length 1,

(3-4) $Q = \Delta_3(1;2,1,1)$, $R = \Delta_3(0;1,1,1)$ and these polygons share an edge of length 1,

(3-5) $Q = \Delta_3(0;2,2,1)$, $R = \Delta_3(0;2,1,1)$ and these polygons share an edge of length 2, or

(3-6) $Q = \Delta_3(0;2,1,1)$, $R = \Delta_3(1;1,1,1)$ and these polygons share an edge of length 1.

Among them, case (3-5) can not occur by Lemma 4.12.

(3-1) If R is a parallelogram, then we can assume that R is

$$\mathrm{Conv}\{(1,0),(2,0),(0,2),(1,2)\}$$

and the common edge of R with Q is its bottom edge. Hence, we get

$$Q = \mathrm{Conv}\{(1,0),(2,0),(2,-1)\}$$

by Pick's formula, but their union does not satisfy the condition of P.

If R is not a parallelogram, then we can assume that R is

$$\mathrm{Conv}\{(0,0),(1,0),(0,1),(2,2)\}$$

and the common edge with Q is either

$$\mathrm{Conv}\{(0,0),(1,0)\} \text{ or } \mathrm{Conv}\{(1,0),(2,2)\}.$$

In the former case, Q is uniquely determined as

$$\mathrm{Conv}\{(0,0),(1,0),(0,-1)\}$$

and the union $Q \cup R = \mathrm{Conv}\{(0,-1),(1,0),(0,1),(2,2)\}$ is isomorphic to P. In the latter case, we can assume that R is

$$\mathrm{Conv}\{(1,0),(2,0),(0,2),(0,3)\}$$

and the common edge is the bottom edge. Then Q must be

$$\mathrm{Conv}\{(1,0),(2,0),(2,1)\},$$

but the union $Q \cup R$ does not satisfy the condition of P.
(3-2) We can assume that R is

$$\mathrm{Conv}\{(0,0),(2,0),(0,1),(1,1)\}$$

and the common edge is the bottom edge. Then Q must be either

$$\mathrm{Conv}\{(0,0),(2,0),(0,-1)\} \text{ or } \mathrm{Conv}\{(0,0),(2,0),(3,-1)\}.$$

In both cases, the union $Q \cup R$ is isomorphic to P.
(3-3) We can assume that R is

$$\mathrm{Conv}\{(0,0),(1,0),(0,1),(1,1)\},$$

but any union with Q does not satisfy the condition of P.
(3-4) We can assume that Q is

$$\mathrm{Conv}\{(0,0),(1,0),(-2,4)\}$$

and the common edge is its bottom edge. Then R must be

$$\mathrm{Conv}\{(0,0),(1,0),(1,-1)\}.$$

Their union $R \cup Q$ is isomorphic to P.

(3-6) We assume that R is

$$\mathrm{Conv}\{(1,0),(2,0),(0,3)\}$$

and the common edge is its bottom edge. Then Q must be either

$$\mathrm{Conv}\{(1,0),(2,0),(2,-2)\} \text{ or } \mathrm{Conv}\{(1,0),(2,0),(3,-2)\}.$$

In both cases, the union $Q \cup R$ is isomorphic to P. □

4.2 Definitions and Lemmata

4.2.1 Existence of 1-tacnodal curves for $\Delta_{\mathrm{I}}, \dots \Delta_{\mathrm{IX}}$

For a polygon P, we set

$$\mathcal{F}(P) := \{f \in \mathbb{C}[x,y]; N_f = P\}.$$

We denote the plane curve defined by $f \in \mathcal{F}(P)$ in $X(P)$ as V_f. We remark that V_f is a member of $|D(P)|$. We consider the following two conditions:

(S1) $V_f \subset X(P)$ is a 1-tacnodal curve whose singular point is contained in the maximal torus of $X(P)$,

(S2) V_f intersects the toric boundary $X(\partial P)$ transversally.

In the rest of this section, except cases (III), (IV), and (V), we only consider polygons whose edges are only of length one. Hence the condition (S2) is automatically satisfied except the three cases.

Lemma 4.4. *For each $i = \mathrm{I}, \mathrm{II}$ and given coefficients c_{ij} on the vertices $(i,j) \in V(P)$, there is a polynomial $f \in \mathcal{F}(\Delta_i)$ which has the fixed coefficients on the vertices and satisfies the conditions (S1), (S2). Furthermore, there is no polynomial $f \in \mathcal{F}(\Delta_i)$ that defines a curve with more complicated singularity than A_3, i.e., the curve does not have an isolated singularity whose Milnor number is more than 3.*

Proof. (I) We first show that we can assume that the coefficients on the vertices of Δ_{I} are 1. We transform the polynomial

$$f = c_{10}x + c_{20}x^2 + Axy + Bxy^2 + Cxy^3 + c_{07}y^7 \in \mathcal{F}(\Delta_{\mathrm{I}})$$

by substituting $x = X^{-1}, y = Y$ and multiplying X^2. Then we get a new polynomial

$$\tilde{f} := c_{20} + c_{10}X + AXY + BXY^2 + CXY^3 + c_{07}X^2Y^7.$$

By multiplying suitable constants to the variables and the whole polynomial, we can assume that $c_{20} = c_{10} = c_{07} = 1$. Transforming $\tilde{f}$ by $x = X^{-1}, y = Y$ again, we get

$$x + x^2 + A'xy + B'xy^2 + C'xy^3 + y^7.$$

We re-denote this polynomial by f.

For a polynomial

$$f = x + x^2 + Axy + Bxy^2 + Cxy^3 + y^7 \in \mathcal{F}(\Delta_{\mathrm{I}}),$$

we apply Lemma 2.5 and eliminate the variables by the system $f = f_x = f_y = \mathrm{Hess}(f) = K(f) = 0$. First, by $f = 0$, we can get A as

$$A = -\frac{x + x^2 + Bxy^2 + Cxy^3 + y^7}{xy}.$$

Therefore the system is reduced as

$$\begin{cases} (1)\ \ x^2 - y^7 = 0, \\ (2)\ \ -x - x^2 + Bxy^2 + 2Cxy^3 + 6y^7 = 0, \\ (3)\ \ \text{substituting } A \text{ for } \mathrm{Hess}(f) = 0, \\ (4)\ \ \text{substituting } A \text{ for } K(f) = 0. \end{cases}$$

Secondly, by equation (2), we can get B as

$$B = \frac{x + x^2 - 2Cxy^3 - 6y^7}{xy^2}.$$

Then the system is reduced as

$$\begin{cases} (1')\ \ x^2 - y^7 = 0, \\ (3')\ \ 4x^3 + 4x^4 + 4Cx^3y^3 + 60x^2y^7 - 49y^{14} = 0, \\ (4')\ \ 2Cx^3 + 7xy^4 + 77x^2y^4 + 7Cxy^7 - 42y^{11} = 0. \end{cases}$$

Thirdly, by equation (3'), we can get C as

$$C = \frac{-4x^3 - 4x^4 - 60x^2y^7 + 49y^{14}}{4x^3y^3}.$$

Then the system is reduced as

$$\begin{cases} x^2 - y^7 = 0, \\ 8x^5 + 8x^6 - 160x^4y^7 + 490x^2y^{14} - 343y^{21} = 0. \end{cases}$$

Hence we obtain $x = 8/5$ and the equation

$$y^7 - (8/5)^2 = 0. \tag{****}$$

Next, we check that the above f satisfies the condition (S1). Let $y_0, y_1, \ldots, y_6$ be the solutions of equation (****) and, for each $i = 1, \ldots, 6$, let $f^{(i)}$ denote the polynomial f with the solution $y = y_i$. By the above calculation, the curve $V_{f^{(i)}}$ defined by $f^{(i)}$ has a tacnode at $(8/5, y_i) \in (\mathbb{C}^*)^2$. Notice that the coefficients A, B and C of $f^{(i)}$ are determined by $x = 8/5$ and $y = y_i$. Let (s, t) be a singular point of $f^{(i)}$ on $V_{f^{(i)}}$. Solving $f_s^{(i)} = 0$, we obtain $s = s_0(t, y_0)$. Set

$$f_1(t, y_0) := f^{(i)}(s_0(t, y_0), t), \quad f_2(t, y_0) := f_t^{(i)}(s_0(t, y_0), t).$$

Eliminating y_0 from f_1, f_2 by $y_0^7 - (8/5)^2 = 0$, we obtain two equations with variable t. We can check that their greatest common divisor is $t^7 - (5/8)^2$. Thus, the singularities of $f^{(i)}$ are only tacnodes.

The coefficient A of $f^{(i)}$ depends only on the solution y_0 of (****) and we can check directly that the coefficients A for $y = y_i$ and $y = y_j$ are different if $i \neq j$. That is, the defining polynomials $f^{(i)}$ and $f^{(j)}$ are different for $i \neq j$. Therefore each $f^{(i)}$ satisfies the condition (S1).

(II) For the polynomial

$$f = x^2 + x^3 + Ax^2y + Bx^2y^2 + Cxy^4 + y^7 \in \mathcal{F}(\Delta_{\mathrm{II}}),$$

we apply Lemma 2.5 and eliminate the variables by the system $f = f_x = f_y = \mathrm{Hess}(f) = K(f) = 0$. First, by $f = 0$, we can get A as

$$A = -\frac{x^2 + x^3 + Bx^2y^2 + Cxy^4 + y^7}{x^2y}.$$

Therefore the system is reduced as

$$\begin{cases} (1)\ \ x^3 - Cxy^4 - 2y^7 = 0, \\ (2)\ \ -x^2 - x^3 + Bx^2y^2 + 3Cxy^4 + 6y^7 = 0, \\ (3)\ \ \text{substituting } A \text{ for } \mathrm{Hess}(f) = 0, \\ (4)\ \ \text{substituting } A \text{ for } K(f) = 0. \end{cases}$$

Secondly, by equation (2), we can get B as

$$B = \frac{x^2 + x^3 - 3Cxy^4 - 6y^7}{x^2y^2}.$$

Then the system is reduced as

$$\begin{cases} (1')\ \ x^3 - Cxy^4 - 2y^7 = 0, \\ (3')\ \ 8x^5 + 8x^6 - 4Cx^3y^4 + 20Cx^4y^4 - 4x^2y^7 + 116x^3y^7 - 28C^2x^2y^8 - 184Cxy^{11} - 256y^{14} = 0, \\ (4')\ \ \text{substituting } B \text{ for } (4)\ = 0. \end{cases}$$

Thirdly, by equation (1'), we can get C as

$$C = \frac{x^3 - 2y^7}{xy^4}.$$

Then the system is reduced as

$$\begin{cases} x^3 + y^7 + xy^7 = 0, \\ 4x^9 + 14x^6y^7 + 5x^7y^7 + 16x^3y^{14} + 11x^4y^{14} + 6y^{21} + 7xy^{21} = 0. \end{cases}$$

By direct computation, we can see that the solution of the above system is

$$(x, y) = (y_0^7, y_0), \tag{*****}$$

where y_0 is a solution of $y^{14} + y^7 + 1 = 0$.

