On algebraic properties of the combinatorial structure of a finite projective geometry

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Abstract

Consider the poset $\mathcal{P}_n(\mathbb{F}_q)$ of all subspaces in an n-dimensional vector space over a finite field $\mathbb{F}_q$ of q elements. It is called a *finite projective geometry*. As an algebraic view of $\mathcal{P}_n(\mathbb{F}_q)$, we consider a matrix algebra, called the *incidence algebra*, defined from the "global" combinatorial structure of $\mathcal{P}_n(\mathbb{F}_q)$. The incidence algebra is known to be a homomorphic image of the quantum algebra $U_{q^{1/2}}(\mathfrak{sl}_2)$. In this thesis, we extend this situation to the level of the *quantum affine algebra* $U_{q^{1/2}}(\widehat{\mathfrak{sl}}_2)$. We introduce two algebras $\mathcal{H}_s$, $\mathcal{H}_f$ from the "local" combinatorial structures of $\mathcal{P}_n(\mathbb{F}_q)$ as well as the "global" one, both of which contain the incidence algebra as a proper subalgebra. We then show that there exist algebra homomorphisms from $U_{q^{1/2}}(\widehat{\mathfrak{sl}}_2)$ to these algebras and that any irreducible module for these algebras is irreducible as a $U_{q^{1/2}}(\widehat{\mathfrak{sl}}_2)$-module.

We next consider a finite projective geometry $\mathcal{P}_n(\mathbb{F})$ over any field $\mathbb{F}$ and discuss the new algebra $\mathcal{H}_f$ from the viewpoint of the association schemes on Schubert cells of a Grassmannian. The Grassmannian $\mathrm{Gr}(m, n)$ is the set of m-dimensional subspaces in $\mathcal{P}_n(\mathbb{F})$, and the general linear group $\mathrm{GL}_n(\mathbb{F})$ acts transitively on it. The Schubert cells of $\mathrm{Gr}(m, n)$ are the orbits of the Borel subgroup $\mathcal{B} \subset \mathrm{GL}_n(\mathbb{F})$ on $\mathrm{Gr}(m, n)$. We consider the association scheme on each Schubert cell defined by the $\mathcal{B}$-action and show it is symmetric and it is the *generalized wreath product* of one-class association schemes, which was introduced by R. A. Bailey [European Journal of Combinatorics 27 (2006) 428–435].

Contents

Chapter 1

Introduction

Let $(P, \leq)$ be a finite graded partially ordered set (poset) of rank N with fibers P_0, P_1, ..., P_N. Consider the *lowering*, *raising* and *projection matrices* L, R, E_i^* $(0 \leq i \leq N)$ with rows and columns indexed by P as follows:

$$L_{x,y} = \begin{cases} 1 & \text{if } y \text{ covers } x, \\ 0 & \text{otherwise,} \end{cases}$$

$$R_{x,y} = \begin{cases} 1 & \text{if } x \text{ covers } y, \\ 0 & \text{otherwise,} \end{cases}$$

$$(E_i^*)_{x,y} = \begin{cases} 1 & \text{if } x = y \in P_i, \\ 0 & \text{otherwise,} \end{cases}$$

for $x, y \in P$. The *incidence algebra* of P is the complex matrix algebra generated by L, R and E_i^* $(0 \leq i \leq N)$, and has been studied in the field of algebraic combinatorics. It is implicit in [23, 25] that the incidence algebras of the following posets are closely related to the Lie algebra $\mathfrak{sl}_2$ or the quantum algebra $U_q(\mathfrak{sl}_2)$:

algebra	posets
$\mathfrak{sl}_2$	subset lattices, Hamming semi-lattices
$U_q(\mathfrak{sl}_2)$	subspace lattices, attenuated spaces, classical polar spaces

For example, for a subset lattice consisting of all subsets of a finite set, the incidence algebra becomes naturally a homomorphic image of the universal enveloping algebra $U(\mathfrak{sl}_2)$. In this thesis, we focus on the *subspace lattices* over finite fields. Our goal is to extend the above situation further to the level of the *quantum affine algebra* $U_q(\widehat{\mathfrak{sl}}_2)$ in a nontrivial manner.

By a subspace lattice, also known as a *finite projective geometry*, we mean the poset of all subspaces of a finite-dimensional vector space over a finite field, where the ordering is given by inclusion. In the field of combinatorics, subspace lattices are regarded as q-analogs of Boolean lattices and therefore they have been studied from many combinatorial points of view, such as Grassmann codes and Grassmann graphs. On the other hand, the quantum affine algebras $U_q(\widehat{\mathfrak{sl}}_2)$ are Hopf algebras that are q-deformations of the universal enveloping algebra of the affine Lie algebra $\widehat{\mathfrak{sl}}_2$ and their representations are developed in [6, Section 5] as trigonometric solutions of the quantum Yang–Baxter equation.

Here we briefly recall some known facts about the subspace lattices. See Section 2.3 for more details. Let N be a positive integer. Let H denote an N-dimensional vector space over a finite field $\mathbb{F}_q$ of q elements and let P denote the subspace lattice consisting of all subspaces of H. The poset P has the grading which is a partition of P into nonempty sets

$$P_i = \{y \in P \mid \dim y = i\} \qquad (0 \leq i \leq N). \tag{1.1}$$

We denote by $\mathcal{I}$ the incidence algebra of P. By some combinatorial counting it is easily verified that in this case $q^{(1-N)/2}L$, R, $K := \sum_{i=0}^{N} q^{N/2-i}E_i^*$ and its inverse, satisfy the defining relations of $U_{q^{1/2}}(\mathfrak{sl}_2)$ in terms of the Chevalley generators. In particular, every irreducible $\mathcal{I}$-submodule of the standard module $V = \mathbb{C}P$ becomes an irreducible $U_{q^{1/2}}(\mathfrak{sl}_2)$-module of type 1.

We summarize the first results of this thesis. See Chapter 2 for more details. The main idea here is to fix one subspace $x \in P$ with $0 < \dim x < N$ and then consider the following new "rectangle" partition of P with respect to x:

$$P_{i,j} = \{y \in P \mid \dim y = i + j, \dim(y \cap x) = i\}, \tag{1.2}$$

for $0 \leq i \leq \dim x$ and for $0 \leq j \leq N - \dim x$. Remark that this is a refinement of the grading. In terms of new partition (1.2), we naturally decompose each of the lowering and raising matrices into the sum of two matrices: $L = L_1 + L_2$ and $R = R_1 + R_2$. Then L_1, L_2, R_1, R_2 and the projection matrices for the new partition, give us a new algebra $\mathcal{H}_s$ which contains $\mathcal{I}$ as a proper subalgebra. We define an action of $U_{q^{1/2}}(\widehat{\mathfrak{sl}}_2)$ on the standard module V using matrices in $\mathcal{H}_s$. With respect to this $U_{q^{1/2}}(\widehat{\mathfrak{sl}}_2)$-module structure on V, we moreover, show that any irreducible $\mathcal{H}_s$-module induces an irreducible $U_{q^{1/2}}(\widehat{\mathfrak{sl}}_2)$-module of type $(1,1)$ which is more precisely a tensor product of two evaluation modules. In particular, it follows that $\mathcal{H}_s$ is generated by the actions of $U_{q^{1/2}}(\widehat{\mathfrak{sl}}_2)$ together with the center of $\mathcal{H}_s$. Our main results are Theorems 2.10.1, 2.10.4, 2.10.5 and 2.10.6. In fact, our approach is quite relevant to Dunkl's study on an addition theorem for some q-Hahn polynomials [10]. Meanwhile, we also describe the center of $\mathcal{H}_s$ (Theorem 2.8.3).

We remark that this work was motivated by the study of the *Terwilliger algebras* [19, 24] of the Grassmann graphs. Indeed, each of the fibers P_i of the subspace lattice P induces a Grassmann graph, and it follows that $E_i^* \mathcal{H}_s E_i^*$ (viewed as a subalgebra of $\mathrm{End}(\mathbb{C}P_i)$) contains its Terwilliger algebra. By first describing the $\mathcal{H}_s$-modules and carefully analyzing their structures, we were able to determine all the irreducible modules of the Terwilliger algebras of the Grassmann graphs. See [20].

We summarize the second results of this thesis. See Chapter 3 for more details. We fix a (full) flag $\{x_i\}_{i=0}^N$ on H instead of the subspace $x \in P$, and consider the following new "hyper-cubic" partition of P with respect to $\{x_i\}_{i=0}^N$:

$$P_\mu = \{y \in P \mid \dim(y \cap x_i) = \mu_1 + \mu_2 + \cdots + \mu_i \ (1 \leq i \leq N)\}, \tag{1.3}$$

for $\mu = (\mu_1, \mu_2, \ldots, \mu_N) \in \{0,1\}^N$. Then for $\mu \in \{0,1\}^N$, we define the *projection matrix* E_μ^* by the diagonal matrix indexed by P whose (y,y)-entry is 1 if $y \in P_\mu$ and 0 otherwise for $y \in P$. We next define the complex matrix algebra $\mathcal{H}_f$ generated by the lowering, raising matrices and these new projection matrices E_μ^*, where $\mu \in \{0,1\}^N$. By the construction, the algebra $\mathcal{H}_f$ contains the incidence algebra as its proper subalgebra. We prove that there exists an algebra homomorphism from the quantum affine algebra $U_{q^{1/2}}(\widehat{\mathfrak{sl}}_2)$ to the algebra $\mathcal{H}_f$, which again extends the above algebra homomorphism from $U_{q^{1/2}}(\mathfrak{sl}_2)$ to the incidence algebra. Moreover, it is also proved that any irreducible module for the algebra $\mathcal{H}_f$ induces an irreducible $U_{q^{1/2}}(\widehat{\mathfrak{sl}}_2)$-module of type $(1,1)$ which is more precisely a tensor product of evaluation modules of dimension 2. Our results are Theorems 3.12.1 and 3.12.5. To prove the main theorems, we classify all the $\mathcal{H}_f$-modules up to isomorphism and determine the multiplicities appearing in the standard module V.

Seen from the viewpoint of the action of the general linear group $\mathrm{GL}_N(\mathbb{F}_q)$ on the subspace lattice P, we may say the results in Chapter 3 are "opposite" to those obtained in Chapter 2. (In Chapters 2 and 3, however, we will not take this point of view in any essential way. We refer the reader to [10] for this viewpoint.) Indeed, the partitions (1.2) and (1.3) turn out to be the orbits of maximal and minimal parabolic subgroups of $\mathrm{GL}_N(\mathbb{F}_q)$, respectively. More precisely, the corresponding subgroups stabilize the fixed subspace x and the fixed flag $\{x_i\}_{i=0}^N$, respectively.

It is worth pointing out that our proofs in Chapter 3 involve a natural and intrinsic combinatorial characterization of the subspace lattice, while the method used in Chapter 2 is rather oriented towards Lie theory and the representation theory of quantum groups. In Chapter 3, we fix a basis $v_1, v_2, \ldots, v_N$ for H such that x_i is spanned by $v_1, v_2, \ldots, v_i$ for $1 \leq i \leq N$. With respect to the basis, we identify each subspace in P with a certain matrix whose entries are in the base field $\mathbb{F}_q$. Then, we relate these matrices to classical combinatorial objects, such as Ferrers boards rook placements, and inversion numbers, and interpret algebraic properties of subspaces in terms of these matrices (and moreover, of other combinatorial objects above). Almost all the problems which we concern in Chapter 3 arrive at problems in such classical combinatorial fields. This type of argument is motivated by Delsarte [9] and the technique used in Chapter 3 is a kind of a generalized version of that in [9].

Comparing the partitions (1.2) and (1.3) again, one may ask whether same kinds of results can still be obtained if we take a more general partition, which is defined by replacing a subspace or a full flag by a general flag. We will not develop this point here because the required computation is expected to be far more complicated. However, we emphasize that we have done for the two extremal and essential cases, and conjecture that similar results still hold in the general case.

We summarize the last results of this thesis. See Chapter 4 for more details. Let n be a positive integer, and let $\mathbb{F}$ be any field. Let $\mathcal{P}_n(\mathbb{F})$ denote the subspace lattice of subspaces of an n-dimensional vector space over $\mathbb{F}$. The general linear group $\mathrm{GL}_n(\mathbb{F})$ acts on $\mathcal{P}_n(\mathbb{F})$. The natural grading structure (1.1) of $\mathcal{P}_n(\mathbb{F})$ is given by $\mathrm{GL}_n(\mathbb{F})$-action, and each fiber (i.e., orbit) is called a *Grassmannian*. In other words, the Grassmannian $\mathrm{Gr}(m,n)$ is the set of m-dimensional subspaces in $\mathbb{F}^n$, where $0 \leq m \leq n$. Let $\mathcal{B}$ denote the Borel subgroup (i.e., minimal parabolic subgroup) of $\mathrm{GL}_n(\mathbb{F})$. Then, the $\mathcal{B}$-action defines a finer "hyper-cubic" grading structure (1.3) of $\mathcal{P}_n(\mathbb{F})$, and each fiber contained in $\mathrm{Gr}(m,n)$ is called a *Schubert cell of* $\mathrm{Gr}(m,n)$. See [15] for details. We showed in Chapter 3 that the algebra defined from the "hyper-cubic" grading structure of $\mathcal{P}_n(\mathbb{F})$ together with its incidence structure has a close relation to the quantum affine algebra $U_q(\widehat{\mathfrak{sl}}_2)$, if $\mathbb{F}$ is a finite field of q^2 elements. In Chapter 4, we study the Schubert cells of a Grassmannian from the combinatorial point of view of *association schemes*. More precisely, we show that the association scheme defined by the $\mathcal{B}$-action on each Schubert cell is a *generalized wreath product* of one-class association schemes with the base set $\mathbb{F}$. The concept of a generalized wreath product of association schemes was introduced by R. A. Bailey [1] in 2006. The (usual) wreath product of association schemes has been actively studied (see e.g., [3, 4, 11, 14, 17, 21, 22, 29]), and we may view the result in Chapter 4 as demonstrating the fundamental importance of Bailey's generalization as well.

In this thesis, we remark that the contents of each Chapters 2, 3, and 4 are based on the author's paper [27], [26] and [28], respectively. For the convenience of the reader, the notation used in each chapter is identical with that in each paper with two exceptions. We replace $\mathcal{H}$ in [27] with $\mathcal{H}_s$ in Chapter 2 and $\mathcal{H}$ in [26] with $\mathcal{H}_f$ in Chapter 3 because they are different and play important roles in this thesis. Thus, some symbols are not consistent between these chapters.

Chapter 2

An algebra associated with a subspace lattice over a finite field and its relation to the quantum affine algebra $U_q(\widehat{\mathfrak{sl}}_2)$

In this chapter, we introduce an algebra $\mathcal{H}_s$ from a subspace lattice with respect to a fixed subspace which contains its incidence algebra as a proper subalgebra and show how it is related to the quantum affine algebra $U_{q^{1/2}}(\widehat{\mathfrak{sl}}_2)$, where q denotes the cardinality of the base field. We show that there is an algebra homomorphism from $U_{q^{1/2}}(\widehat{\mathfrak{sl}}_2)$ to $\mathcal{H}_s$, and that $\mathcal{H}_s$ is generated by its image together with the center. Moreover, we show that any irreducible $\mathcal{H}_s$-module is also irreducible as a $U_{q^{1/2}}(\widehat{\mathfrak{sl}}_2)$-module and is isomorphic to the tensor product of two evaluation modules. We also obtain a small set of generators of the center of $\mathcal{H}_s$. This chapter is based on the author's work [27].

We organize this chapter as follows. In Section 2.1, we recall the basic notation and basic combinatorial structures in a subspace lattice. In Section 2.3, we recall some known facts about the subspace lattices with the quantum algebra $U_q(\mathfrak{sl}_2)$. In Section 2.4, we discuss detailed combinatorial structures in a subspace lattice. In Sections 2.5, 2.6 and 2.7 we introduce the main object of this chapter, the algebra $\mathcal{H}_s$, and discuss the structure of it. In Section 2.8, we describe the center of $\mathcal{H}_s$. In Sections 2.2 and 2.9, for the convenience of the reader, we repeat the relevant material, including the definitions of the quantum algebra $U_q(\mathfrak{sl}_2)$ and the quantum affine algebra $U_q(\widehat{\mathfrak{sl}}_2)$ from [6, 13] without proofs, thus making our exposition self-contained. In Section 2.10, our main results are stated and proved.

2.1 Preliminaries

Recall the integers $\mathbb{Z} = \{0, \pm 1, \pm 2, \ldots\}$, the natural numbers $\mathbb{N} = \{0, 1, 2, \ldots\}$ and the complex field $\mathbb{C}$. Assume a nonzero scalar $q \in \mathbb{C}$ is not a root of unity. Throughout this chapter except in Sections 2.2 and 2.9, we fix positive integers a,b and a finite field $\mathbb{F} = \mathbb{F}_q$ of q elements, so we further assume that q is a prime power. Let H denote a vector space over $\mathbb{F}$ with dimension $a + b$. Let P denote the set of all subspaces of H. We view P as a partially ordered set (poset) with the partial order given by inclusion. For $y, z \in P$, we say z *covers* y whenever $y \subseteq z$ and $\dim z = \dim y + 1$. Two elements in P are called *adjacent* whenever one covers the other. For a nonzero integer $n \in \mathbb{N}$, a *path of length* n is a sequence $y_0, y_1, \ldots, y_n$ in P such that y_{i-1} and y_i are adjacent for every $1 \leq i \leq n$. This path is said to be *from* y_0 *to* y_n.

Let us review some of the basic facts about the poset P. The $\mathbb{Z}$-grading of P is the partition of P into disjoint nonempty sets P_0, P_1, ..., P_{a+b} such that

$$P_i = \{y \in P \mid \dim y = i\} \qquad\qquad (0 \leq i \leq a + b).$$

For notational convenience, for $i \in \mathbb{Z}$ define $P_i = \emptyset$ unless $0 \leq i \leq a + b$. By combinatorial counting we verify the following lemmas.

Lemma 2.1.1. *For $0 \leq i \leq a + b$, the following (i), (ii) hold.*

(i) Given $y \in P_i$, there exist exactly $\frac{q^i-1}{q-1}$ elements $z \in P$ which are covered by y.

(ii) Given $y \in P_i$, there exist exactly $\frac{q^{a+b-i}-1}{q-1}$ elements $z \in P$ which cover y.

Lemma 2.1.2. *The following (i)–(iii) hold.*

(i) Given $y, z \in P$ with $y \subseteq z$ and $\dim z = \dim y + 2$, there exist exactly $q + 1$ elements which are adjacent to both y and z.

(ii) Given $y, z \in P$ with $\dim y = \dim z$ and $y \neq z$, if there exists an element that is covered by y and z, then there exists a unique element that covers y and z.

(iii) Given $y, z \in P$ with $\dim y = \dim z$ and $y \neq z$, if there exists an element that covers y and z, then there exists a unique element that is covered by y and z.

Let $V = \mathbb{C}P$ denote the vector space over $\mathbb{C}$ with a basis P. Let $\mathrm{Mat}_P(\mathbb{C})$ denote the $\mathbb{C}$-algebra consisting of the matrices with entries in $\mathbb{C}$ and rows and columns indexed by P. Observe that $\mathrm{Mat}_P(\mathbb{C})$ acts on V by left multiplication. We call V

the *standard module* for $\mathrm{Mat}_P(\mathbb{C})$. We write $I \in \mathrm{Mat}_P(\mathbb{C})$ for the identity matrix. For any nonzero $n \in \mathbb{N}$, we define

$$[n]_q = \frac{q^n - q^{-n}}{q - q^{-1}}, \qquad [n]^!_q = [n]_q [n-1]_q \cdots [1]_q.$$

Set $[0]_q = 0$ and $[0]^!_q = 1$. For simplicity of notation, we write $[n] = [n]_{q^{1/2}}$ and $[n]! = [n]^!_{q^{1/2}}$ for $n \in \mathbb{N}$. Finally, we recall the Gaussian coefficient

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \prod_{i=0}^{k-1} \frac{q^n - q^i}{q^k - q^i} = q^{k(n-k)/2} \frac{[n]!}{[k]![n-k]!},$$

for $0 < k \leq n$ and $\begin{bmatrix} n \\ 0 \end{bmatrix}_q = 1$.

2.2 The quantum algebra $U_q(\mathfrak{sl}_2)$

In this section, we consider a nonzero scalar $q \in \mathbb{C}$ which is not a root of unity. We introduce the notion of the quantum algebra $U_q(\mathfrak{sl}_2)$.

Definition 2.2.1 ([13, p. 122])**.** Let $U_q(\mathfrak{sl}_2)$ be the associative $\mathbb{C}$-algebra generated by $e, f, k^{\pm 1}$ with the relations

$$\begin{aligned} kk^{-1} &= k^{-1}k = 1, \\ ke &= q^2 ek, \\ kf &= q^{-2} fk, \\ ef - fe &= \frac{k - k^{-1}}{q - q^{-1}}. \end{aligned}$$

The elements $e, f, k^{\pm 1}$ are called the *Chevalley generators* for $U_q(\mathfrak{sl}_2)$.

Lemma 2.2.2 ([13, p. 128])**.** *With reference to Definition 2.2.1, for any finite-dimensional irreducible $U_q(\mathfrak{sl}_2)$-module, there exists $\varepsilon \in \{1, -1\}$ such that a basis $v_0, v_1, \ldots, v_d$ for the module satisfies*

$$\begin{aligned} kv_i &= \varepsilon q^{d-2i} v_i & &(0 \leq i \leq d), & & \\ fv_i &= [i+1]_q v_{i+1} & &(0 \leq i \leq d-1), & fv_d &= 0, \\ ev_i &= \varepsilon [d-i+1]_q v_{i-1} & &(1 \leq i \leq d), & ev_0 &= 0. \end{aligned}$$

We write $V_{\varepsilon,d}$ for the above irreducible $U_q(\mathfrak{sl}_2)$-module.

2.3 The incidence algebra $\mathcal{I}$ of the $\mathbb{Z}$-grading of P

Until further notice, a scalar q is a prime power. Recall the $\mathbb{Z}$-grading of P. For $0 \le i \le a+b$, we define diagonal matrices $E_i^* \in \mathrm{Mat}_P(\mathbb{C})$ with (y,y)-entry

$$(E_i^*)_{y,y} = \begin{cases} 1 & \text{if } y \in P_i, \\ 0 & \text{if } y \notin P_i \end{cases} \qquad (y \in P).$$

We have

$$E_i^* E_j^* = \delta_{i,j} E_i^* \qquad (0 \le i, j \le a+b).$$

Here $\delta_{i,j}$ is the Kronecker delta. Also

$$I = E_0^* + E_1^* + \cdots + E_{a+b}^*.$$

Moreover,

$$V = E_0^* V + E_1^* V + \cdots + E_{a+b}^* V \qquad \text{(direct sum)}.$$

Note that $E_i^* V$ has the basis P_i for $0 \le i \le a+b$. We call E_i^* the *i-th projection matrix*. By the above comments, the projection matrices $E_0^*, E_1^*, \ldots, E_{a+b}^*$ form a basis for a commutative subalgebra of $\mathrm{Mat}_P(\mathbb{C})$. It is easy to check that this subalgebra is generated by the diagonal matrix K whose (y,y)-entry is $q^{(a+b)/2-\dim y}$ for $y \in P$. The matrix K is also defined as

$$K = \sum_{i=0}^{a+b} q^{(a+b)/2-i} E_i^*.$$

We remark that K is invertible from the construction.

Next, we introduce two matrices in $\mathrm{Mat}_P(\mathbb{C})$. The matrices L, R have (y,z)-entries

$$L_{y,z} = \begin{cases} 1 & \text{if } z \text{ covers } y, \\ 0 & \text{otherwise} \end{cases} \qquad (y, z \in P),$$

$$R_{y,z} = \begin{cases} 1 & \text{if } y \text{ covers } z, \\ 0 & \text{otherwise} \end{cases} \qquad (y, z \in P).$$

We remark that L, R are the transpose to each other and we have

$$LE_i^* V \subseteq E_{i-1}^* V \qquad (1 \le i \le a+b), \qquad LE_0^* V = 0, \tag{2.1}$$

$$RE_i^* V \subseteq E_{i+1}^* V \qquad (0 \le i \le a+b-1), \qquad RE_{a+b}^* V = 0. \tag{2.2}$$

By the above inclusions, we call L the *lowering matrix* and R the *raising matrix*. For notational convenience, we adjust

$$\widehat{L} = q^{(1-a-b)/2} L.$$

Definition 2.3.1. Let $\mathcal{I}$ denote the $\mathbb{C}$-subalgebra of $\mathrm{Mat}_P(\mathbb{C})$ generated by $\widehat{L}$, R, K. We call $\mathcal{I}$ the *incidence algebra* of P.

Note that K is invertible so that $K^{-1} \in \mathcal{I}$.

Proposition 2.3.2. *The algebra $\mathcal{I}$ in Definition 2.3.1 is semisimple.*

Proof. This follows since $\mathcal{I}$ is closed under the conjugate-transpose map [7, Chapter 4]. □

Lemma 2.3.3. *With above notation, the following (i)–(iii) hold.*

(i) $K\widehat{L} = q\widehat{L}K$.

(ii) $qKR = RK$.

(iii) $\widehat{L}R - R\widehat{L} = \dfrac{K - K^{-1}}{q^{1/2} - q^{-1/2}}$.

Proof. All formulas follow from Lemmas 2.1.1 and 2.1.2. □

Theorem 2.3.4 ([25, Section 7])**.** *The standard module V supports a $U_{q^{1/2}}(\mathfrak{sl}_2)$-module structure on which the Chevalley generators act as follows:*

generators	e	f	k	k^{-1}
actions on V	$\widehat{L}$	R	K	K^{-1}

Proof. By Lemma 2.3.3, $\widehat{L}$, R, K, K^{-1} satisfy the defining relations in Definition 2.2.1. □

Corollary 2.3.5. *There exists an algebra homomorphism from $U_{q^{1/2}}(\mathfrak{sl}_2)$ to $\mathcal{I}$ that sends*

$$e \mapsto \widehat{L}, \qquad f \mapsto R, \qquad k \mapsto K, \qquad k^{-1} \mapsto K^{-1}.$$

Moreover, this homomorphism is surjective.

Proof. By Theorem 2.3.4, such a homomorphism exists. Since the incidence algebra $\mathcal{I}$ is generated by $\widehat{L}$, R, K, this homomorphism is a surjection. □

From Proposition 2.3.2, the standard module V is decomposed into a direct sum of irreducible $\mathcal{I}$-modules, and every irreducible $\mathcal{I}$-module appears in V up to isomorphism. We now discuss the irreducible $\mathcal{I}$-modules in detail.

Definition 2.3.6. Let W be an irreducible $\mathcal{I}$-module. Define

$$\nu = \min\{i \mid 0 \leq i \leq a+b, E_i^* W \neq 0\},$$
$$d = |\{i \mid 0 \leq i \leq a+b, E_i^* W \neq 0\}| - 1.$$

The integers ν, d are called the *endpoint* and *diameter* of W, respectively.

Proposition 2.3.7. *Let W be an irreducible $\mathcal{I}$-module with endpoint ν and diameter d. Then we have*

$$d = a + b - 2\nu.$$

Proof. This follows from the fact that, for $0 \leq i \leq (a+b)/2$, the linear maps $R^{a+b-2i} : E_i^* V \to E_{a+b-i}^* V$ and $L^{a+b-2i} : E_{a+b-i}^* V \to E_i^* V$ are isomorphisms of $\mathbb{C}$-vector spaces. See for example [8]. □

Proposition 2.3.8. *Let W be an irreducible $\mathcal{I}$-module with endpoint ν and diameter d. Then there exists a basis*

$$w_i \in E_{\nu+i}^* W \qquad (0 \leq i \leq d), \tag{2.3}$$

on which generators $\widehat{L}$, R act as follows:

$$Rw_i = [i+1] w_{i+1} \qquad (0 \leq i \leq d-1), \qquad Rw_d = 0,$$
$$\widehat{L} w_i = [d-i+1] w_{i-1} \qquad (1 \leq i \leq d), \qquad \widehat{L} w_0 = 0.$$

Proof. By the definition of endpoint, $E_\nu^* W \neq 0$. We pick any nonzero vector $w_0 \in E_\nu^* W$. Define

$$w_i = \frac{R^i w_0}{[i]!} \qquad (1 \leq i \leq d).$$

For $0 \leq i \leq d$, observe that $w_i \in E_{\nu+i}^* W$ by (2.2). The actions of $\widehat{L}$ and R on w_i are determined from the relations in Lemma 2.3.3. See for example [13, p. 128]. □

Proposition 2.3.9. *Referring to Proposition 2.3.8, the generator K acts on the basis (2.3) as*

$$K w_i = q^{(a+b-2\nu-2i)/2} w_i \qquad (0 \leq i \leq d).$$

Proof. Recall $w_i \in E_{\nu+i}^* W$. The result follows from the definition of K. □

Corollary 2.3.10. *Let W be an irreducible $\mathcal{I}$-module with endpoint ν. Then we have*

$$\dim W = a + b - 2\nu + 1.$$

In particular, $0 \leq \nu \leq (a+b)/2$.

Proof. Use Propositions 2.3.7 and 2.3.8. □

Proposition 2.3.11. *Let W be an irreducible $\mathcal{I}$-module with diameter d. Then, as $U_{q^{1/2}}(\mathfrak{sl}_2)$-modules, W is isomorphic to $V_{1,d}$ in Lemma 2.2.2.*

Proof. For both modules, compare the actions of the generators. □

Proposition 2.3.12. *For each $0 \leq \nu \leq (a+b)/2$, there exists a unique irreducible $\mathcal{I}$-module W with endpoint ν up to isomorphism. The multiplicity* $\mathrm{mult}(\nu)$ *of W in the decomposition of the standard module V is given by*

$$\mathrm{mult}(\nu) = \begin{bmatrix} a+b \\ \nu \end{bmatrix}_q - \begin{bmatrix} a+b \\ \nu-1 \end{bmatrix}_q \qquad (\nu \geq 1),$$

and $\mathrm{mult}(0) = 1$.

