Dualities and generalized gauge structures in string theory and M-theory

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Doctoral Thesis

Dualities and generalized gauge structures in string theory and M-theory

(弦理論とM理論における双対性と一般化されたゲージ構造)

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Abstract

The present thesis is divided into two parts, which are loosely connected.

Dualities in string theory and M-theory

In the first part, we analyze the underlying mathematical structures of T-duality in toroidal compactifications of string theory and U-duality in M-theory compactifications. The analysis is conducted by using supergeometric methods on graded symplectic manifolds.
We derive a fully twisted Courant algebroid, which encodes all T-dual H- and F- as well as non-geometric Q- and R-fluxes and analyze its underlying cohomology. Then, we construct the underlying graded symplectic manifold, which encodes all generalized fluxes of double field theory, and propose a definition of T-duality in the graded manifold framework. Reductions of the graded manifold correspond to twisted Courant algebroids living on T-dual frames.
After that, we analyze the structure of the Poisson-Courant algebroid, a Courant algebroid on a Poisson manifold, as a model for non-geometric R-flux. We compute its cohomology and relate it to standard Courant algebroid cohomology. Furthermore, we work out its relation to double field theory as a T-duality frame. We construct a topological membrane model, a topological string sigma model and a current algebra based on the Poisson-Courant algebroid with R-flux. Then, we construct a transformation, called *flux duality*, between Courant algebroid with H-flux and the Poisson-Courant algebroid with R-flux, which lifts to an isomorphism of Courant algebroid cohomologies and topological sigma models.
Finally, we investigate the local symmetry L_∞-algebras of higher abelian gerbes underlying several structures related to T-duality and U-duality geometry: B_n-generalized geometry and exceptional generalized geometry.

Higher gauge theory and multiple M5-branes

In the second part, we analyze classes of 2-form higher gauge theories and propose a method to construct higher gauge theories that circumvent the fake curvature condition, called *off-shell covariantization*. Using this method we successfully construct an off-shell covariant 2-form higher gauge theory, which is related to the system of multiple M5-branes compactified on a circle. The method is based on supergeometry via graded symplectic manifolds.

Contents

Notation

Symbol	Meaning
$\mathcal{C}^\infty(M)$	Space of smooth functions on M
$\mathfrak{X}^k(M)$	Space of polyvectors of degree k over M
$\Omega^k(M)$	Space of polyforms of degree k over M
L_X	Lie derivative along the vector X
ι_X	Interior product on differential forms by the vector X
ι_α	Interior product on polyvectors by the 1-form α
d	de Rham differential
d_Π	Lichnerowicz-Poisson differential with respect to the Poisson tensor Π
$\Pi^\sharp$	Interior product of a bivector Π on a form α by $\alpha \mapsto \iota_\alpha \Pi$
$\langle -, - \rangle$	Fiber product
$[-,-]_{\mathrm{Lie}}$	Lie bracket
$[-,-]_{\mathrm{D}}$	Dorfman bracket
$[-,-]_{\mathrm{C}}$	Courant bracket
$[-,-]_{\mathrm{S}}$	Schouten bracket
$[-,-]_{\Pi}$	Koszul bracket with respect to the Poisson bivector Π
$\{-,-\}$	(Graded) Poisson bracket
$\{-,-\}_{\mathrm{BV}}$	BV antibracket
$\{-,-\}_{\mathrm{PB}}$	Ordinary Poisson bracket
ω	(Graded) symplectic structure
X_f	Hamiltonian function with respect to the function f
χ	Superworldvolume
X	Worldvolume
Σ	Worldsheet
$\mathring{\chi}$	Spatial part of superworldsheet
μ_χ	Berezin measure on χ
Ω	Mapping space graded symplectic structure

Hamiltonian function	Induced structure
Θ_H	H-twisted standard Courant algebroid
Θ_R	R-twisted Poisson-Courant algebroid
Θ_β	β-twisted standard Courant algebroid
Θ_f	f-twisted standard Courant algebroid
Θ_0	untwisted standard Courant algebroid
$\Theta_{B\beta e}$	Fully twisted standard Courant algebroid
$\widehat{\Theta}_0$	Double field theory gauge algebra

Elements of supergeometric mapping spaces are denoted by bold face letters, e.g. $\boldsymbol{x}^i$, $\boldsymbol{\xi}^i$, $\boldsymbol{\zeta}_i$ and $\boldsymbol{\zeta}_i$.

Antisymmetrization over indices is denoted by $[\cdots]$, whereas symmetrization over indices is denoted by $(\cdots)$. Both manipulations are accompanied with the appropriate combinatorial factor. For example, the total antisymmetrization and symmetrization of a tensor $M_{i_1\cdots i_n}$ is denoted by

$$M_{[i_1\cdots i_n]} = \frac{1}{n!}\sum_{\sigma} \mathrm{sgn}(\sigma) M_{\sigma(i_1)\cdots\sigma(\imath_n)},$$
$$M_{(i_1\cdots i_n)} = \frac{1}{n!}\sum_{\sigma} M_{\sigma(i_1)\cdots\sigma(\imath_n)}.$$

Chapter 1

Introduction

> *Veritatem inquirenti, semel in vita de omnibus, quantum fieri potest, esse dubitandum.*
>
> – René Descartes, *Principia philosophiae* (1644)

The present thesis *Dualities and generalized gauge structures in string and M-theory* is divided into two main parts, which are thematically loosely connected. For the sake of brevity, the following introduction shall serve the very reader as superficial orientation regarding where the two main parts are allocated in the realm of string theory and M-theory. Each of the two parts provides a specialized introduction into the respective subject.

String theory is the most developed candidate for a theory of quantum gravity as of now. Not only it contains a graviton in its spectrum, but also it is free from UV-divergences. However, supersymmetric string theory, or superstring theory, must be formulated in $9+1$ spacetime dimensions in order to be anomaly-free on quantum level. To make contact with our 4-dimensional observable world, 6 of the 9 spatial dimensions have to be compactified. It turns out that compactified string theory exhibits a special symmetry, which arises due to the fact that its fundamental objects are 1-dimensional strings. This symmetry is called *T-duality*. Strings perceive space and geometry differently compared to point-objects. This fact leads to mysterious compactification spaces, which lie beyond the ordinary concept of manifolds. New mathematical techniques have to be developed in order to capture the characteristics of this novel *stringy geometry*.

It furthermore turned out, that there are many different string theories, which are all incarnations of a single underlying 11-dimensional theory. This theory is called *M-theory*. Although its UV-completion is unknown as of now, its low-energy effective theory is known as 11-dimensional supergravity. In M-theory, T-duality is combined with another (strong-weak coupling) symmetry of string theory, called *S-duality*, to the fundamental symmetry of M-theory, *U-duality*.

There is no notion of strings in M-theory. However, it contains stable soliton solutions, the so-called *M2- and M5-branes*. These are dynamic objects, extended in 2 and 5 spatial dimensions, respectively. As of now, the dynamics of multiple M2-branes are already very well understood. However, the correct description of the dynamics of multiple M5-branes is still unknown. It is believed that it is governed by a non-abelianization of a gerbe. The underlying mathematical theory is believed to be a so-called *higher gauge theory*. A higher gauge theory is a generalization of ordinary Yang-Mills gauge theory by making use of *higher categorification* and contains arbitrary n-form gauge fields and higher field strengths that take values in categorified Lie algebras.

In the first part *Dualities in string theory and M-theory* we head off to analyze the underlying mathematical structures of T-duality in string theory and U-duality in M-theory. It is based on the published papers [1, 2, 3] and the preliminary results of [4], which is work in progress. In the second part *Higher gauge theory and multiple M5-branes* we will investigate higher gauge theories and their relation to multiple M5-brane systems in M-theory. More precisely, we will construct a 2-form higher gauge theory, which can be related to a system of multiple M5-branes compactified on a circle, using our newly proposed method of *off-shell covariantization*. It is based on the published paper [5].

Structure of the thesis

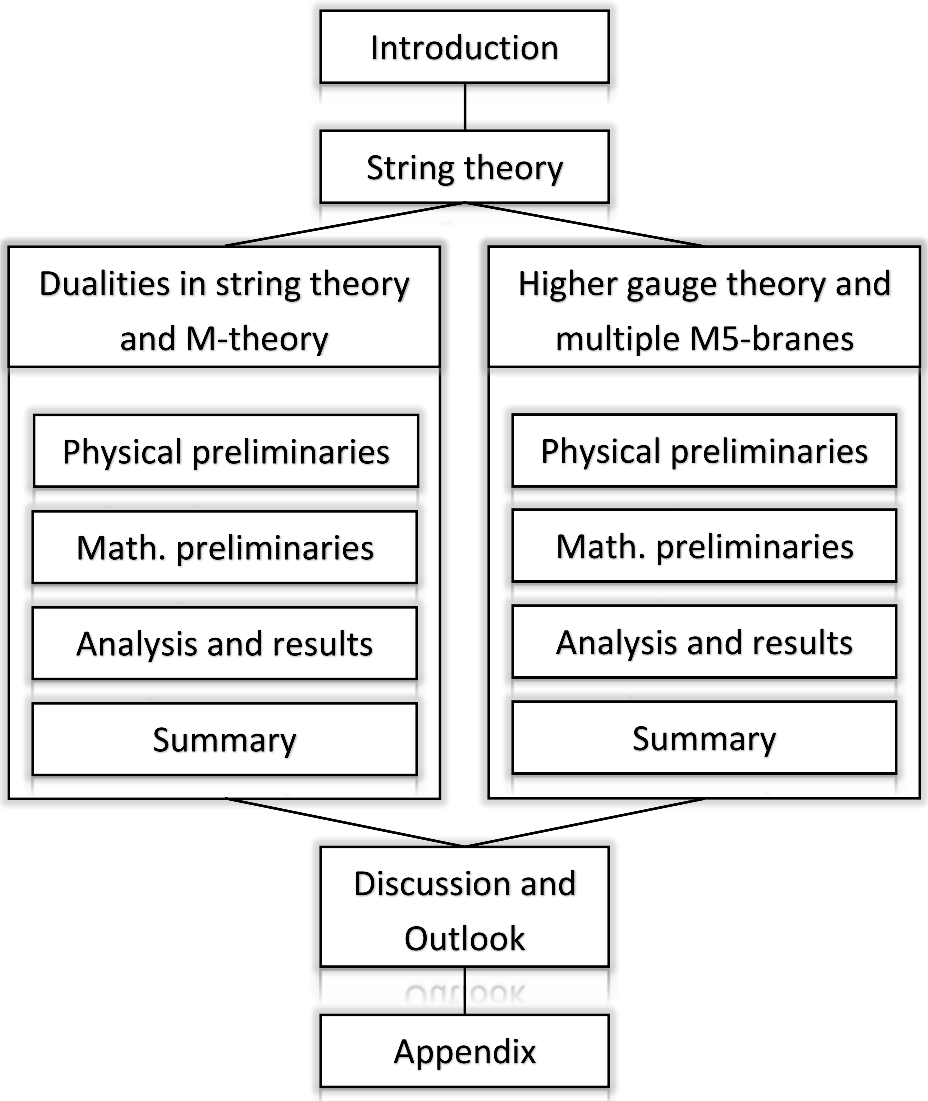
The present thesis is divided into 5 chapters and an appendix.

1.	Introduction
2.	String theory: A synopsis
3.	Dualities in string theory and M-theory
4.	Higher gauge theory and multiple M5-branes
5.	Discussion and outlook
	Appendix

After the *Introduction*, the preliminary chapter on *string theory* shall serve the reader as overview of the foundations on which the two main chapters of this thesis, *Dualities in string theory and M-theory* and *Higher gauge theory and multiple M5-branes*, shall be built. The chapter *Discussion and outlook* recalls and discusses the main results of both main chapters as well as open problems and projects directions of future work.
Each of the main chapters are structured as follows.

1.	Introduction
2.	Physical preliminaries
3.	Mathematical preliminaries
4.	Analysis and results
5.	Summary

The *Introduction* sets the stage for each main chapter and serves as a physical and mathematical orientation. It is followed by the *Physical preliminaries* and *Mathematical preliminaries*, which provide in a self-consistent manner not only the string and M-theoretical setting surrounding the main content, but also the mathematical groundwork. Due to its beauty and charm, the mathematical sections are especially designed to stress the many relations among the various subjects. The sections after the preliminaries contain the *Analysis and results*, which constitute the heart of this thesis. The final *Summary* can be seen as assembly of the various results obtained and shall serve the reader as magnifying glass on the main extract of the respective main chapter.



Structure of the thesis

Chapter 2

String theory: A synopsis

> *Es ist so bequem, unmündig zu sein. Habe ich ein Buch, das für mich Verstand hat, einen Seelsorger, der für mich Gewissen hat, einen Arzt, der für mich die Diät beurtheilt u.s.w., so brauche ich mich ja nicht selbst zu bemühen. Ich habe nicht nöthig zu denken, wenn ich nur bezahlen kann.*
>
> – Immanuel Kant, *Beantwortung der Frage: Was ist Aufklärung?*

In this chapter, we provide a superficial overview of the foundations of string theory and its main ingredients. We start by clarifying why we are in need of a theory of quantum gravity and why string theory is an excellent candidate for such a theory. Then, we introduce string sigma models and comment on bosonic string theory. After that, we provide a small survey on the various incarnations of string theory. This is followed by short expositions on dualities and D-branes. Finally, we touch M-theory, which underlies all string theory incarnations. Standard references are [6, 7, 8, 9]. Further excellent introductions are [10, 11].

Setting the stage

The last century with its many physical breakthroughs and paradigm changes cannot be underestimated in its significance. The emergence of the standard model of particle physics, which governs the physics on tremendously small scales, and the formulation of general relativity, providing us with an amazingly exact theory for unimaginably large scales, brought with it not only practical, but also cultural progress.

The standard model of particle physics is the result of huge developments over decades. Its final form is the renormalizable $SU(3)\times SU(2)\times U(1)$ gauge theory with spin-1 gauge bosons mediating three of the four fundamental forces: electromagnetic force, weak force and strong force. Although it is a remarkably exact theory with an incredible power to predict the physics on very small scales, there are several indications that it is not the final theory.

First, there are between 18 and 25 free parameters in the standard model Lagrangian, which have to be fine-tuned by experiments. It turns out that some of parameters differ by several orders of magnitude posing the so-called *naturalness problem*. For aesthetic reasons, a *natural* theory should possess the property, that ratios of the parameters used in its formulation are of order 1. Second, in the standard model with Higgs boson, the comparison of the strength of the gravitational force and the electro-weak interaction poses the question, why the Planck scale and the electro-weak scale are differing by 16 orders of magnitude. This is called the *hierarchy problem* and is closely related to the naturalness problem. Third, the standard model of particle physics does not incorporate gravity. The naïve approach to quantize gravity leads to a non-renormalizable theory. It contains short-distance divergences.

What we are seeking for is a natural principle from which a consistent theory can be derived, which can explain all the fundamental interactions on quantum level. Such a theory would reproduce and generalize the success of the standard model of particle physics as a quantum field theory and combine it with general relativity in a manner that it stays consistent after quantization. The resulting theory would be a *theory of quantum gravity*. Among other approaches arrive at a consistent theory of quantum gravity, **string theory** turns out to be the best candidate at present. One remarkable property of string theory is that it does not contain any free parameters.

In contrast to ordinary quantum field theory, the fundamental objects of string theory are 1-dimensional strings, which can be open or closed. Due to the fact that the fundamental objects gain one dimension compared to ordinary particles, the interaction vertices between strings are not located at one distinct spacetime point, but smeared out in spacetime. This ultimately cures the UV-divergences. However, we trade locality for this property. So we conclude, that the worldline of ordinary particles in spacetime generalizes to a worldsheet of strings. Ordinary Feynman graphs of particle interactions generalize to interaction surfaces, which can have various topologies.

The zoo of particles, which we try to analyze using the standard model of particle physics, becomes nothing but the many different energy states of open and closed strings traveling through spacetime. Due to their 1-dimensionality, strings contain internal oscillatory excitations, which contribute to their overall energy. To each state, an emergent particle is associated. It turns out, that on quantum level string theory naturally contains a massless spin-2 state, which can be associated with the graviton, the quantum field of gravity. In the low-energy limit, the interaction surfaces can be approximated with Feynman diagrams and ordinary general relativity emerges.

The anomaly-free formulation of the bosonic part of string theory requires $25+1$ spacetime dimensions. In the superstring theory case, $9+1$ spacetime dimensions are required for a consistent formulation on quantum level. In order to relate string theory to our 4-dimensional observable world, 6 dimensions of the spatial 9 dimensions are compactified. This leads to a theory with 4 non-compact and 6 compact dimensions. The topology of the 6-dimensional internal space governs the physics on the non-compact space.

String sigma models

The string as 1-dimensional object sweeps out a 2-dimensional surface, called worldsheet. To get warmed up, let us discuss the embedding of a string worldsheet into a d-dimensional target spacetime M being a smooth manifold with Minkowski metric $\eta_{\mu\nu}$. The embedding functions X^μ depend on local coordinates (τ, σ) of the worldsheet Σ and the index μ runs from 1 to d. Then, the embedding is described by the *Nambu-Goto action*,

$$S_{\text{NG}} = -\frac{1}{2\pi\alpha'} \int_\Sigma d\tau \wedge d\sigma \sqrt{-h}, \tag{2.1}$$

where $h = \det(h_{ab})$ is the determinant of the induced metric,

$$h_{ab} = \eta_{\mu\nu} \frac{\partial X^\mu}{\partial y^a} \frac{\partial X^\nu}{\partial y^b}, \tag{2.2}$$

where a and b run over 1 and 2 and $(y^1, y^2) = (\tau, \sigma)$. The parameter α' denotes the *Regge slope* and is related to the string tension T and string length l_S by

$$T = \frac{1}{2\pi\alpha'}, \quad l_S = \sqrt{\alpha'}. \tag{2.3}$$

This is the simplest invariant action, which describes the embedding of a 2-dimensional string worldsheet into a target space manifold, thus generalizing the worldline embeddings of particle trajectories in general relativity. It is invariant under Poincaré transformations of the target space as well as worldsheet diffeomorphisms. Furthermore, it is equivalent to the so-called *Polyakov action*,

$$S_{\mathrm{P}} = -\frac{1}{4\pi\alpha'}\int_{\Sigma} d\tau \wedge d\sigma \sqrt{-\gamma}\gamma^{ab}\partial_a X^{\mu}\partial_b X^{\nu}\eta_{\mu\nu}, \tag{2.4}$$

where γ^{ab} is an independent metric on the worldsheet. Upon elimination of γ, the Nambu-Goto action can be recovered. The Polyakov action is invariant under Poincaré transformations of the target spacetime, worldsheet diffeomorphisms and local Weyl transformations of the worldsheet metric. After gauge fixing the diffeomorphism and Weyl invariance by setting $\gamma_{ab} = \eta_{ab}$, the Polyakov action gives

$$S'_{\mathrm{P}} = -\frac{1}{4\pi\alpha'}\int_{\Sigma} d\tau \wedge d\sigma \partial_a X^{\mu}\partial^a X_{\mu}, \tag{2.5}$$

which is an action of free scalar fields. It turns out, that the quantum theory associated with the Polyakov action contains a Weyl anomaly if the dimension of the target spacetime is different from 26. In the case of the critical dimension 26, Weyl invariance is restored at quantum level, which leads to an anomaly-free quantum theory of the string.

When investigating the lowest excitations of the quantum string, it turns out that the ground state is tachyonic with mass

$$m^2 = -\frac{1}{\alpha'}\frac{d-2}{6}. \tag{2.6}$$

However, upon introducing supersymmetry, the tachyonic ground state can be cured. This is the starting point of superstring theory. Recall that supersymmetry introduces fermionic excitations such that the freeness of anomalies on quantum level is restored if the target spacetime dimension is 10.

The next higher excitations beyond the tachyon are the massless states living in the irreducible representations of the $\mathbf{24} \otimes \mathbf{24}$ of $SO(24)$. In the symmetric traceless representation, we find a massless spin-2 particle, given by the field $g_{\mu\nu}$, which can be associated with the spacetime metric and yields the graviton. In the anti-symmetric representation we find the so-called Kalb-Ramond field or B-field, given by the antisymmetric tensor $B_{\mu\nu}$. In the trace representation we find a scalar field ϕ, which is called the dilaton.

Let us shortly summarize. The critical dimension of bosonic string theory is 26 and the lowest excitations are a ground state tachyon, the metric g, the B-field and the dilaton ϕ.

When we use the target spacetime metric $g_{\mu\nu}$ instead of the Minkowski metric $\eta_{\mu\nu}$ in the

Polyakov action, we arrive at the action of a *non-linear sigma model*,

$$S_\sigma = \frac{1}{4\pi\alpha'}\int_\Sigma d\tau\wedge d\sigma\sqrt{\gamma}\gamma^{ab}g_{\mu\nu}(X)\partial_a X^\mu\partial_b X^\nu. \tag{2.7}$$

This action can be generalized to incorporate also B-field and dilaton excitations of the string,

$$S_\sigma = \frac{1}{4\pi\alpha'}\int_\Sigma d\tau\wedge d\sigma\sqrt{\gamma}\left(\gamma^{ab}g_{\mu\nu}(X)\partial_a X^\mu\partial_b X^\nu + i\epsilon^{ab}B_{\mu\nu}(X)\partial_a X^\mu\partial_b X^\nu + \alpha'\mathcal{R}\phi(X)\right). \tag{2.8}$$

Here, ϵ^{ab} is the 2-dimensional epsilon-tensor and $\mathcal{R}$ denotes the Ricci scalar with respect to the worldsheet metric. Computation of the β-functions of the worldsheet theory at linear order in α' in critical dimension $d = 26$ gives [6]

$$\beta^g_{\mu\nu} = \alpha' R_{\mu\nu} + 2\alpha'\nabla_\mu\nabla_\nu\phi - \frac{\alpha'}{4}H_{\mu\lambda\rho}H_\nu^{\lambda\rho}, \tag{2.9}$$

$$\beta^B_{\mu\nu} = -\frac{\alpha'}{2}\nabla^\rho H_{\rho\mu\nu} + \alpha'\nabla^\rho\phi H_{\rho\mu\nu}, \tag{2.10}$$

$$\beta^\phi = -\frac{\alpha'}{2}\nabla^2\phi + \alpha'\nabla_\rho\phi\nabla^\rho\phi - \frac{\alpha'}{24}H_{\mu\nu\lambda}H^{\mu\nu\lambda}, \tag{2.11}$$

where ∇ denotes the covariant derivative and $R_{\mu\nu}$ is the Ricci tensor with respect to the spacetime metric. The tensor H is the 3-form field strength of the 2-form B-field, related by $H = dB$. Weyl invariance of the worldsheet theory then implies the vanishing of the β-functions,

$$\beta^g_{\mu\nu} = 0, \quad \beta^B_{\mu\nu} = 0, \quad \beta^\phi = 0. \tag{2.12}$$

The β-functions can also be implied by variation of the following spacetime supergravity action, which describes the so-called NS-NS sector of type II (closed) superstring theory,

$$S_{\mathrm{II}} = \int_M d^d x\sqrt{g}e^{-2\phi}\left(R + 4\partial_\mu\phi\partial^\mu\phi - \frac{1}{12}H_{\mu\nu\lambda}H^{\mu\nu\lambda}\right). \tag{2.13}$$

This action is invariant under diffeomorphisms induced by Lie derivative L_ξ, where ξ^μ is an infinitesimal vector, and B-field gauge transformations by a 1-form λ,

$$B \mapsto B + d\lambda. \tag{2.14}$$

It turns out that in the closed string case, the string coupling constant g_S is related to the dilaton by $g_S = e^\phi$. Therefore, it is not an independent parameter in string theory and can be dynamical.

If the bosonic worldsheet action of the string is extended to include matter fermions, it leads to an $\mathcal{N} = (1,1)$ superconformal 2-dimensional field theory. In this case, there are two different periodicity conditions for the matter fermions. The periodic one refers to the Ramond boundary conditions (R), whereas the antiperiodic one refers to the Neveau-Schwarz boundary conditions (NS). Since there are two different center-of-mass modes associated with the fermionic matter, it leads to $2 \times 2 = 4$ different Hilbert spaces on the circle, referred to as NS-NS, NS-R, R-NS and R-R sectors. In closed string theory, the NS-NS and R-R states are bosons, whereas the NS-R and R-NS states are fermions. As we will see in the following, there are various gauge potentials in superstring theory, which are associated to the R-R sector.

Incarnations of string theory

Let us put the action (2.13) into context by presenting the various incarnations of supersymmetric string theories, or superstring theories for short. As stated before, the use of supersymmetry brings us into the position to add fermions and cures the tachyonic ground state. The critical dimension of superstring theory is 10. However, there are 5 different consistent superstring theories in critical dimension.

The so-called type II string theory contains left- and right-moving fermions, 32 supercharges ($\mathcal{N} = 2$ supersymmetry) and describes the closed oriented string. There are two versions of type II string theories, the non-chiral type IIA string theory with $\mathcal{N} = (1,1)$ supersymmetry and the chiral type IIB string theory with $\mathcal{N} = (2,0)$ supersymmetry. Furthermore, there are the so-called heterotic string theories. They describe the closed string and contain right-moving fermions with 16 supercharges ($\mathcal{N} = 1$ supersymmetry). In contrast to type II string theories, heterotic string theories contain an additional gauge group in 10 dimensions. The vanishing of the quantum anomaly restricts the only possible gauge groups to be $SO(32)$ and $E_8 \times E_8$. We conclude, that two different heterotic string theories, the heterotic $SO(32)$ string theory (HO) and the heterotic $E_8 \times E_8$ string theory (HE) exist. Finally, there is the type I string theory, which contains an additional $SO(32)$ $\mathcal{N} = 1$ supersymmetric Yang-Mills gauge group and describes open and closed unoriented strings in 10 dimensions.

In any case, the low-energy excitation spectrum that we described above, containing metric

g, Kalb-Ramond field B and dilaton ϕ, are part of all superstring theories. However, through the introduction of supersymmetry, the field content grows. The low-energy approximations of above superstring theories are the so-called type II, type I and heterotic supergravities. Let us shortly summarize their massless spectra [11].

Massless sector of the type IIB superstring		
Sector	$SO(8)$ rep. prod.	Field
NS-NS	$\mathbf{1}+\mathbf{28}_V+\mathbf{35}_V$	$(\phi, B_{\mu\nu}, g_{\mu\nu})$
NS-R	$\mathbf{8}_S+\mathbf{56}_S$	$(\chi^1_\alpha, \psi^{1,\alpha}_\mu)$
R-NS	$\mathbf{8}_S+\mathbf{56}_S$	$(\chi^2_\alpha, \psi^{2,\alpha}_\mu)$
R-R	$\mathbf{1}+\mathbf{28}_C+\mathbf{35}_C$	$(C^{(0)}, C^{(2)}, C^{(4)})$

Massless sector of the type IIA superstring		
Sector	$SO(8)$ rep. prod.	Field
NS-NS	$\mathbf{1}+\mathbf{28}_V+\mathbf{35}_V$	$(\phi, B_{\mu\nu}, g_{\mu\nu})$
NS-R	$\mathbf{8}_C+\mathbf{56}_C$	$(\chi^{1\alpha}, \psi^1_{\mu\alpha})$
R-NS	$\mathbf{8}_S+\mathbf{56}_S$	$(\chi^2_\alpha, \psi^{2,\alpha}_\mu)$
R-R	$\mathbf{8}_V+\mathbf{56}_V$	$(C^{(1)}, C^{(3)})$

Massless sector of the heterotic superstring		
Sector	$SO(8)$ rep. prod.	Field
NS	$\mathbf{1}+\mathbf{28}_V+\mathbf{35}_V$	$(\phi, B_{\mu\nu}, g_{\mu\nu})$
R	$\mathbf{8}_S+\mathbf{56}_S$	$(\chi_\alpha, \psi^\alpha_\mu)$
Gauge bosons	In $\mathbf{8}_V$ and $\mathbf{8}_V$	Gauge bosons of $SO(32)$ or $E_8\times E_8$
Gauginos	In $\mathbf{8}_C$ and $\mathbf{8}_C$	Gauge boson superpartners

Massless sector of the type I superstring		
Sector	$SO(8)$ rep. prod.	Field
NS-NS	$\mathbf{1}+\mathbf{35}_V$	$(\phi, g_{\mu\nu})$
NS-R+R-NS	$\mathbf{8}_S+\mathbf{56}_S$	$(\chi^\alpha, \psi^\mu_\alpha)$
R-R	$\mathbf{28}_C$	$C^{(2)}$
Vector boson		A^μ

The representations of the fields are products of $SO(8)$-representations. The closed string sector of the type I superstring is defined by orientifolding the type IIB superstring by the worldsheet parity operator Ω. Then, the vector boson A^μ has to be included to cancel the RR-tadpole at 1-loop level. The vector boson arises by coupling a certain open string sector to the orientifold of type IIB. The resulting type I superstring theory is free of anomalies.

As noted above, g denotes the graviton, B the 2-form Kalb-Ramond field and ϕ the dilaton field. The χ denote dilatini, whereas the ψ denote gravitini. The k-forms $C^{(k)}$ denote the so-called Ramond-Ramond (R-R) gauge potentials with associated field strengths $F^{(k+1)} = dC^{(k)}$. In the case of type IIB supergravity, the 5-form field strength has to be self-dual, $F^{(5)} = \star F^{(5)}$. The field strengths associated with the gauge potentials are gauge invariant under the transformation

$$C^{(k)} \mapsto C^{(k)} + d\lambda^{(k-1)}, \tag{2.15}$$

where $\lambda^{(k-1)}$ is a $(k-1)$-form. The heterotic string theories contain 496 gauge bosons.

Dualities

The various superstring theories are related through a web of dualities, which are called *T-duality* and *S-duality*. In the case of M-theory, S-duality and T-duality combine to *U-duality*, which will be explained in section 3.2.9 in detail. Here, we shortly discuss T- and S-duality.
T-duality is a perturbative target space duality and relates different compactifications of string theory. It arises due to the ability of closed strings to wrap around closed cycles of a string compactification. On the one hand, *wrapping modes* count how often a closed string wraps around a distinct cycle. On the other hand, the momentum of the string in the compact dimension is quantized, leading to *momentum modes*. T-duality is a symmetry that exchanges the winding modes with the momentum modes while inverting the length scale of the compactification. It turns out that type IIA string theory compactified on a circle with radius R is physically equivalent to type IIB string theory compactified on a circle with radius $1/R$. In the case of type IIA-IIB T-duality, the transformation acts as a spacetime parity operator on the fermionic right-movers, changing the chirality of the groundstate. More precisely, it transforms the massless spectra as follows: The graviton and Kalb-Ramond field are transformed into each other according to the Buscher rules [12, 13], which will be explained in detail in section 3.2.2. Furthermore, the R-R gauge potentials of type IIA transform into the R-R gauge potentials of type IIB, and vice versa. The two heterotic string theories HO and HE are related by T-duality to all orders in perturbation theory.
S-duality relates different coupling limits of string theory. It turns out, that type IIB supergravity is invariant under the map

$$(C^{(0)}, C^{(2)}, g, \phi, B) \mapsto (C^{(0)}, B, e^{\phi} g, -\phi, C^{(2)}), \tag{2.16}$$

which changes the string coupling via $g_S \mapsto \frac{1}{g_S}$. The conclusion is, that type IIB string theory at weak coupling is related to a dual type IIB string theory at strong coupling. S-duality is part of a larger $SL(2;\mathbb{Z})$-group under which type IIB supergravity is invariant. It transforms the Kalb-Ramond field B and 2-form gauge potential $C^{(2)}$ as

$$\begin{pmatrix} B \\ C^{(2)} \end{pmatrix} \mapsto \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} B \\ C^{(2)} \end{pmatrix}, \tag{2.17}$$

while acting on the complex coupling $\tau = C^{(0)} + ie^{-\phi}$ by

$$\tau \mapsto \frac{a\tau + b}{c\tau + d}. \tag{2.18}$$

All other fields are kept invariant. Upon quantization, the discrete subgroup of $SL(2,\mathbb{R})$, which is $SL(2,\mathbb{Z})$, becomes an exact symmetry of string theory.
In the case, where the 0-form gauge potential $C^{(0)}$ vanishes, the complex coupling becomes proportional to the string coupling. Then, $SL(2,\mathbb{R})$-invariance of type IIB string theory points at a strong-weak duality of type IIB.

D-branes

We discussed the various incarnations of string theory and their spectra arising from the lowest excited states of closed and open strings traveling through spacetime. Aside from strings, it turns out, that there exist solitonic solutions of the supergravity actions, that have their own dynamics. These so-called D-branes arise as hypersurfaces of spacetime on which the endpoints of open strings can be confined to. Two string endpoints can live on different D-brane worldvolumes, leading to the conclusion, that D-branes interact by open strings extended between them.
The emergence of D-branes can be very well understood, when investigating the boundary conditions of open strings traveling through spacetime. While Neumann boundary conditions imposed on the string endpoints do not break Poincaré invariance, the specification of Dirichlet boundary conditions confines the string endpoints to spatial hypersurfaces. The "D" in D-branes originates from "Dirichlet". The term Dp-brane refers to a D-brane, which extends in p spatial dimensions.
An object, extended in p spatial dimensions can naturally couple to a $(p+1)$-form gauge potential. In this sense, the 1-brane, or fundamental string, couples to the Kalb-Ramond

2-form. In general, let M_{p+1} be the worldvolume of the Dp-brane, then it couples to the $(p+1)$-form gauge potential $C^{(p+1)}$ via

$$S = T_p \int_{M_{p+1}} d^{p+1}\sigma \, C^{(p+1)}_{i_1 \cdots i_{p+1}} \epsilon^{j_1 \cdots j_{p+1}} \partial_{j_1} X^{i_1} \cdots \partial_{j_{p+1}} X^{i_{p+1}}, \tag{2.19}$$

where T_p is the tension of the Dp-brane.

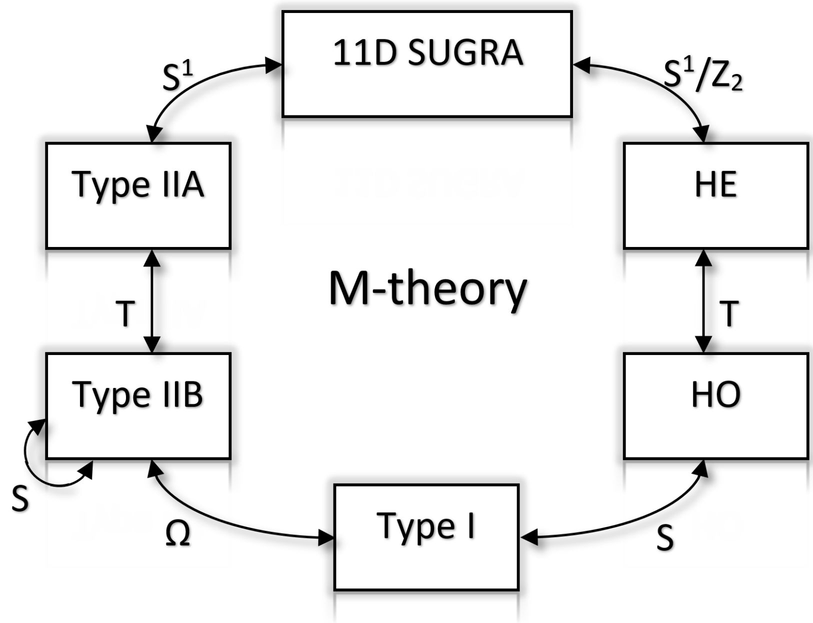
In type II supergravity, D-branes couple to the Ramond-Ramond gauge potentials and their Hodge duals. In type IIA supergravity, there are the gauge fields strengths $F^{(2)}$ and $F^{(4)}$ as well as their Hodge duals $F^{(8)} = \star F^{(2)}$ and $F^{(6)} = \star F^{(4)}$. The Hodge dual field strengths have associated gauge potentials, $C^{(5)}$ and $C^{(7)}$. Associated with the gauge potentials, we find electrically coupled D0-, D2- and magnetically coupled D4- and D6-branes in type IIA supergravity.

Type IIB supergravity contains the gauge field strengths $F^{(1)}$, $F^{(3)}$ and $F^{(5)}$ and their Hodge duals $F^{(9)} = \star F^{(1)}$ and $F^{(7)} = \star F^{(3)}$. Recall that $F^{(5)} = \star F^{(5)}$. So we find a D(-1)-brane associated with the scalar gauge potential $C^{(0)}$, a D1-brane, D3-brane, D5-brane and a D7-brane. The D(-1)-brane is 0-dimensionally extended in spacetime and hence an instanton, the so-called D-instanton. The D3-brane couples electrically to $C^{(4)}$, which is the gauge potential of the self-dual field strength. Therefore, the D3-brane couples also magnetically to the 4-form gauge potential, leading to self-duality.

To summarize, the stable branes in type IIA string theory are Dp-branes for p an even number, whereas the stable branes in type IIB string theory are Dp-branes for p an odd number. T-duality transforms Dirichlet boundary conditions into Neumann boundary conditions, and vice versa. Since D-branes are associated with hypersurfaces on which the open strings can end, T-duality maps IIA and IIB gauge potentials and D-branes into each other.

M-theory

As we pointed out, the various incarnations of string theory are related by a web of dualities. It turned out, that the 10-dimensional superstring theories are limits of one single underlying theory, which is called M-theory. While its low-energy approximation is known as 11-dimensional supergravity containing metric and 3-form tensor, the complete theory remains a mystery. The dynamical objects in M-theory are M2-branes and M5-branes and the string theory S- and T-dualities combine to a U-duality transformation as underlying



String duality web

symmetry structure of M-theory. Type II supergravity arises as circle compactification of 11-dimensional supergravity. The heterotic $E_8 \times E_8$ supergravity theory turns out to arise when 11-dimensional supergravity is compactified on an interval, $S^1/\mathbb{Z}_2$. For the sake of brevity, we postpone the detailed account on M-theory and U-duality to 3.2.9 in the first part of this thesis. The second part of the thesis revolves around M5-branes in M-theory. Therefore, we postpone the detailed account on M-branes to the preliminary section 4.2 in the second part of this thesis.

Chapter 3

Dualities in string theory and M-theory

All is flux, nothing is stationary.
– Heraclitus of Ephesus

3.1 Introduction

This chapter constitutes the first of the two main parts of this thesis. It concerns the investigation of the extended symmetry structures associated with T-duality in string theory and U-duality in M-theory.

Toroidally compactified bosonic type IIB supergravity exhibits a *T-duality* symmetry, which mixes the target space metric and Kalb-Ramond field. Through this mixing, it is a symmetry, which transforms the geometry of the underlying compactification. T-duality transformations of spaces, which exhibit a Killing isometry, have been investigated in the early days in articles of Buscher [12, 13], where the mixing of the metric and B-field components was observed.

The T-duality group in the case of a toroidal string compactification on a D-dimensional torus is given by $O(D, D; \mathbb{Z})$. In this case, the background exhibits several isometry directions and different T-duality transformations are possible. It turns out that after successive T-duality transformations dual theories emerge, which clearly need a mathematical treatment that goes beyond ordinary differential geometry [14, 15, 16].

As we discussed in the beginning, the spectrum of string theory contains various polyform potentials, which can wrap the internal closed cycles of the compactification space. Starting with a toroidal compactification with constant H-flux wrapping an internal 3-torus, after each

T-duality transformation new kinds of fluxes emerge [14, 15] : the geometric f-flux being related to the Weitzenböck connection on the compactification manifold; the non-geometric Q-flux, which is related to compactifications exhibiting monodromies; the non-geometric R-flux related to mysterious compactifications, which cease to be treatable by ordinary manifold techniques. Spaces associated with Q- and R-fluxes are called *non-geometric spaces*. The geometric f-flux being related to the Weitzenböck connection of the internal space directly influences the geometry of the compactified space. The monodromies associated with Q-flux have to be patched by full T-duality transformations. The R-flux case is even beyond monodromies. A geometry, that unifies backgrounds with geometric and non-geometric fluxes has been proposed in [16]. A relation between non-geometric fluxes and so-called *exotic branes* has been examined in [17, 18].

Generalized geometry [19, 20, 21] is a formulation, which treats the metric and the B-field on an equal footing by extending the tangent bundle by a cotangent part to the generalized tangent bundle, $TM \oplus T^*M \to M$, which has natural $O(D, D)$-structure. The metric and B-field are combined into a so-called *generalized metric* on the generalized tangent bundle and it turns out to be the appropriate groundwork for T-duality on H-flux backgrounds. The heart of generalized geometry is the *Courant algebroid* [22, 23] on the generalized tangent bundle, also called *standard Courant algebroid*, which is the underlying structure of T-duality geometry. The standard Courant algebroid exhibits a natural 3-form freedom, which can be associated with the H-flux, and captures the local symmetries of H-flux backgrounds. Its structure is naturally invariant under so-called B-field transformations, for which $dB = 0$. The analysis T-dual supergravity compactifications using generalized geometry also points towards non-geometric backgrounds [24]. However, for the analysis of general non-geometric backgrounds, the standard Courant algebroid with H-flux turns out to be insufficient.

Double field theory [25] is a field theory, which makes the $O(D, D)$-structure manifest by doubling the degrees of freedom. Early ideas underlying double field theory can be found in [26, 27, 28]. In addition to the D-torus of the compactification, a dual D-torus is introduced, which parameterizes the winding sector of the closed strings. Then, the T-duality group acts onto the doubled set of coordinates making the T-duality symmetry manifest. It provides a natural framework to investigate T-duality in toroidal compactifications, which also leads to some understanding of the mysterious non-geometric spaces and their associated non-geometric fluxes [29, 30, 31]. However, since the degrees of freedom are doubled, a so-called *strong constraint* has to be imposed in order to restrict the theory again to a physical subspace. Each solution of the strong constraint is associated with a different T-duality frame.

Contrary to the Courant algebroid on the generalized tangent bundle, the *Poisson-Courant algebroid* [32], a Courant algebroid on a Poisson manifold with Poisson tensor Π, exhibits a natural 3-vector freedom, which can be associated with the non-geometric R-flux. In the Courant algebroid of generalized geometry, H-flux can be naturally induced by B-transformation, giving $H = dB$. In the case of the Poisson-Courant algebroid, a bivector β plays the role of the potential for the R-flux via $d_\Pi \beta = R$, where d_Π is the Lichnerowicz-Poisson differential. The Poisson-Courant algebroid type generalized geometry is called *Poisson-generalized geometry* [33] and is a proposal for a model for non-geometric R-flux. The non-geometric R-flux backgrounds that arise in double field theory also are sourced by the β-potential. However, the mechanism is different. We have to clarify the significance of the Poisson-Courant algebroid with R-flux in light of the double field theory proposal.

In *M-theory* or its low-energy limit, 11-dimensional supergravity, the T-duality symmetry combines with S-duality to *U-duality*. In toroidally compactified 11-dimensional supergravity, the U-duality groups are given by split real forms of exceptional Lie groups $E_{d(d)}$ [34, 35, 36, 37, 38]. Motivated by the success of generalized geometry in the setting of toroidal string theory compactifications, *exceptional generalized geometry* [39, 40, 41] is a rather new research area, which attempts to provide a geometrization of U-duality symmetry of toroidally compactified 11-dimensional supergravity. In the same way as the generalized tangent bundle of generalized geometry encodes the winding modes of strings, the exceptional tangent bundles encode the winding modes of M-branes and symmetries of KK-monopoles in M-theory. Associated with U-duality, new sets of geometric and non-geometric backgrounds are to be discovered. Exceptional generalized geometry is a field in its early stage and the underlying mathematical structures of U-dual backgrounds are still to be explored.

In *Physical preliminaries*, we begin by recalling the emergence of the T-duality symmetry from toroidal compactifications of the string worldsheet theory. After that, we discuss the relation of T-duality in toroidal compactifications of closed string theory to the emergence of non-geometric spaces. Then, we investigate Kaluza-Klein reductions of type II supergravity and their relation to so-called gauged supergravities. This is followed by expositions on Scherk-Schwarz reduction with and without flux and T-duality twisted reductions. After that, we scratch the surface on the emergence of non-geometric fluxes from orbifold CFTs. Then, we go on and survey generalized geometry as underlying geometry of T-duality backgrounds with H-flux. After that, an introduction to double field theory is provided. We discuss how all geometric and non-geometric fluxes naturally appear as components of a so-called *covariant flux*. Then, we step into the realm of M-theory and give an introduction to its fundamental symmetry, U-duality. Finally, we provide an introduction to exceptional generalized geometry as a generalized geometry of M-theory.

The present part consists of 4 main sections. In *Twisted Courant algebroids and fluxes* we construct a twisted Courant algebroid, which naturally encodes the local expressions of all geometric and non-geometric fluxes as well as their Bianchi identities in a simultaneous manner. This is done by making use of *graded symplectic manifolds* [42, 43] and provides the underlying generalized gauge structure associated with T-duality geometry. Furthermore, we provide an analysis of the cohomological structures associated with non-geometric spaces. In *Double field theory and T-duality* we lift our construction up to a $O(D, D)$-manifest reformulation and recover all generalized fluxes that appear in double field theory. Motivated by this result, we provide a consistent definition of T-duality in the graded symplectic manifold setup. In *Poisson-Courant algebroid* we provide a thorough investigation of a Courant algebroid structure on a Poisson manifold. We reconstruct it in the supergeometric setting and analyze its cohomology and relation to double field theory and T-duality. We construct a topological membrane with R-flux along the lines of *AKSZ sigma models* [44]. The boundary theory of the membrane turns out to be a string sigma model with R-flux WZW term. On the loop space of string embeddings, we construct current algebras with R-flux and Poisson-Courant algebroid structure. The resulting current algebras are of Alekseev-Strobl type [45]. Furthermore, we find a duality transformation between the standard Courant algebroid with H-flux and Poisson-Courant algebroid with R-flux, which we call *flux duality*. We analyze the flux duality on 3 levels: as symplectomorphism between graded symplectic manifolds, as isomorphism between cohomologies and as relation between induced membrane sigma models. In *Generalized geometries*, we provide a construction of several higher algebra structures underlying type IIB toroidal compactifications (*generalized geometry*) and heterotic compactifications (*B_n-generalized geometry*). Then, we step into the realm of M-theory constructing higher algebra structures related with several exceptional generalized tangent bundles with U-duality symmetry. Using our reconstruction, we compute the associated local symmetry

L_∞-algebras of the underlying higher gerbes governing the generalized gauge structures.

For the understanding of the *Poisson-Courant algebroid* section, we provide an introduction to *Poisson geometry, cohomology and connections*. *L_∞-algebras* and their algebroid generalizations are the underlying symmetry algebras of all structures in this part. Therefore, a reading of the respective section is strongly recommended. In *Courant algebroids* we discuss all incarnations of Courant algebroids and also their higher generalizations. The section *Graded manifolds and supergeometry* lies at the heart of all calculations and is inevitable for the understanding. In *AKSZ sigma models* we provide an introduction to the construction of AKSZ topological sigma models based on graded manifolds. In *Current algebras*, we discuss how to construct Poisson algebras from graded manifolds and how to promote them to current algebras. In *Deligne cohomology and n-gerbes*, we introduce the description of higher gerbes as Čech-Deligne cocycles, which will be important when we construct the local symmetry L_∞-algebras underlying the higher gerbe structures appearing in T-duality and U-duality symmetric bundles in the section *Generalized geometries*.

3.2 Physical preliminaries

3.2.1 Toroidal compactification of string theory and T-duality

The crucial difference between ordinary particle theories and string theory is that the fundamental objects gain one internal dimension. This leads to the fact that two versions of string exist: open strings and closed strings. The special property of closed strings is that they can wind around compact dimensions. This leads to the conclusions that strings experience geometry and topology in a fundamentally different way compared to ordinary point-particles. Since consistent superstring theory is formulated in $9 + 1$ spacetime dimensions, in order to make contact to our observable world, 6 spatial dimensions have to be compactified. This leads to a high-dimensional internal space, which contains compact cycles around which closed strings can wind and brings us into the position to study the behavior of closed strings traveling in such geometries. In the simplest way, the difference to point-particles can be explored when considering string theory compactified on a circle or higher-dimensional torus. Here, a novel symmetry emerges which is not inherent in ordinary particle theories. This symmetry relates the winding of closed strings and their momentum in these compact dimensions with the geometry of the internal space and is called *T-duality*.

Our investigations revolve around the T-duality symmetry of toroidal string backgrounds. However, before turning to the analysis of its underlying structures, we will have to cover the groundwork. In this section, we discuss general toroidal compactifications of string theory, their invariance properties and the emergence of T-duality. We start with the example of T-duality in the case of a circle compactification and then go on to general D-dimensional torus compactifications. This section is based on [46, 47]. A further excellent introduction is [48]. For information on lattices that appear when considering toroidal compactifications we refer to [49].