Next, we check that the above f satisfies the condition (S1). Notice that the curve V_f defined by f has a tacnode at $(y_0^7, y_0) \in (\mathbb{C}^*)^2$, where y_0 is a solution of (*****). Let $(s, t) \in (\mathbb{C}^*)^2$ be a singular point of V_f. Then, we can easily check that the system $f(s,t) = f_x(s,t) = f_y(s,t) = y_0^{14} + y_0^7 + 1 = 0$ implies $t = y$. After substituting $y = t$ for $f(s,t), f_x(s,t), f_y(s,t)$, we obtain $s - y_0^7$ as their greatest common divisor. That is, the singularities of V_f are only tacnodes. Moreover, we can easily check that for two different solutions y_0 and y_0' of $y^{14} + y^7 + 1 = 0$, the triples (A, B, C) of the coefficients of the polynomial f, which are determined by y_0 and y_0', are different. Therefore, for each solution of $y^{14} + y^7 + 1 = 0$, the polynomial f satisfies the condition (S1). $\square$

Lemma 4.5. *For each $i = \mathrm{VI}, \mathrm{VII}, \mathrm{VIII}, \mathrm{IX}$, and given coefficients c_{ij} on the vertices $(i, j) \in V(P)$, there is a polynomial $f \in \mathcal{F}(\Delta_i)$ which has the fixed coefficients on the vertices and satisfies (S2) and (S2) if and only if*

- $c_{03}c_{20} = 64c_{10}c_{13}$, *if* $i = \mathrm{VI}$,
- $c_{21}c_{00}^2 = -4c_{01}c_{10}^2$, *and* $c_{12}c_{00}^2 = -4c_{10}c_{01}^2$, *if* $i = \mathrm{VII}$,
- $8^6 c_{33}c_{00}^5 = 5^5 c_{10}^3 c_{01}^3$, *if* $i = \mathrm{VIII}$,
- $256c_{42}c_{00}^5 = (41 + 38\sqrt{-1})c_{10}^4 c_{01}^2$, *or* $256c_{42}c_{00}^5 = (41 - 38\sqrt{-1})c_{10}^4 c_{01}^2$, *if* $i = \mathrm{IX}$.

Furthermore, there is no polynomial $f \in \mathcal{F}(\Delta_i)$ that defines a curve with more complicated singularity than A_3.

Proof. (VI) We transform the polynomial

$$f := c_{10}x + c_{20}x^2 + Axy + Bxy^2 + c_{03}y^3 + c_{13}xy^3 \in \mathcal{F}(\Delta_{\mathrm{VI}})$$

by substituting $x = X^{-1}, y = Y$ and multiplying X^2. Then we get the new polynomial

$$\tilde{f} := c_{10}X + c_{20} + A'XY + B'XY^2 + c_{03}X^2Y^3 + c_{13}XY^3.$$

By multiplying suitable constants to the variables and the whole polynomial, we can rewrite $\tilde{f}$ as

$$1 + X + A''XY + B''XY^2 + XY^3 + CX^2Y^3,$$

where

$$C = \frac{c_{03}c_{20}}{c_{10}c_{13}}.$$

For the polynomial

$$1 + x + Axy + Bxy^2 + xy^3 + Cx^2y^3$$

we apply Lemma 2.5 and eliminate the variables by the system $f = f_x = f_y = \mathrm{Hess}(f) = K(f) = 0$. First, by $f = 0$, we can get A as

$$A = -\frac{1 + x + Bxy^2 + xy^3 + Cx^2y^3}{xy}.$$

Therefore the system is reduced as

$$\begin{cases} (1) & -1 + Cx^2y^3 = 0, \\ (2) & -1 - x + Bxy^2 + 2xy^3 + 2Cx^2y^3 = 0, \\ (3) & \text{substituting } A \text{ for } \mathrm{Hess}(f) = 0, \\ (4) & \text{substituting } A \text{ for } K(f) = 0. \end{cases}$$

Secondly, by equation (1), we can get C as

$$C = \frac{1}{x^2y^3}.$$

Then the system is reduced as

$$\begin{cases} (2') & 1 - x + Bxy^2 + 2xy^3 = 0, \\ (3') & -4 + 8x - x^2 - 4Bxy^2 + 2Bx^2y^2 - 4xy^3 + 4x^2y^3 - B^2x^2y^4 - 4Bx^2y^5 - 4x^2y^6 = 0, \\ (4') & 48 - 144x + 36x^2 + 48Bxy^2 + 48xy^3 - 48Bx^2y^2 - 72x^2y^3 + 12B^2x^2y^4 + 24Bx^2y^5 = 0. \end{cases}$$

Thirdly, by equation (2'), we can get B as

$$B = -\frac{1 - x + 2xy^3}{xy^2}.$$

Then the system is reduced as

$$\begin{cases} 4x + 4xy^3 - 1 = 0, \\ 6x + 2xy^3 - 1 = 0. \end{cases}$$

The solution of the above system is

$$(x, y) = (1/8, y_0), \qquad (******)$$

where y_0 is a solution of $y^3 = 1$. Then we obtain

$$A = -9/y_0, \quad B = -9/y_0^2, \quad C = 1/x^2y^3 = 64.$$

Next, we check that the above f satisfies the condition (S1). Notice that the curve V_f defined by f has a tacnode at $(1/8, y_0) \in (\mathbb{C}^*)^2$, where y_0 is a solution of (******). Let $(s, t) \in (\mathbb{C}^*)^2$ be a singular point of V_f. Then, we obtain $t^3 - 1 = 0$ and $s = 1/8$ from the system $f(s, t) = f_x(s, t) = f_y(s, t) = 0$ and the equation $y_0^3 - 1 = 0$. That is, the singularities of V_f are only tacnodes. Moreover, we can easily check that for two different solutions y_0 and y_0' of $y^3 - 1 = 0$, the triples (A, B, C) of the coefficients of the polynomial f, which are determined by y_0 and y_0', are different. Therefore, for each solution of $y^3 - 1 = 0$, the polynomial f satisfies the condition (S1).

(VII) We can rewrite the polynomial

$$f = c_{00} + c_{10}x + c_{01}y + Axy + c_{21}x^2y + c_{12}xy^2 \in \mathcal{F}(\Delta_{\mathrm{VII}})$$

as

$$f = 1 + x + y + Axy + Bx^2y + Cxy^2$$

by the same manner as above, where

$$B = \frac{c_{21}c_{00}^2}{c_{01}c_{10}^2}, \quad C = \frac{c_{12}c_{00}^2}{c_{10}c_{01}^2}.$$

For the polynomial

$$f = 1 + x + y + Axy + Bx^2y + Cxy^2,$$

we apply Lemma 2.5 and eliminate the variables by the system $f = f_x = f_y = \mathrm{Hess}(f) = K(f) = 0$. First, by $f = 0$, we can get A as

$$A = -\frac{1 + x + y + Bx^2y + Cxy^2}{xy}.$$

Therefore the system is reduced as

$$\begin{cases} (1) \quad -1 - y + Bx^2y = 0, \\ (2) \quad -1 - x + Cxy^2 = 0, \\ (3) \quad \text{substituting } A \text{ for } \mathrm{Hess}(f) = 0, \\ (4) \quad \text{substituting } A \text{ for } K(f) = 0. \end{cases}$$

Secondly, by equations (1) and (2), we can get B and C as

$$B = \frac{1+y}{x^2y}, \quad C = \frac{1+x}{xy^2},$$

respectively. Then the system is reduced as

$$\begin{cases} (3') \quad 3 + 4x + 4y + 4xy = 0, \\ (4') \quad (1+y)^2(1+2x) = 0. \end{cases}$$

The solution of the above system is

$$(x, y) = (-1/2, -1/2),$$

and we obtain

$$A = B = C = -4.$$

Next, we check that the above f satisfies the condition (S1). Notice that the curve V_f defined by f has a tacnode at $(-1/2, -1/2) \in (\mathbb{C}^*)^2$. Let $(s,t) \in (\mathbb{C}^*)^2$ be a singular point of V_f. Then, we can solve $f(s,t) = f_x(s,t) = f_y(s,t) = 0$, and the solution is $(s,t) = (-1/2, -1/2)$. That is, the singularity of f is only one point and is a tacnode. Therefore the f satisfies the condition (S1).

(VIII) We can rewrite the polynomial

$$f = c_{00} + c_{10}x + c_{01}y + Axy + Bx^2y^2 + c_{33}x^3y^3 \in \mathcal{F}(\Delta_{\mathrm{VIII}})$$

as

$$f = 1 + x + y + Axy + Bx^2y^2 + Cx^3y^3$$

by the same manner as above, where

$$C = \frac{c_{33}c_{00}^5}{c_{10}^3c_{01}^3}.$$

For the polynomial

$$f = 1 + x + y + Axy + Bx^2y^2 + Cx^3y^3,$$

we apply Lemma 2.5 and eliminate the variables by the system $f = f_x = f_y = \mathrm{Hess}(f) = K(f) = 0$. First, by $f = 0$, we can get A as

$$A = -\frac{1 + x + y + Bx^2y^2 + Cx^3y^3}{xy}.$$

Therefore the system is reduced as

$$\begin{cases} (1) \quad -1 - y + Bx^2y^2 + 2Cx^3y^3 = 0, \\ (2) \quad -1 - x + Bx^2y^2 + 2Cx^3y^3 = 0, \\ (3) \quad \text{substituting } A \text{ for } \mathrm{Hess}(f) = 0, \\ (4) \quad \text{substituting } A \text{ for } K(f) = 0. \end{cases}$$

Secondly, by equation (1), we can get B as

$$B = \frac{1 + y - 2Cx^3y^3}{x^2y^2}.$$

Then the system is reduced as

$$\begin{cases} (2')\ \ x - y = 0, \\ (3')\ \ 4 - x + 4y + 4Cx^3y^3 = 0, \\ (4')\ \ \text{substituting } B \text{ for } (4)\ = 0. \end{cases}$$

Thirdly, by equation (3'), we can get C as

$$C = \frac{-4 + x - 4y}{4x^3y^3}.$$

Then the system is reduced as

$$\begin{cases} x - y = 0, \\ -8 + 3x - 8y = 0. \end{cases}$$

The solution of the above system is

$$(x, y) = (-8/5, -8/5),$$

and we also obtain

$$A = 75/64,\ \ B = -5^4/2^{12},\ \ C = 5^5/8^6.$$

Next, we check that the above f satisfies the condition (S1). Notice that the curve V_f defined by f has a tacnode at $(-8/5, -8/5) \in (\mathbb{C}^*)^2$. Let $(s, t) \in (\mathbb{C}^*)^2$ be a singular point of V_f. Then, we can solve $f(s, t) = f_x(s, t) = f_y(s, t) = 0$, and the solution is $(s, t) = (-8/5, -8/5)$. That is, the singularity of f is only one point and is a tacnode. Therefore the f satisfies the condition (S1).

(IX) We can rewrite the polynomial

$$f = c_{00} + c_{10}x + c_{01}y + Axy + Bx^2y + c_{42}x^4y^2 \in \mathcal{F}(\Delta_{\mathrm{IX}})$$

as

$$f = 1 + x + y + Axy + Bx^2y + Cx^4y^2$$

by the same manner as above, where

$$C = \frac{c_{42}c_{00}^5}{c_{10}^4c_{01}^2}.$$

For the polynomial

$$f = 1 + x + y + Axy + Bx^2y + Cx^4y^2,$$

we apply Lemma 2.5 and eliminate the variables by the system $f = f_x = f_y = \mathrm{Hess}(f) = K(f) = 0$. First, by $f = 0$, we can get A as

$$A = -\frac{1 + x + y + Bx^2y + Cx^4y^2}{xy}.$$

Therefore the system is reduced as

$$\begin{cases} (1) \quad -1 - y + Bx^2y + 3Cx^4y^2 = 0, \\ (2) \quad -1 - x + Cx^4y^2 = 0, \\ (3) \quad \text{substituting } A \text{ for } \mathrm{Hess}(f) = 0, \\ (4) \quad \text{substituting } A \text{ for } K(f) = 0. \end{cases}$$

Secondly, by equation (1), we can get B as

$$B = \frac{1 + y - 3Cx^4y^2}{x^2y}.$$

Then the system is reduced as

$$\begin{cases} (2') \quad -1 - x + Cx^4y^2 = 0, \\ (3') \;\; 1 - 4Cx^2y^2 - 8Cx^3y^2 - 4Cx^2y^3 + 4C^2x^6y^4 = 0, \\ (4') \text{ substituting } B \text{ for } (4) \;\; = 0. \end{cases}$$

Thirdly, by equation (2'), we can get C as

$$C = \frac{1 + x}{x^4y^2}.$$

Then the system is reduced as

$$\begin{cases} 4x + 4y + 3x^2 + 4xy = 0, \\ (4 + 3x)(16x + 8y + 24x^2 + 22xy + 4y^2 + 9x^3 + 12x^2y + 5xy^2) = 0. \end{cases}$$