Proof. The endpoint ν of an irreducible $\mathcal{I}$-module W must satisfy $0 \leq \nu \leq (a+b)/2$ by Corollary 2.3.10. Moreover, the isomorphism class of W is determined by ν by Propositions 2.3.7, 2.3.8 and 2.3.9. For $0 \leq \nu \leq (a+b)/2$, let $(E^*_\nu V)_{\mathrm{new}}$ denote the subspace consisting of the vectors $w \in E^*_\nu V$ with $Lw = 0$. If $\nu = 0$, any $w \in E^*_0 V$ satisfies $Lw = 0$ so that $\dim(E^*_\nu V)_{\mathrm{new}} = \dim E^*_0 V = 1$. If $\nu \neq 0$, it is known that the linear map $L : E^*_\nu V \to E^*_{\nu-1} V$ is surjective (see for example [8]), so that

$$\dim(E^*_\nu V)_{\mathrm{new}} = \dim E^*_\nu V - \dim E^*_{\nu-1} V = \begin{bmatrix} a+b \\ \nu \end{bmatrix}_q - \begin{bmatrix} a+b \\ \nu-1 \end{bmatrix}_q (\geq 1).$$

Thus, the assertion becomes $\mathrm{mult}(\nu) = \dim(E^*_\nu V)_{\mathrm{new}}$ for all $0 \leq \nu \leq (a+b)/2$. For each irreducible $\mathcal{I}$-module W with endpoint ν, there exists a vector of W that belongs to $(E^*_\nu V)_{\mathrm{new}}$ by Proposition 2.3.8. Moreover, the vector generates W. Therefore the multiplicity $\mathrm{mult}(\nu)$ is bounded from above by $\dim(E^*_\nu V)_{\mathrm{new}}$. On the other hand, for each $0 \leq \nu \leq (a+b)/2$, pick any nonzero vector $w \in (E^*_\nu V)_{\mathrm{new}}$. Let $W = \mathcal{I}w$ denote the $\mathcal{I}$-module generated by w. It suffices to show that W is an irreducible $\mathcal{I}$-module with endpoint ν. Indeed, once this holds, $\mathrm{mult}(\nu)$ is bounded from bottom by $\dim(E^*_\nu V)_{\mathrm{new}}$. Let us write the irreducible $\mathcal{I}$-modules decomposition of W as follows:

$$W = W_1 + W_2 + \cdots + W_r \qquad \text{(direct sum)},$$

for some positive integer r. It is sufficient to show that $r = 1$ and W_1 has endpoint ν. According to this decomposition, we write $w = w_1 + w_2 + \cdots + w_r$ such that $w_i \in W_i$ $(1 \leq i \leq r)$. Then, every w_i lies in $(E^*_\nu V)_{\mathrm{new}}$ and moreover, $w_i \neq 0$ since $W_i = \mathcal{I}w_i$. By Propositions 2.3.8 and 2.3.9, for $1 \leq i \leq r$, W_i must have endpoint ν. Moreover, it follows again from Propositions 2.3.8 and 2.3.9 that, for every $M \in \mathcal{I}$, we have $Mw_i = 0$ for *some* i if and only if $Mw_i = 0$ for *all* i $(1 \leq i \leq r)$. This shows $r = 1$, for otherwise $w_1, \ldots, w_r \notin W$, a contradiction. □

2.4 The decomposition $P_{i,j}$

For the rest of the chapter, we fix a subspace $x \in P$ with $\dim x = a$. For $0 \leq i \leq a$ and $0 \leq j \leq b$, define

$$P_{i,j} = \{y \in P \mid \dim(y \cap x) = i, \dim y = i + j\}.$$

For notational convenience, for $i, j \in \mathbb{Z}$ define $P_{i,j} = \emptyset$ unless $0 \leq i \leq a$ and $0 \leq j \leq b$. Note that

$$P = \bigsqcup_{\substack{0 \leq i \leq a \\ 0 \leq j \leq b}} P_{i,j} \qquad \text{(disjoint union)}.$$

We compile some basic properties about the decomposition $P_{i,j}$ whose proofs are straightforward. Related computations can be found in [5, Section 9.3]. Lemma 2.4.6 is obtained by combining other five lemmas. Only for Lemma 2.4.6, we give a partial proof, which includes the most complicated case.

Lemma 2.4.1. *For $0 \leq i \leq a$ and $0 \leq j \leq b$, the following (i)–(iv) hold.*

(i) Given $y \in P_{i,j}$, there exist exactly $\frac{q^j(q^i-1)}{q-1}$ elements in $P_{i-1,j}$ adjacent to y.

(ii) Given $y \in P_{i,j}$, there exist exactly $\frac{q^j-1}{q-1}$ elements in $P_{i,j-1}$ adjacent to y.

(iii) Given $y \in P_{i,j}$, there exist exactly $\frac{q^{a-i}-1}{q-1}$ elements in $P_{i+1,j}$ adjacent to y.

(iv) Given $y \in P_{i,j}$, there exist exactly $\frac{q^{a-i}(q^{b-j}-1)}{q-1}$ elements in $P_{i,j+1}$ adjacent to y.

Lemma 2.4.2. *For $1 \leq i \leq a$ and $1 \leq j \leq b$, the following (i)–(iv) hold.*

(i) Given $y \in P_{i,j}$ and $z \in P_{i-1,j-1}$ with $z \subseteq y$, there exists a unique element in $P_{i,j-1}$ which is adjacent to both y and z.

(ii) Given $y \in P_{i,j}$ and $z \in P_{i-1,j-1}$ with $z \subseteq y$, there exist exactly q elements in $P_{i-1,j}$ which are adjacent to both y and z.

(iii) Given $y \in P_{i-1,j}$ and $z \in P_{i,j-1}$, if there exists an element that is covered by y and z, then there exists a unique element that covers y and z.

(iv) Given $y \in P_{i-1,j}$ and $z \in P_{i,j-1}$, if there exists an element that covers y and z, then there exists a unique element that is covered by y and z.

Lemma 2.4.3. *For $0 \leq i \leq a$ and $0 \leq j \leq b$, the following (i), (ii) hold.*

(i) Assume i is neither 0 nor a. Given $y \in P_{i+1,j}$ and $z \in P_{i-1,j}$ with $z \subseteq y$, there exist exactly $q+1$ elements in $P_{i,j}$ which are adjacent to both y and z.

(ii) *Assume j is neither 0 nor b. Given $y \in P_{i,j+1}$ and $z \in P_{i,j-1}$ with $z \subseteq y$, there exist exactly $q+1$ elements in $P_{i,j}$ which are adjacent to both y and z.*

Lemma 2.4.4. *Let $0 \leq i \leq a$ and $0 \leq j \leq b$. Given $y, z \in P_{i,j}$ with $y \neq z$, the following (i)–(iv) hold.*

(i) *If $y \cap z \in P_{i-1,j}$, then $y + z \in P_{i+1,j}$.*

(ii) *If $y \cap z \in P_{i,j-1}$, then $y + z \in P_{i+1,j}$ or $y + z \in P_{i,j+1}$.*

(iii) *If $y + z \in P_{i,j+1}$, then $y \cap z \in P_{i,j-1}$.*

(iv) *If $y + z \in P_{i+1,j}$, then $y \cap z \in P_{i-1,j}$ or $y \cap z \in P_{i,j-1}$.*

Lemma 2.4.5. *Let $0 \leq i \leq a$ and $0 \leq j \leq b$. Given $y \in P$, $z \in P_{i,j}$, assuming y does not cover z, the following (i), (ii) hold.*

(i) *Suppose $y \in P_{i+1,j}$. If $y \cap z \in P_{i-1,j}$, then $y + z \in P_{i+2,j}$.*

(ii) *Suppose $y \in P_{i,j+1}$. If $y + z \in P_{i,j+2}$, then $y \cap z \in P_{i,j-1}$.*

Lemma 2.4.6. *Fix $y \in P$ and $z \in P_{i,j}$. The tables below give the numbers of paths z, w_1, w_2, y satisfying the specified conditions on w_1, w_2.*

(i) *If $y \in P_{i+1,j}$ and y covers z, then the numbers of paths are given as follows.*

conditions		*assumption*	*the number of paths*
$w_1 \in P_{i-1,j}$	$w_2 \in P_{i,j}$	$1 \leq i \leq a-1$	$\frac{q^j(q+1)(q^i-1)}{q-1}$
$w_1 \in P_{i+1,j}$	$w_2 \in P_{i,j}$	$0 \leq i \leq a-1$	$\frac{q^{a-i}+q^{i+j+1}-q^j-q}{q-1}$
$w_1 \in P_{i+1,j}$	$w_2 \in P_{i+2,j}$	$0 \leq i \leq a-2$	$\frac{(q^{a-i-1}-1)(q+1)}{q-1}$

(ii) *If $y \in P_{i,j+1}$ and y covers z, then the numbers of paths are given as follows.*

conditions		*assumption*	*the number of paths*
$w_1 \in P_{i,j-1}$	$w_2 \in P_{i,j}$	$1 \leq j \leq b-1$	$\frac{(q+1)(q^j-1)}{q-1}$
$w_1 \in P_{i,j+1}$	$w_2 \in P_{i,j}$	$0 \leq j \leq b-1$	$\frac{q^{a+b-i-j}+q^{j+1}-q^{a-i}-q}{q-1}$
$w_1 \in P_{i,j+1}$	$w_2 \in P_{i,j+2}$	$0 \leq j \leq b-2$	$\frac{q^{a-i}(q^{b-j-1}-1)(q+1)}{q-1}$

(iii) If $y \in P_{i+1,j}$ and y does not cover z, then the numbers of paths are given as follows.

conditions	*assumptions*	*the number of paths*
$w_1 \in P_{i-1,j} \quad w_2 \in P_{i,j}$	$y \cap z \in P_{i-1,j} \quad 1 \le i \le a-1$	$q+1$
	otherwise	0
$w_1 \in P_{i+1,j} \quad w_2 \in P_{i,j}$	$y \cap z \in P_{i-1,j} \quad 0 \le i \le a-1$	$q+1$
	$\star \quad 0 \le i \le a-1$	q
	otherwise	0
$w_1 \in P_{i+1,j} \quad w_2 \in P_{i+2,j}$	$y + z \in P_{i+2,j} \quad 0 \le i \le a-2$	$q+1$
	otherwise	0

Here the symbol $\star$ means $y \cap z \in P_{i,j-1}$ and $y + z \in P_{i+2,j}$.

(iv) If $y \in P_{i,j+1}$ and y does not cover z, then the numbers of paths are given as follows.

conditions	*assumptions*	*the number of paths*
$w_1 \in P_{i,j-1} \quad w_2 \in P_{i,j}$	$y \cap z \in P_{i,j-1} \quad 1 \le j \le b-1$	$q+1$
	otherwise	0
$w_1 \in P_{i,j+1} \quad w_2 \in P_{i,j}$	$* \quad 0 \le j \le b-1$	q
	$y + z \in P_{i,j+2} \quad 0 \le j \le b-1$	$q+1$
	otherwise	0
$w_1 \in P_{i,j+1} \quad w_2 \in P_{i,j+2}$	$y + z \in P_{i,j+2} \quad 0 \le j \le b-2$	$q+1$
	otherwise	0

Here the symbol $*$ means $y \cap z \in P_{i,j-1}$ and $y + z \in P_{i+1,j+1}$.

Proof. It is essential to prove (i) and (iii) since the proofs of (ii) and (iv) are similar to those of (i) and (iii), respectively. Here we prove only for the cases of the second condition in (iii) since this is the most complicated one among all the cases in (i) and (iii).

Assume $y \in P_{i+1,j}$ and y does not cover z. Count the paths z, w_1, w_2, y with condition $w_1 \in P_{i+1,j}$ and $w_2 \in P_{i,j}$. Suppose there exists such a path. Then, applying Lemma 2.4.4 (i) to $y, w_1 \in P_{i+1,j}$, we obtain $y + w_1 \in P_{i+2,j}$. By comparing dimensions, we have $y + w_1 = y + z$, so that $y + z \in P_{i+2,j}$. Similarly, applying Lemma 2.4.4 (iv) to $z, w_2 \in P_{i,j}$, we obtain $z \cap w_2 \in P_{i-1,j}$ or $z \cap w_2 \in P_{i,j-1}$. Again we have $z \cap w_2 = y \cap z$. Now we count the number for each of the two cases. Let us first assume $y \cap z \in P_{i-1,j}$. Recall from Lemma 2.4.5 that we have $y + z \in P_{i+2,j}$. Then by Lemma 2.4.3 (i), there are $q + 1$ choices for w_2, an element in $P_{i,j}$ which is adjacent to both $y \cap z$ and y. For any such element w_2, we have $z \cap w_2 = y \cap z$ and then we have $z + w_2 \in P_{i+1,j}$ by Lemma 2.4.4 (i). Thus there exists a unique w_1 satisfying the condition, which is $z + w_2$. Let us next assume $y \cap z \in P_{i,j-1}$ and $y + z \in P_{i+2,j}$. Then by Lemma 2.4.2 (ii), there are q choices for w_2, an element in $P_{i,j}$ which is adjacent to both $y \cap z$ and y. For any such element w_2, we have $z \cap w_2 = y \cap z$ and then we have $z + w_2 \in P_{i+1,j}$ or $z + w_2 \in P_{i,j+1}$ by Lemma 2.4.4 (ii). Since $z + w_2 \subseteq y + z \in P_{i+2,j}$, we must have $z + w_2 \in P_{i+1,j}$. Thus there exists a unique w_1 satisfying the condition, which is $z + w_2$. □

2.5 The algebra $\mathcal{K}$

For $0 \leq i \leq a$ and $0 \leq j \leq b$, define a diagonal matrix $E^*_{i,j} \in \mathrm{Mat}_P(\mathbb{C})$ with (y, y)-entry

$$(E^*_{i,j})_{y,y} = \begin{cases} 1 & \text{if } y \in P_{i,j}, \\ 0 & \text{if } y \notin P_{i,j} \end{cases} \qquad (y \in P).$$

For notational convenience, for $i, j \in \mathbb{Z}$ define $E^*_{i,j} = 0$ unless $0 \leq i \leq a$ and $0 \leq j \leq b$. We have

$$E^*_{i,j} E^*_{s,t} = \delta_{i,s}\delta_{j,t} E^*_{i,j} \qquad (0 \leq i, s \leq a, \qquad 0 \leq j, t \leq b).$$

Also

$$I = \sum_{i=0}^{a} \sum_{j=0}^{b} E^*_{i,j}.$$

Moreover,

$$V = \sum_{i=0}^{a} \sum_{j=0}^{b} E^*_{i,j} V \qquad \text{(direct sum)}.$$

Note that $E^*_{i,j}V$ has the basis $P_{i,j}$ for $0 \le i \le a$ and $0 \le j \le b$. We call $E^*_{i,j}$ the (i,j)*-projection matrix*.

Definition 2.5.1. By the above comments, the matrices

$$E^*_{i,j} \qquad (0 \le i \le a, \qquad 0 \le j \le b),$$

form a basis for a commutative subalgebra of $\mathrm{Mat}_P(\mathbb{C})$. We denote the subalgebra by $\mathcal{K}$.

We now introduce two matrices that generate $\mathcal{K}$. Define diagonal matrices $K_1, K_2 \in \mathrm{Mat}_P(\mathbb{C})$ with (y,y)-entries

$$(K_1)_{y,y} = q^{a/2-i}, \qquad (K_2)_{y,y} = q^{j-b/2},$$

where $y \in P_{i,j}$.

Lemma 2.5.2. *We have*

$$K_1 = \sum_{i=0}^{a} \sum_{j=0}^{b} q^{a/2-i} E^*_{i,j}, \qquad K_2 = \sum_{i=0}^{a} \sum_{j=0}^{b} q^{j-b/2} E^*_{i,j}.$$

Proof. Immediate from the construction. □

Proposition 2.5.3. *The algebra $\mathcal{K}$ in Definition 2.5.1 is generated by K_1, K_2.*

Proof. By Lemma 2.5.2, the elements K_1, K_2 generate a subalgebra $\mathcal{K}'$ of $\mathcal{K}$. For $0 \le i, i' \le a$ and $0 \le j, j' \le b$ with $(i,j) \ne (i',j')$, we have $(q^{a/2-i}, q^{j-b/2}) \ne (q^{a/2-i'}, q^{j'-b/2})$. Therefore $E^*_{i,j}$ is a polynomial in K_1, K_2 for $0 \le i \le a$ and $0 \le j \le b$. Consequently, $\mathcal{K}' = \mathcal{K}$. □

2.6 The algebra $\mathcal{H}_s$

Define matrices $L_1, L_2, R_1, R_2 \in \mathrm{Mat}_P(\mathbb{C})$ with (y,z)-entries

$$(L_1)_{y,z} = \begin{cases} 1 & \text{if } y \in P_{i-1,j},\ z \in P_{i,j},\ y \subset z, \\ 0 & \text{otherwise} \end{cases} \qquad (y, z \in P),$$

$$(L_2)_{y,z} = \begin{cases} 1 & \text{if } y \in P_{i,j-1},\ z \in P_{i,j},\ y \subset z, \\ 0 & \text{otherwise} \end{cases} \qquad (y, z \in P),$$

$$(R_1)_{y,z} = \begin{cases} 1 & \text{if } y \in P_{i+1,j},\ z \in P_{i,j},\ z \subset y, \\ 0 & \text{otherwise} \end{cases} \qquad (y, z \in P),$$

$$(R_2)_{y,z} = \begin{cases} 1 & \text{if } y \in P_{i,j+1},\ z \in P_{i,j},\ z \subset y, \\ 0 & \text{otherwise} \end{cases} \qquad (y, z \in P).$$

Lemma 2.6.1. *For $0 \leq i \leq a$ and $0 \leq j \leq b$,*

$$L_1 E^*_{i,j} V \subseteq E^*_{i-1,j} V, \qquad L_2 E^*_{i,j} V \subseteq E^*_{i,j-1} V,$$
$$R_1 E^*_{i,j} V \subseteq E^*_{i+1,j} V, \qquad R_2 E^*_{i,j} V \subseteq E^*_{i,j+1} V.$$

Proof. Immediate from the construction. □

Because of Lemma 2.6.1, we call L_1, L_2 the *lowering matrices* and R_1, R_2 the *raising matrices*. We remark that $L_1^t = R_1$ and $L_2^t = R_2$, where t denotes the transpose.

Definition 2.6.2. Let $\mathcal{H}_s$ denote the subalgebra of $\mathrm{Mat}_P(\mathbb{C})$ generated by L_1, L_2, R_1, R_2, $\mathcal{K}$.

The algebra $\mathcal{H}_s$ as well as irreducible $\mathcal{H}_s$-modules were discussed in detail in [20], and some of the results in Sections 2.6, 2.7 are given in [20] in different forms. However, since we adopt different generators and their normalization for $\mathcal{H}_s$, and also a different parametrization of the irreducible $\mathcal{H}_s$-modules, we include full proofs of most of these results for the convenience of the reader.

Proposition 2.6.3. *The algebra $\mathcal{H}_s$ in Definition 2.6.2 is semisimple.*

Proof. This follows since $\mathcal{H}_s$ is closed under the conjugate-transpose map [7, Chapter 4]. □

We now consider some relations in $\mathcal{H}_s$. Here we remark that some of the following relations can be obtained by taking the transpose of others. For notational convenience we adjust L_1, L_2 as follows:

$$\widehat{L}_1 = q^{(1-a-b)/2} L_1, \qquad \widehat{L}_2 = q^{(1-a-b)/2} L_2. \tag{2.4}$$

Lemma 2.6.4. *The following (i)–(viii) hold.*

(i) $K_1\widehat{L}_1 = q\widehat{L}_1K_1$.

(ii) $K_1\widehat{L}_2 = \widehat{L}_2K_1$.

(iii) $qK_1R_1 = R_1K_1$.

(iv) $K_1R_2 = R_2K_1$.

(v) $K_2\widehat{L}_1 = \widehat{L}_1K_2$.

(vi) $qK_2\widehat{L}_2 = \widehat{L}_2K_2$.

(vii) $K_2R_1 = R_1K_2$.

(viii) $K_2R_2 = qR_2K_2$.

Proof. Use Lemmas 2.5.2 and 2.6.1. □

Lemma 2.6.5. *With the above notation, the following (i)–(iv) hold.*

(i) $\widehat{L}_1R_2 = R_2\widehat{L}_1$.

(ii) $\widehat{L}_2R_1 = R_1\widehat{L}_2$.

(iii) $q\widehat{L}_1\widehat{L}_2 = \widehat{L}_2\widehat{L}_1$.

(iv) $R_1R_2 = qR_2R_1$.

Proof. This lemma is a matrix reformulation of Lemma 2.4.2. □

Lemma 2.6.6. *With the above notation, the following (i)–(iv) hold.*

(i) $R_1^2\widehat{L}_1 - (q+1)R_1\widehat{L}_1R_1 + q\widehat{L}_1R_1^2 = -q^{-1/2}(q+1)K_1^{-1}K_2R_1$.

(ii) $qR_2^2\widehat{L}_2 - (q+1)R_2\widehat{L}_2R_2 + \widehat{L}_2R_2^2 = -q^{1/2}(q+1)K_1K_2^{-1}R_2$.

(iii) $q\widehat{L}_1^2R_1 - (q+1)\widehat{L}_1R_1\widehat{L}_1 + R_1\widehat{L}_1^2 = -q^{1/2}(q+1)K_1^{-1}K_2\widehat{L}_1$.

(iv) $\widehat{L}_2^2R_2 - (q+1)\widehat{L}_2R_2\widehat{L}_2 + qR_2\widehat{L}_2^2 = -q^{-1/2}(q+1)K_1K_2^{-1}\widehat{L}_2$.

Proof. (i) For $y, z \in P$, we compare (y,z)-entry of both sides of the formula. Let $y \in P_{r,s}$ and $z \in P_{i,j}$. We assume $r = i+1$ and $s = j$, otherwise all (y,z)-entries are 0. For each term in the equation, we compute (y,z)-entry. These entries are obtained from the table below and (2.4).

Case description	$(R_1^2L_1)_{y,z}$	$(R_1L_1R_1)_{y,z}$	$(L_1R_1^2)_{y,z}$	$(K_1^{-1}K_2R_1)_{y,z}$
y covers z	$\frac{q^j(q+1)(q^i-1)}{q-1}$	$\frac{q^{a-i}+q^{i+j+1}-q^j-q}{q-1}$	$\frac{(q^{a-i-1}-1)(q+1)}{q-1}$	$q^{i+j-(a+b)/2+1}$
$y \cap z \in P_{i-1,j}$	$q+1$	$q+1$	$q+1$	0
$y \cap z \in P_{i,j-1}$, $y+z \in P_{i+2,j}$	0	q	$q+1$	0
otherwise	0	0	0	0

The table entries are routinely obtained from Lemma 2.4.6. From the above comments, the result follows.

(ii) Similar to the proof of (i) above.

(iii) Take the transpose of (i), and use Lemma 2.6.4.

(iv) Take the transpose of (ii), and use Lemma 2.6.4.

□

Due to the complexity of the relations in Lemma 2.6.6, we find it useful to introduce a matrix $F \in \mathrm{Mat}_P(\mathbb{C})$ with (y,z)-entry

$$F_{y,z} = \begin{cases} q^{a-i} + q^j - 1 & \text{if } y, z \in P_{i,j},\ y = z, \\ q-1 & \text{if } y,z \in P_{i,j},\ y \cap z \in P_{i,j-1},\ y+z \in P_{i+1,j}, \\ 0 & \text{otherwise,} \end{cases}$$

for $y, z \in P$. For notational convenience, define

$$\widehat{F} = q^{(-a-b)/2} F. \tag{2.5}$$

Lemma 2.6.7. *The following (i),(ii) hold.*

(i) $\widehat{L}_1 R_1 - R_1 \widehat{L}_1 = \dfrac{\widehat{F} - K_1^{-1}K_2}{q^{1/2} - q^{-1/2}}$.

(ii) $\widehat{L}_2 R_2 - R_2 \widehat{L}_2 = \dfrac{K_1K_2^{-1} - \widehat{F}}{q^{1/2} - q^{-1/2}}$.

Moreover, $\widehat{F} \in \mathcal{H}_s$.

Proof. (i) For $y, z \in P$, we compare (y, z)-entry of both sides of the formula. Let $y \in P_{r,s}$ and $z \in P_{i,j}$. We assume $r = i$ and $s = j$, otherwise all (y, z)-entries are 0. For each term in the equation, we compute (y, z)-entry. These entries are obtained from the table below and (2.4), (2.5).

Case description	$(L_1R_1)_{y,z}$	$(R_1L_1)_{y,z}$	$F_{y,z}$	$(K_1^{-1}K_2)_{y,z}$
$y = z$	$\frac{q^{a-i}-1}{q-1}$	$\frac{q^j(q^i-1)}{q-1}$	$q^{a-i} + q^j - 1$	$q^{i+j-(a+b)/2}$
$y \cap z \in P_{i-1,j}$	1	1	0	0
$y \cap z \in P_{i,j-1}$, $y + z \in P_{i+1,j}$	1	0	$q - 1$	0
otherwise	0	0	0	0

The table entries are routinely obtained from Lemmas 2.4.1 and 2.4.4. From the above comments, the result follows.

(ii) Similar to the proof of (i) above.

□

Lemma 2.6.8. *The following (i), (ii) hold.*

(i) $K_1\widehat{F} = \widehat{F}K_1$.

(ii) $K_2\widehat{F} = \widehat{F}K_2$.

Proof. Combine Lemmas 2.6.4 and 2.6.7. □

Lemma 2.6.9. *The following (i)–(iv) hold.*

(i) $\widehat{F}\widehat{L}_1 = q\widehat{L}_1\widehat{F}$.

(ii) $q\widehat{F}\widehat{L}_2 = \widehat{L}_2\widehat{F}$.

(iii) $q\widehat{F}R_1 = R_1\widehat{F}$.

(iv) $\widehat{F}R_2 = qR_2\widehat{F}$.

Proof. Combine Lemmas 2.6.4, 2.6.5 and 2.6.7. □

Our next general goal is to show that $\widehat{F}$ is invertible.

Lemma 2.6.10. *For $0 \le i \le a$ and $0 \le j \le b$, we have*

$$\widehat{F}E^*_{i,j}V \subseteq E^*_{i,j}V.$$

Proof. Immediate from the construction. □

Lemma 2.6.11. *For $0 \leq i \leq a$ and $0 \leq j \leq b$, the matrix $\widehat{F}$ is diagonalizable on $E^*_{i,j}V$. The eigenvalues $\theta_{i,j,l}$ and the multiplicities $m_{i,j,l}$ are given by*

$$\theta_{i,j,l} = q^{(a-b-2i+2j-2l)/2},$$
$$m_{i,j,l} = \begin{bmatrix} a \\ i \end{bmatrix}_q \begin{bmatrix} b \\ j \end{bmatrix}_q \begin{bmatrix} a-i \\ l \end{bmatrix}_q \begin{bmatrix} j \\ l \end{bmatrix}_q \prod_{s=0}^{l-1} \left(q^l - q^s\right),$$

where $0 \leq l \leq \min\{a-i, j\}$.

Proof. Let $F' \in \mathrm{Mat}_P(\mathbb{C})$ be a matrix with (y,z)-entry

$$F'_{y,z} = \begin{cases} (q-1)^{-1} F_{y,z} & \text{if } y \neq z, \\ 0 & \text{if } y = z, \end{cases}$$
$$= \begin{cases} 1 & \text{if } y, z \in P_{i,j},\ y \cap z \in P_{i,j-1},\ y + z \in P_{i+1,j}, \\ 0 & \text{otherwise}, \end{cases}$$

for $y, z \in P$. Then it is sufficient to show that F' is diagonalizable on $E^*_{i,j}V$ with the eigenvalues

$$\theta'_{i,j,l} = \frac{q^{(a+b)/2}\theta_{i,j,l} - q^{a-i} - q^j + 1}{q-1},$$

and the multiplicities $m_{i,j,l}$ for $0 \leq l \leq \min\{a-i, j\}$.

For $0 \leq i \leq a$, $0 \leq j \leq b$, $u \in P_{i,0}$ and $w \in P_{a,j}$, let $(E^*_{i,j}V)_{u,w}$ be a subspace of $E^*_{i,j}V$ spanned by

$$P_{i,j,u,w} = \{y \in P_{i,j} \mid y \cap x = u, y + x = w\}.$$

Then it is easy to check that $F'(E^*_{i,j}V)_{u,w} \subseteq (E^*_{i,j}V)_{u,w}$. Therefore we have a block diagonal form:

$$F' = \bigoplus_{i=0}^{a} \bigoplus_{j=0}^{b} \bigoplus_{u \in P_{i,0}} \bigoplus_{w \in P_{a,j}} F'|_{i,j,u,w},$$

where $|_{i,j,u,w}$ means the restriction to $(E^*_{i,j}V)_{u,w}$.

For each i, j, u, w, observe a bijection from $P_{i,j,u,w}$ to the set of j-subspaces of w/u which intersect with x/u trivially. Under this bijection, $F'|_{i,j,u,w}$ is precisely the adjacency matrix of the *bilinear forms graph* $\mathrm{Bil}_q(a-i, j)$ with eigenvalues $\theta'_{i,j,l}$ and multiplicities $m_{i,j,l}$ (see [5, Section. 9.5 A]). Combining with the block diagonal form of F', we complete the proof of our claim. □

Corollary 2.6.12. *The matrix $\widehat{F}$ is invertible.*

Proof. Observe that the eigenvalues $\theta_{i,j,l}$ of $\widehat{F}$ given in Lemma 2.6.11 are nonzero. □

2.7 The irreducible $\mathcal{H}_s$-modules

Recall from Proposition 2.6.3 that the algebra $\mathcal{H}_s$ is semisimple. Thus the standard module V is a direct sum of irreducible $\mathcal{H}_s$-modules, and every irreducible $\mathcal{H}_s$-module appears in V up to isomorphism. We now discuss the irreducible $\mathcal{H}_s$-modules in V. Recall that for $0 \leq i \leq a$ and $0 \leq j \leq b$ the matrix $\widehat{F}$ acts on $E^*_{i,j}V$. For each eigenvalue $\theta_{i,j,l}$ of this action, let $V_{i,j,l}$ denote the corresponding eigenspace. Thus

$$V_{i,j,l} = \{v \in E^*_{i,j}V \mid \widehat{F}v = \theta_{i,j,l}v\}.$$

Let W be an irreducible $\mathcal{H}_s$-module in V. Then we have

$$W = \sum_{i=0}^{a}\sum_{j=0}^{b} E^*_{i,j}W \qquad \text{(direct sum)}.$$

Definition 2.7.1. Let W denote an irreducible $\mathcal{H}_s$-module. Define

$$\nu = \min\{i \mid 0 \leq i \leq a, E^*_{i,j}W \neq 0 \text{ for some } j\},$$
$$\mu = \min\{j \mid 0 \leq j \leq b, E^*_{i,j}W \neq 0 \text{ for some } i\}.$$

We call the ordered pair (ν, μ) the *lower endpoint* of W. Define

$$\nu' = \max\{i \mid 0 \leq i \leq a, E^*_{i,j}W \neq 0 \text{ for some } j\},$$
$$\mu' = \max\{j \mid 0 \leq j \leq b, E^*_{i,j}W \neq 0 \text{ for some } i\}.$$

We call the ordered pair (ν', μ') the *upper endpoint* of W.

By construction,

$$0 \leq \nu \leq \nu' \leq a, \qquad 0 \leq \mu \leq \mu' \leq b. \tag{2.6}$$

Lemma 2.7.2. *With reference to Definition 2.7.1, the following are equivalent for* $0 \leq i \leq a$ *and* $0 \leq j \leq b$.

(i) $E^*_{i,j}W \neq 0$.

(ii) $\nu \leq i \leq \nu'$ *and* $\mu \leq j \leq \mu'$.

Suppose (i), (ii) hold. Then $\dim E^*_{i,j}W = 1$ *and* $E^*_{i,j}W = (R_1)^{i-\nu}(R_2)^{j-\mu}E^*_{\nu,\mu}W$.