To get started, let us consider the circle compactification with radius R of the bosonic string. For simplicity, we focus on the internal directions. The worldsheet action is given by

$$S = \frac{1}{4\pi\alpha'} \int d\tau d\sigma \eta^{\alpha\beta} \partial_\alpha X \partial_\beta X, \tag{3.1}$$

where we fixed the worldsheet metric to $\eta^{\alpha\beta}$. The embedding functions X are periodic,

$$X \sim X + 2\pi R w, \tag{3.2}$$

where w denotes the so-called *winding number*. The equation of motion of the field X is the free wave equation, which is solved by

$$X(\sigma,\tau) = X_R(\sigma-\tau) + X_L(\sigma+\tau), \tag{3.3}$$

$$X_R(\sigma-\tau) = x_R - \sqrt{\frac{\alpha'}{2}} p_R(\sigma-\tau) + i\sqrt{\frac{\alpha'}{2}} \sum_{k\neq 0} \frac{1}{k} \alpha_k e^{ik(\sigma-\tau)}, \tag{3.4}$$

$$X_L(\sigma+\tau) = x_L - \sqrt{\frac{\alpha'}{2}} p_R(\sigma+\tau) + i\sqrt{\frac{\alpha'}{2}} \sum_{k\neq 0} \frac{1}{k} \widetilde{\alpha}_k e^{-ik(\sigma+\tau)}. \tag{3.5}$$

Here, we have

$$x = x_L + x_R, \quad p_R = \frac{1}{\sqrt{2}} \left(\frac{\sqrt{\alpha'}}{R} n - \frac{R}{\sqrt{\alpha'}} w \right), \quad p_L = \frac{1}{\sqrt{2}} \left(\frac{\sqrt{\alpha'}}{R} n + \frac{R}{\sqrt{\alpha'}} w \right), \tag{3.6}$$

where the integer n results from momentum quantization. The canonical conjugate momentum of X is given by

$$P = \frac{1}{2\pi\sqrt{2\alpha'}} \left(p_L + p_R + \sum_{k\neq 0} \alpha_k e^{ik(\sigma-\tau)} + \sum_{k\neq 0} \widetilde{\alpha}_k e^{-ik(\sigma+\tau)} \right), \tag{3.7}$$

leading to the total momentum

$$P_{\text{tot}} = \int_0^{2\pi} d\sigma P = \frac{1}{\sqrt{2\alpha'}}(p_L + p_R). \tag{3.8}$$

We find the canonical commutation relations,

$$[x_L, p_L] = i\sqrt{\frac{\alpha'}{2}}, \quad [x_R, p_R] = i\sqrt{\frac{\alpha'}{2}}, \quad [\alpha_m, \alpha_n] = m\delta_{m,-n}, \quad [\widetilde{\alpha}_m, \widetilde{\alpha}_n] = m\delta_{m,-n}, \tag{3.9}$$

and the normal ordered Hamiltonian composed of the zero-modes of the left and right Virasoro operators,

$$H = L_0 + \widetilde{L}_0, \tag{3.10}$$

where

$$L_0 = \frac{1}{2}p_R^2 + \sum_{k=1}^{\infty} \alpha_{-k}\alpha_k, \quad \widetilde{L}_0 = \frac{1}{2}p_L^2 + \sum_{k=1}^{\infty} \widetilde{\alpha}_{-k}\widetilde{\alpha}_k. \tag{3.11}$$

The Virasoro operators are invariant under the transformation

$$T : \left(\frac{R}{\sqrt{\alpha'}} \mapsto \frac{\sqrt{\alpha'}}{R}, m \mapsto w, w \mapsto m \right). \tag{3.12}$$

This invariance property is called *target space duality* or *T-duality*. More explicitly, we find

$$T(p_L) = p_L, \quad T(p_R) = -p_R,\ T(\widetilde{\alpha}_k) = \widetilde{\alpha}_k,\ T(\alpha_k) = -\alpha_k,\ T(\dot{X}_L) = \dot{X}_L,\ T(\dot{X}_R) = -\dot{X}_R.$$

The total energy of the string is invariant under T-duality. It can be checked for partition functions with higher-genus contributions and turns out to be a perturbative symmetry of whole string theory. Furthermore, we note that string theory compactified on a small circle with $R/\sqrt{\alpha'} \ll 1$ is equivalent to string theory compactified on a large circle with $\sqrt{\alpha'}/R \ll 1$, when the dilaton VEV is transformed via $\phi \mapsto \phi + 2\log(R)$.

Now let us go on and consider the compactification on a D-dimensional torus T^D parameterized by local coordinates x^i for $i = 1, \ldots, D$. A flat torus T^D can be written by a quotient of $\mathbb{R}^D$ as follows,

$$T^D = \mathbb{R}^D / \sim, \tag{3.13}$$

where the equivalence relation $\sim$ is defined by

$$X^i \sim X^i + 2\pi w^i, \tag{3.14}$$

and the $w^i \in \mathbb{Z}$ denote the winding numbers around the various circles.

The worldsheet action in the case of a D-dimensional toroidal background is given by

$$S = \frac{1}{4\pi}\int d\sigma d\tau\left(\sqrt{\gamma}\gamma^{\alpha\beta}g_{ij}\partial_\alpha X^i\partial_\beta X^j + \epsilon^{\alpha\beta}B_{ij}\partial_\alpha X^i\partial_\beta X^j - \frac{1}{2}\sqrt{\gamma}\phi\mathcal{R}\right). \tag{3.15}$$

Here, $\mathcal{R}$ denotes the worldsheet Ricci scalar and the indices i, j run over $1, \ldots, d$. γ denotes the worldsheet metric and g_{ij} the target space metric. The antisymmetric tensor B_{ij} denotes the Kalb-Ramond B-field. The embedding functions X^i are periodic,

$$X^i \sim X^i + 2\pi w^i. \tag{3.16}$$

The underlying CFT is described by D^2 couplings encoded in the so-called background matrix E, which combines metric g_{ij} and Kalb-Ramond field B_{ij},

$$E = g + B. \tag{3.17}$$

We fix the worldsheet metric to $\eta_{\alpha\beta}$. The gauge fixed action is given by

$$S = \frac{1}{4\pi}\int d\sigma d\tau(\eta^{\alpha\beta}g_{ij}\partial_\alpha X^i\partial_\beta X^j + \epsilon^{\alpha\beta}B_{ij}\partial_\alpha X^i\partial_\beta X^j). \tag{3.18}$$

The mode expansion of X^i is given by

$$X^i(\sigma,\tau) = x^i + w^i\sigma + \tau g^{ij}(p_i - B_{ij}w^j) + \frac{i}{\sqrt{2}}\sum_{k\neq 0}\frac{1}{k}(\alpha^i_k(E)e^{-ik(\tau-\sigma)} + \widetilde{\alpha}^i_k(E)e^{-ik(\tau+\sigma)}) \tag{3.19}$$

with oscillators α^i_k and $\widetilde{\alpha}^i_k$. The canonical momentum density of X^i is given by

$$\begin{aligned} 2\pi P_i(\sigma,\tau) &= \frac{\delta S}{\delta \dot{X}^i} \\ &= g_{ij}\dot{X}^j + B_{ij}X^{j\prime} \\ &= p_i + \frac{1}{\sqrt{2}}\sum_{k\neq 0}(E^T_{ij}\alpha^j_k(E)e^{-ik(\tau-\sigma)} + E_{ij}\widetilde{\alpha}^j_k(E)e^{-ik(\tau+\sigma)}). \end{aligned} \tag{3.20}$$

Here, $E^T_{ij} = g_{ij} - B_{ij}$. We recognize that the canonical momentum can be written as a sum of center of mass momentum p_i and oscillator contributions. The center of mass momentum is quantized $p_i = m_i \in \mathbb{Z}$.
We can evaluate the canonical equal-time commutators

$$[X^i(\sigma,\tau), P_j(\sigma',\tau)] = i\delta^i_j\delta(\sigma-\sigma'), \tag{3.21}$$

leading to

$$[x^i, p_j] = i\delta^i_j, \quad [\alpha(E)^i_n, \alpha(E)^j_m] = mg^{ij}\delta_{m,-n}, \quad [\widetilde{\alpha}(E)^i_n, \widetilde{\alpha}(E)^j_m] = mg^{ij}\delta_{m,-n}. \tag{3.22}$$

The Hamiltonian density is given by

$$\begin{aligned} H &= L_0 + \widetilde{L}_0 \\ &= \frac{1}{4\pi}\int_0^{2\pi} ((2\pi P_i) g^{ij}(2\pi P_j) + X'^i (g - Bg^{-1}B)_{ij} X'^j + X'^i B_{ik} g^{kj} P_j) \\ &= \frac{1}{4\pi}\int_0^{2\pi} d\sigma (P_L^2 + P_R^2), \end{aligned} \tag{3.23}$$

where the left and right momentum have the form

$$P_{La} = (2\pi P_i + (g-B)_{ij} X^{j\prime}) e^{*i}_a, \quad P_{Ra} = (2\pi P_i - (g+B)_{ij} X^{j\prime}) e^{*i}_a. \tag{3.24}$$

Here, we introduced vielbeins $e^a_{\ i}$ and their duals e^{*i}_a. They can be regarded as generators of the non-degenerate D-dimensional lattice Λ and its dual Λ^*, respectively, defined by

$$\Lambda = \{e^a_{\ i} w^i | w^i \in \mathbb{Z}\}, \quad \Lambda^* = \{e^{*ai} m_i | m_i \in \mathbb{Z}\}, \tag{3.25}$$

and obey the relations

$$e^a_{\ i} e^b_{\ j} \delta_{ab} = 2g_{ij}, \quad e^a_{\ i} e^{*j}_b \delta^b_a = \delta^i_j, \quad e^{*i}_a e^{*j}_b \delta^{ab} = \frac{1}{2} g^{ij}. \tag{3.26}$$

Then, the D-dimensional torus can be written as the quotient

$$T^D = \mathbb{R}^D / \pi\Lambda. \tag{3.27}$$

The Hamiltonian can be written in the following form,

$$H = \frac{1}{4\pi}\int_0^{2\pi} d\sigma \begin{pmatrix} X' \\ 2\pi P \end{pmatrix}^T \begin{pmatrix} g - Bg^{-1}B & Bg^{-1} \\ -g^{-1}B & g^{-1} \end{pmatrix} \begin{pmatrix} X' \\ 2\pi P \end{pmatrix}, \tag{3.28}$$

which makes the $O(D,D)$-structure manifest in the following sense. Elements $g \in O(D,D;\mathbb{R})$ act on $2D \times 2D$ by conjugation and leave the matrix

$$J = \begin{pmatrix} 0 & \mathrm{id} \\ \mathrm{id} & 0 \end{pmatrix}, \tag{3.29}$$

invariant, $g^T J g = J$. Here, id is the $D \times D$ identity matrix.

In contrast to the $(D \times D)$-matrix E that appears in the Lagrangian formulation, the matrix appearing in the Hamiltonian formulation,

$$\mathcal{H}(E) = \begin{pmatrix} g - Bg^{-1}B & Bg^{-1} \\ -g^{-1}B & g^{-1} \end{pmatrix}, \tag{3.30}$$

is a $2D \times 2D$-matrix and called *generalized metric*.

The zero-mode component of the Hamiltonian can be expressed as

$$\begin{aligned} H &= L_0 + \widetilde{L}_0 \\ &= \frac{1}{2}(p_L^2 + p_R^2) \\ &= \frac{1}{2}(m_i g^{ij} m_j + w^i (g - BgB)_{ij} w^j + 2 w^i B_{ik} g^{kj} m_j) \\ &= \frac{1}{2} \begin{pmatrix} w^i \\ m_i \end{pmatrix}^T \mathcal{H}(E) \begin{pmatrix} w^i \\ m_i \end{pmatrix}, \end{aligned} \tag{3.31}$$

where the zero-modes of the left and right momenta are given by

$$p_{Ra} = (m_i + w^k (B - g)_{ki}) e_a^{*i}, \quad p_{La} = (m_i + w^k (B + g)_{ki}) e_a^{*i}, \tag{3.32}$$

and the vector

$$Z = \begin{pmatrix} w^i \\ m_i \end{pmatrix} \tag{3.33}$$

combines winding and momentum modes and is called *generalized momentum*. The left and right momenta form an even self-dual Lorentzian lattice, denoted by $\Gamma^{(D,D)}$. It is called *Lorentzian*, since the metric on the lattice has signature (D, D). Furthermore, it is called *even*, since its length is even,

$$p_L^2 - p_R^2 = 2 w^i m_i \in 2\mathbb{Z}. \tag{3.34}$$

Finally, since $\Gamma^{(D,D)} = (\Gamma^{(D,D)})^*$ the lattice is called *self-dual*. All even self-dual Lorentzian lattices $\Gamma^{(D,D)}$ are related by $O(D, D; \mathbb{R})$-transformations and each $\Gamma^{(D,D)}$ is associated to a distinguished toroidal background.

The zero-mode component of the Hamiltonian is invariant under $O(D; \mathbb{R}) \times O(D; \mathbb{R})$-transformations, which rotate p_L and p_R. This corresponds to the internal rotations of the vielbeins accompanying p_L and p_R on which $O(D; \mathbb{R}) \times O(D; \mathbb{R})$ acts from the right. Therefore, it turns out that the moduli space of toroidal compactifications is isomorphic to the coset

$$O(D, D; \mathbb{R}) / (O(D; \mathbb{R}) \times O(D; \mathbb{R})). \tag{3.35}$$

Here, $O(D; \mathbb{R}) \times O(D; \mathbb{R})$ is the maximal compact subgroup of $O(D, D; \mathbb{R})$. Since the dimension of $O(D, D; \mathbb{R})$ is $D(2D-1)$ and the dimension of $O(D; \mathbb{R})$ is $D(D-1)/2$, the dimension of the coset turns out to be D^2, which agrees with the number of scalars from g_{ij} and B_{ij}.

Now we have to investigate the transformation behavior of the oscillators. The complete Hamiltonian has the form

$$H = \frac{1}{2} Z^T M Z + N + \widetilde{N}, \tag{3.36}$$

where N and $\widetilde{N}$ are the number operators,

$$N = \sum_{k>0} \alpha^i_{-k} g_{ij} \alpha^j_k, \quad \widetilde{N} = \sum_{k>0} \widetilde{\alpha}^i_{-k} g_{ij} \widetilde{\alpha}^j_k. \tag{3.37}$$

The coset $O(D,D;\mathbb{R})/(O(D;\mathbb{R}) \times O(D;\mathbb{R}))$ parameterized by the background matrix $E = g + B$ embeds into $O(D,D;\mathbb{R})$ by

$$g_E = \begin{pmatrix} e & B(e^T)^{-1} \\ 0 & (e^T)^{-1} \end{pmatrix}, \tag{3.38}$$

where e is a vielbein obeying $ee^T = g$. Then, $O(D,D;\mathbb{R})$ acts on a $D \times D$ matrix M by *fractional linear transformation*,

$$g(M) = (aM + b)(cM + d)^{-1}, \tag{3.39}$$

where

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in O(D,D;\mathbb{R}). \tag{3.40}$$

We note $g_E(\mathrm{id}) = g + B = E$. Since

$$\mathcal{H}(E) = g_E g_E^T, \tag{3.41}$$

g_E is also referred to as *generalized vielbein*. Then, the background matrix $E = g + B$ transforms under $O(D,D;\mathbb{R})$ as

$$E \mapsto g(E) = (aE + b)(cE + d)^{-1}. \tag{3.42}$$

We remark that $O(D,D;\mathbb{R})$ acts from the left on the background moduli.

The whole spectrum of the theory is invariant under $O(D,D;\mathbb{Z})$-transformations, generated by the following 3 operations. Firstly, it is invariant under the shift of the B-field by an antisymmetric integral $(D \times D)$-matrix Θ by the transformation

$$h_\Theta = \begin{pmatrix} \mathrm{id} & \Theta \\ 0 & \mathrm{id} \end{pmatrix}. \tag{3.43}$$

If the shift is integer, it does not contribute to the path integral. It leads to a total derivative and therefore shifts the B-field by an integer multiple of 2π.

Secondly, the whole spectrum is invariant under the change of the compactification lattice Λ parameterized by an integral matrix $A \in GL(D;\mathbb{Z})$ via

$$h_A = \begin{pmatrix} A & 0 \\ 0 & (A^T)^{-1} \end{pmatrix}. \tag{3.44}$$

It parameterizes a basis change.

Thirdly, the whole spectrum is invariant under so-called *factorized dualities*, parameterized by elements

$$h_{D_k} = \begin{pmatrix} \mathrm{id} - e_k & e_k \\ e_k & \mathrm{id} - e_k \end{pmatrix}, \tag{3.45}$$

where e_k is a $(D \times D)$-matrix with all components zero except for the (k,k)-component, which is 1. This transformation generalizes the transformation of the circle compactification case in the sense that it inverts the radius of the k-th circle in the torus, $R_k \mapsto \frac{1}{R_k}$, interchanging the momentum and winding modes, $m_k \leftrightarrow w^k$. By investigation of the action of a factorized duality h_{D_k} on the metric g_{ij} and the Kalb-Ramond field B_{ij}, we recover the so-called Buscher rules [12, 13],

$$\begin{aligned} g_{kk} &\mapsto \frac{1}{g_{kk}}, \quad g_{ki} \mapsto \frac{B_{ki}}{g_{kk}}, \quad g_{ij} \mapsto g_{ij} - \frac{g_{ki}g_{kj} - B_{ki}B_{kj}}{g_{kk}}, \\ B_{ki} &\mapsto \frac{g_{ki}}{g_{kk}}, \quad B_{ij} \mapsto B_{ij} - \frac{g_{ki}B_{kj} - B_{ki}g_{kj}}{g_{kk}}. \end{aligned} \tag{3.46}$$

Above three transformations generate the discrete group $O(D,D;\mathbb{Z})$.

Let us investigate the transformation behavior of the number operators N and $\widetilde{N}$ under $O(D,D;\mathbb{Z})$-transformations. Under a transformation of the background matrix $E \mapsto g(E) = E'$ we find the following relations between metric g and dual metric g',

$$(d + cE)^T g'(d + cE) = g, \quad (d - cE^T)^T g'(d - cE^T) = g. \tag{3.47}$$

Using the mode expansions of X^i and P_i and the invariance of the canonical commutators under the duality one finds the transformation behavior of the oscillators,

$$\alpha_k(E) \mapsto (d - cE^T)^{-1}\alpha_k(E'), \quad \widetilde{\alpha}_k(E) \mapsto (d + cE)^{-1}\widetilde{\alpha}_k(E'). \tag{3.48}$$

We conclude, that the number operators N and $\widetilde{N}$ are invariant under $O(D,D;\mathbb{Z})$. Together with the invariance of the zero-mode part we find the whole spectrum to be $O(D,D;\mathbb{Z})$-invariant.

The group $O(D,D;\mathbb{Z})$ is called the *T-duality group* for a D-dimensional torus compactification. Dividing out invariance of the spectrum under the right-action of the local Lorentz transformations $O(D;\mathbb{R}) \times O(D;\mathbb{R})$ and the left-action of the T-duality group $O(D,D;\mathbb{Z})$ from the group $O(D,D;\mathbb{R})$, we are left with the moduli space for toroidal compactifications,

$$O(D,D;\mathbb{Z})\backslash O(D,D;\mathbb{R})/(O(D;\mathbb{R}) \times O(D;\mathbb{R})), \tag{3.49}$$

which is called the *Narain moduli space*.

3.2.2 T-duality and non-geometric spaces

Having understood the emergence of T-duality in toroidal string compactifications from the worldsheet point-of-view we now can go on and discuss the relation between T-duality and the mysterious non-geometric spaces. The analysis of these spaces and their underlying geometries play a prominent role in the main part of the present thesis.
In this section, we shortly review T-duality on toroidal string backgrounds in order to discuss the emergence of so-called non-geometric spaces. Details on T-duality can be found in [6, 7]. A very well presented survey is contained in [50]. A good reference for non-geometric flux compactifications and non-geometric backgrounds in general is [14].

T-duality

T-duality is a symmetry of compactified closed string theory that relates the winding modes of closed strings wrapping internal cycles of the compactification with their momentum modes. In the case of an S^1-compactification, where the circle radius is given by R, the mass spectrum of the closed string is given by [6]

$$m^2 = \frac{n^2}{R^2} + \frac{w^2 R^2}{\alpha'^2} + \frac{2}{\alpha'}(N + \widetilde{N} - 2), \tag{3.50}$$

where n is the momentum mode of the closed string in S^1-direction and w is the winding number associated with how often the closed string wraps around the S^1-direction. N and $\widetilde{N}$ are the numbers of oscillators in either direction on the closed string, more precisely, the eigenvalues of the number operators. The first contribution is the energy originating from the momentum quantized in S^1-direction. The second contribution is the potential energy associated with the winding of the closed string. The last contribution consists of the oscillation states of the string in either direction on the string and the vacuum energy. Furthermore, the oscillation states are constrained by the so-called *level matching condition*,

$$nw = N - \widetilde{N}. \tag{3.51}$$

Above two relations are also called the *mass-shell conditions*, which arise from the zero-mode Virasoro operators L_0 and $\widetilde{L}_0$ associated to the oscillator states on the closed string.
Let us investigate some limits of above formula. In the infinite radius limit, $R \to \infty$, we find that the dominant contribution comes from the potential energy of the winding modes. This is natural, since in the decompactification limit of the S^1 a closed string needs an infinite amount of energy to wrap the S^1. However, the quantized momenta become a continuous spectrum.
In the small radius limit, $R \to 0$, the dominant contribution comes from the compact momenta. For very small radia R, the center-of-mass energy of a closed string traveling in the S^1-direction becomes very high. However, the winding state spectrum becomes continuous.
We recognize, that for both opposite limits, the behavior of the compact momentum spectrum and the winding state spectrum is symmetric. The physically interesting observable is the energy. Therefore, both limits are identical from the physical perspective. We furthermore recognize, that the mass formula is invariant under the transformation

$$T : \left(R \mapsto \frac{\alpha'}{R}, \quad n \mapsto w, \quad w \mapsto n \right). \tag{3.52}$$

The transformation T inverts the radius of the S^1-compactification while exchanging winding and momentum modes. Furthermore, two successive transformations give identity, $T^2 = \mathrm{id}$. Due to the invariance of the mass formula under T, we conclude that closed string theory compactified on an S^1 with radius R is physically equivalent to closed string theory compactified on an S^1 with inverse radius and momentum and winding modes exchanged. This duality is called T-duality. T-duality symmetry of string theory is one of the crucial differences to ordinary point-particle theory, where winding modes w are non-existent.
Let us now consider a toroidal compactification on a D-dimensional torus, $T^D = S^1 \times \cdots \times S^1$, with constant internal metric g_{ij} and antisymmetric tensor field B_{ij}. In this case, the T-duality transformation T is generalized to elements of the toroidal T-duality group $O(D, D; \mathbb{Z})$. The elements of the $O(D, D; \mathbb{Z})$-group are given by matrices that keep the metric η_{MN} invariant,

$$h_M^{\ P} \eta_{PQ} h_N^{\ Q} = \eta_{MN}, \tag{3.53}$$

where

$$\eta_{MN} = \begin{pmatrix} 0 & \delta^i{}_j \\ \delta_i{}^j & 0 \end{pmatrix} \tag{3.54}$$

and the capital indices M, N,... run from 1 to $2D$. Note that $\eta^{MN}\eta_{NL} = \delta^M_L$ and the raising and lowering of $O(D,D)$-indices is done by η. Let the momentum modes in the various S^1-directions be denoted by the vector p^i. Furthermore, let the winding mode associated with each of the S^1-directions be denoted by w_i. Then, we can combine p^i and w_i into the so-called *generalized momentum*

$$P^M = \begin{pmatrix} w_i \\ p^i \end{pmatrix}. \tag{3.55}$$

The mass formula of a closed string traveling in the toroidal compactification can be written in terms of the generalized momentum

$$m^2 = P^M P^N \mathcal{H}_{MN} + (N + \widetilde{N} - 2), \tag{3.56}$$

where $\mathcal{H}_{MN}$ denotes the so-called *generalized metric*,

$$\mathcal{H}_{MN} = \begin{pmatrix} g^{ij} & -g^{ik}B_{kj} \\ B_{ik}g^{kj} & g_{ij} - B_{ik}g^{kl}B_{lj} \end{pmatrix}. \tag{3.57}$$

In the mass formula, the last term contains the contribution from the oscillators as well as the vacuum energy. The first term is crucial. It is $O(D,D;\mathbb{Z})$-covariant. The generalized metric contains not only the ordinary metric g_{ij}, but also the B-field B_{ij}. The level matching condition is generalized to

$$N - \widetilde{N} = \frac{1}{2}P^M P_M. \tag{3.58}$$

The generators of $O(D,D)$ can be represented as follows. Diffeomorphisms are block-diagonal matrices,

$$h_M^{\ N} = \begin{pmatrix} E^i{}_j & 0 \\ 0 & E_i{}^j \end{pmatrix}, \tag{3.59}$$

where $E \in \mathrm{GL}(D)$. Off-diagonal transformations, or shifts, are given by B-transformations

$$h_M^{\ N} = \begin{pmatrix} \delta^i{}_j & 0 \\ B_{ij} & \delta_i{}^j \end{pmatrix} \tag{3.60}$$

and β-transformations

$$h_M^{\ N} = \begin{pmatrix} \delta^i{}_j & \beta^{ij} \\ 0 & \delta_i{}^j \end{pmatrix}, \tag{3.61}$$

where B_{ij} and β^{ij} are antisymmetric tensors. Finally, since D different S^1-directions are available, for each direction there is an associated T-duality transformation. These are called factorized T-dualities,

$$h^{(k)}{}_M{}^N = \begin{pmatrix} \delta^i{}_j - t^i{}_j & t^{ij} \\ t_{ij} & \delta_i{}^j - t_i{}^j \end{pmatrix}, \tag{3.62}$$

where $(t_{ij}) = \mathrm{diag}(0, \ldots, 0, 1, 0, \ldots, 0)$. The 1 in (t_{ij}) is at the k-th position. The factorized T-duality transformations recover the so-called Buscher rules relating T-dual metric and B-field on toroidal compactification under T-duality along an isometry direction, in which metric and B-field are constant. The Buscher rules for a T-duality transformation along an isometry direction x^k is given by [12, 13]

$$\begin{aligned} g_{kk} &\mapsto \frac{1}{g_{kk}}, \quad g_{ki} \mapsto \frac{B_{ki}}{g_{kk}}, \quad g_{ij} \mapsto g_{ij} - \frac{g_{ki}g_{kj} - B_{ki}B_{kj}}{g_{kk}}, \\ B_{ki} &\mapsto \frac{g_{ki}}{g_{kk}}, \quad B_{ij} \mapsto B_{ij} - \frac{g_{ki}B_{kj} - B_{ki}g_{kj}}{g_{kk}}. \end{aligned} \tag{3.63}$$

For a simple S^1-compactification, the Buscher rules condense to the T-duality transformation T in (3.52).

The diffeomorphisms generate basis changes of the torus lattice, whereas the factorized T-duality transformations exchange the circle radius with its dual. Diffeomorphisms and B-transformations generate the so-called geometric subgroup of the T-duality group, given by $O(D,D)_{\text{geometric}} \subset O(D,D)$. Obviously, the mass formula is invariant under the T-duality transformations,

$$\mathcal{H} \mapsto h_M^P h_N^Q \mathcal{H}_{PQ}, \quad P^M \mapsto h^M_N P^N, \tag{3.64}$$

where the matrices h_M^N are elements of $O(D,D;\mathbb{Z})$.

As we explained above, the space of inequivalent torus lattices is given by the so-called *Narain moduli space* [51, 52]

$$O(D,D;\mathbb{Z})\backslash O(D,D;\mathbb{R})/(O(D;\mathbb{R}) \times O(D;\mathbb{R})), \tag{3.65}$$

where the remaining Lorentz transformations $O(D,\mathbb{R}) \times O(D,\mathbb{R})$ as well as the discrete subgroup $O(D,D,\mathbb{Z})$ are factorized out. The subgroup $O(D,D,\mathbb{Z})$ takes the torus lattice into itself, while permuting the lattice points.

The D-dimensional string coupling constant is given by $g_S^{(D)} = e^{\phi}$, where ϕ is the dilaton. Under S^1-compactification we find the $(D-1)$-dimensional string coupling constant given by

$g_S^{(D-1)} = \frac{R}{l_S} g_S^{(D)}$. Invariance of the dilaton scattering amplitudes under T-duality then gives the transformation property of the dilaton,

$$\phi \mapsto \phi - \log\left(\frac{R}{l_S}\right). \tag{3.66}$$

We can construct a T-duality invariant d from the dilaton and the metric by

$$e^{-2d} = \sqrt{g}e^{-2\phi}. \tag{3.67}$$

Non-geometric spaces

The simplest compactifications of string theory are toroidal compactifications. However, as made clear above, in the case of the compactification on a D-torus, several successive T-duality transformations are be possible, in general. We follow the description of [14, 15]. Furthermore, in the case of a B-field generating an NS-NS 3-form H-flux, which is a class in third integral de Rham cohomology over the compactification M, $H \in H^3_{\text{de Rham}}(M;\mathbb{Z})$, the H-flux can wrap any compact 3-cycle of M. It is in integral cohomology due to the quantization condition.

Let us discuss the example of a 3-torus T^3, parameterized by local coordinates x^i, where $i = 1, 2, 3$, and put N units of H-flux on the whole T^3,

$$H = \frac{1}{3!} H_{ijk} dx^i \wedge dx^j \wedge dx^k = N dx^1 \wedge dx^2 \wedge dx^3, \tag{3.68}$$

so that $H_{123} = N \in \mathbb{Z}$. Let the periods of the torus be 1. Then we choose the identification of local coordinates via

$$(x^1, x^2, x^3) \sim (x^1 + 1, x^2, x^3) \sim (x^1, x^2 + 1, x^3) \sim (x^1, x^2, x^3 + 1). \tag{3.69}$$

The torus is now given by the quotient space $T^3 = \mathbb{R}^3/\sim$. Now, we assume that the internal metric on the 3-torus is flat,

$$g = g_{ij} dx^i \otimes dx^j = dx^1 \otimes dx^1 + dx^2 \otimes dx^2 + dx^3 \otimes dx^3. \tag{3.70}$$

Having a look at the string equations of motion associated with this setup, we recognize, that a flat metric with non-zero H-flux is not a solution. However, as a conceptual example to clarify the emergence of non-geometric spaces it will suffice. If the 3-torus is complemented by a fibration of over a more intricate space, one can generate setups that obey the string equations of motion. In the next step, we choose a gauging of the H-flux,

$$B = \frac{1}{2} B_{ij} dx^i \wedge dx^j = N x^3 dx^1 \wedge dx^2. \tag{3.71}$$

Note that $dB = H$ is satisfied. The metric is obviously globally defined. It does not produce an inconsistency as we go around the torus. The B-field also does not change if we go around the x^1- or x^2-directions. However, We need a local patching of the B-field if we go around x^3-direction in order to get a globally well-defined H-flux. Therefore, this space can also be seen as fiber bundle $T^3 \to S^1$ with fiber T^2.

As is clear by now, we have two isometry directions at hand, x^1 and x^2. Therefore, we can apply the Buscher rules and compute the T-dual in x^1-direction. This magically eliminates the H-flux in favor of a metric twist,

$$g = (dx^1 - N x^3 dx^2) \otimes (dx^1 - N x^3 dx^2) + dx^2 \otimes dx^2 + dx^3 \otimes dx^3, \tag{3.72}$$

and $B = 0$. This space is an example of a so-called <u>twisted torus</u> or <u>nilmanifold</u>. If we try to think of the resulting torus in the same way as before, as the quotient $T^3 = \mathbb{R}^3/\sim$, we recognize that the metric is globally ill-defined. The topology of the torus has changed and with it the equivalence relation, which becomes

$$\begin{aligned}
(x^1, x^2, x^3) &\sim' (x^1 + 1, x^2, x^3), \\
(x^1, x^2, x^3) &\sim' (x^1, x^2 + 1, x^3), \\
(x^1, x^2, x^3) &\sim' (x^1 + N x^2, x^2, x^3 + 1).
\end{aligned} \tag{3.73}$$

The torus after T-duality is given by $T^3 = \mathbb{R}^3/\sim'$ with metric (3.72) and zero H-flux. Due to the twist in the equivalence relation, this torus is called a twisted torus.

The N units of H-flux have been transformed into N units of a "metric twist". How can we describe it formally? A formalization is possible by employing vielbein fields, that diagonalize the metric. Let us introduce the covectors of the tetrad formalism by $e^a = e^a{}_i dx^i$, where the index $a = 1, 2, 3$ denotes the Lorentz frame. Then the metric can be rewritten as

$$g_{ij} = \delta_{ab} e^a{}_i e^b{}_j. \tag{3.74}$$

Obviously, the covectors are given by

$$e^1 = dx^1 - N x^3 dx^2, \quad e^2 = dx^2, \quad e^3 = dx^3. \tag{3.75}$$

The 1-forms e^a are called Maurer-Cartan coframe. They are subject to the Maurer-Cartan equation,

$$de^a = \frac{1}{2} f^a_{bc} e^b \wedge e^c, \tag{3.76}$$

where f^a_{bc} are the structure constants of the Lie algebra associated with the Maurer-Cartan frame $e_a = e_a{}^i \partial_i$. Acting with the de Rham differential on (3.76), we find the Jacobi identity $f^e_{d[a} f^d_{bc]} = 0$. Recall, that the vielbein compatibility is given by

$$\nabla_i e_a{}^j = \partial_i e_a{}^j + \Gamma^j_{ik} e_a{}^k - W^b_{ia} e_b{}^j = 0, \tag{3.77}$$

where Γ^j_{ik} is the Christoffel connection and W^b_{ia} the spin connection. We find

$$W^b_{ia} = \Omega^b_{ca} e^c{}_i + \Gamma^k_{ij} e_a{}^j e^b{}_k, \tag{3.78}$$

where Ω^b_{ca} is called the Weitzenböck connection and is given by

$$\Omega^b_{ca} = e_c{}^i \partial_i e_a{}^j e^c{}_j. \tag{3.79}$$

The main property of the Weitzenböck connection is that it has vanishing curvature, but in general non-vanishing torsion. In contrast to that, the Christoffel connection has vanishing torsion, but in general non-vanishing curvature. Then, the structure constants f^a_{bc} are related to the antisymmetric part of the Weitzenböck connection via

$$f^a_{bc} = 2\Omega^a_{[bc]}. \tag{3.80}$$

In this way, f^a_{bc} can be associated with the antisymmetrization of the projected part of the torsion-less spin connection.

In our example, $f^1_{23} = N$ so that we recognize that the H-flux got transformed to the antisymmetric part of the Weitzenböck connection, which is responsible for the twist in the torus geometry. Therefore, the coefficients f^a_{bc} are also called geometric f-flux and one says that H-flux transforms to f-flux.

Investigating the isometry directions of the twisted torus, we still find x^1 and x^2 as isometries. A T-duality transformation along x^1 would take us back to the flat T^3 with H-flux. Therefore, we choose to T-duality transform in x^2-direction. Applying the Buscher rules, we find an intricate structure for the metric and the B-field,

$$g = \frac{1}{1+(Nx^3)^2}(dx^1 \otimes dx^1 + dx^2 \otimes dx^2) + dx^3 \otimes dx^3, \tag{3.81}$$

$$B = \frac{Nx^3}{1+(Nx^3)^2} dx^1 \wedge dx^2. \tag{3.82}$$

Upon going around the x^3-circle, both fields require to be patched by a transformation. This transformation is not an ordinary gauge transformation anymore, but the whole T-duality group comes into play as structure group. In order to arrive at a globally well-defined metric and B-field, a T-duality transformation has to be applied when going around the x^3-direction. This ultimately mixes metric and B-field and constitutes the crucial difference to ordinary compactifications. Such spaces, where monodromies appear, that need to be patched by the full T-duality group, are called T-folds or globally non-geometric and to them, a new, globally non-geometric Q-flux is associated. In our case, the metric f-flux has been transformed to the non-geometric Q-flux in the sense $N = f^1_{23} \mapsto Q^{12}_3$.

In [14] it was argued that although in the Q-flux background there is no isometry direction left, in which T-duality could be applied, the T-duality invariance of the type IIA superpotential in an orientifold compactification on a twisted torus hints to new locally non-geometric flux coefficients and associated non-geometric backgrounds. Application of several T-duality transformations lead to the emergence of coefficients in the superpotential, which are argued to belong to the T-dual of the Q-flux background. Requiring duality-invariance of the superpotential is directly related to the existence of these coefficients. The associated flux is named R-flux and is related to locally non-geometric backgrounds, where even locally a geometric description of the background fails. Arguing along this line, in our example, a further application of T-duality in x^3-direction transforms the Q-flux into R-flux, $N = Q^{12}_3 \mapsto R^{123}$. However, the mathematical groundwork of this R-flux is still highly mysterious and will be one crucial element to be discussed in the main part of this thesis.

Summarizing all backgrounds, that occur through the successive application of T-duality transformations in our toroidal background, we are lead to the well-known T-duality chain [14],

$$H_{123} \overset{T_1}{\longleftrightarrow} f^1_{23} \overset{T_2}{\longleftrightarrow} Q^{12}_3 \overset{T_3}{\longleftrightarrow} R^{123}.$$

From the perspective of topology, investigations of T-duality on circle or more generally principal torus bundles with H-flux have been conducted in [53, 54]. It turned out that T-duality exchanges the first Chern class $c_1(E)$ of the circle bundle E with the 3-form H-flux. T-duality action on this configuration leads to a dual bundle $\widehat{E}$, with dual Kalb-Ramond field $\widehat{H}$. Both bundles are related by $c_1(\widehat{E}) = \pi_* H$ and $c_1(E) = \pi_* \widehat{H}$. It can be shown, that T-duality in this setting is an isomorphism of Courant algebroids $\Gamma(TE \oplus T^*E)_{S^1} \to \Gamma(T\widehat{E} \oplus T^*\widehat{E})_{S^1}$, where the subscript S^1 means S^1-equivariance [55].

Conformal field theory calculations involving T-dual backgrounds with R-flux have been conducted in [56]. There, non-geometric R-flux backgrounds are believed to be related to non-associative geometries leading to a 3-bracket

$$[X^i, X^j, X^k] = R^{ijk}(X). \tag{3.83}$$

3.2.3 Kaluza-Klein reduction and gauged supergravity

We recalled how the T-duality group appears in the case of toroidal compactifications of string theory from the worldsheet perspective and constructed the Narain moduli space of inequivalent compactifications. Now, we can go on and consider the toroidal compactification of the low-energy effective field theory of string theory. More precisely, we reduce the low-energy field theory of string theory on a D-dimensional torus along the lines of Kaluza-Klein, which is called *Kaluza-Klein reduction*. We will observe how a global continuous $O(D, D; \mathbb{R})$-symmetry arises in the reduced theory. Part of the symmetry can be gauged using a gauge group G. This is done by the method of the so-called *embedding tensor* and leads to what is known as *gauged supergravities*. The structure constants of the gauge algebra of gauged supergravity turn out to be related to the integrated geometric as well as non-geometric fluxes. In the main part of the present thesis we will be able to construct the most general non-abelian gauged supergravity algebra by making use of twisted Courant algebroids. These twisted Courant algebroids encode the underlying local symmetries of backgrounds with flux. In this section, we discuss the Kaluza-Klein reduction of the bosonic part of the low-energy effective field theory theory of string theory on a D-dimensional torus and the emergence of gauged supergravities. It is based on [57, 58, 59]. More information on gauged supergravities can be found in [60].

The bosonic part of the low-energy effective field theory of string theory in d dimensions is given by

$$S = \int d^d x \sqrt{-\mathcal{G}} e^{-\phi} \left(\mathcal{R} + (\nabla\phi)^2 - \frac{1}{12} \mathcal{H}_{\mu\nu\rho} \mathcal{H}^{\mu\nu\rho} \right), \tag{3.84}$$

where ϕ denotes the dilaton, $\mathcal{H}$ the field strength of the Kalb-Ramond field $\mathcal{B}$ and $\mathcal{G}$ the spacetime metric. $\mathcal{R}$ denotes the Ricci scalar of the spacetime metric. The Kaluza-Klein reduction of this theory on a D-dimensional torus, T^D, is given by

$$S = \int d^{d-D} x \sqrt{-g} \left(R + (\nabla\phi)^2 - \frac{1}{12} H_{\mu\nu\rho} H^{\mu\nu\rho} + \frac{1}{8} L_{ab} \nabla_\mu K^{bc} L_{cd} \nabla^\mu K^{da} - \frac{1}{4} F^a_{\mu\nu} L_{ab} K^{bc} L_{cd} F^{d\mu\nu} \right). \tag{3.85}$$

Here, the vector fields V_μ^M and $B_{\mu M}$ from the reduction of the metric and B-field respectively have been assembled in the vector field $A_\mu^a = (V_\mu^M, B_{\mu M})$ with abelian field strength $F^a = dA^a$. The capital indices M, N run over $1, \ldots, D$ and the indices a, b run over $1, \ldots, 2D$. The vector field A_μ^a transforms in the fundamental representation of $O(D,D)$ and L_{ab} denotes the $O(D,D)$-invariant metric. The scalar fields that arise in the reduced theory take values in the coset $O(D,D)/(O(D)\times O(D))$ and are assembled in the symmetric traceless matrix K^{ab}, $L_{ab}K^{ab}=0$. The gauge bosons and the scalars transform under $O(D,D)$ as

$$A^a \mapsto M^a_{\ b}A^b, \quad K^{ab} \mapsto M^a_{\ c}M^b_{\ d}K^{cd}. \tag{3.86}$$

We conclude, that the resulting theory has global $O(D,D)$-symmetry and $U(1)^{2D}$ gauge symmetry.

In order to arrive at a formulation of gauged supergravities, we introduce a $2D$-dimensional subgroup of $O(D,D)$ and promote it to a local symmetry of the theory. When G is embedded into $O(D,D)$, the fundamental representation of $O(D,D)$ has to become the adjoint representation of G. Then, the gauge bosons transform in the adjoint representation. We can denote the generators of $O(D,D)$ as antisymmetric matrices t_{ab}. Let us furthermore denote the generators of G by T^a. They obey the relation

$$[T_a, T_b] = i f^c_{ab} T_c \tag{3.87}$$

with structure constants f^c_{ab}. Then, the so-called *embedding tensor method* is used to embed the group G into $O(D,D)$ via

$$T_a = \frac{1}{2}\Theta_a^{\ bc} t_{bc}, \tag{3.88}$$

where $\Theta_a^{\ bc}$ is called the *embedding tensor*. Then, the deformed action is given by

$$\begin{aligned} S = \int d^{d-D}x\sqrt{-g}e^{-\phi}\bigg(&R + (\nabla\phi)^2 + \frac{1}{8}L_{ab}D_\mu K^{bc}L_{cd}D^\mu K^{da} - \frac{1}{4}F^a_{\mu\nu}L_{ab}K^{bc}L_{cd}F^{d\mu\nu} \\ &-\frac{1}{12}H_{\mu\nu\rho}H^{\mu\nu\rho} - g^2 W(K)\bigg), \end{aligned} \tag{3.89}$$

where g is the gauge coupling constant and $W(K)$ is a gauge-invariant scalar potential. D_μ denotes the covariant derivative and F is the non-abelian field strength of A. Furthermore, the H-flux is shifted by a non-abelian Chern-Simons term of the gauge field A. The gauged action becomes invariant under $O(D,D)$-transformations if the structure constants of G transform as

$$f^c_{ab} \mapsto M_a^{\ d} M_b^{\ e} M^c_{\ f} f^f_{de}, \tag{3.90}$$

changing the embedding by $O(D,D)$-transformations.

We can decompose T^a into generators Z_M and X^M relate it to the gauge bosons V^M and B_M, respectively. The most general gauge algebra is then given by

$$[Z_M, Z_N]_{\text{Lie}} = f^P_{MN} Z_P + H_{MNP} X^P, \tag{3.91}$$

$$[Z_M, X^N]_{\text{Lie}} = -\widetilde{f}^N_{MP} X^P + Q^{NP}_M Z_P, \tag{3.92}$$

$$[X^M, X^N]_{\text{Lie}} = \widetilde{Q}^{MN}_P X^P + R^{MNP} Z_P. \tag{3.93}$$

The structure constants are subject to Jacobi identities and are interpreted as integrated geometric as well as non-geometric fluxes from the perspective of compactification.

When considering a compactification of string theory on a background with isometries, the resulting theory is expected to inherit the transformation properties of the uncompactified theory under T-duality. T-duality covariance of the non-abelian gauge algebra above then requires the existence of the full set of structure constants.

Ordinary compactifications of supergravity correspond to geometric compactifications, where only the geometric flux f and the H-flux are turned on. However, general gaugings cannot be obtained by an ordinary compactification of a higher-dimensional supergravity theory. Nevertheless, seen from the invariance property of the lower-dimensional theory, general gaugings should exists and correspond to realizations intrinsically possible only in string theory. Such realizations arise as asymmetric orbifold compactifications or elliptically twisted reductions to be introduced below.

3.2.4 Scherk-Schwarz reduction

In the former section we discussed ordinary Kaluza-Klein reduction of the low-energy effective action of string theory. If the internal metric is taken to depend on the internal coordinates in a certain way parameterized through a so-called twist matrix, then the reduction process is generalized to what is called *Scherk-Schwarz reduction*. So in general, one can say that Scherk-Schwarz reduction is a twisted reduction scheme which leads to non-abelian gauge symmetries. The compactification manifold becomes a group manifold. In the case of a torus reduction, it leads to a so-called *twisted torus*. It turns out that twisted torus reductions of string theory lead to consistent backgrounds with non-zero geometric f-flux. In the main part of the thesis, we will show how to invoke the geometric f-flux by a certain twist of a Courant algebroid with generalized frame bundle. A reduction of the Courant algebroid to a

Lie algebra then derives the associated non-abelian gauge algebra. This section is based on [61, 58, 59, 57]. Also see [62].
We start with the Scherk-Schwarz reduction on a D-dimensional torus T^D, which switches on geometric f-flux in the gauge algebra. It is a generalization of ordinary Kaluza-Klein reduction. For convenience, we divide our discussion into Scherk-Schwarz reduction without and with flux. Scherk-Schwarz reduction in the presence of 3-form flux will be interesting from the string theory perspective.

Without flux

Let the torus be parameterized by local coordinates y^I, whereas the non-compact direction is parameterized by coordinates x^μ. In the Scherk-Schwarz reduction scheme, a twist matrix ${\sigma^I}_J(y)$ is defined, which depends on the internal coordinates y. Then, the internal metric is decomposed by

$$g_{KL}(x,y) = g'_{IJ}(x){\sigma^I}_K(y){\sigma^J}_L(y). \tag{3.94}$$

The whole metric is then described in terms of vielbein fields $e^I = {\sigma^I}_J(y)dy^J$ and given by

$$ds^2 = g_{\mu\nu}dx^\mu \otimes dx^\nu + g'_{IJ}(x)(e^I + A^I) \otimes (e^J + A^J). \tag{3.95}$$

Here, the 1-forms A^I correspond to the gauge bosons of the reduction of the metric. When taking the twist matrices to be identity, we come back to an ordinary Kaluza-Klein reduction. Therefore, Scherk-Schwarz reduction is also called twisted reduction, since the vielbein frames e^I are twisted with respect to the ordinary frames dy^I. A general twisted torus reduction can be described as an ordinary torus reduction with non-trivial spin-connection, or torsion. The vielbein fields obey the relation

$$de^I = -\frac{1}{2}f^I_{JK}e^J \wedge e^K, \tag{3.96}$$

with structure constant f^I_{JK} and therefore model a group manifold. The gauge group of the lower-dimensional theory corresponds group of right-translations. Applying the de Rham differential to above equation yields the Jacobi identity for the structure constants. Dually, they correspond to the structure constants of the gauge group of generators Z_I corresponding to the gauge bosons of the reduction of the metric,

$$[Z_I, Z_J]_{\text{Lie}} = f^K_{IJ}Z_K. \tag{3.97}$$

With flux

Now, let us include a 3-form flux in our discussion. Again, for the internal manifold, we specify a basis of vielbein fields e^M so that

$$de^M = -\frac{1}{2} f^M_{JK} e^J \wedge e^K. \tag{3.98}$$

Then, we make an ansatz for the internal metric

$$ds^2 = G_{MN} e^M e^N \tag{3.99}$$

and internal B-field

$$B = \frac{1}{2} B_{MN} e^M \wedge e^N + \varphi \tag{3.100}$$

so that B_{MN} and G_{MN} are independent of the internal coordinates y^M. The 2-form φ gives rise to the constant 3-form

$$d\varphi = -\frac{1}{3!} H_{MNK} e^M \wedge e^N \wedge e^K. \tag{3.101}$$

The constants f^N_{MK} and H_{MNK} are structure constants of the resulting non-abelian gauge algebra given by

$$[X^M, X^N]_{\text{Lie}} = 0, \tag{3.102}$$

$$[Z_M, X^N]_{\text{Lie}} = -f^N_{MK} X^K, \tag{3.103}$$

$$[Z_M, Z_N]_{\text{Lie}} = f^K_{MN} Z_K + H_{MNK} X^K. \tag{3.104}$$

Again, the generator X^M arises from the B-field reduction, whereas the generator Z_M arises from the metric reduction. The moduli B_{MN} and G_{MN} give rise to D^2 scalar fields.