The solutions of the above system are

$$(x_0, y_0) = \Big(-\frac{6}{5} + \frac{2}{5}\sqrt{-1}, \frac{2}{5} - \frac{4}{5}\sqrt{-1}\Big), \;\; (x_1, y_1) = \Big(-\frac{6}{5} - \frac{2}{5}\sqrt{-1}, \frac{2}{5} + \frac{4}{5}\sqrt{-1}\Big),$$

and we obtain

$$C = -\frac{41}{256} + \frac{19}{128}\sqrt{-1} \quad \text{if} \quad x = x_0, \quad C = \frac{41}{256} + \frac{19}{128}\sqrt{-1} \quad \text{if} \quad x = x_1.$$

Next, we check that the above f satisfies the condition (S1). Notice that the curve V_f defined by f has a tacnode at $(x_0, y_0) \in (\mathbb{C}^*)^2$. Let $(s,t) \in (\mathbb{C}^*)^2$ be a singular point of V_f. Then, we can solve $f(s,t) = f_x(s,t) = f_y(s,t) = 0$, and the solution is $(s,t) = (x_0, y_0)$. That is, the singularity of f is only one point and is a tacnode. Therefore the f satisfies the condition (S1). Also, we can check the condition (S1) for (x_1, y_1) by the same manner. □

Lemma 4.6. *For each $i = \mathrm{III}, \mathrm{IV}, \mathrm{V}$ and given coefficients c_{ij} on the vertices $(i,j) \in V(P)$, there is a polynomial $f \in \mathcal{F}(\Delta_i)$ which has the fixed coefficients on the vertices such that f defines a curve which has*

(III) an A_2-singularity on the toric divisor corresponding to the edge of length 2,

(IV) an A_1-singularity on the toric divisor corresponding to the edge of length 2,

(V) an intersection with the toric divisor corresponding to the edge of length 4 whose multiplicity is 4.

Proof. (III) We set

$$f := 1 + Ax + x^2 + Bxy + Cxy^2 + xy^3 \in \mathcal{F}(\Delta_{\mathrm{III}}).$$

Let $\sigma \subset \Delta_{\mathrm{III}}$ be the edge of length 2. The intersection point of $X(\sigma)$ and the curve defined by f is an A_2-singularity and this implies $A = \pm 2$.

We assume $A = 2$ and the singularity is at $(-1, 0)$. For $f = (1+x)^2 + Bxy + Cxy^2 + y^3$, the solution of $f(-1,0) = f_x(-1,0) = f_y(-1,0) = \mathrm{Hess}(f)(-1,0) = 0$ is $B = C = 0$. Therefore we obtain the polynomial $f := 1 + 2x + x^2 + xy^3 \in \mathcal{F}(\Delta_{\mathrm{III}})$.

(IV) We set

$$f := 1 + Ax + x^2 + Bxy + xy^2 \in \mathcal{F}(\Delta_{\mathrm{IV}}).$$

Let $\sigma \subset \Delta_{\mathrm{IV}}$ be the edge of length 2. The intersection point of $X(\sigma)$ and the curve defined by f is an A_1-singularity and this implies $A = \pm 2$.

We assume $A = 2$ and the singularity is at $(-1, 0)$. For $f = (1+x)^2 + Bxy + xy^2$, the solution of $f(-1,0) = f_x(-1,0) = f_y(-1,0) = 0$ is $B = 0$. Therefore we obtain the polynomial $f := 1 + 2x + x^2 + xy^2 \in \mathcal{F}(\Delta_{\mathrm{IV}})$.

(V) We can prove that the polynomial

$$f := (1 \pm x)^4 + y \in \mathcal{F}(\Delta_{\mathrm{V}})$$

satisfies the condition. □

Set

$$\hat{\Delta}_{\mathrm{III}} := \mathrm{Conv}\{(0,-1),(2,0),(0,3)\},$$
$$\hat{\Delta}_{\mathrm{IV}} := \mathrm{Conv}\{(0,-2),(2,0),(0,2)\},$$
$$\hat{\Delta}_{\mathrm{V}} := \mathrm{Conv}\{(0,-1),(4,0),(0,1)\},$$

see Figure H.

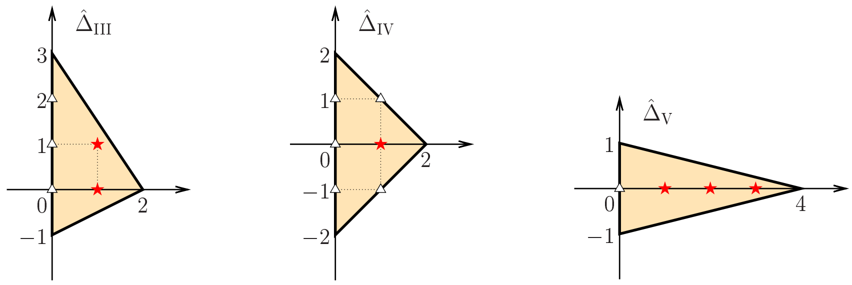

Figure H: Polygons $\hat{\Delta}_{\mathrm{III}}$, $\hat{\Delta}_{\mathrm{IV}}$ and $\hat{\Delta}_{\mathrm{V}}$. The notation $\vartriangle$ means a lattice point on the boundary which is not a vertex and the notation $\star$ means an interior lattice point.

For the polygons Δ_{III} and $\Delta_3(0;2,1,1)$ appearing in Definition 4.1 (III), the polynomial on Δ_{III} obtained in Lemma 4.6 induces the polynomial on $\Delta_3(0;2,1,1)$ as

$$1 + Ax + x^2 + y,$$

where $A = \pm 2$. Therefore the exceptional polygon in this case is $\hat{\Delta}_{\mathrm{III}}$.

For the polygons Δ_{IV} and $\Delta_3(1;2,1,1)$ appearing in Definition 4.1 (IV), the polynomial on Δ_{IV} obtained in Lemma 4.6 induces the polynomial on $\Delta_3(1;2,1,1)$ as

$$1 + Ax + x^2 + Bxy + xy^2,$$

where $A = \pm 2$. If $B = 0$, the exceptional polygon compatible with the data is $\hat{\Delta}_{\mathrm{IV}}$. Note that, if $B \neq 0$, the exceptional polygon compatible with the data is

$$\mathrm{Conv}\{(2,0),(0,2),(0,-1)\},$$

and it has no deformation pattern which defines an 1-tacnodal curve, see the discussion in Lemma 4.17.

For the polygons Δ_{V} and $\Delta_3(0;4,1,1)$ appearing in Definition 4.1 (V), the polynomial on Δ_{V} obtained in Lemma 4.6 induces the same polynomial on $\Delta_3(0;4,1,1)$. Therefore, the exceptional polygon compatible with the data is $\hat{\Delta}_{\mathrm{V}}$.

Lemma 4.7. *For each $i = \mathrm{III}, \mathrm{IV}, \mathrm{V}$, there is a deformation pattern $\phi \in \mathcal{F}(\hat{\Delta}_i)$ compatible* *with given data in Lemma 4.6 which has the fixed coefficients on the vertices such that the curve defined by ϕ in $X(\hat{\Delta}_i)$ is a 1-tacnodal curve.*

Proof. (III) For the polynomial

$$\phi := 1 + Ay + x^2y + By^2 + Cxy^2 + Dy^3 + y^4 \in \mathcal{F}(\hat{\Delta}_{\mathrm{III}})$$

we apply Lemma 2.5 and eliminate the variables by the system $\phi = \phi_x = \phi_y = \mathrm{Hess}(\phi) = K(\phi) = 0$. Notice that y is nonzero. First, by $\phi_x = 0$, we can get C as

$$C = -\frac{2x}{y}.$$

Therefore the system is reduced as

$$\begin{cases} (1)\ \ 1 + Ay - x^2y + By^2 + Dy^3 + y^4 = 0, \\ (2)\ \ A - 3x^2 + 2By + 3Dy^2 + 4y^3 = 0, \\ (3)\ \ 4By - 12x^2 + 12Dy^2 + 24y^3 = 0, \\ (4)\ \ -x^2 + Dy^2 + 4y^3 = 0. \end{cases}$$

Secondly, by equation (4), we obtain

$$x^2 = y^2(D + 4y).$$

Then the system is reduced as

$$\begin{cases} (1')\ \ 1 + Ay - 3y^4 + By^2 = 0, \\ (2')\ \ A - 8y^3 + 2By = 0, \\ (3')\ \ -B + 6y^2 = 0. \end{cases}$$

Thirdly, by equation (3'), we can get B as

$$B = 6y^2.$$

Then the system is reduced as

$$\begin{cases} (1'')\ \ 1 + Ay + 3y^4 = 0, \\ (2'')\ \ A + 4y^3 = 0. \end{cases}$$

Hence we obtain $A = -4y^3$ and then the equation

$$y^4 - 1 = 0.$$

The solution is

$$(A, B, C, D, x, y) = (-4y_0^3, 6y_0^2, -2x_0/y_0, D, x_0, y_0),$$

where y_0 is a solution of $y^4 - 1 = 0$ and x_0 is a solution of $x^2 = y_0^2(D + 4y_0)$.

Next, we check that the above ϕ has only one singularity and it is a tacnode. Notice that the curve V_ϕ defined by ϕ has a tacnode at $(x_0, y_0) \in \mathbb{C}^2$. Let $(s, t) \in \mathbb{C}^2$ be a singular point of V_ϕ. Then we solve $\phi(s, t) = \phi_x(s, t) = \phi_y(s, t) = 0$ and we check that the solution is only $(s, t) = (x_0, y_0)$. That is, the singularity of ϕ is only one point and is a tacnode.

(IV) We consider the following polynomial

$$\phi := 1 + Ay + By^2 + Cy^3 + y^4 + c_{11}xy + c_{13}xy^3 + x^2y^2 \in \mathcal{F}(\hat{\Delta}_{\mathrm{IV}}).$$

Note that c_{11}, c_{13} are both zero because of the form of the polynomials derived by Lemma 4.6 (IV).

For the polynomial

$$\phi := 1 + Ay + By^2 + Cy^3 + y^4 + x^2y^2 \in \mathcal{F}(\hat{\Delta}_{\mathrm{IV}}),$$

we eliminate the variables by the system $\phi = \phi_x = \phi_y = \mathrm{Hess}(\phi) = K(\phi) = 0$ by Lemma 2.5. Notice that y is nonzero. First, by $\phi_x = 0$, we obtain $x = 0$. Therefore the system is reduced as

$$\begin{cases} (1)\ \ 1 + Ay + By^2 + Cy^3 + y^4 = 0, \\ (2)\ \ A + 2By + 3Cy^2 + 4y^3 = 0, \\ (3)\ \ B + 3Cy + 6y^2 = 0, \\ (4)\ \ C + 4y = 0. \end{cases}$$

Secondly, by equation (4), we obtain

$$C = -4y.$$

Then the system is reduced as

$$\begin{cases} (1')\ \ 1 + 4y + By^2 - 3y^4 = 0, \\ (2')\ \ A + 2By - 8y^3 = 0, \\ (3')\ \ B - 6y^2 = 0. \end{cases}$$

Thirdly, by equation (3'), we can get B as

$$B = 6y^2.$$

Then the system is reduced as

$$\begin{cases} (1") \ \ 1 + Ay + 3y^4 = 0, \\ (2") \ \ A + 4y^3 = 0. \end{cases}$$

Hence we obtain $A = -4y^3$ and then the equation

$$y^4 - 1 = 0.$$

The solution is

$$(A, B, C, x, y) = (-4y_0^3, 6y_0^2, -4y_0, 0, y_0),$$

where y_0 is a solution of $y^4 - 1 = 0$.