Proof. By the definition of the lower endpoint, there exists a nonzero vector $w \in E^*_{\nu,\mu}W$ such that $\widehat{L}_1 w = \widehat{L}_2 w = 0$. For $n, m \in \mathbb{N}$, a vector $(R_1)^n(R_2)^m w$ is in $E^*_{\nu+n,\mu+m}W$ by Lemma 2.6.1. The assertion is equivalent to showing that the module W has a basis

$$(R_1)^n(R_2)^m w, \qquad 0 \leq n \leq \nu' - \nu, \quad 0 \leq m \leq \mu' - \mu. \tag{2.7}$$

From the irreducibility of W, we can write $W = \mathcal{H}_s w$. Since $\mathcal{H}_s$ is generated by R_1, R_2, $\widehat{L}_1$, $\widehat{L}_2$, K_1, K_2 and from the relations in Lemmas 2.6.4, 2.6.5, 2.6.7, 2.6.8 and 2.6.9, the module W is spanned by

$$(R_1)^n(R_2)^m(K_1)^s(K_2)^t(\widehat{F})^u(\widehat{L}_1)^i(\widehat{L}_2)^j w \qquad (n, m, s, t, u, i, j \in \mathbb{N}). \tag{2.8}$$

Since w vanishes by the actions of $\widehat{L}_1$ and $\widehat{L}_2$, the vector in (2.8) is zero unless $i = j = 0$. Moreover, since K_1, K_2, $\widehat{F}$ are diagonalizable on $E^*_{\nu,\mu}W$, we may take vectors with $s = t = u = 0$ for a spanning set of W. By the definition of the upper endpoint, $(R_1)^n(R_2)^m w \neq 0$ if $n = \nu' - \nu$ and $m = \mu' - \mu$, while $(R_1)^n(R_2)^m w = 0$ if $n > \nu' - \nu$ or $m > \mu' - \mu$. It remains to prove that the vectors in (2.7) are nonzero. Suppose there exist $0 \leq n' \leq \nu' - \nu$, $0 \leq m' \leq \mu' - \mu$ such that $(R_1)^{n'}(R_2)^{m'} w = 0$. Then by Lemma 2.6.5 (iv),

$$(R_1)^{\nu'-\nu}(R_2)^{\mu'-\mu} w = q^{n'(\mu'-\mu-m')}(R_1)^{\nu'-\nu-n'}(R_2)^{\mu'-\mu-m'}\left((R_1)^{n'}(R_2)^{m'} w\right). \tag{2.9}$$

Then the left-hand side of (2.9) is nonzero, while the right-hand side of (2.9) is zero. This is a contradiction and completes the proof. □

Lemma 2.7.3. *With reference to Definition 2.7.1, we have*

$$\nu + \mu + \nu' + \mu' = a + b.$$

Proof. Let $W = W_1 + W_2 + \cdots + W_r$ denote a direct sum decomposition into irreducible $\mathcal{I}$-modules, and let ν_i denote the endpoint of W_i for $1 \leq i \leq r$. Observe that $\nu + \mu = \min\{\nu_i \mid 1 \leq i \leq r\}$ by Lemma 2.7.2. On the other hand, it follows from Proposition 2.3.7 and Lemma 2.7.2 that

$$\nu' + \mu' = \max\{a + b - \nu_i \mid 1 \leq i \leq r\} = a + b - \min\{\nu_i \mid 1 \leq i \leq r\} = a + b - \nu - \mu.$$

The result follows. □

Definition 2.7.4. Let W denote an irreducible $\mathcal{H}_s$-module with lower endpoint (ν, μ) and upper endpoint (ν', μ'). By Lemma 2.7.3, we have $a - \nu - \nu' = -b + \mu + \mu'$. We denote this common value by ρ and call it the *index* of W.

Lemma 2.7.5. *Let W denote an irreducible $\mathcal{H}_s$-module with lower endpoint (ν, μ) and index ρ. Then*

$$2\mu - b \leq \rho \leq a - 2\nu.$$

Proof. This follows from (2.6). □

Lemma 2.7.6. *Let W denote an irreducible $\mathcal{H}_s$-module with lower endpoint (ν, μ) and index ρ. Then*

$$\dim W = (a - 2\nu - \rho + 1)(b - 2\mu + \rho + 1).$$

Proof. Let (ν', μ') denote the upper endpoint of W. By Lemma 2.7.2, the dimension of W is $(\nu' - \nu + 1)(\mu' - \mu + 1)$. By the definition of ρ, this is the same as the desired formula. □

Proposition 2.7.7. *Let W be an irreducible $\mathcal{H}_s$-module in V with lower endpoint (ν, μ) and index ρ. There exists a basis*

$$w_{n,m} \qquad (0 \leq n \leq a - 2\nu - \rho, \qquad 0 \leq m \leq b - 2\mu + \rho), \tag{2.10}$$

on which the generators $\widehat{L}_1$, $\widehat{L}_2$, R_1, R_2 act as follows:

$$\begin{aligned}
\widehat{L}_1 w_{n,m} &= q^{(-b+2\mu-\rho+2m)/2}[a - 2\nu - \rho - n + 1] w_{n-1,m}, \\
\widehat{L}_2 w_{n,m} &= q^{(a-2\nu-\rho)/2}[b - 2\mu + \rho - m + 1] w_{n,m-1}, \\
R_1 w_{n,m} &= [n+1] w_{n+1,m}, \\
R_2 w_{n,m} &= q^{-n}[m+1] w_{n,m+1}.
\end{aligned}$$

Here we set $w_{n,m} = 0$ unless n and m satisfy the inequalities in (2.10).

Proof. Let (ν', μ') denote the upper endpoint of W. Remark that $\nu' = a - \nu - \rho$ and $\mu' = b - \mu + \rho$. Pick any nonzero vector $w \in E^*_{\nu,\mu} W$. Set

$$w_{n,m} = \frac{(R_1)^n (R_2)^m}{[n]![m]!} w \qquad (0 \leq n \leq \nu' - \nu, \quad 0 \leq m \leq \mu' - \mu). \tag{2.11}$$

Remark that $w_{0,0} = w$. The set (2.11) of vectors forms a basis for the vector space W by Lemma 2.7.2. We show that every vector in (2.11) satisfies the desired actions. By the construction and Lemma 2.6.5 (iv), it is easily seen that the desired actions of R_1 and R_2 hold. For $\widehat{L}_1$, recall Lemma 2.6.6 (i). For $0 \leq n \leq \nu' - \nu$ and $0 \leq m \leq \mu' - \mu$, applying both sides of the equation in Lemma 2.6.6 (i) to $w_{n,m}$, we have

$$\begin{aligned}
&(R_1)^2 \widehat{L}_1 w_{n,m} - (q+1)[n+1] R_1 \widehat{L}_1 w_{n+1,m} + q[n+1][n+2] \widehat{L}_1 w_{n+2,m} \\
&= -q^{-1/2}(q+1)[n+1] K_1^{-1} K_2 w_{n+1,m}.
\end{aligned} \tag{2.12}$$

By the definitions of K_1 and K_2, the scaler of $w_{n+1,m}$ in the right hand side of (2.12) is known. By Lemmas 2.6.1 and 2.7.2, we know $\widehat{L}_1 w_{n,m}$ is a scalar multiple of $w_{n-1,m}$ for $1 \leq n \leq \nu' - \nu$ and $0 \leq m \leq \mu' - \mu$. Set the scaler by $c_{n-1,m}$ and set $c_{-1,m} = c_{d+1,m} = 0$ for $0 \le m \le \mu' - \mu$. Substituting this in (2.12), we have for $0 \le n \le d$ and $0 \le m \le \delta$,

$$\begin{aligned} &c_{n-1,m} - (q+1)[n-\nu+1]c_{n,m} + q[n-\nu+1][n-\nu+2]c_{n+1,m} \\ &\quad = -(q+1)[n-\nu+1]q^{(-a-b+2n+2m+1)/2}. \end{aligned}$$

It turns out $c_{n,m} = q^{(-b-\rho+2m)/2}[a-\nu-\rho-n]$ $(\nu \le n \le \nu', \mu \le m \le \mu')$ is the unique solution to this system. This determines the action of $\widehat{L}_1$. The proof of the action of $\widehat{L}_2$ is similar. □

Proposition 2.7.8. *Referring to Proposition 2.7.7, the elements K_1, K_2, $\widehat{F}$ act on the basis (2.10) in the following way:*

$$\begin{aligned} K_1 w_{n,m} &= q^{(a-2n-2\nu)/2} w_{n,m}, \\ K_2 w_{n,m} &= q^{(-b+2m+2\mu)/2} w_{n,m}, \\ \widehat{F} w_{n,m} &= q^{(a-b-2n+2m-2\nu+2\mu-2\rho)/2} w_{n,m}. \end{aligned}$$

Proof. From the required actions in Proposition 2.7.7 and from Lemmas 2.6.1 and 2.7.2, it turns out that $w_{n,m} \in E^*_{\nu+n,\mu+m}W$ for $0 \le n \le a-2\nu-\rho$ and $0 \le m \le b-2\mu+\rho$. Then the actions of K_1, K_2 on each $w_{n,m}$ are given by their definitions. As for the action of $\widehat{F}$, we use the relations in Lemma 2.6.7 (i), combined with the other actions of $\widehat{L}_1$, R_1, K_1, K_2. □

Theorem 2.7.9. *Let W denote an irreducible $\mathcal{H}_s$-module. Then W is determined up to isomorphism by its lower endpoint and index.*

Proof. From Propositions 2.7.7 and 2.7.8, the actions of $\widehat{L}_1$, $\widehat{L}_2$, R_1, R_2, K_1, K_2 on W are determined by its lower endpoint and index. The result follows since $\mathcal{H}_s$ is generated by $\widehat{L}_1$, $\widehat{L}_2$, R_1, R_2, K_1, K_2. □

Definition 2.7.10. Let W denote an irreducible $\mathcal{H}_s$-module with lower endpoint (ν, μ) and index ρ. We call the triple (ν, μ, ρ) the *type* of W.

Lemma 2.7.11. *Let W be an irreducible $\mathcal{H}_s$-module of type (ν, μ, ρ). Then*

$$0 \le \rho \le \mu.$$

Proof. From Proposition 2.7.8, the value $q^{(a-b-2i+2j-2\rho)/2}$ is an eigenvalue of the $\widehat{F}$-action on $E^*_{i,j}V$ for $\nu \le i \le a-\nu-\rho$ and $\mu \le j \le b-\mu+\rho$. Then, by Lemma 2.6.11, the index ρ must satisfy $0 \le \rho \le \min\{a-i, j\}$. This implies the desired inequality. □

Theorem 2.7.12. *The following (i), (ii) hold.*

(i) Let W denote an irreducible $\mathcal{H}_s$-module in V of type (i,j,l). Then

$$E^*_{i,j}W \subseteq V_{i,j,l} \cap \operatorname{Ker} \widehat{L}_1 \cap \operatorname{Ker} \widehat{L}_2.$$

(ii) For (ν,μ,ρ) satisfying $0 \leq \nu \leq a$, $0 \leq \mu \leq b$, $0 \leq \rho \leq \mu$, $2\nu - b \leq \rho \leq a - 2\nu$ and for a nonzero vector $w \in V_{\nu,\mu,\rho} \cap \operatorname{Ker} \widehat{L}_1 \cap \operatorname{Ker} \widehat{L}_2$, the $\mathcal{H}_s$-module $\mathcal{H}_s w$ is irreducible of type (ν,μ,ρ).

Here we see each $\widehat{L}_i$ as a linear operator and $\operatorname{Ker} \widehat{L}_i$ denotes the kernel of it, i.e., $\operatorname{Ker} \widehat{L}_i = \{v \in V \mid \widehat{L}_i v = 0\}$.

Proof. (i) Since (i,j) is the lower endpoint of W, the space $E^*_{i,j}W$ is a subspace of $\operatorname{Ker} \widehat{L}_1$ and $\operatorname{Ker} \widehat{L}_2$. By Proposition 2.7.8, there exists a nonzero vector $w \in E^*_{i,j}W$ such that

$$\widehat{F}w = q^{(a-b-2i+2j-2l)/2}w.$$

This proves $w \in V_{i,j,l}$. By Lemma 2.7.2, the subspace $E^*_{i,j}W$ is spanned by one vector w. Hence the result follows.

(ii) Set $W = \mathcal{H}_s w$. We write the irreducible $\mathcal{H}_s$-modules decomposition of W as follows:

$$W = W_1 + W_2 + \cdots + W_r \qquad \text{(direct sum)},$$

for some positive integer r. It is sufficient to show that $r = 1$ and W_1 is of type (ν,μ,ρ). According to this decomposition, we write $w = w_1 + w_2 + \cdots + w_r$ such that $w_i \in W_i$ $(1 \leq i \leq r)$. Then, every w_i lies in $V_{\nu,\mu,\rho} \cap \operatorname{Ker} \widehat{L}_1 \cap \operatorname{Ker} \widehat{L}_2$ and moreover, $w_i \neq 0$ since $W_i = \mathcal{H}_s w_i$. By Propositions 2.7.7 and 2.7.8, for $1 \leq i \leq r$, W_i must be of type (ν,μ,ρ). Moreover, it follows again from Propositions 2.7.7 and 2.7.8 that, for every $M \in \mathcal{H}_s$, we have $Mw_i = 0$ for *some* i if and only if $Mw_i = 0$ for *all* i $(1 \leq i \leq r)$. This shows $r = 1$, for otherwise $w_1, \ldots, w_r \notin W$, a contradiction.

□

Corollary 2.7.13. *For integers ν, μ, ρ, the following are equivalent.*

(i) There exists an irreducible $\mathcal{H}_s$-module of type (ν,μ,ρ).

(ii) $0 \leq \nu \leq a$ and $0 \leq \mu \leq b$ and $0 \leq \rho \leq \mu$ and $2\mu - b \leq \rho \leq a - 2\nu$.

Proof. We first assume (i) holds. Then (ii) follows from (2.6) and Lemmas 2.7.5 and 2.7.11. On the other hand, assume (ii) holds. If there exists a nonzero vector

$w \in V_{\nu,\mu,\rho}$ such that $\widehat{L}_1 w = \widehat{L}_2 w = 0$, then by Theorem 2.7.12, $\mathcal{H}_s w$ is an irreducible $\mathcal{H}_s$-module of type (ν, μ, ρ). So all we need to show is that the vector space $V_{\nu,\mu,\rho} \cap \operatorname{Ker} \widehat{L}_1 \cap \operatorname{Ker} \widehat{L}_2$ is nonzero under the assumption (ii). This is shown in [20, Corollary 7.12]. □

Theorem 2.7.12 shows that the irreducible $\mathcal{H}_s$-modules in V are classified by their types up to isomorphism. Now, in addition to types, we introduce a different characterization, which also has a combinatorial importance.

Definition 2.7.14. Let W be an irreducible $\mathcal{H}_s$-module with lower endpoint (ν, μ) and upper endpoint (ν', μ'). Define

$$d = \nu' - \nu, \qquad \delta = \mu' - \mu.$$

We call the ordered pair (d, δ) the *diameter* of W.

Lemma 2.7.15. *Let W denote an irreducible $\mathcal{H}_s$-module of index ρ. Its lower endpoint (ν, μ) and diameter (d, δ) are related as follows:*

$$\nu = \frac{a - d - \rho}{2}, \qquad \mu = \frac{b - \delta + \rho}{2},$$
$$d = a - 2\nu - \rho, \qquad \delta = b - 2\mu + \rho.$$

Moreover,

$$\nu' = a - \nu - \rho = \frac{a + d - \rho}{2}, \qquad \mu' = b - \mu + \rho = \frac{b + \delta + \rho}{2}.$$

Proof. Use Definitions 2.7.4 and 2.7.14. □

Corollary 2.7.16. *Let W denote an irreducible $\mathcal{H}_s$-module. Then W is determined up to isomorphism by diameter and index.*

Proof. Use Theorem 2.7.9 and Lemma 2.7.15. □

Definition 2.7.17. Let W denote an irreducible $\mathcal{H}_s$-module with diameter (d, δ) and index ρ. We call the triple (d, δ, ρ) the *shape* of W.

We now show how Corollary 2.7.13 looks in terms of shape. We find it convenient to work with shapes instead of types.

Corollary 2.7.18. *For integers d, δ, ρ, the following are equivalent.*

(i) There exists an irreducible $\mathcal{H}_s$-module of shape (d, δ, ρ).

(ii) $0 \le d \le a$ and $0 \le \delta \le b$ and $0 \le \rho \le \min\{a - d, b - \delta\}$.

Proof. Use Corollary 2.7.13 and Lemma 2.7.15. □

Let $\mathbb{S}$ denote the set of triples (d, δ, ρ) of integers that satisfy Corollary 2.7.18 (ii). For $s \in \mathbb{S}$, we denote by $E_s \in \mathrm{Mat}_P(\mathbb{C})$ the projection from V onto the sum of all irreducible $\mathcal{H}_s$-modules of shape s. We have

$$E_s E_{s'} = \delta_{s,s'} E_s \qquad (s, s' \in \mathbb{S}),$$
$$I = \sum_{s \in \mathbb{S}} E_s.$$

Moreover,

$$V = \sum_{s \in \mathbb{S}} E_s V \qquad \text{(direct sum)}.$$

In Propositions 2.7.7 and 2.7.8, we described the actions of $\widehat{L}_1$, $\widehat{L}_2$, R_1, R_2, K_1, K_2, $\widehat{F}$ on a basis for an irreducible $\mathcal{H}_s$-module. We now describe these actions in terms of shapes.

Corollary 2.7.19. *Let W be an irreducible $\mathcal{H}_s$-module in V of shape (d, δ, l). On the basis (2.10) the generators $\widehat{L}_1$, $\widehat{L}_2$, R_1, R_2 act as follows, where the inequalities in (2.10) become $0 \le n \le d$ and $0 \le m \le \delta$:*

$$\widehat{L}_1 w_{n,m} = q^{(-\delta+2m)/2}[d - n + 1] w_{n-1,m}$$
$$\widehat{L}_2 w_{n,m} = q^{d/2}[\delta - m + 1] w_{n,m-1},$$
$$R_1 w_{n,m} = [n + 1] w_{n+1,m},$$
$$R_2 w_{n,m} = q^{-n}[m + 1] w_{n,m+1}.$$

Here, recall that we set $w_{n,m} = 0$ unless n and m satisfy the inequalities in (2.10).

Proof. This is a restatement of Proposition 2.7.7 in terms of shapes. □

Corollary 2.7.20. *Referring to Corollary 2.7.19, the elements K_1, K_2, $\widehat{F}$ act on the basis (2.10) in the following way:*

$$K_1 w_{n,m} = q^{(d+l-2n)/2} w_{n,m},$$
$$K_2 w_{n,m} = q^{(-\delta+l+2m)/2} w_{n,m},$$
$$\widehat{F} w_{n,m} = q^{(d-\delta-2n+2m)/2} w_{n,m}.$$

Proof. This is a restatement of Proposition 2.7.8 in terms of shapes. □

Corollary 2.7.21. *Let W be an irreducible $\mathcal{H}_s$-module in V with index l. Then for $0 \le i \le a$, $0 \le j \le b$ we have*

$$E^*_{i,j} W \subseteq V_{i,j,l}.$$

Moreover, $\sum_{i=0}^{a} \sum_{j=0}^{b} V_{i,j,l}$ is an $\mathcal{H}_s$-module.

Proof. Let (ν,μ) be the lower endpoint of W. Assume $\nu \le i \le a-\nu-l$ and $\mu \le j \le b-\mu+l$; otherwise the first assertion is trivial from Lemma 2.7.2. From Corollary 2.7.8, there is a vector $w = w_{i-\nu,j-\mu} \in E^*_{i,j}W$ such that

$$\widehat{F}w = q^{(a-b-2i+2j-2\rho)/2}w.$$

By the definition of $V_{i,j,l}$, this shows us that $w \in V_{i,j,l}$. From Lemma 2.7.2, we know the vector space $E^*_{i,j}W$ is spanned by one vector w so that the first assertion holds.

Next let $\widetilde{V}_l$ denote the sum of all irreducible $\mathcal{H}_s$-modules in V of index l. By the first assertion, we have $\widetilde{V}_l \subseteq \sum_{i=0}^{a}\sum_{j=0}^{b} V_{i,j,l}$. Recall the two direct sum decompositions of V:

$$V = \sum_{l} \widetilde{V}_l \qquad \text{(direct sum)},$$
$$V = \sum_{l}\left(\sum_{i=0}^{a}\sum_{j=0}^{b} V_{i,j,l}\right) \qquad \text{(direct sum)},$$

where the two sums are over all indices l. Then by the above comment, the corresponding summands must coincide, i.e., we have $\widetilde{V}_l = \sum_{i=0}^{a}\sum_{j=0}^{b} V_{i,j,l}$ for every l. This proves the second assertion. □

2.8 The center of the algebra $\mathcal{H}_s$

In this section, we show central elements which generate the center of the algebra $\mathcal{H}_s$. Recall from Corollary 2.6.12, we have $\widehat{F}^{-1} \in \mathcal{H}_s$. Define $\Lambda_0, \Lambda_1, \Lambda_2 \in \mathcal{H}_s$ by

$$\Lambda_0 = K_1K_2\widehat{F}^{-1},$$
$$\Lambda_1 = \widehat{L}_1R_1K_2^{-1} + \frac{q^{1/2}K_1^{-1} + q^{-1/2}\widehat{F}K_2^{-1}}{(q^{1/2}-q^{-1/2})^2},$$
$$\Lambda_2 = \widehat{L}_2R_2K_1^{-1} + \frac{q^{-1/2}K_2^{-1} + q^{1/2}\widehat{F}K_1^{-1}}{(q^{1/2}-q^{-1/2})^2}.$$

Lemma 2.8.1. *The above elements Λ_0, Λ_1, Λ_2 are in the center of the algebra $\mathcal{H}_s$.*

Proof. It is sufficient to check that each of Λ_0, Λ_1, Λ_2 commutes with the generators. This follows from Lemmas 2.6.4, 2.6.7, 2.6.8 and 2.6.9. One may use the properties that K_1, K_2, F are symmetric and that L_1, L_2 are transpose to R_1, R_2, respectively. □

Lemma 2.8.2. *The following hold.*

(i) The complete set of the eigenvalues for Λ_0 is

$$q^l \qquad\qquad (0 \le l \le a).$$

Moreover, $\sum_{d,\delta} E_{(d,\delta,l)}V$ is the eigenspace with respect to the above eigenvalue.

(ii) The complete set of the eigenvalues for Λ_1 is

$$\frac{q^{(d-l+1)/2} + q^{(-d-l-1)/2}}{(q^{1/2} - q^{-1/2})^2} \qquad\qquad (0 \le d \le a, \quad 0 \le l \le a-d).$$

Moreover, $\sum_{\delta} E_{(d,\delta,l)}V$ is the eigenspace with respect to the above eigenvalue.

(iii) The complete set of the eigenvalues for Λ_2 is

$$\frac{q^{(\delta-l+1)/2} + q^{(-\delta-l-1)/2}}{(q^{1/2} - q^{-1/2})^2} \qquad\qquad (0 \le \delta \le b, \quad 0 \le l \le a).$$

Moreover, $\sum_{d} E_{(d,\delta,l)}V$ is the eigenspace with respect to the above eigenvalue.

In particular, each of Λ_0, Λ_1, Λ_2 is diagonalizable in the standard module V. Here the sum ranges all possible shapes in $\mathbb{S}$.

Proof. Let $\lambda_0(l)$, $\lambda_1(d,l)$, $\lambda_2(\delta,l)$ denote the desired eigenvalues of Λ_0, Λ_1, Λ_2 respectively. We fix a shape $(d,\delta,l) \in \mathbb{S}$. Let W be an irreducible $\mathcal{H}_s$-module of shape (d,δ,l). Let $\{w_{n,m}\}$ be the basis for W in (2.10). For given one vector $w_{n,m}$, it is sufficient to show that

$$\begin{aligned}
\Lambda_0 w_{n,m} &= \lambda_0(l) w_{n,m}, \\
\Lambda_1 w_{n,m} &= \lambda_1(d,l) w_{n,m}, \\
\Lambda_2 w_{n,m} &= \lambda_2(\delta,l) w_{n,m}.
\end{aligned}$$

These formulas are obtained by using the actions in Corollaries 2.7.19 and 2.7.20. □

Theorem 2.8.3. *The center of the algebra $\mathcal{H}_s$ is generated by Λ_0, Λ_1 and Λ_2.*

Proof. Let $\lambda_0(l)$, $\lambda_1(d,l)$, $\lambda_2(\delta,l)$ denote the eigenvalues of Λ_0, Λ_1, Λ_2, respectively, from Lemma 2.8.2. Recall that for $s \in \mathbb{S}$, we denote by E_s the projection matrix from V onto the sum of irreducible $\mathcal{H}_s$-modules in V of shape s. By Lemma 2.8.2, we have

$$\Lambda_0 = \sum_{(d,\delta,l)\in\mathbb{S}} \lambda_0(l) E_{(d,\delta,l)}, \quad \Lambda_1 = \sum_{(d,\delta,l)\in\mathbb{S}} \lambda_1(d,l) E_{(d,\delta,l)}, \quad \Lambda_2 = \sum_{(d,\delta,l)\in\mathbb{S}} \lambda_2(\delta,l) E_{(d,\delta,l)}.$$

Moreover, we have $(\lambda_0(l), \lambda_1(d,l), \lambda_2(\delta,l)) = (\lambda_0(l'), \lambda_1(d',l'), \lambda_2(\delta',l'))$ if and only if $(d,\delta,l) = (d',\delta',l')$. Hence, for $s \in \mathbb{S}$, we see E_s as a polynomial in Λ_0, Λ_1, Λ_2. Since the elements E_s $(s \in \mathbb{S})$ are the central primitive idempotents of $\mathcal{H}_s$, the result follows. □

2.9 The quantum affine algebra $U_q(\widehat{\mathfrak{sl}}_2)$

In this section, we fix a nonzero scalar $q \in \mathbb{C}$ which is not a root of unity.

Definition 2.9.1 ([6, Section 2])**.** The quantum affine algebra $U_q(\widehat{\mathfrak{sl}}_2)$ is the associative $\mathbb{C}$-algebra generated by $e_i^{\pm}, k_i^{\pm 1}$, $(i = 0, 1)$ with the relations

$$\begin{aligned}
k_i k_i^{-1} &= k_i^{-1} k_i = 1, \\
k_0 k_1 &= k_1 k_0, \\
k_i e_i^{\pm} &= q^{\pm 2} e_i^{\pm} k_i, \\
k_i e_j^{\pm} &= q^{\mp 2} e_j^{\pm} k_i \qquad (i \neq j), \\
e_i^{+} e_i^{-} - e_i^{-} e_i^{+} &= \frac{k_i - k_i^{-1}}{q - q^{-1}}, \\
e_0^{\pm} e_1^{\mp} - e_1^{\mp} e_0^{\pm} &= 0, \\
(e_i^{\pm})^3 e_j^{\pm} - [3]_q (e_i^{\pm})^2 e_j^{\pm} e_i^{\pm} + [3]_q e_i^{\pm} e_j^{\pm} (e_i^{\pm})^2 - e_j^{\pm} (e_i^{\pm})^3 &= 0 \qquad (i \neq j).
\end{aligned}$$

We call $e_i^{\pm}, k_i^{\pm 1}$ the *Chevalley generators* for $U_q(\widehat{\mathfrak{sl}}_2)$.

Lemma 2.9.2 ([6, Section 2])**.** *The quantum affine algebra $U_q(\widehat{\mathfrak{sl}}_2)$ has the following Hopf algebra structure. The comultiplication Δ satisfies*

$$\begin{aligned}
\Delta(e_i^{+}) &= e_i^{+} \otimes k_i + 1 \otimes e_i^{+}, \\
\Delta(e_i^{-}) &= e_i^{-} \otimes 1 + k_i^{-1} \otimes e_i^{-}, \\
\Delta(k_i) &= k_i \otimes k_i.
\end{aligned}$$

The counit ε satisfies

$$\varepsilon(e_i^{\pm}) = 0, \quad \varepsilon(k_i) = 1.$$

The antipode S satisfies

$$S(k_i) = k_i^{-1}, \quad S(e_i^{+}) = -e_i^{+} k_i^{-1}, \quad S(e_i^{-}) = -k_i e_i^{-}.$$

Lemma 2.9.3 ([6, Section 4])**.** *For any nonzero scalar $\alpha \in \mathbb{C}$, there is an algebra homomorphism $\mathrm{ev}_\alpha : U_q(\widehat{\mathfrak{sl}}_2) \to U_q(\mathfrak{sl}_2)$ that sends*

$$\begin{aligned}
e_0^{+} &\mapsto \alpha f, & e_0^{-} &\mapsto \alpha^{-1} e, & k_0 &\mapsto k^{-1}, \\
e_1^{+} &\mapsto e, & e_1^{-} &\mapsto f, & k_1 &\mapsto k.
\end{aligned}$$

We call the algebra homomorphism ev_α in Lemma 2.9.3 the *evaluation homomorphism with evaluation parameter α*. Modules for $U_q(\widehat{\mathfrak{sl}}_2)$ can be obtained from modules for $U_q(\mathfrak{sl}_2)$ along evaluation homomorphism ev_α.

Definition 2.9.4 ([6, Section 4]). For $d \in \mathbb{N}$ and a nonzero scalar $\alpha \in \mathbb{C}$, the *evaluation module* $V_d(\alpha)$ for $U_q(\widehat{\mathfrak{sl}}_2)$ is the pull-back of the $U_q(\mathfrak{sl}_2)$-module $V_{1,d}$ along the evaluation homomorphism $\mathrm{ev}_\alpha : U_q(\widehat{\mathfrak{sl}}_2) \to U_q(\mathfrak{sl}_2)$.

Lemma 2.9.5. *For $d \in \mathbb{N}$ and a nonzero scalar $\alpha \in \mathbb{C}$, the evaluation $U_q(\widehat{\mathfrak{sl}}_2)$-module $V_d(\alpha)$ has a basis $\{v_i\}_{i=0}^d$ on which the Chevalley generators act as follows:*

$$\begin{aligned}
e_0^+ v_i &= \alpha[i+1]_q v_{i+1} && (0 \le i \le d-1), && e_0^+ v_d = 0,\\
e_1^+ v_i &= [d-i+1]_q v_{i-1} && (1 \le i \le d), && e_1^+ v_0 = 0,\\
e_0^- v_i &= \alpha^{-1}[d-i+1]_q v_{i-1} && (1 \le i \le d), && e_1^+ v_0 = 0,\\
e_1^- v_i &= [i+1]_q v_{i+1} && (0 \le i \le d-1), && e_0^+ v_d = 0,\\
k_0 v_i &= q^{2i-d} v_i && (0 \le i \le d), &&\\
k_1 v_i &= q^{d-2i} v_i && (0 \le i \le d). &&
\end{aligned}$$

Proof. Use Lemmas 2.2.2 and 2.9.3. □

Lemma 2.9.6. *For $d, \delta \in \mathbb{N}$ and nonzero scalars $\alpha, \beta \in \mathbb{C}$, the tensor product of two evaluation $U_q(\widehat{\mathfrak{sl}}_2)$-modules $V_d(\alpha) \otimes V_\delta(\beta)$ is again a $U_q(\widehat{\mathfrak{sl}}_2)$-module. This module has a basis*

$$v_n \otimes w_m \qquad (0 \le n \le d, \quad 0 \le m \le \delta), \tag{2.13}$$

on which the Chevalley generators act as follows:

$$\begin{aligned}
e_0^+(v_n \otimes w_m) &= \alpha[n+1]_q q^{2m-\delta}(v_{n+1} \otimes w_m) + \beta[m+1]_q(v_n \otimes w_{m+1}),\\
e_1^+(v_n \otimes w_m) &= [d-n+1]_q q^{\delta-2m}(v_{n-1} \otimes w_m) + [\delta-m+1]_q(v_n \otimes w_{m-1}),\\
e_0^-(v_n \otimes w_m) &= \alpha^{-1}[d-n+1]_q(v_{n-1} \otimes w_m) + \beta^{-1}[\delta-m+1]_q q^{d-2n}(v_n \otimes w_{m-1}),\\
e_1^-(v_n \otimes w_m) &= [n+1]_q(v_{n+1} \otimes w_m) + [m+1]_q q^{2n-d}(v_n \otimes w_{m+1}),\\
k_0(v_n \otimes w_m) &= q^{2n+2m-d-\delta}(v_n \otimes w_m),\\
k_1(v_n \otimes w_m) &= q^{d+\delta-2n-2m}(v_n \otimes w_m).
\end{aligned}$$

Here we set $v_n \otimes w_m = 0$ unless n and m satisfy the inequalities in (2.13).