3.2.5 Reduction with duality twists

We understood that Kaluza-Klein reduction of the bosonic part of type II supergravity in general leads to gauged supergravities with a non-abelian gauge algebra in which the integrated fluxes appear as structure constants. Furthermore, we discussed how to generate geometric flux by Scherk-Schwarz reduction on a group manifold, and how additional contributions to the gauge algebra arise if the reduction of the 3-form flux is also considered.

Here, we show the simplest example of a reduction, where the so-called non-geometric Q-flux arises as non-trivial structure constant of the gauge algebra. This reduction is called *reduction with duality twists* and its possibility is an intrinsic feature of string theory. In the main section of the present thesis, we will show how to generate the twisted Courant algebroid, which reduces to the non-abelian gauge Lie algebra underlying such backgrounds. For this part, we refer to the articles [61, 58, 59, 57].

Let us reduce string theory in two steps. In the first step, we reduce string theory on a D-dimensional torus T^D. Then, we know that the resulting theory exhibits global $O(D, D; \mathbb{Z})$-symmetry. The next step leads to the crucial difference compared to the formerly described reductions. We reduce the theory again on a circle, S^1. However, the reduction is accompanied by a twist by the T-duality group $O(D, D; \mathbb{Z})$. So we can say, we consider a reduction on $T^D \times S^1$ with a twist. The result is a fiber bundle T^D over the base S^1.

Let us decompose the local coordinates by $x^M = (x^I, y)$ and let the radius of the base circle to be R. Then the coordinate y has the periodicity

$$y \sim y + 2\pi R. \tag{3.105}$$

The crucial difference is now, that compared to the other reductions the gauge bosons from the metric and the B-field are twisted by a monodromy matrix taking values in toroidal T-duality group. Any field in the fundamental of $O(D, D; \mathbb{Z})$, Ψ, depends on the internal circle coordinate y via

$$\Psi(x^\mu, y) = \exp\left(\frac{My}{2\pi R}\right)\Psi(x^\mu). \tag{3.106}$$

Obviously, the resulting fields exhibit a monodromy expressed by

$$\mathcal{M} = \exp(M) \in O(D, D; \mathbb{Z}), \tag{3.107}$$

which is called *monodromy matrix*.

In the following, let us investigate the resulting non-abelian gauge group. For this, we decompose the gauge bosons from the reduction of the metric and B-field according to $T^D \times S^1$ via $V^M = (V^I, V^y)$ and $B_M = (B_I, B_y)$. The corresponding generators of the gauge group decompose as $Z_M = (Z_I, Z_y)$ and $X^M = (X^I, X^y)$. Let us regroup the gauge generators by

$$T_a = (Z_y, X^y, T_\alpha), \quad T_\alpha = (Z_I, X^I). \tag{3.108}$$

Then, the generators T_α are $O(D, D)$-fundamentals. The resulting gauge algebra is given by

$$[Z_y, T_\alpha]_{\text{Lie}} = M_\alpha{}^\beta T_\beta, \tag{3.109}$$

where the mass matrix decomposes as

$$M_\alpha{}^\beta = \begin{pmatrix} W_I{}^J & U_{IJ} \\ V^{IJ} & -(W^T)^I{}_J \end{pmatrix}. \tag{3.110}$$

Here, U_{IJ} and V^{IJ} are antisymmetric matrices. The matrix M takes values in the Lie algebra of $O(D,D)$. The non-abelian gauge algebra then decomposes as

$$[Z_y, Z_I]_{\text{Lie}} = W_I{}^J Z_J + U_{IJ} X^J, \tag{3.111}$$

$$[Z_y, X^I]_{\text{Lie}} = -W_J{}^I X^J + V^{IJ} Z_J, \tag{3.112}$$

and all other commutators zero. In the special case for vanishing V^{IJ} we recover the gauge algebra of Scherk-Schwarz reductions in the presence of 3-form flux. We find the following geometric and non-geometric fluxes as gaugings of the algebra,

$$H_{yIJ} = U_{IJ}, \quad f_{yI}^J = \widetilde{f}_{yI}^J = W_I{}^J, \quad Q_y^{IJ} = V^{IJ}. \tag{3.113}$$

It turns out that V^{IJ} is related to non-geometric Q-flux. In the case where $V^{IJ} \neq 0$, the reduction is non-geometrically twisted and the resulting internal space is called a *T-fold*. The generator $W_I{}^I$ is in the Lie algebra of $GL(D;\mathbb{R})$. It corresponds to a geometric twist. The generator U_{IJ} corresponds to B-field shifts. In conclusion, the possibility of a reduction on a T-fold is what distinguishes string theory from ordinary particle theory. In the next section, we discuss reductions on T-folds and even more involved non-geometric reductions from the viewpoint of conformal field theories.

3.2.6 Non-geometric fluxes from orbifold CFTs

It turns out that the non-abelian gauge algebras with geometric and non-geometric fluxes can be derived from exact CFT descriptions at the orbifold point. Such a construction is possible since compactifications with elliptic monodromies admit freely-acting symmetric as well as asymmetric orbifold descriptions at special points of the moduli space [59, 57]. In this section, we will only scratch the surface and refer to details on the CFT computations to [57]. For further CFT computations involving non-geometric fluxes see [63, 64]. In the main section of the present thesis, it will turn out that certain non-geometrically twisted Courant algebroids reduce to the non-abelian gauge Lie algebras described in this section.

Let us go back to the example of a reduction on $T^D \times S^1$, where the reduction on S^1 is twisted by a flux matrix M. In this case, the metric g and Kalb-Ramond field B depend on the S^1-coordinate y. Therefore, a T-duality transformation in circle direction along the lines of Buscher is not possible due to the lack of isometry. However, it turns out that at the orbifold point T-duality can exactly be computed on the level of conformal field theories [57]. This is because the dependence on y is only through the boundary conditions. The former gauge algebra

$$[Z_y, T_\alpha]_{\text{Lie}} = M_\alpha{}^\beta T_\beta \tag{3.114}$$

transforms via T-duality along y at the orbifold point exchanging $Z_y \mapsto X^y$ and $X^y \mapsto Z_y$ to

$$[X^y, T_\alpha]_{\text{Lie}} = M_\alpha{}^\beta T_\beta. \tag{3.115}$$

In this way, the resulting integrated fluxes, which are switched on, are $\widetilde{f}$, Q, $\widetilde{Q}$ and R leading to a non-geometric background. However, this is not a true R-flux background, since it emerges from a Q-flux background by T-duality.

A true non-geometric R-flux background requires an additional asymmetry in the base S^1. This is done by considering an $Z_n \times Z_m$ orbifold action, where the Z_n action is supplemented by a y coordinate shift and the Z_m action is supplemented by a dual coordinate $\widetilde{y}$ shift. The resulting gauge algebra is given by

$$[X^y, T_\alpha]_{\text{Lie}} = \widetilde{M}_\alpha{}^\beta T_\beta, \tag{3.116}$$

$$[Z_y, T_\alpha]_{\text{Lie}} = M_\alpha{}^\beta T_\beta, \tag{3.117}$$

and it has been shown that the corresponding twist of all involved fields Ψ is given by

$$\Psi(x^\mu, y, \widetilde{y}) = \exp\left(\frac{My}{2\pi R}\right)\exp\left(\frac{\widetilde{M}\widetilde{y}}{2\pi \widetilde{R}}\right)\Psi(x^\mu). \tag{3.118}$$

When we decompose the flux matrices by

$$M_\alpha{}^\beta = \begin{pmatrix} W_I{}^J & U_{IJ} \\ V^{IJ} & -(W^T)^I{}_J \end{pmatrix}, \quad \widetilde{M}_\alpha{}^\beta = \begin{pmatrix} \widetilde{W}_I{}^J & \widetilde{U}_{IJ} \\ \widetilde{V}^{IJ} & -(\widetilde{W}^T)^I{}_J \end{pmatrix}, \tag{3.119}$$

then we find the decomposed non-abelian gauge algebra

$$[X^y, Z_I]_{\text{Lie}} = \widetilde{W}_I{}^J Z_J + \widetilde{U}_{IJ} X^J, \tag{3.120}$$

$$[X^y, X^I]_{\text{Lie}} = -\widetilde{W}_J{}^I X^J + \widetilde{V}^{IJ} Z_J, \tag{3.121}$$

$$[Z_y, Z_I]_{\text{Lie}} = W_I{}^J Z_J + U_{IJ} X^J, \tag{3.122}$$

$$[Z_y, X^I]_{\text{Lie}} = -W_J{}^I X^J + V^{IJ} Z_J, \tag{3.123}$$

and the associated integrated fluxes

$$\begin{aligned} H_{yIJ} = U_{IJ}, \quad R^{yIJ} = \widetilde{V}^{IJ}, \\ f_{yI}^{J} = \widetilde{f}_{yI}^{J} = W_I{}^J, \quad \widetilde{f}_{IJ}^{y} = -\widetilde{U}_{IJ}, \\ Q_I^{yJ} = -\widetilde{Q}_I^{yJ} = -\widetilde{W}_I{}^J, \quad Q_y^{IJ} = V^{IJ}. \end{aligned} \tag{3.124}$$

Such a background has been reconstructed as a freely-acting asymmetric $\mathbb{Z}_4 \times \mathbb{Z}_2$ orbifold in [57] to which we refer for details on the conformal field theory calculations. The conclusion is that true non-geometric R-flux backgrounds require the introduction of a dependence on the dual variables $\widetilde{y}$ in the twist.

3.2.7 Generalized geometry

Since our analysis is based on generalized geometry and Courant algebroids, in this section, we will provide an introduction to generalized geometry, which treats the metric g and the B-field on an equal footing and therefore is a formidable tool to analyze T-duality. More generally, it can be used to study flux compactifications of type II supergravity. We will also clarify its relation to Courant algebroids. The section devoted to Courant algebroids can be found in the mathematical preliminaries. Generalized geometry was proposed by Hitchin in [21]. It turned out to be the appropriate tool to capture the geometry faced by T-duality. It was developed further in [20]. Good reviews are [19, 65]. From the mathematical perspective, the lecture notes [55] can also be recommended. The relation between generalized geometry and T-duality has been discussed in [66]. Investigations towards non-geometric spaces can be found in [24].

Let M be a smooth manifold of dimension D. In contrast to ordinary differential geometry, where the metric is defined with respect to the tangent bundle $TM \to M$, generalized geometry follows the ansatz to combine tangent and cotangent vectors into one object. The resulting vector bundle is then given by the so-called generalized tangent bundle,

$$TM \oplus T^*M \to M. \tag{3.125}$$

More precisely, it can be described of an extension of the tangent bundle by the cotangent bundle. It fits into the following short exact sequence,

$$0 \to T^*M \to E \xrightarrow{\rho} TM \to 0. \tag{3.126}$$

A section of the generalized tangent bundle is given by $X + \alpha \in \Gamma(TM \oplus T^*M)$. We can define a natural pairing on the generalized tangent bundle, given by the fiber metric

$$\langle X + \alpha, Y + \beta \rangle = \frac{1}{2}(\iota_X \beta + \iota_Y \alpha). \tag{3.127}$$

This fiber metric has a natural $O(D, D)$-structure, which makes it useful for the purpose of T-duality, as we will see below. As described above, the generators of the $O(D, D)$-group are given by diffeomorphisms, B- and β-transformations. Under finite B- and β-transformations, the sections transform as

$$e^B(X + \alpha) = X + \alpha + \iota_X B, \quad e^\beta(X + \alpha) = X + \iota_\alpha \beta + \alpha. \tag{3.128}$$

We will now show, how this structure is related to a gerbe with connective structure. Let $\{U_\alpha\}$ be a good cover of M, then the transition from one patch U_α to another patch U_β on the double-overlap $U_{\alpha\beta} = U_\alpha \cap U_\beta$ is given by

$$X_{(\alpha)} + \alpha_{(\alpha)} = A_{(\alpha\beta)} X_{(\beta)} + A^{-T}_{(\alpha\beta)} \alpha_{(\beta)} - \iota_{A_{(\alpha\beta)} X_{(\beta)}} B_{(\alpha\beta)}, \tag{3.129}$$

where $A_{(\alpha\beta)}$ and $A^{-T}_{(\alpha\beta)}$ denote the diffeomorphisms on the tangent and cotangent bundles, respectively. Due to the extension by the cotangent bundle, the transition function is supported by a B-transformation by $B_{(\alpha\beta)}$. This reduces the initial $O(D, D)$-structure to its geometric subgroup $G_B \rtimes GL(D)$, where G_B is generated by the B-transformations. Since $B_{(\alpha\beta)} = -d\Lambda_{(\alpha\beta)}$ on double-overlaps, one finds after several applications of the Poincaré theorem

$$\Lambda_{(\alpha\beta)} + \Lambda_{(\beta\gamma)} + \Lambda_{(\gamma\alpha)} = g_{(\alpha\beta\gamma)} dg_{(\alpha\beta\gamma)} \tag{3.130}$$

on the triple-overlap $U_{\alpha\beta\gamma} = U_{(\alpha)} \cap U_{(\beta)} \cap U_{(\gamma)}$, where $g_{(\alpha\beta\gamma)} = e^{i\phi} \in U(1)$. This is a classical example of an abelian gerbe with globally defined 3-form field strength $H = dB$.

The gauge structure on the generalized tangent bundle is resembled by the Courant bracket,

$$[X + \alpha, Y + \beta]_{\mathrm{C}} = [X, Y]_{\mathrm{Lie}} + L_X \beta - L_Y \alpha - \frac{1}{2} d(\iota_X \beta - \iota_Y \alpha), \tag{3.131}$$

which is invariant under the geometric subgroup of $O(D, D)$, if the B-transformation is induced by a closed form, $dB = 0$. In the case, where $dB = H$, the Courant bracket is twisted by the 3-form H,

$$e^{-B}[X + \alpha, Y + \beta]_{\mathrm{C}} = [e^{-B}(X + \alpha), e^{-B}(Y + \beta)]_{\mathrm{C}} - \iota_X \iota_Y H. \tag{3.132}$$

The resulting bracket is called the H-twisted Courant bracket. Let $\rho : TM \oplus T^*M \to TM$ be the projection to the tangent bundle component. We find, that it preserves the bracket structure,

$$\rho([X+\alpha, Y+\beta]_{\mathrm{C}}) = [\rho(X+\alpha), \rho(Y+\beta)]_{\mathrm{Lie}}. \tag{3.133}$$

Such a map is called bundle morphism. The 4-tuple $(TM \oplus T^*M \to M, \langle -,-\rangle, \rho, [-,-]_{\mathrm{C}})$ is called an exact Courant algebroid.
Since above short sequence is exact, we find an isotropic splitting $\rho^* : TM \to E$ obeying $\langle \rho^* X, \rho^* Y\rangle = 0$ and $\rho \circ \rho^* = \mathrm{id}$. Let a generalized metric on $TM \oplus T^*M$ be a self-adjoint orthogonal endomorphism $\mathcal{H}$, such that $\langle \mathcal{H}(X+\alpha), X+\alpha\rangle > 0$ for all sections [55]. The most general $\mathcal{H}$ can be defined by two orthogonal isotropic splittings,

$$s_+(X) = X + (g+B)(X), \tag{3.134}$$

$$s_-(X) = X + (g-B)(X), \tag{3.135}$$

where g is a symmetric matrix and B an antisymmetric matrix. This leads to the two orthogonal graphs

$$C_+ = \{X + (g+B)(X) | X \in \Gamma(TM)\}, \tag{3.136}$$

$$C_- = \{X + (g-B)(X) | X \in \Gamma(TM)\}. \tag{3.137}$$

Then, we can construct the generalized metric by imposing the eigenvalue equation $C_\pm = \ker(\mathcal{H} \mp 1)$, which is possible since $\mathcal{H}$ is self-adjoint. This gives the well-known generalized metric on the generalized tangent bundle, if we understand g as metric on M and B as B-field of the H-flux.
The metric and the B-field can also be seen as parameterizing the coset space

$$\frac{O(D,D)}{O(D) \times O(D)}, \tag{3.138}$$

since bundles with $O(D,D)$-structure are reducible to $O(D) \times O(D)$-bundles.

3.2.8 Double field theory

In this section, we will provide an introduction to double field theory, a T-duality manifest formulation of the closed string traveling in a torus background. In the main section, we will derive the double field theory gauge algebra and its twisted versions by making use of graded symplectic manifolds.

The T-duality group of closed string theory compactified on a D-dimensional torus T^D is given by $O(D, D; \mathbb{Z})$. Double field theory [25] is formulated such that it contains the $O(D, D; \mathbb{R})$-symmetry on the field theory level. This is done by introduction of a doubled set of coordinates, which are assigned to the winding modes of closed strings wrapping the D-torus. Related early ideas can already be found in [28, 27, 26]. A review can be found in [50]. The so-called doubled formalism was pioneered in [16].

Let us consider n-dimensional closed string theory compactified on a D-torus, T^D. Let the torus be parameterized by local coordinates x^i, where the index i runs from 1 to D. The non-compact directions may be denoted by x^μ, where the index μ runs from $D+1$ to 10.

Double field theory works in the regime of modes of the closed string, which become massless in the decompactification limit. This is given by the cancellation of the oscillator contributions with the vacuum energy in the mass formula,

$$N + \widetilde{N} = 2. \tag{3.139}$$

Taking the level matching condition into account, the solution that works for $P^M P_M = 0$ is given by $N = \widetilde{N} = 1$, which corresponds to the field content of metric g_{ij}, B-field B_{ij} and dilaton ϕ. The generalized momenta have to be orthogonal.

Since double field theory employs $O(D, D)$-invariance on the field-theoretical level, the generalized metric $\mathcal{H}_{MN}$ now consists of non-constant metric g_{ij} and B-field B_{ij}. The raising and lowering of indices is done by using the $O(D, D)$-invariant metric η_{MN}. The metric and the dilaton together give the $O(D, D; \mathbb{Z})$-invariant scalar d via (3.67).

The first crucial step to arrive at an $O(D, D)$-invariant field theory is to introduce a dual torus $\widetilde{T}^D$ in addition to the torus of the compactification, T^D. This dual torus is locally parameterized by coordinates $\widetilde{x}_i$, which are the conjugate coordinates of the winding modes w_i. Recall that the ordinary coordinates x^i are conjugate to the momentum modes p^i. Let us for consistency denote the winding modes by $\widetilde{p}_i = w_i$. As the momentum and winding modes combine into a generalized momentum, the ordinary and dual torus coordinates can be combined into a *generalized coordinate*,

$$X^M = \begin{pmatrix} \widetilde{x}_i \\ x^i \end{pmatrix}. \tag{3.140}$$

Now, we have written everything in terms of $O(D, D)$-fundamentals.

The second crucial step is to let all fields depend on the double set of coordinates, $\mathcal{H}(x) \to \mathcal{H}(X)$ and $d(x) \to d(X)$. The generalized coordinates transform under T-duality via

$$X^M \mapsto h^M_{\ N} X^N, \tag{3.141}$$

where $h \in O(D, D; \mathbb{Z})$, so that the generalized metric and the dilaton, which depends now on the double set of coordinates, transforms according to

$$\mathcal{H}(X) \mapsto h_M^{\ P} h_N^{\ Q} \mathcal{H}_{PQ}(hX), \quad d(X) \mapsto d(hX), \tag{3.142}$$

where $h \in O(D, D; \mathbb{Z})$. In this formulation, T-duality transformations in isometry as well as non-isometry directions are possible.

The introduction of the dual coordinates $\widetilde{x}_i$ on the dual torus $\widetilde{T}^D$ is a means of producing elements that are $O(D, D; \mathbb{Z})$-fundamentals. However, physically this doubling of the degrees of freedom is artificial. In order to return to a physical subspace in the double configuration space $T^D \times \widetilde{T}^D$, the number of degrees of freedom has to be cut in half. This process is performed by demanding that the physical fields live on the subspace specified by the solution of the so-called weak section condition,

$$\eta^{MN} \partial_M \partial_N \Psi(X) = 0, \tag{3.143}$$

where Ψ is any field depending on the generalized coordinates. The generalized derivative is given by $\partial_N = (\widetilde{\partial}^i, \partial_i)$, where $\widetilde{\partial}^i = \frac{\partial}{\partial \widetilde{x}_i}$. This condition is implied by the level matching condition

$$(L_0 - \widetilde{L}_0)\Psi = 0, \tag{3.144}$$

where L_0 and $\widetilde{L}_0$ are the Virasoro operators. The weak section condition is also called weak constraint. If we demand the weak constraint for products of fields,

$$\eta^{MN} \partial_M \partial_N (\Psi(X)\Phi(X)) = 0,$$

we arrive at the strong constraint,

$$\eta^{MN} (\partial_M \Psi(X))(\partial_N \Phi(X)) = 0, \tag{3.145}$$

where Ψ and Φ are any fields. The strong constraint implies the weak constraint. Note that the weak section condition is $O(D, D; \mathbb{Z})$-invariant. It can be rewritten in the form

$$\partial_i \widetilde{\partial}^i \Psi(X) = 0. \tag{3.146}$$

A field theory that lives on a D-dimensional subspace specified by the constraint is called a frame. An obvious reduction of double field theory to a physical subspace is by elimination of the dependence on all the dual coordinates $\widetilde{x}_i$. In this D-dimensional subspace, the supergravity coordinates x^i survive and the associated frame is called the supergravity frame. Contrary to that, the reduction by elimination of the dependence on all ordinary coordinates x^i leaves us with the winding frame, which may contain intricate non-geometric structures. Of course, mixed choices of dependence are also possible and each solution to the section constraint corresponds to a different T-duality frame.

The metric g and the B-field have been combined into the generalized metric. Since the metric is diffeomorphism invariant and the B-field as 2-form gauge field inherits a gauge invariance, we combine both transformations into one generalized gauge transformation $\mathcal{L}_\xi$ with generalized gauge parameter $\xi^M = (\widetilde{\xi}_i, \xi^i)$. A generalized vector V^M transforms under generalized Lie derivative via

$$\mathcal{L}_\xi V^M = \xi^P \partial_P V^M + (\partial^M \xi_P - \partial_P \xi^M) V^P + \omega(V) \partial_P \xi^P V^M, \tag{3.147}$$

where $\omega(V)$ is the weight of the generalized vector V^M. The generalized diffeomorphism action on the generalized metric $\mathcal{H}_{MN}$ and the scalar d is given by

$$\mathcal{L}_\xi \mathcal{H}_{MN} = X^Q \partial_Q \mathcal{H}_{MN} + (\partial_M X^Q - \partial^Q X_M)\mathcal{H}_{QN} + (\partial_N X^Q - \partial^Q X_N)\mathcal{H}_{NQ}, \tag{3.148}$$

$$\mathcal{L}_\xi (e^{-2d}) = \partial_M (\xi^M e^{-2d}). \tag{3.149}$$

The gauge invariant action of double field theory is given by [67]

$$\begin{aligned} S_{\mathrm{DFT}} = \int d^D x d^D \widetilde{x}\, e^{-2d} \big(& 4\mathcal{H}^{MN} \partial_M d \partial_N d - 2\partial_M d \partial_N \mathcal{H}^{MN} \\ & -\frac{1}{2}\mathcal{H}^{MN}\partial_N \mathcal{H}^{KL} \partial_L \mathcal{H}_{MK} + \frac{1}{8}\mathcal{H}^{MN} \partial_M \mathcal{H}^{KL} \partial_N \mathcal{H}_{KL}\Big) . \end{aligned} \tag{3.150}$$

It reduces to the bosonic NS-NS type II supergravity action (2.13), when the section condition is solved in favor of the ordinary coordinates x^i.

The closure of the gauge transformations,

$$[\mathcal{L}_{\xi_1}, \mathcal{L}_{\xi_2}] = \mathcal{L}_{[\xi_1,\xi_2]_{\mathrm{C}}}, \tag{3.151}$$

is obeyed when, the strong constraint is solved. Here, the so-called C-bracket is defined by

$$[\xi_1, \xi_2]^M_{\mathrm{C}} = [\xi_1, \xi_2]^M_{\mathrm{Lie}} + \eta^{MP} \eta_{NQ} \xi^Q_{[1} \partial_P \xi^N_{2]} . \tag{3.152}$$

The C-bracket can be found via antisymmetrization of the D-bracket,

$$[\xi_1, \xi_2]_{\mathrm{D}} = L_{\xi_1}\xi_2. \tag{3.153}$$

The generalized metric $\mathcal{H}$ can be decomposed using generalized vielbeins $E^A{}_M$,

$$\mathcal{H}_{MN} = E^A{}_M S_{AB} E^B{}_N, \tag{3.154}$$

where the indices A,B,... denote flat indices and run from 1 to $2D$ and

$$S_{AB} = \begin{pmatrix} \eta^{ab} & 0 \\ 0 & \eta_{ab} \end{pmatrix}. \tag{3.155}$$

The generalized vielbeins satisfy

$$\eta_{MN} = E^A{}_M \eta_{AB} E^B{}_N, \tag{3.156}$$

where η_{AB} is used to raise and lower flat $O(D, D)$-indices. More precisely, η_{AB} is given by

$$\eta_{AB} = \begin{pmatrix} 0 & \delta_a{}^b \\ \delta^a{}_b & 0 \end{pmatrix}. \tag{3.157}$$

In terms of D-dimensional vielbeins $e_a{}^i$ and B-field, the generalized vielbeins are written as

$$E^A{}_M = \begin{pmatrix} e_a{}^i & e_a{}^j B_{ji} \\ 0 & e^a{}_i \end{pmatrix}, \tag{3.158}$$

where $g_{ij} = e^a{}_i \eta_{ab} e^b{}_j$. The transformation behavior of the generalized vielbeins under generalized diffeomorphisms is given by

$$L_\xi E^A{}_M = \xi^P \partial_P E^A{}_M + (\partial_M \xi^P - \partial^P \xi_M) E^A{}_P. \tag{3.159}$$

If we switch on the β-field, possible non-geometries can be accommodated and the generalized vielbein is given by [68]

$$E^A{}_M = \begin{pmatrix} e_a{}^i & e_a{}^j B_{ji} \\ e^a{}_j \beta^{ji} & e^a{}_i + e^a{}_j \beta^{jk} B_{ki} \end{pmatrix}. \tag{3.160}$$

Finally, the generalized fluxes are defined using the scalar d and the generalized vielbein via [50]

$$\mathcal{F}_{ABC} = E_{CN} L_{E_A} E_B{}^N = 3\Omega_{[ABC]}, \tag{3.161}$$

$$\mathcal{F}_A = -e^{2d} L_{E_A} e^{-2d} = \Omega^B{}_{BA} + 2E_A{}^N \partial_N d, \tag{3.162}$$

where

$$\Omega_{ABC} = E_A{}^M \partial_M E_B{}^N E_{CN} = -\Omega_{ACB} \tag{3.163}$$

is the generalized Weitzenböck connection. The geometric as well as non-geometric fluxes arise as components of $\mathcal{F}_{ABC}$ [69, 68],

$$\begin{aligned} H_{abc} &= \mathcal{F}_{abc} = 3\left(\nabla_{[a}B_{bc]} - B_{d[a}\widetilde{\nabla}^d B_{bc]}\right), &(3.164)\\ F^c_{ab} &= \mathcal{F}^c_{ab} = 2\Gamma^c_{[ab]} + \widetilde{\nabla}^c B_{ab} + 2\Gamma^{mc}_{[a} B_{b]m} + \beta^{cm} H_{mab}, &(3.165)\\ Q^{ab}_c &= \mathcal{F}^{ab}_c = 2\Gamma^{[ab]}_c + \partial_c\beta^{ab} + B_{cm}\widetilde{\partial}^m\beta^{ab} + 2F^{[a}_{mc}\beta^{b]m} - H_{mnc}\beta^{ma}\beta^{nb}, &(3.166)\\ R^{abc} &= \mathcal{F}^{abc} = 3\left(\beta^{[am}\nabla_m\beta^{bc]} + \widetilde{\nabla}^{[a}\beta^{bc]} + B_{mn}\widetilde{\nabla}^n\beta^{[ab}\beta^{c]m} + \beta^{[am}\beta^{bn}\widetilde{\nabla}^{c]}B_{mn}\right)\\ &\quad + \beta^{am}\beta^{bn}\beta^{cl}H_{mnl}, &(3.167) \end{aligned}$$

where the covariant and contravariant derivatives are given by

$$\begin{aligned} \nabla_a B_{bc} &= \partial_a B_{bc} - \Gamma^d_{ab} B_{dc} - \Gamma^d_{ac} B_{bd}, &(3.168)\\ \nabla_a \beta^{bc} &= \partial_a \beta^{bc} + \Gamma^b_{ad}\beta_{dc} + \Gamma^c_{ad}\beta^{bd}, &(3.169)\\ \widetilde{\nabla}^a B_{bc} &= \widetilde{\partial}^a B_{bc} + \Gamma^{ad}_b B_{dc} + \Gamma^{ad}_c B_{bd}, &(3.170)\\ \widetilde{\nabla}^a \beta^{bc} &= \widetilde{\partial}^a \beta^{bc} - \Gamma^{ab}_d \beta^{dc} - \Gamma^{ac}_d \beta^{bd}, &(3.171) \end{aligned}$$

and the covariant and contravariant connections defined by

$$\Gamma^c_{ab} = e_a{}^i \partial_i e_b{}^j e^c{}_j, \quad \Gamma^{ab}_c = e^a{}_i \widetilde{\partial}^i e^b{}_j e_c{}^j. \tag{3.172}$$

Solving the strong constraint reduces the generalized fluxes to the fluxes on the physical subspace associated to a T-dual configuration. The T-duality chain can be recovered by rotation of the internal components of $\mathcal{F}_{ABC}$.

3.2.9 M-theory and U-duality

In this section, we provide an account in M-theory and U-duality. Together with the next section on exceptional generalized geometry, this will set the stage for our investigation of higher gerbe structures in M-theory in section 3.7. The underlying structure of this section consists of the very rich survey [70] and the excellent lecture notes [71].

All superstring theories are defined as asymptotic expansions in the string coupling constant g_S. In other words, each theory is an asymptotic expansion around a different vacuum of an 11-dimensional non-perturbative theory, which is named **M-theory** and whose low-energy limit is 11-dimensional supergravity [34]. 11-dimensional supergravity contains 32 supercharges. Spacetime dimensions lower or equal to 11 allow for supersymmetry with spin-n fields, where $n \leq 2$. However, for higher spacetime dimensions undesired higher spin fields appear. Naively, M-theory can be understood as patch-wise defined, so that on each patch there lives a different superstring theory. These superstring theories are related by dualities, more precisely, T- and S-dualities. When lifted to 11-dimensional supergravity, they combine to U-duality, which is the underlying symmetry of M-theory.

M-theory unifies all superstring theories. We learned that the IIB string theory is self-dual under the inversion of the coupling constant, which is called the strong-weak duality, or S-duality. However, although type IIB and type IIA are related by T-duality, it is not true that for type IIA string theory there exists a dual theory in the strong coupling limit. This is the starting point of M-theory, the strong coupling limit of the type IIA superstring. The argument is very nicely outlined in [11]: Type IIA superstring contains Dp-branes for p even. The D0-brane is charged under the $C^{(1)}$ gauge potential leading to a mass

$$m_{D0} = \frac{k}{g_S}, \tag{3.173}$$

where k is the charge. We observe, that the mass is inverse to the string coupling, so that in the strong coupling limit produces an infinite tower of massless states. The reinterpretation is, that the string coupling is nothing but the radius of the S^1-compactification of an 11-dimensional theory, $R = g_S$. In this way, the type IIA superstring in the strong coupling limit corresponds to the decompactification limit of the 11-dimensional supergravity theory and the m_{D0} states arise via Kaluza-Klein compactification.

The spectrum of 11-dimensional supergravity is given by the 11-dimensional metric g, the 3-form gauge potential C_3, its dual C_6, and a gravitino ψ_μ. The bosonic action of 11-dimensional supergravity is given by [72]

$$S = \frac{1}{2\kappa^2} \int d^{11}x \left(\sqrt{-g}R - \frac{1}{2} F_4 \wedge \star F_4 - \frac{1}{6} C_3 \wedge dC_3 \wedge dC_3 \right), \tag{3.174}$$

where $C_3 \in \Omega^3(M)$ is the 3-form gauge potential with curvature $F_4 = dC_3$ and κ^2 is a constant. R denotes the Ricci scalar. 11-dimensional supergravity does not contain any scalar fields and all couplings are dimensionful. Therefore, M-theory does not have a perturbative expansion, with which to prove UV-finiteness. 11-dimensional supergravity should be seen as low-energy effective theory, whose underlying theory is not known at present time. However, the existence of a UV-completion of 11-dimensional supergravity is conjectured and proposed to be related to the strong coupling limit of the full type IIA superstring. The equations of motion of (3.174) are given by [72]

$$R_{\mu\nu} = \frac{1}{12}\left(F_{4,\mu\epsilon\rho\sigma}F_{4,\nu}^{\ \epsilon\rho\sigma} - \frac{1}{12}g_{\mu\nu}F_4^2\right), \tag{3.175}$$

$$0 = d(\star F_4) + \frac{1}{2}F_4 \wedge F_4. \tag{3.176}$$

The object $R_{\mu\nu}$ denotes the Ricci tensor.

Type IIB supergravity can be reached from M-theory in the following limit. Consider T^2-compactification of M-theory, where the two radia are denoted by R_{10} and R_{11}. The coupling constant of type IIA string theory, $g_{11}^{(A)}$, is recovered by $g_{11}^{(A)} = R_{11}^{\frac{3}{2}}$ in the decompactification limit $R_{11} \to \infty$. Since type IIA on S^1 with radius R_{10} is related by T-duality to type IIB on S^1 with inverse radius, one finds the uncompactified type IIB string theory with coupling constant $g_S^{(B)} = \frac{R_{11}}{R_{10}}$ in the singular limit $(R_{11} \to 0, R_{10} \to 0)$.

Finally, the strong coupling limit of $E_8 \times E_8$ heterotic supergravity is related to a compactification of 11-dimensional supergravity on an interval, $I = S^1/\mathbb{Z}_2$. This compactification is called *Hořava-Witten theory*. The radius of the interval is related to the string coupling of heterotic supergravity via $R = g_S$. As in the type IIA string theory case, the strong coupling limit corresponds to the decompactification limit of Hořava-Witten theory.

There exist BPS-solutions of 11-dimensional supergravity, which are the M2- and M5-branes. We postpone the detailed description of these objects to the second part of this thesis, *Higher gauge theory and multiple M5-branes.*

When 11-dimensional supergravity is Kaluza-Klein compactified on a d-dimensional torus T^d, continuous non-compact global symmetries emerge, which have the structure of split real forms of the exceptional Lie groups $E_{d(d)}$ [35, 36, 37, 38]. In contrast to T-duality, these so-called exceptional symmetries mix weak and strong coupling regimes. It has been proposed, that the discrete subgroup of the exceptional symmetry groups remain an exact non-perturbative symmetry on quantum level of the whole M-theory. This discrete symmetry is named **U-duality** and the associated discrete subgroups $E_{d(d)}(\mathbb{Z})$ are called U-duality groups, which combine the T-duality and S-duality groups. After commenting on the emergence of U-duality in string theory in more detail in the next section, we will describe the program of geometrization of U-duality, which is called *exceptional generalized geometry*, also called

M-geometry. It can be regarded as parallel to generalized geometry, which geometrizes the T-duality symmetry in toroidal string compactifications of type II string theory.

3.2.10 On the emergence of U-duality

M-theory is the underlying theory of all 5 consistent superstring theories. Although its complete description is still unknown, its low-energy effective description is given by 11-dimensional supergravity. It turns out that toroidal compactification of 11-dimensional supergravity exhibits global continuous symmetries that combine S-duality and T-duality groups in a non-commutative way. The resulting groups turn out to be exceptional Lie groups and are termed *U-duality*. The appearance of U-duality symmetries can already be observed when compactifying type IIB string theory on a torus. In this section, we comment on the emergence of exceptional symmetries in toroidal compactifications of type IIB string theory and M-theory. This section is based on [11, 70].

Let us consider type IIB string theory compactified on a 6-dimensional torus, T^6. From T-duality investigations, we know that there emerge $6^2 = 36$ scalars from the metric g_{ij} and Kalb-Ramond field B_{ij}, which parameterize the Narain moduli space.

Furthermore, there are the dilaton ϕ and 0-form gauge field $C^{(0)}$ from the 10-dimensional theory. They parameterize the coset

$$\frac{SL(2;\mathbb{R})}{U(1) \times SL(2;\mathbb{Z})}. \tag{3.177}$$

Then, the Ramond-Ramond 2-form and 4-form gauge fields $C^{(2)}$ and $C^{(4)}$ contribute 15 scalars, respectively. Finally, the duals of the 2-forms $B_{\mu\nu}$ and $C^{(2)}_{\mu\nu}$ in the non-compact 4 dimensions contribute 2 scalars. In sum, we have $36 + 2 + 15 + 15 + 2 = 70$ moduli. It turns out that they locally parameterize the coset

$$E_7/SU(8). \tag{3.178}$$

The group E_7 is generated by $SO(6, 6)$ and $SL(2)$. We conclude, that in the supergravity approximation, the action has a continuous symmetry of E_7 acting on above coset. Furthermore, there arise 56 gauge bosons from the metric and forms, which transform in the **56** of the classical symmetry group E_7. Due to charge quantization, the continuous symmetry of E_7 cannot be an exact symmetry of the quantum theory. However, it is conjectured that its discrete subgroup, which leaves the charge lattice invariant, is the exact symmetry of the complete theory. It is denoted by $E_7(\mathbb{Z})$ and termed *U-duality group*. It contains the T-duality group and S-duality group $O(6,6;\mathbb{Z})$ and $SL(2;\mathbb{Z})$. So the moduli space is in the end given by

$$\frac{E_7}{SU(8)\times E_7(\mathbb{Z})}, \tag{3.179}$$

the U-duality analog of the Narain moduli space.

Let us now go on an discuss the symmetries, which emerge from toroidal compactification of M-theory. Upon compactification on a D-dimensional torus, the 11-dimensional $\mathcal{N}=1$ supersymmetry algebra decomposes under $SO(1,11)\to SO(1,10-D)\times SO(D)$. Here, $SO(1,10-D)$ is the Lorentz group of the non-compact directions and $SO(D)$ is the so-called R-symmetry. On field theory level, the R-symmetry is enhanced.

The fields of the NS and R sectors of type IIA string theory combine when going to the 11-dimensional supergravity description. In the case of a D-dimensional torus compactification, the NS and R sectors mix under the emerging symmetry group $SL(D;\mathbb{R})$. The continuous symmetry group of the low-energy effective theory is given by

$$E_{D(D)} = SO(D-1,D-1;\mathbb{R}) \bowtie SL(D;\mathbb{R}). \tag{3.180}$$

Here, $\bowtie$ expresses the fact, that $E_{D(D)}$ is generated by two non-commuting subgroups. The symbol $E_{D(D)}$ is used since it turned out that the emerging groups correspond to the so-called $E_{D(D)}$-series, where $E_{D(D)}$ is the so-called *normal real form* of the exceptional group E_D. In the normal real form, all Cartan generators and positive roots are non-compact. The scalars of the toroidally compactified theory, which emerge from the metric, the 3-form and its dual, parameterize the coset $E_{D(D)}/H_D$, where H_D denotes the maximal compact subgroup of $E_{D(D)}$. It corresponds to the enhanced R-symmetry. As we stated already in the case of type IIB string theory, the continuous symmetry $E_{D(D)}(\mathbb{R})$ cannot be a symmetry of the quantum theory due to charge quantization. The discrete subgroup of $E_{D(D)}(\mathbb{R})$, which leaves the charge lattice invariant, is denoted by $E_{D(D)}(\mathbb{Z})$. The charges arise from the Kaluza-Klein momenta along the torus and the wrapping modes of M2- and M5-branes on internal cycles of the torus. It is conjectured, that the U-duality group of M-theory on a D-dimensional torus is generated by T-duality of type IIA string theory on a $(D-1)$-dimensional torus and the modular group of T^D, leading to the conjectured structure

$$E_{D(D)}(\mathbb{Z}) = SO(D-1,D-1;\mathbb{Z}) \bowtie SL(D,\mathbb{Z}). \tag{3.181}$$

The second component emerges from the reparameterization invariance on the D-dimensional torus.

3.2.11 Exceptional generalized geometry

This section describes the idea of exceptional generalized geometry and its relation to M-theory in an economical manner. It serves as orientation for the analysis conducted in 3.7. For the sake of brevity, we cannot be exhaustive at this point. Selected background information will be provided during the analysis in 3.7. This section is partly based on the pioneering article of *M-geometry* [40]. Further important development can be found in the articles [73, 74, 75, 76].

As described before, the bosonic field content of 11-dimensional supergravity consists of an 11-dimensional metric and a bosonic matter 3-form. When compactified on a d-torus, the field theory exhibits a natural symmetry under the discrete action of split real forms of exceptional Lie groups $E_{d(d)}(\mathbb{Z})$. This symmetry is called U-duality and can be shown to be composed of S-duality and T-duality.

When we considered T-duality in the context of toroidally compactified string theory, it turned out that the generalized tangent bundle $E = TM \oplus T^*M$ together with the generalized metric made out of g and B can capture T-dual geometry astonishingly well due to its natural $O(D, D)$-structure. The cotangent bundle part captures the wrapping modes of closed strings on the torus.

In M-theory, there no strings are present. However, depending on the dimension of the torus, on which 11-dimensional supergravity is compactified, M2-brane charges, M5-brane charges or even KK6-monopole charges can arise. This suggests a generalized tangent bundle, which contains an additional 2-form cotangent part to accommodate the M2-brane wrapping modes, a 5-form cotangent part to accommodate the M5-brane wrapping modes and a 6-vector part for the Kaluza-Klein monopole charge,

$$E = TM \oplus \wedge^2 T^*M \oplus \wedge^5 T^*M \oplus \wedge^6 TM. \tag{3.182}$$

This bundle is an example of a so-called exceptional generalized tangent bundle and transforms in the adjoint of $E_{d(d)}$. The U-duality action mixes momentum and winding modes of M2- and M5-branes. The 2-form twists of generalized geometry become 3- and 6-form twists in exceptional generalized geometry.

In generalized geometry, the metric and the B-field parameterize the coset space (3.138). Let H_d be the maximal compact subgroup of $E_{d(d)}$. In exceptional generalized geometry, the coset space is given by $E_{d(d)}/H_d$, which is naturally parameterized by the 11-dimensional metric g, the 3-form gauge field C_3 and its Hodge dual C_6, depending on the dimension of the torus compactification.

We understand that non-geometric spaces like the T-folds associated with Q-flux and even more intricate spaces associated with R-flux arise in toroidal string compactifications by T-duality transformation. The transition functions of T-folds take values in the full $O(D, D)$-group, not just its geometric subgroup, and lead to non-geometric fluxes. In the same manner, non-geometric fluxes and non-geometric spaces can arise from U-duality transformations on toroidal compactifications of 11-dimensional supergravity. Such spaces, whose transition functions take values in the full U-duality group are named *U-folds*.

Exceptional generalized geometry attempts to construct an $E_{d(d)}$-covariant generalized formulation of geometry, which captures the underlying symmetry of 11-dimensional supergravity backgrounds, along the lines of generalized geometry. In the same way as generalized geometry provides a geometry for generic T-dual flux backgrounds, the exceptional generalized geometry shall provide a geometric perspective on general U-dual flux compactifications of M-theory. This is done by the introduction of so-called *exceptional generalized metrics* on the exceptional generalized tangent bundle.

The exceptional generalized tangent bundles that transform in the adjoint representation under the respective exceptional Lie groups are summarized in table 3.1. It is partly composed of information taken from [70, 40, 77]. "Dec." means decomposition under $SL(d;\mathbb{R})$. In the cases $d = 2, 3, 4$, there are no M5-brane modes, since they cannot be accommodated by a d-torus for $d < 5$. However, the appropriate $\wedge^5 T^*M$-component emerges for $d = 5$ and higher. For the case $d = 7$ and higher, an additional symmetry due to dual diffeomorphisms emerges, accommodated by the intricate $\wedge^6 TM \cong (T^*M \otimes \wedge^7 T^*M)$-part. In the case $d = 8$, the structure becomes more intricate, due to the emergence of dual components associated with so-called *exotic branes*, which are sourced by non-geometric fluxes. On each exceptional tangent bundle, an exceptional generalized metric is defined, which contains both the original metric and the gauge potentials.

We follow the argument in [78] relating M-theory with exceptional generalized geometry on the algebraic level. The equations of motion of 11-dimensional supergravity incorporate a

Exceptional groups and generalized tangent bundles

d	Lie Group	H_d	Dec.	Irreps.	Exceptional tangent bundle
2	$SL(2,\mathbb{R}) \times \mathbb{R}$	$SO(2)$	$\mathbf{3}$	$\mathbf{2}+\mathbf{1}$	$TM \oplus \wedge^2 T^*M$
3	$SL(3,\mathbb{R}) \times SL(2,\mathbb{R})$	$SO(3) \times SO(2)$	$\mathbf{(3,2)}$	$\mathbf{3}+\mathbf{3}$	$TM \oplus \wedge^2 T^*M$
4	$SL(5,\mathbb{R})$	$SO(5)$	$\mathbf{10}$	$\mathbf{4}+\mathbf{6}$	$TM \oplus \wedge^2 T^*M$
5	$\mathrm{Spin}(5,5)$	$\frac{\mathrm{Sp}(2)\times\mathrm{Sp}(2))}{\mathbb{Z}_2}$	$\mathbf{16}$	$\mathbf{5}+\mathbf{10}+\mathbf{1}$	$TM \oplus \wedge^2 T^*M \oplus \wedge^5 T^*M$
6	$E_{6(6)}$	$\frac{\mathrm{Sp}(4)}{\mathbb{Z}_2}$	$\mathbf{27}$	$\mathbf{6}+\mathbf{15}+\mathbf{6}$	$TM \oplus \wedge^2 T^*M \oplus \wedge^5 T^*M$
7	$E_{7(7)}$	$\frac{SU(8)}{\mathbb{Z}_2}$	$\mathbf{56}$	$\mathbf{7}+\mathbf{21}+\mathbf{21}+\mathbf{7}$	$TM \oplus \wedge^2 T^*M \oplus \wedge^5 T^*M$ $\oplus(T^*M \otimes \wedge^7 T^*M)$
8	$E_{8(8)}$	$\frac{\mathrm{Spin}(16)}{\mathbb{Z}_2}$	$\mathbf{248}$	$2(\mathbf{8}+\mathbf{28}+\mathbf{56})+\mathbf{63}+\mathbf{1}$	$TM \oplus \wedge^2 T^*M \oplus \wedge^5 T^*M$ $\oplus(T^*M \otimes \wedge^7 T^*M)$ $\oplus \wedge^5 TM \oplus \wedge^2 TM \oplus T^*M$

Table 3.1

relation between the 4-form field strength F_4 of the gauge potential C_3 and its Hodge dual $F_7 = \star F_4$,

$$dF_4 = 0, \tag{3.183}$$

$$d(F_7) + \frac{1}{2}F_4 \wedge F_4 = d\left(F_7 + \frac{1}{2}C_3 \wedge F_4\right) = 0. \tag{3.184}$$

Introducing the local potential C_6 with

$$dC_6 = F_7 + \frac{1}{2}C_3 \wedge F_4, \tag{3.185}$$

we find

$$F_4 = dC_3, \quad F_7 = dC_6 - \frac{1}{2}C_3 \wedge dC_3. \tag{3.186}$$

This system of equations in invariant under action of the semi-direct product of the diffeomorphism group with a group that parameterizes closed 3- and 6-form twists. It turns out, that the correct generalized tangent bundle, which transforms in the adjoint under the invariance group of the equations of motion is given by

$$E = TM \oplus \wedge^2 T^*M \oplus \wedge^5 T^*M. \tag{3.187}$$

The sections of the bundle transform infinitesimally under 3-form and 6-form twists as

$$A_3 \triangleright (X + \sigma + \overline{\sigma}) = \iota_X A_3 - A_3 \wedge \sigma, \tag{3.188}$$

$$A_6 \triangleright (X + \sigma + \overline{\sigma}) = -\iota_X A_6 \wedge \sigma, \tag{3.189}$$

where $X + \sigma + \overline{\sigma} \in \Gamma(TM \oplus \wedge^2 T^*M \oplus \wedge^5 T^*M)$. In analogy to the Dorfman bracket in generalized geometry, there is a natural bracket on the generalized tangent bundle, which is invariant under diffeomorphisms and closed form twists. It generates the generalized symmetries of the bundle and is given by

$$[X + \sigma + \overline{\sigma}, Y + \gamma + \overline{\gamma}]_{\mathrm{D}} = [X, Y]_{\mathrm{Lie}} + L_X\gamma - \iota_Y d\sigma + L_X\overline{\gamma} - \iota_Y d\overline{\sigma} + d\sigma \wedge \gamma, \tag{3.190}$$

where $X + \sigma + \overline{\sigma}, X + \gamma + \overline{\gamma} \in \Gamma(TM \oplus \wedge^2 T^*M \oplus \wedge^5 T^*M)$. As the Dorfman bracket in generalized geometry can be twisted by a 3-form H-flux, the exceptional Dorfman bracket can be twisted by the 4-form and 7-form field strengths F_4 and F_7,

$$\begin{aligned}[X + \sigma + \overline{\sigma}, Y + \gamma + \overline{\gamma}]_{\mathrm{D},F_4,F_7} = [X, Y]_{\mathrm{Lie}} &+ L_X\gamma - \iota_Y d\sigma + L_X\overline{\gamma} - \iota_Y d\overline{\sigma} + d\sigma \wedge \gamma \\ &+ \iota_X\iota_Y F_4 + \iota_X\iota_Y F_4 \wedge \gamma,\end{aligned} \tag{3.191}$$

where $F_4 \in \Omega^4(M)$ and $F_7 \in \Omega^5(M)$. The Leibniz identity of the twisted Dorfman bracket then requires the equations of motion (3.183) and (3.184). The resulting algebroid captures underlying local symmetries of toroidally compactified 11-dimensional supergravity with M2- and M5-branes wrapping closed cycles. From the perspective of dg-Leibniz algebras induced by $\mathbb{R}[n]$-bundles, the structure of the generalized tangent bundles of $E_{6(6)}$-generalized geometry has been studied in [79]. The structure of $E_{7(7)}$-generalized geometry has been studied in [39] and investigations on associated U-dual non-geometric fluxes can be found in [41].