Next, we check that the above ϕ has only one singularity and it is a tacnode. Notice that the curve V_ϕ defined by ϕ has a tacnode at $(0, y_0) \in \mathbb{C}^2$. Let $(s,t) \in \mathbb{C}^2$ be a singular point of V_ϕ. Then, we solve $\phi(s,t) = \phi_x(s,t) = \phi_y(s,t) = 0$ and check that the solution is only $(s,t) = (0, y_0)$. That is, the singularity of ϕ is only one point and is a tacnode.

(V) In this case, in order to achieve $\phi_{xx} \neq 0$, we exchange the variables x and y in ϕ.

For the polynomial

$$\phi := 1 + Ax + Bxy + Cxy^2 + xy^4 + x^2 \in \mathcal{F}(\hat{\Delta}_{\mathrm{V}}),$$

we eliminate the variables by the system $\phi = \phi_x = \phi_y = \mathrm{Hess}(\phi) = K(\phi) = 0$ by Lemma 2.5. Notice that x is nonzero. First, by $\phi = 0$, we obtain

$$A = -\frac{1 + x^2 + Bxy + Cxy^2 + xy^4}{x}.$$

Therefore the system is reduced as

$$\begin{cases} (1) \ \ (x-1)(x+1) = 0, \\ (2) \ \ B + 2Cy + 4y^3 = 0, \\ (3) \ \ \text{substituting } A \text{ for } \mathrm{Hess}(\phi) = 0, \\ (4) \ \ \text{substituting } A \text{ for } K(\phi) = 0. \end{cases}$$

Secondly, by equation (2), we obtain

$$B = -2y(C + 2y).$$

Then the system is reduced as

$$\begin{cases} (1') & (x-1)(x+1)=0, \\ (3') & 4x(C+6y^2)=0, \\ (4') & 192xy=0. \end{cases}$$

Thirdly, by equation (3'), we can get C as

$$C=-6y^2.$$

The solution is

$$(A,B,C,x,y)=(\mp 2,0,0,\pm 1,0).$$

Next, we check that the above ϕ has only one singularity and it is a tacnode. Suppose that the tacnode is at $(1,0)$. Let $(s,t)\in\mathbb{C}^2$ be a singular point of V_ϕ. Then we solve $\phi(s,t)=\phi_x(s,t)=\phi_y(s,t)=0$ and check that the solution is only $(s,t)=(1,0)$. That is, the singularity of ϕ is only one point and is a tacnode. We can check the condition for the case where the tacnode is at $(-1,0)$ by the same manner. □

Remark 4.8. Among the calculation in this section, there are finitely many polynomials which define 1-tacnodal curves except case (III) in Lemma 4.7. In case (III) in Lemma 4.7, we can get the conclusion without eliminating the variable D. This means that there exists a one-parameter family of deformation patterns which define 1-tacnodal curves.

4.2.2 Remarks on the polygon Δ_{E}

By the above discussion, for each tropical 1-tacnodal curve, except case (E), there is a "*degenerate model of a 1-tacnodal curve*" whose tropical amoeba is the tropical 1-tacnodal curve. In this subsection, we discuss what happens in case (E).

Lemma 4.9. *There is NO polynomial $f\in\mathcal{F}(\Delta_{\mathrm{E}})$ which defines a 1-tacnodal curve on $X(\Delta_{\mathrm{E}})$.*

Proof. We assume that a polynomial

$$f := c_{00}+Ax+c_{20}x^2+c_{01}y+Bxy+c_{12}xy^2$$

defines a 1-tacnodal curve. Then, since f_{xx} is non-zero, we can apply Lemma 2.5 and obtain $y=-B/2c_{12}$. Substituting it for $f_y=c_{01}+Bx+2c_{12}xy=0$, we get $c_{01}=0$, but this is a contradiction. □

On the other hand, there is a polynomial $f\in\mathcal{F}(\Delta_{\mathrm{E}})$ which has a Newton degenerate singularity on $X(\sigma)\subset X(\Delta_{\mathrm{E}})$, where $\sigma\subset\Delta_{\mathrm{E}}$ is the edge of length 2. Actually, we can calculate as follows: Set $P := \Delta_4(1;2,1,1,1), Q := \Delta_3(0;2,1,1)$. We consider the polynomial

$$f := c_{00} + Ax + c_{01}y + c_{20}x^2 + Bxy + c_{12}xy^2 \in \mathcal{F}(P).$$

By multiplying suitable constants to the variables and the whole polynomial, we can rewrite f as

$$1 + Ax + y + x^2 + Bxy + Cxy^2 \in \mathcal{F}(P).$$

If the curve $V_f \subset X(P)$ defined by f intersects $X(\sigma)$ at two points, we can easily check that these points are smooth points of V_f and the intersection $V_f \cap X(\sigma)$ is transversal. Therefore $V_f \cap X(\sigma)$ is exactly one point. Then, f can be rewritten as follows:

$$f = (\epsilon + x)^2 + y + Bxy + Cxy^2 \in \mathcal{F}(P),$$

where $\epsilon = \pm 1$. Set $(X, Y) := (x + \epsilon, y)$. Then f is rewritten as follows:

$$\tilde{f}(X, Y) := X^2 + BXY + (1 \mp B)Y + CXY^2 \mp CY^2.$$

Thus the most complicated isolated singular point defined by this polynomial at the origin (under the condition that the form of the polynomial does not change) is given as

$$X^2 \pm XY + \frac{1}{4}Y^2 + (\text{higher terms}).$$

By a direct computation, the curve defined by f has no singularity more complicated than A_3. Since f is irreducible, the curve has only a cusp as the singularity.

Applying the refinement argument mechanically in this case, we find that the edge $\Delta_4(1;2,1,1,1) \cap \Delta_3(0;2,1,1)$ does not correspond to a 1-tacnodal curve as follows: By the above discussion, the exceptional polygon in this case is $\hat{\Delta}_2$. We only consider the case of $\epsilon = 1$. The other case can be proved by the same argument. According to the explanation of a deformation pattern in Definition 1.16, we set

$$\phi := 1 + A'y + x^2y + B'y^2 + xy^2 + \frac{1}{4}y^3 \in \mathcal{F}(\hat{\Delta}_2).$$

By a direct computation, we get $\phi_{xx} \neq 0$. Using Lemma 2.5, we obtain $8B'x = 0$. Both cases $x = 0$ and $B' = 0$ contradict $\phi = 0$.

In [19], it is assumed that each polynomial f_i has only semi-quasi-homogeneous singularity since the paper only deals with the case of nodal or 1-cuspidal curves. This assumption may not be reasonable in the case of 1-tacnodal curves. Actually, when we list the possible polygons for 1-tacnodal curves we cannot ignore case (E). This is the reason why this case is included in the definition of tropical 1-tacnodal curves. Note that, in fact, by the above discussion, there is no degenerate model of 1-tacnodal curve corresponding to case (E).

4.3 Proof of Main Theorem

The main theorem of this thesis is the following:

Main Theorem. *Let $F \in K[z,w]$ be a polynomial which defines an irreducible 1-tacnodal curve. If the rank of the tropical amoeba T_F defined by F is more than or equal to the number of the lattice points of the Newton polygon of F minus four and the tropicalization of the curve defined by F in $X(N_F)$ has only isolated singularities, then T_F is a tropical 1-tacnodal curve.*

Let F be a polynomial in the assertion, T_F be the tropical amoeba defined by F, whose rank satisfies

$$\sharp\Delta_{\mathbb{Z}} - 1 \geq \mathrm{rk}(T_F) \geq \sharp\Delta_{\mathbb{Z}} - 4,$$

and S be the dual subdivision of T_F. We remark that, from the discussion in [19, Section 4], if $\sharp\Delta_{\mathbb{Z}} - 1 \geq \mathrm{rk}(T_F) \geq \sharp\Delta_{\mathbb{Z}} - 3$, then T_F is smooth, nodal or 1-cuspidal. Moreover, by Remark 4.14 below, we can assume that the rank of T_F is $\sharp\Delta_{\mathbb{Z}} - 4$.

From the discussion in [19, Subsection 3.3] and the equality $g(C^{(t)}) = \sharp\mathrm{Int}\Delta_{\mathbb{Z}} - 2$, we can see that

$$\sharp\partial\Delta_{\mathbb{Z}} - \sharp(V(S) \cap \partial\Delta) = 0 \text{ or } 1.$$

We decompose the proof into four cases

(A) S is a TP-subdivision and satisfies $\sharp\partial\Delta_{\mathbb{Z}} - \sharp(V(S) \cap \partial\Delta) = 0$,

(B) S is a TP-subdivision and satisfies $\sharp\partial\Delta_{\mathbb{Z}} - \sharp(V(S) \cap \partial\Delta) = 1$,

(C) S is NOT a TP-subdivision and satisfies $\sharp\partial\Delta_{\mathbb{Z}} - \sharp(V(S) \cap \partial\Delta) = 0$,

(D) S is NOT a TP-subdivision and satisfies $\sharp\partial\Delta_{\mathbb{Z}} - \sharp(V(S) \cap \partial\Delta) = 1$.

For each case, we remove polygons which cannot correspond to a 1-tacnodal curve and show that the remaining polygons are exactly tropical 1-tacnodal curves in Definition 4.1.

To explain the removing process more precisely, we prepare some terminologies.

Definition 4.10. A 2-dimensional polygon P is *1-tacnodal* if there is a polynomial $f \in \mathcal{F}(P)$ which defines a 1-tacnodal curve $V_f \in |D(P)|$ satisfying the conditions (S1) and (S2) in Subsection 4.2.1.

Let $\sigma := P_1 \cap P_2$ be an edge which is the intersection of 2-dimensional polygons P_1 and P_2. The edge σ is *1-tacnodal* if there is a pair of polynomials $(f_1, f_2) \in \mathcal{F}(P_1) \times \mathcal{F}(P_2)$ such that

- their truncation polynomials f_1^σ, f_2^σ on σ are same,
- each of the curves C_1 and C_2 defined by f_1 and f_2 has a smooth point or an isolated singular point at z in $X(\sigma)$,

- there exists a deformation pattern $\phi \in \mathcal{F}(\Delta_z)$ compatible with the above data which defines a 1-tacnodal curve in $X(\Delta_z)$.

It can be seen from the discussion in Subsection 4.2.1 that the polygons and the pairs of polygons appearing in Definition 4.1 are 1-tacnodal. To prove the theorem, for each of cases (A), (B), (C) and (D), we carry out the following arguments.

(1) Remove configurations of edges and interior lattice points of polygons which do not exist.

(2) Classify polygons that are not 1-tacnodal.

(3) From the list in (2), remove polygons which do not have 1-tacnodal edges.

In Subsection 4.3.1, we prepare lemmata for the non-existence of polygons in (1), and then prove the theorem for case (A), (B), (C) and (D) in Subsection 4.3.2, 4.3.3, 4.3.4 and 4.3.5, respectively.

4.3.1 Auxiliary definitions and lemmata

Lemma 4.11 (On interior lattice points)**.** *(1) The number of interior lattice points of a non-parallel quadrangle whose edges are length* 1 *is larger than* 0.
(2) For an integer $m \geq 5$*, the number of interior lattice points of an* m*-gon is larger than* 0.

Proof. (1) If a non-parallel $\Delta_4(0; 1, 1, 1, 1)$ exists, it can be decomposed into two triangles of area $1/2$. Thus, we can map this polygon to

$$\mathrm{Conv}\{(0,0),(1,0),(0,1),(p,q)\}$$

by some isomorphism. Then, from Pick's formula, we obtain

$$\frac{p+q}{2} = 1.$$

Hence $p = q = 1$. This is a parallelogram.
(2) It is obvious from the facts that the minimum pentagon is Δ_{VII} and any m-gon can be decomposed into polygons including a pentagon. □

Lemma 4.12 (Non-existence of some polygons)**.** *(1) The following polygons do NOT exist:*

$$\Delta_3(1;2,2,1),\ \ \Delta_3(1;3,1,1),\ \ \Delta_3(0;2,2,1),\ \ \Delta_3(0;3,2,1),\ \ \Delta_5(0;2,1,1,1,1).$$

(2) There is NO non-parallel quadrangle $\Delta_4(0; 2, 2, 1, 1)$.