Proof. Recall that $U_q(\widehat{\mathfrak{sl}}_2)$ has a Hopf algebra structure. The comultiplication

$$\Delta : U_q(\widehat{\mathfrak{sl}}_2) \to U_q(\widehat{\mathfrak{sl}}_2) \otimes U_q(\widehat{\mathfrak{sl}}_2),$$

induces the $U_q(\widehat{\mathfrak{sl}}_2)$-module structure on $V_d(\alpha) \otimes V_\delta(\beta)$. The actions are obtained from Lemmas 2.9.2 and 2.9.5. □

With an evaluation module $V_d(\alpha)$, we associate the set of scalars

$$S_d(\alpha) = \{\alpha q^{d-1}, \alpha q^{d-3}, \ldots, \alpha q^{-d+1}\}.$$

The set $S_d(\alpha)$ is called a *q-string* of length d. Two q-strings $S_d(\alpha)$, $S_\delta(\beta)$ are said to be in *general position* if one of the following occurs:

(i) $S_d(\alpha) \cup S_\delta(\beta)$ is not a q-string,

(ii) $S_d(\alpha) \subseteq S_\delta(\beta)$ or $S_\delta(\beta) \subseteq S_d(\alpha)$.

Moreover, q-strings $S_{d_1}(\alpha_1), \ldots, S_{d_r}(\alpha_r)$ are said to be in *general position* if every two q-strings are in general position.

Theorem 2.9.7 ([6, Section 4]). *A tensor product $V_{d_1}(\alpha_1) \otimes \cdots \otimes V_{d_r}(\alpha_r)$ of evaluation modules for $U_q(\widehat{\mathfrak{sl}}_2)$ is irreducible if and only if the associated q-strings $S_{d_1}(\alpha_1)$, $\ldots$, $S_{d_r}(\alpha_r)$ are in general position.*

Theorem 2.9.8 ([6, Section 4]). *Two tensor products of evaluation modules for $U_q(\widehat{\mathfrak{sl}}_2)$ are isomorphic if and only if one is obtained from the other by permuting the factors in the tensor product.*

2.10 $\mathcal{H}_s$ and $U_{q^{1/2}}(\widehat{\mathfrak{sl}}_2)$

In this section, we get back to the subspace lattice. Let q be a fixed prime power.

Theorem 2.10.1. *The standard module V supports a $U_{q^{1/2}}(\widehat{\mathfrak{sl}}_2)$-module structure on which the Chevalley generators act as follows:*

generators	e_0^+	e_1^+	e_0^-	e_1^-
actions on V	$R_1 + R_2$	$\widehat{L}_1 X + Y \widehat{L}_2$	$\widehat{L}_1 + \widehat{L}_2$	$R_1 Z + W R_2$

generators	k_0	k_1	k_0^{-1}	k_1^{-1}
actions on V	$K_1^{-1} K_2$	$K_1 K_2^{-1}$	$K_1 K_2^{-1}$	$K_1^{-1} K_2$

Here $X = K_1 \widehat{F}^{-1}$, $Y = K_2 \widehat{F}^{-1}$, $Z = K_2^{-1}$, $W = K_1^{-1}$.

Proof. The proof is straightforward. We check if the actions on V corresponding to the Chevalley generators satisfy the defining relations of $U_{q^{1/2}}(\widehat{\mathfrak{sl}}_2)$ in Definition 2.9.1. These follow from the relations in Lemmas 2.6.4–2.6.9. □

Corollary 2.10.2. *There exists an algebra homomorphism from $U_{q^{1/2}}(\widehat{\mathfrak{sl}}_2)$ to $\mathcal{H}_s$ that sends*

$$
\begin{aligned}
e_0^+ &\mapsto R_1 + R_2, & e_1^+ &\mapsto \widehat{L}_1 X + Y \widehat{L}_2,\\
e_0^- &\mapsto \widehat{L}_1 + \widehat{L}_2, & e_1^- &\mapsto R_1 Z + W R_2,\\
k_0 &\mapsto K_1^{-1} K_2, & k_1 &\mapsto K_1 K_2^{-1}.
\end{aligned}
$$

Here $X = K_1 \widehat{F}^{-1}$, $Y = K_2 \widehat{F}^{-1}$, $Z = K_2^{-1}$, $W = K_1^{-1}$.

Proof. Immediate from Theorem 2.10.1. □

Remark that Theorem 2.10.1 and Corollary 2.10.2 are still true if we swap the values of X and Z and/or those of Y and W. However, this does not cause any essential difference for our discussion.

By the homomorphism in Corollary 2.10.2, we can see any module for $\mathcal{H}_s$ as a module for $U_{q^{1/2}}(\widehat{\mathfrak{sl}}_2)$. In particular, the standard module V becomes a $U_{q^{1/2}}(\widehat{\mathfrak{sl}}_2)$-module.

Lemma 2.10.3. *Referring to Proposition 2.7.7 and Lemma 2.9.6, let $W_{d,\delta,l}$ denote an irreducible $\mathcal{H}_s$-module of shape (d, δ, l) and $V_d(\alpha) \otimes V_\delta(\beta)$ denote the tensor product of two evaluation $U_{q^{1/2}}(\widehat{\mathfrak{sl}}_2)$-modules with evaluation parameters α and β. For nonzero scalars $c_{n,m} \in \mathbb{C}$ $(0 \le n \le d, 0 \le m \le \delta)$, the following are equivalent.*

(i) There exists a $U_{q^{1/2}}(\widehat{\mathfrak{sl}}_2)$-module isomorphism $\varphi : V_d(\alpha) \otimes V_\delta(\beta) \to W_{d,\delta,l}$ such that

$$\varphi(v_n \otimes u_m) = c_{n,m} w_{n,m} \qquad (0 \le n \le d, \qquad 0 \le m \le \delta).$$

(ii) $\alpha = \beta = q^{l/2}$ and there exists a nonzero scalar $\gamma \in \mathbb{C}$ such that

$$c_{n,m} = \gamma q^{n(\delta - l)/2 - ml/2 - nm} \qquad (0 \le n \le d, \qquad 0 \le m \le \delta).$$

Proof. First we assume (i), and show (ii). It is independent of the scalars $c_{n,m}$ that the map φ preserves the actions of k_0, k_1. Thus, we check the map φ preserves the actions of the other Chevalley generators.

Comparing e_0^+ action on $v_n \otimes u_m$ in Lemma 2.9.6 with $R_1 + R_2$ action on $w_{n,m}$ in Corollary 2.7.19, we have

$$\alpha q^{(2m-\delta)/2} c_{n+1,m} = c_{n,m}, \qquad \beta c_{n,m+1} = q^{-n} c_{n,m}.$$

Similarly, from e_1^+ action and $\widehat{L}_1 X + Y \widehat{L}_2$ action, we obtain

$$q^{(\delta-2m)/2} c_{n-1,m} = q^{l/2} c_{n,m}, \qquad c_{n,m-1} = q^{(2n+l)/2} c_{n,m}.$$

By the above formulas, we get $\alpha = \beta = q^{l/2}$. Moreover, we have

$$c_{n,m} = q^{n(\delta-2m-l)/2} c_{0,m} = q^{n(\delta-2m-l)/2-lm/2} c_{0,0},$$

for any $0 \le n \le d$, $0 \le m \le \delta$.

On the other hand, the scalars $c_{n,m}$ define the map φ. It is a direct calculation to show that φ becomes an isomorphism of $U_{q^{1/2}}(\widehat{\mathfrak{sl}}_2)$-modules. □

Theorem 2.10.4. *Referring to Lemma 2.10.3, we have the following isomorphism for $U_{q^{1/2}}(\widehat{\mathfrak{sl}}_2)$-modules.*

$$V_d(q^{l/2}) \otimes V_\delta(q^{l/2}) \simeq W_{d,\delta,l},$$

for $(d, \delta, l) \in \mathbb{S}$.

Proof. The proof is immediate from Lemma 2.10.3. □

Theorem 2.10.5. *Any irreducible $\mathcal{H}_s$-module is irreducible as a $U_{q^{1/2}}(\widehat{\mathfrak{sl}}_2)$-module.*

Proof. Let W be an irreducible $\mathcal{H}_s$-module of shape (d, δ, l). Then from Theorem 2.10.4, this module W is isomorphic to

$$V_d(q^{l/2}) \otimes V_\delta(q^{l/2}),$$

which is an irreducible $U_{q^{1/2}}(\widehat{\mathfrak{sl}}_2)$-module by Theorem 2.9.7. □

Theorem 2.10.6. *The algebra $\mathcal{H}_s$ is generated by both the image of $U_{q^{1/2}}(\widehat{\mathfrak{sl}}_2)$ of the algebra homomorphism defined in Corollary 2.10.2 and its center.*

Proof. Let $\mathcal{H}'_s$ denote the subalgebra of $\mathcal{H}_s$ generated by the homomorphic image of $U_{q^{1/2}}(\widehat{\mathfrak{sl}}_2)$ and the center of $\mathcal{H}_s$. By Theorem 2.10.5, any irreducible $\mathcal{H}_s$-module is also irreducible as an $\mathcal{H}'_s$-module. We now show that $\mathcal{H}'_s$ coincides with $\mathcal{H}_s$ by comparing the $\mathcal{H}'_s$-isomorphism classes of irreducible $\mathcal{H}_s$-modules in V.

Let W_1, W_2 be irreducible $\mathcal{H}_s$-modules in V. If W_1, W_2 are $\mathcal{H}_s$-isomorphic, then they are clearly $\mathcal{H}'_s$-isomorphic. On the other hand, suppose that W_1, W_2 are $\mathcal{H}'_s$-isomorphic. Observe that the projections E_s $(s \in \mathbb{S})$ belong to the center of $\mathcal{H}_s$. Hence there is a unique $s \in \mathbb{S}$ such that $E_s W_1$ and $E_s W_2$ are both nonzero. In other words, W_1, W_2 are $\mathcal{H}_s$-isomorphic. The result follows. □

Chapter 3

An algebra associated with a flag in a subspace lattice over a finite field and the quantum affine algebra $U_q(\widehat{\mathfrak{sl}}_2)$

In this chapter, we introduce an algebra $\mathcal{H}_f$ from a subspace lattice with respect to a fixed flag which contains its incidence algebra as a proper subalgebra. We then establish a relation between the algebra $\mathcal{H}_f$ and the quantum affine algebra $U_{q^{1/2}}(\widehat{\mathfrak{sl}}_2)$, where q denotes the cardinality of the base field. It is an extension of the well-known relation between the incidence algebra of a subspace lattice and the quantum algebra $U_{q^{1/2}}(\mathfrak{sl}_2)$. We show that there exists an algebra homomorphism from $U_{q^{1/2}}(\widehat{\mathfrak{sl}}_2)$ to $\mathcal{H}_f$ and that any irreducible module for $\mathcal{H}_f$ is irreducible as an $U_{q^{1/2}}(\widehat{\mathfrak{sl}}_2)$-module. This chapter is based on the author's work [26].

We organize this chapter as follows. In Section 3.1, we recall the basic notation and introduce a hyper-cubic structure in a subspace lattice. In Section 3.2, we recall some notation on Ferrers boards, rook placements and inversion numbers which is used in this chapter. In Sections 3.3 and 3.4, we introduce a matrix representation of P and interpret some properties of matrices in terms of rook placements and inversion numbers. In Sections 3.5 and 3.6, we introduce the main object of this chapter, the algebra $\mathcal{H}_f$, and discuss the structure of it. In Sections 3.7, 3.8, 3.9 and 3.10, we discuss the $\mathcal{H}_f$-action on the standard module and classify all the irreducible $\mathcal{H}_f$-modules up to isomorphism. In Section 3.11, for the convenience of the reader, we repeat the relevant material, including the definition of the quantum affine algebra $U_q(\widehat{\mathfrak{sl}}_2)$, from [6] without proofs, thus making our exposition self-contained. In Section 3.12, our main results are stated and proved.

3.1 A subspace lattice and its hyper-cubic structure

We now begin our formal argument. Recall the integers $\mathbb{Z} = \{0, \pm 1, \pm 2, \ldots\}$ and the natural numbers $\mathbb{N} = \{0, 1, 2, \ldots\}$ and let $\mathbb{C}$ denote the complex field. The Kronecker delta is denoted by δ. Throughout this chapter except Section 3.11, we fix $N \in \mathbb{N} \setminus \{0\}$. Throughout this chapter except Sections 3.2, 3.9 and 3.11, we fix a prime power q. Let $\mathbb{F}_q$ denote a finite field of q elements and let H denote a vector space over $\mathbb{F}_q$ with dimension N. Let P denote the set of all subspaces of H. We view P as a poset with the partial order given by inclusion. The poset P is a graded lattice of rank N where the rank function is defined by its dimension and called the *subspace lattice*. For two subspaces $y, z \in P$, we say y *covers* z whenever $z \subseteq y$ and $\dim z = \dim y - 1$. By a *(full) flag* on H we mean a sequence $\{x_i\}_{i=0}^N$ of subspaces in P such that $\dim x_i = i$ for $0 \leq i \leq N$ and $x_{i-1} \subsetneq x_i$ for $1 \leq i \leq N$. For the rest of this chapter, we fix a flag $\{x_i\}_{i=0}^N$ on H. A basis $v_1, v_2, \ldots, v_N$ for H is said to be *adapted to the flag* $\{x_i\}_{i=0}^N$ whenever each x_i is spanned by $v_1, v_2, \ldots, v_i$ for $1 \leq i \leq N$.

By the N*-cube* we mean the poset consisting of all N-tuples in $\{0,1\}^N$ with the partial order $\mu \leq \nu$ defined by $\mu_m \leq \nu_m$ for all $1 \leq m \leq N$, where $\mu = (\mu_1, \mu_2, \ldots, \mu_N), \nu = (\nu_1, \nu_2, \ldots, \nu_N) \in \{0,1\}^N$. (We note that it is isomorphic to the Boolean lattice of all subsets of an N-set.) The N-cube is a graded lattice of rank N with the rank function defined by

$$|\mu| = \mu_1 + \mu_2 + \cdots + \mu_N,$$

for $\mu = (\mu_1, \mu_2, \ldots, \mu_N) \in \{0,1\}^N$.

Proposition 3.1.1. *There exists an order-preserving map from the subspace lattice P to the N-cube which sends $y \in P$ to $(\mu_1, \mu_2, \ldots, \mu_N) \in \{0,1\}^N$ where*

$$\dim(y \cap x_m) = \mu_1 + \mu_2 + \cdots + \mu_m,$$

for $1 \leq m \leq N$. Moreover, this map is surjective.

Proof. For $y \in P$ and $1 \leq m \leq N$, observe that $\mu_m = \dim(y \cap x_m) - \dim(y \cap x_{m-1})$ is either 0 or 1 since $x_{m-1} \subsetneq x_m$ and $\dim x_m - \dim x_{m-1} = 1$. Therefore this correspondence becomes a map from P to the N-cube. It is clear that this map preserves the ordering. To show its surjectivity, let $v_1, v_2, \ldots, v_N$ denote a basis for H adapted to the flag $\{x_i\}_{i=0}^N$. For any $\mu = (\mu_1, \mu_2, \ldots, \mu_N) \in \{0,1\}^N$, consider the subspace $y \in P$ spanned by the vectors $\{v_i \mid 1 \leq i \leq N, \mu_i = 1\}$ and check that y is mapped to μ. Therefore it is surjective. □

Definition 3.1.2. If $\mu \in \{0,1\}^N$ is the image of $y \in P$ by the map in Proposition 3.1.1, we call μ the *location* of y. For $\mu \in \{0,1\}^N$, let P_μ denote the set of all subspaces at location μ. For notational convenience, for $\mu \in \mathbb{Z}^N$ we set $P_\mu = \emptyset$ unless $\mu \in \{0,1\}^N$.

Note that P is the disjoint union of P_μ, where $\mu \in \{0,1\}^N$. Observe that $\dim y = |\mu|$ for $y \in P_\mu$.

Definition 3.1.3. Let $1 \le m \le N$. For $\mu = (\mu_1, \mu_2, \ldots, \mu_N), \nu = (\nu_1, \nu_2, \ldots, \nu_N) \in \{0,1\}^N$, we say μ *m-covers* ν whenever $\nu_m < \mu_m$ and $\nu_n = \mu_n$ for $1 \le n \le N$ with $n \neq m$. Similarly, for $y, z \in P$, we say y *m-covers* z whenever y covers z and the location of y m-covers the location of z.

For each $1 \le m \le N$, let $\widehat{m}$ denote the N-tuple in $\{0,1\}^N$ with a 1 in m-th coordinate and 0 elsewhere. To simplify the notation, we consider the coordinate-wise addition in $\mathbb{Z}^N$ so that μ m-covers ν if and only if $\mu = \nu + \widehat{m}$ for $\mu, \nu \in \{0,1\}^N$.

Lemma 3.1.4. *For $\mu = (\mu_1, \mu_2, \ldots, \mu_N) \in \{0,1\}^N$ and for $1 \le m \le N$, the following (i), (ii) hold.*

(i) Given $y \in P_\mu$, the number of subspaces m-covered by y is

$$\delta_{\mu_m,1} q^{\mu_{m+1}+\mu_{m+2}+\cdots+\mu_N}.$$

(ii) Given $y \in P_\mu$, the number of subspaces which m-cover y is

$$\delta_{\mu_m,0} q^{(m-1)-(\mu_1+\mu_2+\cdots+\mu_{m-1})}.$$

Proof. (i) Assume $\mu_m = 1$, otherwise, the assertion is clear. Set $n = \dim y - \dim(y \cap x_m) = \mu_{m+1} + \cdots + \mu_N$. Take linearly independent vectors $u_1, \ldots, u_n \in y \setminus (y \cap x_m)$ such that $\mathrm{Span}\{u_1, \ldots, u_n\} \cap (y \cap x_m) = 0$. We define $w = \mathrm{Span}\{u_1, \ldots, u_n\} + (y \cap x_{m-1})$. Then we have $w \subseteq y$ and $w \in P_{\mu-\widehat{m}}$. On the other hand, for any $w' \in P_{\mu-\widehat{m}}$ with $w' \subseteq y$, there exist linearly independent vectors $u'_1, \ldots, u'_n \in w' \setminus (w' \cap x_m)$ such that $\mathrm{Span}\{u'_1, \ldots, u'_n\} \cap (w' \cap x_m) = 0$. Then we have $\mathrm{Span}\{u'_1, \ldots, u'_n\} \cap (y \cap x_m) = 0$ and $w' = \mathrm{Span}\{u'_1, \ldots, u'_n\} + (y \cap x_{m-1})$. Thus, there are

$$\frac{(q^{\mu_1+\cdots+\mu_N} - q^{\mu_1+\cdots+\mu_m}) \cdots (q^{\mu_1+\cdots+\mu_N} - q^{\mu_1+\cdots+\mu_m+n-1})}{(q^{\mu_1+\cdots+\mu_N-1} - q^{\mu_1+\cdots+\mu_{m-1}}) \cdots (q^{\mu_1+\cdots+\mu_N-1} - q^{\mu_1+\cdots+\mu_{m-1}+n-1})} = q^n$$

subspaces $w \in P_{\mu-\widehat{m}}$ with $w \subseteq y$.

(ii) Assume $\mu_m = 0$, otherwise, the assertion is clear. Take a vector $u \in x_m \setminus x_{m-1}$. Then $\mathrm{Span}\{u\} \cap y = 0$. Set $w = \mathrm{Span}\{u\} + y$. Then $y \subseteq w$ and we have $w \in P_{\mu+\widehat{m}}$. On the other hand, for any $w' \in P_{\mu+\widehat{m}}$ with $y \subseteq w'$, we can write as $w' = \mathrm{Span}\{u'\} + y$ for any $u' \in (w' \cap x_m) \setminus (w' \cap x_{m-1}) \subseteq x_m \setminus x_{m-1}$. Thus, there are

$$\frac{q^m - q^{m-1}}{q^{\mu_1+\cdots+\mu_{m-1}+1} - q^{\mu_1+\cdots+\mu_{m-1}}} = q^{(m-1)-(\mu_1+\cdots+\mu_{m-1})}$$

subspaces $w \in P_{\mu+\widehat{m}}$ with $y \subseteq w$.

□

Lemma 3.1.5. *Let $1 \leq m < n \leq N$. For $\mu = (\mu_1, \mu_2, \ldots, \mu_N) \in \{0,1\}^N$ with $\mu_m = \mu_n = 1$, the following hold.*

(i) Given $z \in P_\mu$ and $y \in P_{\mu-\widehat{m}-\widehat{n}}$ with $y \subseteq z$, there exists a unique element in $P_{\mu-\widehat{n}}$ which m-covers y and which is n-covered by z.

(ii) Given $z \in P_\mu$ and $y \in P_{\mu-\widehat{m}-\widehat{n}}$ with $y \subseteq z$, there exist exactly q elements in $P_{\mu-\widehat{m}}$ which n-cover y and which are m-covered by z.

(iii) Given $y \in P_{\mu-\widehat{m}}$ and $z \in P_{\mu-\widehat{n}}$, if there exists an element that is covered by both y and z, then there exists a unique element that covers both y and z.

(iv) Given $y \in P_{\mu-\widehat{m}}$ and $z \in P_{\mu-\widehat{n}}$, if there exists an element that covers both y and z, then there exists a unique element that is covered by both y and z.

Proof. (i) Set $w = y + (z \cap x_{n-1})$. It is easy to check that w covers y and w is covered by z. Observe that the location of w is $\mu - \widehat{n}$. On the other hand, any $w' \in P_{\mu-\widehat{n}}$ which covers y and which is covered by z contains both y and $z \cap x_{n-1}$. So $w \subseteq w'$. By computing dimensions, w and w' must coincide. The result follows.

(ii) There exist exactly $q + 1$ elements which cover y and which are covered by z since $\dim z - \dim y = 2$. Observe that they must belong to either $P_{\mu-\widehat{n}}$ or $P_{\mu-\widehat{m}}$. Therefore the result follows from (i).

(iii) Since y and z are distinct, the element that is covered by both y and z must be $y \cap z$. Therefore, $y + z$ is a unique element which covers both y and z.

(iv) Similar to (iii).

□

3.2 Ferrers boards

We introduce the notion of Ferrers boards. For the general theory on this topic, we refer the reader to [18, Chapters 1 and 2]. Note that we modify the notation in [18] to fit our setting.

Let $\mu = (\mu_1, \mu_2, \ldots, \mu_N) \in \{0,1\}^N$. Then μ has a natural correspondence with a bipartition of $\{1, 2, \ldots, N\}$, which is defined by

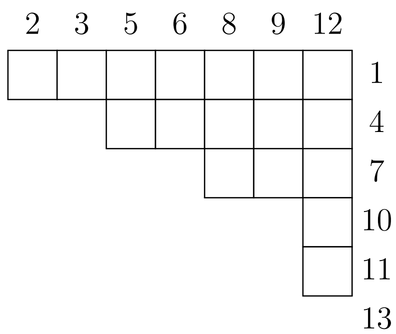
$$S_\mu = \{s \in \mathbb{N} \mid 1 \le s \le N, \mu_s = 0\}, \qquad T_\mu = \{t \in \mathbb{N} \mid 1 \le t \le N, \mu_t = 1\}. \tag{3.1}$$

We remark that S_μ and T_μ are empty if and only if $\mu = \mathbf{1} = (1, 1, \ldots, 1)$ and $\mu = \mathbf{0} = (0, 0, \ldots, 0)$, respectively. The *Ferrers board* of shape μ is defined by

$$B_\mu = \{(s,t) \in S_\mu \times T_\mu \mid s < t\}. \tag{3.2}$$

If both S_μ and T_μ are not empty, i.e. if $\mu \ne \mathbf{0}, \mathbf{1}$, we can draw a Ferrers board as a two-dimensional subarray of a matrix whose rows indexed by S_μ and columns indexed by T_μ, whose (s,t)-entry has a box for all $(s,t) \in B_\mu$. This subarray is also known as a *Young diagram* of shape μ.

Example 3.2.1 ($N = 13$)**.** *Let* $\mu = (0,1,1,0,1,1,0,1,1,0,0,1,0) \in \{0,1\}^{13}$. *Then the corresponding Ferrers board* B_μ *has the following subarray form:*



Take a nonempty Ferrers board B_μ of shape μ. For $(s_0, t_0) \in B_\mu$, the *rectangle* in B_μ with respect to (s_0, t_0), denoted by $B_\mu(s_0, t_0)$, is defined by

$$B_\mu(s_0, t_0) = \{(s,t) \in B_\mu \mid s \le s_0, t \ge t_0\}. \tag{3.3}$$

It is actually the rectangle in the corresponding Young diagram which includes the top-right corner and the (s_0, t_0)-th box as its bottom-left corner. We remark that such a rectangle is called the *Durfee square* if it is the largest square in B_μ. To see the rectangle structure, we use the following notation:

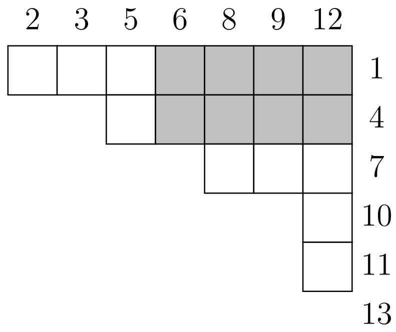
$$S_\mu(m) = \{s \in S_\mu \mid s \le m\}, \qquad T_\mu(m) = \{t \in T_\mu \mid t \ge m\}, \tag{3.4}$$

for $1 \le m \le N$ so that we can write $B_\mu(s_0, t_0) = S_\mu(s_0) \times T_\mu(t_0)$.

Example 3.2.2 ($N = 13$)**.** *Take $\mu \in \{0,1\}^{13}$ as in Example 3.2.1. Then $(4,6) \in B_\mu$ and the rectangle $B_\mu(4,6)$ is the set of the following eight elements:*

$$(1,6), \quad (1,8), \quad (1,9), \quad (1,12), \quad (4,6), \quad (4,8), \quad (4,9), \quad (4,12).$$

In the corresponding Young diagram, $B_\mu(4,6)$ is the following gray rectangle:



Take a nonempty Ferrers board B_μ of shape μ. A subset of B_μ such that no two elements have a common entry is called a *rook placement* on B_μ. Let σ denote a rook placement on B_μ. The *row index set* $\pi_1(\sigma)$ and the *column index set* $\pi_2(\sigma)$ of σ are defined by

$$\pi_1(\sigma) = \{s \in S_\mu \mid (s,t) \in \sigma \text{ for some } t\}, \tag{3.5}$$

$$\pi_2(\sigma) = \{t \in T_\mu \mid (s,t) \in \sigma \text{ for some } s\}, \tag{3.6}$$

respectively. Remark that $|\pi_1(\sigma)| = |\pi_2(\sigma)| = |\sigma|$. Assume $\sigma \neq \emptyset$. For $1 \leq i \leq |\sigma|$, we denote by s_i and by t_i the i-th smallest element in $\pi_1(\sigma)$ and in $\pi_2(\sigma)$, respectively. Then σ gives rise to a permutation of $\{1, 2, \ldots, |\sigma|\}$ which sends i to j where $(s_i, t_j) \in \sigma$.

Lemma 3.2.3. *Let $\mu \in \{0,1\}^N$ and let σ be a rook placement on B_μ with the row/column index sets $\pi_1 = \pi_1(\sigma)$, $\pi_2 = \pi_2(\sigma)$, respectively. Then the pair (π_1, π_2) satisfies the following:*

(i) $|\pi_1| = |\pi_2|$.

(ii) Let n denote the common value in (i). For $1 \leq i \leq n$, the i-th smallest element in π_1 is strictly smaller than the i-th smallest element in π_2.

Proof. (i) It is clear.

(ii) We may assume $\sigma \neq \emptyset$ since otherwise the assertion is clear. Let $\widetilde{\sigma}$ denote the permutation of $\{1, 2, \ldots, n\}$ corresponding to σ. For $1 \leq i \leq n$, we write s_i, t_i for the i-th smallest element in π_1, π_2, respectively. Fix $1 \leq i \leq n$. Since

$\widetilde{\sigma}$ is a permutation, there exists $i \le k \le n$ such that $\widetilde{\sigma}(k) \le i$. So we have $(s_k, t_{\widetilde{\sigma}(k)}) \in \sigma$. Therefore $s_i \le s_k < t_{\widetilde{\sigma}(k)} \le t_i$ as desired.

□

Proposition 3.2.4. *Let $\mu \in \{0,1\}^N$. For a pair (π_1, π_2) such that $\pi_1 \subseteq S_\mu$ and $\pi_2 \subseteq T_\mu$, the following are equivalent.*

(i) There exists a rook placement σ on B_μ such that $\pi_1 = \pi_1(\sigma)$ and $\pi_2 = \pi_2(\sigma)$.

(ii) It satisfies (i), (ii) in Lemma 3.2.3.

Proof. We have shown in Lemma 3.2.3 that (i) implies (ii).

Suppose we are given $\pi_1 \subseteq S_\mu$ and $\pi_2 \subseteq T_\mu$ satisfying (i), (ii) in Lemma 3.2.3. By the condition (i) in Lemma 3.2.3, we set $n = |\pi_1| = |\pi_2|$. Let $\sigma = \{(s_i, t_i) \mid 1 \le i \le n\}$, where each s_i, t_i is the i-th smallest element in π_1, π_2, respectively. By the condition (ii) in Lemma 3.2.3, we have $\sigma \subseteq B_\mu$ and so σ is a rook placement on B_μ. By construction, it is clear that $\pi_1 = \pi_1(\sigma)$ and $\pi_2 = \pi_2(\sigma)$. So (ii) implies (i). □

Definition 3.2.5. Let $\mu \in \{0,1\}^N$ and consider the Ferrers board B_μ of shape μ. Then the *type* of a rook placement σ on B_μ is defined by the disjoint union

$$\pi_1(\sigma) \cup \pi_2(\sigma) \subseteq \{1, 2, \ldots, N\},$$

where $\pi_1(\sigma)$, $\pi_2(\sigma)$ are the row/column index sets of σ defined in (3.5), (3.6).

Lemma 3.2.6. *Let $\mu \in \{0,1\}^N$. For $\lambda \subseteq \{1, 2, \ldots, N\}$, the following are equivalent.*

(i) There exists a rook placement on B_μ of type λ.

(ii) The pair $(\lambda \cap S_\mu, \lambda \cap T_\mu)$ satisfies (i), (ii) in Lemma 3.2.3.

Proof. Immediate from Proposition 3.2.4. □

Lemma 3.2.7. *For $\lambda \subseteq \{1, 2, \ldots, N\}$, the following are equivalent.*

(i) There exists a rook placement on B_μ of type λ for some $\mu \in \{0,1\}^N$.

(ii) The cardinality of λ is even.