3.3 Mathematical preliminaries

This section serves as an introduction to the various mathematical fields touched in the first part of this thesis. Due to fact, that all these fields are intertwined, we provide additional information on how certain mathematical structures can be seen from different perspectives using different mathematical tools. The study of such relations is interesting in its own right. However, our approach should provide adequate knowledge so that the very reader does not get lost in the vast number of mathematical structures.

3.3.1 Poisson geometry, cohomology and connections

In this section, we give an introduction to Poisson manifolds, their Lichnerowicz-Poisson cohomology and contravariant connections. Furthermore, we discuss the isomorphism between de Rham and Lichnerowicz-Poisson cohomology in the case of a non-degenerate Poisson structure. The background will become important in the analysis of the Poisson-Courant algebroid and its relation to the standard Courant algebroid of generalized geometry. It is mainly based on the excellent lecture notes [80] and the article [81]. Useful information on the Lichnerowicz-Poisson cohomology associated to a Poisson manifold can be found in [82]. Standard results are presented in the textbook [83]. A very good review that relates various incarnations of Poisson geometry to Lie algebroid theory can be found in [84].
We start with the definition of a Poisson algebra.

Definition 3.3.1 (Poisson algebra) *Let A be an associative algebra over a field k. Furthermore, let $\{-,-\} : A \otimes A \to A$ be a Lie bracket, that satisfies the Leibniz identity,*

$$\{f, \{gh\}\} = g\{f, h\} + \{f, g\}h. \tag{3.192}$$

The bracket $\{-,-\}$ *is called a* <u>*Poisson bracket*</u> *and the* 2*-tuple* $(A, \{-,-\})$ *is a* <u>*Poisson algebra*</u>*.*

Through the Leibniz identity, the Poisson bracket acts as a derivation of the product in A. The Poisson manifold is a Poisson algebra on a space of smooth functions.

Definition 3.3.2 (Poisson manifold) *Let* M *be a smooth manifold. Let* $\{-,-\} : \mathcal{C}^\infty(M) \otimes \mathcal{C}^\infty(M) \to \mathcal{C}^\infty(M)$ *be a Poisson bracket on the associative algebra of smooth functions on* M *with point-wise multiplication. The bracket is called a* <u>*Poisson structure*</u> *and the* 2*-tuple* $(M, \{-,-\})$ *is called a* <u>*Poisson manifold*</u>*.*

Two Poisson manifolds are related by a Poisson map.

Definition 3.3.3 (Poisson map) *Let* $(M_1, \{-,-\}_1)$ *and* $(M_2, \{-,-\}_2)$ *be two Poisson manifolds. A map* $\Psi : M_1 \to M_2$ *is called a* <u>*Poisson map*</u>*, if it preserves the Poisson bracket.*

If we have a Poisson manifold, we can define a Hamiltonian vector field with associated Hamiltonian function.

Definition 3.3.4 (Hamiltonian vector field) *Let* $(M, \{-,-\})$ *be a Poisson manifold. Furthermore, let* $\Theta \in \mathcal{C}^\infty(M)$ *be a smooth function on* M*. We define the* <u>*Hamiltonian vector field*</u> *associated to* Θ *as*

$$X_\Theta(f) = \{\Theta, f\}, \tag{3.193}$$

for all $f \in \mathcal{C}^\infty(M)$*. In this case,* Θ *is called a* <u>*Hamiltonian function*</u>*.*

There is a natural bracket on the space of polyvector fields, the *Schouten bracket*.

Definition 3.3.5 (Schouten bracket) *Let* M *be a smooth manifold. Furthermore, let* $X \in \mathfrak{X}^i(M)$ *and* $Y \in \mathfrak{X}^j(M)$ *be* i*- and* j*-vector fields. The* <u>*Schouten bracket*</u> *between polyvector fields is defined as*

$$[X, Y]_S = X \circ Y - (-1)^{(i-1)(j-1)} Y \circ X, \tag{3.194}$$

where

$$Y \circ X(df_1, \ldots, df_{i+j-1}) = \sum_\sigma (-1)^\sigma Y(d(X(df_{\sigma(1)}, \ldots, df_{\sigma(i)})), df_{\sigma(i+1)}, \ldots, df_{\sigma(i+j-1)}), \tag{3.195}$$

where σ *are* $(i, j-1)$*-shuffles.*

As we will see in the following, a choice of bivector field Π on M, $\Pi \in \mathfrak{X}^2(M)$, can be used to induce a Poisson structure. Interpreting general k-vectors Ξ on a smooth manifold M, $\Xi \in \mathfrak{X}^k(M)$, as totally antisymmetric multilinear maps on the space of 1-forms by contraction,

$$\Xi : \Omega^1(M)^{\otimes_k} \to \mathcal{C}^\infty(M), \tag{3.196}$$

one can induce the following multilinear map on the space of smooth functions on M

$$\widehat{\Xi} : \mathcal{C}^\infty(M)^{\otimes_k} \to \mathcal{C}^\infty(M) \tag{3.197}$$

via the identification

$$\widehat{\Xi}(f_1, \dots, f_k) = \Xi(df_1, \dots, df_k), \tag{3.198}$$

where $f_i \in \mathcal{C}^\infty(M)$ for $i = 1, \dots, k$. Then, with a choice of a 2-vector field $\Pi \in \mathfrak{X}^2(M)$, the following bracket bracket can be induced,

$$\{f, g\}_\Pi = \Pi(df, dg), \tag{3.199}$$

where $f, g \in \mathcal{C}^\infty(M)$. The Schouten bracket of Π with itself gives the Jacobi identity of the induced bracket,

$$\frac{1}{2}[\Pi, \Pi]_\text{S}(df, dg, dh) = \{\{f, g\}_\Pi, h\}_\Pi + \{\{h, f\}_\Pi, g\}_\Pi + \{\{g, h\}_\Pi, f\}_\Pi. \tag{3.200}$$

We are lead to the following proposition.

Proposition 3.3.6 *Let M be a smooth manifold. Furthermore, let $\Pi \in \mathfrak{X}^2(M)$ be a bivector field, that induces the bracket $\{-,-\}_\Pi : \mathcal{C}^\infty(M) \otimes \mathcal{C}^\infty(M) \to \mathcal{C}^\infty(M)$. Then, the 2-tuple $(M, \{-,-\}_\Pi)$ is a Poisson manifold, if $[\Pi, \Pi]_S = 0$.*

Such a Π is called Poisson structure or Poisson tensor. A Poisson manifold $(M, \{-,-\}_\Pi)$ can also be denoted by (M, Π).

Using a bivector field $\Pi \in \mathfrak{X}^2(M)$ we can define a map between the cotangent and tangent bundles over M via the musical isomorphism,

$$\Pi^\sharp : T^*M \to TM, \tag{3.201}$$

defined by $\Pi^\sharp : \alpha \mapsto \iota_\alpha \Pi$. Then, the Hamiltonian vector field X_Θ associated to a Hamiltonian function Θ can be written by

$$X_\Theta = \Pi^\sharp(d\Theta). \tag{3.202}$$

Definition 3.3.7 (Non-degenerate bivector field) *Let* $\Pi \in \mathfrak{X}^2(M)$ *be a bivector field.* Π *is called* <u>*non-degenerate*</u>*, if* $\Pi^\sharp_x$ *is an isomorphism for all* $x \in M$*.*

If Π is non-degenerate, then the matrix (Π^{ij}) is invertible, and vice versa.
In a symmetric manner, we can associate a map between the tangent and cotangent bundles over M with a 2-form $\omega \in \Omega^2(M)$ using via

$$\omega^\flat : TM \to T^*M, \tag{3.203}$$

defined by $\omega^\flat : X \mapsto \iota_X \omega$.

Definition 3.3.8 (Non-degenerate 2-form) *Let* $\omega \in \Omega^2(M)$ *be a 2-form on* M*.* ω *is called* <u>*non-degenerate*</u>*, if* $\omega^\flat_x$ *is an isomorphism for all* $x \in M$*.*

We find an important relation between Π and ω in the non-degeneracy case.

Proposition 3.3.9 *Let* $\Pi \in \mathfrak{X}^2(M)$ *and* $\omega \in \Omega^2(M)$ *be a non-degenerate bivector field and a non-degenerate 2-form. Then, we have the one-to-one correspondence,*

$$\omega^\flat = (\Pi^\sharp)^{-1} \leftrightarrow \Pi^\sharp = (\omega^\flat)^{-1}. \tag{3.204}$$

In this case, we have the relation

$$[\Pi, \Pi](\alpha, \beta, \gamma) = -2d\omega(\Pi^\sharp(\alpha), \Pi^\sharp(\beta), \Pi^\sharp(\gamma)), \tag{3.205}$$

where $\alpha, \beta, \gamma \in T^*M$*.*

We now establish a relation between symplectic structures and Poisson structures. First recall some definitions regarding symplectic structures.

Definition 3.3.10 (Symplectic structure) *Let* M *be a smooth manifold. Furthermore, let* $\omega \in \Omega^2(M)$ *be a 2-form on* M*.* ω *is called a* <u>*symplectic structure*</u>*, if it is closed,* $d\omega = 0$*, and non-degenerate.*

Definition 3.3.11 (Symplectic manifold) *Let* M *be a smooth manifold equipped with a symplectic structure* ω*. The 2-tuple* (M, ω) *is called a* <u>*symplectic manifold*</u>*.*

The following proposition relates symplectic structures with Poisson structures. It is important for the analysis of the Poisson-Courant algebroid.

Proposition 3.3.12 *Let M be a smooth manifold. Then, non-degenerate Poisson structures on M are in one-to-one correspondence to symplectic structures on M.*

Their matrix elements are related by inversion, $(\omega_{ij}(x)) = (\Pi^{ij}(x))^{-1}$, or $\omega = \Pi^{-1}$ for short. Having understood Poisson manifolds and their relation to symplectic manifolds, we can now go on and define the cohomology associated to Poisson manifolds. With a choice of Poisson structure Π, we can define a natural differential on the complex of polyvectors.

Definition 3.3.13 (Lichnerowicz-Poisson cohomology) *Let Π be a Poisson structure on a smooth manifold M. We define the so-called Lichnerowicz-Poisson differential as contraction of the Poisson structure with the Schouten bracket,*

$$d_\Pi = [\Pi, -]_S, \tag{3.206}$$

acting on polyvector fields $d_\Pi : \mathfrak{X}^k(M) \to \mathfrak{X}^{k+1}(M)$. The nilpotency of the differential is equivalent to the Poisson condition of Π,

$$d_\Pi^2 = 0 \Leftrightarrow [\Pi, \Pi]_S = 0. \tag{3.207}$$

Therefore, we can define the cohomology of the complex $(d_\Pi, \mathfrak{X}^\bullet(M))$,

$$H^k_{LP}(M, \Pi) = \frac{ker(d_\Pi : \mathfrak{X}^k(M) \to \mathfrak{X}^{k+1}(M))}{im(d_\Pi : \mathfrak{X}^{k-1}(M) \to \mathfrak{X}^k(M))}. \tag{3.208}$$

This cohomology is called the Lichnerowicz-Poisson cohomology.

Obviously, Π defines a fundamental class $[\Pi]$ in $H^2_{\mathrm{LP}}(M, \Pi)$.
The *Koszul bracket* can be regarded as the Lie bracket on 1-forms. It plays a crucial role in the definition of the Poisson-Courant algebroid.

Definition 3.3.14 (Koszul bracket) *Let (M, Π) be a Poisson manifold. We can define a natural bracket on the space of 1-forms by*

$$[\alpha, \beta]_\Pi = L_{\Pi^\sharp(\alpha)}\beta - L_{\Pi^\sharp(\beta)}\alpha - d(\Pi(\alpha, \beta)), \tag{3.209}$$

where $\alpha, \beta \in \Omega^1(M)$ and $L_\bullet$ denotes the Lie derivative. The bracket $[-,-]_\Pi$ is called the Koszul bracket and becomes a Lie bracket on $\Omega^1(M)$, since Π is a Poisson structure.

We conclude, that for a Poisson manifold (M, Π), the space $(\Omega^1(M), [-,-]_\Pi)$ is a Lie algebra. The musical isomorphism $\Pi^\sharp : T^*M \to TM$ can be extended to polyforms via

$$\Pi^\sharp : \Omega^k(M) \to \mathfrak{X}^k(M), \tag{3.210}$$

in local coordinates

$$\begin{aligned} \Pi^\sharp\alpha &= \frac{1}{k!}\Pi^\sharp\alpha_{i_1\cdots i_k} dx^{i_1} \wedge \cdots \wedge dx^{i_k} \\ &= \frac{1}{k!}\alpha_{j_1\cdots j_k}\Pi^{j_1 i_1}\cdots\Pi^{j_k i_k} dx^{i_1} \wedge \cdots \wedge dx^{i_k}. \end{aligned} \tag{3.211}$$

Note that the use the same symbol $\Pi^\sharp$ for any degree k. In the literature it is sometimes denoted by $\wedge^k\Pi^\sharp$. This provides us with a relation between de Rham cohomology and Lichnerowicz-Poisson cohomology over M, given by

$$d_\Pi(\Pi^\sharp\alpha) = \Pi^\sharp(d\alpha), \tag{3.212}$$

for any k-form $\alpha \in \Omega^k(M)$. We recognize, that d-coboundaries map to d_Π-coboundaries and d-cocycles map to d_Π-cocycles. In the case of a non-degenerate Poisson tensor, the inverse defines a symplectic manifold and this relation becomes an isomorphism, which lifts to an isomorphism between de Rham and Lichnerowicz-Poisson cohomology,

$$\Pi^\sharp : H^\bullet_{\text{de Rham}}(M) \to H^\bullet_{\text{LP}}(M, \Pi). \tag{3.213}$$

For completeness, let us define a deformation of the Poisson manifold in presence of a closed 3-form H [85, 86].

Definition 3.3.15 (Twisted Poisson manifold) *Let M be a smooth manifold and let $\Pi \in \mathfrak{X}^2(M)$ a bivector such that*

$$\frac{1}{2}[\Pi, \Pi]_S = \Pi^\sharp H, \tag{3.214}$$

where $H \in \Omega^3(M)$ is a closed 3-form, $dH = 0$. Then, the 3-tuple (M, Π, H) is called a twisted Poisson manifold and Π is called a twisted Poisson structure.

Finally, let us discuss connections on a Poisson manifold associated with the Poisson tensor. Such connections are called contravariant connections. We start by defining the Poisson module according to [87].

Definition 3.3.16 (Poisson module) *Let A be a Poisson algebra and k a field. Let E be an A-module equipped with a k-linear map $\lambda : A \times E \to E$ that satisfies*

$$\lambda(\{a,b\},e) = \lambda(a,\lambda(b,e)) - \lambda(b,\lambda(a,e)), \tag{3.215}$$

$$\{a,b\}e = a\lambda(b,e) - \lambda(b,ae), \tag{3.216}$$

where $a, b \in A$ and $e \in E$. The 2-tuple (E,λ) is called a Poisson module.

Then, we can define a Poisson vector bundle according to [88].

Definition 3.3.17 (Poisson vector bundle) *Let $E \to M$ be a vector bundle and let $\Gamma^\infty(E)$ be the space of smooth sections of E seen as projective $\mathcal{C}^\infty(M)$-module. Then, $E \to M$ has the structure of a Poisson vector bundle, if $\Gamma^\infty(E)$ is a Poisson module.*

We arrive at the definition of a contravariant connection.

Definition 3.3.18 (Contravariant connection) *Let $E \to M$ be a vector bundle. Furthermore, let $D : \Omega^1(M) \times \Gamma^\infty(E) \to \Gamma^\infty(E)$ be a bilinear map such that*

$$D_{f\alpha}s = fD_\alpha s, \tag{3.217}$$

$$D_\alpha(fs) = fD_\alpha s - \alpha(X_f)s, \tag{3.218}$$

where $\alpha \in \Omega^1(M)$, $f \in \mathcal{C}^\infty(M)$ and $s \in \Gamma^\infty(E)$. Then, D is a contravariant connection on the vector bundle $E \to M$.

Recall, that $\alpha(X_f) = \alpha(\Pi^\sharp(df)) = -(\Pi^\sharp\alpha)(f)$.
An important special case of the contravariant connection is the *linear contravariant connection* appearing in relation with the Poisson-Courant algebroid.

Definition 3.3.19 (Linear contravariant connection) *Let $F^*(M) \to M$ be a vector bundle over an n-dimensional manifold M with structure group $GL(n)$, so that $F^*(M)$ is the coframe bundle over M.*
A contravariant connection on $F^*(M) \to M$ is called a linear contravariant connection.

In the case of a linear contravariant connection, $\Gamma^\infty(E)$ is given by the space of sections on T^*M and the linear operator is given by $D : \Omega^1(M) \times \Omega^1(M) \to \Omega^1(M)$.

Let $U \subset M$ be an open subset of M. Then, the contravariant Christoffel symbols are locally defined by

$$D_{dx^i} dx^j = \Gamma^{ij}_k dx^k. \tag{3.219}$$

Under coordinate transformations, the contravariant Christoffel symbols transform as

$$\widetilde{\Gamma}^{lm}_n = \frac{\partial y^l}{\partial x^i}\frac{\partial y^m}{\partial x^j}\frac{\partial x^k}{\partial y^n}\Gamma^{ij}_k + \frac{\partial y^l}{\partial x^i}\frac{\partial^2 y^m}{\partial x^j \partial x^k}\frac{\partial x^j}{\partial y^n}\Pi^{ik}. \tag{3.220}$$

One recognizes the contribution by the Poisson tensor Π, which is in contrast to ordinary Christoffel connections on frame bundles. The action of the contravariant derivative D_α along a 1-form α onto a general tensor $T^{i_1\cdots i_r}_{k_1\cdots k_s}$ is given by

$$(D_\alpha T)^{i_1\cdots i_r}_{k_1\cdots k_s} = \Pi^{kl}\alpha_k T^{i_1\cdots i_r}_{k_1\cdots k_s} - \sum_{a=1}^{r}\Gamma^{ki_a}_n \alpha_k T^{i_1\cdots n\cdots i_r}_{k_1\cdots k_s} + \sum_{a=1}^{r}\Gamma^{kn}_{k_a}\alpha_k T^{i_1\cdots i_r}_{k_1\cdots n\cdots k_s}. \tag{3.221}$$

In the case, where we have a vector bundle $E \to M$ equipped with an ordinary connection ∇, we can define a contravariant connection by

$$D_{df}\alpha = \nabla_{X_f}\alpha, \tag{3.222}$$

where $\alpha \in \Omega^1(M)$. The curvature and torsion tensors are defined by

$$R(\alpha,\beta)\gamma = D_\alpha D_\beta \gamma - D_\beta D_\alpha \gamma - D_{[\alpha,\beta]_\Pi}\gamma, \tag{3.223}$$

$$T(\alpha,\beta) = D_\alpha \beta - D_\beta \alpha - [\alpha,\beta]_\Pi. \tag{3.224}$$

Locally, they can be written by

$$R^{ijk}_l = 2\Gamma^{[i|r|}_l \Gamma^{j]k}_r + 2\Pi^{[i|r|}\partial_r \Gamma^{j]k}_l - \partial_r \Pi^{ij}\Gamma^{rk}_l, \tag{3.225}$$

$$T^{ij}_k = 2\Gamma^{[ij]}_k - \partial_k \Pi^{ij}. \tag{3.226}$$

Finally, we define a special linear contravariant connection, the *Poisson connection.*

Definition 3.3.20 (Poisson connection) *Let (M,Π) be a Poisson manifold. A Poisson connection is a linear contravariant connection on M, such that $D\Pi = 0$, where D is the contravariant derivative.*

A Poisson tensor, for which $D\Pi = 0$, is called parallel.

3.3.2 L_∞-algebras

The structures of L_∞-algebras lie at the heart of many analyses conducted in this thesis. Especially, in section 3.7 we construct L_∞-algebras, which encode the local symmetries of higher gerbes as they appear in the generalized geometries associated with T- and U-duality, by making use of graded symplectic manifolds. Therefore, this section provides a short introduction to the realm of L_∞-algebras supplemented by some pedagogical examples.
The concept of an L_∞-algebras was first introduced in [89]. An introduction to this subject can be found in [90]. We start by defining the notion of a graded vector space.

Definition 3.3.21 (Graded vector space) *A graded vector space V is a vector space, which is a direct sum decomposition, given by*

$$V = \bigoplus_k V_k, \tag{3.227}$$

such that elements $v \in V_k$ have degree $|v| = k$.

An L_∞-algebra is then a graded vector space with certain maps, that satisfy certain conditions.

Definition 3.3.22 (L_∞-algebra) *Let $L = \bigoplus_k L_k$ be a graded vector space. Furthermore, for $n \leq 1$, let $\mu_n : \bigotimes^n L \to L$ be graded antisymmetric multilinear maps of degree $|\mu_n| = n-2$. The graded antisymmetry of μ_n is expressed by the appearance of the Koszul sign,*

$$\mu_n(l_{\sigma(1)}, \ldots, l_{\sigma(n)}) = \chi(\sigma; l_1, \ldots, l_n)\mu_n(l_1, \ldots, l_n). \tag{3.228}$$

Then, for homogeneously graded elements $l_1, \ldots, l_n$, such that $n \leq 1$, the maps μ_n obey the so-called homotopy Jacobi identities,

$$\sum_{i+j=n} \sum_{\sigma} \chi(\sigma; l_1, \ldots, l_n)(-1)^{ij} \mu_{j+1}(\mu_i(l_{\sigma(1)}, \ldots, l_{\sigma(i)}), l_{\sigma(i+1)}, \ldots, l_{\sigma(n)}) = 0. \tag{3.229}$$

The permutations σ are so-called (i,j)-unshuffles, which are permutations of $(i+j)$ elements such that $\sigma(1) < \cdots < \sigma(i)$ and $\sigma(i+1) < \cdots < \sigma(i+j)$.
The graded vector space L together with the maps μ_n is called an L_∞-algebra.

An L_∞-algebra is also referred to as strong homotopy Lie algebra. One calls an L_∞-algebra concentrated in degrees $\{i_1, \ldots, i_m\}$, if all L_k are trivial except for the degrees $\{i_1, \ldots, i_m\}$.

Definition 3.3.23 (n-term L_∞-algebra) *An n-term L_∞-algebra is an L_∞-algebra, which is concentrated in degrees* $\{0, \dots, n-1\}$.

An n-term L_∞-algebra is categorically equivalent to a semistrict Lie n-algebra. If the highest multilinear map μ_{n+1} vanishes, then it is called a strict Lie n-algebra. Let us now give some examples.

Example 3.3.1 (Lie algebra) Let $L = L_0$ be an L_∞-algebra concentrated in degree 1. Since $|\mu_1| = -1$, obviously $\mu_1 = 0$. The only non-trivial map is μ_2, which is of degree zero,

$$\mu_2 : L_0 \otimes L_0 \to L_0. \tag{3.230}$$

The map μ_2 is antisymmetric by construction and the only non-trivial homotopy Jacob identity is given by

$$\mu_2(\mu_2(l_1, l_2), l_3) = \mu_2(\mu_2(l_1, l_3), l_2) - \mu_2(\mu_2(l_2, l_3), l_1), \tag{3.231}$$

which is the usual Jacobi identity. We conclude, that the L_∞-algebra concentrated in degree 1, (L_0, μ_2), is a Lie algebra. It can also be called a 1-term L_∞-algebra or a Lie 1-algebra.

Example 3.3.2 (dg-Lie algebra) Let $L = \bigoplus_i L_i$ be an L_∞-algebra, such that all maps μ_k, for $k \geq 3$, are trivial. The homotopy Jacobi identities implies the nilpotency of μ_1,

$$\mu_1 \circ \mu_1 = 0, \tag{3.232}$$

the graded Jacobi identity of μ_2,

$$\mu_2(l_1, \mu_2(l_2, l_3)) = \mu_2(\mu_2(l_1, l_2), l_3) + (-1)^{|l_1||l_2|}\mu_2(l_2, \mu_2(l_1, l_3)), \tag{3.233}$$

and that μ_1 is a graded derivation of μ_2,

$$\mu_1(\mu_2(l_1, l_2)) = \mu_2(\mu_1(l_1), l_2) + (-1)^{|l_1|}\mu_2(l_1, \mu_1(l_2)). \tag{3.234}$$

The graded Jacobi identity can also be read in the sense, that the adjoint action $\mu_2(l_1, -)$ is a graded derivation of μ_2. The resulting structure $(L = \bigoplus_i L_i, \mu_1, \mu_2)$ is called a differential graded Lie algebra, or dg-Lie algebra. We conclude, that L_∞-algebras, which have all maps μ_k for $k \geq 3$ trivial, are dg-Lie algebras.

The Lie crossed module is the governing structure of parallel transport of 1-dimensional objects and the starting point of our analysis the second part of this thesis. Let us therefore in the following discuss how the differential crossed module, the infinitesimal approximation of the Lie crossed module, fits into the realm of L_∞-algebras.

Example 3.3.3 (Differential crossed module as dg-Lie algebra) Let $(\mathfrak{g}, \mathfrak{h}, \underline{t}, \underline{\alpha})$ be a differential crossed module. We can construct a dg-Lie algebra $L = \left(\mathfrak{h}[1] \xrightarrow{t} \mathfrak{g}\right)$, where μ_1 is identified with $\underline{t}$, $\mu_2 : \mathfrak{g} \otimes \mathfrak{g} \to \mathfrak{g}$ is the Lie bracket on $\mathfrak{g}$, $\mu_2 : \mathfrak{g} \otimes \mathfrak{h}[1] \to \mathfrak{h}[1]$ is identified with $\underline{\alpha}$ by

$$\mu_2(g, h) = \underline{\alpha}(g)(h). \tag{3.235}$$

The object $(\mathfrak{h}[1] \xrightarrow{t} \mathfrak{g}, \mu_2)$ defines a dg-Lie algebra. We conclude, that the homotopy Jacobi identities of the associated dg-Lie algebra are equivalent to the conditions of the differential crossed module.

We furthermore conclude, that a differential crossed module is an example of a strict L_∞-algebra, more precisely, a strict Lie 2-algebra. In order to clarify the difference between strict and semistrict Lie n-algebras, we discuss the example of a semistrict Lie 2-algebra.

Example 3.3.4 (Semistrict Lie 2-algebra) Let $L = L_0 \oplus L_1$ be an L_∞-algebra concentrated in degrees 0 and 1. In this case, the maps $\mu_1 : L_1 \to L_0$, $\mu_2 : L_0 \otimes L_0 \to L_0$, $\mu_2 : L_0 \otimes L_1 \to L_1$ and $\mu_3 : L_0 \otimes L_0 \otimes L_0 \to L_1$ are non-trivial. Similar to the strict case, the unary bracket μ_1 (differential) is a nilpotent derivation of the binary bracket μ_2, which in form of $\mu_2 : L_0 \otimes L_0 \to L_0$ is a Lie bracket. The part $\mu_2 : L_0 \otimes L_1 \to L_1$ encodes an action of L_0 on L_1. The non-trivial 3-bracket μ_3, called the Jacobiator "Jac", distinguishes the strict Lie 2-algebra from a semistrict Lie 2-algebra. For non-trivial μ_3, we get the following deformed or additional homotopy Jacobi identities,

$$\mu_1(\mu_3(x_1, x_2, x_3)) = -\mu_2(\mu_2(x_1, x_2), x_3) + \mu_2(\mu_2(x_1, x_3), x_2) - \mu_2(\mu_2(x_2, x_3), x_1) \tag{3.236}$$

$$\mu_3(\mu_1(y), x_1, x_2) = -\mu_2(\mu_2(x_1, x_2), y) - \mu_2(\mu_2(y, x_1), x_2) + \mu_2(\mu_2(y, x_2), x_1) \tag{3.237}$$

$$\begin{aligned}
&- \mu_2(\mu_3(x_1, x_2, x_3), x_4) + \mu_2(\mu_3(x_1, x_2, x_4), x_3) - \mu_2(\mu_3(x_1, x_3, x_4), x_2)\\
&+ \mu_2(\mu_3(x_2, x_3, x_4), x_1) = -\mu_3(\mu_2(x_1, x_2), x_3, x_4) - \mu_3(\mu_2(x_2, x_3), x_1, x_4) - \mu_3(\mu_2(x_3, x_4), x_1, x_2)\\
&- \mu_3(\mu_2(x_1, x_4), x_2, x_3) + \mu_3(\mu_2(x_1, x_3), x_2, x_4) + \mu_3(\mu_2(x_2, x_4), x_1, x_3).
\end{aligned} \tag{3.238}$$

The first equation is the Jacobi identity broken by the differential of the Jacobiator. The second equation is called the action property. The last equation is the so-called coherence property of the Jacobiator.

For vanishing Jacobiator the semistrict Lie 2-algebra condenses to a strict Lie 2-algebra.

3.3.3 Courant algebroids

In this section, we will give an introduction into the realm of Courant algebroids. They will serve as a crucial ingredient in the analysis of the Poisson-Courant algebroid, T-duality presentation in the graded symplectic manifold setup and T-dual non-geometric flux backgrounds. Their higher analogues appear in the analysis of exceptional generalized geometry with M2-branes.
We start with the definition of a general Courant algebroid.

Definition 3.3.24 (Courant algebroid) *Let $E \to M$ be a vector bundle over E, where M is a smooth manifold. Furthermore, let the vector bundle E be equipped with a fiber metric $\langle -,- \rangle : E \otimes E \to \mathbb{R}$ and a binary bracket $[-,-]_D : E \otimes E \to E$. Finally, let $\rho : E \to TM$ be a bundle map to the tangent bundle over M. The bundle map is also referred to as anchor map. Then, the 4-tuple $(E \to M, \langle -,- \rangle, \rho, [-,-]_D)$ constitutes a Courant algebroid, if the operations obey the conditions*

$$[e^1,[e^2,e^3]_D]_D = [[e^1,e^2]_D,e^3]_D + [e^2,[e^1,e^3]_D]_D, \tag{3.239}$$

$$\rho(e^1)\langle e^2,e^3\rangle = \langle [e^1,e^2]_D,e^3\rangle + \langle e^2,[e^1,e^3]_D\rangle, \tag{3.240}$$

$$\rho(e^1)\langle e^2,e^3\rangle = \langle e^1,[e^2,e^3]_D + [e^3,e^2]_D\rangle, \tag{3.241}$$

where e^1, e^2, $e^3 \in \Gamma(E)$. The binary bracket $[-,-]_D$ is called Dorfman bracket.

The antisymmetrization of the Dorfman bracket is called the Courant bracket,

$$[e^1,e^2]_\mathrm{C} = \frac{1}{2}([e^1,e^2]_\mathrm{D} - [e^2,e^1]_\mathrm{D}). \tag{3.242}$$

Special subclasses of Courant algebroids are the transitive Courant algebroid [91] and the exact Courant algebroid. The transitive Courant algebroid appears in heterotic string theory, where in addition to the metric and a 2-form field a non-abelian gauge field emerges. The exact Courant algebroid has been discussed in relation with generalized geometry. For the relation of Courant algebroids to Lie bialgebroids see [22].

Definition 3.3.25 (Transitive Courant algebroid) *Let $(E \to M, \langle -,- \rangle, \rho, [-,-]_D)$ be a Courant algebroid. The Courant algebroid is called transitive, if the bundle map ρ is surjective.*

Definition 3.3.26 (Exact Courant algebroid) *A Courant algebroid on E is called exact, if it fits into the short exact sequence,*

$$0 \to T^*M \to E \xrightarrow{\rho} TM \to 0. \tag{3.243}$$

Exact Courant algebroids are examples of transitive Courant algebroids.
Let us in the following discuss important examples of Courant algebroids: the standard Courant algebroid on $TM \oplus T^*M$, the H-twisted standard Courant algebroid, the Poisson-Courant algebroid and its R-twisted version.

Example 3.3.5 (Standard Courant algebroid) Let M be a smooth manifold. Let us take as the vector bundle the direct product of tangent and cotangent bundle, $E = TM \oplus T^*M$. This vector bundle is called the generalized tangent bundle. Any section of the generalized tangent bundle, $e \in \Gamma(TM \oplus T^*M)$, can be written in the form $e = X + \alpha$, where $X \in TM$ and $\alpha \in T^*M$. The fiber metric $\langle -,- \rangle$ is the paring between the form and vector components,

$$\langle X + \alpha, Y + \beta \rangle = \iota_X \beta + \iota_Y \alpha, \tag{3.244}$$

where $X + \alpha, Y + \beta \in \Gamma(TM \oplus T^*M)$. The bundle map is chosen to be the natural projection to the tangent component,

$$\rho(X + \alpha) = X, \tag{3.245}$$

where $X + \alpha \in \Gamma(TM \oplus T^*M)$. Finally, the Dorfman bracket is given by

$$[X + \alpha, Y + \beta]_{\mathrm{D}} = [X, Y]_{\mathrm{Lie}} + L_X \beta - \iota_Y d\alpha. \tag{3.246}$$

Above operations obey the conditions for a Courant algebroid and the 4-tuple $(TM \oplus T^*M \to M, \langle -,- \rangle, \rho, [-,-]_{\mathrm{D}})$ is called the standard Courant algebroid.

It turns out that standard Courant algebroids are classified by a 3-form, $H \in H^3(M, \mathbb{R})$ [92]. This class is also referred to as Ševera class of the Courant algebroid. This independent degree of freedom can be introduced as a twist of the standard Courant algebroid as follows.

Example 3.3.6 (H-twisted standard Courant algebroid) Let $TM \oplus T^*M \to M$ be the generalized tangent bundle. The fiber metric as well as the anchor map are the same as in the case of the untwisted standard Courant algebroid. However, the Dorfman bracket gains an additional term,

$$\begin{aligned}[X+\alpha, Y+\beta]_{\mathrm{D},H} &= [X+\alpha, Y+\beta]_{\mathrm{D}} + \iota_X \iota_Y H \\ &= [X,Y] + L_X\beta - \iota_Y d\alpha + \iota_X\iota_Y H, \end{aligned} \tag{3.247}$$

where $X+\alpha, Y+\beta \in \Gamma(TM \oplus T^*M)$ and the closed 3-form $H \in \Omega^3(M)$ with $dH = 0$ is the Ševera class. Note that we introduced the <u>H-twisted Dorfman bracket</u> $[-,-]_{\mathrm{D},H}$ in order to distinguish it from the untwisted Dorfman bracket, $[-,-]_{\mathrm{D}}$. Then, the 4-tuple $(TM \oplus T^*M \to M, \langle -,-\rangle, \rho, [-,-]_{\mathrm{D},H})$ is called the <u>H-twisted standard Courant algebroid</u>.

The <u>H-twisted Courant bracket</u> is defined by antisymmetrization of the H-twisted Dorfman bracket,

$$[X+\alpha, Y+\beta]_{C,H} = \frac{1}{2}([X+\alpha, Y+\beta]_{\mathrm{D},H} - [Y+\beta, X+\alpha]_{\mathrm{D},H}). \tag{3.248}$$

A particularly useful type of Courant algebroid for the analysis of non-geometric R-flux geometries in string theory is the so-called Poisson-Courant algebroid, which is a Courant algebroid defined on a Poisson manifold. It has been used in the definition of Poisson-generalized geometry as a model for non-geometric R-flux [32]. A contravariant version of topological T-duality based on the Poisson-Courant algebroid has been developed in [33].

Definition 3.3.27 (Poisson-Courant algebroid, [32]) *Let M be a smooth manifold, that is equipped with a Poisson tensor $\Pi \in \Gamma(\wedge^2 TM)$, so that (M, Π) becomes a Poisson manifold. Furthermore, let $E = TM \oplus T^*M \to M$ be the generalized tangent bundle over (M, Π). We define the three Courant algebroid operations as follows. The fiber metric on E is given as usual,*

$$\langle X+\alpha, Y+\beta\rangle = \iota_X\beta + \iota_Y\alpha, \tag{3.249}$$

where $X+\alpha, Y+\beta \in \Gamma(E)$. The bundle map $\rho : E \to TM$ is defined by

$$\rho(X+\alpha) = \Pi^\sharp(\alpha) = \Pi^{ij}\alpha_i\partial_j, \tag{3.250}$$

where $\Pi^\sharp : T^*M \to TM$ denotes the musical isomorphism. Furthermore, the Dorfman bracket is defined by

$$[X+\alpha, Y+\beta]^\pi_D = [\alpha,\beta]_\Pi + L^\Pi_\alpha Y - \iota_\beta d_\Pi X, \tag{3.251}$$

where $X+\alpha, Y+\beta \in \Gamma(E)$. The bracket $[-,-]_\Pi : T^*M \otimes T^*M \to T^*M$ denotes the Koszul bracket with respect to the Poisson tensor Π, $L^\Pi_\alpha = \iota_\alpha d_\Pi + d_\Pi \iota_\alpha$ denotes the contravariant Lie derivative associated to Π and ι_α is the contravariant interior product acting on polyvectors. Finally, $d_\Pi = [\Pi, -]_S$ denotes the Lichnerowicz-Poisson differential.
The 4-tuple $(TM \oplus T^*M \to (M,\Pi), \langle -,- \rangle, \rho, [-,-]^\Pi_D)$ is called a Poisson-Courant algebroid.

Theorem 3.3.28 *The Poisson-Courant algebroid is a Courant algebroid.*

The physically and string theoretically appealing feature of the Poisson-Courant algebroid is the natural 3-vector freedom. In the same sense, as standard Courant algebroids can be twisted and are classified by a d-closed 3-form H, Poisson-Courant algebroids can be twisted by and are classified by a d_Π-closed 3-vector R. This fact leads to the following definition.

Definition 3.3.29 (R-twisted Poisson-Courant algebroid, [32]) *Let $(TM \oplus T^*M \to (M,\Pi), \langle -,- \rangle, \rho, [-,-]^\Pi_D)$ be a Poisson-Courant algebroid. Furthermore, let $R \in \Gamma(\wedge^3 TM)$ be a 3-vector field on M, which is d_Π-closed, where d_Π is the Lichnerowicz-Poisson differential associated with Π.*
The fiber metric on $TM \oplus T^*M$ and the anchor map are the same as in the Poisson-Courant algebroid case. However, the Dorfman bracket is twisted by the 3-vector R,

$$\begin{aligned}[X+\alpha, Y+\beta]^\pi_{D,R} &= [X+\alpha, Y+\beta]^\pi_D + \iota_\alpha \iota_\beta R \\ &= [\alpha,\beta]_\Pi + L^\Pi_\alpha Y - \iota_\beta d_\Pi X + \iota_\alpha \iota_\beta R, \end{aligned} \tag{3.252}$$

where $X+\alpha, Y+\beta \in \Gamma(E)$. The 4-tuple $(TM \oplus T^*M \to (M,\Pi), \langle -,- \rangle, \rho, [-,-]^\Pi_{D,R})$ is called an R-twisted Poisson-Courant algebroid.

Theorem 3.3.30 *The R-twisted Poisson-Courant algebroid is a Courant algebroid with Ševera class R.*

A direct comparison of the standard Courant algebroid and Poisson-Courant algebroid leads to the insight that due to the existence of the Poisson structure the 1-forms can be lifted to 1-vectors in such a manner that the Courant algebroid structure is preserved. In other words, the Poisson-Courant algebroid is the standard Courant algebroid of contravariant geometry in which the roles of tangent and cotangent bundles are exchanged. In this sense, we can call the Poisson-Courant algebroid a contravariant Courant algebroid. We will elucidate this relation during the main analysis of this part.

For convenience, we define Dirac structures associated to standard Courant algebroids.

Definition 3.3.31 (Dirac structure) *Let $(TM \oplus T^*M \to M, \langle -,- \rangle, \rho, [-,-]_D)$ be a standard Courant algebroid. A Dirac structure L is a half-rank subbundle $L \to M$ of $TM \oplus T^*M$, which satisfies*

$$[\Gamma(L), \Gamma(L)]_D \subset \Gamma(L) \quad \textit{Integrability}, \tag{3.253}$$

$$\langle L, L \rangle = 0 \quad \textit{Maximal isotropicity.} \tag{3.254}$$

Finally, let us discuss higher Courant algebroids [93, 94], which are important for the analysis of exceptional generalized geometry. We follow the definition according to [94].

Definition 3.3.32 (Courant algebroid of degree n) *Let M be a smooth manifold. Let $E_n = TM \oplus \wedge^n T^*M \to M$ be a vector bundle over M. Furthermore, let $\langle -,- \rangle : E_n \otimes E_n \to \wedge^{n-1} T^*M$ be a symmetric fiber metric. Let $[-,-]_D : E_n \otimes E_n \to E_n$ be a bilinear map, which we call the Dorfman bracket of degree n. Finally, let $\rho : E_n \to TM$ be the projection to TM. If the fiber metric and Dorfman bracket are defined as*

$$\langle X + \alpha, Y + \beta \rangle = \iota_X \beta + \iota_Y \alpha, \tag{3.255}$$

$$[X + \alpha, Y + \beta]_D = [X, Y]_{Lie} + L_X \beta - \iota_Y d\alpha, \tag{3.256}$$

then they satisfy the relations

$$[e^1, [e^2, e^3]_D]_D = [[e^1, e^2]_D, e^3]_D + [e^2, [e^1, e^3]_D]_D, \tag{3.257}$$

$$[e^1, f e^2] = f[e^1, e^2] + \rho(e^1)(f) e^2, \tag{3.258}$$

$$[f e^1, e^2] = f[e^1, e^2] - \rho(e^2)(f) e^1 + df \wedge \langle e^1, e^2 \rangle, \tag{3.259}$$

where $e^1, e^2, e^3 \in E_n$, $f \in \mathcal{C}^\infty(M)$, and define a Courant algebroid of degree n.

The associated higher Courant bracket is given by antisymmetrization of the higher Dorfman bracket,

$$\begin{aligned} [X + \alpha, Y + \beta]_{\mathrm{C}} &= \frac{1}{2}([X + \alpha, Y + \beta]_{\mathrm{D}} - [Y + \beta, X + \alpha]_{\mathrm{D}}) \\ &= [X, Y]_{\mathrm{Lie}} + L_X \beta - L_Y \alpha - \frac{1}{2} d(\iota_X \beta - \iota_Y \alpha). \end{aligned} \tag{3.260}$$

We recognize, that the crucial difference between the standard Courant algebroid and higher Courant algebroid is that they accommodate forms of different degree. The 1-form component

in the standard Courant algebroid serving as a freedom to accommodate the winding modes of closed strings in toroidal compactifications and closely related to generalized geometry is generalized to an n-form component. For $n = 2$ the generalized tangent bundle can accommodate the wrapping modes of M2-branes, and for $n = 5$ even the wrapping modes for M5-branes. However, in the toroidally compactified 11-dimensional supergravity setting, both M2- and M5-branes might be present simultaneously. In this case, a higher Courant algebroid is insufficient. We will analyze this situation in this part of the thesis from the perspective of graded symplectic manifolds.
We end this section by stating the following theorem that relates higher Courant algebroids to L_∞-algebras. It makes use of and generalizes the construction in [95].

Theorem 3.3.33 ([94]) *Let M be a smooth manifold. The complex*

$$\mathcal{C}^\infty(M) \xrightarrow{d} \Omega^1(M) \xrightarrow{d} \cdots \xrightarrow{d} \Omega^{n-2} \xrightarrow{d} \Gamma(E_{n-1}) = TM \oplus \wedge^{n-1}T^*M \tag{3.261}$$

has a Lie n-algebra structure with the following operations. μ_1 is given by the de Rham differential. μ_2 is given by the Courant bracket of degree $n-1$ on sections of $\Gamma(E_{n-1})$,

$$\mu_2(-,-) = [-,-]_D : E_{n-1} \otimes E_{n-1} \to E_{n-1} \tag{3.262}$$

and for $X + \alpha \in \Gamma(E_{n-1})$ and $\beta \in \Omega^{\bullet<n-1}(M)$ given by the Lie derivative,

$$\mu_2(X + \alpha, \xi) = \frac{1}{2} L_X \xi. \tag{3.263}$$

Then, μ_3 is given by

$$\mu_3(X + \alpha, Y + \beta, Z + \gamma) = -\frac{1}{3!}(\langle [X + \alpha, Y + \beta]_C, Z + \gamma\rangle \pm \textit{perm.}), \tag{3.264}$$

where $X + \alpha, Y + \beta, Z + \gamma \in \Gamma(E_{n-1})$ and by

$$\mu_3(\xi, X + \alpha, Y + \beta) = -\frac{1}{6}\left(\frac{1}{2}(\iota_X L_Y - \iota_Y L_X) + \iota_{[X,Y]_{Lie}}\right)\xi, \tag{3.265}$$

where $X + \alpha, Y + \beta \in \Gamma(E_{n-1})$ and $\xi \in \Omega^{\bullet<n-1}(M)$. The brackets μ_n for $n \geq 4$ are given by

$$\mu_n(X_1 + \alpha_1, \ldots, X_n + \alpha_n) = \sum_i [X_1, \ldots, \alpha_i, \ldots, X_n], \tag{3.266}$$

where

$$[\alpha, X_1, \ldots, X_{n-1}] = \frac{(-1)^{\frac{n+1}{2}} 12 B_{n-1}}{(n-1)(n-2)} \sum_{1\leq i<j\leq n-1} (-1)^{i+j+1} \iota_{X_{n-1}} \cdots \widehat{\iota_{X_j}} \cdots \widehat{\iota_{X_i}} \iota_{X_1} [\alpha, X_i, X_j], \tag{3.267}$$

for sections of $\Gamma(E_{n-1})$*. For* $\xi \in \Omega^{\bullet<n-1}(M)$*, the maps are given by*

$$[\xi, X_1+\alpha_1, \dots, X_{n-1}+\alpha_{n-1}] = \frac{(-1)^{\frac{n+1}{2}} 12 B_{n-1}}{(n-1)(n-2)} \sum_{1\leq i<j\leq n-1} (-1)^{i+j+1} \iota_{X_{n-1}} \cdots \widehat{\iota_{X_j}} \cdots \widehat{\iota_{X_i}} \iota_{X_1} [\xi, X_i, X_j]. \tag{3.268}$$

Here, B_n *denote the Bernoulli numbers.*

3.3.4 Graded manifolds and supergeometry

In this section, we will give an introduction in graded manifolds and supergeometry. An introduction to supergeometry from the mathematical perspective can be found in [96]. The lecture notes [97] cover also the physical perspective. A good introduction with helpful examples, which also covers the relation to so-called $\mathbb{R}[n]$-bundles can be found in [79]. Furthermore, we recommend [98, 99] for more details on the underlying structures to be introduced.

Supergeometry is an extremely useful method to rewrite mathematical structures and study their properties. We will make use of it in order to gain understanding of the underlying structures of T-duality and U-duality. The presentation of this mathematical subject is supplemented by many introductory examples always stressing the relations to other important mathematical constructions. In the end, this section culminates in the analysis of Courant algebroids, their higher generalizations and twists of Courant algebroids from the supergeometric point of view. It will bring us into the position to generate deformed versions of Courant algebroids, which are related to double field theory and T-duality.

Let us start with the definition of a graded manifold and explain what grading is.

Definition 3.3.34 (Graded Manifold) *Let* M *be a smooth manifold. A* graded manifold $\mathcal{M}$ *is a locally ringed space* $(M, \mathcal{O}_M)$*, which is locally isomorphic to* $(U, \mathcal{C}^\infty(U) \wedge W^*)$*, where* U *is an open subset of* $\mathbb{R}^n$ *and* W *is a vector space of finite dimension. The isomorphism is such that the* parity *is preserved,*

$$\bigoplus_k \mathcal{C}^\infty(U) \otimes \wedge^k W^* \to \mathbb{Z}_2, \quad f \otimes x \mapsto |f \otimes x| = |x| = k \mod 2. \tag{3.269}$$

A graded manifold with $\mathbb{Z}_2$-grading is called a supermanifold. Locally, a graded manifold consists of open subsets of $\mathbb{R}^n$ complemented with odd coordinates from W^*. The algebra of smooth functions over $\mathcal{M}$ is denoted by $\mathcal{C}^\infty(\mathcal{M})$. Homogeneous elements $f, g \in \mathcal{C}^\infty(\mathcal{M})$ have an associated degree, denoted by $|f|$ and $|g|$, such that

$$fg = (-1)^{|f||g|} gf. \tag{3.270}$$

Having the graded manifold at hand, we can equip it with a differential, denoted by Q.