Proof. (1) The first triangle is equivalent to

$$\mathrm{Conv}\{(p,0),(p+2,0),(0,q)\}.$$

By Pick's formula, we obtain $q = 5/2$. But this contradicts $q \in \mathbb{Z}$. We can easily check the non-existence of the second, third and fourth triangles. If there exists a pentagon $\Delta_5(0;2,1,1,1,1)$, we can split it into two quadrangles $\Delta_4(0;1,1,1,1)$ and $\Delta_4'(0;1,1,1,1)$. But these quadrangles are parallelograms by the fact that is proved in Lemma 4.11 (1). Thus the union can not be a pentagon.

(2) If it exists, then the edges of length 2 are either adjacent or in opposite sides. The former case can not occur since a triangle $\Delta_3(0;2,2,1)$ does not exist. In the latter case, we can split it into two triangles $\Delta_3(0;2,1,1)$ and $\Delta_3'(0;2,1,1)$. We can assume that one of the triangles is isomorphic to $\mathrm{Conv}\{(0,0),(1,0),(0,2)\}$ and the common edge is the bottom edge. Then, by Pick's formula, the last vertex of $\Delta_4(0;2,2,1,1)$ must be one of the following lattice points

$$(0,-2),\quad (1,-2),\quad (2,-2),$$

but all of them do not satisfy the required conditions. □

Lemma 4.13. *For the polygon*

$$P := \mathrm{Conv}\{(0,0),(2,0),(0,1),(2,1)\},$$

the polynomial

$$f := c_{00} + Ax + c_{20}x^2 + c_{01}y + Bxy + c_{21}x^2y \in \mathcal{F}(P)$$

satisfies $f = f_x = f_y = \mathrm{Hess}(f) = 0$ *if and only if*

$$c_{21}c_{00} = c_{20}c_{01}.$$

Moreover, if f satisfies $f = f_x = f_y = \mathrm{Hess}(f) = 0$, i.e., $V_f \subset X(P)$ has a singularity more complicated than A_1, then f has the form

$$(y+1)(x\pm 1)^2$$

up to multiplication of a non-zero constant. In particular, the set of singularities of V_f is non-isolated.

Proof. By direct computation. □

Remark 4.14 (Known Results)**.** (1) Let $I \geq 0$, $s, t, u \geq 1$ be integers such that

$$0 \leq I + (s-1) + (t-1) + (u-1) \leq 2.$$

For each $(I; s, t, u)$, a triangle $\Delta_3(I; s, t, u)$ is uniquely determined up to the equivalence as follows:

$$\begin{aligned}
\Delta_3(2;1,1,1) &\simeq \mathrm{Conv}\{(0,0),(3,2),(2,3)\},\\
\Delta_3(1;2,1,1) &\simeq \mathrm{Conv}\{(0,0),(2,0),(1,2)\},\\
\Delta_3(1;1,1,1) &\simeq \mathrm{Conv}\{(0,0),(1,2),(2,1)\},\\
\Delta_3(0;3,1,1) &\simeq \mathrm{Conv}\{(0,0),(3,0),(0,1)\},\\
\Delta_3(0;2,1,1) &\simeq \mathrm{Conv}\{(0,0),(2,0),(0,1)\},\\
\Delta_3(0;1,1,1) &\simeq \mathrm{Conv}\{(0,0),(1,0),(0,1)\}.
\end{aligned}$$

(2) For integers $I \in \{0, 1\}$, $s, t \geq 1$ such that

$$0 \leq I + 2(s-1) + 2(t-1) \leq 2,$$

a parallelogram $\Delta_4^{\mathrm{par}}(I; s, t)$ is uniquely determined up to the equivalence as follows:

$$\begin{aligned}
\Delta_4^{\mathrm{par}}(1;1,1) &\simeq \mathrm{Conv}\{(0,0),(1,0),(1,2),(2,2)\},\\
\Delta_4^{\mathrm{par}}(0;2,1) &\simeq \mathrm{Conv}\{(0,0),(2,0),(0,1),(2,1)\},\\
\Delta_4^{\mathrm{par}}(0;1,1) &\simeq \mathrm{Conv}\{(0,0),(1,0),(0,1),(1,1)\}.
\end{aligned}$$

(3) The polygons in this remark are not 1-tacnodal (By [19, Lemma 4.2] and Lemma 4.13, or direct computation).

Lemma 4.15 (Describing some polygons)**.** *(1) Let $I \geq 0$, $s, t, u \geq 1$ be integers such that*

$$I + (s-1) + (t-1) + (u-1) = 3.$$

For each $(I; s, t, u)$, a triangle $\Delta_3(I; s, t, u)$ has the following isomorphisms:

$$\begin{aligned}
\Delta_3(3;1,1,1) &\simeq \Delta_{\mathrm{I}}, \Delta_{\mathrm{II}},\\
\Delta_3(2;2,1,1) &\simeq \mathrm{Conv}\{(0,0),(2,0),(1,3)\},\\
\Delta_3(0;4,1,1) &\simeq \mathrm{Conv}\{(0,0),(0,1),(4,0)\},\\
\Delta_3(0;2,2,2) &\simeq \mathrm{Conv}\{(0,0),(2,0),(0,2)\}.
\end{aligned}$$

(2) A quadrangle $\Delta_4(0; 2, 1, 1, 1)$ is uniquely determined as $\mathrm{Conv}\{(0,0),(2,0),(0,1),(1,1)\}$ *up to the equivalence.*

Proof. (1) These claims, except the last case, are the same as Lemma 4.2. We prove the last one. Without loss of generality, the polygon can be assumed to be

$$\mathrm{Conv}\{(p,0),(p+2,0),(0,q)\}.$$

From Pick's formula, we obtain $q = 2$ and $p = 2k$ for some $k \in \mathbb{Z}$. Thus, by the isomorphism

$$\begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix} : \mathbb{Z}^2 \to \mathbb{Z}^2,$$

it is mapped to the polygon $\mathrm{Conv}\{(0,0),(2,0),(0,2)\}$.

(2) We can split $P = \Delta_4(0;2,1,1,1)$ into two polygons Q, R which are either

- $Q = \Delta_3(0;2,1,1)$, $R = \Delta_3(0;1,1,1)$ and these polygons share an edge of length 1, or
- $Q = \Delta_3(0;1,1,1)$, $R = \Delta_4(0;1,1,1,1)$ and these polygons share an edge of length 1.

In the former case, we can assume that Q is

$$\mathrm{Conv}\{(0,0),(1,0),(0,2)\}$$

and the common edge is its bottom edge. Then the last vertex of P must be $(1,-1)$. In the latter case, we can assume that R is

$$\mathrm{Conv}\{(0,0),(1,0),(0,1),(1,1)\}$$

and the common edge is its bottom edge. Then the last vertex of P must be either $(0,-1)$, or $(1,-1)$. All of them are equivalent to

$$\mathrm{Conv}\{(0,0),(2,0),(0,1),(1,1)\}.$$

□

Lemma 4.16 (Non 1-tacnodal polygons)**.** *The following polygons are NOT 1-tacnodal polygons:*

(1) $\Delta_3(0;2,2,2)$,
(2) $\Delta_3(0;4,1,1)$,
(3) $\Delta_3(2;2,1,1)$,
(4) $\Delta_4(0;2,1,1,1)$,
(5) $\mathrm{Conv}\{(1,0),(0,1),(2,1),(1,3)\}$.

Proof. (1) This is by the fact that the Milnor number of an isolated singularity of a projective conic does not exceed 1.

(2) Notice that this polygon is uniquely determined as $\mathrm{Conv}\{(0,0),(0,1),(4,0)\}$. Then a polynomial f with this Newton polygon has no singularity since f_y is a non-zero constant.

(3) We assume that a polynomial

$$f := 1 + Ax + x^2 + Bxy + Cxy^2 + xy^3 \in \mathcal{F}(\Delta_3(2;2,1,1))$$

satisfies the condition (S1). Since the polynomial f satisfies $f_{xx} \neq 0$, the system $f = f_x = f_y = \mathrm{Hess}(f) = K(f) = 0$ must have a solution. But, we obtain $K(f) = 48x$. This is a contradiction.

(4) Notice that this polygon is uniquely determined as $\mathrm{Conv}\{(0,0),(2,0),(0,1),(1,1)\}$. We set a polynomial f as

$$f := c_{00} + Ax + c_{20}x^2 + c_{01}y + c_{11}xy \in \mathcal{F}(\mathrm{Conv}\{(0,0),(2,0),(0,1),(1,1)\}).$$

Then we have $\mathrm{Hess}(f) = -c_{11}^2 \neq 0$.

(5) We assume that a polynomial

$$f := c_{10}x + c_{01}y + Axy + c_{21}x^2y + Bxy^2 + c_{13}xy^3$$

defines a 1-tacnodal curve. Then, since f_{xx} is non-zero, we can apply Lemma 2.5 and obtain

$$4c_{01}x(c_{13}y^3 + c_{10}) = -4c_{01}c_{13}xy^3 = 0.$$

This is a contradiction. □

Set

$$\begin{aligned}
\hat{\Delta}_1 &= \mathrm{Conv}\{(2,0),(0,1),(0,-1)\},\\
\hat{\Delta}_2 &= \mathrm{Conv}\{(2,0),(0,2),(0,-1)\},\\
\hat{\Delta}_3 &= \mathrm{Conv}\{(3,0),(0,1),(0,-1)\},
\end{aligned}$$

see Figure I.

Lemma 4.17 (Non 1-tacnodal edges). *The following edges σ are not 1-tacnodal edges:*
(1) the edge $\Delta_3(0;2,1,1) \cap \Delta_3(0;2,1,1)$ of length 2,
(2) the edge $\Delta_3(1;2,1,1) \cap \Delta_3(0;2,1,1)$ of length 2 and the edge $\Delta_3(1;2,1,1) \cap \Delta_4(0;2,1,1,1)$ of length 2,
(3) the edge $\Delta_3(0;3,1,1) \cap \Delta_3(0;3,1,1)$ of length 3,
(4) the edge $\Delta_4(0;2,1,1,1) \cap \Delta_3(0;2,1,1)$ of length 2,
(5) the edge $\Delta_4(0;2,1,1,1) \cap \Delta_4(0;2,1,1,1)$ of length 2,

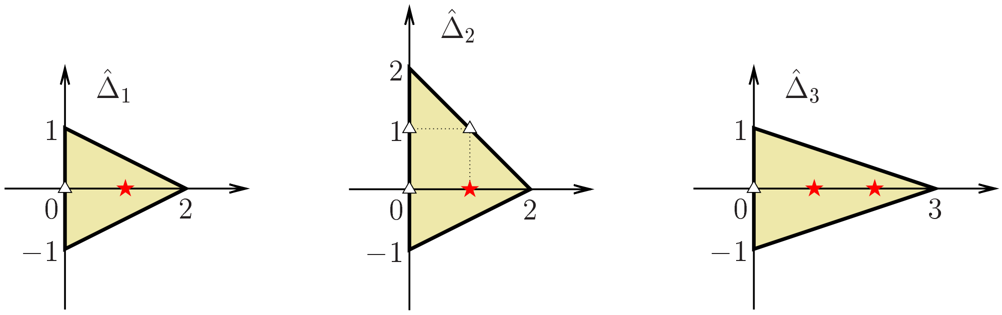

Figure I: Polygons $\hat{\Delta}_1, \hat{\Delta}_2$ and $\hat{\Delta}_3$. The notation $\triangle$ means a lattice point on the boundary which is not a vertex and the notation $\star$ means an interior lattice point.