Proof. Fix $\lambda \subseteq \{1, 2, \ldots, N\}$. Suppose there exists a rook placement σ on B_μ of type λ for some $\mu \in \{0,1\}^N$. Then by Lemma 3.2.6, the pair $(\lambda \cap S_\mu, \lambda \cap T_\mu)$ satisfies (i), (ii) in Lemma 3.2.3. In particular, $|\lambda| = |\lambda \cap S_\mu| + |\lambda \cap T_\mu|$ is even. So (ii) holds.

Conversely, we suppose $|\lambda| = 2n$ for some $n \in \mathbb{N}$ and show (i) holds. Let (π_1, π_2) denote the bipartition of λ where π_1 contains the first n smallest elements in λ and π_2 contains the remaining n elements in λ. Take any $\mu \in \{0,1\}^N$ such that $\pi_1 \subseteq S_\mu$

and $\pi_2 \subseteq T_\mu$. Then we have $\pi_1 = \lambda \cap S_\mu$ and $\pi_2 = \lambda \cap T_\mu$. Observe that the pair (π_1, π_2) satisfies (i), (ii) in Lemma 3.2.3. So by Lemma 3.2.6, there exists a rook placement on B_μ of type λ. In particular, (i) holds. □

Since rook placements can be seen as permutations, we define the concept of inversions. Let σ be a nonempty rook placement on a Ferrers board B_μ of shape μ. For $(s_0, t_0) \in \sigma$, the *local inversion number of σ at (s_0, t_0)*, denoted by $\mathrm{inv}(\sigma, s_0, t_0)$, is defined by

$$\mathrm{inv}(\sigma, s_0, t_0) = |\{(s,t) \in \sigma \mid s < s_0, t > t_0\}| = |\sigma \cap B_\mu(s_0, t_0)| - 1. \tag{3.7}$$

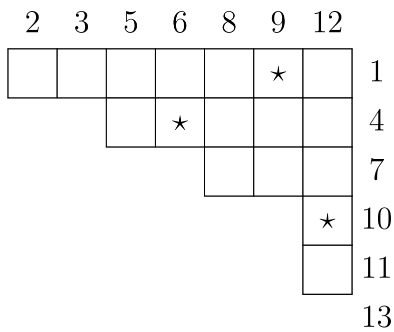
For a rook placement σ, the *(total) inversion number of σ*, denoted by $\mathrm{inv}(\sigma)$, is defined by

$$\mathrm{inv}(\sigma) = \sum_{(s,t)\in\sigma} \mathrm{inv}(\sigma, s, t).$$

Example 3.2.8 ($N = 13$)**.** *Take $\mu \in \{0,1\}^{13}$ as in Example 3.2.1. Consider the following rook placement σ on B_μ:*

$$\sigma = \{(1,9), (4,6), (10,12)\}.$$

Then we have $\mathrm{inv}(\sigma, 1, 9) = \mathrm{inv}(\sigma, 10, 12) = 0$ *and* $\mathrm{inv}(\sigma, 4, 6) = 1$*. Thus* $\mathrm{inv}(\sigma) = 1$.



The next lemma is a generalization of [18, Corollary 1.3.10] and the proof of the next lemma is motivated by that of [18, Corollary 1.3.10].

Lemma 3.2.9. *Let $\mu \in \{0,1\}^N$ and let $\lambda \subseteq \{1, 2, \ldots, N\}$ satisfy (ii) in Lemma 3.2.6. For $1 \le m \le N$, set*

$$\rho(m, \mu, \lambda) = |\lambda \cap S_\mu(m)| + |\lambda \cap T_\mu(m)| - \frac{|\lambda|}{2}.$$

Then for $q \in \mathbb{C}$ with $q \neq 0, 1$, we have

$$\sum_\sigma q^{\mathrm{inv}(\sigma)} = \prod_{s \in \lambda \cap S_\mu} \frac{q^{\rho(s,\mu,\lambda)} - 1}{q - 1},$$

where the sum is taken over all rook placements σ on B_μ of type λ.

Proof. If $\lambda = \emptyset$, the assertion is clear. We assume $\lambda \neq \emptyset$. We claim that there exists a bijection between the following two sets:

(i) The set of rook placements σ on B_μ of type λ,

(ii) The set of integer sequences $(a_s)_{s \in \lambda \cap S_\mu}$ such that $0 \leq a_s \leq \rho(s, \mu, \lambda) - 1$ for $s \in \lambda \cap S_\mu$,

such that $\mathrm{inv}(\sigma) = \sum_{s \in \lambda \cap S_\mu} a_s$. Suppose for the moment that the claim is true. Then we have

$$\sum_{\sigma} q^{\mathrm{inv}(\sigma)} = \prod_{s \in \lambda \cap S_\mu} \left(\sum_{a_s = 0}^{\rho(s,\mu,\lambda)-1} q^{a_s} \right) = \prod_{s \in \lambda \cap S_\mu} \frac{q^{\rho(s,\mu,\lambda)} - 1}{q - 1}.$$

So the result follows.

Therefore, it remains to prove the claim. For a given rook placement σ on B_μ of type λ and for $s \in \lambda \cap S_\mu$, there exists a unique $t \in \lambda \cap T_\mu$ such that $(s, t) \in \sigma$. Then we set $a_s = \mathrm{inv}(\sigma, s, t)$. Then for $s \in \lambda \cap S_\mu$, we have

$$\begin{aligned} 0 \leq a_s &= |\sigma \cap (S_\mu(s) \times T_\mu(t))| - 1 \\ &\leq |\sigma \cap (S_\mu(s) \times T_\mu(s))| - 1 \\ &= \rho(s, \mu, \lambda) - 1. \end{aligned}$$

The inequality follows from the fact that $s \leq t$ and the last equality follows by the direct caluculation. Conversely, set $r = |\lambda \cap S_\mu|$ and for $1 \leq i \leq r$, we write s_i the i-th smallest element in $\lambda \cap S_\mu$. By definition, we remark that $\rho(m, \mu, \lambda) \leq |\lambda \cap S_\mu(m)|$ for $1 \leq m \leq N$. For a given integer sequence $(a_1, a_2, \ldots, a_r)$ with $0 \leq a_i \leq \rho(s_i, \mu, \lambda) - 1$ for $1 \leq i \leq r$, since $0 \leq a_i \leq i - 1$, there exists a unique permutation $\widetilde{\sigma}$ of $\{1, 2, \ldots, r\}$ such that

$$a_i = |\{j \mid 1 \leq j < i, \widetilde{\sigma}(i) < \widetilde{\sigma}(j)\}|. \tag{3.8}$$

Then consider the set $\sigma = \{(s_i, t_{\widetilde{\sigma}(i)}) \mid 1 \leq i \leq r\}$, where t_i is the i-th smallest element in $\lambda \cap T_\mu$. Fix $1 \leq i \leq r$. By (3.8), we have $\widetilde{\sigma}(i) \geq i - a_i$ and so we have

$$\widetilde{\sigma}(i) \geq i - a_i \geq i - \rho(s_i, \mu, \lambda) + 1 = |\{t \in T_\mu \mid t < s_i\}| + 1.$$

The above equality follows from the definition of $\rho(s_i, \mu, \lambda)$. This implies that $s_i < t_{\widetilde{\sigma}(i)}$. This holds for any $1 \leq i \leq r$ and so σ becomes a rook placement on B_μ. It is clear that σ is of type λ. Therefore our claim holds. □

3.3 The matrix representation of P

For a field $\mathbb{K}$ and for two finite nonempty sets S and T, let $\mathrm{Mat}_{S,T}(\mathbb{K})$ denote the set of all matrices with rows indexed by S and columns indexed by T whose entries are in $\mathbb{K}$. If $S = T$, we write it $\mathrm{Mat}_S(\mathbb{K})$ for short. For $M \in \mathrm{Mat}_{S,T}(\mathbb{K})$, the *support* of M, denoted by $\mathrm{Supp}(M)$, is the set of indices containing nonzero entries:

$$\mathrm{Supp}(M) = \{(s,t) \in S \times T \mid M_{s,t} \neq 0\}.$$

For $\mu \in \{0,1\}^N$, recall the corresponding bipartition S_μ, T_μ from (3.1) and the Ferrers board B_μ of shape μ from (3.2). We will assume $\mu \neq \mathbf{0}$, $\mathbf{1}$ in this section so that both S_μ and T_μ are nonempty.

Definition 3.3.1. Let $\mu \in \{0,1\}^N$ with $\mu \neq \mathbf{0}$, $\mathbf{1}$. Let $\mathcal{M}_\mu(\mathbb{F}_q)$ denote the set of matrices in $\mathrm{Mat}_{S_\mu,T_\mu}(\mathbb{F}_q)$ such that $\mathrm{Supp}(M) \subseteq B_\mu$.

Recall the set P_μ of subspaces at location $\mu \in \{0,1\}^N$ from Definition 3.1.2.

Proposition 3.3.2. *Let $\mu \in \{0,1\}^N$ with $\mu \neq \mathbf{0}$, $\mathbf{1}$. Fix a basis $v_1, v_2, \dots, v_N$ for H adapted to the flag $\{x_i\}_{i=0}^N$. There exists a bijection from P_μ to the set $\mathcal{M}_\mu(\mathbb{F}_q)$ in Definition 3.3.1 that sends $y \in P_\mu$ to $Y \in \mathcal{M}_\mu(\mathbb{F}_q)$, where y has a basis*

$$\sum_{s \in S_\mu} Y_{s,t} v_s + v_t \qquad\qquad (t \in T_\mu).$$

Proof. For $y \in P_\mu$, there exists a basis w_t $(t \in T_\mu)$ for y such that $w_t \in x_t \setminus x_{t-1}$ for each $t \in T_\mu$. Write w_t as a linear combination of the fixed basis $v_1, v_2, \dots, v_t$ for x_t. Without loss of generality, we may assume the coefficient of v_t is 1. Use linear operations on the basis w_t to make the coefficient of $v_{t'}$ 0 for any $t' \in T_\mu$ with $t \neq t'$. Observe that the resulting basis w'_t $(t \in T_\mu)$ is uniquely determined by y. Then from the basis w'_t, we construct the matrix $Y \in \mathrm{Mat}_{S_\mu,T_\mu}(\mathbb{F}_q)$ such that $Y_{s,t}$ is the coefficient of v_s in w'_t. Then we have $Y \in \mathcal{M}_\mu(\mathbb{F}_q)$ since $w'_t \in x_t$. On the other hand, let $Y \in \mathcal{M}_\mu(\mathbb{F}_q)$. For $t \in T_\mu$, we write $w_t = \sum_{s \in S_\mu} Y_{s,t} v_s + v_t$. Since $\mathrm{Supp}(Y) \subseteq B_\mu$, the vector w_t is a linear combination of $v_1, v_2, \dots, v_t$, that means $w_t \in x_t \setminus x_{t-1}$. Therefore the subspace y spanned by the vectors w_t must belong to P_μ. □

Definition 3.3.3. Let $\mu \in \{0,1\}^N$ with $\mu \neq \mathbf{0}$, $\mathbf{1}$. Take $y \in P_\mu$. By the *matrix form* of y, we mean the matrix $Y \in \mathcal{M}_\mu(\mathbb{F}_q)$ which is the image of y under the bijection in Proposition 3.3.2. We note that the matrix form of y depends on the basis $v_1, v_2, \dots, v_N$ for H.

Let $\mu \in \{0,1\}^N$ with $\mu \neq \mathbf{0}$, $\mathbf{1}$. For $s \in S_\mu$, we denote by s^- the one smaller element in S_μ. If there is no such element, we set $s^- = 0$. For $t \in T_\mu$, we denote by t^+ the one larger element in T_μ. If there is no such element, we set $t^+ = N+1$. Observe that for $(s,t) \in B_\mu$, we have $(s^-, t) \in B_\mu$ if $s^- \neq 0$ and we have $(s, t^+) \in B_\mu$ if $t^+ \neq N+1$. For $M \in \mathcal{M}_\mu(\mathbb{F}_q)$ and for $(s,t) \in B_\mu$, let $M(s,t)$ denote the submatrix of M indexed by the rectangle with respect to (s,t) in (3.3). Moreover, we set

$$r^-(M,s,t) = \begin{cases} \operatorname{rank}(M(s^-,t)) & \text{if } s^- \neq 0, \\ 0 & \text{if } s^- = 0, \end{cases} \tag{3.9}$$

$$r^+(M,s,t) = \begin{cases} \operatorname{rank}(M(s,t^+)) & \text{if } t^+ \neq N+1, \\ 0 & \text{if } t^+ = N+1, \end{cases} \tag{3.10}$$

$$r^{-+}(M,s,t) = \begin{cases} \operatorname{rank}(M(s^-,t^+)) & \text{if } s^- \neq 0 \text{ and } t^+ \neq N+1, \\ 0 & \text{if } s^- = 0 \text{ or } t^+ = N+1. \end{cases} \tag{3.11}$$

Definition 3.3.4. Let $\mu \in \{0,1\}^N$ with $\mu \neq \mathbf{0}, \mathbf{1}$. For $M \in \mathcal{M}_\mu(\mathbb{F}_q)$, we define the set $\sigma(M)$ consisting of all indices $(s,t) \in B_\mu$ such that

$$r^\epsilon(M,s,t) = \operatorname{rank}(M(s,t)) - 1,$$

for all $\epsilon \in \{-,+,-+\}$.

Lemma 3.3.5. *Let $\mu \in \{0,1\}^N$ with $\mu \neq \mathbf{0}, \mathbf{1}$. For $M \in \mathcal{M}_\mu(\mathbb{F}_q)$, the set $\sigma(M)$ in Definition 3.3.4 is a rook placement on B_μ.*

Proof. Fix $M \in \mathcal{M}_\mu(\mathbb{F}_q)$. Since $\sigma(M)$ is a subset of B_μ, it suffices to show that no two elements in $\sigma(M)$ have a common entry. To do this, we take $(s_1,t),(s_2,t) \in \sigma(M)$ and assume $s_1 < s_2$. Observe that $s_2^- \neq 0$. Since $(s_1,t) \in \sigma(M)$, we have

$$r^+(M,s_1,t) = \operatorname{rank}(M(s_1,t)) - 1.$$

Since $(s_2,t) \in \sigma(M)$, we have

$$r^+(M,s_2^-,t) = \operatorname{rank}\left(M(s_2^-,t)\right).$$

By the two equalities above, we have $s_2^- < s_1$, which contradicts to $s_1 < s_2$. Therefore we must have $s_1 = s_2$. Similarly, if we take $(s,t_1),(s,t_2) \in \sigma(M)$, then one can show that $t_1 = t_2$. So the result follows. ☐

Recall the local inversion numbers of a rook placement from (3.7).

Lemma 3.3.6. *Let $\mu \in \{0,1\}^N$ with $\mu \neq \mathbf{0}, \mathbf{1}$. For $M \in \mathcal{M}_\mu(\mathbb{F}_q)$, we have*

$$\operatorname{rank}(M(s,t)) = \operatorname{inv}(\sigma(M),s,t) + 1,$$

for $(s,t) \in \sigma(M)$.

Proof. Fix $(s,t) \in \sigma(M)$. Observe that $\operatorname{rank}(M(s,t))$ can be computed as follows:

$$\sum_{(s',t') \in B_\mu(s,t)} \left(\operatorname{rank}(M(s',t')) - r^-(M,s',t') - r^+(M,s',t') + r^{-+}(M,s',t') \right).$$

Then by the definition of $\sigma(M)$, each summand is 0 if $(s',t') \notin \sigma(M)$ and it is 1 if $(s',t') \in \sigma(M)$. So, $\operatorname{rank}(M(s,t))$ is equal to the cardinality of $\sigma(M) \cap B_\mu(s,t)$. The result follows from the definition of local inversion numbers. □

Lemma 3.3.7. *Let $\mu \in \{0,1\}^N$ with $\mu \neq \mathbf{0}$, $\mathbf{1}$. For a subset $\sigma \subseteq B_\mu$, the following are equivalent.*

(i) There exists $M \in \mathcal{M}_\mu(\mathbb{F}_q)$ such that $\sigma(M) = \sigma$.

(ii) It is a rook placement on B_μ.

Proof. Lemma 3.3.5 shows that (i) implies (ii).

Assume we are given a rook placement σ on B_μ. Consider the matrix $M_\sigma \in \mathcal{M}_\mu(\mathbb{F}_q)$ defined by

$$(M_\sigma)_{s,t} = \begin{cases} 1 & \text{if } (s,t) \in \sigma, \\ 0 & \text{otherwise,} \end{cases}$$

for $s \in S_\mu$, $t \in T_\mu$. Then it is easy to check that $\sigma(M) = \sigma$. So (ii) implies (i). □

3.4 The number of matrices with given parameter

Let $\mu \in \{0,1\}^N$ with $\mu \neq \mathbf{0}, \mathbf{1}$. Recall from Lemma 3.3.7 that each matrix $\mathcal{M}_\mu(\mathbb{F}_q)$ corresponds to a rook placement on the Ferrers board B_μ of shape μ. Recall the sets from (3.1) and (3.4). To simplify the notation, we set

$$n(\pi_1) = \sum_{s \in \pi_1} |S_\mu(s)|, \tag{3.12}$$

for a subset $\pi_1 \subseteq S_\mu$.

Definition 3.4.1. Let $\mu \in \{0,1\}^N$. A subset $\lambda \subseteq \{1,2,\ldots,N\}$ is said to be *column-full* with respect to μ whenever $T_\mu \subseteq \lambda$. Moreover, a rook placement σ on B_μ is said to be *column-full* whenever the type of σ is column-full.

Let $\mu \in \{0,1\}^N$. We remark that a rook placement σ on B_μ is column-full if and only if the column index set $\pi_2(\sigma)$, defined in (3.6), is maximal.

Proposition 3.4.2. *Let $\mu \in \{0,1\}^N$ with $\mu \neq \mathbf{0}, \mathbf{1}$ and let σ denote a rook placement on B_μ. Assume σ is column-full in Definition 3.4.1. Then the number of matrices $M \in \mathcal{M}_\mu(\mathbb{F}_q)$ such that $\sigma = \sigma(M)$ in Definition 3.3.4 is given by*

$$(q-1)^{|\mu|} q^{\mathrm{inv}(\sigma)+|B_\mu|-n(\pi_1(\sigma))}.$$

Proof. Let $t \in T_\mu$. We count the number of possibilities for the t-th column of M with $\sigma = \sigma(M)$. Since σ is a column-full rook placement, there uniquely exists $s \in S_\mu$ such that $(s,t) \in \sigma$. Recall from the definition of σ that we have $r^{-+}(M,s,t) = r^-(M,s,t)$ and $r^+(M,s,t) = \mathrm{rank}\,(M(s,t)) - 1$ in (3.9), (3.10) and (3.11). Then the number of possibilities for the t-th column of $M(s,t)$ is

$$(q-1)q^{r^{-+}(M,s,t)} = (q-1)q^{\mathrm{inv}(\sigma,s,t)}.$$

Here the equality follows from the definition of σ and Lemma 3.3.6. Since $M \in \mathcal{M}_\mu(\mathbb{F}_q)$, or equivalently $\mathrm{Supp}(M) \subseteq B_\mu$, the (s',t)-entries are 0 if $(s',t) \notin B_\mu$. For the remaining entries, the choices are arbitrary. Therefore the total number of possibilities for the t-th column of M is

$$(q-1)q^{\mathrm{inv}(\sigma,s,t)} \times q^{|S_\mu(t)|-|S_\mu(s)|}.$$

The result follows from the definition of $\mathrm{inv}(\sigma)$, the column-full property and

$$\sum_{t \in T_\mu} |S_\mu(t)| = |\{(s,t) \in S_\mu \times T_\mu \mid s < t\}| = |B_\mu|.$$

□

Corollary 3.4.3. *Let $\mu \in \{0,1\}^N$ with $\mu \neq \mathbf{0}, \mathbf{1}$ and let $\lambda \subseteq \{1,2,\ldots,N\}$ satisfy (ii) in Lemma 3.2.6. Assume σ is column-full in Definition 3.4.1. Then the number of matrices $M \in \mathcal{M}_\mu(\mathbb{F}_q)$ such that $\sigma(M)$ is of type λ in Definitions 3.2.5 and 3.3.4 is given by*

$$q^{|B_\mu|-n(\lambda \cap S_\mu)} \prod_{s \in \lambda \cap S_\mu} \left(q^{\rho(s,\mu,\lambda)} - 1\right),$$

where $\rho(s,\mu,\lambda)$ is defined in Lemma 3.2.9.

Proof. Use Lemma 3.2.9 and Proposition 3.4.2. □

3.5 The algebra $\mathcal{H}_f$

Recall $\mathrm{Mat}_P(\mathbb{C})$, the set of all matrices whose rows and columns are indexed by P and whose entries are in $\mathbb{C}$. We see it as a $\mathbb{C}$-algebra. We write $I \in \mathrm{Mat}_P(\mathbb{C})$ for the identity matrix and $O \in \mathrm{Mat}_P(\mathbb{C})$ for the zero matrix. In this section, we introduce a subalgebra $\mathcal{H}_f$ of $\mathrm{Mat}_P(\mathbb{C})$ which represents the N-cube structure in P.

Let $V = \mathbb{C}P$ denote the vector space over $\mathbb{C}$ consisting of the column vectors whose coordinates are indexed by P and whose entries are in $\mathbb{C}$. Observe that $\mathrm{Mat}_P(\mathbb{C})$ acts on V by left multiplication. We call V the *standard module* for $\mathrm{Mat}_P(\mathbb{C})$. We equip V with the standard Hermitian inner product defined by $\langle u, v\rangle = u^t\bar{v}$ for $u, v \in V$, where ^{t} denotes transpose and $^-$ denotes complex conjugate.

Recall from Definition 3.1.2 that we have partitioned P into the sets P_μ of all subspaces at location μ for $\mu \in \{0,1\}^N$. For $\mu \in \mathbb{Z}^N$, define a diagonal matrix $E^*_\mu \in \mathrm{Mat}_P(\mathbb{C})$ by

$$(E^*_\mu)_{y,y} = \begin{cases} 1 & \text{if } y \in P_\mu, \\ 0 & \text{if } y \notin P_\mu \end{cases} \qquad (y \in P).$$

Observe that $E^*_\mu = O$ unless $\mu \in \{0,1\}^N$. By construction, we have

$$E^*_\mu E^*_\nu = \delta_{\mu,\nu} E^*_\mu \qquad (\mu, \nu \in \{0,1\}^N),$$

$$I = \sum_{\mu\in\{0,1\}^N} E^*_\mu.$$

Moreover, we have a decomposition of V:

$$V = \sum_{\mu\in\{0,1\}^N} E^*_\mu V \qquad \text{(direct sum)},$$

where $E^*_\mu V$ is the subspace of V with the basis P_μ. Thus, the matrix E^*_μ is the projection from V onto $E^*_\mu V$ and we call it the *projection matrix*.

Definition 3.5.1. By the above comments, the matrices E^*_μ, where $\mu \in \{0,1\}^N$ form a basis for a commutative subalgebra of $\mathrm{Mat}_P(\mathbb{C})$. We denote this subalgebra by $\mathcal{K}$.

We now introduce matrices that generate $\mathcal{K}$. For $1 \le m \le N$, we define diagonal matrices $K_m \in \mathrm{Mat}_P(\mathbb{C})$ by

$$(K_m)_{y,y} = q^{1/2-\mu_m} \qquad (y \in P_\mu),$$

where $\mu = (\mu_1, \mu_2, \dots, \mu_N)$.

Lemma 3.5.2. *For $1 \le m \le N$, we have*

$$K_m = \sum_{\mu\in\{0,1\}^N} q^{1/2-\mu_m} E^*_\mu,$$

where $\mu = (\mu_1, \mu_2, \dots, \mu_N)$.

Proof. Immediate from the construction. □

Proposition 3.5.3. *The algebra $\mathcal{K}$ in Definition 3.5.1 is generated by K_m for $1 \leq m \leq N$.*

Proof. By Lemma 3.5.2, the matrices K_m $(1 \leq m \leq N)$ generate a subalgebra $\mathcal{K}'$ of $\mathcal{K}$. For distinct indices $\mu = (\mu_1, \mu_2, \dots, \mu_N), \nu = (\nu_1, \nu_2, \dots, \nu_N) \in \{0,1\}^N$, we have $q^{1/2-\mu_m} \neq q^{1/2-\nu_m}$ for some $1 \leq m \leq N$. Therefore E^*_μ is a polynomial in K_m $(1 \leq m \leq N)$ for every $\mu \in \{0,1\}^N$. Consequently, $\mathcal{K}' = \mathcal{K}$. □

Next we introduce two kinds of matrices from covering relations in Definition 3.1.3. For $1 \leq m \leq N$, the matrices $L_m, R_m \in \mathrm{Mat}_P(\mathbb{C})$ are defined by

$$(L_m)_{y,z} = \begin{cases} 1 & \text{if } z \ m\text{-covers } y, \\ 0 & \text{otherwise,} \end{cases} \qquad (R_m)_{y,z} = \begin{cases} 1 & \text{if } y \ m\text{-covers } z, \\ 0 & \text{otherwise,} \end{cases}$$

for $y, z \in P$. We remark that for each $1 \leq m \leq N$, the matrices L_m and R_m are transpose to each other. Recall the comment in the above of Lemma 3.1.4.

Lemma 3.5.4. *For $1 \leq m \leq N$ and $\mu \in \{0,1\}^N$, we have the following:*

(i) $L_m E^*_\mu = E^*_{\mu-\widehat{m}} L_m$ *and* $R_m E^*_\mu = E^*_{\mu+\widehat{m}} R_m$.

(ii) $L_m E^*_\mu V \subseteq E^*_{\mu-\widehat{m}} V$ *and* $R_m E^*_\mu V \subseteq E^*_{\mu+\widehat{m}} V$.

Proof. Immediate from the construction. □

Because of Lemma 3.5.4 (ii), we call L_m the *lowering matrices* and R_m the *raising matrices*.

Definition 3.5.5. Let $\mathcal{H}_f$ denote the subalgebra of $\mathrm{Mat}_P(\mathbb{C})$ generated by L_m, R_m $(1 \leq m \leq N)$ and the algebra $\mathcal{K}$ in Definition 3.5.1.

Proposition 3.5.6. *The algebra $\mathcal{H}_f$ in Definition 3.5.5 is semisimple.*

Proof. This follows since $\mathcal{H}_f$ is closed under the conjugate-transpose map. □

We recall the incidence algebra, which is generated by L, R and E^*_i $(0 \leq i \leq N)$ in Definition 2.3.1. We remark that $\mathcal{H}_f$ contains the incidence algebra as its subalgebra because $L = \sum_{m=1}^N L_m$, $R = \sum_{m=1}^N R_m$ and $E^*_i = \sum_{\mu \in \{0,1\}^N, |\mu|=i} E^*_\mu$. Moreover, if $N \geq 2$, the incidence algebra is a proper subalgebra of $\mathcal{H}_f$.

3.6 The structure of the algebra $\mathcal{H}_f$

In this section, we discuss the relations among the generators L_m, R_m, K_m of the algebra $\mathcal{H}_f$.

Proposition 3.6.1. *For $1 \leq m, n \leq N$ with $m \neq n$, the following hold.*

(i) $L_m K_n = K_n L_m$.

(ii) $R_m K_n = K_n R_m$.

(iii) $q L_m K_m = K_m L_m$.

(iv) $R_m K_m = q K_m R_m$.

Proof. This lemma follows by combining Lemmas 3.5.2 and 3.5.4 (i). □

Proposition 3.6.2. *For $1 \leq m, n \leq N$, we have the following:*

(i) $L_m^2 = R_m^2 = 0$.

(ii) $q L_m L_n = L_n L_m$ *if* $m < n$.

(iii) $R_m R_n = q R_n R_m$ *if* $m < n$.

(iv) $L_m R_n = R_n L_m$ *if* $m \neq n$.

Proof. (i) It follows from the definition of L_m and R_m.

(ii), (iii) These are matrix reformulations of Lemma 3.1.5 (i), (ii).

(iv) This is a matrix reformulation of Lemma 3.1.5 (iii), (iv).

□

3.7 The L_m- and R_m-actions on V

We now describe a basis for V, which is the key in this chapter. In this section, we fix a basis $v_1, v_2, \ldots, v_N$ for H adapted to the flag $\{x_i\}_{i=0}^N$ and assume that the matrix forms in Definition 3.3.3 are always taken with respect to this basis $v_1, v_2, \ldots, v_N$.

Definition 3.7.1. Let χ denote a nontrivial character of the additive group $\mathbb{F}_q$ and let $\mu \in \{0,1\}^N$. For $y \in P_\mu$, define a vector $\chi_y \in V$ as follows.

(i) If $\mu = \mathbf{0}$ or $\mathbf{1}$, then for $z \in P$, the z-th entry of χ_y is 1 if $y = z$ and 0 otherwise.

(ii) If $\mu \neq \mathbf{0}, \mathbf{1}$, then for $z \in P$, the z-th entry of χ_y is defined by

$$\begin{cases} \chi\left(\mathrm{tr}(YZ^t)\right) & \text{if } z \in P_\mu, \\ 0 & \text{if } z \notin P_\mu, \end{cases}$$

where $Y, Z \in \mathcal{M}_\mu(\mathbb{F}_q)$ are the matrix forms of y, z, respectively in Definition 3.3.3. Here ^{t} denotes transpose and tr denotes the trace map of matrices.

For the rest of this section, we fix a nontrivial character χ of the additive group $\mathbb{F}_q$.

Lemma 3.7.2. *For $\mu \in \{0,1\}^N$, the set of vectors $\chi_y \in V$ for $y \in P_\mu$ in Definition 3.7.1 forms an orthogonal basis for the vector space $E^*_\mu V$.*

Proof. Let $\mu \in \{0,1\}^N$. For $y \in P_\mu$, observe that $\chi_y \in E^*_\mu V$ from the construction. If $\mu = \mathbf{0}$ or $\mathbf{1}$, then the assertion is trivial since $\dim E^*_\mu V = 1$. Assume $\mu \neq \mathbf{0}, \mathbf{1}$ and take $y, y' \in P_\mu$. Consider the Hermitian inner product

$$\langle \chi_y, \chi_{y'} \rangle = \sum_{z \in P} \chi_y(z)\overline{\chi_{y'}(z)},$$

where $\chi_y(z)$, $\chi_{y'}(z)$ denote the z-th entries of χ_y, $\chi_{y'}$, respectively. By the definitions of $\chi_y(z)$, $\chi_{y'}(z)$ and by the orthogonality of character χ, we obtain $\langle \chi_y, \chi_{y'} \rangle = q^{|B_\mu|} = |P_\mu|$ if $y = y'$ and 0 otherwise. Therefore the set of vectors χ_y for $y \in P_\mu$ becomes an orthogonal basis for a subspace V_μ of $E^*_\mu V$. By comparing their dimensions, we have $V_\mu = E^*_\mu V$ and the result follows. □

Recall the m-covering relation from Definition 3.1.3.

Lemma 3.7.3. *Let $1 \leq m \leq N$ and let $\mu, \nu \in \{0,1\}^N$ with $\mu, \nu \neq \mathbf{0}, \mathbf{1}$ such that μ m-covers ν. Take $y \in P_\mu$, $z \in P_\nu$ and let $Y \in \mathcal{M}_\mu(\mathbb{F}_q)$ and $Z \in \mathcal{M}_\nu(\mathbb{F}_q)$ denote the matrix forms of y, z, respectively in Definition 3.3.3. Then y m-covers z if and only if*

$$Z_{s,t} = Y_{s,t} + Y_{s,m}Z_{m,t},$$

for $s \in S_\mu$ and for $t \in T_\nu$.