Definition 3.3.35 (Differential graded manifold) *Let $\mathcal{M}$ be a graded manifold. Furthermore, let Q be a vector field of degree 1 on $\mathcal{M}$, $Q \in \mathfrak{X}^1(\mathcal{M})$, so that it is homological, $Q^2 = 0$. Then, the resulting structure $(\mathcal{M}, Q)$ is called differential graded manifold.*

In general, a vector field $Q \in \mathfrak{X}^1(\mathcal{M})$ is called homological, if $Q^2 = 0$. A Differential graded manifolds is also called dg-manifold or Q-manifold.
An important class of Q-manifolds are the non-negatively graded ones. This leads to the following definition.

Definition 3.3.36 (NQ-manifold) *Let $(\mathcal{M}, Q)$ be a dg-manifold. The 2-tuple $(\mathcal{M}, Q)$ is called an NQ-manifold, if $\mathcal{M}$ is non-negatively graded.*

Let us consider a simple example of an NQ-manifold, which encodes the structure of a de Rham complex.

Example 3.3.7 (De Rham complex) Let M be an ordinary smooth manifold. Let us consider the graded manifold $\mathcal{M} = T[1]M$. The object $T[1]$ takes the tangent bundle, so that the degree of the tangent fiber coordinates is shifted by 1. Then, $\mathcal{M}$ can locally be described by coordinates (x^i, ξ^i) of degrees $(0, 1)$, where the index runs over $i = 1, 2, \ldots \dim(M)$. The coordinates x^i denote the local coordinates on M, whereas ξ^i denote the local coordinates on the fiber. Furthermore, we choose the homological vector field $Q = \xi^i \frac{\partial}{\partial x^i}$. The space of smooth functions on $\mathcal{M}$ can be decomposed by degree,

$$\mathcal{C}^\infty(\mathcal{M}) = \bigoplus_{k=0}^{\infty} \mathcal{C}_k^\infty(\mathcal{M}), \tag{3.271}$$

where $\mathcal{C}_k^\infty(\mathcal{M})$ denotes the subset of smooth functions on $\mathcal{M}$, which are of degree k. More precisely, these spaces are given by

$$\mathcal{C}_k^\infty(\mathcal{M}) = \left\{ \frac{1}{k!} \alpha_{i_1 \cdots i_k} \xi^{i_1} \cdots \xi^{i_k} | \alpha_{i_1 \cdots i_k} \in \mathcal{C}^\infty(M) \right\}. \tag{3.272}$$

The homological vector field Q raises the degree by 1,

$$Q\left(\frac{1}{k!}\alpha_{i_1\cdots i_k}\xi^{i_1}\cdots\xi^{i_k}\right)=\frac{1}{k!}\partial_{i_1}\alpha_{i_2\cdots i_{k+1}}\xi^{i_1}\cdots\xi^{i_{k+1}} \tag{3.273}$$

and the homological condition, $Q^2=0$, is trivially satisfied. Therefore, we find the complex

$$0\to\mathcal{C}_0^\infty(\mathcal{M})\xrightarrow{Q}\mathcal{C}_1^\infty(\mathcal{M})\xrightarrow{Q}\mathcal{C}_2^\infty(\mathcal{M})\xrightarrow{Q}\cdots\xrightarrow{Q}\mathcal{C}_{\dim(M)}^\infty(\mathcal{M})\xrightarrow{Q}0. \tag{3.274}$$

Let us define an injection map $j:T^*M\to\mathcal{M}$ from the cotangent bundle over M to the graded manifold $\mathcal{M}$ by

$$j:(x^i,dx^i)\mapsto(x^i,\xi^i), \tag{3.275}$$

so that the pullback of an element $\alpha\in\mathcal{C}_k^\infty(\mathcal{M})$ along j is given by

$$\alpha=\frac{1}{k!}\alpha_{i_1\cdots i_k}\xi^{i_1}\cdots\xi^{i_k}\mapsto j^*(\alpha)=\frac{1}{k!}\alpha_{i_1\cdots i_k}dx^{i_1}\wedge\cdots\wedge dx^{i_k}\in\Omega^k(M). \tag{3.276}$$

Furthermore, the homological vector field is related to the de Rham differential d on $\Omega^\bullet(M)$ via $j^*\circ Q=d\circ j^*$, so that the whole complex (3.274) can be pulled back to the de Rham complex over M,

$$0\to\Omega^0(M)\xrightarrow{d}\Omega^1(M)\xrightarrow{d}\Omega^2(M)\xrightarrow{d}\cdots\xrightarrow{d}\Omega^{\dim(M)}(M)\xrightarrow{d}0. \tag{3.277}$$

We conclude, that the structure induced by the Q-manifold $(T[1]M,Q=\xi^i\frac{\partial}{\partial x^i})$ encodes the de Rham complex over M.

In general, the so-called *shift functor* $[n]$ shifts the degrees of objects it is applied to. For example, an element v of an ordinary vector space V has degree $|v|=0$. However, if we apply the shift functor, we find the graded vector space $V[n]$ with elements v of degree $|v|=n$. In the same sense, if we have a vector bundle $E\to M$ and apply the shift functor to give $E[n]\to M$, then the local coordinates of the fiber of E are shifted in degree by n. Locally, the shifted vector bundle is given by $F[n]\times U$, where U is an open subset of M and $F[n]$ is the degree shifted fiber of $E[n]$. The shift of the tangent bundle is denoted by $T[n]M$ and the local fiber coordinates gain the degree n. The shift of the cotangent bundle, which is denoted by $T^*[n]M$, produces local fiber coordinates of degree n.

Finally, let us consider the double fibration, $T^*[n]T[m]M$. In this case, local coordinates x^i on M are of degree 0. When treating the first fiber $T[m]$, we find local fiber coordinates ξ^i of degree m. Now, we take a second fiber $T^*[n]$ over $T[m]M$. To each local coordinate

(x^i, ξ^i) on $T[m]M$ we assign a dual coordinate on the fiber, where the degree is shifted by m. Therefore, we find coordinates p_i dual to x^i of degree $m - 0 = m$ and coordinates ζ_i dual to ξ^i of degree $m - n$. The degree of the initial coordinate has to be subtracted, since we take the **co**tangent fiber, which is **dual** to the tangent fiber.
Now let us endow the Q-manifold with an additional graded symplectic structure.

Definition 3.3.37 (Symplectic NQ-manifold) *Let $(\mathcal{M}, Q)$ be an NQ-manifold. Equip $\mathcal{M}$ with a graded symplectic structure ω of degree n and demand the compatibility condition $L_Q\omega = 0$. Then, $(\mathcal{M}, Q, \omega)$ is called a symplectic NQ-manifold of degree n.*

A symplectic NQ-manifold is also called QP-manifold, where "Q" refers to the homological vector field Q and "P" to the graded Poisson structure, which is induced by the graded symplectic structure as follows. The graded symplectic structure induces a graded Poisson bracket on the space of smooth functions on $\mathcal{M}$, $\{-,-\} : \mathcal{C}^\infty(\mathcal{M}) \otimes \mathcal{C}^\infty(\mathcal{M}) \to \mathcal{C}^\infty(\mathcal{M})$, via

$$\{f, g\} = (-1)^{|f|+n+1} \iota_{X_f} \iota_{X_g} \omega, \tag{3.278}$$

where $f, g \in \mathcal{C}^\infty(\mathcal{M})$ and $|f|$ denotes the degree of f. The object X_f denotes the Hamiltonian vector field of f and is defined by

$$\iota_{X_f} \omega = -\delta f, \tag{3.279}$$

where δ is the de Rham differential on $\mathcal{M}$. If we are given a QP-manifold $(\mathcal{M}, Q, \omega)$ of degree n, then there exists a function $\Theta \in \mathcal{C}^\infty(\mathcal{M})$ of degree $n + 1$, such that

$$\{\Theta, f\} = Qf, \tag{3.280}$$

for any $f \in \mathcal{C}^\infty(\mathcal{M})$. This function Θ is called Hamiltonian function associated to the homological vector field Q. The property of the vector field to be homological is then expressed by the classical master equation,

$$Q^2 = 0 \Leftrightarrow \{\Theta, \Theta\} = 0. \tag{3.281}$$

The Hamiltonian function is also called homological function. The 2-tuple $(\mathcal{M}, \omega)$ can be referred to as P-manifold, where ω is the so-called P-structure. The vector field Q is referred to as Q-structure.
QP-manifolds lie at the heart of this thesis. They serve as the underlying structure in almost all investigations conducted in this thesis. QP-manifolds are deeply related with the

Batalin-Vilkoviski formalism [42, 43]. For details on the mathematics surrounding graded mathematics, we refer to appendix B.

It will turn out, that the analysis of double field theory using the supergeometric point of view needs a slight generalization of the QP-manifold. Let us give the definition of this weaker version of the QP-manifold, which has been first established in [100].

Definition 3.3.38 (Pre-QP-manifold) *Let $\mathcal{M}$ be a graded manifold. Let furthermore $\mathcal{M}$ be equipped with a graded symplectic structure ω of degree n. Finally, let Q be a vector field defined on $\mathcal{M}$ such that $L_Q\omega = 0$. The 3-tuple $(\mathcal{M}, Q, \omega)$ is called pre-QP-manifold of degree n.*

A pre-QP-manifold can also be referred to as symplectic pre-NQ-manifold. The vector field Q can arise from a Hamiltonian function Θ via $Q = \{\Theta, -\}$. However, the classical master equation is not satisfied,

$$\{\Theta, \Theta\} \neq 0. \tag{3.282}$$

This is in contrast to the ordinary QP-manifold.

We define Lagrangian submanifolds in the graded symplectic manifold setting for convenience.

Definition 3.3.39 (Lagrangian submanifold) *Let $(\mathcal{M}, \omega)$ be a graded symplectic manifold. A Lagrangian submanifold $\mathcal{L}$ of $(\mathcal{M}, \omega)$ is a submanifold of $\mathcal{M}$ such that the restriction of the graded symplectic structure to $\mathcal{L}$ is vanishing, $\omega|_{\mathcal{L}} = 0$, and the dimension of $\mathcal{L}$ is half of the dimension of $\mathcal{M}$.*

The strength of QP-manifolds is that they capture the structure of intricate mathematical objects in a clean and convenient manner. Let us start with the simplest example an observe how a QP-manifold encodes a Lie algebra structure.

Example 3.3.8 (Lie algebra) Let V be a finite-dimensional vector space with generators e_i over a field k. Let e^i be the generators of its dual V^*. Let us consider the graded manifold $\mathcal{M} = T^*[n]V[1] \cong V[1] \oplus V^*[n-1]$, locally parameterized by coordinates (v^i, v_i) of degrees $(1, n-1)$, where $n \in \mathbb{N}$. Furthermore, may the graded manifold be equipped with the graded symplectic structure

$$\omega = (-1)^n \delta v^i \wedge \delta v_i. \tag{3.283}$$

Finally, let the homological function be

$$\Theta = -\frac{1}{2} f^i_{jk} v^j v^k v_i, \tag{3.284}$$

where f^i_{jk} is a constant. The classical master equation, $\{\Theta, \Theta\} = 0$, is equivalent to the Jacobi identity, $f^i_{[jk} f^m_{l]i} = 0$. Using *derived brackets* we find the Lie bracket,

$$\begin{aligned} [X, Y]_{\text{Lie}} &= j^* \{\{\Theta, j_*(X)\}, j_*(Y)\} \\ &= X^i Y^j f^k_{ij} e_k, \end{aligned} \tag{3.285}$$

where the injection map j is defined as

$$\begin{aligned} j &: V \oplus V^* \to V[1] \oplus V^*[n-1], \\ j &: (e_i, e^i) \mapsto (v_i, v^i). \end{aligned} \tag{3.286}$$

So we have the assignment

$$j_*(X^i e_i) = X^i v_i, \quad j_*(Y_i e^i) = Y_i v^i, \tag{3.287}$$

where X^i and Y_i are coefficients are elements of the field k. We conclude that the QP-structure $(\mathcal{M}, \Theta, \omega)$ induces a Lie algebra structure on V, $(V, [-,-])$.

The crucial point in reconstructing the mathematical object from the QP-manifold is the use of *derived brackets* [101] and an injection map that identifies coordinates on the graded manifold with elements of the mathematical object, which shall be reconstructed. Derived brackets involve a contraction of elements of the graded manifold with the Hamiltonian function using two or more graded Poisson brackets. Therefore, the structure of the QP-manifold, which is encoded in the Hamiltonian function, directly influences the structure of the mathematical object to be reconstructed.
A slight change in perspective allows us to reconstruct the dual model of a Lie algebra, the Lie coalgebra, using the same underlying QP-manifold.

Example 3.3.9 (Lie coalgebra) Let V be a finite-dimensional vector space with generators e_i over a field k. Let V^* denote its dual with generators e^i. The QP-structure $(\mathcal{M}, \Theta, \omega)$ is the same as in the former example. From the dual vector space being encoded in $\mathcal{M} = T^*[n]V[1] \cong V[1] \oplus V^*[n-1]$ and the fact that we could induce a Lie algebra structure on V, we will be able to find a Lie coalgebra structure on the dual.

The homological vector field $Q = \{\Theta, -\}$ induces a nilpotent derivation d of degree 1 on the exterior algebra of V^* in the following way,

$$d = (-1)^{n+1} j^* \circ Q \circ j_*, \tag{3.288}$$

so that

$$dY = j^*(\{\Theta, Y_i v^i\}) = \frac{1}{2} Y_i f^i_{jk} e^j e^k, \tag{3.289}$$

for $Y = Y_i e^i \in V^*$ and the coefficient Y_i is element of the field k.

Another important example is the following generalization of a Lie algebra.

Definition 3.3.40 (Lie algebroid) *Let $E \to M$ be a vector bundle over a smooth manifold M. May E be endowed with a Lie algebra structure $[-,-] : \Gamma(E) \otimes \Gamma(E) \to \Gamma(E)$ and a bundle map from the vector bundle E to the tangent bundle over M, $\rho : E \to TM$. If the two operations satisfy the relations*

$$[\rho(e_1), \rho(e_2)] = \rho([e_1, e_2]_{Lie}), \tag{3.290}$$

$$[e_1, f e_2] = f[e_1, e_2] + (\rho(e_1)f)e_2, \tag{3.291}$$

where $e_1, e_2 \in \Gamma(E)$ and $f \in \mathcal{C}^\infty(M)$, then the 3-tuple $(E \to M, [-,-], \rho)$ is called a Lie algebroid.

It can be easily reconstructed using a QP-manifold.

Example 3.3.10 (Lie algebroid) Let $\mathcal{M} = T^*[n]E[1]$ be a graded manifold, where $E \to M$ is a vector bundle over a smooth manifold M. The degree shift in $E[1]$ refers to the fiber coordinates being shifted in degree by 1. Therefore, the local coordinates on $\mathcal{M}$ are given by $(x^i, \xi^a, \zeta_a, p_i)$ of degrees $(0, 1, n-1, n)$. x^i is a local coordinate on M, ξ^a a local coordinate on the fiber of $E[1]$, and ζ_a as well as p_i the conjugate coordinates associated with ξ^a and x^i. Let the graded symplectic structure on $\mathcal{M}$ be defined as

$$\omega = -\delta x^i \wedge \delta p_i + (-1)^n \delta\xi^a \wedge \delta\zeta_a, \tag{3.292}$$

and the Hamiltonian function given by

$$\Theta = \rho^i_a p_i \xi^a - \frac{1}{2} f^a_{bc} \xi^b \xi^c \zeta_a, \tag{3.293}$$

where $\rho_a^i, f_{bc}^a \in \mathcal{C}^\infty(M)$. The crucial difference to the Lie algebra is the existence of the function ρ_a^i and the fact that the structure constants become structure functions. We define the injection map j as follows,

$$\begin{aligned} j &: E \oplus TM \to \mathcal{M}, \\ j &: (x^i, e_a, \partial_i) \mapsto (x^i, \zeta_a, p_i), \end{aligned} \tag{3.294}$$

where the coordinate ξ^a is in the kernel of j^*. A section $X = X^a e_a \in \Gamma(E)$ can then be pushed forward to

$$X = X^a e_a \mapsto j_*(X^a e_a) = X^a \zeta_a \tag{3.295}$$

and corresponds to an element of degree $n+1$ on $\mathcal{M}$. Then, *derived brackets* induce the bundle map and the bracket of the Lie algebroid,

$$\rho(X)f = j_*\{\{\Theta, j_*(X)\}, j_*(f)\}, \tag{3.296}$$

$$[X, Y] = j_*\{\{\Theta, j_*(X)\}, j_*(Y)\}, \tag{3.297}$$

and the classical master equation, $\{\Theta, \Theta\} = 0$, is equivalent to the conditions on the two operations. We conclude, that the QP-manifold $(\mathcal{M}, \Theta, \omega)$ induces the structure of a Lie algebroid on $E \to M$.

The most important example for a QP-manifold of degree 1 is the *Poisson-Lie algebroid*.

Example 3.3.11 (Poisson-Lie algebroid) Let $\mathcal{M} = T^*[1]M$ be a graded manifold, which consists of the cotangent bundle over a smooth manifold M with fiber degree shifted by 1. Locally, $\mathcal{M}$ can be described by coordinates (x^i, ζ_i) of degrees $(0, 1)$. Coordinates on M are denoted by x^i and coordinates on the fiber are denoted by ζ_i. The decomposition of the space of smooth functions on $\mathcal{M}$ by degree is given by

$$\mathcal{C}^\infty(\mathcal{M}) = \bigoplus_{k=0}^{\infty} \mathcal{C}_k^\infty(\mathcal{M}), \tag{3.298}$$

where

$$\mathcal{C}_k^\infty(\mathcal{M}) = \left\{ \frac{1}{k!} X^{i_1 \cdots i_k} \zeta_{i_1} \cdots \zeta_{i_k} | X^{i_1 \cdots i_k} \in \mathcal{C}^\infty(M) \right\}. \tag{3.299}$$

We introduce a symplectic structure

$$\omega = \delta x^i \wedge \delta \zeta_i, \tag{3.300}$$

which induces the graded Poisson bracket

$$\{f,g\} = \frac{f\overleftarrow{\partial}}{\partial x^i}\frac{\overrightarrow{\partial} g}{\partial \zeta_i} - \frac{f\overleftarrow{\partial}}{\partial \zeta_i}\frac{\overrightarrow{\partial} g}{\partial x^i}. \tag{3.301}$$

Since the QP-manifold is of degree 1, the Hamiltonian function has to be of degree 2. The most general form of the Hamiltonian function is given by

$$\Theta = \frac{1}{2}\Pi^{ij}\zeta_i\zeta_j, \tag{3.302}$$

where $\Pi^{ij} \in \mathcal{C}^\infty(M)$. Since the coordinates ζ_i are Grassmann odd, $\zeta_i\zeta_j = -\zeta_j\zeta_i$, the matrix (Π^{ij}) is antisymmetric. We can define an injection map $j : TM \to \mathcal{M}$ from the tangent bundle over M to the graded manifold $\mathcal{M}$,

$$j : (x^i, \partial_i) \mapsto (x^i, \zeta_i). \tag{3.303}$$

The pullback of the degree k part in the decomposition of $\mathcal{C}^\infty(\mathcal{M})$ along j is the space of k-vectors over M, $j^*(\mathcal{C}^\infty_k(\mathcal{M})) = \wedge^k TM$. Then, the pullback of the Hamiltonian function Θ along j is a bivector $j^*(\Theta) = \frac{1}{2}\Pi^{ij}\partial_i\partial_j \in \wedge^2 TM$. The homological vector field is given by contraction of the Hamiltonian function with the graded Poisson bracket,

$$Q = \{\Theta, -\} = \frac{1}{2}\partial_k\Pi^{ij}\zeta_i\zeta_j\frac{\overrightarrow{\partial}}{\partial \zeta_k} - \Pi^{ij}\zeta_i\frac{\overrightarrow{\partial}}{\partial x^j}. \tag{3.304}$$

Note that the degree of the Poisson bracket is (-1), so that Q has degree $2-1=1$ as expected. In this case, the classical master equation is not trivially solved,

$$\{\Theta, \Theta\} = Q\Theta = -\Pi^{in}\partial_n\Pi^{jk}\zeta_i\zeta_j\zeta_k. \tag{3.305}$$

Since the coordinates ζ_i are Grassmann odd, the classical master equation is solved, if the product $(\Pi^{in}\partial_n\Pi^{jk})$ is totally antisymmetric in the indices $[ijk]$, leading to the condition

$$\Pi^{[i|n|}\partial_n\Pi^{jk]} = 0. \tag{3.306}$$

The equation (3.306) can be rewritten in the form

$$[\Pi, \Pi]_{\mathrm{S}} = 0, \tag{3.307}$$

where $[-,-]_{\mathrm{S}}$ denotes the Schouten bracket on the space of polyvectors over M, $[-,-]_{\mathrm{S}} : \wedge^i TM \otimes \wedge^j TM \to \wedge^{i+j-1} TM$. This equation can be seen as the pullback of the classical master equation along j. We conclude, that $(\mathcal{M}, Q, \omega)$ is a QP-manifold if the bivector Π is a Poisson tensor. In other words, (M, Π) is a Poisson manifold. The Poisson bracket on M induced by Π is reconstructed by a *derived bracket*,

$$\{f, g\}_\Pi = \Pi^{ij}\partial_i f \partial_j g \equiv -j^*\{\{j_*(f), \Theta\}, j_*(g)\}, \tag{3.308}$$

where $f, g \in \mathcal{C}^\infty(M)$ and j_* denotes the pushforward along j. The classical master equation then demands that $\{-,-\}_\Pi$ obeys the Jacobi identity. We conclude, that the structure on the QP-manifold $(\mathcal{M}, Q, \omega)$ is transported to $(TM, \{-,-\}_\Pi)$, the Poisson bracket $\{-,-\}_\Pi$ can be seen as (local) Lie bracket on the tangent bundle TM. Therefore, this structure is called Poisson-Lie algebroid.

Finally, let us investigate the pullback of the complex

$$0 \to \mathcal{C}_0^\infty(\mathcal{M}) \xrightarrow{Q} \mathcal{C}_1^\infty(\mathcal{M}) \xrightarrow{Q} \mathcal{C}_2^\infty(\mathcal{M}) \xrightarrow{Q} \cdots \xrightarrow{Q} \mathcal{C}_{\dim(M)}^\infty(\mathcal{M}) \xrightarrow{Q} 0 \tag{3.309}$$

along j, where Q is a nilpotent differential if the classical master equation is satisfied. The pullback of Q along j relates to the Lichnerowicz-Poisson differential $d_\Pi = [\Pi, -]_{\rm S}$ via $j^* \circ Q = -d_\Pi \circ j^*$, so that the complex (3.309) is pulled back to the Lichnerowicz-Poisson complex of (M, Π),

$$0 \to \wedge^0 TM \xrightarrow{d_\Pi} \wedge^1 TM \xrightarrow{d_\Pi} \wedge^2 TM \xrightarrow{d_\Pi} \cdots \xrightarrow{d_\Pi} \wedge^{\dim(M)} TM \xrightarrow{d_\Pi} 0. \tag{3.310}$$

The nilpotency of the Lichnerowicz-Poisson differential, $d_\Pi^2 = 0$, is guaranteed by the pullback of the classical master equation, $[\Pi, \Pi]_{\rm S} = 0$. We conclude, that the QP-manifold $(\mathcal{M}, Q, \omega)$ encodes the Lichnerowicz-Poisson complex of polyvectors over (M, Π).

Let us state the following important theorem about QP-manifolds of degree 1.

Theorem 3.3.41 *([102]) QP-manifolds of degree* 1 *are in one-to-one correspondence with Poisson manifolds.*

Now we step into the realm of Courant algebroids. For this, we have to consider QP-manifolds of degree 2. To get started, we state two important theorems relating Lie 2-algebras, QP-manifolds of degree 2 and Courant algebroids.

Theorem 3.3.42 *([103]) A Courant algebroid gives rise to a Lie* 2*-algebra.*

Theorem 3.3.43 *([102]) QP-manifolds of degree* 2 *are in one-to-one correspondence with Courant algebroids.*

In the next example, we directly show the equivalence between QP-manifolds of degree 2 and Courant algebroids. Courant algebroids from the perspective of graded manifolds and derived brackets have been introduced in [104]. A very recommendable survey on Courant algebroids and related objects can be found in [84].

Example 3.3.12 (Courant algebroid) Let $\mathcal{M} = T^*[2]E[1]$ be a graded manifold, where $E \to M$ is a vector bundle over the smooth manifold M. The local fibers of E are denoted by F. The object $E[1]$ then expresses that the fiber coordinates of E are shifted in degree by 1, so that the local fibers can be written by $F[1]$. Let us denote local coordinates on M by x^i, which have degree 0. The coordinates on $F[1]$ are denoted as ξ^a and have degree 1. The index a runs from 1 to rank(E). Now, we fiber the vector bundle again over $T^*[2]$. We introduce fiber coordinates on $T^*[2]$ with respect to x^i and denote them by p_i. The degree of p_i is 2. Finally, the fiber coordinates in $T^*[2]$ with respect to ξ^a live in $F^*[1]$ and are denoted by ζ_a. Their degree is 1. We introduce a fiber metric $\langle -, - \rangle$ on $E[1]$, so that we can identify $F[1]$ with $F^*[1]$ and combine ξ^a and ζ_a by introduction of a new variable $\eta^a = (\xi^a, \zeta_a)$. Then, $\mathcal{M}$ can be locally described by coordinates (x^i, η^a, p_i) of degrees $(0, 1, 2)$.
In the next step, we introduce the graded symplectic structure by

$$\omega = -\delta x^i \wedge \delta p_i + \frac{1}{2} k_{ab} \delta\eta^a \wedge \delta\eta^b, \tag{3.311}$$

where the coefficient k_{ab} is related to the fiber metric $\langle \eta^a, \eta^b \rangle = k^{ab}$ by $(k_{ab}) = (k^{ab})^{-1}$. The induced graded Poisson bracket is given by

$$\{f, g\} = -\frac{f\overleftarrow{\partial}}{\partial x^i}\frac{\overrightarrow{\partial} g}{\partial p_i} + \frac{f\overleftarrow{\partial}}{\partial p_i}\frac{\overrightarrow{\partial} g}{\partial x^i} + \frac{f\overleftarrow{\partial}}{\partial \eta^a} k^{ab} \frac{\overrightarrow{\partial} g}{\partial \eta^b}. \tag{3.312}$$

Since we are constructing a QP-manifold of degree 2, the Hamiltonian function is of degree 3. The most general Hamiltonian function on this space is given by

$$\Theta = \rho^i{}_a p_i \eta^a + \frac{1}{3!} C_{abc} \eta^a \eta^b \eta^c, \tag{3.313}$$

where $\rho^i{}_a, C_{abc} \in \mathcal{C}^\infty(M)$. The function $\rho^i{}_a$ will be related to the anchor map of the Courant algebroid, whereas the function C_{abc} will turn out to be the Ševera class. Let us investigate the degree-decomposition of $\mathcal{C}^\infty(\mathcal{M})$ first,

$$\mathcal{C}^\infty(\mathcal{M}) = \bigoplus_{k=0}^{\infty} \mathcal{C}^\infty_k(\mathcal{M}). \tag{3.314}$$

We are interested in the first two spaces, which are given by

$$\mathcal{C}_0^\infty(\mathcal{M}) \cong \mathcal{C}^\infty(M), \tag{3.315}$$

$$\mathcal{C}_1^\infty(\mathcal{M}) = \{X_a\eta^a | X_a \in \mathcal{C}^\infty(M)\}. \tag{3.316}$$

We introduce an injection map, that relates the Courant algebroid to the graded manifold $\mathcal{M}$,

$$\begin{aligned} j &: E \oplus TM \to \mathcal{M}, \\ j &: (x^i, e^a, \partial_i) \mapsto (x^i, \eta^a, p_i), \end{aligned} \tag{3.317}$$

where e^a denote sections of E. We conclude, that the pullback of $\mathcal{C}_1^\infty(\mathcal{M})$ along j is the space of sections of E,

$$X = X_a\eta^a \mapsto j^*(X) = X_a e^a \in \Gamma(E). \tag{3.318}$$

We reconstruct the Courant algebroid structure in terms of the three operations, fiber metric, anchor map and Dorfman bracket, via pullback along j and derived brackets as follows. The fiber metric on E, $\langle -, - \rangle : E \otimes E \to \mathbb{R}$, is given by the pullback of the graded Poisson structure along j,

$$\langle X, Y \rangle \equiv j^*\{j_*(X), j_*(Y)\}, \tag{3.319}$$

where $X, Y \in \Gamma(E)$. The anchor map, $\rho : E \to TM$, is reconstructed via derived bracket,

$$\rho(X)f \equiv j^*\{\{\Theta, j_*(X)\}, j_*(f)\}, \tag{3.320}$$

where $X \in \Gamma(E)$ and $f \in \mathcal{C}^\infty(M)$. Finally, the C-twisted Dorfman bracket is reconstructed by

$$[X, Y]_{\mathrm{D},C} \equiv j^*\{\{\Theta, j_*(X)\}, j_*(Y)\}. \tag{3.321}$$

The C-twisted Courant bracket is then given by the antisymmetrization,

$$[X, Y]_{\mathrm{C},C} \equiv \frac{1}{2}\left(j^*\{\{\Theta, j_*(X)\}j_*(Y)\} - j^*\{\{\Theta, j_*(Y)\}j_*(X)\}\right). \tag{3.322}$$

The classical master equation, $\{\Theta, \Theta\} = 0$, leads to relations among the two functions $\rho^i{}_a$ and C_{abc}, which are equivalent to the Courant algebroid conditions. We conclude, that the QP-manifold of degree 2, $(\mathcal{M}, \Theta, \omega)$, induces a Courant algebroid structure on E with Ševera class C.

We pointed out the importance of the H-twisted standard Courant algebroid in generalized geometry and the analysis of T-duality geometry. Let us therefore discuss how the H-twisted standard Courant algebroid fits into the framework of QP-manifolds of degree 2. This will bring us into the position to introduce twist operations on the QP-manifold, to be described below.

Example 3.3.13 (H-twisted standard Courant algebroid) The H-twisted Courant algebroid is defined on the generalized tangent bundle $E = TM \oplus T^*M$. For the reconstruction, we start with the graded manifold $\mathcal{M} = T^*[2]T[1]M$, where M is a smooth manifold. Locally, $\mathcal{M}$ can be described by coordinates $(x^i, \xi^i, \zeta_i, p_i)$ of degrees $(0, 1, 1, 2)$. Compared to the example on the general Courant algebroid, we split $\eta^a = (\xi^i, \zeta_i)$ and take the fiber metric as $k = \begin{pmatrix} 0 & \mathrm{id} \\ \mathrm{id} & 0 \end{pmatrix}$. The graded symplectic form then condenses to

$$\omega = -\delta x^i \wedge \delta p_i + \delta \xi^i \wedge \delta \zeta_i, \tag{3.323}$$

inducing the graded Poisson bracket

$$\{f, g\} = -\frac{f\overleftarrow{\partial}}{\partial x^i}\frac{\overrightarrow{\partial} g}{\partial p_i} + \frac{f\overleftarrow{\partial}}{\partial p_i}\frac{\overrightarrow{\partial} g}{\partial x^i} + \frac{f\overleftarrow{\partial}}{\partial \xi^i}\frac{\overrightarrow{\partial} g}{\partial \zeta_i} + \frac{f\overleftarrow{\partial}}{\partial \zeta_i}\frac{\overrightarrow{\partial} g}{\partial \xi^i}. \tag{3.324}$$

Furthermore, we choose ${\rho^i}_a = ({\delta^i}_j, 0)$ and only the component H_{ijk} in C_{abc} non-vanishing. The Hamiltonian function then condenses to

$$\Theta_H = p_i \xi^i + \frac{1}{3!} H_{ijk} \xi^i \xi^j \xi^k. \tag{3.325}$$

Since we will make use of this function in the analysis of the main part of this thesis, we will denote the Hamiltonian function by Θ_H. It is directly related to the H-twisted standard Courant algebroid. The derived brackets induce the anchor map as the natural projection to the tangent bundle part and the H-twisted Dorfman bracket. The pullback of the graded Poisson bracket induces the correct form of the fiber metric. The classical master equation leads to

$$\{\Theta, \Theta\} = 0 \Rightarrow dH = 0, \tag{3.326}$$

demanding that H be a closed 3-form on M, $H \in H^3(M, \mathbb{R})$, as expected from the Courant algebroid structure.

For the special case of $H = 0$ we find the standard Courant algebroid.

Example 3.3.14 (Standard Courant algebroid) Let M be a smooth manifold and $\mathcal{M} = T^*[2]T[1]M$ a graded manifold locally described by coordinates $(x^i, \xi^i, \zeta_i, p_i)$ of degrees $(0, 1, 1, 2)$. If the graded symplectic structure and Hamiltonian function are defined as

$$\omega = -\delta x^i \wedge \delta p_i + \delta\xi^i \wedge \delta\zeta_i, \tag{3.327}$$

$$\Theta_0 = p_i \xi^i, \tag{3.328}$$

then the classical master equation is trivially solved and the derived bracket construction recovers the standard Courant algebroid $(TM \oplus T^*M \to M, \langle -, - \rangle, \rho, [-, -]_{\mathrm{D}})$.

Note that, we denote the Hamiltonian function, that induces the untwisted standard Courant algebroid, by Θ_0.

Higher Courant algebroids turn out to be important in the study of exceptional general generalized geometry, where the M2-brane wrapping modes become integral part of the tangent bundle construction. We will introduce special QP-manifolds of n, which induce higher Courant algebroids, since they will play the main role in our analysis of exceptional generalized geometry with M2-branes. The crucial structures are called *Vinogradov algebroids*, which have been studied in [105] and are integral part in the analysis of [106].

Definition 3.3.44 (Vinogradov Lie n-algebroid) *Let M be a smooth manifold. Let V_n be the graded manifold defined by*

$$V_n = T^*[n]T[1]M, \tag{3.329}$$

locally parameterized by coordinates $(x^i, \xi^i, \zeta_i, p_i)$ of degrees $(0, 1, n-1, n)$. Equip V_n with a graded symplectic structure,

$$\omega = -\delta x^i \wedge \delta p_i + (-1)^n \delta\xi^i \wedge \delta\zeta_i. \tag{3.330}$$

Finally, define the Hamiltonian function $\Theta \in V_n$ via

$$\Theta = \xi^i p_i. \tag{3.331}$$

Then, the classical master equation, $\{\Theta, \Theta\} = 0$, is automatically satisfied. The 3-tuple (V_n, Θ, ω) is called a Vinogradov Lie n-algebroid.

Obviously, a Vinogradov Lie 2-algebroid induces a standard Courant algebroid structure. In general, the degree $n-1$ subspace of $\mathcal{C}^\infty(V_n)$ can be identified with the space of sections of a Courant algebroid of degree $n-1$ by the injection map j,

$$\begin{aligned} j &: (TM \oplus \wedge^{n-1}T^*M) \oplus TM \to V_n, \\ j &: (x^i, \partial_i, dx^{i_1} \wedge \cdots \wedge dx^{i_{n-1}}, \partial_i) \mapsto (x^i, \zeta_i, \xi^{i_1} \cdots \xi^{i_{n-1}}, p_i), \end{aligned} \tag{3.332}$$

leading to

$$j^*\left(X^i\zeta_i + \frac{1}{(n-1)!}\alpha_{i_1\cdots i_{n-1}}\xi^{i_1}\cdots\xi^{i_{n-1}}\right) = X^i\partial_i + \frac{1}{(n-1)!}\alpha_{i_1\cdots i_{n-1}}dx^{i_1}\wedge\cdots\wedge dx^{i_{n-1}}. \tag{3.333}$$

The higher Courant bracket, higher Dorfman bracket and projection map are reconstructed via derived brackets. The fiber metric is given as the pullback of the symplectic structure. We are lead to the following theorem.

Theorem 3.3.45 *A Vinogradov Lie n-algebroid induces the structure of a Courant algebroid of degree $(n-1)$.*

In the following we define an operation that can be used to manipulate general QP-manifolds.

Definition 3.3.46 (Canonical transformation) *Let $(\mathcal{M}, \Theta, \omega)$ be a QP-manifold of degree n. Fix a function $\alpha \in \mathcal{C}^\infty(\mathcal{M})$ so that $|\alpha| = n$. The canonical transformation by α is defined by exponential adjoint action,*

$$\exp(\delta_\alpha)f = f + \{f, \alpha\} + \frac{1}{2}\{\{f, \alpha\}, \alpha\} + \cdots, \tag{3.334}$$

where $f \in \mathcal{C}^\infty(\mathcal{M})$ of any degree. Since the degree of the symplectic structure is $|\omega| = n$, and therefore the degree of the induced Poisson bracket is $|\{-,-\}| = -n$, this action is degree-preserving. Furthermore, The exponential adjoint action preserves the graded Poisson bracket,

$$\{\exp(\delta_\alpha)f, \exp(\delta_\alpha)g\} = \exp(\delta_\alpha)\{f, g\}, \tag{3.335}$$

where $f, g \in \mathcal{C}^\infty(\mathcal{M})$.

Since the canonical transformation preserves the graded Poisson bracket, it can be called a symplectomorphism. A different terminology is to refer to canonical transformations as twists or twisting [107], as will become evident below. Canonical transformations are an important tool to explore the classes of solutions of classical master equations, since

$$\{\Theta, \Theta\} = 0 \Rightarrow \{\exp(\delta_\alpha)\Theta, \exp(\delta_\alpha)\Theta\} = \exp(\delta_\alpha)\{\Theta, \Theta\} = 0. \tag{3.336}$$

Let us consider the twist of the standard Courant algebroid by a 2-form B-field. We will see that no additional condition arises due to the property (3.336).

Example 3.3.15 (B-Twist of the standard Courant algebroid) Let us start by considering the QP-manifold ($\mathcal{M} = T^*[2]T[1]M, \Theta = p_i\xi^i, \omega$), which induces the standard Courant algebroid. The symplectic structure is of degree 2. Therefore, any function $\alpha \in \mathcal{C}^\infty(T^*[2]T[1]M)$, for which $|\alpha| = 2$, defines a twist of the QP-manifold.
Let us for simplicity take the function $B = \frac{1}{2}B_{ij}\xi^i\xi^j$. Pullback along j shows that B is a 2-form, $j^*(B) = \frac{1}{2}B_{ij}dx^i dx^j$. The twist of the Hamiltonian Θ by $\exp(\delta_B)$ leads to

$$\begin{aligned}\exp(\delta_B)\Theta &= \Theta + \{\Theta, B\} + \cdots \\ &= p_i\xi^i + \frac{1}{2}\partial_i B_{jk}\xi^i\xi^j\xi^k. \end{aligned} \tag{3.337}$$

Higher order adjoint actions are vanishing so that the series converges. Comparison to the H-twisted Courant algebroid shows that we actually induced a dB-twisted Courant algebroid. Naturally, the classical master equation is still trivially solved, since $d(dB) = 0$ due to $d^2 = 0$.

The 2-form can be associated with the Kalb-Ramond field, while the H-flux is the associated field strength. In the case of the standard Courant algebroid there are more twists available, which become important in our analysis.

3.3.5 AKSZ sigma models

The AKSZ method [44] due to Alexandrov, Kontsevich, Schwarz and Zaboronsky is a prescription to generate topological sigma models from QP-manifold structures. The resulting topological sigma models are called AKSZ sigma models and induce a *BV-BRST formalism* of quantum field theories. The target space of the topological sigma models constructed in this way inherit the structure induced by the QP-manifold. The BV-BRST formalism will be conveniently described using superfield formalism. The superfields are expanded in Grassmann odd coordinates and the various components encode not only the physical fields, but also the ghosts and antifields. We recommend the lecture notes [97], which provides an introduction to the subject of AKSZ sigma models from the physical point of view.
Let us assume, we want to construct a topological sigma model. For this, we have to specify the worldvolume manifold of the dynamical object to be described. This is the manifold, which will be embedded. Then, we have to specify the target space manifold, in which

the worldvolume shall be embedded. The embedding functions, that determine how the embedding is realized and which structure the target space has to have, live on the embedding space.

We now introduce an abstract language to formalize this process. Let X be the worldvolume on which the sigma model will be defined. We promote X to a graded manifold $\chi = T[1]X$, which locally is parameterized by coordinates (σ^μ, θ^μ) of degrees $(0,1)$. The coordinates σ^μ are Grassmann even, whereas the coordinates θ^μ are Grassmann odd. Furthermore, we introduce a differential $\boldsymbol{d} = \theta^\mu \partial_\mu$ on χ as well as a $\boldsymbol{d}$-invariant non-degenerate measure μ_χ. Here, the derivative is denoted by $\partial_\mu = \frac{\partial}{\partial \sigma^\mu}$. Then, the 3-tuple $(\chi, \boldsymbol{d}, \mu_\chi)$ is a dg-manifold with $\boldsymbol{d}$-invariant non-degenerate measure μ_χ. We showed above, that a dg-manifold of type $(T[1]X, \theta^\mu \partial_\mu)$ serves as a model of the de Rham complex over X. We conclude, we constructed a superworldvolume dg-manifold $(\chi, \boldsymbol{d}, \mu_\chi)$, which we embed into a super target space in the next step. Recall, that the expression **super** refers to the superfield formalism of the associated BV-BRST formalism of topological sigma models. It is **not** related to supersymmetry.

Let the target space be denoted by M. The space of embeddings of X to M is denoted by $\mathrm{Map}(X, M)$. We promote M to a QP-manifold $(\mathcal{M}, Q, \omega)$ so that the target space inherits the structure of the QP-manifold. In turn, we lift the space of embeddings of X to M to the space of superembeddings $\mathrm{Map}(\chi, \mathcal{M})$. Using the AKSZ method, we will be able to construct a BV action on the mapping space between the two graded manifolds χ and $\mathcal{M}$, which is subject to a classical master equation on the mapping space. The respective differentials $\boldsymbol{d}$ on χ and Q on $\mathcal{M}$ lift to $\check{\boldsymbol{d}}$ and $\check{Q}$ on $\mathrm{Map}(\chi, \mathcal{M})$ via

$$\check{\boldsymbol{d}}(z, f) = \boldsymbol{d}(z)\delta f(z), \quad \check{Q}(z, f) = Qf(z), \tag{3.338}$$

where $z \in \chi$ and $f \in \mathrm{Map}(\chi, \mathcal{M})$ a superembedding function.

In the next step, we lift structures from the target space QP-manifold to the mapping space. The mapping space will become a QP-manifold itself and the Hamiltonian function is given by the BV action of the associated BV-BRST formalism. The lift of the graded Poisson bracket becomes the BV antibracket. In order to construct the lift, we have to define two maps, the evaluation map and the chain map. The combination of both gives the desired lift in terms of the so-called *transgression map.*

Definition 3.3.47 (Evaluation map) *Let $(\chi, \boldsymbol{d}, \mu)$ and $(\mathcal{M}, Q, \omega)$ be as defined above. We* *define the evaluation map ev that evaluates a function $f \in Map(\chi, \mathcal{M})$ at $z \in \chi$ via*

$$ev : \chi \times Map(\chi, \mathcal{M}) \to \mathcal{M}, \quad (z, f) \mapsto f(z). \tag{3.339}$$

Definition 3.3.48 (Chain map) *Let $(\chi, \boldsymbol{d}, \mu)$ and $(\mathcal{M}, Q, \omega)$ be as defined above. We define the chain map μ_* as follows,*

$$\begin{aligned} \mu_* : \Omega^\bullet(\chi \times Map(\chi, \mathcal{M})) &\to \Omega^\bullet(Map(\chi, \mathcal{M})), \\ \mu_*\omega(f)(v_1, \dots, v_k) &= \int_\chi \mu_\chi(z)\omega(z, f)(v_1, \dots, v_k), \end{aligned} \tag{3.340}$$

where $v_i \in \mathfrak{X}^1(\chi)$. The object $\int_\chi \mu_\chi$ denotes the Berezin integration on χ.

Having understood the evaluation map and the chain map, we combine them to the transgression map.

Definition 3.3.49 (Transgression map) *The transgression map is defined via composition of the evaluation map and chain map,*

$$\mu_* ev^* : \Omega^\bullet(\mathcal{M}) \to \Omega(Map(\chi, \mathcal{M})). \tag{3.341}$$

The transgression map is used to transport the QP-structure on $\mathcal{M}$ to the mapping space $\mathrm{Map}(\chi, \mathcal{M})$. The graded symplectic structure on the mapping space is the direct transgression of the graded symplectic structure on $\mathcal{M}$,

$$\Omega = \mu_* \mathrm{ev}^* \omega. \tag{3.342}$$

Since ω is non-degenerate and closed and the transgression map preserves these properties, Ω is non-degenerate and closed. Ω induces a graded Poisson bracket on $\mathrm{Map}(\chi, \mathcal{M})$, which we denote by $\{-,-\}_\Omega$. It gives the BV antibracket of the associated BV-BRST formalism. The homological function on $\mathrm{Map}(\chi, \mathcal{M})$ is defined as

$$S = S_0 + S_1 = \iota_{\check{\boldsymbol{d}}} \mu_* \mathrm{ev}^* \vartheta + \mu_* \mathrm{ev}^* \Theta, \tag{3.343}$$

where $\vartheta = -\delta\omega$ denotes the Liouville 1-form with respect to the graded symplectic structure on $\mathcal{M}$ and Θ is the homological function on $\mathcal{M}$. The homological function S gives the BV action of the associated BV-BRST formalism and consist of two parts. The first part is the transgression of the Liouville 1-form, where the de Rham differential is exchanged by the differential $\boldsymbol{d}$ on the worldvolume graded manifold. It gives the kinetic term in the BV action. The second part is the transgression of the homological function on $\mathcal{M}$ and gives the interaction term. Using the homological function S, we can define a degree 1 homological vector field on $\mathrm{Map}(\chi, \mathcal{M})$,

$$\mathcal{Q} = \{S, -\}_{\Omega}. \tag{3.344}$$

The homological vector field $\mathcal{Q}$ is the BRST operator of the BV-BRST formalism, or the BRST-BV charge. We can show the following theorem.

Theorem 3.3.50 ([44]) *The 3-tuple $(Map(\chi, \mathcal{M}), \mathcal{Q}, \Omega)$ is a QP-manifold, i.e., the vector field $\mathcal{Q}$ is homological,*

$$\mathcal{Q}^2 = 0. \tag{3.345}$$

Since the BRST operator is nilpotent, it induces a BRST complex and the associated BRST cohomology of the BV-BRST formalism. The result is a topological field theory on the mapping space.

Let us analyze the degree of the mapping space QP-manifold $(\mathrm{Map}(\chi, \mathcal{M}), \mathcal{Q}, \Omega)$. If the worldvolume X is d-dimensional, then the chain map μ_* has degree $(-d)$. The evaluation map is degree-preserving. Let the target space QP-manifold $(\mathcal{M}, Q, \omega)$ be of degree n. By degree-counting, the transgressed graded symplectic structure Ω is of degree $(n-d)$. Therefore, the mapping space QP-manifold $(\mathrm{Map}(\chi, \mathcal{M}), \mathcal{Q}, \Omega)$ is of degree $(n-d)$.

Now, there are two cases of importance. In the case, where the dimension of the worldvolume X matches the degree of the target space QP-manifold, the mapping space QP-manifold has vanishing degree. The induced structure is equivalent to a Batalin-Fradkin-Vilkovisky (BFV) formalism [108, 109] and the bracket induced by Ω is an ordinary Poisson bracket, which we will denote by $\{-,-\}_{\Omega} \to \{-,-\}_{\mathrm{PB}}$.

The second case comes into play, when the worldvolume X has dimension $d = n+1$, where n is the degree of the target space QP-manifold. In this case, the mapping space QP-manifold has degree $n - (n+1) = -1$. The induced structure is equivalent to the Batalin-Vilkovisky (BV) formalism [110, 111] modeling a topological sigma model and the bracket induced by Ω is the BV bracket, which we denote by $\{-,-\}_{\Omega} \to \{-,-\}_{\mathrm{BV}}$. The associated ghosts and antifields are naturally included in the graded objects on the mapping space as we will see in the following.

Definition 3.3.51 (ASKZ sigma model) *The* 3*-tuple* $(Map(\chi, \mathcal{M}), \mathcal{Q}, \Omega)$ *constructed out of the data above, where the target space QP-manifold has degree* n *and the worldvolume dimension is* $(n+1)$*, is called an* *ASKZ sigma model.*

Therefore, a AKSZ sigma model is a BV formalism on the superembedding space.

Let us discuss the construction of the Poisson sigma model as an introductory example. The underlying structure is the Poisson-Lie algebroid, which is a QP-manifold of degree 1. Therefore, the resulting topological sigma model is a topological string. For details on the mapping space calculus we refer to appendix B.