(6) the edge $\Delta_4^{\mathrm{par}}(0;2,1) \cap \Delta_3(0;2,1,1)$ *of length* 2 *and the edge* $\Delta_4^{\mathrm{par}}(0;2,1) \cap \Delta_4(0;2,1,1,1)$ *of length* 2*,*
(7) the edge $\Delta_3(0;2,2,2) \cap \Delta_3(0;2,1,1)$ *of length* 2*.*

Proof. The assertion for cases (1), (3), (4) and (5) are already proved in [19, Lemma 3.9, 3.10 and 4.4]. Here we only prove (2), (6) and (7).
(2) Set $P := \Delta_3(1;2,1,1), Q := \Delta_3(0;2,1,1)$. It is easy to check that a curve in $|D(P)|$ cannot have an isolated singularity more complicated than A_1. Also, we can easily check that if a curve V_f intersects $X(\sigma)$ at two points then the points are smooth points of V_f and those intersections are transversal. Therefore we can set

$$f := (x+\epsilon)^2 + Axy + xy^2 \in \mathcal{F}(P),$$

where $\epsilon = \pm 1$ and suppose that f defines a curve which has an A_1-singularity on $X(\sigma) \subset X(P)$. With a simple calculation, we obtain $A = 0$. The polynomial corresponding to the polygon Q becomes

$$f' := (x+\epsilon)^2 + y \in \mathcal{F}(Q).$$

Then the exceptional polygon in this case is $\hat{\Delta}_2$. According to the explanation of a deformation pattern in Definition 1.16, we set

$$\phi := 1 + A'y + \epsilon x^2 y + B'y^2 + y^3 \in \mathcal{F}(\hat{\Delta}_2).$$

In the case $\epsilon = 1$, we get $\phi_{xx} \neq 0$ by $y \neq 0$. Using Lemma 2.5, we obtain $48y^3 = 0$, but this is a contradiction. We also have a contradiction in the case $\epsilon = -1$.

(6) Set $P := \Delta_4^{\mathrm{par}}(0;2,1), Q := \Delta_3(0;2,1,1)$. For P, we set

$$f := (\epsilon + x)^2 + (1 + Ax + x^2)y \in \mathcal{F}(P),$$

where $\epsilon = \pm 1$. Then a polynomial corresponding to Q must be

$$f' := (\epsilon + x)^2 + y \in \mathcal{F}(Q).$$

Then the exceptional polygon in this case is $\hat{\Delta}_1$.

If $\epsilon = 1$, ϕ is given as

$$\phi := 1 + x^2 y + y^2 + A'y \in \mathcal{F}(\hat{\Delta}_1),$$

and we can easily check that the solution of the system $\phi = \phi_x = \phi_y = \mathrm{Hess}(\phi) = 0$ does not exist. The case $\epsilon = -1$ can be proved by the same argument.

(7) Set $P := \Delta_3(0; 2, 2, 2), Q := \Delta_3(0; 2, 1, 1)$. Without loss of generality, we can assume that P and Q are

$$P = \mathrm{Conv}\{(0,0),(2,0),(0,2)\}, \quad Q = \mathrm{Conv}\{(0,0),(2,0),(0,-1)\}.$$

For P, we set

$$f := 1 + 2\epsilon x + x^2 + By + y^2 + Cxy \in \mathcal{F}(P),$$

where $\epsilon = \pm 1$. Applying the new coordinates $(X, Y) := (x + \epsilon, y)$ for f, we obtain

$$f = X^2 + (B - C\epsilon)Y + CXY + Y^2.$$

Notice that, if $\mathrm{Hess}(f) = C^2 - 4 = 0$, f defines a line of multiplicity 2, that is, f has non-isolated singularity. Therefore we may assume $C^2 - 4 \neq 0$. If $B - C\epsilon \neq 0$, the exceptional polygon in this case is $\hat{\Delta}_1$. If $B - C\epsilon = 0$, then $(\epsilon, 0) \in \mathbb{C}^2$ is an A_1-singularity, i.e., f has the form $f = X^2 + CXY + Y^2$. Hence, the exceptional polygon in this case is $\hat{\Delta}_2$. The conclusion is derived by the same calculation as in (7) for the former case and in (2) for the latter case, respectively. □

Remark 4.18 (On an edge of length 1)**.** Let Δ_1, Δ_2 be polygons such that their intersection $\sigma := \Delta_1 \cap \Delta_2$ is an edge of length 1. The edge σ is NOT an 1-tacnodal edge. Actually, we can prove it as follows: For integers $m_1, m_2 > 0$ and the triangle

$$\hat{\Delta} := \mathrm{Conv}\{(1,0),(0,m_1),(0,-m_2)\},$$

a polynomial $\phi \in \mathcal{F}(\hat{\Delta})$ can be given as

$$\phi = 1 + \psi(y) + xy^{m_2},$$

where $\psi \in \mathbb{C}[y]$ is a polynomial in y which satisfies $\psi(0) = 0$. If the polynomial ϕ defines a singular curve, then $\phi = \phi_x = \phi_y = 0$ at the singular point. By $\phi_x = y^{m_2} = 0$, the singular point satisfies $y = 0$. However it satisfies $\phi(x, 0) \neq 0$ and this is a contradiction. Therefore, any deformation pattern cannot define a 1-tacnodal curve.

To prevent complication of the proof of the main theorem, we give the following auxiliary definition.

Definition 4.19. The notation $\mathbb{T}_{-1}$ means the set of polygons equivalent to $\Delta_3(1; 1, 1, 1)$ and pairs of polygons equivalent to the pair of $\Delta_3(0; 2, 1, 1)$ and $\Delta_3'(0; 2, 1, 1)$ such that their intersection $\Delta_3(0; 2, 1, 1) \cap \Delta_3'(0; 2, 1, 1)$ is a segment of length 2.

The notation $\mathbb{T}_{-2}$ means the set of polygons equivalent to $\Delta_3(2; 1, 1, 1)$ and pairs of polygons equivalent to either

- the pair of $\Delta_3(1; 2, 1, 1)$ and $\Delta_3(0; 2, 1, 1)$ such that their intersection $\Delta_3(1; 2, 1, 1) \cap \Delta_3(0; 2, 1, 1)$ is a segment of length 2,
- the pair of $\Delta_3(0; 3, 1, 1)$ and $\Delta_3'(0; 3, 1, 1)$ such that their intersection $\Delta_3(0; 3, 1, 1) \cap \Delta_3'(0; 3, 1, 1)$ is a segment of length 3.

The triple $\Delta_3(0; 2, 2, 1)$, $\Delta_3(0; 2, 1, 1)$ and $\Delta_3'(0; 2, 1, 1)$ such that the intersections $\Delta_3(0; 2, 2, 1) \cap \Delta_3(0; 2, 1, 1)$ and $\Delta_3(0; 2, 2, 1) \cap \Delta_3'(0; 2, 1, 1)$ are segments of length 2 does not exist by Lemma 4.12.

Note that, from the above discussion, these polygons and their sharing edges are not 1-tacnodal.

4.3.2 Case (A)

Let S be the dual subdivision of T_F. We assume that S is a TP-subdivision and satisfies $\sharp\partial\Delta_{\mathbb{Z}} - \sharp(V(S) \cap \partial\Delta) = 0$. Then $d(S) = 0$ by Lemma 1.15. Thus

$$\mathrm{rk}(T_F) = \mathrm{rk}_{\exp}(T_F) = \sharp\Delta_{\mathbb{Z}} - 4.$$

By the definition of $\mathrm{rk}_{\exp}(T_F)$, we get

$$\begin{aligned} \sharp\Delta_{\mathbb{Z}} - 4 &= \sharp V(S) - 1 - \sum_{k=1}^{N} (\sharp V(\Delta_k) - 3) \\ &= \sharp V(S) - 1 - N_4'. \end{aligned}$$

Since $\sharp V(S) \leq \sharp\Delta_{\mathbb{Z}}$, we obtain $0 \leq N_4' \leq 3$.

(A-0) If S satisfies $N_4' = 0$, then it satisfies $\sharp V(S) = \sharp\Delta_{\mathbb{Z}} - 3$ and consists of triangles. Then, the subdivision S must contain exactly one of the following polygons:

(i) $\Delta_3(3; 1, 1, 1)$,

(ii) $\Delta_3(2;1,1,1)$ with one of $\mathbb{T}_{-1}$,

(iii) $\Delta_3(2;2,1,1)$ and $\Delta_3(0;2,1,1)$ such that their intersection is a segment whose length is 2,

(iv) $\Delta_3(1;2,1,1)$ and $\Delta_3(1;2,1,1)$ such that their intersection is a segment whose length is 2,

(v) $\Delta_3(1;2,1,1)$ and $\Delta_3(0;2,1,1)$ such that their intersection is a segment whose length is 2 with one of $\mathbb{T}_{-1}$,

(vi) $\Delta_3(1;2,2,1)$, $\Delta_3(0;2,1,1)$ and $\Delta'_3(0;2,1,1)$ such that their intersections $\Delta_3(1;2,2,1)\cap \Delta_3(0;2,1,1)$ and $\Delta_3(1;2,2,1)\cap\Delta'_3(0;2,1,1)$ are segments whose lengths are 2,

(vii) $\Delta_3(0;2,2,1)$, $\Delta_3(0;2,1,1)$ and $\Delta_3(1;2,1,1)$ such that their intersections $\Delta_3(0;2,2,1)\cap \Delta_3(0;2,1,1)$ and $\Delta_3(0;2,2,1)\cap\Delta_3(1;2,1,1)$ are segments whose lengths are 2,

(viii) $\Delta_3(0;2,2,2)$, $\Delta_3(0;2,1,1)$, $\Delta'_3(0;2,1,1)$ and $\Delta''_3(0;2,1,1)$ such that their intersections $\Delta_3(0;2,2,2)\cap\Delta_3(0;2,1,1)$, $\Delta_3(0;2,2,2)\cap\Delta'_3(0;2,1,1)$ and $\Delta_3(0;2,2,2)\cap \Delta''_3(0;2,1,1)$ are segments whose lengths are 2,

(ix) $\Delta_3(1;3,1,1)$ and $\Delta_3(0;3,1,1)$ such that their intersection $\Delta_3(1;3,1,1)\cap\Delta_3(0;3,1,1)$ is a segment whose length is 3,

(x) $\Delta_3(0;3,1,1)$ and $\Delta'_3(0;3,1,1)$ such that their intersection $\Delta_3(0;3,1,1)\cap\Delta_3(0;3,1,1)$ is a segment whose length is 3, with one of $\mathbb{T}_{-1}$,

(xi) $\Delta_3(0;3,2,1)$, $\Delta_3(0;3,1,1)$ and $\Delta_3(0;2,1,1)$ such that their intersections $\Delta_3(0;3,2,1)\cap \Delta_3(0;3,1,1)$ and $\Delta_3(0;3,2,1)\cap\Delta_3(0;2,1,1)$ are segments whose lengths are 3 and 2, respectively,

(xii) $\Delta_3(0;4,1,1)$ and $\Delta'_3(0;4,1,1)$ such that their intersection $\Delta_3(0;4,1,1)\cap\Delta'_3(0;4,1,1)$ is a segment whose length is 4,

(xiii) three of $\mathbb{T}_{-1}$,

(xiv) one of $\mathbb{T}_{-2}$ and one of $\mathbb{T}_{-1}$.

(A-1) If S satisfies $N'_4=1$, then it satisfies $\sharp V(S)=\sharp\Delta_{\mathbb{Z}}-2$ and contains only one parallelogram in the following list and the rest of S consists of triangles:

(i) $\Delta_4^{\mathrm{par}}(2;1,1)$,

(ii) $\Delta_4^{\mathrm{par}}(0;2,1)$, $\Delta_3(0;2,1,1)$ and $\Delta'_3(0;2,1,1)$ such that their intersections $\Delta_4^{\mathrm{par}}(0;2,1)\cap \Delta_3(0;2,1,1)$ and $\Delta_4^{\mathrm{par}}(0;2,1)\cap\Delta'_3(0;2,1,1)$ are segments whose lengths are 2,

(iii) $\Delta_4^{\mathrm{par}}(1;1,1)$ with one of $\mathbb{T}_{-1}$,

(iv) $\Delta_4^{\mathrm{par}}(0;1,1)$ with two of $\mathbb{T}_{-1}$,

(v) $\Delta_4^{\mathrm{par}}(0;1,1)$ with one of $\mathbb{T}_{-2}$.