Proof. Recalling the bijection of Proposition 3.3.2, for $t \in T_\mu$ and $t' \in T_\nu$, we write

$$w_t(Y) = \sum_{s \in S_\mu} Y_{s,t} v_s + v_t, \qquad w_{t'}(Z) = \sum_{s' \in S_\nu} Z_{s',t'} v_{s'} + v_{t'}.$$

Assume y covers z. For each $t' \in T_\nu$, since $z \subseteq y$, the vector $w_{t'}(Z)$ is a linear combination of $w_t(Y)$, where $t \in T_\mu$. Comparing the coefficients of v_t for $t \in T_\mu$, we have $w_{t'}(Z) = Z_{m,t'}w_m(Y) + w_{t'}(Y)$. Then comparing the coefficients of v_s for $s \in S_\mu$, we obtain the desired equality. On the other hand, assume the equality $Z_{s,t'} = Y_{s,t'} + Z_{m,t'}Y_{s,m}$ for $s \in S_\mu$ and $t' \in T_\nu$. By the same argument above, we have $w_{t'}(Z) \in y$ for all $t' \in T_\nu$. This implies y covers z, as desired. □

Lemma 3.7.4. *Let $1 \leq m \leq N$ and let $\mu, \nu \in \{0,1\}^N$ with $\mu, \nu \neq \mathbf{0}, \mathbf{1}$ such that μ m-covers ν. Take $y \in P_\mu$, $z \in P_\nu$ and let $Y \in \mathcal{M}_\mu(\mathbb{F}_q)$, $Z \in \mathcal{M}_\nu(\mathbb{F}_q)$ denote the matrix forms of y, z, respectively in Definition 3.3.3. Then the z-th entry of $L_m\chi_y$ is given by*

$$L_m\chi_y(z) = q^{|S_\mu(m-1)|}\chi\left(\sum_{s\in S_\mu}\sum_{t\in T_\nu} Y_{s,t}Z_{s,t}\right),$$

if $Y_{s,m} = \sum_{t\in T_\nu} Y_{s,t}Z_{m,t}$ for all $s \in S_\mu$ with $s < m$ and 0 otherwise.

Proof. By the definition of L_m, the z-th entry of $L_m\chi_y$ is defined by

$$L_m\chi_y(z) = \sum_{y'} \chi_y(y'),$$

where the sum is taken over all $y' \in P_\mu$ such that y' m-covers z. Then by Definition 3.7.1 and Lemma 3.7.3, we have

$$L_m\chi_y(z) = \chi\left(\sum_{s\in S_\mu}\sum_{t\in T_\nu} Y_{s,t}Z_{s,t}\right)\sum \chi\left(\sum_{s\in S_\mu}\left(Y_{s,m} - \sum_{t\in T_\nu} Y_{s,t}Z_{m,t}\right)Y'_{s,m}\right),$$

where the third sum is taken over all $Y'_{s,m} \in \mathbb{F}_q$ for $s \in S_\mu$ with $s < m$, and where we set $Y'_{s,m} = 0$ for $s \in S_\mu$ with $s > m$. By the orthogonality of characters, the third sum does not vanish if and only if

$$Y_{s,m} = \sum_{t\in T_\nu} Y_{s,t}Z_{m,t},$$

for all $s \in S_\mu$ with $s < m$. Moreover, in this case, the sum is the number of choices for $Y'_{s,m} \in \mathbb{F}_q$ for $s \in S_\mu$ with $s < m$, which is $q^{|S_\mu(m-1)|}$. □

Lemma 3.7.5. *Let $1 \leq m \leq N$ and let $\mu, \nu \in \{0,1\}^N$ with $\mu, \nu \neq \mathbf{0}, \mathbf{1}$ such that μ m-covers ν. Take $y \in P_\mu$, $z \in P_\nu$ and let $Y \in \mathcal{M}_\mu(\mathbb{F}_q)$, $Z \in \mathcal{M}_\nu(\mathbb{F}_q)$ denote the matrix forms of y, z, respectively in Definition 3.3.3. Then the y-th entry of $R_m\chi_z$ is given by*

$$R_m\chi_z(y) = q^{|T_\nu(m+1)|}\chi\left(\sum_{s\in S_\mu}\sum_{t\in T_\nu} Y_{s,t}Z_{s,t}\right),$$

if $Z_{m,t} = -\sum_{s\in S_\mu} Z_{s,t}Y_{s,m}$ for all $t \in T_\nu$ with $t > m$ and 0 otherwise.

Proof. Similar to the proof of Lemma 3.7.4. □

Lemma 3.7.6. *Referring to Lemma 3.7.4, let λ denote the type of $\sigma(Y)$ in Definitions 3.2.5 and 3.3.4. Then the number of $Z \in \mathcal{M}_\nu(\mathbb{F}_q)$ such that $Y_{s,m} = \sum_{t\in T_\nu} Y_{s,t}Z_{m,t}$ for all $s \in S_\mu$ with $s < m$ is given by q^l, where*

$$l = |B_\nu| - |\lambda \cap S_\mu(m-1)| - |\lambda \cap T_\mu(m+1)| + |\lambda|/2,$$

if $m \notin \lambda$, and 0 otherwise.

Proof. We count the number of possibilities for $Z_{s,t} \in \mathbb{F}_q$ for $s \in S_\nu$ and $t \in T_\nu$. If $s > t$, then $Z_{s,t} = 0$ since $\mathrm{Supp}(Z) \subseteq B_\nu$. If $s \neq m$ and $s < t$, then $Z_{s,t}$ is arbitrary and therefore the number of possibilities is q. The number of such pairs (s,t) is given by

$$|\{(s,t) \in B_\nu \mid s \neq m\}| = |B_\nu| - |T_\nu(m+1)|.$$

For the case $s = m$ and $m < t$, by the constraint, the sequence $(Z_{m,t})_{t\in T_\nu, t>m}$ must be a solution of the system of linear equations over $\mathbb{F}_q$:

$$C\mathbf{u} = \mathbf{c},$$

where $C = (Y_{s,t})_{s\in S_\mu, s<m, t\in T_\nu, t>m}$ is the coefficient matrix, $\mathbf{u} = (u_t)_{t\in T_\nu, t>m}$ is the unknown vector and $\mathbf{c} = (Y_{s,m})_{s\in S_\mu, s<m}$ is the constant vector. By linear algebra, the system $C\mathbf{u} = \mathbf{c}$ has a solution if and only if the rank of the augmented matrix $[C, \mathbf{c}]$ is equal to the rank of the coefficient matrix C. By Definition 3.3.4, it is also equivalent to $(s,m) \notin \sigma(Y)$ for all $s \in S_\mu$ with $s < m$, which means $m \notin \lambda$. Moreover, suppose there is a solution of the system $C\mathbf{u} = \mathbf{c}$. Since there are $|T_\nu(m+1)|$ columns in C, the number of solutions is given by

$$q^{|T_\nu(m+1)| - \mathrm{rank}\, C}.$$

By Lemma 3.3.6, the rank of C is computed as follows:

$$\begin{aligned}\mathrm{rank}\, C &= |\{(s,t) \in \sigma(Y) \mid s \leq m-1, t \geq m+1\}| \\ &= |\{(s,t) \in \sigma(Y) \mid s \leq m-1\}| + |\{(s,t) \in \sigma(Y) \mid t \geq m+1\}| - |\sigma(Y)| \\ &= |\lambda \cap S_\mu(m-1)| + |\lambda \cap T_\mu(m+1)| - |\lambda|/2.\end{aligned}$$

Therefore the result follows. □

Lemma 3.7.7. *Referring to Lemma 3.7.5, let λ denote the type of $\sigma(Z)$ in Definitions 3.2.5 and 3.3.4. Then the number of $Y \in \mathcal{M}_\mu(\mathbb{F}_q)$ such that $Z_{m,t} = -\sum_{s\in S_\mu} Z_{s,t}Y_{s,m}$ for all $t \in T_\nu$ with $t > m$ is given by q^l where*

$$l = |B_\mu| - |\lambda \cap S_\mu(m-1)| - |\lambda \cap T_\mu(m+1)| + |\lambda|/2,$$

if $m \notin \lambda$, and 0 otherwise.

Proof. Similar to the proof of Lemma 3.7.6. ☐

Definition 3.7.8. Let $\mu \in \{0,1\}^N$ and take $y \in P_\mu$. If $\mu \neq \mathbf{0}, \mathbf{1}$, then let $Y \in \mathcal{M}_\mu(\mathbb{F}_q)$ denote the matrix form of y in Definition 3.3.3. Then the *type* of y is defined to be the type of $\sigma(Y)$ in Definitions 3.2.5 and 3.3.4. If $\mu = \mathbf{0}$ or $\mathbf{1}$, then the *type* of y is defined to be the empty set. We note that the type of y depends on the basis $v_1, v_2, \dots, v_N$ for H since the matrix form does.

Lemma 3.7.9. *Let $\mu \in \{0,1\}^N$ and let $\lambda \subseteq \{1,2,\dots,N\}$ satisfy (ii) in Lemma 3.2.6. For $1 \le m \le N$, the following are equivalent.*

(i) For any $y \in P_\mu$ of type λ, we have $L_m \chi_y = 0$.

(ii) $m \in S_\mu$ or $m \in \lambda$.

Proof. Set $\nu = \mu - \widehat{m}$ so that μ m-covers ν. For $y \in P_\mu$, observe that $L_m\chi_y \in E^*_\nu V$ by Lemma 3.5.4 (ii). If $m \in S_\mu$, then $E^*_\nu V = 0$ and so both (i) and (ii) hold. If $m \in T_\mu$ and $\mu = \mathbf{1}$, then $L_m\chi_y \neq 0$ by Definition 3.7.1 and $\lambda = \emptyset$ by Definition 3.7.8. So both (i) and (ii) fail to hold. If $m \in T_\mu$ and $\nu = \mathbf{0}$, then $P_\nu = \{0\}$ and subspaces $y' \in P_\mu$ m-cover 0. So we have

$$L_m\chi_y(0) = \sum_{Y' \in \mathcal{M}_\mu(\mathbb{F}_q)} \chi\left(\operatorname{tr}(Y(Y')^t)\right),$$

where $Y \in \mathcal{M}_\mu(\mathbb{F}_q)$ is the matrix form of y in Definition 3.3.3. By the orthogonality of χ, it vanishes if and only if Y is not the zero matrix, which means m is in the type λ of $\sigma(Y)$. If $m \in T_\mu$, $\mu \neq \mathbf{1}$ and $\nu \neq \mathbf{0}$, then the result follows from Lemmas 3.7.4 and 3.7.6. ☐

Lemma 3.7.10. *Let $\nu \in \{0,1\}^N$ and let $\lambda \subseteq \{1,2,\dots,N\}$ satisfy (ii) in Lemma 3.2.6. For $1 \le m \le N$, the following are equivalent.*

(i) For any $z \in P_\nu$ of type λ, we have $R_m\chi_z = 0$.

(ii) $m \in T_\nu$ or $m \in \lambda$.

Proof. Similar to the proof of Lemma 3.7.9. ☐

Recall from Lemma 3.2.7, a subset $\lambda \subseteq \{1,2,\dots,N\}$ becomes a type if and only if it has even cardinality. For $\lambda \subseteq \{1,2,\dots,N\}$ with even cardinality, let V_λ denote the subspace of V spanned by the vectors $\chi_y \in V$ for all $y \in P$ of type λ in Definitions 3.7.1 and 3.7.8. Then for $\lambda \subseteq \{1,2,\dots,N\}$ with even cardinality, we define a matrix $E_\lambda \in \mathrm{Mat}_P(\mathbb{C})$ such that

$$\begin{aligned} (E_\lambda - I)V_\lambda &= 0, \\ E_\lambda V_{\lambda'} &= 0 \qquad \text{if } \lambda \neq \lambda', \end{aligned}$$

where $\lambda' \subseteq \{1, 2, \ldots, N\}$ with even cardinality. In other words, E_λ is the projection from V onto V_λ. Observe that E^*_μ and E_λ commute for all $\mu \in \{0,1\}^N$ and $\lambda \subseteq \{1, 2, \ldots, N\}$ with even cardinality.

Lemma 3.7.11. *For $\mu \in \{0,1\}^N$ and for $\lambda \subseteq \{1, 2, \ldots, N\}$ with even cardinality, the following are equivalent.*

(i) $E^*_\mu E_\lambda = E_\lambda E^*_\mu \neq 0$.

(ii) The pair $(\lambda \cap S_\mu, \lambda \cap T_\mu)$ satisfies (i), (ii) in Lemma 3.2.3.

Proof. This is a matrix interpretation of Lemma 3.2.6. □

3.8 The $L_m R_m$- and $R_m L_m$-actions on V

In this section, we fix a basis $v_1, v_2, \ldots, v_N$ for H adapted to the flag $\{x_i\}_{i=0}^N$ and assume that the matrix forms in Definition 3.3.3 and the types in Definition 3.7.8 are always taken with respect to this basis $v_1, v_2, \ldots, v_N$. We also fix a nontrivial character χ of the additive group $\mathbb{F}_q$. Recall from Section 3.7, the definition of E_λ for $\lambda \subseteq \{1, 2, \ldots, N\}$ with even cardinality depends on the basis $v_1, v_2, \ldots, v_N$ and on the character χ. We show in this section, that E_λ is independent of the basis $v_1, v_2, \ldots, v_N$ for H adapted to the flag $\{x_i\}_{i=0}^N$ and the nontrivial character χ of the additive group $\mathbb{F}_q$.

Lemma 3.8.1. *Let $1 \leq m \leq N$, and let $\mu \in \{0,1\}^N$ and $\lambda \subseteq \{1, 2, \ldots, N\}$ satisfy (ii) in Lemma 3.2.6. Set*

$$\kappa(m, \mu, \lambda) = |S_\mu(m-1) \setminus \lambda| + |T_\mu(m+1) \setminus \lambda| + |\lambda|/2. \tag{3.13}$$

Then for $v \in E^*_\mu E_\lambda V$, we have the following:

$$R_m L_m v = \begin{cases} q^{\kappa(m,\mu,\lambda)} v & \text{if } m \in T_\mu \text{ and } m \notin \lambda, \\ 0 & \text{if } m \in S_\mu \text{ or } m \in \lambda. \end{cases}$$

Proof. Observe that $R_m L_m$ acts on $E^*_\mu V$ by Lemma 3.5.4 (ii). Fix $y \in P_\mu$ of type λ in Definition 3.7.8. We show that χ_y is an eigenvector for $R_m L_m$. If $m \in S_\mu$ or $m \in \lambda$, then by Lemma 3.7.9, we have $L_m \chi_y = 0$ and so χ_y is an eigenvector for $R_m L_m$ with respect to the eigenvalue 0. If $\mu = \mathbf{1}$, then $P_\mu = \{H\}$ and $\lambda = \emptyset$. So we have $\dim E^*_\mu V = 1$. Therefore, χ_y is an eigenvector of $R_m L_m$ and the corresponding eigenvalue is the number of subspaces which are m-covered by $y = H$, which is equal to $q^{N-m} = q^{\kappa(m,\mathbf{1},\emptyset)}$ by Lemma 3.1.4 (i). Set $\nu = \mu - \widehat{m}$ so that μ m-covers ν. If

$m \in T_\mu$, $m \notin \lambda$ and $\nu = \mathbf{0}$, then $P_\nu = \{0\}$ and $\lambda = \emptyset$. In other words, the matrix form of y in Definition 3.3.3 equals to the zero matrix O, and so y'-th entry $\chi_y(y')$ of χ_y is 1 if $y' \in P_\mu$ and 0 if $y' \notin P_\mu$. Since $P_\nu = \{0\}$, χ_y is an eigenvector of $R_m L_m$ and the corresponding eigenvalue is the number of subspaces which m-covers $z = 0$, which is equal to $q^{m-1} = q^{\kappa(m,\widehat{m},\emptyset)}$ by Lemma 3.1.4 (ii). If $m \in T_\mu$, $m \notin \lambda$, $\mu \neq \mathbf{1}$ and $\nu \neq \mathbf{0}$, then we have

$$R_m L_m \chi_y = \frac{1}{|P_\mu|} \sum_{y' \in P_\mu} \langle R_m L_m \chi_y, \chi_{y'} \rangle \chi_{y'}.$$

Let $y' \in P_\mu$. Since L_m and R_m are (conjugate-)transpose to each other, we have

$$\begin{aligned} \langle R_m L_m \chi_y, \chi_{y'} \rangle &= \langle L_m \chi_y, L_m \chi_{y'} \rangle \\ &= \sum_{z \in P_\nu} L_m \chi_y(z) \overline{L_m \chi_{y'}(z)}. \end{aligned}$$

Let $Y, Y' \in \mathcal{M}_\mu(\mathbb{F}_q)$ and $Z \in \mathcal{M}_\nu(\mathbb{F}_q)$ be the matrix forms of y, y', z, respectively in Definition 3.3.3. Then by Lemma 3.7.4, it becomes

$$\sum_{z \in P_\nu} L_m \chi_y(z) \overline{L_m \chi_{y'}(z)} = q^{2|S_\mu(m-1)|} \sum \chi \left(\sum_{s \in S_\mu} \sum_{t \in T_\nu} \left(Y_{s,t} - Y'_{s,t}\right) Z_{s,t} \right),$$

where the sum is taken over all $Z \in \mathcal{M}_\nu(\mathbb{F}_q)$ such that

$$\sum_{t \in T_\nu} Y_{s,t} Z_{m,t} = Y_{s,m}, \qquad \sum_{t \in T_\nu} Y'_{s,t} Z_{m,t} = Y'_{s,m}, \tag{3.14}$$

for all $s \in S_\mu$ with $s < m$. Then since $\mathrm{Supp}(Z) \subseteq B_\nu$, by the orthogonality of the characters, the sum vanishes unless $Y_{s,t} = Y'_{s,t}$ for all $s \in S_\mu$ and $t \in T_\nu$ with $s < t$, which by (3.14) and Lemma 3.7.6 implies $Y = Y'$ and so $y = y'$. In particular, χ_y is an eigenvector of $R_m L_m$. Moreover, using Lemma 3.7.6 and $|P_\mu| = q^{|B_\mu|}$, we can easily show that the corresponding eigenvalues is $q^{\kappa(m,\mu,\lambda)}$. □

Lemma 3.8.2. *Let $1 \le m \le N$, and let $\mu \in \{0,1\}^N$ and $\lambda \subseteq \{1, 2, \ldots, N\}$ satisfy (ii) in Lemma 3.2.6. Recall $\kappa(m, \mu, \lambda)$ from (3.13). Then for $v \in E^*_\mu E_\lambda V$, we have the following:*

$$L_m R_m v = \begin{cases} q^{\kappa(m,\mu,\lambda)} v & \text{if } m \in S_\mu \text{ and } m \notin \lambda, \\ 0 & \text{if } m \in T_\mu \text{ or } m \in \lambda. \end{cases}$$

Proof. Similar to the proof of Lemma 3.8.1. □

Proposition 3.8.3. *For $\lambda \subseteq \{1, 2, \ldots, N\}$ with even cardinality, the matrix E_λ belongs to the algebra $\mathcal{H}_f$ in Definition 3.5.5.*

Proof. Referring to (3.13), we set

$$\theta(m, \mu, \lambda) = \begin{cases} q^{\kappa(m,\mu,\lambda)} & \text{if } m \notin \lambda, \\ 0 & \text{if } m \in \lambda \end{cases}$$

for $1 \leq m \leq N$, $\mu \in \{0,1\}^N$ and $\lambda \subseteq \{1, 2, \ldots, N\}$ satisfying (ii) in Lemma 3.2.6. Then by Lemmas 3.8.1 and 3.8.2, we have

$$R_m L_m + L_m R_m = \sum_{\mu,\lambda} \theta(m, \mu, \lambda) E_\mu^* E_\lambda,$$

where the sum is taken over all pairs $\mu \in \{0,1\}^N$ and $\lambda \subseteq \{1, 2, \ldots, N\}$ satisfying (ii) in Lemma 3.2.6. Pick $\mu \in \{0,1\}^N$ and multiply each term on the left of the above equation, by E_μ^*. Then we obtain

$$E_\mu^* R_m L_m + E_\mu^* L_m R_m = \sum_{\lambda} \theta(m, \mu, \lambda) E_\mu^* E_\lambda,$$

where the sum is taken over $\lambda \subseteq \{1, 2, \ldots, N\}$ satisfying (ii) in Lemma 3.2.6. For distinct λ, λ' in the sum, there exists $1 \leq m \leq N$ such that $\theta(m, \mu, \lambda) \neq \theta(m, \mu, \lambda')$. Therefore each $E_\mu^* E_\lambda$ is a polynomial in $E_\mu^* R_m L_m + E_\mu^* L_m R_m$ $(1 \leq m \leq N)$. Observe that for $\lambda \subseteq \{1, 2, \ldots, N\}$ with even cardinality, we have

$$E_\lambda = \sum_{\mu} E_\mu^* E_\lambda,$$

where the sum is taken over all $\mu \in \{0,1\}^N$ such that the pair $(\lambda \cap S_\mu, \lambda \cap T_\mu)$ satisfies (i), (ii) in Lemma 3.2.3. Then the result follows. □

We remark that the above proof of Proposition 3.8.3 also shows that the matrices E_λ are independent of the basis $v_1, v_2, \ldots, v_N$ for H adapted to the flag $\{x_i\}_{i=0}^N$ and the nontrivial character χ of the additive group $\mathbb{F}_q$.

Lemma 3.8.4. *Let V_{new} denote the set of all $v \in V$ such that $L_m v = 0$ for all $1 \leq m \leq N$. Then we have*

$$V_{\text{new}} = \sum_{\mu,\lambda} E_\mu^* E_\lambda V \qquad \text{(direct sum)},$$

where the sum is taken over all pairs (μ, λ) with $\mu \in \{0,1\}^N$ and $\lambda \subseteq \{1, 2, \ldots, N\}$ satisfying (ii) in Lemma 3.2.6 such that λ is column-full with respect to μ in Definition 3.4.1.

Proof. Take $\mu \in \{0,1\}^N$ and $\lambda \subseteq \{1, 2, \ldots, N\}$ satisfying (ii) in Lemma 3.2.6. Observe that the following are equivalent.

(i) For $1 \leq m \leq N$, we have either $m \in S_\mu$ or $m \in \lambda$.

(ii) λ is column-full with respect to μ.

Then by Lemma 3.7.9, if λ is column-full with respect to μ, we have $E^*_\mu E_\lambda V \subseteq V_{\text{new}}$. Suppose λ is not column-full with respect to μ. Then since we assume λ satisfies Lemma 3.2.6 (ii), there exists $1 \leq m \leq N$ such that $m \in T_\mu$ and $m \notin \lambda$. By Lemma 3.8.1, for any $v \in E^*_\mu E_\lambda V$, $R_m L_m v$ is a nonzero scalar multiple of v. In particular, $L_m v \neq 0$ and so $v \notin V_{\text{new}}$. By above comments and by the fact that V is the direct sum of $E^*_\mu E_\lambda V$, the result follows. □

Recall the column-full property in Definition 3.4.1. For $\mu \in \{0,1\}^N$ and $\lambda \subseteq \{1, 2, \ldots, N\}$ satisfying (ii) in Lemma 3.2.6, we say λ is *row-full* with respect to μ if $S_\mu \subseteq \lambda$.

Lemma 3.8.5. *Let V_{old} denote the set of all $v \in V$ such that $R_m v = 0$ for all $1 \leq m \leq N$. Then we have*

$$V_{\text{old}} = \sum_{\mu,\lambda} E^*_\mu E_\lambda V \qquad \textit{(direct sum)},$$

where the sum is taken over all pairs (μ, λ) with $\mu \in \{0,1\}^N$ and $\lambda \subseteq \{1, 2, \ldots, N\}$ satisfying (ii) in Lemma 3.2.6 such that λ is row-full with respect to μ.

Proof. Similar to the proof of Lemma 3.8.4. □

3.9 The scalar $\kappa(m, \mu, \lambda)$

In this section, we discuss on the scalar $\kappa(m, \mu, \lambda)$ in (3.13).

Lemma 3.9.1. *Let $\mu \in \{0,1\}^N$ and $\lambda \subseteq \{1, 2, \ldots, N\}$ satisfy (ii) in Lemma 3.2.6. Referring to (3.13), we have the following:*

$$\sum_m (-1)^{\mu_m} \kappa(m, \mu, \lambda) = \frac{(N-1)(N-2|\mu|)}{2},$$

where the sum is taken over all $1 \leq m \leq N$ with $m \notin \lambda$.

Proof. Fix $\mu \in \{0,1\}^N$ and we prove the assertion by induction on the cardinality of λ. Let $F(\lambda)$ denote the left-hand side of the equation. Observe that

$$\begin{aligned} F(\lambda) = &\left(\sum_{s \in S_\mu \setminus \lambda} |S_\mu(s-1) \setminus \lambda| + \sum_{s \in S_\mu \setminus \lambda} |T_\mu(s+1) \setminus \lambda| + \sum_{s \in S_\mu \setminus \lambda} \frac{|\lambda|}{2} \right) \\ &- \left(\sum_{t \in T_\mu \setminus \lambda} |S_\mu(t-1) \setminus \lambda| + \sum_{t \in T_\mu \setminus \lambda} |T_\mu(t+1) \setminus \lambda| + \sum_{t \in T_\mu \setminus \lambda} \frac{|\lambda|}{2} \right). \end{aligned}$$

Then the second and fourth terms cancel out, i.e., $F(\lambda)$ equals

$$\left(\sum_{s\in S_\mu\setminus\lambda} |S_\mu(s-1)\setminus\lambda|\right) - \left(\sum_{t\in T_\mu\setminus\lambda} |T_\mu(t+1)\setminus\lambda|\right) + \frac{|\lambda|}{2}\left(|S_\mu\setminus\lambda| - |T_\mu\setminus\lambda|\right).$$

If $\lambda = \emptyset$ then we have

$$\sum_{s\in S_\mu} |S_\mu(s-1)| = 0 + 1 + \cdots + (N - |\mu| - 1) = \frac{(N-|\mu|)(N-|\mu|-1)}{2},$$

and

$$\sum_{t\in T_\mu} |T_\mu(t+1)| = 0 + 1 + \cdots + (|\mu| - 1) = \frac{|\mu|(|\mu|-1)}{2}.$$

Therefore, we have

$$F(\emptyset) = \frac{(N-|\mu|)(N-|\mu|-1)}{2} - \frac{|\mu|(|\mu|-1)}{2} = \frac{(N-1)(N-2|\mu|)}{2},$$

and the result follows.

If $|\lambda| \geq 1$, there exist $s = \max(\lambda \cap S_\mu)$ and $t = \max(\lambda \cap T_\mu)$ since the pair $(\lambda \cap S_\mu, \lambda \cap T_\mu)$ satisfies (i) in Lemma 3.2.3. Set $\lambda' = \lambda \setminus \{s, t\}$ and observe that λ' satisfies (ii) in Lemma 3.2.6 and we have

$$\sum_{s'\in S_\mu\setminus\lambda} |S_\mu(s'-1)\setminus\lambda| = \left(\sum_{s'\in S_\mu\setminus\lambda'} |S_\mu(s'-1)\setminus\lambda'|\right) - |S_\mu\setminus\lambda|,$$

and

$$\sum_{t'\in T_\mu\setminus\lambda} |T_\mu(t'+1)\setminus\lambda| = \left(\sum_{t'\in T_\mu\setminus\lambda'} |T_\mu(t'+1)\setminus\lambda'|\right) - |T_\mu\setminus\lambda|.$$

Therefore, since $|\lambda| = |\lambda'| + 2$, we have

$$F(\lambda) = F(\lambda'),$$

and by the inductive hypothesis, the result follows. □

In the next lemma, we do not assume q to be a prime power.

Lemma 3.9.2. *Let $\mu = (\mu_1, \mu_2, \ldots, \mu_N) \in \{0,1\}^N$ and $\lambda \subseteq \{1, 2, \ldots, N\}$ satisfy (ii) in Lemma 3.2.6. Referring to (3.13), for $q \in \mathbb{C}$ with $q \neq 0, 1$, we have the following:*

$$\sum_m (-1)^{\mu_m} q^{\kappa(m,\mu,\lambda)} = \frac{q^{N-|\mu|} - q^{|\mu|}}{q-1},$$

where the sum is taken over all $1 \leq m \leq N$ with $m \notin \lambda$.

Proof. For notational convenience, in this proof we use the following notation. Take $n \in \mathbb{N} \setminus \{0\}$. For $\nu = (\nu_1, \nu_2, \dots, \nu_n) \in \{0,1\}^n$, a sequence $\mathfrak{a} = (\mathfrak{a}_1, \mathfrak{a}_2, \dots, \mathfrak{a}_n) \in \mathbb{Z}^n$ is called a *κ-sequence with respect to ν* whenever it satisfies

$$\mathfrak{a}_i = \begin{cases} \mathfrak{a}_{i-1} + 1 & \text{if } \nu_{i-1} = \nu_i, \\ -\mathfrak{a}_{i-1} & \text{if } \nu_{i-1} \neq \nu_i, \end{cases}$$

for $2 \leq i \leq n$. We call $\nu \in \{0,1\}^n$ *reduced* if $n \leq 2$ or ν is either $\mathbf{0}$ or $\mathbf{1}$. Let $\mathfrak{a} = (\mathfrak{a}_1, \mathfrak{a}_2, \dots, \mathfrak{a}_n) \in \mathbb{Z}^n$ be a κ-sequence with respect to a non-reduced $\nu = (\nu_1, \nu_2, \dots, \nu_n) \in \{0,1\}^n$. Then we have $\nu_{i-1} \neq \nu_i$ for some $2 \leq i \leq n$. Let $\nu' \in \{0,1\}^{n-2}$ be the sequence obtained from ν by removing the coordinates $i-1$ and i, and let $\mathfrak{a}' \in \mathbb{Z}^{n-2}$ denote the sequence obtained from $\mathfrak{a}$ by removing the same pair of coordinates. Then it is easy to show that the sequence $\mathfrak{a}'$ is again a κ-sequence with respect to ν'. Moreover, by continuing this process, any κ-sequence reaches a κ-sequence with respect to a reduced tuple ν. More precisely, a κ-sequence $\mathfrak{a}$ with respect to $\nu \in \{0,1\}^n$ becomes

(i) a κ-sequence of length 2 with respect to $(0,1)$ or $(1,0)$ if $2|\nu| = n$,

(ii) a κ-sequence of length $n - 2|\nu|$ with respect to $\mathbf{0} \in \{0,1\}^{n-2|\nu|}$ if $2|\nu| < n$,

(iii) a κ-sequence of length $2|\nu| - n$ with respect to $\mathbf{1} \in \{0,1\}^{2|\nu|-n}$ if $2|\nu| > n$.

We call this a *reduced κ-sequence from* $\mathfrak{a}$. For a κ-sequence $\mathfrak{a} = (\mathfrak{a}_1, \mathfrak{a}_2, \dots, \mathfrak{a}_n) \in \mathbb{Z}^n$ with respect to $\nu = (\nu_1, \nu_2, \dots, \nu_n) \in \{0,1\}^n$, we define

$$f(\nu, \mathfrak{a}; q) = \sum_{i=1}^{n} (-1)^{\nu_i} q^{(-1)^{\nu_i} \mathfrak{a}_i}.$$

Observe that the value $f(\nu, \mathfrak{a}; q)$ is invariant under the reducing process above. In particular, if $\mathfrak{a}'$ is a reduced κ-sequence with respect to ν' from a κ-sequence $\mathfrak{a}$ with respect to ν, then we have $f(\nu, \mathfrak{a}; q) = f(\nu', \mathfrak{a}'; q)$.