Example 3.3.16 (Poisson sigma model) Let $(\mathcal{M} = T^*[1]M, \Theta, \omega)$ be the target space QP-manifold inducing the Poisson-Lie algebroid, defined by

$$\omega = \delta x^i \wedge \delta\zeta_i, \quad \Theta = \frac{1}{2}\Pi^{ij}\zeta_i\zeta_j, \tag{3.346}$$

where the local coordinates (x^i, ζ_i) are of degrees $(0, 1)$. The Liouville 1-form ϑ is given by

$$\vartheta = -\zeta_i \delta x^i. \tag{3.347}$$

Let X be the 2-dimensional worldsurface, so that the superworldsurface becomes $\chi = T[1]X$. The kinetic term of the BV action is the lift of the Liouville 1-form to the mapping space,

$$S_0 = \int_\chi \mu_\chi \boldsymbol{\zeta}_i \boldsymbol{dx}^i, \tag{3.348}$$

where $\boldsymbol{\zeta}_i = \boldsymbol{\zeta}_i(\sigma, \theta)$ and $\boldsymbol{x}^i = \boldsymbol{x}^i(\sigma, \theta)$ are mapping space superfields associated with the local coordinates on $\mathcal{M}$ via the pullback along the evaluation map ev. The measure μ_χ is given by $\mu_\chi = d\sigma^1 d\sigma^2 d\theta^2 d\theta^1$. Note the reverse order of the Berezin measure. The interaction term of the BV action is given by transgression of the Hamiltonian function on $\mathcal{M}$,

$$S_1 = \int_\chi \mu_\chi \frac{1}{2}\Pi^{ij}(\boldsymbol{x})\boldsymbol{\zeta}_i\boldsymbol{\zeta}_j. \tag{3.349}$$

We find the BV action of the Poisson sigma model on χ by adding up the kinetic and the interaction term,

$$S = \int_\chi \mu_\chi \left(\boldsymbol{\zeta}_i \boldsymbol{dx}^i + \frac{1}{2}\Pi^{ij}(\boldsymbol{x})\boldsymbol{\zeta}_i\boldsymbol{\zeta}_j \right). \tag{3.350}$$

The physical fields, ghost, ghost-of-ghosts and antifields are encoded in the components of the superfields.

Recall that QP-manifolds of degree 1 are in one-to-one correspondence with Poisson manifolds. We conclude, that the most general AKSZ sigma model in 2 dimensions is given by the Poisson sigma model.

Since it will be important for our analysis, let us introduce the ASKZ sigma model associated with the H-twisted Courant algebroid, the Courant sigma model with H-flux Wess-Zumino term. We start with a standard Courant algebroid structure. Let M be the target space manifold, which we promote to the QP-manifold structure of a H-twisted standard Courant algebroid on the graded manifold $\mathcal{M} = T^*[2]T[1]M$. We take a Liouville 1-form ϑ associated with the graded symplectic structure

$$\omega = -\delta x^i \wedge \delta p_i + \delta\xi^i \wedge \delta\zeta_i, \tag{3.351}$$

via $-\delta\vartheta = \omega$ by

$$\vartheta = -p_i \delta x^i - \xi^i \delta\zeta_i. \tag{3.352}$$

The Hamiltonian function is given by

$$\Theta = p_i\xi^i + \frac{1}{3!} H_{ijk}\xi^i\xi^j\xi^k. \tag{3.353}$$

We consider the embedding of the membrane superworldvolume $\chi = T[1]X$ into $\mathcal{M}$, where X is the 3-dimensional membrane worldvolume. The local coordinates on χ are denoted by (σ^μ, θ^μ) of degrees $(0, 1)$. The associated sigma model on the mapping space $\mathrm{Map}(\chi, \mathcal{M})$ is defined via transgression of the target space data. The Hamiltonian on $\mathrm{Map}(\chi, \mathcal{M})$ consists of two parts,

$$S = S_0 + S_1 = \iota_{\check{\boldsymbol{d}}}\mu_* \mathrm{ev}^*\vartheta + \mu_*\mathrm{ev}^*\Theta. \tag{3.354}$$

The transgression of the graded symplectic structure ω is given by

$$\Omega = \int_\chi \mu_\chi(-\delta\boldsymbol{x}^i \wedge \delta\boldsymbol{p}_i + \delta\boldsymbol{\xi}^i \wedge \delta\boldsymbol{\zeta}_i), \tag{3.355}$$

where $\mu_\chi = d\sigma^1 d\sigma^2 d\sigma^3 d\theta^3 d\theta^2 d\theta^1$. Using the relation $-\boldsymbol{\delta\vartheta} = \Omega$ on the mapping space, we find the transgression of the Liouville 1-form,

$$\boldsymbol{\vartheta} = \int_\chi \mu_\chi(\boldsymbol{p}_i \delta\boldsymbol{x}^i + \boldsymbol{\xi}^i \delta\boldsymbol{\zeta}_i). \tag{3.356}$$

Then, the kinetic term of the BV action is given by

$$S_0 = \int_\chi \mu_\chi(-\boldsymbol{p}_i \boldsymbol{d}\boldsymbol{x}^i - \boldsymbol{\xi}^i \boldsymbol{d}\boldsymbol{\zeta}_i). \tag{3.357}$$

The interaction part of the BV action is the transgression of the Hamiltonian function of the target space,

$$S_1 = \int_{\chi} \mu_{\chi}(\boldsymbol{p}_i\boldsymbol{\xi}^i + \frac{1}{3!}H_{ijk}(\boldsymbol{x})\boldsymbol{\xi}^i\boldsymbol{\xi}^j\boldsymbol{\xi}^k), \tag{3.358}$$

so that the full BV action on the mapping space is given by

$$S = \int_{\chi} \mu_{\chi}(-\boldsymbol{p}_i\boldsymbol{d}\boldsymbol{x}^i - \boldsymbol{\xi}^i\boldsymbol{d}\boldsymbol{\zeta}_i + \boldsymbol{p}_i\boldsymbol{\xi}^i + \frac{1}{3!}H_{ijk}(\boldsymbol{x})\boldsymbol{\xi}^i\boldsymbol{\xi}^j\boldsymbol{\xi}^k). \tag{3.359}$$

The variation of (3.359) gives

$$\begin{aligned}
\delta S &= \int_{\chi} \mu_{\chi}\left(-\delta\boldsymbol{p}_i\boldsymbol{d}\boldsymbol{x}^i - \boldsymbol{p}_i\boldsymbol{d}\delta\boldsymbol{x}^i - \delta\boldsymbol{\xi}^i\boldsymbol{d}\boldsymbol{\zeta}_i - \boldsymbol{\xi}^i\boldsymbol{d}\delta\boldsymbol{\zeta}_i + \delta\left(\boldsymbol{p}_i\boldsymbol{\xi}^i + \frac{1}{3!}H_{ijk}\boldsymbol{\xi}^i\boldsymbol{\xi}^j\boldsymbol{\xi}^k\right)\right)\\
&= \int_{\partial\chi} \mu_{\partial\chi}(\boldsymbol{\xi}^i\delta\boldsymbol{\zeta}_i - \boldsymbol{p}_i\delta\boldsymbol{x}^i) \qquad (3.360)\\
&\quad + \int_{\chi} \mu_{\chi}\left(-\delta\boldsymbol{p}_i\boldsymbol{d}\boldsymbol{x}^i + \boldsymbol{d}\boldsymbol{p}_i\delta\boldsymbol{x}^i - \delta\boldsymbol{\xi}^i\boldsymbol{d}\boldsymbol{\zeta}_i - \boldsymbol{d}\boldsymbol{\xi}^i\delta\boldsymbol{\zeta}_i + \delta\left(\boldsymbol{p}_i\boldsymbol{\xi}^i + \frac{1}{3!}H_{ijk}\boldsymbol{\xi}^i\boldsymbol{\xi}^j\boldsymbol{\xi}^k\right)\right).
\end{aligned}$$

The vanishing of the boundary variation,

$$\int_{\partial\chi} \mu_{\partial\chi}(\boldsymbol{\xi}^i\delta\boldsymbol{\zeta}_i - \boldsymbol{p}_i\delta\boldsymbol{x}^i), = 0 \tag{3.361}$$

fixes the boundary conditions, $\boldsymbol{\xi}^i|_{\partial\chi} = \boldsymbol{p}_i|_{\partial\chi} = 0$, corresponding to the Lagrangian submanifold of the zero locus of the Liouville 1-form ϑ. The classical master equation gives

$$\{S, S\}_{\mathrm{BV}} = \int_{\partial\chi} \mu_{\partial\chi}\left(-\boldsymbol{p}_i\boldsymbol{d}\boldsymbol{x}^i - \boldsymbol{\xi}^i\boldsymbol{d}\boldsymbol{\zeta}_i + \boldsymbol{p}_i\boldsymbol{\xi}^i + \frac{1}{3!}H_{ijk}\boldsymbol{\xi}^i\boldsymbol{\xi}^j\boldsymbol{\xi}^k\right), \tag{3.362}$$

which is satisfied on the Lagrangian submanifold. Let us derive the boundary sigma model by integrating out the auxiliary variable $\boldsymbol{p}_i$ from (3.359), giving the relation $\boldsymbol{d}\boldsymbol{x}^i = \boldsymbol{\xi}^i$ and leading to

$$\begin{aligned}
S &= \int_{\chi} \mu_{\chi}\left(-\boldsymbol{d}\boldsymbol{x}^i\boldsymbol{d}\boldsymbol{\zeta}_i + \frac{1}{3!}H_{ijk}\boldsymbol{d}\boldsymbol{x}^i\boldsymbol{d}\boldsymbol{x}^j\boldsymbol{d}\boldsymbol{x}^k\right)\\
&= \int_{\partial\chi} \mu_{\partial\chi}(\boldsymbol{\zeta}_i\boldsymbol{d}\boldsymbol{x}^i) + \frac{1}{3!}\int_{\chi} \mu_{\chi}H_{ijk}\boldsymbol{d}\boldsymbol{x}^i\boldsymbol{d}\boldsymbol{x}^j\boldsymbol{d}\boldsymbol{x}^k, \qquad (3.363)
\end{aligned}$$

which is a Poisson sigma model with H-flux Wess-Zumino term and vanishing Poisson structure. In the following, we extract the ghost-free action by expanding the superfields in the Grassmann odd coordinates and integrating over the Berezin measure while keeping only the physical components. The superfield expansion is given by

$$\boldsymbol{x}^i(\sigma,\theta) = \boldsymbol{x}^{i,(0)}(\sigma) + \boldsymbol{x}^{i,(1)}_{\mu}(\sigma)\theta^{\mu} + \boldsymbol{x}^{i,(2)}_{\mu\nu}(\sigma)\theta^{\mu}\theta^{\nu}, \tag{3.364}$$

$$\boldsymbol{\zeta}_i(\sigma,\theta) = \boldsymbol{\zeta}^{(0)}_i(\sigma) + \boldsymbol{\zeta}^{(1)}_{i,\mu}(\sigma)\theta^{\mu} + \frac{1}{2}\boldsymbol{\zeta}^{(2)}_{i,\mu\nu}(\sigma)\theta^{\mu}\theta^{\nu}. \tag{3.365}$$

After projection to the physical components by integration over the Grassmann odd coordinates we find

$$S_{\text{ghostfree}} = \int_{\partial X} \zeta_i \wedge dx^i + \int_X H, \tag{3.366}$$

where we denote $\zeta_i = \boldsymbol{\zeta}^{(1)}_{i,\mu} d\sigma^\mu$ and $x^i = \boldsymbol{x}^{i,(0)}$. In our analysis we will show that this sigma model is dual to the contravariant Poisson sigma model with R-flux Wess-Zumino term, which will be derived in the main calculations.

In the remainder of this section, we discuss the case, where the worldvolume X has a boundary, $\partial X \neq 0$. In this case, boundary terms can be generated by twist of the BV action using special classes of canonical transformations, the *canonical functions.*

Definition 3.3.52 (Canonical function) *Let $(\mathcal{M}, Q, \omega)$ be a QP-manifold and $\mathcal{L}$ be a Lagrangian submanifold of $(\mathcal{M}, \omega)$. Furthermore, let α be a twist defined on $(\mathcal{M}, Q, \omega)$. The twist α is a canonical function with respect to $\mathcal{L}$ if* $\exp(\delta_\alpha)\Theta|_{\mathcal{L}} = 0$.

The function α is also called a Poisson function [112].

Definition 3.3.53 (AKSZ sigma model with boundary) *Let $(Map(\chi, \mathcal{M}), \mathcal{Q}, \Omega)$ be a QP-manifold which arises from $(\mathcal{M}, Q, \omega)$ by transgression. Furthermore, let the superworldvolume χ have a boundary, $\partial\chi \neq 0$. Finally, let $\mathcal{L}$ be the Lagrangian submanifold defined by the zero locus of the Liouville 1-form ϑ and α be a canonical function with respect to $\mathcal{L}$. We call the 5-tuple $(Map(\chi, \mathcal{M}), \mathcal{Q}, \Omega, \mathcal{L}, \alpha)$ an AKSZ sigma model with boundary.*

Theorem 3.3.54 ([113]) *The twisted BV-action $S = \mu_{\check{\boldsymbol{d}}} \mu_* ev^* \vartheta + \mu_* ev^* \exp(\delta_\alpha)\Theta$ satisfies the classical master equation, $\{S, S\}_{BV} = 0$.*

The BV action of the AKSZ sigma model with boundary is considered to be a twisted version of the initial BV action $S = S_0 + S_1$,

$$\begin{aligned} S_\alpha &= \exp(-\delta_{\widehat{\alpha}})(S_0 + \mu_* \text{ev}^* \exp(\delta_\alpha)\Theta) \\ &= \exp(-\delta_{\widehat{\alpha}}) S_0 + S_1, \end{aligned} \tag{3.367}$$

where $\widehat{\alpha}$ denotes the associated twist on the mapping space by transgression, $\widehat{\alpha} = \mu_* \text{ev}^* \alpha$. The classical master equation is deformed in the following way,

$$\{S_\alpha, S_\alpha\} = \exp(-\delta_{\widehat{\alpha}})(\iota_{\check{\boldsymbol{d}}} \mu_{\partial\chi*}(i_\partial \times \text{id})^* \text{ev}^* \vartheta + \mu_{\partial\chi*}(i_\partial \times \text{id})^* \text{ev}^* \exp(\delta_\alpha)\Theta), \tag{3.368}$$

where $\mu_{\partial\chi}$ denotes the boundary measure on $\partial\chi$. The map i_∂ denotes the inclusion of the boundary, $i_\partial : \partial\chi \to \chi$.
In the special case, where $\{\alpha, \alpha\} = 0$, we find

$$S_\alpha = S - \mu_{\partial\chi*}(i_\partial \times \mathrm{id})^*\mathrm{ev}^*\alpha = S_0 + S_1 - \{S_0, \widehat{\alpha}\}_{\mathrm{BV}}. \tag{3.369}$$

It turns out that the additional term introduces a boundary term,

$$-\{S_0, \widehat{\alpha}\}_{\mathrm{BV}} = \int_{\partial\chi} \mu_{\partial\chi}\mathrm{ev}^*(\alpha). \tag{3.370}$$

We conclude, that the twist of a BV action of a AKSZ sigma model with boundary by a canonical function $\widehat{\alpha}$, for which $\{\alpha, \alpha\} = 0$, introduces a boundary term proportional to the twist $\widehat{\alpha}$.

3.3.6 Current algebras

This section concerns the construction of current algebras from general target space QP-manifolds. This is done in two steps. First, we derive a Poisson algebra from the respective QP-manifold transgression. We describe the method in *Poisson algebras from QP-manifolds* below. Second, the current algebra is the Poisson algebra on a special subspace of smooth functions on the QP-manifold. This step is described in *Current algebras from QP-manifolds*. The method has been described in the published paper [1].

Poisson algebras from QP-manifolds

We start with a target space QP-manifold $(\mathcal{M}, \Theta, \omega)$ of degree n. Furthermore, we denote the embedded superworldvolume as $\chi = T[1]X$, where X is the d-dimensional worldvolume, $\dim(X) = d$. As already discussed above, we derive a Poisson algebra with Poisson bracket $\{-,-\}_{\mathrm{PB}}$ on the mapping space $(\mathrm{Map}(\chi, \mathcal{M}), \mathcal{Q}, \Omega)$ by transgression $\mu_*\mathrm{ev}^*$ if the dimension of the worldvolume and the degree of the target space QP-manifold are related by $d = n$. This leads to the fact, that the (graded) Poisson bracket on the mapping space has zero degree, i.e., it condenses to a usual Poisson bracket. One recognizes that the resulting Poisson bracket is constructed by transgression of the graded symplectic structure on the target space without making use of the Hamiltonian function Θ. Therefore, we employ a different method that also incorporates the Q-structure generating twisted versions of Poisson brackets.

Let $\mathcal{L}$ be a Lagrangian submanifold with respect to ω, which we want to consider as target space of physical canonical quantities. Furthermore, we denote the natural projection from $\mathcal{M}$ to $\mathcal{L}$ as $\mathrm{pr}_{\mathcal{L}} : \mathcal{M} \to \mathcal{L}$. The derived bracket on $(\mathcal{M}, \Theta, \omega)$ given by $\{\{f, \Theta\}, g\}$, where $f, g \in \mathcal{C}^\infty(\mathcal{M})$, is of degree $n + 1 - n - n = -n + 1$. Let us define the restriction of the derived bracket to $\mathcal{L}$ by

$$\{f, g\}_{\mathcal{L}} \equiv \{\{\mathrm{pr}^* f, \Theta\}, \mathrm{pr}^* g\}|_{\mathcal{L}}, \tag{3.371}$$

where $f, g \in \mathcal{C}^\infty(\mathcal{L})$ and pr^* denotes the pullback along the projection to the Lagrangian submanifold $\mathcal{L}$. The degree of $\{-,-\}_{\mathcal{L}}$ is $(-n+1)$. We can prove the following theorem.

Theorem 3.3.55 *Let $(\mathcal{M}, \Theta, \omega)$ be a QP-manifold and let $\mathcal{L}$ be a Lagrangian submanifold with respect to the graded symplectic structure ω. Furthermore, let $pr_{\mathcal{L}} : \mathcal{M} \to \mathcal{L}$ be the natural projection to the Lagrangian submanifold. Then, the bracket $\{-,-\}_{\mathcal{L}}$ defined on $\mathcal{L}$ by the derived bracket construction is a graded Poisson bracket.*

In the next step, we consider the transgression of $\{-,-\}_{\mathcal{L}}$ to the mapping space $\mathrm{Map}(\chi, \mathcal{M})$, where $\chi = T[1]X$ is the supervolume associated with a $(d = n-1)$-dimensional worldvolume X. The transgression is given by

$$\begin{aligned}
\mu_* \mathrm{ev}^* \{f, g\}_{\mathcal{L}} &= \mu_* \mathrm{ev}^* \{\{\mathrm{pr}^* f, \Theta\}, \mathrm{pr}^* g\}|_{\mathcal{L}} \\
&= \{\{\widehat{\mathrm{pr}}^* \mu_* \mathrm{ev}^* f, \mu_* \mathrm{ev}^* \Theta\}_\Omega, \widehat{\mathrm{pr}}^* \mu_* \mathrm{ev}^* g\}_\Omega|_{\widehat{\mathcal{L}}} \\
&= \{\{\widehat{\mathrm{pr}}^* \mu_* \mathrm{ev}^* f, S_1\}_\Omega, \widehat{\mathrm{pr}}^* \mu_* \mathrm{ev}^* g\}_\Omega|_{\widehat{\mathcal{L}}},
\end{aligned} \tag{3.372}$$

where $\widehat{\mathcal{L}}$ is the transgression of the Langrangian submanifold to the mapping space, i.e., a Lagrangian submanifold of the transgressed graded symplectic structure $\Omega = \mu_* \mathrm{ev}^* \omega$. Furthermore, $\widehat{\mathrm{pr}}^*$ denotes the pullback along the projection from the mapping space to the transgressed Lagrangian submanifold, $\widehat{\mathrm{pr}} : \mathrm{Map}(\chi, \mathcal{M}) \to \widehat{\mathcal{L}}$.

The resulting bracket on the mapping space is extended to be defined on arbitrary functions on $\mathrm{Map}(\chi, \mathcal{M})$, that are not in the image of the transgression $\mu_* \mathrm{ev}^*$. Finally, we arrive at the desired Poisson bracket,

$$\{F, G\}_{\mathrm{PB}, \widehat{\mathcal{L}}} = \{\{F, S_1\}_\Omega, G\}_\Omega|_{\widehat{\mathcal{L}}}, \tag{3.373}$$

where $F, G \in \mathrm{Map}(\chi, \mathcal{M})$.

Let us confirm, that the resulting bracket indeed has degree zero. The bracket $\{-,-\}_\Omega$ is the transgression of the graded Poisson bracket on the target QP-manifold. Its degree can be found by investigation of the degrees of

$$\mu_* \mathrm{ev}^* \{f, g\} = \{\mu_* \mathrm{ev}^* f, \mu_* \mathrm{ev}^* g\}_\Omega, \tag{3.374}$$

where $f, g \in \mathcal{C}^\infty(\mathcal{M})$. The map μ_* introduces the Berezin measure over $\chi = T[1]X$, where $\dim(X) = n-1$, so that $|\mu_*| = -(n-1)$. So we find that $|\{-,-\}_\Omega| = -(n-1) - n + (n-1) - (n-1) = 1$. Furthermore, the degree of the transgression of the Hamiltonian function, $S_1 = \mu_* \mathrm{ev}^* \Theta$, is given by $-(n-1) + (n+1) = 2$. Therefore, the degree of the resulting bracket is $|\{-,-\}_{\mathrm{PB},\hat{\mathcal{L}}}| = 2 - 1 - 1 = 0$. We are lead to the following theorem.

Theorem 3.3.56 *The bracket* $\{-,-\}_{PB,\widehat{\mathcal{L}}}$ *on Map*$(\chi, \mathcal{M})$ *is an ordinary Poisson bracket.*

We now have a Poisson bracket at hand that is directly related to the Q-structure of the target space QP-manifold. In the following, we will use it to generate current algebras associated to various sigma models.

First, let us discuss an easy example.

Example 3.3.17 (H-flux Poisson bracket) This example makes use of the QP-manifold of degree 2 that induces the standard Courant algebroid with H-flux, with graded symplectic structure

$$\omega = \delta x^i \wedge \delta p_i + \delta \xi^i \wedge \delta \zeta_i \tag{3.375}$$

and Hamiltonian function

$$\Theta_H = p_i \xi^i + \frac{1}{3!} H_{ijk} \xi^i \xi^j \xi^k. \tag{3.376}$$

The resulting Poisson bracket is defined on the cotangent space of the loop space $\mathrm{Map}(S^1, T^*M)$. The superworldvolume of the boundary theory of the membrane is $T[1]X$, where we choose $X = S^1 \times \mathbb{R}$ to be decomposed into spatial and temporal direction. We define $\mathring{\chi} = T[1]S^1$, so that the mapping space of superloops into the target space graded manifold is given by $\mathrm{Map}(\mathring{\chi}, \mathcal{M})$. Since Lagrangian submanifolds with respect to the graded symplectic structure are candidates for boundary theories, we consider the transgression of the target space graded Poisson bracket restricted to a Lagrangian submanifold $\mathcal{L}$ to the mapping space $\mathrm{Map}(\mathring{\chi}, \mathcal{M})$. This results in the Poisson bracket of canonical quantities.

For simplicity, we start with the Lagrangian submanifold $\mathcal{L}$ defined by $p_i = \xi^i = 0$. The derived brackets of non-zero quantities on $\mathcal{L}$ are given by

$$\{\{x^i, \Theta\}, x^j\} = 0, \tag{3.377}$$

$$\{\{x^i, \Theta\}, \zeta_j\} = \delta^i_j, \tag{3.378}$$

$$\{\{\zeta_i, \Theta\}, \zeta_j\} = -H_{ijk}\xi^k. \tag{3.379}$$

Obviously, the pullback of the derived bracket to $\mathcal{L}$ projects out the contribution by the H-flux. Therefore, a twist of $\mathcal{L}$ by a canonical transformation will be used on the mapping space in order to generate the correct Poisson brackets with H-flux. First, we find the transgressed derived bracket on the mapping space to be

$$\{\{\boldsymbol{x}^i(z), S_1\}_\Omega, \boldsymbol{x}^j(z')\}_\Omega = 0, \tag{3.380}$$

$$\{\{\boldsymbol{x}^i(z), S_1\}_\Omega, \boldsymbol{\zeta}_j(z')\}_\Omega = \delta^i_j\delta(z - z'), \tag{3.381}$$

$$\{\{\boldsymbol{\zeta}_i(z), S_1\}_\Omega, \boldsymbol{\zeta}_j(z')\}_\Omega = -H_{ijk}(\boldsymbol{x}(z))\boldsymbol{\xi}^k(z)\delta(z - z'), \tag{3.382}$$

where $\delta(z - z') = \delta(\sigma - \sigma')\delta(\theta - \theta')$ and $z = (\sigma, \theta)$. For details on the calculations please refer to the functional analytic conventions on the mapping space, which can be found in the appendix.

In order to generate the Poisson bracket with H-flux, we have to deform the initial Lagrangian submanifold $\mathcal{L}$ by a twist. The symplectic structure that induces the derived brackets on $\mathcal{L}$ is given by

$$\omega_{\mathcal{L}} = \delta x^i \wedge \delta\zeta_i. \tag{3.383}$$

The Liouville 1-form of $\omega_{\mathcal{L}}$ such that $\mathcal{L}$ is in the zero locus of ϑ is given via $-\delta\vartheta_{\mathcal{L}} = \omega_{\mathcal{L}}$,

$$\vartheta_{\mathcal{L}} = -\zeta_i\delta x^i. \tag{3.384}$$

Twist of the coordinates by the transgression of the Liouville 1-form,

$$\alpha \equiv \iota_{\check{\boldsymbol{d}}}\mu_*\mathrm{ev}^*\vartheta_{\mathcal{L}} = \int_{\mathring{X}} \mu_{\mathring{X}}\boldsymbol{\zeta}_i\boldsymbol{dx}^i, \tag{3.385}$$

deforms

$$\boldsymbol{\xi}^i \mapsto \exp(\alpha)\boldsymbol{\xi}^i = \boldsymbol{\xi}^i - \boldsymbol{dx}^i, \tag{3.386}$$

keeping the other coordinates invariant. The exponential adjoint action is defined with respect to the transgressed graded Poisson bracket, $\{-,-\}_\Omega$. On the α-twisted Lagrangian submanifold $\widehat{\mathcal{L}}_\alpha$, the condition $\boldsymbol{\xi}^i = 0$ is deformed to $\boldsymbol{\xi}^i - \boldsymbol{dx}^i = 0$. This gives the derived brackets pulled back on $\widehat{\mathcal{L}}_\alpha$,

$$\{\{\boldsymbol{x}^i(z), S_1\}_\Omega, \boldsymbol{x}^j(z')\}_\Omega|_{\widehat{\mathcal{L}}_\alpha} = 0, \tag{3.387}$$

$$\{\{\boldsymbol{x}^i(z), S_1\}_\Omega, \boldsymbol{\zeta}_j(z')\}_\Omega|_{\widehat{\mathcal{L}}_\alpha} = \delta^i_j\delta(z-z'), \tag{3.388}$$

$$\{\{\boldsymbol{\zeta}_i(z), S_1\}_\Omega, \boldsymbol{\zeta}_j(z')\}_\Omega|_{\widehat{\mathcal{L}}_\alpha} = -H_{ijk}\boldsymbol{dx}^k\delta(z-z'), \tag{3.389}$$

and therefore the desired Poisson brackets on the superfields,

$$\{\boldsymbol{x}^i(z), \boldsymbol{x}^j(z')\}_{\mathrm{PB},\widehat{\mathcal{L}}_\alpha} = 0, \tag{3.390}$$

$$\{\boldsymbol{x}^i(z), \boldsymbol{\zeta}_j(z')\}_{\mathrm{PB},\widehat{\mathcal{L}}_\alpha} = \delta^i_j\delta(z-z'), \tag{3.391}$$

$$\{\boldsymbol{\zeta}_i(z), \boldsymbol{\zeta}_j(z')\}_{\mathrm{PB},\widehat{\mathcal{L}}_\alpha} = -H_{ijk}\boldsymbol{dx}^k\delta(z-z'). \tag{3.392}$$

In the final step, we project onto the physical components via Berezin integration. For this, we expand the superfields in the supercoordinate θ,

$$\boldsymbol{x}^i(\sigma,\theta) = \boldsymbol{x}^{(0),i}(\sigma) + \boldsymbol{x}^{(1),i}(\sigma)\theta, \tag{3.393}$$

$$\boldsymbol{\zeta}_i(\sigma,\theta) = \boldsymbol{\zeta}^{(0)}_i(\sigma) + \boldsymbol{\zeta}^{(1)}_i(\sigma)\theta. \tag{3.394}$$

The ghost-degree zero components are the physical fields, which we denote by $x^i = \boldsymbol{x}^{(0),i}$ and $\zeta_i = \boldsymbol{\zeta}^{(1)}_i$. The Poisson brackets of the physical canonical quantities are the ghost-number zero components of above relations,

$$\{x^i(\sigma), x^j(\sigma')\}_{\mathrm{PB},\widehat{\mathcal{L}}_\alpha} = 0, \tag{3.395}$$

$$\{x^i(\sigma), \zeta_j(\sigma')\}_{\mathrm{PB},\widehat{\mathcal{L}}_\alpha} = \delta^i_j\delta(\sigma-\sigma'), \tag{3.396}$$

$$\{\zeta_i(\sigma), \zeta_j(\sigma')\}_{\mathrm{PB},\widehat{\mathcal{L}}_\alpha} = -H_{ijk}\partial_\sigma x^k\delta(\sigma-\sigma'). \tag{3.397}$$

The symplectic structure, which induces these relations, is given by

$$\omega_H = \int_{S^1} d\sigma \delta x^i \wedge \delta\zeta_i + \frac{1}{2}\int_{S^1} d\sigma H_{ijk}(x)\partial_\sigma x^i \delta x^j \wedge \delta x^k. \tag{3.398}$$

We derived the Alekseev-Strobl type symplectic structure of a Poisson bracket twisted by H-flux [45].

Current algebras from QP-manifolds

Above, we showed how to construct consistent Poisson algebras with QP-manifold structure. In this section, we use these Poisson algebras to construct current algebras of Noether currents on the loop space associated with QP-manifolds.

Let $(\mathcal{M}, \Theta, \omega)$ be a QP-manifold of degree n. Let $\mathcal{L}$ be a Lagrangian submanifold with respect to ω. Finally, let $\chi = T[1]X$ be the superworldvolume, where $\dim(X) = n-1$. Then, $\mathrm{Map}(\chi, \mathcal{M})$ is the space of maps from χ to $\mathcal{M}$. The transgression of $\mathcal{L}$ to the mapping space be denoted by $\widehat{\mathcal{L}}$. Let us furthermore recall, that the transgression of the derived bracket on the target space to the mapping space and restricted to $\widehat{\mathcal{L}}$ is an ordinary Poisson bracket, $|\{-,-\}_{\mathrm{PB},\widehat{\mathcal{L}}}| = 0$.

We can decompose the space of smooth functions on $\mathcal{M}$ by degree,

$$\mathcal{C}^\infty(\mathcal{M}) = \bigoplus_{i=0}^{\infty} \mathcal{C}^\infty_i(\mathcal{M}). \tag{3.399}$$

The current algebra associated to the target space QP-manifold emerges from the transgression of $\bigoplus_{i=0}^{n-1} \mathcal{C}^\infty_i(\mathcal{M})$.

Definition 3.3.57 (Current algebra) *The current algebra associated with a target space QP-manifold $(\mathcal{M}, \Theta, \omega)$ of degree n and transgressed Lagrangian submanifold $\widehat{\mathcal{L}}$ is the Poisson algebra on the space of transgressed elements of $\bigoplus_{i=0}^{n-1} \mathcal{C}^\infty_i(\mathcal{M})$ equipped with Poisson bracket $\{-,-\}_{PB,\widehat{\mathcal{L}}}$.*

Let us compute the current algebra associated with the H-twisted standard Courant algebroid.

Example 3.3.18 (H-twisted Courant algebroid current algebra) Let $(\mathcal{M}, \Theta, \omega)$ be the QP-manifold of the H-twisted Courant algebroid. For constructing the current algebra, the degree zero and 1 subspaces of the space of smooth functions on the graded manifold $\mathcal{M}$ are sufficient,

$$\mathcal{C}^\infty_0(\mathcal{M}) \cong \mathcal{C}^\infty(M), \quad \mathcal{C}^\infty_1(\mathcal{M}) \cong TM \oplus T^*M. \tag{3.400}$$

To each element of a subspace, we associate a precursor for a current via

$$j_{[(0),f]} = f, \quad j_{[(1),X+\alpha]} = X^i \zeta_i + \alpha_i \xi^i, \tag{3.401}$$

where $f, X^i, \alpha_i \in \mathcal{C}^\infty(M)$. The derived brackets between the associated precursors are given by

$$\{\{j_{[(0),f]}, \Theta\}, j_{[(0),g]}\} = 0, \tag{3.402}$$

$$\{\{j_{[(1),X+\alpha]}, \Theta\}, j_{[(0),g]}\} = -\rho(X) j_{[(0),g]}, \tag{3.403}$$

$$\{\{j_{[(1),X+\alpha]}, \Theta\}, j_{[(1),Y+\beta]}\} = -j_{[(1),[X+\alpha,Y+\beta]_{\mathrm{D},H}]}. \tag{3.404}$$

Here, $[X+\alpha, Y+\beta]_{\mathrm{D},H}$ denotes the H-twisted Dorfman bracket on the generalized tangent bundle of the standard Courant algebroid. From these elements, the current algebra on the mapping space is constructed via transgression. We take the Lagrangian submanifold $\mathcal{L}$ as the hypersurface where $p_i = \xi^i = 0$. Define as usual $\mathring{\chi} = T[1]S^1$. The transgression of the precursor currents give the supercurrents on the superloopspace $\mathrm{Map}(T[1]S^1, T^*[2]T[1]M)$,

$$J_{[(0),f]}(\varepsilon_{(1)}) = \mu_* \varepsilon_{(1)} \mathrm{ev}^* j_{[(0),f]} = \int_{\mathring{\chi}} \mu_{\mathring{\chi}} \varepsilon_{(1)} f(\boldsymbol{x}(\sigma)), \tag{3.405}$$

$$J_{[(1),X+\alpha]}(\varepsilon_{(0)}) = \mu_* \varepsilon_{(0)} \mathrm{ev}^* j_{[(1),X+\alpha]} = \int_{\mathring{\chi}} \mu_{\mathring{\chi}} \varepsilon_{(0)} (X^i(\boldsymbol{x}(\sigma))\boldsymbol{\zeta}_i + \alpha_i(\boldsymbol{x}(\sigma))\boldsymbol{\xi}^i), \tag{3.406}$$

where we introduced test functions ε_i of degree i on $\mathring{\chi}$. At this step, we encounter the same problem as previously. Namely, that on the Lagrangian submanifold $\widehat{\mathcal{L}}$ some information gets lost. In order to anticipate this problem and generate a current algebra twisted by H-flux, we twist by $\alpha = \iota_{\check{\boldsymbol{d}}} \mu_* \mathrm{ev}^* \vartheta_{\mathcal{L}}$ to deform the Lagrangian submanifold $\widehat{\mathcal{L}} \to \widehat{\mathcal{L}}_\alpha$. The supercurrents on $\widehat{\mathcal{L}}_\alpha$ become

$$J_{[(0),f]}(\varepsilon_{(1)}) = \int_{\mathring{\chi}} \mu_{\mathring{\chi}} \varepsilon_{(1)} f(\boldsymbol{x}(\sigma)), \tag{3.407}$$

$$J_{[(1),X+\alpha]}(\varepsilon_{(0)}) = \int_{\mathring{\chi}} \mu_{\mathring{\chi}} \varepsilon_{(0)} (X^i(\boldsymbol{x}(\sigma))\boldsymbol{\zeta}_i + \alpha_i(\boldsymbol{x}(\sigma))\boldsymbol{d}\boldsymbol{x}^i). \tag{3.408}$$

The Poisson brackets of the supercurrents can then be computed in a similar fashion as above

$$\{J_{[(0),f]}(\varepsilon), J_{[(0),g]}(\varepsilon')\}_{\mathrm{PB},\widehat{\mathcal{L}}_\alpha} = 0, \tag{3.409}$$

$$\{J_{[(1),X+\alpha]}(\varepsilon), J_{[(0),g]}(\varepsilon')\}_{\mathrm{PB},\widehat{\mathcal{L}}_\alpha} = \rho(X) J_{[(0),g]}(\varepsilon\varepsilon'), \tag{3.410}$$

$$\{J_{[(1),X+\alpha]}(\varepsilon), J_{[(1),Y+\beta]}(\varepsilon')\}_{\mathrm{PB},\widehat{\mathcal{L}}_\alpha} = J_{[(1),[X+\alpha,Y+\beta]_{\mathrm{D},H}]}(\varepsilon\varepsilon') + \int_{\mathring{\chi}} \mu_{\mathring{\chi}} \boldsymbol{d}\varepsilon_{(0)} \varepsilon'_{(0)} \langle X+\alpha, Y+\beta \rangle. \tag{3.411}$$

Finally, we expand the superfields in θ and project to the degree zero component by Berezin integration, giving the physical current algebra on loop space twisted by H-flux,

$$\{j_{[(0),f]}(\sigma), j_{[(0),g]}(\sigma')\}_{\mathrm{PB},\widehat{\mathcal{L}}_\alpha} = 0, \tag{3.412}$$

$$\{j_{[(1),X+\alpha]}(\sigma), j_{[(0),g]}(\sigma')\}_{\mathrm{PB},\widehat{\mathcal{L}}_\alpha} = -\rho(X) j_{[(0),g]}(\sigma)\delta(\sigma-\sigma'), \tag{3.413}$$

$$\{j_{[(1),X+\alpha]}(\sigma), j_{[(1),Y+\beta]}(\sigma')\}_{\mathrm{PB},\widehat{\mathcal{L}}_\alpha} = -j_{[(1),[X+\alpha,Y+\beta]_{\mathrm{D},H}]}(\sigma)\delta(\sigma-\sigma') + \langle X+\alpha, Y+\beta\rangle(\sigma')\partial_\sigma\delta(\sigma-\sigma'). \tag{3.414}$$

We derived the Alekseev-Strobl type generalized currents, which were described in [45].

3.3.7 Deligne cohomology and n-gerbes

(Non-)abelian n-gerbes with connective structure appear in various areas of string theory and M-theory. The 3-form field strength H of the Kalb-Ramond B-field is the curvature of an abelian 1-gerbe. Non-abelian higher gerbes with connective structures are closely related with multiple M5-brane systems, as we will elucidate in the second main part *Higher gauge theory and multiple M5-branes* of this thesis. Higher gauge theories of M-brane systems as they appear in M-theory are governed by non-abelian n-gerbes. We are interested in the local symmetry L_∞-algebra associated with abelian higher gerbes with connective structure associated with the generalized as well as exceptional generalized geometries. Therefore, we provide a straightforward introduction in abelian n-gerbes as Čech-Deligne cocycles in Deligne cohomology. This will provide the background for our calculations in *Generalized geometries*. A standard reference, in which Deligne cohomology is discussed as a hypercohomology, is [114].

We start by constructing the Deline complex from the total complex of the Čech-de Rham double complex. Let M be a smooth manifold of dimension m. Furthermore, let $\{U_\alpha\}$ be a good open cover of M. Then, let $Y = \sqcup_\alpha U_\alpha$ be the disjoint union of open sets of the cover. The k-th fibered product of Y with itself is defined by

$$Y^{[k]} = Y \times_M \cdots \times_M Y = \{(x^1,\ldots,x^k) \in Y \times \cdots \times Y | \sigma(x^1) = \cdots = \sigma(x^k)\}, \tag{3.415}$$

where $\sigma : Y \to M$ is a surjective submersion. We can define face maps $f_j^{k-1} : Y^{[k]} \to Y^{[k-1]}$ for $j = 0,\ldots,k$ via

$$f_{i-1}^{k-1}(x^1,\ldots,x^k) = (x^1,\ldots,\widehat{x}^i,\ldots,x^k), \tag{3.416}$$

where hat means omission. Note that $(\{Y^{[k]}\},\{f_j^{k-1}\})$ forms a simplicial manifold, a simplicial object in the category of differentiable manifolds. We can define the Čech cohomology over M using the coboundary operator $\check{\delta} : \Omega^n(Y^{[k]}) \to \Omega^n(Y^{[k+1]})$, defined by

$$(\check{\delta}\alpha)(x) = \sum_{j=0}^{k-1} (-1)^j ((f_j^{k-1})^* \alpha)(x). \tag{3.417}$$

Here, $x \in Y^{[k+1]}$ and $\alpha \in \Omega^n(Y^{[k]})$. Since the coboundary operator is nilpotent, $\check{\delta}^2 = 0$, we can define the associated Čech cohomology.

Now we let us consider forms taking values in an abelian group, denoted by $U(1)$, and complement the Čech complex with the de Rham complex with de Rham differential d to the Čech-de Rham double complex.

$$\begin{array}{ccccccccccc}
\vdots & & \vdots & & \vdots & & \vdots & & & & \vdots \\
\uparrow{\scriptstyle\check{\delta}} & & \uparrow{\scriptstyle\check{\delta}} & & \uparrow{\scriptstyle\check{\delta}} & & \uparrow{\scriptstyle\check{\delta}} & & & & \uparrow{\scriptstyle\check{\delta}} \\
\mathcal{C}^\infty(Y^{[4]}, U(1)) & \xrightarrow{d\log} & \Omega^1(Y^{[4]}) & \xrightarrow{d} & \Omega^2(Y^{[4]}) & \xrightarrow{d} & \Omega^3(Y^{[4]}) & \xrightarrow{d} & \cdots & \xrightarrow{d} & \Omega^m(Y^{[4]}) \\
\uparrow{\scriptstyle\check{\delta}} & & \uparrow{\scriptstyle\check{\delta}} & & \uparrow{\scriptstyle\check{\delta}} & & \uparrow{\scriptstyle\check{\delta}} & & & & \uparrow{\scriptstyle\check{\delta}} \\
\mathcal{C}^\infty(Y^{[3]}, U(1)) & \xrightarrow{d\log} & \Omega^1(Y^{[3]}) & \xrightarrow{d} & \Omega^2(Y^{[3]}) & \xrightarrow{d} & \Omega^3(Y^{[3]}) & \xrightarrow{d} & \cdots & \xrightarrow{d} & \Omega^m(Y^{[3]}) \\
\uparrow{\scriptstyle\check{\delta}} & & \uparrow{\scriptstyle\check{\delta}} & & \uparrow{\scriptstyle\check{\delta}} & & \uparrow{\scriptstyle\check{\delta}} & & & & \uparrow{\scriptstyle\check{\delta}} \\
\mathcal{C}^\infty(Y^{[2]}, U(1)) & \xrightarrow{d\log} & \Omega^1(Y^{[2]}) & \xrightarrow{d} & \Omega^2(Y^{[2]}) & \xrightarrow{d} & \Omega^3(Y^{[2]}) & \xrightarrow{d} & \cdots & \xrightarrow{d} & \Omega^m(Y^{[2]}) \\
\uparrow{\scriptstyle\check{\delta}} & & \uparrow{\scriptstyle\check{\delta}} & & \uparrow{\scriptstyle\check{\delta}} & & \uparrow{\scriptstyle\check{\delta}} & & & & \uparrow{\scriptstyle\check{\delta}} \\
\mathcal{C}^\infty(Y, U(1)) & \xrightarrow{d\log} & \Omega^1(Y) & \xrightarrow{d} & \Omega^2(Y) & \xrightarrow{d} & \Omega^3(Y) & \xrightarrow{d} & \cdots & \xrightarrow{d} & \Omega^m(Y)
\end{array}$$

The Deligne complex is constructed out of the total complex of the Čech-de Rham double complex. Let us define $C_{i,0} = \mathcal{C}^\infty(Y^{[i+1]}, U(1))$ and $C_{i,j} = \Omega^j(Y^{[i+1]})$ for $j = 2, \ldots, m$, where $Y^{[1]} = Y$. Then, the total complex $\mathrm{Tot}(C)_\bullet$ is given by the chain complex

$$\mathrm{Tot}(C)_n = \bigoplus_{i+j=n} C_{i,j}. \tag{3.418}$$

The differential on the total complex is given by the total differential D, which is the sum of the de Rham differential and Čech differential, defined by

$$D = d + (-1)^k \check{\delta}, \tag{3.419}$$

where k denotes the form degree of the object on which D is acting. For any function $g \in \mathcal{C}^\infty(Y^{[i]}, U(1))$, we define

$$Dg = \frac{1}{2\pi i} d \log g + \check{\delta}(g). \tag{3.420}$$

Obviously, D is nilpotent. We are lead to the following definition.

Definition 3.3.58 (Deligne complex) *The complex* $(Tot(C)_\bullet, D)$ *is called the* *Deligne complex* *of degree* m.

The associated cohomology is called the Deligne cohomology of degree m, which is denoted by $H^{m+1}_{\text{conn}}(M,\mathbb{Z})$. We can now define the notion of a n-gerbe.

Definition 3.3.59 (n-gerbe with connective structure) *Let* $H^{m+1}_{conn}(M,\mathbb{Z})$ *be the Deligne cohomology of degree* m *over* M. *An* *n-gerbe with connective structure* *is a Čech-Deligne cocycle in degree* $n+2$.

Deligne cohomology encodes gauge transformations of abelian n-gerbes as we will see in the following. The equivalence classes of n-gerbes are governed by Čech-Deligne coboundaries. The curvature of an n-gerbe is given by the gauge-invariant $F = dA$, where $A \in \Omega^{n+1}(Y)$. Let us compute some examples.

Example 3.3.19 ((-1)-gerbe) A (-1)-gerbe is a Čech-Deligne cocycle in degree 1. It defines a class in $H^1_{\text{conn}}(M,\mathbb{Z})$. It consists of functions $\{g_a\} \in \mathcal{C}^\infty(Y, U(1))$, such that $\check{\delta}(g) = g_b g_a^{-1} = 1$ on double-overlaps U_{ab}. Its curvature is given by the 1-form $G = \frac{1}{2\pi i} d\log g$.

Example 3.3.20 (0-gerbe) A 0-gerbe is a Čech-Deligne cocycle in degree 2. In this case, it defines a class in $H^2_{\text{conn}}(M,\mathbb{Z})$ and consists of functions $\{g_{ab}\} \in \mathcal{C}^\infty(Y^{[2]}, U(1))$ and 1-forms $\{A_a\} \in \Omega^1(Y)$, that satisfy

$$\check{\delta}(g) = 1, \quad \check{\delta}(A) - \frac{1}{2\pi i} d\log g_{ab} = 0, \tag{3.421}$$

leading to

$$g_{ab}g_{bc}g_{ca} = 1, \quad A_a - A_b = \frac{1}{2\pi i} d\log g_{ab}, \tag{3.422}$$

on triple overlaps U_{abc} and double-overlaps U_{ab}, respectively. The curvature of the 0-gerbe is given by the 2-form $F = dA$. We conclude, that a 0-gerbe encodes the connective structure of ordinary gauge theory with abelian structure group, a principal $U(1)$-bundle with connection.

Two 0-gerbes G and $\widetilde{G}$ are related by a coboundary $\gamma \in \text{Tot}_0 = \mathcal{C}^\infty(Y, U(1))$ such that $D\gamma = G - \widetilde{G}$. The equation can be rewritten by

$$\check{\delta}(\gamma) = g - \widetilde{g}, \quad d\gamma = A - \widetilde{A}, \tag{3.423}$$

leading to

$$\gamma_a^{-1}\widetilde{g}_{ab}\gamma_b = g_{ab}, \quad d\gamma_a + \widetilde{A}_a = A_a, \tag{3.424}$$

on double-overlaps U_{ab} and on open subsets U_a, respectively.

Example 3.3.21 (1-gerbe) A 1-gerbe, or just gerbe, is a Čech-Deligne cocycle in degree 3. It is a class in $H^3_{\text{conn}}(M, \mathbb{Z})$. It consists of functions $\{g_{abc}\} \in \mathcal{C}^\infty(Y^{[3]}, U(1))$, 1-forms $\{A_{ab}\} \in \Omega^1(Y^{[2]})$ and 2-forms $\{B_a\} \in \Omega^2(Y)$, that satisfy

$$\check{\delta}(g) = 1, \quad \check{\delta}(A) - \frac{1}{2\pi i} d\log g = 0, \quad \check{\delta}(B) + dA = 0. \tag{3.425}$$

This gives

$$g_{abc}g_{abd}^{-1}g_{acd}g_{bcd}^{-1} = 1, \quad A_{ab} + A_{bc} + A_{ca} = \frac{1}{2\pi i} d\log g_{abc}, \quad B_b - B_a = dA_{ab}, \tag{3.426}$$

on 4-overlaps U_{abcd}, triple-overlaps U_{abc} and double-overlaps U_{ab}, respectively. The curvature of the gerbe is given by the 3-form $H = dB$. This structure is also called bundle gerbe with connection.