(A-2) If S satisfies $N'_4 = 2$, then it satisfies $\sharp V(S) = \sharp\Delta_{\mathbb{Z}} - 1$ and contains exactly two parallelograms in the following list and the rest of S consists of triangles:

(i) $\Delta_4^{\mathrm{par}}(1;1,1)$, $\Delta_4^{\mathrm{par}}(0;1,1)$

(ii) two $\Delta_4^{\mathrm{par}}(0;1,1)$ with one of $\mathbb{T}_{-1}$.

(A-3) If S satisfies $N'_4 = 3$, then $\sharp V(S) = \sharp\Delta_{\mathbb{Z}}$ holds and S contains exactly three parallelograms. Thus S has three $\Delta_4^{\mathrm{par}}(0;1,1)$ and the rest of S consists of triangles whose area is 1/2.

In the above list, by Remark 4.14 and Lemma 4.12, cases (vi), (vii), (ix), (xi) in (A-0) does NOT occur. Furthermore, the following cases do NOT have a regular singularity by Lemma 4.16:

- (ii), (v), (viii), (x), (xiii), (xiv) in (A-0),
- (ii), (iii), (iv), (v) in (A-1),
- (i), (ii) in (A-2),
- (A-3).

Among them, the refinement of the following cases do NOT have an irregular singularity by Lemma 4.17 and Remark 4.18:

- (ii), (v), (viii), (x), (xiii), (xiv) in (A-0),
- (ii), (iii), (iv), (v) in (A-1),
- (i), (ii) in (A-2),
- (A-3).

The remaining cases are (i), (iii), (iv) and (xii) in (A-0) and (i) in (A-1), and they correspond to the polygons $\Delta_{\mathrm{I}}, \Delta_{\mathrm{II}}, \Delta_{\mathrm{III}}, \Delta_{\mathrm{IV}}, \Delta_{\mathrm{V}}$ and Δ_{VI}, respectively, by Lemma 4.2. Moreover, by Lemma 4.4, 4.5, 4.6 and 4.7, these polygons are 1-tacnodal.

4.3.3 Case (B)

We assume that S is a TP-subdivision and satisfies $\sharp\partial\Delta_{\mathbb{Z}} - \sharp(V(S) \cap \partial\Delta) = 1$. By the latter condition, S must have exactly one polygon $P \in S$ such that $P \cap \partial\Delta$ is a segment of length 2. By Lemma 1.15, we get

$$\mathrm{rk}(T_F) = \mathrm{rk}_{\exp}(T_F) = \sharp\Delta_{\mathbb{Z}} - 4.$$

By the definition of $\mathrm{rk}_{\exp}(T_F)$, we obtain

$$\begin{aligned}\sharp\Delta_{\mathbb{Z}} - 4 &= \sharp V(S) - 1 - \sum_{k=1}^{N}(\sharp V(\Delta_k) - 3) \\ &= \sharp V(S) - 1 - N_4'.\end{aligned}$$

Since $\sharp V(S) \leq \sharp\Delta_{\mathbb{Z}} - 1$, we have $0 \leq N_4' \leq 2$.

(B-0) If S satisfies $N_4' = 0$, then S satisfies $\sharp V(S) = \sharp\Delta_{\mathbb{Z}} - 3$ and consists of triangles. Let $P \in S$ be a polygon which intersects $\partial\Delta$ as a segment of length 2. Then S satisfies one of the following:

- (i) $P = \Delta_3(0; 2, 1, 1)$ and S contains two of $\mathbb{T}_{-1}$ or one of $\mathbb{T}_{-2}$,
- (ii) $P = \Delta_3(1; 2, 1, 1)$ and S contains one of $\mathbb{T}_{-1}$,
- (iii) $P = \Delta_3(2; 2, 1, 1)$,
- (iv) $P = \Delta_3(0; 2, 2, 2)$,
- (v) $P = \Delta_3(0; 2, 2, 1)$, and S contains one of $\mathbb{T}_{-1}$,
- (vi) $P = \Delta_3(1; 2, 2, 1)$,
- (vii) $P = \Delta_3(0; 2, 3, 1)$.

(B-1) If S satisfies $N_4' = 1$, then S satisfies $\sharp V(S) = \sharp\Delta_{\mathbb{Z}} - 2$. Let $P \in S$ be a polygon which intersects $\partial\Delta$ as a segment of length 2. Then S satisfies one of the following:

- (i) $P = \Delta_4^{\mathrm{par}}(0; 2, 1)$ and $\Delta_3(0; 2, 1, 1)$ such that their intersection $P \cap \Delta_3(0; 2, 1, 1)$ is a segment of length 2,
- (ii) $P = \Delta_3(0; 2, 1, 1)$ and S contains $\Delta_4^{\mathrm{par}}(1; 1, 1)$,
- (iii) $P = \Delta_3(1; 2, 1, 1)$ and S contains $\Delta_4^{\mathrm{par}}(0; 1, 1)$,
- (iv) $P = \Delta_3(0; 2, 2, 1)$ and S contains $\Delta_4^{\mathrm{par}}(0; 1, 1)$,
- (v) $P = \Delta_3(0; 2, 1, 1)$ and S contains $\Delta_4^{\mathrm{par}}(0; 1, 1)$, and one of $\mathbb{T}_{-1}$.

(B-2) If S satisfies $N_4' = 2$, then S satisfies $\sharp V(S) = \sharp\Delta_{\mathbb{Z}}$ and contains exactly two parallelograms. Thus $P = \Delta_3(0; 2, 1, 1)$ and S contains two $\Delta_4^{\mathrm{par}}(0; 1, 1)$.

In the above list, by Lemma 4.12, the following cases do NOT occur:

- (v), (vi), (vii) in (B-0),
- (iv) in (B-1).

Furthermore, the following cases do NOT have a regular singularity by Remark 4.14 and Lemma 4.16:

- (i), (ii), (iv) in (B-0),
- (i), (ii), (iii), (v) in (B-1),
- (B-2).

Among them, (iv) in (B-0) does NOT have an irregular singularity by Lemma 4.17 and the other polygons except (iii) in (B-0) also do NOT have it since they have only one edge of length more than 1, which should be on the boundary $\partial\Delta$, and this edge cannot be a 1-tacnodal edge. The remaining case is (iii) in (B-0) and this corresponds to the polygon Δ_{III} by Lemma 4.2. Moreover, by Lemma 4.16 (3), this polygon is NOT 1-tacnodal.

4.3.4 Case (C)

We assume that S is NOT a TP-subdivision. Then

$$\begin{aligned} d(S) &= \sharp\Delta_{\mathbb{Z}} - 4 - \Big\{\sharp V(S) - 1 - \sum_{k=1}^{N}(\sharp V(\Delta_k) - 3)\Big\} \\ &= \sharp\Delta_{\mathbb{Z}} - \sharp V(S) - 3 + \sum_{k=1}^{N}(\sharp V(\Delta_k) - 3) \\ &\geq -3 + \sum_{k=1}^{N}(\sharp V(\Delta_k) - 3) \\ &= \sum_{m\geq 3}(m-3)N_m - 3. \end{aligned}$$

By $0 \leq d(S) \leq \mathcal{N}_S/2$ due to Lemma 1.15, we get

$$\sum_{m\geq 3}(m-3)N_m \leq -\sum_{m\geq 2} N'_{2m} + 5 \quad \text{and} \quad \sum_{m\geq 2} N'_{2m} \leq 2.$$

We decompose the proof into the following three cases:

(C-0) $\sum_{m\geq 2} N'_{2m} = 0$ and $\sum_{m\geq 3}(m-3)N_m \leq 5$,

(C-1) $\sum_{m\geq 2} N'_{2m} = 1$ and $\sum_{m\geq 3}(m-3)N_m \leq 4$,

(C-2) $\sum_{m\geq 2} N'_{2m} = 2$ and $\sum_{m\geq 3}(m-3)N_m \leq 3$.

(C-0) In this case, since $N_4 + 2N_5 + 3N_6 + 4N_7 + 5N_8 \leq 5$ and $\sum_{m\geq 2} N'_{2m} = 0$, the possible patterns are the following:

(i) $N_8 = 1$ and $N'_8 = 0$,

(ii) $N_7 = 1$, $N_4 = 0, 1$ and $N'_4 = 0$,

(iii) $N_6 = 1$, $N_4', N_6' = 0$ and $(N_4, N_5) = (0,0), (1,0), (2,0), (0,1)$,

(iv) $N_5 = 2$, $N_4 = 0, 1$ and $N_4' = 0$,

(v) $N_5 = 1$, $N_4 = 0, 1, 2, 3$ and $N_4' = 0$,

(vi) $N_4 = 1, 2, 3, 4, 5$ and $N_4' = 0$.

In case (i), $N_8 = 1$ and $N_8' = 0$. Since $\mathcal{N}_S = 4$, we get $0 \leq d(S) \leq 2$. On the other hand, any octagon has two or more inner lattice points (Lemma 4.11), so

$$\begin{aligned} d(S) &= \sharp\Delta_{\mathbb{Z}} - 4 - \{\sharp V(S) - 1 - 5\} \\ &= \sharp\Delta_{\mathbb{Z}} - \sharp V(S) + 2 \\ &\geq 4. \end{aligned}$$

This is a contradiction. Therefore case (i) does not occur. We can prove that the above cases except the cases (v) with $N_4 = 0$ and (vi) with $N_4 = 1, 2$ do NOT occur by the same argument.

Next, we observe the remaining cases.

Case (v) with $N_4 = 0$. S has exactly one pentagon and the rest of S consists of triangles. Then $\mathrm{rk}_{\mathrm{exp}}(S) = \sharp V(S) - 3$ holds. Therefore, the set $(\Delta \cap \mathbb{Z}^2) \setminus V(S)$ is exactly one lattice point. By Lemma 4.11 (2), the pentagon is $\Delta_5(1; 1, 1, 1, 1, 1)$. This polygon is equivalent to Δ_{VII} by Lemma 4.2 (6). Moreover, by Lemma 4.5, the pentagon is a 1-tacnodal polygon.

Case (vi) with $N_4 = 1$. S has exactly one non-parallel quadrangle and the rest of S consists of triangles. Since $\mathrm{rk}_{\mathrm{exp}}(S) = \sharp V(S) - 2$, the set $(\Delta \cap \mathbb{Z}^2) \setminus V(S)$ consists of two lattice points. Therefore, a possible non-parallel quadrangle $\Delta_4(I; s, t, u, v)$ is one of the following list:

(a) $\Delta_4(0; 1, 1, 1, 1)$,

(b) $\Delta_4(0; 2, 1, 1, 1)$,

(c) $\Delta_4(0; 2, 2, 1, 1)$,

(d) $\Delta_4(1; 1, 1, 1, 1)$,

(e) $\Delta_4(1; 2, 1, 1, 1)$,

(f) $\Delta_4(2; 1, 1, 1, 1)$.

Cases (a) and (c) do NOT occur by Lemma 4.11 and Lemma 4.12, respectively. The polygons in (b) and (e) are NOT 1-tacnodal polygons by (4) of Lemma 4.16 and Lemma 4.9, respectively. Also the polygon in (d) is NOT a 1-tacnodal polygon by

[19, Lemma 4.2 (i)]. Notice that, by Remark 4.18, the polygon in (d) does NOT have a 1-tacnodal edge.