Set $n = N - |\lambda|$. Let $\nu = \nu(\mu, \lambda) \in \{0,1\}^n$ be the sequence obtained from μ by removing all the coordinates indexed by λ. Consider the sequence $\mathfrak{a} \in \mathbb{Z}^n$ defined by

$$\mathfrak{a} = ((-1)^{\mu_m} \kappa(m, \mu, \lambda))_{m \in \{1,2,\dots,N\} \setminus \lambda},$$

where the index m increases from left to right. For $1 \leq m < n \leq N$ with $m, n \notin \lambda$, observe that

$$\kappa(m, \mu, \lambda) - \kappa(n, \mu, \lambda) = |\{t \in T_\mu \setminus \lambda \mid m < t \leq n\}| - |\{s \in S_\mu \setminus \lambda \mid m \leq s < n\}|.$$

Therefore, the sequence $\mathfrak{a}$ is a κ-sequence with respect to ν. Let $\mathfrak{a}'$ be a reduced κ-sequence with respect to ν' from $\mathfrak{a}$. Then the left-hand side of the desired identity becomes $f(\nu', \mathfrak{a}'; q)$.

We first consider the case $2|\mu| = N$. Then we have $|S_\mu| = |T_\mu|$ and so $2|\nu| = n$ since the pair $(\lambda \cap S_\mu, \lambda \cap T_\mu)$ satisfies (i) in Lemma 3.2.3. Thus, $\mathfrak{a}'$ is a κ-sequence of length 2 with respect to $(0,1)$ or $(1,0)$ and so $f(\nu', \mathfrak{a}'; q) = 0$ and the result follows. We next consider the case $2|\mu| < N$. Then by the similar argument above, we have $2|\nu| < n$. Thus, $\mathfrak{a}'$ is a κ-sequence of length $n - 2|\nu| = N - 2|\mu|$ with respect to $\mathbf{0} \in \{0,1\}^{n-2|\nu|}$. By the definition of κ-sequence, $\mathfrak{a}'$ is an arithmetic sequence with common difference 1. We claim that

$$\mathfrak{a}' = (|\mu|, |\mu| + 1, \ldots, N - |\mu| - 1).$$

To show this, since it is an arithmetic sequence, it suffices to show that

$$\sum_{a' \in \mathfrak{a}'} a' = \frac{(N-1)(N-2|\mu|)}{2}.$$

This follows from Lemma 3.9.1 since $\sum_{a' \in \mathfrak{a}'} a' = \sum_{a \in \mathfrak{a}} a$. For the case $2|\mu| > N$, the proof is similar to that for the case $2|\mu| < N$. Hence the result follows. □

3.10 The $\mathcal{H}_f$-modules

Recall from Proposition 3.5.6 that the algebra $\mathcal{H}_f$ is semisimple. Thus the standard module V is a direct sum of irreducible $\mathcal{H}_f$-modules, and every irreducible $\mathcal{H}_f$-module appears in V up to isomorphism. We now discuss the $\mathcal{H}_f$-submodules of V, which from now on we call $\mathcal{H}_f$-modules for short.

Proposition 3.10.1. *Any irreducible $\mathcal{H}_f$-module is generated by a nonzero vector $v \in V$ such that $L_m v = 0$ for all $1 \leq m \leq N$.*

Proof. Set $\Phi(v) = \{m \mid 1 \leq m \leq N, L_m v \neq 0\}$ for $v \in V$. Let W denote an irreducible $\mathcal{H}_f$-module and take a nonzero vector $w \in W$. If $\Phi(w) = \emptyset$, then $L_m w = 0$ for all $1 \leq m \leq N$ and by the irreducibility of W, the module W is generated by w and so the result follows. Suppose $\Phi(w) \neq \emptyset$. Let $m = \min \Phi(w)$ and set $w' = L_m w \in W$. By Proposition 3.6.2 (i) and (ii), we have $\Phi(w') \subsetneq \Phi(w)$. By continuing this process at most $|\Phi(w)|$ times, we get a nonzero vector $v \in W$ such that $\Phi(v) = \emptyset$. By the same argument above, the assertion holds. □

Recall from Sections 3.7 and 3.8, that there are the matrices E_λ in $\mathcal{H}_f$ and that they turn out to be independent of the basis $v_1, v_2, \ldots, v_N$ for H and the nontrivial character χ of the additive group $\mathbb{F}_q$. By Lemma 3.8.4 and Proposition 3.10.1, it suffices to consider the module $\mathcal{H}_f v$ for $v \in \sum_{\mu,\lambda} E^*_\mu E_\lambda V$, where the sum is taken over all pairs (μ, λ) with $\mu \in \{0,1\}^N$ and $\lambda \subseteq \{1, 2, \ldots, N\}$ satisfying (ii) in Lemma 3.2.6 such that λ is column-full with respect to μ in Definition 3.4.1.

Proposition 3.10.2. *Let $\mu \in \{0,1\}^N$ and $\lambda \subseteq \{1, 2, \ldots, N\}$ satisfy (ii) in Lemma 3.2.6, and assume that λ is column-full with respect to μ in Definition 3.4.1. Recall $\kappa(m, \mu, \lambda)$ in (3.13). For a nonzero vector $v \in E^*_\mu E_\lambda V$, the $\mathcal{H}_f$-module $\mathcal{H}_f v$ has a basis*

$$w(\varepsilon) \in E^*_{\mu+\varepsilon} V \qquad \left(\varepsilon = (\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_N), \quad \varepsilon_m = \begin{cases} 0 & \text{if } m \in \lambda, \\ 0 \text{ or } 1 & \text{if } m \notin \lambda \end{cases} \right), \tag{3.15}$$

on which the generators L_m, R_m $(1 \leq m \leq N)$ act as follows:

$$L_m w(\varepsilon) = q^{\kappa(m,\mu,\lambda) - (\varepsilon_1 + \cdots + \varepsilon_{m-1})} w(\varepsilon - \widehat{m}),$$
$$R_m w(\varepsilon) = q^{\varepsilon_{m+1} + \cdots + \varepsilon_N} w(\varepsilon + \widehat{m}),$$

where we set $w(\varepsilon) = 0$ if ε is not of the form in (3.15).

Proof. Let $\mathcal{H}_f^+$ denote the subalgebra of $\mathcal{H}_f$ generated by $R_1, R_2, \ldots, R_N$. Consider $\mathcal{H}_f^+ v$, the $\mathcal{H}_f^+$-module generated by v. We show that $\mathcal{H}_f^+ v$ is an $\mathcal{H}_f$-module. Let $1 \leq m \leq N$. Then $\mathcal{H}_f^+ v$ is R_m-invariant by the construction and K_m-invariant by Proposition 3.6.1 (ii), (iv). In addition, $\mathcal{H}_f^+ v$ is L_m-invariant by Proposition 3.6.2 (i), (iii), (iv), Lemma 3.8.2 and since $L_m v = 0$ by Lemma 3.8.4. Since $\mathcal{H}_f$ is generated by R_m, L_m and K_m, for $1 \leq m \leq N$, $\mathcal{H}_f^+ v$ is an $\mathcal{H}_f$-module. Thus we have $\mathcal{H}_f^+ v = \mathcal{H}_f v$. By Proposition 3.6.2 (i), (iii), $\mathcal{H}_f^+ v$ is spanned by

$$w(\varepsilon) = R_N^{\varepsilon_N} R_{N-1}^{\varepsilon_{N-1}} \cdots R_1^{\varepsilon_1} v,$$

for $\varepsilon = (\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_N) \in \{0,1\}^N$. By Lemma 3.5.4 (ii), $w(\varepsilon) \in E^*_{\mu+\varepsilon} V$. By Lemmas 3.7.10 and 3.8.5, $w(\varepsilon) \neq 0$ if and only if $m \in S_\mu$ and $m \notin \lambda$ for all $1 \leq m \leq N$ with $\varepsilon_m = 1$. Thus (3.15) forms a basis for $\mathcal{H}_f v$. For $1 \leq m \leq N$, the L_m-actions on $w(\varepsilon)$ follow from Proposition 3.6.2 (iii), (iv), Lemma 3.8.2 and $L_m v = 0$. Similarly, for $1 \leq m \leq N$, the R_m-actions on $w(\varepsilon)$ follow from Proposition 3.6.2 (i), (iii). The result follows. □

Proposition 3.10.3. *Referring to Proposition 3.10.2, the basis (3.15) for $\mathcal{H}_f v$ satisfies the following:*

$$K_m w(\varepsilon) = q^{1/2 - (\mu_m + \varepsilon_m)} w(\varepsilon),$$

for $1 \leq m \leq N$, where $\mu = (\mu_1, \mu_2, \ldots, \mu_N)$ and $\varepsilon = (\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_N)$.

Proof. By Proposition 3.10.2, we have $w(\varepsilon) \in E^*_{\mu+\varepsilon}V$. The result follows from the definition of K_m. □

Theorem 3.10.4. *For any irreducible $\mathcal{H}_f$-module W, there uniquely exist $\mu \in \{0,1\}^N$ and $\lambda \subseteq \{1,2,\ldots,N\}$ satisfying (ii) in Lemma 3.2.6 where λ is column-full with respect to μ, such that W is generated by a nonzero vector in $E^*_\mu E_\lambda V$. Moreover, W is determined up to isomorphism by μ and λ.*

Proof. By Proposition 3.10.1, there exists a nonzero vector $v \in W$ with $L_m v = 0$ for all $1 \le m \le N$ such that $W = \mathcal{H}_f v$. According to the direct sum decomposition in Lemma 3.8.4, we write

$$v = \sum_{\mu,\lambda} E^*_\mu E_\lambda v.$$

Since v is nonzero, there exists a pair (μ, λ) such that $E^*_\mu E_\lambda v \neq 0$. By Proposition 3.8.3, $E^*_\mu E_\lambda v$ belongs to W and so by the irreducibility of W, $E^*_\mu E_\lambda v$ generates W. Suppose there exists another pair (μ', λ') such that $E^*_{\mu'} E_{\lambda'} v \neq 0$. Then $E^*_{\mu'} E_{\lambda'} v$ also generates W. Thus we have the two bases (3.15) for W. However, by comparing them, we obtain $(\mu', \lambda') = (\mu, \lambda)$ and the result follows. □

Definition 3.10.5. Referring to Theorem 3.10.4, we call $\mu \in \{0,1\}^N$ the *endpoint* of W and $\lambda \subseteq \{1,2,\ldots,N\}$ the *shape* of W.

Corollary 3.10.6. *Let $\lambda \subseteq \{1,2,\ldots,N\}$ with even cardinality. For an irreducible $\mathcal{H}_f$-module W of shape λ, we have*

$$\dim W = 2^{N-|\lambda|}.$$

Proof. Count the vectors in the basis (3.15) for W. □

Theorem 3.10.7. *For $\mu \in \{0,1\}^N$ and $\lambda \subseteq \{1,2,\ldots,N\}$ satisfying (ii) in Lemma 3.2.6 where λ is column-full with respect to μ, there exists an irreducible $\mathcal{H}_f$-module of endpoint μ and shape λ. Moreover, the multiplicity in V is given by*

$$q^{|B_\mu| - n(\lambda \cap S_\mu)} \prod_{s \in \lambda \cap S_\mu} \left(q^{\rho(s,\mu,\lambda)} - 1\right),$$

where $n(\lambda \cap S_\mu)$ is defined in (3.12) and $\rho(s,\mu,\lambda)$ is defined in Lemma 3.2.9.

Proof. Take a nonzero vector $v \in E^*_\mu E_\lambda V$. We show that $W = \mathcal{H}_f v$ is irreducible. Consider an irreducible $\mathcal{H}_f$-module decomposition of W as follows.

$$W = W_1 + W_2 + \cdots + W_r \qquad \text{(direct sum)},$$

for some positive integer $r \geq 1$. According to this decomposition, we write $v = w_1 + w_2 + \cdots + w_r$ such that $w_n \in W_n$ $(1 \leq n \leq r)$. Since this sum is direct and $v \in E^*_\mu E_\lambda W$, we find that w_n is nonzero and $w_n \in E^*_\mu E_\lambda W$ for $1 \leq n \leq r$. However, by Proposition 3.10.2, we have $\dim E^*_\mu E_\lambda W = 1$. Since the vectors w_n $(1 \leq n \leq r)$ are linearly independent, this forces $r = 1$, i.e., W is irreducible.

The multiplicity of W in V is $\dim E^*_\mu E_\lambda V$, which is determined in Corollary 3.4.3. □

3.11 The quantum affine algebra $U_q(\widehat{\mathfrak{sl}}_2)$

In this section, we fix a nonzero scalar $q \in \mathbb{C}$ which is not a root of unity. For $n \in \mathbb{N}$, we define

$$[n]_q = \frac{q^n - q^{-n}}{q - q^{-1}}.$$

We recall the definition of $U_q(\widehat{\mathfrak{sl}}_2)$ from [6] in terms of Chevalley generators.

Definition 3.11.1 ([6, Section 2]). The quantum affine algebra $U_q(\widehat{\mathfrak{sl}}_2)$ is the associative $\mathbb{C}$-algebra generated by $e_i^{\pm}, k_i, k_i^{-1}$ $(i = 0, 1)$ with the relations

$$k_i k_i^{-1} = k_i^{-1} k_i = 1, \qquad k_0 k_1 = k_1 k_0, \tag{3.16}$$

$$k_i e_i^{\pm} = q^{\pm 2} e_i^{\pm} k_i, \qquad k_i e_j^{\pm} = q^{\mp 2} e_j^{\pm} k_i \quad (i \neq j), \tag{3.17}$$

$$e_i^+ e_i^- - e_i^- e_i^+ = \frac{k_i - k_i^{-1}}{q - q^{-1}}, \qquad e_0^{\pm} e_1^{\mp} - e_1^{\mp} e_0^{\pm} = 0, \tag{3.18}$$

$$(e_i^{\pm})^3 e_j^{\pm} - [3]_q (e_i^{\pm})^2 e_j^{\pm} e_i^{\pm} + [3]_q e_i^{\pm} e_j^{\pm} (e_i^{\pm})^2 - e_j^{\pm} (e_i^{\pm})^3 = 0 \quad (i \neq j). \tag{3.19}$$

We call $e_i^{\pm}, k_i, k_i^{-1}$ $(i = 0, 1)$ the *Chevalley generators* for $U_q(\widehat{\mathfrak{sl}}_2)$.

It is known that the quantum affine algebra $U_q(\widehat{\mathfrak{sl}}_2)$ has the following Hopf algebra structure. The comultiplication Δ satisfies

$$\Delta(e_i^+) = e_i^+ \otimes k_i + 1 \otimes e_i^+, \qquad \Delta(e_i^-) = e_i^- \otimes 1 + k_i^{-1} \otimes e_i^-, \qquad \Delta(k_i) = k_i \otimes k_i.$$

It is also known that there exists a family of finite-dimensional irreducible $U_q(\widehat{\mathfrak{sl}}_2)$-modules $V_d(\alpha)$ for $d \in \mathbb{N}$, $\alpha \in \mathbb{C} \setminus \{0\}$, where $V_d(\alpha)$ has a basis $\{u_i\}_{i=0}^d$ satisfying

$$\begin{aligned}
e_0^+ u_i &= \alpha [i+1]_q u_{i+1} && (0 \leq i \leq d-1), && e_0^+ u_d = 0,\\
e_1^+ u_i &= [d-i+1]_q u_{i-1} && (1 \leq i \leq d), && e_1^+ u_0 = 0,\\
e_0^- u_i &= \alpha^{-1} [d-i+1]_q u_{i-1} && (1 \leq \varepsilon \leq d), && e_0^- u_0 = 0,\\
e_1^- u_i &= [i+1]_q u_{i+1} && (0 \leq i \leq d-1), && e_1^- u_d = 0,\\
k_0 u_i &= q^{2i-d} u_i && (0 \leq i \leq d), &&\\
k_1 u_i &= q^{d-2i} u_i && (0 \leq i \leq d). &&
\end{aligned}$$

We call $V_d(\alpha)$ the *evaluation module* for $U_q(\widehat{\mathfrak{sl}}_2)$ with the *evaluation parameter* α. We recurrently define the algebra homomorphism

$$\Delta^{(N)} : U_q(\widehat{\mathfrak{sl}}_2) \to \underbrace{U_q(\widehat{\mathfrak{sl}}_2) \otimes \cdots \otimes U_q(\widehat{\mathfrak{sl}}_2)}_{(N+1) \text{ times}},$$

for $N \in \mathbb{N}$ by

$$\Delta^{(0)} = \mathrm{id},$$
$$\Delta^{(1)} = \Delta,$$
$$\Delta^{(N)} = (\underbrace{\mathrm{id} \otimes \cdots \otimes \mathrm{id}}_{(N-2) \text{ times}} \otimes \Delta) \circ \Delta^{(N-1)} \qquad (N \geq 2).$$

This algebra homomorphism $\Delta^{(N)}$ is called the *N-fold comultiplication*. For each $N \geq 1$, by the $(N-1)$-fold comultiplication $\Delta^{(N-1)}$, a tensor product of N evaluation modules again becomes a $U_q(\widehat{\mathfrak{sl}}_2)$-module. More precisely, a tensor product $V_{d_1}(\alpha_1) \otimes \cdots \otimes V_{d_N}(\alpha_N)$ has a basis

$$u(\varepsilon) = u_{\varepsilon_1} \otimes \cdots \otimes u_{\varepsilon_N} \qquad (0 \leq \varepsilon_1 \leq d_1, \quad \ldots, \quad 0 \leq \varepsilon_N \leq d_N), \tag{3.20}$$

on which the Chevalley generators act as follows:

$$e_0^+ u(\varepsilon) = \sum_{m=1}^{N} \alpha_m [\varepsilon_m + 1]_q q^{2(\varepsilon_{m+1}+\cdots+\varepsilon_N)-(d_{m+1}+\cdots+d_N)} u(\varepsilon + \widehat{m}), \tag{3.21}$$

$$e_1^+ u(\varepsilon) = \sum_{m=1}^{N} [d_m - \varepsilon_m + 1]_q q^{(d_{m+1}+\cdots+d_N)-2(\varepsilon_{m+1}+\cdots+\varepsilon_N)} u(\varepsilon - \widehat{m}), \tag{3.22}$$

$$e_0^- u(\varepsilon) = \sum_{m=1}^{N} \alpha_m^{-1} [d_m - \varepsilon_m + 1]_q q^{(d_1+\cdots+d_{m-1})-2(\varepsilon_1+\cdots+\varepsilon_{m-1})} u(\varepsilon - \widehat{m}), \tag{3.23}$$

$$e_1^- u(\varepsilon) = \sum_{m=1}^{N} [\varepsilon_m + 1]_q q^{2(\varepsilon_1+\cdots+\varepsilon_{m-1})-(d_1+\cdots+d_{m-1})} u(\varepsilon + \widehat{m}), \tag{3.24}$$

$$k_0 u(\varepsilon) = q^{2(\varepsilon_1+\cdots+\varepsilon_N)-(d_1+\cdots+d_N)} u(\varepsilon), \tag{3.25}$$

$$k_1 u(\varepsilon) = q^{(d_1+\cdots+d_N)-2(\varepsilon_1+\cdots+\varepsilon_N)} u(\varepsilon), \tag{3.26}$$

where $\varepsilon = (\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_N) \in \mathbb{Z}^N$ and we define $u(\varepsilon) = 0$ if ε is not of the form in (3.20).

Let W denote a finite-dimensional irreducible $U_q(\widehat{\mathfrak{sl}}_2)$-module. By [6, Proposition 3.2], there exist scalars $\epsilon_0, \epsilon_1 \in \{-1, 1\}$ such that each eigenvalue of k_i on W is ϵ_i times an integral power of q for $i = 0, 1$. The pair (ϵ_0, ϵ_1) is called the *type* of W. For each pair $\epsilon_0, \epsilon_1 \in \{-1, 1\}$, there exists an algebra automorphism of $U_q(\widehat{\mathfrak{sl}}_2)$ that sends

$$k_i \mapsto \epsilon_i k_i, \qquad e_i^+ \mapsto \epsilon_i e_i^+, \qquad e_i^- \mapsto e_i^- \qquad (i = 0, 1).$$

By this automorphism, any finite-dimensional irreducible $U_q(\widehat{\mathfrak{sl}}_2)$-module of type (ϵ_0, ϵ_1) becomes that of type $(1, 1)$.

Theorem 3.11.2 ([6, Theorem 4.11]). *Every finite-dimensional irreducible $U_q(\widehat{\mathfrak{sl}}_2)$-module of type $(1, 1)$ is isomorphic to a tensor product of evaluation modules. Moreover, two such tensor products are isomorphic if and only if one is obtained from the other by permuting the factors in the tensor product.*

With an evaluation module $V_d(\alpha)$, we associate the set of scalars

$$S_d(\alpha) = \{\alpha q^{d-1}, \alpha q^{d-3}, \dots, \alpha q^{-d+1}\}.$$

The set $S_d(\alpha)$ is called a *q-string* of length d. Two q-strings $S_{d_1}(\alpha_1)$, $S_{d_2}(\alpha_2)$ are said to be in *general position* if one of the following occurs:

(i) $S_{d_1}(\alpha_1) \cup S_{d_2}(\alpha_2)$ is not a q-string,

(ii) $S_{d_1}(\alpha_1) \subseteq S_{d_2}(\alpha_2)$ or $S_{d_2}(\alpha_2) \subseteq S_{d_1}(\alpha_1)$.

Moreover, several q-strings are said to be in *general position* if every two q-strings are in general position.

Theorem 3.11.3 ([6, Theorem 4.8]). *A tensor product of evaluation modules for $U_q(\widehat{\mathfrak{sl}}_2)$ is irreducible if and only if the associated q-strings are in general position.*

3.12 The algebra $\mathcal{H}_f$ and the quantum affine algebra $U_{q^{1/2}}(\widehat{\mathfrak{sl}}_2)$

In this section, we get back to the subspace lattice P over $\mathbb{F}_q$. Recall the matrices $E_\lambda \in \mathcal{H}_f$ in Sections 3.7 and 3.8. Let $\mu \in \{0,1\}^N$ and $\lambda \subseteq \{1, 2, \dots, N\}$ satisfy (ii) in Lemma 3.2.6. For $v \in E^*_\mu E_\lambda V$ and $1 \leq m \leq N$, if $L_m v \neq 0$, then we have $m \in T_\mu$ and $m \notin \lambda$ by Lemma 3.7.9 and so $(L_m R_m) L_m v = q^{\kappa(m,\mu,\lambda)} L_m v$ by Lemma 3.8.1. Therefore, we define the matrix $(L_m R_m)^{-1} L_m$ by

$$(L_m R_m)^{-1} L_m v = \begin{cases} q^{-\kappa(m,\mu,\lambda)} L_m v & \text{if } L_m v \neq 0, \\ 0 & \text{if } L_m v = 0, \end{cases} \tag{3.27}$$

for $v \in V$. We remark that $(L_m R_m)^{-1} L_m$ does not mean the product of $(L_m R_m)^{-1}$ and L_m since $L_m R_m$ is not invertible by Lemma 3.8.1. Similarly, we define the matrix $(R_m L_m)^{-1} R_m$ by

$$(R_m L_m)^{-1} R_m v = \begin{cases} q^{-\kappa(m,\mu,\lambda)} R_m v & \text{if } R_m v \neq 0, \\ 0 & \text{if } R_m v = 0, \end{cases} \tag{3.28}$$

for $v \in V$.

Theorem 3.12.1. *Let $\alpha_1, \alpha_2, \ldots, \alpha_N$ denote nonzero scalars. The standard module V supports a $U_{q^{1/2}}(\widehat{\mathfrak{sl}}_2)$-module structure on which the Chevalley generators act as follows:*

generators	*actions on* V
e_0^+	$q^{(1-N)/2}\sum_{m=1}^N \alpha_m R_m$
e_1^+	$q^{(N-1)/2}\sum_{m=1}^N (L_m R_m)^{-1} L_m$
e_0^-	$\sum_{m=1}^N \alpha_m^{-1} L_m$
e_1^-	$\sum_{m=1}^N (R_m L_m)^{-1} R_m$
k_0	$\prod_{m=1}^N K_m^{-1}$
k_0^{-1}	$\prod_{m=1}^N K_m$
k_1	$\prod_{m=1}^N K_m$
k_1^{-1}	$\prod_{m=1}^N K_m^{-1}$

Here the matrices $(L_m R_m)^{-1} L_m$ and $(R_m L_m)^{-1} R_m$ are defined in (3.27) and in (3.28), respectively.

Proof. Referring to the above table, for $i = 0, 1$ let $\widehat{e_i^+}, \widehat{e_i^-}, \widehat{k_i}, \widehat{k_i}^{-1}$ denote the expressions to the right of $e_i^+, e_i^-, k_i, k_i^{-1}$ respectively. We show these elements $\widehat{e_i^+}$, $\widehat{e_i^-}$, $\widehat{k_i}$, $\widehat{k_i}^{-1}$ $(i = 0, 1)$ satisfy the defining relations (3.16)–(3.19) of $U_{q^{1/2}}(\widehat{\mathfrak{sl}}_2)$ on V.

We first show $\widehat{e_i^+}, \widehat{e_i^-}, \widehat{k_i}, \widehat{k_i}^{-1}$ $(i = 0, 1)$ satisfy the relations except the first relation in (3.18). They satisfy the relations in (3.16) by the definitions of $\widehat{k_i}, \widehat{k_i}^{-1}$ $(i = 0, 1)$. They satisfy the first relation in (3.17) with $i = 0$ by Proposition 3.6.1. They satisfy the second relation in (3.17) with $(i, j) = (1, 0)$ by Proposition 3.6.1. Since the other relations involve $\widehat{e_1^+}, \widehat{e_1^-}$, we show them as follows. Fix a nonzero vector $v \in V$. Then we apply both sides of each defining relation to v and check the results are the same. These elements $\widehat{e_i^+}, \widehat{e_i^-}, \widehat{k_i}, \widehat{k_i}^{-1}$ $(i = 0, 1)$ satisfy the first relation in (3.17) with $i = 1$ by Proposition 3.6.1. They satisfy the second relation in (3.17) with $(i, j) = (0, 1)$ by Proposition 3.6.1. They satisfy the second relation in (3.18) and the relations in (3.19) by Proposition 3.6.2.

It remains to show that they satisfy the first relation in (3.18). Take a nonzero vector $v \in E^*_\mu E_\lambda V$ for some $\mu = (\mu_1, \mu_2, \ldots, \mu_N) \in \{0, 1\}^N$, $\lambda \subseteq \{1, 2, \ldots, N\}$. By

Lemmas 3.8.1 and 3.8.2, we have

$$\left(\widehat{e}_0^+\widehat{e}_0^- - \widehat{e}_0^-\widehat{e}_0^+\right) v = -\left(q^{(1-N)/2}\sum_m (-1)^{\mu_m} q^{\kappa(m,\mu,\lambda)}\right) v,$$

where the sum is taken over all $1 \le m \le N$ with $m \notin \lambda$. On the other hand, by the definition of K_m, we have

$$\left(\frac{\widehat{k}_0 - \widehat{k}_0^{-1}}{q^{1/2} - q^{-1/2}}\right) v = \left(\frac{q^{|\mu|-N/2} - q^{N/2-|\mu|}}{q^{1/2} - q^{-1/2}}\right) v.$$

By Lemma 3.9.2, it turns out that both scalars are the same and so $\widehat{e}_0^+, \widehat{e}_0^-, \widehat{k}_0, \widehat{k}_0^{-1}$ satisfy the first relation in (3.18). Similarly, $\widehat{e}_1^+, \widehat{e}_1^-, \widehat{k}_1, \widehat{k}_1^{-1}$ satisfy the first relation in (3.18). □

Corollary 3.12.2. *Let $\alpha_1, \alpha_2, \dots, \alpha_N$ denote nonzero scalars. There exists an algebra homomorphism from $U_{q^{1/2}}(\widehat{\mathfrak{sl}}_2)$ to $\mathcal{H}_f$ that sends*

$$e_0^+ \mapsto q^{(1-N)/2}\sum_{m=1}^N \alpha_m R_m, \qquad e_1^+ \mapsto q^{(N-1)/2}\sum_{m=1}^N (L_m R_m)^{-1} L_m,$$

$$e_0^- \mapsto \sum_{m=1}^N \alpha_m^{-1} L_m, \qquad e_1^- \mapsto \sum_{m=1}^N (R_m L_m)^{-1} R_m,$$

$$k_0 \mapsto \prod_{m=1}^N K_m^{-1}, \qquad k_1 \mapsto \prod_{m=1}^N K_m.$$

Proof. Immediate from Proposition 3.12.1. □

The algebra homomorphism in Corollary 3.12.2 turns an $\mathcal{H}_f$-module into a $U_{q^{1/2}}(\widehat{\mathfrak{sl}}_2)$-module.

Lemma 3.12.3. *Let $\mu \in \{0,1\}^N$ and $\lambda \subseteq \{1, 2, \dots, N\}$ satisfy (ii) in Lemma 3.2.6 where λ is column-full with respect to μ in Definition 3.4.1. Let $W_{\mu,\lambda}$ denote an irreducible $\mathcal{H}_f$-module with endpoint μ and shape λ. The basis (3.15) for $W_{\mu,\lambda}$ has the following actions of Chevalley generators via the algebra homomorphism in*

Corollary 3.12.2.

$$e_0^+ w(\varepsilon) = q^{(1-N)/2} \sum_{m=1}^{N} \alpha_m q^{\varepsilon_{m+1}+\cdots+\varepsilon_N} w(\varepsilon + \widehat{m}), \tag{3.29}$$

$$e_1^+ w(\varepsilon) = q^{(N-1)/2} \sum_{m=1}^{N} q^{-(\varepsilon_{m+1}+\cdots+\varepsilon_N)} w(\varepsilon - \widehat{m}), \tag{3.30}$$

$$e_0^- w(\varepsilon) = \sum_{m=1}^{N} \alpha_m^{-1} \theta_m(\mu,\lambda) q^{-(\varepsilon_1+\cdots+\varepsilon_{m-1})} w(\varepsilon - \widehat{m}), \tag{3.31}$$

$$e_1^- w(\varepsilon) = \sum_{m=1}^{N} \theta_m(\mu,\lambda)^{-1} q^{\varepsilon_1+\cdots+\varepsilon_{m-1}} w(\varepsilon + \widehat{m}), \tag{3.32}$$

$$k_0 w(\varepsilon) = q^{-N/2+|\mu|+|\varepsilon|} w(\varepsilon), \tag{3.33}$$

$$k_1 w(\varepsilon) = q^{N/2-|\mu|-|\varepsilon|} w(\varepsilon), \tag{3.34}$$

where $\varepsilon = (\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_N) \in \{0,1\}^N$. Here we define $w(\varepsilon) = 0$ if ε is not of the form in (3.15).

Proof. Use Propositions 3.10.2, 3.10.3 and Corollary 3.12.2. □

Lemma 3.12.4. *Let $\mu \in \{0,1\}^N$ and $\lambda \subseteq \{1,2,\ldots,N\}$ satisfy (ii) in Lemma 3.2.6 where λ is column-full with respect to μ in Definition 3.4.1. We define $d = (d_1, d_2, \ldots, d_N) \in \{0,1\}^N$ by*

$$d_m = \begin{cases} 1 & \text{if } m \notin \lambda, \\ 0 & \text{if } m \in \lambda \end{cases} \qquad (1 \le m \le N).$$

Then we have the following:

(i) $|d| = N - 2|\mu|$.