This ends the preliminary sections. From now, we will enter the main calculations.

3.4 Twisted Courant algebroids and fluxes

We understand, which important role the standard Courant algebroid plays in generalized geometry as a geometrization of T-duality H-flux backgrounds. However, we also understand, that other crucial fluxes are to expected, when trying to find a unified description of all toroidal T-duality backgrounds. Aside from the H-flux, we encounter the geometric f-flux and the mysterious non-geometric Q- and R-fluxes. In this section, we want to answer the question of *How far can we exhaust the Courant algebroid structure to serve as an object that unifies all these fluxes?* The method to arrive at an answer to this question is to deform the untwisted Courant algebroid using representations of the generators of $O(D, D)$. This will introduce not only geometric but also non-geometric flux freedom into the Courant algebroid in a covariant way and even provide the correct generalized flux Bianchi identities as consistency equations of the Courant algebroid. More precisely, since the analysis is done using graded symplectic manifolds, the generalized flux Bianchi identities naturally arise from the classical master equation. The resulting *fully fluxed Courant algebroid*, which we will construct, then encodes the local symmetries of toroidal compactifications with H-, f-, Q- and R-fluxes.

In 3.4.1, we recall the graded symplectic manifold setup underlying the untwisted Courant algebroid. This serves as the base for the analysis conducted in this section. In 3.4.2, we discuss the insufficiencies of the Courant algebroid where a 3-vector R-flux is introduced by hand. It will motivate us to dig deeper and investigate all possible twists in this setup in 3.4.3. In 3.4.4, we produce an intermediate result by the construction of a β-twisted Courant algebroid and analyze its cohomology. For the introduction of the geometric f-flux, we need to introduce a generalization of the Courant algebroid structure by a frame bundle, constructed in 3.4.5 and used in 3.4.6 to generate geometric f-flux. In 3.4.7, we construct the final result of the Courant algebroid twisted by all fluxes appearing in toroidal closed string compactifications. As a nice feature, we find the generalized flux Bianchi identities directly by the classical master equation of the underlying graded symplectic manifold. This section is based on the published papers [2, 3].

3.4.1 Graded symplectic manifold

During the entire discussion, we will merely be making use of graded symplectic manifolds of degree 2. These are equivalent to Courant algebroids.

Let $\mathcal{M} = T^*[2]T[1]M$ be a graded manifold, where M is a smooth manifold. M plays the role of the target spacetime. We use the same local parameterization as above, coordinates $(x^i, \xi^i, \zeta_i, p_i)$ of degrees $(0, 1, 1, 2)$. Let us choose the graded symplectic structure as

$$\omega = -\delta x^i \wedge \delta p_i + \delta\xi^i \wedge \delta\zeta_i. \tag{3.427}$$

A general Hamiltonian on $\mathcal{M}$ is of degree 3. The simplest non-trivial Hamiltonian is given by

$$\Theta_0 = \xi^i p_i. \tag{3.428}$$

It has been shown above, how the derived bracket construction using this Hamiltonian leads to the untwisted standard Courant algebroid on the generalized tangent bundle $TM \oplus T^*M \to M$. The introduction of H-flux to this setup is given by a small modification of the Hamiltonian function,

$$\Theta_H = \xi^i p_i + \frac{1}{3!} H_{ijk}(x)\xi^i\xi^j\xi^k, \tag{3.429}$$

which gives the H-twisted standard Courant algebroid on the generalized tangent bundle via the derived bracket construction.

3.4.2 Courant algebroid with trivial R-flux

In a similar way as H-flux can be introduced in Θ_0, a 3-vector freedom in form of an R-flux is also possible by

$$\Theta = \xi^i p_i + \frac{1}{3!} R^{ijk}(x)\zeta_i\zeta_j\zeta_k, \tag{3.430}$$

where $R \in \mathfrak{X}^3(M)$. However, the classical master equation requires the R-flux to be trivial. The authors of [115] used the AKSZ procedure to generate a sigma model based on above Hamiltonian function, which according to their discussion provides a model for a string propagating in non-geometric R-space. The resulting action is a sigma model of a string embedding into the contangent bundle $T^*M \to M$, whose phase space Poisson structure becomes quasi-Poisson due to the R-flux. Quantization of such a phase space structure leads to a non-associative star product.

3.4.3 Classification of twists

We showed above that an H-flux $dB = H$ can be locally induced by a B-twist of the Hamiltonian function. On a QP-manifold of degree 2, any degree-preserving twist is of degree 2. In the next step, we investigate all possible twists of our Hamiltonian function and we will recognize that we can generate all geometric as well as non-geometric fluxes by a certain succession of transformations.

The possible twists are given by

$$\begin{aligned} \exp(\delta_B) &= \exp\left(\frac{1}{2}B_{ij}(x)\xi^i\xi^j\right), \quad \exp(\delta_\beta) = \exp\left(\frac{1}{2}\beta^{ij}(x)\zeta_i\zeta_j\right), \\ \exp(\delta_f) &= \exp\left(\frac{1}{2}{f_j}^i(x)\xi^j\zeta_i\right), \quad \exp(\delta_a) = \exp\left(a^i(x)p_i\right), \end{aligned} \tag{3.431}$$

where $B_{ij}, \beta^{ij}, {f_j}^i, a^i \in \mathcal{C}^\infty(M)$. The first and second exponential actions generate B-twist, β-twist. The third turns out to generate diffeomorphisms. The fourth generates local translations and will therefore by ignored in the further analysis. The first three twists leave the inner product on the Courant algebroid invariant and therefore are in one-to-one correspondence with generators of local $O(D,D)$-transformations.

3.4.4 Non-geometric β-twisted Courant algebroid and cohomology

Let us now discuss the β-twisted Hamiltonian, which turns out to be

$$\exp(\delta_\beta)\Theta_0 = p_i\xi^i + \beta^{ij}p_i\zeta_j + \frac{1}{2}\partial_i\beta^{jk}\xi^i\zeta_j\zeta_k - \frac{1}{2}\beta^{in}\partial_n\beta^{jk}\zeta_i\zeta_j\zeta_k. \tag{3.432}$$

We can rewrite the resulting Hamiltonian by

$$\Theta_\beta = p_i\xi^i + \beta^{ij}p_i\zeta_j + \frac{1}{2}Q_i^{jk}\xi^i\zeta_j\zeta_k - \frac{1}{3!}R^{ijk}\zeta_i\zeta_j\zeta_k, \tag{3.433}$$

so that the classical master equation fixes $R = \frac{1}{2}[\beta,\beta]_{\mathrm{S}}$ or $R^{ijk} = 3\beta^{[i|n|}\partial_n\beta^{jk]}$ and $Q_i^{jk} = \partial_i\beta^{jk}$. We recognize that the β-twist of the standard Courant algebroid naturally introduces Q-flux and R-flux contributions.

Furthermore, we can split the Hamiltonian into two parts

$$\Theta_\beta = \Theta_0 + \Theta_{\mathrm{Poisson}}, \tag{3.434}$$

so that under the assumption that $[\beta,\beta]_{\mathrm{S}} = 0$,

$$\Theta_{\mathrm{Poisson}} = \beta^{ij}p_i\zeta_j + \frac{1}{2}\partial_i\beta^{jk}\xi^i\zeta_j\zeta_k \tag{3.435}$$

induces a Poisson cohomology on $\wedge^\bullet TM$ with Lichnerowicz-Poisson differential given by the β-bivector field, $d_\beta = [\beta, -]_{\mathrm{S}}$, via

$$d_\beta = j^* \circ Q_{\mathrm{Poisson}} \circ j_*, \tag{3.436}$$

where $Q_{\mathrm{Poisson}} = \{\Theta_{\mathrm{Poisson}}, -\}$. The condition $[\beta,\beta]_{\mathrm{S}} = 0$ requires β to be a Poisson tensor. Then, the differential d_β is nilpotent, $d_\beta^2 = 0$. The fact that $\{\Theta_0, \Theta_{\mathrm{Poisson}}\} = 0$ then leads to a factorization of the classical master equation,

$$Q_\beta^2 = (Q_0 + Q_{\mathrm{Poisson}})^2 = Q_0^2 + Q_{\mathrm{Poisson}}^2 = 0, \tag{3.437}$$

where $Q_0 = \{\Theta_0, -\}$, so that Q_β^2 induces the total cohomology of the Poisson-de Rham double complex with total differential

$$D = d + d_\beta = j^* \circ (Q_0 + Q_{\mathrm{Poisson}}) \circ j_*, \tag{3.438}$$

acting on elements of $\wedge^\bullet(TM \oplus T^*M)$. The resulting cohomology can be generalized to the so-called Courant algebroid cohomology on $\mathcal{M}$, [102].

If the R-flux contribution is non-zero, then the Poisson condition is deformed to $\frac{1}{2}[\beta,\beta]_{\mathrm{S}} = R$. In other words, the Poisson bracket associated to β is given by

$$\{f,g\}_\beta = \beta^{ij}\partial_i f \partial_j g, \tag{3.439}$$

where $f, g \in \mathcal{C}^\infty(M)$ does not satisfy the Jacobi identity,

$$\{\{f,g\}_\beta, h\}_\beta + \{\{h,f\}_\beta, g\}_\beta + \{\{g,h\}_\beta, f\}_\beta = R^{ijk}\partial_i f \partial_j g \partial_k h. \tag{3.440}$$

Such a structure is called quasi-Poisson structure. The Q-flux contribution introduced by the β-twist can then be rewritten as [116]

$$\{x^i, x^j\}_\beta = \int Q_k^{ij} dx^k, \tag{3.441}$$

leading to an interpretation as non-commutativity that the closed string perceives traveling through non-geometric space.

The classical master equation of the β-twisted Hamiltonian, $\{\Theta_\beta, \Theta_\beta\} = 0$, leads to the generalized flux Bianchi identities,

$$\partial_{[m} Q_{i]}^{[jk]} = 0, \tag{3.442}$$

$$3\beta^{[i|m|}\partial_m Q_n^{jk]} - \partial_n R^{[ijk]} + 3Q_n^{[i|m|}Q_m^{jk]} = 0, \tag{3.443}$$

$$\beta^{[i|m|}\partial_m R^{jkl]} - \frac{3}{2}R^{[ij|m|}Q_m^{kl]} = 0. \tag{3.444}$$

Introducing bases of the cotangent bundle $e^i \in T^*M$ and their image under $\beta^\sharp$ in the tangent bundle, $e_\sharp^i = \beta^\sharp e^i = \beta^{ij}\partial_j$, and $\partial_i = e_i \in TM$ being the dual of e^i, we find that above relations coincide with the Jacobi identities of the generalized commutation relations,

$$[e_i, e_j]_{\mathrm{Lie}} = 0, \tag{3.445}$$

$$[e_i, e_\sharp^j]_{\mathrm{Lie}} = Q_i^{jn} e_n, \tag{3.446}$$

$$[e_\sharp^i, e_\sharp^j]_{\mathrm{Lie}} = R^{ijn} e_n + Q_n^{ij} e_\sharp^n. \tag{3.447}$$

We follow the notation of [116]. We can compute the associated β-twisted standard Courant algebroid on the generalized tangent bundle via the derived bracket construction. The induced anchor map is given by

$$\rho(X+\alpha)f = j^*\{\{\Theta_\beta, j_*(X+\alpha)\}, j_*(f)\} = (X - \beta^\sharp(\alpha))(f), \tag{3.448}$$

where $X+\alpha \in \Gamma(TM \oplus T^*M)$ and $f \in \mathcal{C}^\infty(M)$. We find the β-twisted Dorfman bracket via

$$\begin{aligned}[X+\alpha, Y+\gamma]_{\mathrm{D}} &= j^*\{\{\Theta_\beta, j_*(X+\alpha)\}, j_*(Y+\gamma)\} \\ &= [X,Y]_{\mathrm{Lie}} + L_X\gamma - \iota_Y d\alpha - [\alpha,\gamma]_\beta - L^\beta_\alpha Y + \iota_\gamma d_\beta X - \iota_\alpha \iota_\gamma R. \end{aligned} \tag{3.449}$$

Here, $L^\beta_\alpha Y$ denotes the Poisson-Lie derivative along the 1-form α, defined by

$$L^\beta_\alpha Y = d_\beta \iota_\alpha Y + \iota_\alpha d_\beta Y, \tag{3.450}$$

and $[\alpha,\gamma]_\beta$ denotes the Koszul bracket, given by

$$[\alpha,\gamma]_\beta = L_{\beta^\sharp(\alpha)}\iota_\alpha Y + \iota_\alpha d_\beta Y. \tag{3.451}$$

The fiber metric is invariant under the β-twist, since it emerges from the graded symplectic form,

$$\begin{aligned}\langle X+\alpha, Y+\gamma\rangle &= j^*\{j_*(X+\alpha), j_*(Y+\gamma)\} \\ &= \iota_X\gamma + \iota_Y\alpha. \end{aligned} \tag{3.452}$$

Let us summarize our findings in the following theorem.

Theorem 3.4.1 (β-twisted standard Courant algebroid) *The QP-manifold $(\mathcal{M}, \Theta_\beta, \omega)$ induces the generalized flux Bianchi identities as well as the local description of the Q- and R-fluxes in terms of the β-potential. Furthermore, its associated Courant algebroid $(TM \oplus T^*M, \langle -,-\rangle, \rho, [-,-]_D)$ realizes the operations*

$$\begin{aligned}\langle X+\alpha, Y+\gamma\rangle &= \iota_X\gamma + \iota_Y\alpha, \\ \rho(X+\alpha) &= X - \beta^\sharp(\alpha), \\ [X+\alpha, Y+\gamma]_D &= [X,Y]_{Lie} + L_X\gamma - \iota_Y d\alpha - [\alpha,\gamma]_\beta - L^\beta_\alpha Y + \iota_\gamma d_\beta X - \iota_\alpha\iota_\gamma R. \end{aligned}$$

In the special case, where the R-flux is vanishing, the Q-structure induces the total cohomology of the Poisson-de Rham double complex on $\Gamma(\wedge^\bullet(TM \oplus T^*M))$.

3.4.5 Courant algebroid with frame bundle

In order to discuss spaces with non-trivial curvature or torsion, we introduce a frame bundle over our graded manifold $\mathcal{M}$. We showed that under the injection map j, the coordinates

(ξ^i, ζ_j) correspond to the basis of the generalized tangent bundle, (dx^i, ∂_j). We introduce a frame bundle of $T[1]M \oplus T^*[1]M$ for (ξ^i, ζ_j). Let $V = \mathbb{R}^D$ be a flat vector space of the same dimension as M. Then, the frame bundle is given by $V[1] \oplus V^*[1]$ with local coordinates (ξ^a, ζ_b) corresponding to a flat frame. A general frame is then given by (ξ^i, ζ_j) on $T[1]M \oplus T^*[1]M$. Twists by functions that locally relate flat frames with general frames are made out of the coordinates $(\xi^i, \zeta_j, \xi^a, \zeta_b)$ on the direct product $T[1]M \oplus T^*[1]M \oplus V[1] \oplus V^*[1]$. Such twists have the form

$$\exp(\delta_e) = \exp\left(e_a{}^i(x)\xi^a\zeta_i\right), \quad \exp(\delta_{e^{-1}}) = \exp\left(e^a{}_i(x)\xi^i\zeta_a\right), \tag{3.453}$$

where $e_a{}^i, e_i{}^a \in \mathcal{C}^\infty(M)$. The graded Poisson brackets are given by $\{\xi^a, \zeta_b\} = \delta^a_b$ and $\{\xi^i, \zeta_j\} = \delta^i_j$. All other combinations among the coordinates vanish. It turns out that a certain order of twists acts as a transition from flat to curved frame and vice versa. Therefore, we can think of the functions $e_a{}^i$ as sections of the frame bundle over M, or vielbeins.

Having enlarged our space of local coordinates by the flat frame, we introduce an enlarged injection map from the generalized tangent bundle with frame bundle to the corresponding graded manifold,

$$\begin{aligned} j : TM \oplus (TM \oplus T^*M) \oplus V \oplus V^* &\to T^*[2]T[1]M \oplus V[1] \oplus V^*[1] \\ \left(\frac{\partial}{\partial x^i}, x^i, dx^i, \partial_i, u^a, u_a\right) &\mapsto (p_i, x^i, \xi^i, \zeta_i, \xi^a, \zeta_a). \end{aligned} \tag{3.454}$$

3.4.6 Courant algebroid with geometric flux

We shall now discuss, how we can make use of the additional coordinates due to the introduction of the frame bundle in order to induce geometric f-flux. This flux is associated with the torsion-less part of the projected spin connection of the compactification the string travels through.

The T-dual of a 3-dimensional torus on which H-flux is wrapped is given by a so-called nilmanifold or twisted torus, formulated as S^1-bundle over S^2 realizing a non-trivial spin connection. In the next step, we will show how to induce the geometric flux associated with such a spin connection by manipulation of the Hamiltonian function of our QP-manifold.

In terms of a vielbein $e_a{}^i$, the f-flux is given by

$$f^a_{bc} = 2e_{[c}{}^i\partial_i e^a{}_j e_{b]}{}^j. \tag{3.455}$$

Recall, that the indices $i, j, k, \dots$ denote curved coordinates, whereas indices $a, b, c, \dots$ denote flat coordinates. It is easy to show, that the Hamiltonian

$$\Theta_f = e_c{}^i p_i \xi^c - \frac{1}{2} f^c_{ab} \xi^a \xi^b \zeta_c \tag{3.456}$$

induces the appropriate relations by the classical master equation,

$$e_{[b}^{\ j} \partial_j e_{a]}^{\ i} = -\frac{1}{2} e_c{}^i f^c_{ab}, \quad e_{[d}^{\ j} \partial_j f^c_{ab]} = f^c_{[ab} f^c_{|e|d]}. \tag{3.457}$$

If we introduce an inverse $e^a{}_i$ of $e_b{}^i$, such that $e^a{}_i e_b{}^i = \delta^a_b$, then we can find the defining relation of the f-flux.

Until now, we introduced the local expressions for the fluxes by twist of the Hamiltonian, whereas for the Hamiltonian, that realizes f-flux, the local expression is encoded in the classical master equation. Therefore, our next task is to find an appropriate twist such that the correct form of the f-flux emerges. For doing so, we make use of the transformations corresponding to the diffeomorphism generators of $O(D, D)$. It turns out that the appropriate twist is of the form

$$\mathfrak{D} \equiv \exp(-\delta_e) \exp(\delta_{e^{-1}}) \exp(-\delta_e). \tag{3.458}$$

We find

$$\mathfrak{D}\Theta_0 = e_a{}^n \xi^a p_n - e_a{}^n \partial_n e_b{}^i e^b{}_j \xi^j \xi^a \zeta_i - e_a{}^n \partial_n e_b{}^i e^c{}_i \xi^a \xi^b \zeta_c. \tag{3.459}$$

Since

$$\mathfrak{D}\xi^i = e_a{}^i \xi^a, \qquad \mathfrak{D}\zeta_i = e^a{}_i \zeta_a, \tag{3.460}$$

we understand that the part which directly contributes to the anchor map received a coefficient proportional to the vielbein. We can read off the f-flux from the twisted Hamiltonian,

$$\frac{1}{2} f^c_{ab} = e_{[a}^{\ i} \partial_i e_{b]}^{\ j} e^c{}_j = e_{[b}^{\ i} \partial_i e^c{}_j e_{a]}^{\ j}. \tag{3.461}$$

The second term in the twisted Hamiltonian corresponds to a connection on the frame bundle. In terms of a vielbein basis on the tangent bundle $e_a{}^i \partial_i \in TM$, the f-flux can be expressed by the Lie commutator,

$$[e_a, e_b]_{\text{Lie}} = f^c_{ab} e_c. \tag{3.462}$$

Let us summarize our findings in the following theorem.

Theorem 3.4.2 (f-twisted standard Courant algebroid) *The QP-manifold $(\mathcal{M}, \mathfrak{D}\Theta_0, \omega)$ realizes a local formulation of an f-flux background.*

By successive deformation of the Hamiltonian Θ_0 by B-transformation and diffeomorphic twist we find the covariant formulation of H-flux on a background twisted by f-flux,

$$\mathfrak{D}\exp(\delta_B)\Theta_0 = e_a{}^n\xi^a p_n - e_a{}^n\partial_n e_b{}^i e^b{}_j\xi^j\xi^a\zeta_i - e_a{}^n\partial_n e_b{}^i e^c{}_i\xi^a\xi^b\zeta_c + \frac{1}{2}\partial_i B_{jk} e_a{}^i e_b{}^j e_c{}^k\xi^a\xi^b\xi^c. \quad (3.463)$$

We find the correct local expression of H-flux in flat coordinates,

$$H_{abc} = 3\partial_i B_{jk} e_{[a}{}^i e_b{}^j e_{c]}{}^k = 3(\partial_{[a}B_{bc]} - f^d_{[ab}B_{|d|c]}) \equiv 3\nabla_{[a}B_{bc]}, \quad (3.464)$$

where we introduced the covariant derivative ∇.

3.4.7 Fully fluxed Courant algebroid

In this section, we derive the Courant algebroid that realizes all geometric as well as non-geometric fluxes in a covariant manner from a twisted QP-manifold. Consistency of the QP-manifold directly leads to the generalized flux Bianchi identities. The procedure also derives the correct local expressions for all fluxes.

In order to find the appropriate Hamiltonian function, we first twist by B- and β-transformation, giving

$$\begin{aligned}
\exp(\delta_\beta)\exp(\delta_B)\Theta_0 &= \exp(\delta_\beta)\left(p_i\xi^i + \frac{1}{2}\partial_i B_{jk}\xi^i\xi^j\xi^k\right) \\
&= p_i\xi^i + p_m\beta^{mi}\zeta_i + \frac{1}{2}\partial_n B_{rs}\xi^n\xi^r\xi^s + (\partial_m B_{ns} + \frac{1}{2}\partial_s B_{mn})\beta^{si}\zeta_i\xi^m\xi^n \\
&\quad + \left[\frac{1}{2}\partial_i\beta^{hk} - \frac{1}{2}\partial_i B_{rs}\beta^{sh}\beta^{rk} + \partial_r B_{is}\beta^{sh}\beta^{rk}\right]\xi^i\zeta_h\zeta_k \\
&\quad - \left[-\frac{1}{2}\partial_l\beta^{ih}\beta^{lk} + \frac{1}{2}\partial_n B_{rs}\beta^{si}\beta^{rh}\beta^{nk}\right]\zeta_i\zeta_h\zeta_k. \quad (3.465)
\end{aligned}$$

The vielbein is introduced by twisting by $\mathfrak{D}$,

$$
\begin{aligned}
&\mathfrak{D}\exp(\delta_\beta)\exp(\delta_B)\Theta_0 \\
&\quad= p_i e_b{}^i\xi^b + p_m\beta^{ml}e^b{}_l\zeta_b - e_b{}^m\partial_m e_a{}^j e^a{}_i\zeta_j\xi^i\xi^b - \beta^{ml}e^b{}_l\partial_m e_a{}^j e^a{}_i\zeta_j\xi^i\zeta_b \\
&\qquad - e_c{}^m\partial_m e_a{}^j e^b{}_j\xi^a\zeta_b\xi^c - \beta^{ml}e^c{}_l\partial_m e_a{}^j e^b{}_j\xi^a\zeta_b\zeta_c \\
&\qquad + \frac{1}{2}\partial_n B_{rs}e_a{}^n e_b{}^r e_c{}^s\xi^a\xi^b\xi^c + (\partial_m B_{ns} + \frac{1}{2}\partial_s B_{mn})\beta^{si}e^a{}_i e_b{}^m e_c{}^n\zeta_a\xi^b\xi^c \\
&\qquad + \left[\frac{1}{2}\partial_i\beta^{hk} - \frac{1}{2}\partial_i B_{rs}\beta^{sh}\beta^{rk} + \partial_r B_{is}\beta^{sh}\beta^{rk}\right] e_a{}^i e^b{}_h e^c{}_k\xi^a\zeta_b\zeta_c \\
&\qquad - \left[-\frac{1}{2}\partial_l\beta^{ih}\beta^{lk} + \frac{1}{2}\partial_n B_{rs}\beta^{si}\beta^{rh}\beta^{nk}\right] e^a{}_i e^b{}_h e^c{}_k\zeta_a\zeta_b\zeta_c \\
&\quad= e_b{}^i p_i\xi^b + \beta^{ml}e^b{}_l p_m\zeta_b + \beta^{ml}e^b{}_l\partial_m e_a{}^j e^a_i\xi^i\zeta_j\zeta_b - e_b{}^m\partial_m e_a{}^j e^a{}_i\xi^i\xi^b\zeta_j \\
&\qquad + \frac{1}{2}\partial_n B_{rs}e_a{}^n e_b{}^r e_c{}^s\xi^a\xi^b\xi^c - \left[e_b{}^m\partial_m e_c{}^j e^a{}_j - (\partial_m B_{ns} + \frac{1}{2}\partial_s B_{mn})\beta^{si}e^a{}_i e_b{}^m e_c{}^n\right]\zeta_a\xi^b\xi^c \\
&\qquad + \left[-\beta^{ml}e^c{}_l\partial_m e_a{}^j e^b{}_j + \left[\frac{1}{2}\partial_i\beta^{hk} - \frac{1}{2}\partial_i B_{rs}\beta^{sh}\beta^{rk} + \partial_r B_{is}\beta^{sh}\beta^{rk}\right] e_a{}^i e^b{}_h e^c{}_k\right]\xi^a\zeta_b\zeta_c \\
&\qquad - \left[-\frac{1}{2}\partial_l\beta^{ih}\beta^{lk} + \frac{1}{2}\partial_n B_{rs}\beta^{si}\beta^{rh}\beta^{nk}\right] e^a{}_i e^b{}_h e^c{}_k\zeta_a\zeta_b\zeta_c. \qquad (3.466)
\end{aligned}
$$

We can rewrite the resulting Hamiltonian via

$$
\begin{aligned}
\Theta_{B\beta e} &= e_b{}^i\xi^b p_i - e^b{}_l\beta^{lm}\zeta_b p_m - e^b{}_l\beta^{lm}\partial_m e_a{}^j e^a{}_i\xi^i\zeta_j\zeta_b - e_b{}^m\partial_m e_a{}^j e^a{}_i\xi^i\xi^b\zeta_j \\
&\quad + \frac{1}{3!}H_{abc}\xi^a\xi^b\xi^c - \frac{1}{2}F^a_{bc}\zeta_a\xi^b\xi^c + \frac{1}{2}Q^{bc}_a q^a\zeta_b\zeta_c - \frac{1}{3!}R^{abc}\zeta_a\zeta_b\zeta_c, \qquad (3.467)
\end{aligned}
$$

by defining

$$
\begin{aligned}
H_{abc} &= 3\nabla_{[a}B_{bc]}, && (3.468)\\
F^a_{bc} &= f^a_{bc} - H_{mns}\beta^{si}e^a{}_i e_b{}^m e_c{}^n, && (3.469)\\
f^a_{bc} &= 2e_{[b}{}^m\partial_m e_{c]}{}^j e^a{}_j, && (3.470)\\
H_{mns} &= 3\partial_{[m}B_{ns]}, && (3.471)\\
Q^{bc}_a &= \partial_a\beta^{bc} + f^b_{ad}\beta^{dc} - f^c_{ad}\beta^{db} + H_{isr}\beta^{sh}\beta^{rk}e_a{}^i e^b{}_h e^c{}_k, && (3.472)\\
R^{abc} &= 3(\beta^{[a|m|}\partial_m\beta^{bc]} + f^{[a}_{mn}\beta^{b|m|}\beta^{c]n}) - H_{mns}\beta^{mi}\beta^{nh}\beta^{sk}e^a{}_i e^b{}_h e^c{}_k. && (3.473)
\end{aligned}
$$

The resulting Hamiltonian encodes the local expressions of all fluxes in terms of the potentials B, β and e. Furthermore, the classical master equation of the Hamiltonian encodes the Jacobi identities for the commutators

$$[e_a, e_b]_{\text{Lie}} = F^c_{ab} e_c + H_{abc} e^c_\sharp, \tag{3.474}$$

$$[e_a, e^b_\sharp]_{\text{Lie}} = Q^{bc}_a e_c - F^b_{ac} e^c_\sharp, \tag{3.475}$$

$$[e^a_\sharp, e^b_\sharp]_{\text{Lie}} = R^{abc} e_c + Q^{ab}_c e^c_\sharp, \tag{3.476}$$

and the H-flux Bianchi identity. The basis elements in flat coordinates are defined by $e_a \equiv {e_a}^i \partial_i$ and $e^a_\sharp = \beta^\sharp e^a = \beta^{ab} e_b$ with $\beta^{ab} \equiv {e^a}_i {e^b}_j \beta^{ij}$ following [116]. The full generalized flux Bianchi identities, that follow from the classical master equation, are given by

$$e_{[a}{}^m \partial_{|m|} H_{bcd]} - \frac{3}{2} F^e_{[ab} H_{|e|cd]} = 0, \tag{3.477}$$

$$e^{[a}{}_l \beta^{|lm|} \partial_m R^{bcd]} - \frac{3}{2} Q^{[ab}_e R^{|e|cd]} = 0, \tag{3.478}$$

$$e^d{}_l \beta^{ln} \partial_n H_{[abc]} - 3 e_{[a}{}^n \partial_n F^d_{bc]} - 3 H_{e[ab} Q^{ed}_{c]} + 3 F^d_{e[a} F^e_{bc]} = 0, \tag{3.479}$$

$$-2 e^{[c}{}_l \beta^{|ln|} \partial_n F^{d]}_{[ab]} - 2 e_{[a}{}^n \partial_n Q^{[cd]}_{b]} + H_{e[ab]} R^{e[cd]} + Q^{[cd]}_e F^e_{[ab]} + F^{[c}_{e[a} Q^{|e|d]}_{b]} = 0, \tag{3.480}$$

$$3 e^{[b}{}_l \beta^{|ln|} \partial_n Q^{cd]}_a - e_a{}^n \partial_n R^{[bcd]} + 3 F^{[b}_{ea} R^{|e|cd]} - 3 Q^{[bc}_e Q^{|e|d]}_a = 0. \tag{3.481}$$

They correspond to the closure conditions of the non-abelian gauge algebra of gauged supergravities. Note that the condition of vanishing R-flux is equivalent to a twisted Poisson structure, $[\Pi, \Pi]_S = \Pi^\sharp H$. On the space of polyvectors, the resulting homological function induces a twisted Poisson differential, $d_{\Pi,H} = d_\Pi + (\wedge^2 \Pi \otimes 1)(H)$, [85, 86]. In the next step, we will compute the associated Courant algebroid realizing the entire flux freedom at once in a consistent manner. Since we are now working on the frame bundle associated with the generalized tangent bundle, we take sections of the generalized frame bundle, $X + \alpha = X^a \partial_a + \alpha_a dx^a$.

Since by construction all the twists leave the Courant algebroid inner product invariant, we are still left with the standard fiber metric on the generalized tangent bundle,

$$\begin{aligned} \langle X + \alpha, Y + \gamma \rangle &= j^* \{ j_*(X + \alpha), j_*(Y + \gamma) \} \\ &= \iota_X \gamma + \iota_Y \alpha, \end{aligned} \tag{3.482}$$

where $X + \alpha, Y + \gamma \in TM \oplus T^*M$. The anchor map is given by

$$\begin{aligned} \rho(X + \alpha) f &\equiv j^* \{ \{ \Theta_{B\beta e}, j_*(X + \alpha) \}, j_*(f) \} \\ &= (X^a e_a{}^m \partial_m - \alpha_a \beta^{am} \partial_m) f \\ &= (X - \beta^\sharp(\alpha)) f, \end{aligned} \tag{3.483}$$

where $X+\alpha \in TM \oplus T^*M$ and $f \in \mathcal{C}^\infty(M)$. Finally, we derive the Dorfman bracket step by step. The Dorfman bracket of two vectors is given by

$$\begin{aligned}[X,Y]_{\mathrm{D}} &\equiv j^*\{\{\Theta_{B\beta e}, j_*(X)\}, j_*(Y)\} \\ &= [X,Y]^{\nabla}_{\mathrm{Lie}} - \beta^\sharp(\iota_X\iota_Y H) + \iota_X\iota_Y H \\ &= [X,Y]^{\nabla}_{H} + \iota_X\iota_Y H. \end{aligned} \tag{3.484}$$

Here, we find that the introduction of the f-flux leads to a covariant Lie bracket $[X,Y]^{\nabla}$ with covariant derivative $\nabla_a X^b = \partial_a X^b + \Gamma^b_{ae}X^e$. The Weitzenböck connection Γ^b_{ae} is related to the geometric f-flux via $f^b_{ae} = 2\Gamma^b_{[ae]}$. The bracket

$$[X,Y]^{\nabla}_{H} \equiv [X,Y]^{\nabla}_{\mathrm{Lie}} - \beta^\sharp(\iota_X\iota_Y H) \tag{3.485}$$

denotes the covariant H-twisted Lie bracket. In the next step, we compute the Dorfman bracket on two 1-forms, giving

$$\begin{aligned}[\alpha,\gamma]_{\mathrm{D}} &\equiv j^*\{\{\Theta_{B\beta e}, j_*(\alpha)\}, j_*(\gamma)\} \\ &= -L^{\nabla}_{\beta^\sharp(\alpha)}\gamma + \iota_{\beta^\sharp(\gamma)}\nabla\alpha + \iota_{\beta^\sharp(\alpha)}\iota_{\beta^\sharp(\gamma)}H - \iota_\alpha\iota_\gamma R \\ &= -[\alpha,\gamma]^{\nabla}_{\beta,H} - \iota_\alpha\iota_\gamma R. \end{aligned} \tag{3.486}$$

Above equations involve a covariant H-twisted Koszul bracket, which we define by

$$[\alpha,\gamma]^{\nabla}_{\beta,H} \equiv [\alpha,\gamma]^{\nabla}_{\beta} - \iota_{\beta^\sharp(\alpha)}\iota_{\beta^\sharp(\gamma)}H. \tag{3.487}$$

Furthermore, the covariant Koszul bracket is defined by

$$[\alpha,\gamma]^{\nabla}_{\beta} \equiv L^{\nabla}_{\beta^\sharp(\alpha)}\gamma - \iota_{\beta^\sharp(\gamma)}\nabla\alpha, \tag{3.488}$$

where L^{∇}_X denotes the covariant Lie derivative along the vector X acting on forms, given by

$$L^{\nabla}_X = \nabla\iota_X + \iota_X\nabla, \tag{3.489}$$

where ∇ acts on a 1-form γ by $\nabla\gamma = \partial_a\gamma_b dx^a\wedge dx^b - \Gamma^d_{ab}\gamma_d dx^a\wedge dx^b$. Finally, the Dorfman brackets of vector and 1-form are given by

$$\begin{aligned}[\alpha,Y]_{\mathrm{D}} &\equiv j^*\{\{\Theta_{B\beta e}, j_*(\alpha)\}, j_*(Y)\} \\ &= -\iota_Y\nabla\alpha + \iota_{\beta^\sharp(\alpha)}\iota_Y H - L^{\nabla,\beta}_{\alpha}Y - \beta^\sharp(\iota_{\beta^\sharp(\alpha)}\iota_Y H), \end{aligned} \tag{3.490}$$

$$\begin{aligned}[X,\gamma]_{\mathrm{D}} &\equiv j^*\{\{\Theta_{B\beta e}, j_*(X)\}, j_*(\gamma)\} \\ &= L^{\nabla}_X\gamma + \iota_X\iota_{\beta^\sharp(\gamma)}H + \iota_\gamma\nabla_\beta X - \beta^\sharp(\iota_X\iota_{\beta^\sharp(\gamma)}H). \end{aligned} \tag{3.491}$$

Here, the symbol $L_\alpha^{\nabla,\beta}$ denotes the covariant Poisson-Lie derivative defined by

$$L_\alpha^{\nabla,\beta} \equiv \nabla_\beta \iota_\alpha + \iota_\alpha \nabla_\beta, \tag{3.492}$$

where ∇_β is the covariant Lichnerowicz-Poisson differential. Having understood all constituents of the Dorfman bracket separately, we can now write down the full induced Dorfman bracket on the generalized tangent bundle supported by geometric and non-geometric flux,

$$\begin{aligned}[X+\alpha, Y+\gamma]_{\mathrm{D}} = {} & [X,Y]_H^\nabla - [\alpha,\gamma]_{\beta,H}^\nabla + \iota_\gamma \nabla_\beta X - \iota_Y \nabla\alpha + L_X^\nabla \gamma - L_\alpha^{\nabla,\beta} Y + \iota_X \iota_Y H \\ & + \iota_{\beta^\sharp(\alpha)} \iota_Y H + \iota_X \iota_{\beta^\sharp(\gamma)} H - \beta^\sharp(\iota_{\beta^\sharp(\alpha)} \iota_Y H) - \beta^\sharp(\iota_X \iota_{\beta^\sharp(\gamma)} H) - \iota_\alpha \iota_\gamma R. \end{aligned} \tag{3.493}$$

It turns out that the reduction of the twisted Courant algebroid along the twisted anchor ρ for integer fluxes leads to the most general non-abelian gauge algebra of gauged supergravities described in the preliminary sections.
Let us summarized our findings in a theorem.

Theorem 3.4.3 (Fully twisted standard Courant algebroid) *The QP-manifold $(\mathcal{M}, \Theta_{B\beta e}, \omega)$ realizes a Courant algebroid structure $(TM \oplus T^*M, \langle -,- \rangle, \rho, [-,-]_D)$ on the generalized tangent bundle with geometric H- and F- as well as non-geometric Q- and R-flux contribution, with operations*

$$\begin{aligned} \langle X+\alpha, Y+\gamma \rangle &= \iota_X \gamma + \iota_Y \alpha, \\ \rho(X+\alpha) &= X - \beta^\sharp(\alpha), \\ [X+\alpha, Y+\gamma]_D &= [X,Y]_H^\nabla - [\alpha,\gamma]_{\beta,H}^\nabla + \iota_\gamma \nabla_\beta X - \iota_Y \nabla\alpha + L_X^\nabla \gamma - L_\alpha^{\nabla,\beta} Y + \iota_X \iota_Y H \\ & \quad + \iota_{\beta^\sharp(\alpha)} \iota_Y H + \iota_X \iota_{\beta^\sharp(\gamma)} H - \beta^\sharp(\iota_{\beta^\sharp(\alpha)} \iota_Y H) - \beta^\sharp(\iota_X \iota_{\beta^\sharp(\gamma)} H) - \iota_\alpha \iota_\gamma R. \end{aligned}$$

Furthermore, the classical master equation of the QP-manifold induces the full generalized flux Bianchi identities. The reduction of this structure along ρ yields the non-abelian gauge algebra of gauged supergravities. Then, the classical master equation is equivalent to the closure condition.

We derived the Courant algebroid, that encodes the local symmetries of toroidal string compactifications that exhibit NS-NS H-flux, geometric f-flux and non-geometric Q- and R-fluxes.

3.5 Double field theory and T-duality

Double field theory is a manifestly T-duality invariant field theory under actions of the toroidal T-duality group. To achieve this feature, double field theory introduces a dual torus $\widetilde{T}$ in addition to the torus T of the compactification. The dual torus is parameterized by double coordinates associated with the winding modes of closed strings. In this setting, T-duality becomes a transformation on the generalized coordinates.

In this section, we take our former analysis of the fully twisted Courant algebroid as a stepping stone to lift our graded symplectic manifold structure to a construction from which we can derive the local gauge algebra, the generalized fluxes, the generalized Bianchi identities and the T-duality presentation of double field theory. The section is based on the published papers [2, 3].

We introduce the graded symplectic manifold setup, which will need for a generalization of the notion of a QP-manifold to a pre-QP-manifold in 3.5.1. There, we derive the generalized gauge algebra of double field theory from the pre-QP-manifold. In section 3.5.2, we derive the double field theory fluxes and generalized Bianchi identities from the twisted pre-QP-manifold. In section 3.5.3, we derive the fluxes and generalized Bianchi identities on the physical winding space after projecting out the ordinary coordinate dependence. Finally, in section 3.5.4 we present a formulation of T-duality in the graded symplectic manifold setup and compute some examples associated to toroidal compactifications of closed string theory.

To accommodate double field theory, we double the dimension of our underlying manifold M to $\widehat{M} = M \times \widetilde{M}$ in order to incorporate the winding space. For this, we introduce a new notation. Elements $\widehat{V}$ with a hat will be related to the full double space, whereas elements $\widetilde{V}$ with a tilde will be related purely to the winding space. Elements V without hat or tilde will remain ordinary.

3.5.1 Pre-QP-manifold

Graded symplectic manifold constructions associated with double field theory have first been discussed in [117]. Further analysis along the lines of extended Riemannian geometry has been conducted in [100]. Our initial pre-QP-manifold construction is based on these investigations. Let $\widehat{M} = M \times \widetilde{M}$ be the double space manifold, where M is the usual space manifold and $\widetilde{M}$ denotes the winding space manifold. Let both manifolds M and $\widetilde{M}$ be D-dimensional, so

that $\dim(\widehat{M}) = 2D$. Local coordinates on M are denoted by x^i and local coordinates on $\widetilde{M}$ are denoted by $\widetilde{x}_i$. Note that the index structure is opposite. On $\widehat{M}$, we can combine both local coordinates via $x^M = (x^i, \widetilde{x}_i)$. We introduce capital $O(D, D; \mathbb{R})$-indices $M, N, K, \ldots$ which run over the whole $2D$-dimensional space. In the next step, we define the double space graded manifold that will inherit a Courant algebroid structure under certain conditions by $\widehat{\mathcal{M}} = T^*[2]T[1]\widehat{M}$. The local coordinates on $\widehat{\mathcal{M}}$ are denoted by ($x^M = (x^i, \widetilde{x}_i)$, $\xi^M = (\xi^i, \widetilde{\xi}_i)$, $\zeta_M = (\zeta_i, \widetilde{\zeta}^i)$, $p_M = (p_i, p^i)$) and are of degrees (0, 1, 1, 2).
We take the graded symplectic structure $\widehat{\omega}$ such that

$$\widehat{\omega} = -\delta x^i \wedge \delta p_i + \delta \widetilde{x}_i \wedge \delta \widetilde{p}^i + \delta \xi^i \wedge \delta \zeta_i + \delta \widetilde{\xi}_i \wedge \delta \widetilde{\zeta}^i. \tag{3.494}$$

The resulting structure is defined on the doubled generalized tangent bundle,

$$\widehat{E} = T\widehat{M} \oplus T^*\widehat{M} = T(M \times \widetilde{M}) \oplus T^*(M \times \widetilde{M}). \tag{3.495}$$

Therefore, a general section is given by

$$X + \alpha = X^i(x, \widetilde{x})\partial_i + X_i(x, \widetilde{x})\widetilde{\partial}^i + \alpha_i(x, \widetilde{x})dx^i + \alpha^i(x, \widetilde{x})d\widetilde{x}_i. \tag{3.496}$$

Recall that $\widetilde{\partial}^i = \frac{\partial}{\partial \widetilde{x}_i}$ denotes the basis of the tangent bundle over the winding space manifold, $T\widetilde{M} \to \widetilde{M}$. We introduce an injection map $\widehat{j}$ relating the doubled generalized tangent bundle with the graded manifold $\widehat{\mathcal{M}}$, given by

$$\begin{aligned} &\widehat{j} : T\widehat{M} \oplus (T\widehat{M} \oplus T^*\widehat{M}) \to \widehat{\mathcal{M}}, \\ \widehat{j} : &\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial \widetilde{x}_i}, x^i, \widetilde{x}_i, dx^i, d\widetilde{x}_i, \partial_i, \widetilde{\partial}^i\right) \mapsto (p_i, \widetilde{p}^i, x^i, \widetilde{x}_i, \xi^i, -\widetilde{\xi}_i, \zeta_i, -\widetilde{\zeta}^i). \end{aligned} \tag{3.497}$$

Please note the signs, that appear for technical reasons. Finally, we define the untwisted double field theory Hamiltonian function via

$$\begin{aligned} \widehat{\Theta}_0 &= p_M(\xi^M + \eta^{MN}\zeta_M) \\ &= p_i(\xi^i + \widetilde{\zeta}^i) + \widetilde{p}^i(\zeta_i + \widetilde{\xi}_i). \end{aligned} \tag{3.498}$$

Let us introduce a polarized section, which plays the role of a double field theory generalized vector, as

$$X = X^N\partial_N = X^i(x, \widetilde{x})\partial_i + X_i(x, \widetilde{x})\widetilde{\partial}^i. \tag{3.499}$$

Its pushforward along $\widehat{j}$ is given by

$$\widehat{j}_*(X) = X^i(x, \widetilde{x})\zeta_i - X_i(x, \widetilde{x})\widetilde{\zeta}^i, \tag{3.500}$$

where the minus sign appears due to our convention. It turns out that the derived bracket construction on this pre-QP-manifold yields the double field theory D-bracket [117],

$$[X,Y]_{\mathrm{D}} = \widehat{j}^*\{\{\widehat{\Theta}_0, \widehat{j}_*(X)\}, \widehat{j}_*(Y)\}. \tag{3.501}$$

The C-bracket is then simply the antisymmetrization,

$$[X,Y]_{\mathrm{C}} = \frac{1}{2}\big(\widehat{j}^*\{\{\widehat{\Theta}_0, \widehat{j}_*(X)\}, \widehat{j}_*(Y)\} - \widehat{j}^*\{\widehat{\Theta}_0, \widehat{j}_*(Y)\}, \widehat{j}_*(X)\}\big). \tag{3.502}$$

An easy calculation shows that the classical master equation, $\{\widehat{\Theta}_0, \widehat{\Theta}_0\} = 0$, is not solved trivially [117, 100],

$$\{\widehat{\Theta}_0, \widehat{\Theta}_0\} \sim p_i \widetilde{p}^i = 0. \tag{3.503}$$

Therefore, we can induce the double field theory strong constraint by

$$\{\{\widehat{\Theta}_0, \widehat{\Theta}_0\}, f\}, g\} \sim \partial_i f \widetilde{\partial}^i g + \widetilde{\partial}^i f \partial_i g = 0, \tag{3.504}$$

where $f, g \in \mathcal{C}^\infty(\widehat{M})$. We recognize, that the algebroid structure induced by this Hamiltonian function does not directly constitute a Courant algebroid on the whole space, but merely upon solving the strong constraint. Solving the strong constraint can be done by reducing the graded manifold to a half-rank graded submanifold by a projection map. A simple example would be the projection to the supergravity frame via

$$\begin{aligned} \pi &: \widehat{\mathcal{M}} \to \mathcal{M}, \\ \pi &: (p_i, \widetilde{p}^i, x^i, \widetilde{x}_i, \xi^i, \widetilde{\xi}_i, \zeta_i, \widetilde{\zeta}^i) \mapsto (p_i, 0, x^i, 0, \xi^i, 0, \zeta_i, 0), \end{aligned} \tag{3.505}$$

reducing the graded symplectic structure as well as the Hamiltonian to familiar objects,

$$\widehat{\Theta}_0|_{\pi(\widehat{\mathcal{M}})} = \Theta_0, \tag{3.506}$$

$$\widehat{\omega}|_{\pi(\widehat{\mathcal{M}})} = \omega, \tag{3.507}$$

on $\pi(\mathcal{C}^\infty(\widehat{\mathcal{M}})) = \mathcal{C}^\infty(\mathcal{M}) \subset \mathcal{C}^\infty(\widehat{\mathcal{M}})$. Clearly, the projected Hamiltonian solves the classical master equation, $\{\widehat{\Theta}_0|_{\pi(\mathcal{M})}, \widehat{\Theta}_0|_{\pi(\mathcal{M})}\} = 0$.