By Lemma 4.2, the polygon (f) is equivalent to one of

$$\Delta_{\mathrm{VIII}}, \quad \Delta_{\mathrm{IX}} \quad \text{and} \quad \mathrm{Conv}\{(1,0),(0,1),(2,1),(1,3)\}.$$

The polygons Δ_{VIII}, Δ_{IX} are 1-tacnodal polygons by Lemma 4.5. On the other hand, the polygon $\mathrm{Conv}\{(1,0),(0,1),(2,1),(1,3)\}$ is NOT a 1-tacnodal polygon by Lemma 4.16 (5) and does NOT have a 1-tacnodal edge by Remark 4.18.

If S contains the polygon in (b), since $\mathrm{rk}_{\mathrm{exp}}(S) = \sharp\Delta_{\mathbb{Z}} - 3$, the adjacent polygon which shares the edge of length 2 of $\Delta_4(1;2,1,1,1)$ must be either $\Delta_3(0;2,1,1)$ or $\Delta_3(1;2,1,1)$. Each of their intersection with $\Delta_4(1;2,1,1,1)$ is NOT a 1-tacnodal edge by (2) and (4) of Lemma 4.17. Therefore, any edge contained in S is NOT a 1-tacnodal edge.

If S contains the polygon (e), since $\mathrm{rk}_{\mathrm{exp}}(S) = \sharp\Delta_{\mathbb{Z}} - 4 = \mathrm{rk}(S)$, the adjacent polygon which shares the edge of length 2 of $\Delta_4(1;2,1,1,1)$ must be $\Delta_3(0;2,1,1)$. This is a dual subdivision of a tropical 1-tacnodal curve of type (E).

Case (vi) with $N_4 = 2$. S has exactly two non-parallel quadrangles and the rest of S consists of triangles. Since $\mathrm{rk}_{\mathrm{exp}}(S) = \sharp V(S) - 3$, the set $(\Delta \cap \mathbb{Z}^2) \setminus V(S)$ consists of exactly one lattice point. Therefore S contains $\Delta_4(0;2,1,1,1)$ and $\Delta'_4(0;2,1,1,1)$ such that their intersection is a segment whose length is 2. This is because a non-parallel quadrangle must satisfy either "the number of interior lattice points is non-zero" or "the polygon has an edge of length ≥ 2", by Lemma 4.11. These polygons are NOT 1-tacnodal polygons by Lemma 4.16. Also their intersection is NOT a 1-tacnodal edge by Lemma 4.17 (5).

(C-1) In this case, since $N_4 + 2N_5 + 3N_6 + 4N_7 \leq 4$ and $\sum_{m\geq 2} N'_{2m} = 1$, the following patterns can occur:

(i) $N_6 = N'_6 = 1$, $N_4 = 0, 1$ and $N'_4 = 0$,

(ii) $N_4 = N'_4 = 1$, $N_6 = 1$ and $N'_6 = 0$,

(iii) $N_4 = 2$, $N'_4 = 1$ and $N_5 = 1$,

(iv) $N_4 = N'_4 = 1$ and $N_5 = 1$,

(v) $N_4 = 2, 3, 4$ and $N'_4 = 1$.

However, we can check that the cases, except case (v) with $N_4 = 2$, are impossible by the same argument as in case (i) in (C-0).

We observe case (v) with $N_4 = 2$. S contains a non-parallel quadrangle P and a parallelogram Q, and the rest of S consists of triangles. Notice that, by Lemma 4.11, P must satisfy either "the number of interior lattice points is non-zero" or "the polygon has an edge of length ≥ 2". Since $\mathrm{rk}_{\exp}(S) = \sharp V(S) - 3$, the set $(\Delta \cap \mathbb{Z}^2) \setminus V(S)$ consists of exactly one lattice point. Therefore P and Q must be either

- $P = \Delta_4(1; 1, 1, 1, 1)$ and $Q = \Delta_4^{\mathrm{par}}(0; 1, 1)$, or
- $P = \Delta_4(0; 2, 1, 1, 1)$ and $Q = \Delta_4^{\mathrm{par}}(0; 1, 1)$ such that the edge of length 2 of P intersects a triangle $\Delta_3(0; 2, 1, 1)$.

In both cases, the polygons are not 1-tacnodal by Lemma 4.16, Lemma 4.17 and Remark 4.18.

(C-2) In this case, since $N_4 + 2N_5 + 3N_6 \leq 3$ and $\sum_{m\geq 2} N'_{2m} = 2$, any possible subdivision satisfies $N_4 = 3$ and $N'_4 = 2$. Since $\mathcal{N}_S = 0$, we get $d(S) = 0$. On the other hand, since $\sharp V(S) \leq \sharp\Delta_{\mathbb{Z}} - 1$ by Lemma 4.11,

$$d(S) = \sharp\Delta_{\mathbb{Z}} - \sharp V(S) \geq 1.$$

This is a contradiction.

4.3.5 Case (D)

We assume that S is NOT a TP-subdivision and satisfies $\sharp\partial\Delta_{\mathbb{Z}} - \sharp(V(S) \cap \partial\Delta) = 1$. By the former condition, we can apply the argument in case (C) to case (D) and obtain the list of possible subdivisions as follows:

(1) (v) with $N_4 = 0$ in (C-0),

(2) (vi) with $N_4 = 1$ in (C-0),

(3) (vi) with $N_4 = 2$ in (C-0),

(4) (v) with $N_4 = 2$ in (C-1).

Case (1). S has exactly one pentagon and the rest of S consists of triangles. Then $\mathrm{rk}_{\exp}(S) = \sharp V(S) - 3$ holds. By the boundary condition, the set $(\Delta \cap \mathbb{Z}^2) \setminus V(S)$ is empty. If S contains a triangle P whose intersection with $\partial\Delta$ is an edge of length 2, then, by Lemma 4.11, S does NOT have a pentagon. Therefore, the possible pentagon is $\Delta_5(0; 2, 1, 1, 1, 1)$, whose intersection with $\partial\Delta$ is an edge of length 2. However, the pentagon does NOT exist by Lemma 4.12.

Case (2). S has exactly one non-parallel quadrangle and the rest of S consists of triangles. By $\mathrm{rk}_{\exp}(S) = \sharp V(S) - 2$ and the boundary condition, the set $(\Delta \cap \mathbb{Z}^2) \setminus V(S)$ consists of one lattice point. Therefore, the possible non-parallel quadrangle $\Delta_4(I; s, t, u, v)$ is one of the following list:

(a) $\Delta_4(0; 1, 1, 1, 1)$,

(b) $\Delta_4(0; 2, 1, 1, 1)$,

(c) $\Delta_4(1; 1, 1, 1, 1)$,

Case (a) does NOT occur by Lemma 4.11. The polygon in (b) is NOT a 1-tacnodal polygon by Lemma 4.16 (4). Also the polygon in (c) is NOT a 1-tacnodal polygon by [19, Lemma 4.2 (i)]. Notice that, by Remark 4.18, the polygon in (c) does NOT have a 1-tacnodal edge.

If S contains the polygon in (b), since $\mathrm{rk}_{\exp}(S) = \sharp\Delta_{\mathbb{Z}} - 4$, the intersection of the quadrangle $\Delta_4(0; 2, 1, 1, 1)$ and $\partial\Delta$ is an edge of length 2. Thus the edge is NOT a 1-tacnodal edge.

Case (3) and (4). S has exactly two non-parallel quadrangles and the rest of S consists of triangles. By $\mathrm{rk}_{\exp}(S) = \sharp V(S) - 3$ and the boundary condition, the set $(\Delta \cap \mathbb{Z}^2) \setminus V(S)$ is empty. Therefore, such a subdivision S does NOT exist by the fact that a non-parallel quadrangle must satisfy either "the number of interior lattice points is non-zero" or "the polygon has an edge of length ≥ 2" in Lemma 4.11. □

Remark 4.20 (The case of tropicalization having a non-isolated singularity)**.** By the upper semi-continuity of the Milnor number, there is a possibility that a tropicalization has non-isolated singularities. By Lemma 4.13 and the assumption of the rank, we can make a list of polygons whose tropicalizations may have non-isolated singularities. For example, the polygon Δ_{X} in Figure J satisfies these conditions. In [19, Section 3.6], the refinement of tropicalization along a non-isolated singularity is defined. Applying the method mechanically, by simple calculation, we get that the exceptional polygon is $\hat{\Delta}_{\mathrm{X}}$ shown on the right in Figure J.

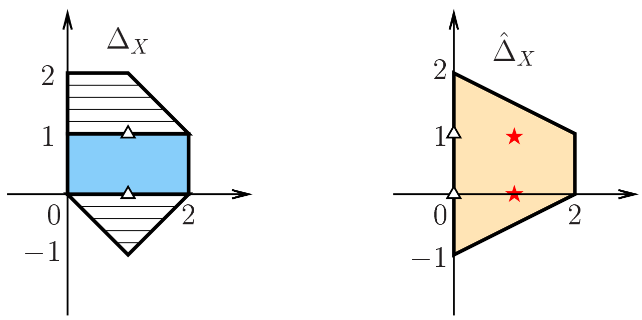

Figure J: Polygons Δ_{X} and $\hat{\Delta}_{\mathrm{X}}$. The notation $\triangle$ means a lattice point on the boundary which is not a vertex and the notation $\star$ means an interior lattice point.

Remark 4.21 (On a possibility of patchworking)**.** As mentioned in the introduction, this research aims to construct the tropical version of enumerative geometry of 1-tacnodal curves. Therefore, we would like to lift the 1-tacnodal curve from a given degenerate 1-tacnodal curve by patchworking. It is known that there is no obstruction if the singular point is A_1, and this is still true even if it is A_2, which can be checked by a numerical criterion of the vanishing of the obstruction constructed by Shustin (See [18, Theorem 4.1], or [19, Lemma 5.4] for a tropical version). But, unfortunately, this criterion does not work if it is A_3 because of the following reason:

We recall a sufficient condition to apply patchworking [19, Lemma 5.5 (*ii*)], called *transversality*. Let S be the dual subdivision of a tropical curve T, $\Delta_1, \ldots, \Delta_N$ be the 2-dimensional polygons of S and $(C_1, \ldots, C_N)$ be a collection of complex curves such that the Newton polygon of the defining polynomial f_i of C_i is $\Delta_i \in S$ and, if $\sigma_{ij} := \Delta_i \cap \Delta_j \neq \emptyset$, $f_i^{\sigma_{ij}} = f_j^{\sigma_{ij}}$.

For an irreducible curve C_k for some $k \in \{1, \ldots N\}$, if there is a union Δ_k^- of edges of Δ_k such that C_k satisfies the following inequality

$$\sum{}' b(C_k, \xi) + \sum{}'' \tilde{b}(C_k, Q) + \sum{}''' ((C_k \cdot X(\sigma)) - \epsilon) < \sum_{\sigma \subset \partial\Delta} (C_k \cdot X(\sigma)),$$

where

- if C has a tacnode, then $b(C, \xi) = 1$ for both branches, if C is locally given by $\{x^{pr} + y^{qr} = 0\}$ for coprime integers p, q, then $\tilde{b}(C, \xi) = p + q - 1$ for each branch,
- $\sum'$ ranges over all local branches ξ of C_k, centered at the points $z \in \mathrm{Sing}(C_k) \cap (\mathbb{C}^*)^2$,
- $\sum''$ ranges over all local branches Q of C_k, centered at the points $z \in \mathrm{Sing}(C_k) \cap X(\partial\Delta_k)$, and
- $\sum'''$ ranges over all non-singular points z of C_k on $X(\partial\Delta_k)$ with $\epsilon = 0$ if $\sigma \subset \Delta_k^-$ and $\epsilon = 1$ otherwise,

then C_k is transversal.

Let $V \subset X(\Delta_{\mathrm{III}})$ be a curve which is constructed in Lemma 4.6. We can easily check

$$\sum{}' b(V, \xi) = 0, \quad \sum{}'' \tilde{b}(V, Q) = 4, \quad \sum{}''' \big((V \cdot X(\sigma)) - \epsilon\big) \geq 0 \quad \text{and} \sum_{\sigma \subset \partial\Delta} (V \cdot X(\sigma)) = 4.$$

Therefore V does not satisfy the above inequality.

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