(ii) *If $m \notin \lambda$, then $\kappa(m,\mu,\lambda) = (N-1)/2 + (d_1+\cdots+d_{m-1})/2 - (d_{m+1}+\cdots+d_N)/2$ defined in (3.13).*

Proof. (i) By the definition of d, we have $|d| = N - |\lambda|$. By the assumption, we have $|\lambda| = 2|\mu|$ and so the result follows.

(ii) Assume $m \notin \lambda$. Observe that

$$|S_\mu(m-1) \setminus \lambda| = d_1 + \cdots + d_{m-1}, \qquad |T_\mu(m+1) \setminus \lambda| = 0.$$

By the definition of d,

$$|\lambda|/2 = N/2 - (d_1 + \cdots + d_N)/2.$$

Hence the result follows from the above comments and $d_m = 1$.

□

Theorem 3.12.5. *Let $\mu \in \{0,1\}^N$ and $\lambda \subseteq \{1,2,\ldots,N\}$ satisfy (ii) in Lemma 3.2.6 where λ is column-full with respect to μ in Definition 3.4.1. Let $W_{\mu,\lambda}$ denote an irreducible $\mathcal{H}_f$-module with endpoint μ and shape λ. Then by the algebra homomorphism in Corollary 3.12.2, $W_{\mu,\lambda}$ becomes a $U_{q^{1/2}}(\widehat{\mathfrak{sl}}_2)$-module and we have the following:*

(i) $W_{\mu,\lambda}$ has type $(1,1)$.

(ii) $W_{\mu,\lambda}$ is isomorphic to the tensor product of $V_1(\alpha_m)$, where $1 \le m \le N$ such that $m \notin \lambda$.

Proof. (i) This follows from (3.33) and (3.34).

(ii) Recall $(d_1, d_2, \ldots, d_N) \in \{0,1\}^N$ from Lemma 3.12.4. It suffices to show that

$$W_{\mu,\lambda} \simeq V_{d_1}(\alpha_1) \otimes \cdots \otimes V_{d_N}(\alpha_N).$$

Recall the basis $w(\varepsilon)$ in (3.15) for $W_{\mu,\lambda}$ and the basis $u(\varepsilon)$ in (3.20) for $V_{d_1}(\alpha_1)\otimes \cdots \otimes V_{d_N}(\alpha_N)$, where $\varepsilon = (\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_N) \in \{0,1\}^N$ such that $w(\varepsilon) = 0$ and $u(\varepsilon) = 0$ if $d_m < \varepsilon_m$ for some $1 \le m \le N$. We define a linear map φ from $V_{d_1}(\alpha_1) \otimes \cdots \otimes V_{d_N}(\alpha_N)$ to $W_{\mu,\lambda}$ that sends $u(\varepsilon)$ to $\gamma(\varepsilon)w(\varepsilon)$, where

$$\gamma(\varepsilon) = q^{|\varepsilon|(1-N)/2} \prod_{m \in T_\varepsilon} q^{(d_{m+1}+\cdots+d_N)/2}.$$

We check φ preserves the actions of Chevalley generators. Observe that

$$\gamma(\varepsilon) = q^{(N-1)/2} q^{-(d_{m+1}+\cdots+d_N)/2} \gamma(\varepsilon + \widehat{m}), \tag{3.35}$$

for $\varepsilon \in \{0,1\}^N$.

By (3.25) and (3.33) and Lemma 3.12.4 (i), φ preserves the action of k_0. By (3.26) and (3.34) and Lemma 3.12.4 (i), φ preserves the action of k_1. By (3.21), (3.29) and (3.35), the map φ preserves the action of e_0^+. By (3.22), (3.30) and (3.35), the map φ preserves the action of e_1^+. By (3.23), (3.31), (3.35) and Lemma 3.12.4 (ii), the map φ preserves the action of e_0^-. By (3.24), (3.32), (3.35) and Lemma 3.12.4 (ii), the map φ preserves the action of e_1^-.

□

Chapter 4

Association schemes on the Schubert cells of a Grassmannian

In this chapter, let $\mathbb{F}$ be any field. The Grassmannian $\mathrm{Gr}(m, n)$ is the set of m-dimensional subspaces in $\mathbb{F}^n$, and the general linear group $\mathrm{GL}_n(\mathbb{F})$ acts transitively on it. The Schubert cells of $\mathrm{Gr}(m, n)$ are the orbits of the Borel subgroup $\mathcal{B} \subset \mathrm{GL}_n(\mathbb{F})$ on $\mathrm{Gr}(m, n)$. We consider the association scheme on each Schubert cell defined by the $\mathcal{B}$-action and show it is symmetric and it is the *generalized wreath product* of one-class association schemes, which was introduced by R. A. Bailey in [1]. This chapter is based on the author's work [28].

4.1 Preliminaries

We briefly recall the notion of the generalized wreath product of association schemes. For the definition of association schemes, see [2, 16, 30], and for the theory of posets, see [18]. Let $(X, \leq)$ be a nonempty finite poset. A subset Y in X is called an *anti-chain* if any two elements in Y is incomparable. For an anti-chain Y in X, define the *down-set* (also known as the *order ideal*) by

$$\mathrm{Down}(Y) = \{x \in X \mid x < y \text{ for some } y \in Y\}.$$

We note that this definition follows from [1] and it is different from [18], where $\mathrm{Down}(Y) \cup Y$ is called the down-set of Y. For each $x \in X$, let $\mathcal{Q}_x$ denote an r_x-class association scheme on a set Ω_x. We do not assume either $\mathcal{Q}_x$ is symmetric or Ω_x is finite. Let $R_{x,i}$ denote the i-th associate class for $i \in \{0, 1, \ldots, r_x\}$. By convention, we choose the index so that $R_{x,0} = \{(\omega, \omega) \mid \omega \in \Omega_x\}$. We set $\Omega = \prod_{x \in X} \Omega_x$. For each anti-chain Y in X and for each $(i_x)_{x\in Y} \in \prod_{x\in Y}\{1, 2, \ldots, r_x\}$, let $R(Y, (i_x)_{x\in Y})$ denote the set of $((\alpha_x)_{x\in X}, (\beta_x)_{x\in X}) \in \Omega \times \Omega$ satisfying (i) $\alpha_x = \beta_x$ if

$x \in X \setminus (Y \cup \mathrm{Down}(Y))$, and (ii) $(\alpha_x, \beta_x) \in R_{x,i_x}$ if $x \in Y$. Let $\mathcal{R}$ denote the set of $R(Y, (i_x)_{x \in Y})$ for all anti-chains Y in X and $(i_x)_{x \in Y} \in \prod_{x \in Y} \{1, 2, \ldots, r_x\}$.

Theorem 4.1.1 (cf. [1, Theorem 3])**.** *The pair $(\Omega, \mathcal{R})$ is an association scheme.*

The association scheme $(\Omega, \mathcal{R})$ in Theorem 4.1.1 is called the *generalized wreath product of $\mathcal{Q}_x$ over the poset X*. We remark that Bailey [1, Theorem 3] assumes that each base set Ω_x is finite and each association scheme $\mathcal{Q}_x$ is symmetric. However, the theorem is still true if we drop both of these assumptions.

4.2 Subspace lattices

Throughout this chapter, we fix a positive integer n and a field $\mathbb{F}$. Let $\mathbb{F}^n$ denote the n-dimensional column vector space over $\mathbb{F}$. By the *subspace lattice*, denoted by $\mathcal{P}_n(\mathbb{F})$, we mean the poset consisting of all subspaces in $\mathbb{F}^n$ with partial order given by inclusion. We fix the sequence $\{V_i\}_{i=0}^n$ in $\mathcal{P}_n(\mathbb{F})$ such that each V_i consists of vectors whose bottom $n-i$ entries are zero. We remark that $\{V_i\}_{i=0}^n$ is a *(complete) flag* (i.e., a maximal chain) in $\mathcal{P}_n(\mathbb{F})$.

Let $\mathrm{Mat}_n(\mathbb{F})$ denote the set of $n \times n$ matrices with entries in $\mathbb{F}$. Let $\mathrm{GL}_n(\mathbb{F})$ denote the set of invertible matrices in $\mathrm{Mat}_n(\mathbb{F})$. Observe that $\mathrm{GL}_n(\mathbb{F})$ acts on $\mathbb{F}^n$ by left multiplication, and hence it acts on $\mathcal{P}_n(\mathbb{F})$. Let $\mathcal{B}$ denote the (Borel) subgroup in $\mathrm{GL}_n(\mathbb{F})$ stabilizing $\{V_i\}_{i=0}^n$. In other words, $\mathcal{B}$ consists of all upper triangular invertible matrices in $\mathrm{Mat}_n(\mathbb{F})$. In this chapter, we consider the $\mathcal{B}$-action on $\mathcal{P}_n(\mathbb{F})$.

A matrix in $\mathrm{Mat}_n(\mathbb{F})$ is said to be in *reverse column echelon form* if the following two conditions are met:

(CE1) Any zero columns are right of all nonzero columns.

(CE2) The last nonzero entry of a nonzero column is always strictly below of the last nonzero entry of its right column.

A matrix in $\mathrm{Mat}_n(\mathbb{F})$ is said to be in *reduced reverse column echelon form* if it is in reverse column echelon form and the following third condition is also met:

(CE3) Every last nonzero entry of a nonzero column is 1 and is the only nonzero entry in its row.

By elementary linear algebra, we have the following:

Proposition 4.2.1. *There exists a bijection between the following two sets:*

(i) The set of all matrices in $\mathrm{Mat}_n(\mathbb{F})$ *in reduced reverse column echelon form,*

(ii) The set $\mathcal{P}_n(\mathbb{F})$ *of all subspaces in* $\mathbb{F}^n$*,*

that sends a matrix in $\mathrm{Mat}_n(\mathbb{F})$ *to its column space.*

Proof. See for instance [12]. □

Example 4.2.2 ($n = 7$)**.** *Let* $e_1, e_2, \ldots, e_7$ *denote the standard basis for* $\mathbb{F}^7$*. Suppose* U *is the* 4*-dimensional subspace in* $\mathbb{F}^7$ *given by* $U = \mathrm{Span}\{8e_1 + 6e_3 + 4e_6 + 2e_7, 8e_1 + 9e_3 + e_4 + e_5, 4e_1 + e_2 + 5e_3 + e_4, 3e_1 + e_2\}$*. Then the following is the matrix corresponding to* U *by the bijection in Proposition 4.2.1:*

$$M = \begin{pmatrix} 4 & 7 & 1 & 3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 3 & 4 & 5 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Observe that for $M, N \in \mathrm{Mat}_n(\mathbb{F})$ in reduced reverse column echelon form and for $G \in \mathcal{B}$, the column space of N moves to that of M by the G-action if and only if M and GN are column equivalent. Since GN is in reverse column echelon form (but not necessarily reduced), these conditions are equivalent to $M = GNH^t$ for some $H \in \mathcal{B}$. For notational convenience, we write $M \sim N$ if there exist $G, H \in \mathcal{B}$ such that $M = GNH^t$. Observe that $\sim$ is an equivalence relation on $\mathrm{Mat}_n(\mathbb{F})$.

For the rest of this chapter, we will identify $\mathcal{P}_n(\mathbb{F})$ with the set of all matrices in $\mathrm{Mat}_n(\mathbb{F})$ in reduced reverse column echelon form by the bijection in Proposition 4.2.1.

4.3 The $\mathcal{B}$-action on $\mathcal{P}_n(\mathbb{F})$

For a positive integer m, we write $[m] = \{1, 2, \ldots, m\}$. We define a partial order in the index set $[n] \times [n]$ of matrices in $\mathrm{Mat}_n(\mathbb{F})$ by $(i, j) \le (k, l)$ if $i \le k$ and $j \le l$. This is known as the *direct product order* in [18, Section 3.2]. For $M \in \mathrm{Mat}_n(\mathbb{F})$, by the *support* of M, denoted by $\mathrm{Supp}(M)$, we mean the subposet of $[n] \times [n]$ consisting of all indices $(i, j) \in [n] \times [n]$ with $M_{i,j} \neq 0$. The *pivot-set* of M, denoted by $\mathrm{Piv}(M)$, is the set of all maximal elements in $\mathrm{Supp}(M)$. Each element in the pivot-set is called a *pivot*. Observe that $(i, j) \in \mathrm{Piv}(M)$ if and only if $M_{i,j} \neq 0$ and $M_{k,l} = 0$ if

$(k,l) > (i,j)$. We remark that every entry indexed by a pivot of a matrix in $\mathcal{P}_n(\mathbb{F})$ must be 1 by the condition (CE3).

Lemma 4.3.1. *For $M, N \in \mathcal{P}_n(\mathbb{F})$, the following are equivalent.*

(i) $M \sim N$,

(ii) $\mathrm{Piv}(M) = \mathrm{Piv}(N)$.

Proof. (i) ⇒ (ii) Suppose $M \sim N$. There exist $G, H \in \mathcal{B}$ such that $M = GNH^t$. It suffices to show $\mathrm{Piv}(N) \subseteq \mathrm{Piv}(M)$. For $(i,j) \in \mathrm{Piv}(N)$, we have $N_{i,j} = 1$ and $N_{k,l} = 0$ if $(k,l) > (i,j)$. Since G, H are upper triangular, we have

$$M_{k,l} = \sum_{s=k}^{n} \sum_{t=l}^{n} G_{k,s} N_{s,t} H_{l,t} = \begin{cases} G_{i,i} H_{j,j} & \text{if } (k,l) = (i,j), \\ 0 & \text{if } (k,l) > (i,j). \end{cases}$$

Since G, H are invertible, $G_{i,i} H_{j,j} \neq 0$. These imply $(i,j) \in \mathrm{Piv}(M)$ and hence $\mathrm{Piv}(N) \subseteq \mathrm{Piv}(M)$.

(ii) ⇒ (i) Suppose $\mathrm{Piv}(M) = \mathrm{Piv}(N)$. Take $X \in \mathcal{P}_n(\mathbb{F})$ with $X_{i,j} = 1$ if $(i,j) \in \mathrm{Piv}(M)$ and $X_{i,j} = 0$ otherwise. Observe that for each $j \in [n]$, there exists at most one k such that $(j,k) \in \mathrm{Piv}(M)$ and then we define $G \in \mathrm{Mat}_n(\mathbb{F})$ by

$$G_{i,j} = \begin{cases} M_{i,k} & \text{if } (j,k) \in \mathrm{Piv}(M) \text{ for some } k, \\ \delta_{i,j} & \text{if there is no } k \text{ such that } (j,k) \in \mathrm{Piv}(M), \end{cases}$$

for $i, j \in [n]$. Then we have $G_{i,i} = 1$ for $i \in [n]$ and $G_{i,j} = 0$ if $j < i$ for $i, j \in [n]$. Thus $G \in \mathcal{B}$. By the direct calculation, we have $M = GX$ and hence, $M \sim X$. Similarly we have $N \sim X$ and so $M \sim N$. □

Let $1 \leq m \leq n-1$ and $M \in \mathcal{P}_n(\mathbb{F})$ with $\mathrm{rank}\, M = m$. Note that we avoid the trivial cases $m = 0$ and $m = n$. Since the pivots of M lie in the first m columns, $\mathrm{Piv}(M)$ is an anti-chain in $[n] \times [m]$ of size m. For $1 \leq m \leq n-1$ and for an anti-chain α in $[n] \times [m]$ of size m, we set

$$\mathcal{O}_\alpha = \{M \in \mathcal{P}_n(\mathbb{F}) \mid \mathrm{Piv}(M) = \alpha\}. \tag{4.1}$$

For each $1 \leq m \leq n-1$ and each anti-chain α in $[n] \times [m]$ of size m, consider $M \in \mathcal{P}_n(\mathbb{F})$ with $M_{i,j} = 1$ if $(i,j) \in \alpha$ and $M_{i,j} = 0$ otherwise. Then we have $M \in \mathcal{O}_\alpha$ and in particular $\mathcal{O}_\alpha \neq \emptyset$.

Proposition 4.3.2. *The rank of any matrix in (4.1) is $m = |\alpha|$. Moreover, each subset (4.1) is an orbit of the $\mathcal{B}$-action on $\mathcal{P}_n(\mathbb{F})$.*

Proof. Immediate from the construction and Lemma 4.3.1. □

Recall the Grassmannian $\mathrm{Gr}(m,n)$ and we identify $\mathrm{Gr}(m,n)$ with a set of matrices by the bijection in Proposition 4.2.1. In other words,

$$\mathrm{Gr}(m,n) = \{M \in \mathcal{P}_n(\mathbb{F}) \mid \operatorname{rank} M = m\}.$$

By Proposition 4.3.2, each $\mathcal{O}_\alpha$ in (4.1) is a $\mathcal{B}$-orbit in $\mathrm{Gr}(m,n)$, where $m = |\alpha|$. It is called a *Schubert cell of a Grassmannian* [15].

Example 4.3.3 ($n = 7$, $m = 4$)**.** *Take $M \in \mathcal{P}_7(\mathbb{F})$ as in Example 4.2.2. Then we have $\mathrm{Piv}(M) = \{(2,4),(4,3),(5,2),(7,1)\}$. Moreover, $\mathcal{O}_{\mathrm{Piv}(M)}$ is the set of matrices of the form*

$$\begin{pmatrix} * & * & * & * & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ * & * & * & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ * & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \tag{4.2}$$

where the symbol $*$ denotes an arbitrary element in $\mathbb{F}$.

Lemma 4.3.4. *Let $1 \le m \le n-1$ and let α denote an anti-chain in $[n] \times [m]$ of size m. For $M, N, M', N' \in \mathcal{O}_\alpha$, the following are equivalent.*

(i) *(M,N) moves to (M',N') by the diagonal $\mathcal{B}$-action,*

(ii) $\mathrm{Piv}(M-N) = \mathrm{Piv}(M'-N')$.

Proof. (i) $\Rightarrow$ (ii) Suppose there exist $G, H, K \in \mathcal{B}$ such that $M' = GMH^T$ and $N' = GNK^t$. Then we have

$$\mathrm{Piv}(GM) = \mathrm{Piv}(GN) = \alpha, \tag{4.3}$$

$$\mathrm{Piv}(GM - GN) = \mathrm{Piv}(M - N), \tag{4.4}$$

since $G \in \mathcal{B}$ (cf. Lemma 4.3.1). We write $\alpha = \{(k_r, r) \mid r \in [m]\}$ and observe that $k_1 > k_2 > \cdots > k_m$ and that $\mathrm{Piv}(M-N) \subseteq \mathrm{Down}(\alpha)$.

Take $(i,j) \in \mathrm{Piv}(M-N)$. Observe that $j \in [m]$ and $k_j > i$ and hence there exists $m' = \max\{r \in [m] \mid k_r > i\}$. For $r, l \in [m]$ with $j \le r \le m'$ and $j \le l \le m'$, since H, K are upper triangular, we have

$$M'_{k_r,l} = \sum_{t=l}^{n} (GM)_{k_r,t} H_{l,t},$$

$$N'_{k_r,l} = \sum_{t=l}^{n} (GN)_{k_r,t} K_{l,t}.$$

In the above equations, we have the following: For each r, l, we have $M'_{k_r,l} = N'_{k_r,l} = \delta_{r,l}$ by (CE3); For each r, t, we have $(GM)_{k_r,t} = (GN)_{k_r,t}$ since $(k_r, t) > (i, j)$ and by (4.4); For each r, t with $r < t$, we have $(GM)_{k_r,r} = (GN)_{k_r,r} = G_{k_r,k_r} \neq 0$ and $(GM)_{k_r,t} = (GN)_{k_r,t} = 0$ since $(k_r, r) \in \alpha$ and by (4.3). By these comments, for each $l \in [m]$ with $j \leq l \leq m'$, both $(H_{l,l}, H_{l,l+1}, \ldots, H_{l,m'})$ and $(K_{l,l}, K_{l,l+1}, \ldots, K_{l,m'})$ are solutions to the same system of $m' - j + 1$ independent linear equations. Hence, $H_{l,t} = K_{l,t}$ for $j, t \in [m]$ with $j \leq l \leq t \leq m'$.

For $(k, l) \in [n] \times [n]$ with $(k, l) > (i, j)$, we have $(GM)_{k,t} = (GN)_{k,t}$ if $t \geq l$ since $(k, t) > (i, j)$ and by (4.4), and we also have $(GM)_{k,t} = (GN)_{k,t} = 0$ if $t > m'$ by the definition of m' and by (4.3). Recall that we have shown $H_{l,t} = K_{l,t}$ if $j \leq l \leq t \leq m'$. By these comments, we have

$$M'_{k,l} = \sum_{t=l}^{n} (GM)_{k,t} H_{l,t} = \sum_{t=l}^{n} (GN)_{k,t} K_{l,t} = N'_{k,l}.$$

Similarly, we have

$$M'_{i,j} - N'_{i,j} = \sum_{t=j}^{n} (GM)_{i,t} H_{j,t} - \sum_{t=j}^{n} (GN)_{i,t} K_{j,t} = ((GM)_{i,j} - (GN)_{i,j}) H_{j,j}.$$

In the above equations, we have $(GM)_{i,j} \neq (GN)_{i,j}$ by (4.4), and we also have $H_{j,j} \neq 0$ since $H \in \mathcal{B}$. Therefore we obtain $M'_{i,j} \neq N'_{i,j}$. These imply $(i, j) \in \operatorname{Piv}(M' - N')$ and hence $\operatorname{Piv}(M - N) \subseteq \operatorname{Piv}(M' - N')$. Since G, H, K are invertible, we also have $\operatorname{Piv}(M' - N') \subseteq \operatorname{Piv}(M - N)$. Consequently, we have $\operatorname{Piv}(M - N) = \operatorname{Piv}(M' - N')$.

(ii) $\Rightarrow$ (i) Let $M, N, M', N' \in \mathcal{O}_\alpha$ with $\operatorname{Piv}(M - N) = \operatorname{Piv}(M' - N')$. Take $X \in \mathcal{O}_\alpha$ with $X_{i,j} = 1$ if $(i, j) \in \alpha \cup \operatorname{Piv}(M - N)$ and $X_{i,j} = 0$ otherwise, and $Y \in \mathcal{O}_\alpha$ with $Y_{i,j} = 1$ if $(i, j) \in \alpha$ and $Y_{i,j} = 0$ otherwise. Define $G \in \operatorname{Mat}_n(\mathbb{F})$ by

$$G_{i,j} = \begin{cases} N_{i,k} & \text{if } (j,k) \in \alpha \text{ for some } k, \\ M_{i,k} - N_{i,k} & \text{if } (j,k) \in \operatorname{Piv}(M-N) \text{ for some } k, \\ \delta_{i,j} & \text{if there is no } k \text{ such that } (j,k) \in \alpha \cup \operatorname{Piv}(M-N), \end{cases}$$

for $i, j \in [n]$. Then we have $G \in \mathcal{B}$ and $M = GX$ and $N = GY$. Similarly there exists $G' \in \mathcal{B}$ such that $M' = G'X$ and $N' = G'Y$. Therefore (M, N) moves to (M', N') by the diagonal action of $G'G^{-1}$. □

Let $1 \leq m \leq n - 1$ and let α denote an anti-chain in $[n] \times [m]$ of size m. Let $M, N \in \mathcal{O}_\alpha$. Observe that $\operatorname{Piv}(M - N)$ is an anti-chain in

$$\mathcal{D}(\alpha) = \{(i, j) \in \operatorname{Down}(\alpha) \mid \text{there is no } k \text{ such that } (i, k) \in \alpha\}. \tag{4.5}$$

For $1 \le m \le n-1$ and for an anti-chain α in $[n]\times[m]$ of size m and for an anti-chain β in $\mathcal{D}(\alpha)$, we set

$$\mathcal{R}_{\alpha,\beta} = \{(M,N) \in \mathcal{O}_\alpha \times \mathcal{O}_\alpha \mid \mathrm{Piv}(M-N) = \beta\}. \tag{4.6}$$

For each $1 \le m \le n-1$, each anti-chain α in $[n]\times[m]$ of size m and each anti-chain β in $\mathcal{D}(\alpha)$, consider $M \in \mathcal{P}_n(\mathbb{F})$ with $M_{i,j} = 1$ if $(i,j) \in \alpha \cup \beta$ and $M_{i,j} = 0$ otherwise, and $N \in \mathcal{P}_n(\mathbb{F})$ with $N_{i,j} = 1$ if $(i,j) \in \alpha$ and $N_{i,j} = 0$ otherwise. Then we have $(M,N) \in \mathcal{R}_{\alpha,\beta}$ and in particular $\mathcal{R}_{\alpha,\beta} \ne \emptyset$.

Proposition 4.3.5. *Let $1 \le m \le n-1$ and let α denote an anti-chain in $[n]\times[m]$ of size m. Each subset (4.6) is an orbital of the $\mathcal{B}$-action on $\mathcal{O}_\alpha$.*

Proof. Immediate from Lemma 4.3.4. □

Let $1 \le m \le n-1$. For an anti-chain α in $[n]\times[m]$ of size m, consider

$$\mathcal{D}_1(\alpha) = \{i \mid \text{there is no } k \text{ such that } (i,k) \in \alpha\}. \tag{4.7}$$

Then we have $|\mathcal{D}_1(\alpha)| = n-m$ since $|\alpha| = m$. For $1 \le i \le n-m$, we define $\lambda_i = |\{j \mid (d_i, j) \in \mathcal{D}(\alpha)\}|$, where d_i denotes the i-th smallest element in $\mathcal{D}_1(\alpha)$. Then $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_{n-m}) \in \mathbb{N}^{n-m}$ is an integer partition (i.e., a non-increasing sequence) with largest part at most m, where

$$\mathbb{N} = \{0, 1, \ldots\}.$$

Consider the map φ_m which sends α to λ.

For an integer partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_l)$, the *Ferrers board* of shape λ is defined by

$$\{(i,j) \in \mathbb{N}\times\mathbb{N} \mid 1 \le i \le l, 1 \le j \le \lambda_i\}.$$

We endow the Ferrers board with direct product order in $\mathbb{N}\times\mathbb{N}$.

Lemma 4.3.6. *For $1 \le m \le n-1$, the map φ_m is a bijection between the following two sets:*

(i) The set of anti-chains in $[n]\times[m]$ of size m.

(ii) The set of integer partitions in $\mathbb{N}^{n-m}$ with largest part at most m.

Proof. Let $1 \le m \le n-1$. It is clear that φ_m is a map from (i) to (ii). We define the map φ'_m from (ii) to (i) as follows. For a given integer partition λ in $\mathbb{N}^{n-m}$ with largest part at most m, we define α as the set of maximal elements in the Ferrers board of shape $\mu = \lambda \cup (m, m-1, \ldots, 1)$, which is the integer partition obtained by rearranging parts of both λ and $(m, m-1, \ldots, 1)$ in non-increasing order. Since μ is in $\mathbb{N}^n$ and its largest part is m, α is an anti-chain in $[n] \times [m]$ of size m. The map φ'_m is defined to send λ to α. By construction, φ_m and φ'_m are inverses and hence bijections. □

Lemma 4.3.7. *Let $1 \leq m \leq n-1$ and let α denote an anti-chain in $[n] \times [m]$ of size m. The poset $\mathcal{D}(\alpha)$ in (4.5) is isomorphic to the Ferrers board of shape $\varphi_m(\alpha)$. Moreover, there is a one-to-one correspondence between the anti-chains in $\mathcal{D}(\alpha)$ and the subpartitions of $\varphi_m(\alpha)$.*

Proof. Recall the set $\mathcal{D}_1(\alpha)$ in (4.7). Then define the map ψ from the Ferrers board of shape $\varphi_m(\alpha)$ to $\mathcal{D}(\alpha)$ by $\psi(i,j) = (d_i, j)$, where d_i denotes the i-th smallest element in $\mathcal{D}_1(\alpha)$. It is obvious that ψ is an order-preserving bijection. The first assertion follows. To show the second assertion, we define the map ρ from the subpartitions of $\varphi_m(\alpha)$ to the anti-chains in $\mathcal{D}(\alpha)$ by $\rho(\mu) = \psi(\max(\mu))$, where $\max(\mu)$ is the set of all maximal elements in the Ferrers board of shape μ. From the construction, the map ρ is also a bijection. The second assertion follows. □

Example 4.3.8 $(n = 7, m = 4)$**.** *Take the anti-chain $\alpha = \{(2,4), (4,3), (5,2), (7,1)\}$ as in Example 4.3.3. Recall that $\mathcal{O}_\alpha$ is the set of matrices of the form (4.2). Then $\varphi_4(\alpha) = (4,3,1)$. We remark that each number in $(4,3,1)$ equals the number of $*$'s in each row without a pivot.*

4.4 The association scheme on each Schubert cell

For $1 \leq m \leq n-1$ and for an anti-chain α in $[n] \times [m]$ of size m, by Propositions 4.3.2 and 4.3.5, the pair

$$\mathfrak{X}_\alpha = (\mathcal{O}_\alpha, \{\mathcal{R}_{\alpha,\beta}\}_\beta), \tag{4.8}$$

becomes an association scheme, where β runs over all anti-chains in $\mathcal{D}(\alpha)$ in (4.5). See [30, Preface]. We remark that by Lemma 4.3.6, the family of association schemes $\{\mathfrak{X}_\alpha\}_\alpha$ can be indexed by integer partitions $\lambda \in \mathbb{N}^{n-m}$ with largest part at most m. In this case, the associate classes of $\mathfrak{X}_\alpha = \mathfrak{X}_\lambda$ are indexed by the subpartitions of λ by Lemma 4.3.7.

Theorem 4.4.1. *Let $1 \leq m \leq n-1$ and let α denote an anti-chain in $[n] \times [m]$ of size m. The association scheme $\mathfrak{X}_\alpha$ in (4.8) is symmetric.*

Proof. Immediate from the definition of $\mathcal{R}_{\alpha,\beta}$. □

Theorem 4.4.2. *For $1 \leq m \leq n-1$ and for an anti-chain α in $[n] \times [m]$ of size m, the association scheme $\mathfrak{X}_\alpha$ in (4.8) is the generalized wreath product of the one-class association schemes with the base set $\mathbb{F}$ over the poset $\mathcal{D}(\alpha)$ in (4.5).*

Proof. For $M, N \in \mathcal{O}_\alpha$ and for an anti-chain β in $\mathcal{D}(\alpha)$, we have $(M, N) \in \mathcal{R}_{\alpha,\beta}$ if and only if $M_{i,j} = N_{i,j}$ if $(i, j) \notin \beta \cup \mathrm{Down}(\beta)$, $M_{i,j} \neq N_{i,j}$ if $(i, j) \in \beta$. Therefore, this associate relation is the same as that of the generalized wreath product of the one-class association schemes with the base set $\mathbb{F}$ over the poset $\mathcal{D}(\alpha)$. So the result follows. □

4.5 Concluding remarks

This chapter focuses on the Schubert cells of a Grassmannian. It would be an interesting problem to find similar results on Schubert cells for other types of BN-pairs.

The Terwilliger algebra, introduced by P. Terwilliger [24], of the wreath product of one-class association schemes is discussed in several papers [4, 21, 29]. We will consider the Terwilliger algebra of the generalized wreath product of one-class association schemes in a future paper.

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