Another extreme choice for a projection would be to the graded winding submanifold via

$$\begin{aligned} \widetilde{\pi} &: \widehat{\mathcal{M}} \to \widetilde{\mathcal{M}} = T^*[2]T[1]\widetilde{M}, \\ \widetilde{\pi} &: (p_i, \widetilde{p}^i, x^i, \widetilde{x}_i, \xi^i, \widetilde{\xi}_i, \zeta_i, \widetilde{\zeta}^i) \mapsto (0, \widetilde{p}^i, 0, \widetilde{x}_i, 0, \widetilde{\xi}_i, 0, \widetilde{\zeta}^i). \end{aligned} \tag{3.508}$$

The resulting projected objects are given by

$$\widehat{\Theta}_0|_{\widetilde{\pi}(\widehat{\mathcal{M}})} = \widetilde{\Theta}_0 = \widetilde{\xi}_i \widetilde{p}^i, \tag{3.509}$$

$$\widehat{\omega}|_{\widetilde{\pi}(\widehat{\mathcal{M}})} = \widetilde{\omega} = \delta\widetilde{x}_i \wedge \delta\widetilde{p}^i + \delta\widetilde{\xi}_i \wedge \widetilde{\zeta}^i. \tag{3.510}$$

This structure induces an untwisted Courant algebroid on the winding space. The induced differential,

$$\widetilde{d} = \widetilde{j}^* \circ \widetilde{Q}_0 \circ \widetilde{j}_*, \tag{3.511}$$

acting on k-forms $\alpha = \frac{1}{k!}\alpha^{i_1\cdots i_k}(\widetilde{x})d\widetilde{x}_{i_1} \wedge \cdots \wedge d\widetilde{x}_{i_k} \in \Omega^k(\widetilde{M})$ on the winding space, is the de Rham differential on the winding space, where $\widetilde{j} = \widetilde{\pi} \circ \widehat{j}$. More precisely, we have

$$\begin{aligned} \widetilde{j} &: T\widetilde{M} \oplus (T\widetilde{M} \oplus T^*\widetilde{M}) \to \widetilde{\mathcal{M}}, \\ \widetilde{j} &: \left(\frac{\partial}{\partial\widetilde{x}_i}, \widetilde{x}_i, d\widetilde{x}_i, \widetilde{\partial}^i\right) \mapsto (\widetilde{p}^i, \widetilde{x}_i, -\widetilde{\xi}_i, -\widetilde{\zeta}^i). \end{aligned} \tag{3.512}$$

Furthermore, the vector field $\widetilde{Q}_0$ is defined by $\widetilde{Q}_0 = \{\widetilde{\Theta}_0, -\}$. More involved half-rank projections that solve the strong constraint are possible. We will study the projection to the Poisson-Courant algebroid in section 3.6. We summarize the extract of this section in the following theorem.

Theorem 3.5.1 (Double field theory pre-QP-manifold) *The 3-tuple $(\widehat{\mathcal{M}}, \widehat{\Theta}, \widehat{\omega})$ is a pre-QP-manifold, which induces the gauge structure of double field theory. A strong constraint solving projection to a half-rank submanifold induces a Courant algebroid structure, which lives on a T-duality frame.*

3.5.2 Generalized fluxes

In this section, we show how to introduce geometric as well as non-geometric fluxes into the graded manifold formalism of double field theory by twist of the Hamiltonian function $\widehat{\Theta}_0$. The result leads to H-, F-, Q- and R-fluxes treated from an $O(D,D;\mathbb{R})$-covariant perspective and therefore on the same footing.

First, we start with a classification of the possible twists. Since we doubled all coordinates of the graded symplectic manifold, more transformations are possible. The twists can be classified in transformations, that entirely work in the supergravity frame,

$$\begin{aligned}\exp(\delta_B) &= \exp\left(\frac{1}{2}B_{ij}(x,\widetilde{x})\xi^i\xi^j\right), \quad \exp(\delta_\beta) = \exp\left(\frac{1}{2}\beta^{ij}(x,\widetilde{x})\zeta_i\zeta_j\right),\\ \exp(\delta_f) &= \exp\left(\frac{1}{2}f_j{}^i(x,\widetilde{x})\xi^j\zeta_i\right), \quad \exp(\delta_a) = \exp\left(\frac{1}{2}a^i(x,\widetilde{x})p_i\right),\end{aligned} \tag{3.513}$$

transformations, that entirely work in the winding frame,

$$\begin{aligned}\exp(\widetilde{\delta}_B) &= \exp\left(\frac{1}{2}B_{ij}(x,\widetilde{x})\widetilde{\zeta}^i\widetilde{\zeta}^j\right), \quad \exp(\widetilde{\delta}_\beta) = \exp\left(\frac{1}{2}\beta^{ij}(x,\widetilde{x})\widetilde{\xi}_i\widetilde{\xi}_j\right),\\ \exp(\widetilde{\delta}_f) &= \exp\left(\frac{1}{2}f^j{}_i(x,\widetilde{x})\widetilde{\xi}_j\widetilde{\zeta}^i\right), \quad \exp(\widetilde{\delta}_a) = \exp\left(\frac{1}{2}a_i(x,\widetilde{x})\widetilde{p}^i\right),\end{aligned} \tag{3.514}$$

and finally transformations, that mix both frames,

$$\begin{aligned}\exp(\widehat{\delta}_B) &= \exp\left(\frac{1}{2}B_{ij}(x,\widetilde{x})\widetilde{\zeta}^i\xi^j\right), \quad \exp(\widehat{\delta}_\beta) = \exp\left(\frac{1}{2}\beta^{ij}(x,\widetilde{x})\zeta_i\widetilde{\xi}_j\right),\\ \exp(\widehat{\delta}_f) &= \exp\left(\frac{1}{2}f^j{}_i(x,\widetilde{x})\zeta_j\widetilde{\zeta}^i\right), \quad \exp(\widehat{\delta}_g) = \exp\left(\frac{1}{2}g^j{}_i(x,\widetilde{x})\widetilde{\xi}_j\xi^i\right).\end{aligned} \tag{3.515}$$

For now, it will be sufficient to perform twists in the supergravity frame and investigate the new contributions due to the $\widetilde{x}_i$-dependence. After that, we develop a notion of T-duality, with which to isomorphically map the supergravity frame result into any other.

In order to be able to introduce also geometric f-flux contributions depeding on the double space, we enlarge the doubled generalized tangent bundle by a frame bundle. Let $(\xi^i,\widetilde{\xi}_i,\zeta_i,\widetilde{\zeta}^i) \in T[1]\widehat{M} \oplus T^*[1]\widehat{M} \subset \widehat{\mathcal{M}}$ be associated with a general frame. We introduce flat vector spaces $\widehat{V} = V \oplus \widetilde{V}$, where $V = \mathbb{R}^D$ and $\widetilde{V} = \mathbb{R}^D$, such that an associated flat frame is given by $(\xi^a,\widetilde{\xi}_a,\zeta_a,\widetilde{\zeta}^a) \in \widehat{V}[1] \oplus \widehat{V}^*[1]$. Then, we can define the following injection map of the generalized frame bundle into the graded manifold supplied with the flat vector spaces,

$$\begin{gathered}\widehat{j} : T\widehat{M} \oplus (T\widehat{M} \oplus T^*\widehat{M}) \oplus \widehat{V} \oplus \widehat{V}^* \to T^*[2]T[1]\widehat{M} \oplus \widehat{V}[1] \oplus \widehat{V}^*[1],\\ \left(\frac{\partial}{\partial x^i},\frac{\partial}{\partial \widetilde{x}_i},x^i,\widetilde{x}_i,dx^i,d\widetilde{x}_i,\partial_i,\widetilde{\partial}^i,u^a,u_a,\widetilde{u}_a,\widetilde{u}^a\right) \mapsto (p_i,\widetilde{p}^i,x^i,\widetilde{x}_i,\xi^i,\widetilde{\xi}_i,\zeta_i,\widetilde{\zeta}^i,\xi^a,\zeta_a,\widetilde{\xi}_a,\widetilde{\zeta}^a).\end{gathered} \tag{3.516}$$

Now we have access to twists introducing vielbein fields,

$$\begin{aligned}\exp(\delta_e) &= \exp\left(e_a{}^i(x,\widetilde{x})\xi^a\zeta_i\right), \quad \exp(\delta_{e^{-1}}) = \exp\left(e^a{}_i(x,\widetilde{x})\xi^i\zeta_a\right),\\ \exp(\widetilde{\delta}_e) &= \exp\left(e_a{}^i(x,\widetilde{x})\widetilde{\zeta}^a\widetilde{\xi}_i\right), \quad \exp(\widetilde{\delta}_{e^{-1}}) = \exp\left(e^a{}_i(x,\widetilde{x})\widetilde{\zeta}^i\widetilde{\xi}_a\right),\\ \mathfrak{D} &\equiv \exp(-\delta_e)\exp(\delta_{e^{-1}})\exp(-\delta_e), \quad \widetilde{\mathfrak{D}} \equiv \exp(-\widetilde{\delta}_e)\exp(\widetilde{\delta}_{e^{-1}})\exp(-\widetilde{\delta}_e).\end{aligned} \tag{3.517}$$

In the following, we perform the B-, β and $\mathfrak{D}$-twists in the same order as we did in section 3.4. This leads to the local expressions of the H-, F-, Q- and R-fluxes in terms of the potentials B, β and vielbein in the doubled space. Direct computation of the B- and β-twist gives

$$
\begin{aligned}
&\exp(\delta_\beta)\exp(\delta_B)\widehat{\Theta}_0 \\
&\quad = (p_i + B_{mi}\widetilde{p}^m)\xi^i + (\widetilde{p}^i + p_m\beta^{mi} + \widetilde{p}^n B_{nm}\beta^{mi})\zeta_i \\
&\qquad + \frac{1}{2}\left[-B_{in}\widetilde{\partial}^i B_{rs} + \partial_n B_{rs}\right]\xi^n\xi^r\xi^s \\
&\qquad - \left[\frac{1}{2}\widetilde{\partial}^i B_{mn} + (B_{lm}\widetilde{\partial}^l B_{ns} - \partial_m B_{ns} + \frac{1}{2}B_{ls}\widetilde{\partial}^l B_{mn} - \frac{1}{2}\partial_s B_{mn})\beta^{si}\right]\zeta_i\xi^m\xi^n \\
&\qquad + \left[\frac{1}{2}\partial_i\beta^{hk} - \frac{1}{2}B_{li}\widetilde{\partial}^l\beta^{hk} + \widetilde{\partial}^h B_{in}\beta^{nk}\right. \\
&\qquad \left. - \frac{1}{2}\left[-B_{li}\widetilde{\partial}^l B_{rs} + \partial_i B_{rs} - B_{ls}\widetilde{\partial}^l B_{ir} + \partial_s B_{ir} + B_{lr}\widetilde{\partial}^l B_{is} - \partial_r B_{is}\right]\beta^{sh}\beta^{rk}\right]\xi^i\zeta_h\zeta_k \\
&\qquad - \left[\frac{1}{2}\widetilde{\partial}^i\beta^{hk} - \frac{1}{4}\partial_l\beta^{ih}\beta^{lk} - \frac{1}{4}\beta^{li}\partial_l\beta^{hk} + \frac{1}{4}B_{ln}\widetilde{\partial}^l\beta^{ih}\beta^{nk}\right. \\
&\qquad + \frac{1}{4}B_{ln}\beta^{ni}\widetilde{\partial}^l\beta^{hk} - \frac{1}{2}\widetilde{\partial}^i B_{mn}\beta^{nh}\beta^{mk} \\
&\qquad \left. + \frac{1}{3!}(-B_{ln}\widetilde{\partial}^l B_{rs} + \partial_n B_{rs} - B_{ls}\widetilde{\partial}^l B_{nr} + \partial_s B_{nr} + B_{lr}\widetilde{\partial}^l B_{ns} - \partial_r B_{ns})\beta^{si}\beta^{rh}\beta^{nk}\right]\zeta_i\zeta_h\zeta_k \\
&\qquad + p_i\widetilde{\zeta}^i + \widetilde{p}^i\widetilde{\xi}_i + \frac{1}{2}(\partial_i B_{jk}\widetilde{\zeta}^i - \widetilde{\partial}^i B_{jk}\widetilde{\xi}_i)\xi^j\xi^k + \frac{1}{2}(\partial_i\beta^{jk}\widetilde{\zeta}^i - \widetilde{\partial}^i\beta^{jk}\widetilde{\xi}_i)\zeta_j\zeta_k \\
&\qquad + \partial_i B_{jk}\beta^{km}\widetilde{\zeta}^i\xi^j\zeta_m - \widetilde{\partial}^i B_{jk}\beta^{km}\widetilde{\xi}_i\xi^j\zeta_m + \frac{1}{2}\partial_i B_{jk}\beta^{jm}\beta^{kn}\widetilde{\zeta}^i\zeta_m\zeta_n - \frac{1}{2}\widetilde{\partial}^i B_{jk}\beta^{jm}\beta^{kn}\widetilde{\xi}_i\zeta_m\zeta_n.
\end{aligned}
\tag{3.518}
$$

Further twist by $\mathfrak{D}$ leads to

$$
\begin{aligned}
&\mathfrak{D}\exp(\delta_\beta)\exp(\delta_B)\widehat{\Theta}_0\\
&\quad= e_d{}^i p_i \xi^d + e_d{}^i B_{mi}\widetilde{p}^m\xi^d + e^c{}_l\widetilde{p}^l\zeta_c + \beta^{ml}e^c{}_l p_m\zeta_c + e^c{}_l B_{nm}\beta^{ml}\widetilde{p}^n\zeta_c\\
&\qquad - e_d{}^i(\partial_i + B_{im}\widetilde{\partial}^m)e_a{}^j e^a{}_k\zeta_j\xi^k\xi^d + e^c{}_l(\widetilde{\partial}^l + \beta^{lm}\partial_m + \beta^{lm}B_{mn}\widetilde{\partial}^n)e_a{}^j e^a{}_k\zeta_j\xi^k\zeta_c\\
&\qquad + \frac{1}{2}\left[-B_{in}\widetilde{\partial}^i B_{rs} + \partial_n B_{rs}\right]e_a{}^n e_b{}^r e_c{}^s\xi^a\xi^b\xi^c\\
&\qquad - \Bigg[e_b{}^i(\partial_i + B_{im}\widetilde{\partial}^m)e_c{}^j e^a{}_j + \frac{1}{2}\widetilde{\partial}^i B_{mn} +\\
&\qquad +(B_{lm}\widetilde{\partial}^l B_{ns} - \partial_m B_{ns} + \frac{1}{2}B_{ls}\widetilde{\partial}^l B_{mn} - \frac{1}{2}\partial_s B_{mn})\beta^{si}\Bigg]e^a{}_i e_b{}^m e_c{}^n\zeta_a\xi^b\xi^c\\
&\qquad + \Bigg[e^c{}_l(\widetilde{\partial}^l + \beta^{lm}\partial_m + \beta^{lm}B_{mn}\widetilde{\partial}^n)e_a{}^j e^b{}_j + \frac{1}{2}\partial_i\beta^{hk} - \frac{1}{2}B_{li}\widetilde{\partial}^l\beta^{hk} + \widetilde{\partial}^h B_{in}\beta^{nk}\\
&\qquad - \frac{1}{2}\left[-B_{li}\widetilde{\partial}^l B_{rs} + \partial_i B_{rs} - B_{ls}\widetilde{\partial}^l B_{ir} + \partial_s B_{ir} + B_{lr}\widetilde{\partial}^l B_{is} - \partial_r B_{is}\right]\beta^{sh}\beta^{rk}\Bigg]e_a{}^i e^b{}_h e^c{}_k\xi^a\zeta_b\zeta_c\\
&\qquad - \Bigg[\frac{1}{2}\widetilde{\partial}^i\beta^{hk} - \frac{1}{4}\partial_l\beta^{ih}\beta^{lk} - \frac{1}{4}\beta^{li}\partial_l\beta^{hk} + \frac{1}{4}B_{ln}\widetilde{\partial}^l\beta^{ih}\beta^{nk} + \frac{1}{4}B_{ln}\beta^{ni}\widetilde{\partial}^l\beta^{hk} - \frac{1}{2}\widetilde{\partial}^i B_{mn}\beta^{nh}\beta^{mk}\\
&\qquad +\frac{1}{3!}(-B_{ln}\widetilde{\partial}^l B_{rs} + \partial_n B_{rs} - B_{ls}\widetilde{\partial}^l B_{nr} + \partial_s B_{nr} + B_{lr}\widetilde{\partial}^l B_{ns} - \partial_r B_{ns})\beta^{si}\beta^{rh}\beta^{nk}\Bigg]e^a{}_i e^b{}_h e^c{}_k\zeta_a\zeta_b\zeta_c\\
&\qquad + (p_i - \partial_i e_a{}^j e^a{}_k\zeta_j\xi^k - \partial_i e_a{}^j e^b{}_j\xi^a\zeta_b)\widetilde{\zeta}^i + (\widetilde{p}^i + \widetilde{\partial}^i e_a{}^j e^a{}_k\zeta_j\xi^k + \widetilde{\partial}^i e_a{}^j e^b{}_j\xi^a\zeta_b)\widetilde{\xi}_i\\
&\qquad + \frac{1}{2}(\partial_i B_{jk}\widetilde{\zeta}^i - \widetilde{\partial}^i B_{jk}\widetilde{\xi}_i)e_a{}^j e_b{}^k\xi^a\xi^b + \frac{1}{2}(\partial_i\beta^{jk}\widetilde{\zeta}^i - \widetilde{\partial}^i\beta^{jk}\widetilde{\xi}_i)e^b{}_j e^c{}_k\zeta_b\zeta_c\\
&\qquad + \partial_i B_{jk}\beta^{km}e_b{}^j e^c{}_m\widetilde{\zeta}^i\xi^b\zeta_c - \widetilde{\partial}^i B_{jk}\beta^{km}e_b{}^j e^c{}_m\widetilde{\xi}_i\xi^b\zeta_c + \frac{1}{2}\partial_i B_{jk}\beta^{jm}\beta^{kn}e^b{}_m e^c{}_n\widetilde{\zeta}^i\zeta_b\zeta_c\\
&\qquad - \frac{1}{2}\widetilde{\partial}^i B_{jk}\beta^{jm}\beta^{kn}e^b{}_m e^c{}_n\widetilde{\xi}_i\zeta_b\zeta_c,
\end{aligned}
\tag{3.519}
$$

which can be rewritten by

$$
\begin{aligned}
\widehat{\Theta}_{B\beta e} &= e_d{}^i p_i \xi^d + e_d{}^i B_{mi}\widetilde{p}^m\xi^d + e^c{}_l\widetilde{p}^l\zeta_c + \beta^{ml}e^c{}_l p_m\zeta_c + e^c{}_l B_{nm}\beta^{ml}\widetilde{p}^n\zeta_c\\
&\quad - e_d{}^i(\partial_i + B_{im}\widetilde{\partial}^m)e_a{}^j e^a{}_k\zeta_j\xi^k\xi^d + e^c{}_l(\widetilde{\partial}^l + \beta^{lm}\partial_m + \beta^{lm}B_{mn}\widetilde{\partial}^n)e_a{}^j e^a{}_k\zeta_j\xi^k\zeta_c\\
&\quad + (p_i - \partial_i e_a{}^j e^a{}_k\zeta_j\xi^k - \partial_i e_a{}^j e^b{}_j\xi^a\zeta_b)\widetilde{\zeta}^i + (\widetilde{p}^i + \widetilde{\partial}^i e_a{}^j e^a{}_k\zeta_j\xi^k + \widetilde{\partial}^i e_a{}^j e^b{}_j\xi^a\zeta_b)\widetilde{\xi}_i\\
&\quad + \frac{1}{2}(\partial_i B_{jk}\widetilde{\zeta}^i - \widetilde{\partial}^i B_{jk}\widetilde{\xi}_i)e_a{}^j e_b{}^k\xi^a\xi^b + \frac{1}{2}(\partial_i\beta^{jk}\widetilde{\zeta}^i - \widetilde{\partial}^i\beta^{jk}\widetilde{\xi}_i)e^b{}_j e^c{}_k\zeta_b\zeta_c\\
&\quad + \partial_i B_{jk}\beta^{km}e_b{}^j e^c{}_m\widetilde{\zeta}^i\xi^b\zeta_c - \widetilde{\partial}^i B_{jk}\beta^{km}e_b{}^j e^c{}_m\widetilde{\xi}_i\xi^b\zeta_c + \frac{1}{2}\partial_i B_{jk}\beta^{jm}\beta^{kn}e^b{}_m e^c{}_n\widetilde{\zeta}^i\zeta_b\zeta_c\\
&\quad - \frac{1}{2}\widetilde{\partial}^i B_{jk}\beta^{jm}\beta^{kn}e^b{}_m e^c{}_n\widetilde{\xi}_i\zeta_b\zeta_c\\
&\quad + \frac{1}{3!}H_{abc}\xi^a\xi^b\xi^c - \frac{1}{2}F^a_{bc}\zeta_a\xi^b\xi^c + \frac{1}{2}Q^{bc}_a\xi^a\zeta_b\zeta_c - \frac{1}{3!}R^{abc}\zeta_a\zeta_b\zeta_c,
\end{aligned}
\tag{3.520}
$$

by defining

$$H_{abc} = 3(\nabla_{[a}B_{bc]} + B_{[a|m|}\widetilde{\partial}^m B_{bc]} + \tilde{f}^{mn}_{[a}B_{b|m|}B_{c]n}), \tag{3.521}$$

$$F^a_{bc} = f^a_{bc} - H_{mns}\beta^{si}e^a{}_i e_b{}^m e_c{}^n + \widetilde{\partial}^a B_{bc} + \tilde{f}^{ad}_b B_{dc} - \tilde{f}^{ad}_c B_{db}, \tag{3.522}$$

$$\begin{aligned} Q^{bc}_a = {} & \tilde{f}^{bc}_a + \partial_a\beta^{bc} + f^b_{ad}\beta^{dc} - f^c_{ad}\beta^{db} + H_{isr}\beta^{sh}\beta^{rk}e_a{}^i e^b{}_h e^c{}_k \\ & + B_{am}\widetilde{\partial}^m\beta^{bc} + \widetilde{\partial}^{[b}B_{ae}\beta^{e|c]} + 2B_{[a|e}\tilde{f}^{be}_{d]}\beta^{dc} - 2B_{[a|e}\tilde{f}^{ce}_{d]}\beta^{db}, \end{aligned} \tag{3.523}$$

$$\begin{aligned} R^{abc} = {} & 3(\beta^{[a|m|}\partial_m\beta^{bc]} + f^{[a}_{mn}\beta^{b|m|}\beta^{c]n} + \widetilde{\partial}^{[a}\beta^{bc]} - \tilde{f}^{[ab}_d\beta^{|d|c]} \\ & + B_{ln}\widetilde{\partial}^l\beta^{[ab}\beta^{|n|c]} + \widetilde{\partial}^{[a}B_{ed}\beta^{|e|b}\beta^{|d|c]} + \tilde{f}^{[a|e|}_n B_{ed}\beta^{|n|b|}\beta^{|d|c]}) \\ & - H_{mns}\beta^{mi}\beta^{nh}\beta^{sk}e^a{}_i e^b{}_h e^c{}_k, \end{aligned} \tag{3.524}$$

$$H_{mns} = 3(\partial_{[m}B_{ns]} + B_{[m|l|}\widetilde{\partial}^l B_{ns]}), \tag{3.525}$$

$$\tilde{f}^{ab}_c = 2e^{[a}{}_m\widetilde{\partial}^m e^{b]}{}_j e_c{}^j. \tag{3.526}$$

We conclude, that the above procedure induces all the local geometric and non-geometric fluxes of double field theory. The classical master equation induces the following identities between the fluxes,

$$e_{[a}{}^i B_{in}\widetilde{\partial}^n H_{bcd]} + e_{[a}{}^m\partial_{|m|}H_{bcd]} - \frac{3}{2}F^e_{[ab}H_{|e|cd]} = 0, \tag{3.527}$$

$$(e^{[a}{}_n + e^{[a}{}_l\beta^{lm}B_{mn})\widetilde{\partial}^n R^{bcd]} + e^{[a}{}_l\beta^{|lm|}\partial_m R^{bcd]} - \frac{3}{2}Q^{[ab}_e R^{|e|cd]} = 0, \tag{3.528}$$

$$\begin{aligned} & (e^d{}_n + e^d{}_l\beta^{lm}B_{mn})\widetilde{\partial}^n H_{[abc]} - 3e_a{}^i B_{in}\widetilde{\partial}^n F^d_{bc} + e^d{}_l\beta^{ln}\partial_n H_{[abc]} \\ & \qquad -3e_{[a}{}^n\partial_n F^d_{bc]} - 3H_{e[ab}Q^{ed}_{c]} + 3F^d_{e[a}F^e_{bc]} = 0, \end{aligned} \tag{3.529}$$

$$\begin{aligned} & -2(e^{[c}{}_n + e^{[c}{}_l\beta^{lm}B_{mn})\widetilde{\partial}^n F^{d]}_{[ab]} - 2e_{[a}{}^i B_{in}\widetilde{\partial}^n Q^{[cd]}_{b]} - 2e^{[c}{}_l\beta^{|ln|}\partial_n F_{[ab]} \\ & \qquad -2e_{[a}{}^n\partial_n Q^{[cd]}_{b]} + H_{e[ab]}R^{e[cd]} + Q^{[cd]}_e F^e_{[ab]} + F^{[c}_{e[a}Q^{|e|d]}_{b]} = 0, \end{aligned} \tag{3.530}$$

$$\begin{aligned} & 3(e^{[b}{}_n + e^{[b}{}_l\beta^{lm}B_{mn})\widetilde{\partial}^n Q^{cd]}_a - e_a{}^i B_{in}\widetilde{\partial}^n R^{[bcd]} \\ & \qquad +3e^{[b}{}_l\beta^{ln}\partial_n Q^{cd]} - e_a{}^n\partial_n R^{[bcd]} + 3F^{[b}_{ea}R^{|e|cd]} - 3Q^{[bc}_e Q^{|e|d]}_a = 0, \end{aligned} \tag{3.531}$$

which are obeyed on a half-rank submanifold on which the strong constraint is solved. In general, these equations are not solved, since they emerge from equations that were not satisfied from the beginning, the classical master equation $\{\widehat{\Theta}_0, \widehat{\Theta}_0\} \neq 0$. Let us summarize our findings in the following theorem.

Theorem 3.5.2 *The twist of the double field theory Hamiltonian by B-field, β-bivector and diffeomorphism leads to the local expressions of all geometric and non-geometric fluxes en-*

coded in the pre-QP-manifold structure. A half-rank projection that solves the strong constraint leads to an associated twisted Courant algebroid on the respective T-duality frame that encodes the allowed fluxes and their Bianchi identities.

3.5.3 Non-geometric Courant algebroids

In this section, we construct two examples of twisted Courant algebroids on non-geometric T-duality frames that arise as half-rank projections of the twisted double field theory Hamiltonian.

We start with the winding space Courant algebroid which emerges from a half-rank projection of the Hamiltonian function $\widehat{\Theta}_0$ after several twists in the sector spanned by $\widetilde{\xi}_i, \widetilde{\zeta}^i, \widetilde{\xi}_a, \widetilde{\zeta}^a \in T[1]\widetilde{M} \oplus T^*[1]\widetilde{M} \oplus \widetilde{V}[1] \oplus \widetilde{V}^*[1]$. The twisted Hamiltonian has the structure

$$\widetilde{\mathfrak{D}} \exp(\widetilde{\delta}_\beta) \exp(\widetilde{\delta}_B) \widehat{\Theta}_0|_{\widetilde{\pi}(\widehat{\mathcal{M}})} = \cdots + \frac{1}{3!} H_{abc} \widetilde{\zeta}^a \widetilde{\zeta}^b \widetilde{\zeta}^c - \frac{1}{2} F^a_{bc} \widetilde{\xi}_a \widetilde{\zeta}^b \widetilde{\zeta}^c + \frac{1}{2} Q^{bc}_a \widetilde{\zeta}^a \widetilde{\xi}_b \widetilde{\xi}_c - \frac{1}{3!} R^{abc} \widetilde{\xi}_a \widetilde{\xi}_b \widetilde{\xi}_c, \tag{3.532}$$

which encodes the local expressions for the fluxes,

$$H_{abc} = 3(B_{[a|m} \widetilde{\partial}^m B_{bc]} + \tilde{f}^{mn}_{[a} B_{b|m|} B_{c]n}), \tag{3.533}$$

$$F^a_{bc} = H_{mns} \beta^{si} e^a{}_i e_b{}^m e_c{}^n + \widetilde{\partial}^a B_{bc} + \tilde{f}^{ad}_b B_{dc} - \tilde{f}^{ad}_c B_{db}, \tag{3.534}$$

$$\begin{aligned} Q^{bc}_a &= \tilde{f}^{bc}_a + H_{isr} \beta^{sh} \beta^{rk} e_a{}^i e^b{}_h e^c{}_k + B_{am} \widetilde{\partial}^m \beta^{bc} + \widetilde{\partial}^{[b} B_{ae} \beta^{e|c]} \\ &\quad + 2B_{[a|e} \tilde{f}^{be}_{d]} \beta^{dc} - 2B_{[a|e} \tilde{f}^{ce}_{d]} \beta^{db}, \end{aligned} \tag{3.535}$$

$$\begin{aligned} R^{abc} &= 3(\widetilde{\partial}^{[a} \beta^{bc]} - \tilde{f}^{[ab}_d \beta^{|d|c]} + B_{ln} \widetilde{\partial}^l \beta^{[ab} \beta^{|n|c]} + \widetilde{\partial}^{[a} B_{ed} \beta^{|e|b} \beta^{|d|c]} \\ &\quad + \tilde{f}^{[a|e|}_n B_{ed} \beta^{|n|b} \beta^{|d|c]}) - H_{mns} \beta^{mi} \beta^{nh} \beta^{sk} e^a{}_i e^b{}_h e^c{}_k, \end{aligned} \tag{3.536}$$

$$H_{mns} = 3B_{[m|l} \widetilde{\partial}^l B_{ns]}, \tag{3.537}$$

$$\tilde{f}^{ab}_c = 2e^{[a}{}_m \widetilde{\partial}^m e^{b]}{}_j e_c{}^j, \tag{3.538}$$

on the winding frame. The classical master equation of the twisted projected Hamiltonian induces the generalized flux Bianchi identities on the winding frame,

$$e_{[a}^{\ i} B_{in} \widetilde{\partial}^n H_{bcd]} - \frac{3}{2} F^e_{[ab} H_{|e|cd]} = 0, \tag{3.539}$$

$$(e^{[a}_{\ n} + e^{[a}_{\ l} \beta^{lm} B_{mn}) \widetilde{\partial}^n R^{bcd]} - \frac{3}{2} Q_e^{[ab} R^{|e|cd]} = 0, \tag{3.540}$$

$$(e^{d}_{\ n} + e^{d}_{\ l} \beta^{lm} B_{mn}) \widetilde{\partial}^n H_{[abc]} - 3 e_{[a}^{\ i} B_{in} \widetilde{\partial}^n F^d_{bc]} - 3 H_{e[ab} Q_{c]}^{ed} + 3 F^d_{e[a} F^e_{bc]} = 0, \tag{3.541}$$

$$\begin{aligned} &-2(e^{[c}_{\ n} + e^{[c}_{\ l} \beta^{lm} B_{mn}) \widetilde{\partial}^n F^{d]}_{[ab]} \\ &\quad -2 e_{[a}^{\ i} B_{in} \widetilde{\partial}^n Q_{b]}^{[cd]} + H_{e[ab]} R^{e[cd]} + Q_e^{[cd]} F^e_{[ab]} + F^{[c}_{e[a} Q_{b]}^{|e|d]} = 0, \end{aligned} \tag{3.542}$$

$$3(e^{[b}_{\ n} + e^{[b}_{\ l} \beta^{lm} B_{mn}) \widetilde{\partial}^n Q_a^{cd]} - e_a^{\ i} B_{in} \widetilde{\partial}^n R^{[bcd]} + 3 F^{[b}_{ea} R^{|e|cd]} - 3 Q_e^{[bc} Q_a^{|e|d]} = 0. \tag{3.543}$$

By construction, this QP-manifold induces a consistent twisted Courant algebroid encoding the local symmetries of the respective T-dual frame.

The second example is given by projecting (3.520) to the supergravity frame ($\widetilde{p}^i = 0$) with $B_{ij} = 0$ leading to the fluxes

$$H_{abc} = 0, \tag{3.544}$$

$$F^a_{bc} = f^a_{bc}, \tag{3.545}$$

$$Q_a^{bc} = \partial_a \beta^{bc} + f^b_{ad} \beta^{dc} - f^c_{ad} \beta^{db}, \tag{3.546}$$

$$R^{abc} = 3(\beta^{[a|m|} \partial_m \beta^{bc]} + f^{[a}_{mn} \beta^{b|m|} \beta^{c]n}). \tag{3.547}$$

The associated twisted Courant algebroid encodes the local symmetries of a non-geometric supergravity frame with vanishing B-field, which is relevant for β-supergravity [118]. This example has also been discussed in [119].

3.5.4 T-duality

In this section, we discuss the presentation of T-duality in the setting of QP-manifolds and Courant algebroids from the perspective of the Hamiltonian function $\widehat{\Theta}_0$ and twists thereof. First, we note that any Hamiltonian function can be split into two parts $\Theta = \Theta_{\text{Diff}} + \Theta_{\text{Flux}}$. The first part Θ_{Diff} induces the generalized derivatives and is first order in p_i or $\widetilde{p}^i$, respectively. The second part Θ_{Flux} is zeroth order in p_i or $\widetilde{p}^i$ and contains the Ševera class. In our case it contains the local information of the geometric as well as non-geometric fluxes.

For the understanding of this section it will be sufficient to only consider the generalized derivative inducing part. It turns out, that the twist of the Hamiltonian function $\widehat{\Theta}_0$ in the geometric as well as the non-geometric sectors, i.e., variables $(\xi^i, \widetilde{\xi}_i, \zeta_i, \widetilde{\zeta}^i, \xi^a, \widetilde{\xi}_a, \zeta_a, \widetilde{\zeta}^a) \in \widehat{\mathcal{M}}$ leads to a manifest expression in terms of the generalized vielbein depending on all potentials B, β and ordinary vielbein. To see this, we perform the twist

$$\widetilde{\mathfrak{D}} \exp(\widetilde{\delta}_\beta) \exp(\widetilde{\delta}_B) \mathfrak{D} \exp(\delta_\beta) \exp(\delta_B) \widehat{\Theta}_0 = \widehat{\Theta}_{\text{Diff}} + \widehat{\Theta}_{\text{Flux}}, \tag{3.548}$$

which we split as described above. Further inspection of Θ_{Diff}, we find that it can be rewritten in the following form,

$$\begin{aligned}\widehat{\Theta}_{\text{Diff}} &= e_a{}^i p_i(\xi^a + \widetilde{\zeta}^a) - e_a{}^l B_{li} \widetilde{p}^i (\xi^a + \widetilde{\zeta}^a) + (e^a{}_i + e^a{}_l B_{im} \beta^{ml}) \widetilde{p}^i (\zeta_a + \widetilde{\xi}_a) - e^a{}_l \beta^{li} p_i (\zeta_a + \widetilde{\xi}_a) \\ &= E_a{}^i p_i(\xi^a + \widetilde{\zeta}^a) + E_{ai} \widetilde{p}^i (\xi^a + \widetilde{\zeta}^a) + E^a{}_i \widetilde{p}^i (\zeta_a + \widetilde{\xi}_a) + E^{ai} p_i (\zeta_a + \widetilde{\xi}_a),\end{aligned} \tag{3.549}$$

where we recognize that the generalized vielbein components emerge,

$$E_a{}^i \equiv e_a{}^i, \quad E_{ai} \equiv -e_a{}^l B_{li}, \quad E^a{}_i \equiv e^a{}_i + e^a{}_l B_{im} \beta^{ml}, \quad E^{ai} \equiv -e^a{}_l \beta^{li}. \tag{3.550}$$

We can reassemble them into the generalized vielbein

$$E^A{}_M = \begin{pmatrix} E_a{}^i & E_{ai} \\ E^{ai} & E^a{}_i \end{pmatrix} = \begin{pmatrix} e_a{}^i & -e_a{}^l B_{li} \\ -e^a{}_l \beta^{li} & e^a{}_i + e^a{}_l B_{im} \beta^{ml} \end{pmatrix}, \tag{3.551}$$

and rewrite the Hamiltonian in an $O(D, D)$-covariant form,

$$\widehat{\Theta}_{\text{Diff}} = E^A{}_I(e, \beta, B) P^I (Z_A + \widetilde{\Xi}_A), \tag{3.552}$$

where we introduced $P^I = (p_i, \widetilde{p}^i)$, $Z_A = (\zeta_a, \widetilde{\zeta}^a)$ and $\widetilde{\Xi}_A = (\xi^a, \widetilde{\xi}_a)$. A physical background is described by the functions B, β and vielbein. Therefore, a choice of a physical background fixes $\widehat{\Theta}_{\text{Diff}}$, which in turn fixes $\widehat{\Theta}_{\text{Flux}}$ by the classical master equation after half-rank projection. Therefore, it is sufficient to manipulate only the generalized differential inducing part of a twisted Hamiltonian defined on the double space in order to relate it T-dual frames. We associate a background with a choice of $\widehat{\Theta}_{\text{Diff}}$. Having understood the structure of the Hamiltonian we can give a definition of T-duality transformations in the graded manifold setting.

Definition 3.5.3 (T-duality) *Let $(\widehat{\mathcal{M}}, \widehat{\omega})$ be the graded symplectic manifold as defined above. Furthermore, let*

$$\widehat{\Theta} = \widehat{\Theta}_{Diff}(B, \beta, e) + \widehat{\Theta}_{Flux}(H, F, Q, R) \tag{3.553}$$

be any Hamiltonian function fixed by the background potentials B, β and vielbein.
A *T-duality transformation* $\mathfrak{T}_i$ *along the direction i is the discrete transformation*

$$\mathfrak{T}_i : (x^i, p_i, \xi^i, \zeta_i) \leftrightarrow (\widetilde{x}_i, -\widetilde{p}^i, -\widetilde{\xi}_i, -\widetilde{\zeta}^i). \tag{3.554}$$

The T-duality transformation along i-direction of any Hamiltonian describing a distinct background,

$$\mathfrak{T}_i\widehat{\Theta} = \widehat{\Theta}' = \widehat{\Theta}_{Diff}(B', \beta', e') + \widehat{\Theta}_{Flux}(H', F', Q', R'), \tag{3.555}$$

describes the background which emerges by T-duality transformation along i-direction.

Note that $\mathfrak{T}_i^2 = 1$ as expected. In order to get accommodated with the meaning of above definition, let us compute two examples of T-duality in this setting: T-duality on backgrounds with S^1-isometry and the well-known T-duality chain on T^3.

Example 3.5.1 (S^1-isometry) The first example concerns T-duality transformation of an S^1-compactification without any background potential. Let R be the radius of the S^1-isometry. We can write down the associated Hamiltonian function

$$\begin{aligned}\widehat{\Theta} &= e_1{}^1 p_1(\xi^1 + \widetilde{\zeta}^1) + e^1{}_1\widetilde{p}^1(\zeta_1 + \widetilde{\xi}_1)\\ &= Rp_1(\xi^1 + \widetilde{\zeta}^1) + R^{-1}\widetilde{p}^1(\zeta_1 + \widetilde{\xi}_1).\end{aligned} \tag{3.556}$$

The projections to the supergravity frame and its T-dual are given by

$$\widehat{\Theta}|_{\pi(\widehat{\mathcal{M}})} = Rp_1\xi^1, \quad \widehat{\Theta}|_{\widetilde{\pi}(\widehat{\mathcal{M}})} = R^{-1}\widetilde{p}^1\widetilde{\xi}_1. \tag{3.557}$$

We therefore first note that $\widehat{\Theta}$ contains information on all T-dual hypersurfaces at once. Depending on how we read off the information decides which T-duality frame we are treating. We further notice, that both projections are related by the T-duality transformation $\mathfrak{T}_1$,

$$\mathfrak{T}_1\left(\widehat{\Theta}|_{\pi(\widehat{\mathcal{M}})}\right) = \widehat{\Theta}|_{\widetilde{\pi}(\widehat{\mathcal{M}})}. \tag{3.558}$$

We can also see it by comparison of

$$\widehat{\Theta}|_{\pi(\widehat{\mathcal{M}})} = Rp_1\xi^1 \tag{3.559}$$

with

$$(\mathfrak{T}_1\widehat{\Theta})|_{\pi(\widehat{\mathcal{M}})} = R^{-1}p_1\xi^1 \tag{3.560}$$

We conclude, that T-duality inverts the radius $R \leftrightarrow R^{-1}$, which is consistent.

The next example concerns the well-known T-duality chain on a 3-torus with H-flux.

Example 3.5.2 (T-duality chain on T^3) Let T^3 be a 3-torus locally parameterized by coordinates x^i, where $i = 1, 2, 3$. Let furthermore be $H \in \Omega^3(T^3)$ the H-flux wrapping the 3-torus such that $H_{123} = N \in \mathbb{Z}$. Finally, let the metric on the 3-torus be flat. Therefore, the background data is given by

$$e^a{}_i = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad B_{12} = Nx^3 = -B_{21}, \quad H_{123} = \partial_3 B_{12} = N,$$

leading to the following differential inducing part,

$$\begin{aligned} \widehat{\Theta}_{\text{Diff}} &= e_i{}^i p_i(\xi^i + \widetilde{\zeta}^i) - e_1{}^1 B_{12}\widetilde{p}^2(\xi^1 + \widetilde{\zeta}^1) - e_2{}^2 B_{21}\widetilde{p}^1(\xi^2 + \widetilde{\zeta}^2) + e^i{}_i\widetilde{p}^i(\xi_i + \widetilde{\zeta}_i) \\ &= p_i(\xi^i + \widetilde{\zeta}^i) - Nx^3\widetilde{p}^2(\xi^1 + \widetilde{\zeta}^1) + Nx^3\widetilde{p}^1(\xi^2 + \widetilde{\zeta}^2) + \widetilde{p}^i(\zeta_i + \widetilde{\xi}_i). \end{aligned} \tag{3.561}$$

Note that we restrict the discussion to the differential inducing part. In this case, the flux part is non-zero and consists of the H-flux contribution.
There are two isometry directions along which we can T-duality transform, x^1 and x^2. We start with a transformation in x^1-direction, giving

$$\mathfrak{T}_1\widehat{\Theta} = p_i(\xi^i + \widetilde{\zeta}^i) - Nx^3 p_1(\xi^2 + \widetilde{\zeta}^2) + \widetilde{p}^i(\zeta_i + \widetilde{\xi}_i) + Nx^3\widetilde{p}^2(\zeta_1 + \widetilde{\xi}_1). \tag{3.562}$$

The background data of the transformed background is given by

$$B = 0, \quad e^a{}_i = \begin{pmatrix} 1 & Nx^3 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad g_{ij} = \begin{pmatrix} 1 & Nx^3 & 0 \\ Nx^3 & 1 + (Nx^3)^2 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad f^1_{23} = 2e_{[2}{}^m\partial_m e_{3]}{}^j e^1{}_j = N,$$

and describes a so-called twisted torus. In the next step we transform along x^2. We find

$$\mathfrak{T}_2\mathfrak{T}_1\widehat{\Theta} = p_i(\xi^i + \widetilde{\zeta}^i) + \widetilde{p}^i(\zeta_i + \widetilde{\xi}_i) + Nx^3 p_1(\zeta_2 + \widetilde{\xi}_2) - Nx^3 p_2(\zeta_1 + \widetilde{\xi}_1), \tag{3.563}$$

realizing the transformed background data

$$e^a{}_i = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \beta^{12} = Nx^3, \quad Q_3^{12} = \partial_3\beta^{12} = N.$$

The metric twist turned into the non-geometric potential β by the second T-duality transformation. Finally, transformation in x^3-direction leads to

$$\mathfrak{T}_3\mathfrak{T}_2\mathfrak{T}_1\widehat{\Theta} = p_i(\xi^i + \widetilde{\zeta}^i) + \widetilde{p}^i(\zeta_i + \widetilde{\xi}_i) + N\widetilde{x}_3 p_1(\zeta_2 + \widetilde{\xi}_2) + N\widetilde{x}_3 p_2(\zeta_1 + \widetilde{\xi}_1), \tag{3.564}$$

realizing the background data

$$e^a{}_i = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \beta^{12} = N\widetilde{x}_3, \quad R^{123} = \widetilde{\partial}^3 \beta^{12} = N.$$

We conclude, that the chain of transformations

$$\widehat{\Theta} \overset{\mathfrak{T}_1}{\longleftrightarrow} \widehat{\Theta}' \overset{\mathfrak{T}_2}{\longleftrightarrow} \widehat{\Theta}'' \overset{\mathfrak{T}_3}{\longleftrightarrow} \widehat{\Theta}'''$$

realizes the well-known T-duality chain [14]

$$H_{123} \overset{T_1}{\longleftrightarrow} f^1_{23} \overset{T_2}{\longleftrightarrow} Q^{12}_3 \overset{T_3}{\longleftrightarrow} R^{123}$$

in the graded symplectic manifold setting.

Arbitrary non-geometric backgrounds and their T-duals can be computed and investigated in the setup presented above. Furthermore, any half-rank projection introduced above can be interpreted as realizing a Courant algebroid on a T-dual hypersurface. Then, the T-duality transformation is nothing but an isomorphism of QP-manifolds of degree 2 realizing different (non-)geometric background data.

Any half-rank projection leads to a twisted Courant algebroid associated with a T-duality frame. We conclude, that on each T-duality frame there lives a twisted Courant algebroid that encodes all allowed geometric and non-geometric fluxes and their flux Bianchi identities. Let us summarize our findings in the following theorem.

Theorem 3.5.4 *The fully twisted double field theory Hamiltonian encodes the local expressions for all geometric as well as non-geometric fluxes H, F, Q and R in a T-duality covariant way. A half-rank projection that solves the strong constraint thus leads to a twisted Courant algebroid on an associated T-duality frame that encodes not only all fluxes that live on the respective frame in a consistent way, but also the flux Bianchi identities.*

Having understood the underlying gauge algebra of the T-duality manifest formulation of double field theory twisted by geometric as well as non-geometric fluxes, we will go on in the following section to investigate the significance of the Poisson-Courant algebroid as model for non-geometric flux.

3.6 Poisson-Courant algebroid

This section concerns the analysis of the Poisson-Courant algebroid as an object that exhibits natural trivector freedom through the introduction of a Poisson tensor Π. When the trivector freedom is associated to non-geometric R-flux, the Poisson-Courant algebroid may serve as model for T-dual non-geometric spaces. Among several investigations concerning the Poisson-Courant algebroid itself and its induced cohomology, we will analyze its relation to the standard Courant algebroid and double field theory. Along the way, we derive a topological membrane sigma model with R-flux, that exhibits a sigma model of a string traveling in Poisson-Courant algebroid background on its boundary. Furthermore, we derive the associated Poisson algebra and R-twisted current algebra on the loopspace of string embeddings into a target space with Poisson-Courant algebroid structure.

In 3.6.1, we will show how to reconstruct the Poisson-Courant algebroid from a QP-manifold. In 3.6.2, we compute the Poisson-Courant algebroid cohomology as well as the standard Courant algebroid cohomology on special subspaces. In 3.6.3, a duality transformation between the QP-manifolds of the Poisson-Courant algebroid with R-flux and the standard Courant algebroid with H-flux is derived and the symmetry of their construction is investigated. This duality transformation is named **flux duality**. Section 3.6.4 lifts the flux duality of the associated QP-manifolds to an isomorphism of associated cohomologies. The embedding of the Poisson-Courant algebroid into double field theory as a non-trivial T-duality subspace realizing R-flux freedom is computed in 3.6.5. In 3.6.6, a topological membrane sigma model on a Poisson-Courant algebroid background is constructed. From that, a topological sigma model of a string traveling in R-flux background with Poisson-Courant algebroid structure is derived. Through the introduction of a kinetic term, a string sigma model action with R-flux is constructed. Section 3.6.7 concerns the investigation of flux duality on the level of topological sigma models. The Poisson algebra of observables on the loop space with Poisson-Courant algebroid structure is computed in 3.6.8. From that, the current algebra on the loop space with Poisson-Courant algebroid structure is derived in section 3.6.9. This section is based on the published papers [1, 2, 3].

3.6.1 Supergeometry of the Poisson-Courant algebroid

In this section, we work out the supergeometric description of the Poisson-Courant algebroid.

Let M be a smooth manifold. Furthermore, let $\mathcal{M} = T^*[2]T[1]M$ be a graded manifold locally described by coordinates $(x^i, \xi^i, \zeta_i, p_i)$ of degrees $(0,1,1,2)$. We equip $\mathcal{M}$ with the graded symplectic structure

$$\omega = -\delta x^i \wedge \delta p_i + \delta\xi^i \wedge \delta\zeta_i, \tag{3.565}$$

which induces the graded Poisson structure

$$\{f,g\} = -\frac{f\overleftarrow{\partial}}{\partial x^i}\frac{\overrightarrow{\partial}g}{\partial p_i} + \frac{f\overleftarrow{\partial}}{\partial p_i}\frac{\overrightarrow{\partial}g}{\partial x^i} + \frac{f\overleftarrow{\partial}}{\partial \xi^i}\frac{\overrightarrow{\partial}g}{\partial \zeta_i} + \frac{f\overleftarrow{\partial}}{\partial \zeta_i}\frac{\overrightarrow{\partial}g}{\partial \xi^i}, \tag{3.566}$$

where $f, g \in \mathcal{C}^\infty(\mathcal{M})$. Finally, we define the Hamiltonian function by

$$\Theta_R = \Pi^{ij}\zeta_i p_j - \frac{1}{2}\frac{\partial \Pi^{ij}}{\partial x^k}\xi^k\zeta_i\zeta_j + \frac{1}{3!}R^{ijk}\zeta_i\zeta_j\zeta_k, \tag{3.567}$$

where $\Pi^{ij}, R^{ijk} \in \mathcal{C}^\infty(\mathcal{M})$. Obviously, (Π^{ij}) and (R^{ijk}) are totally antisymmetric tensors. Furthermore, the injection map $j : (TM \oplus T^*M) \oplus TM \to \mathcal{M}$ is given by

$$j : \left(x^i, \partial_i, dx^i, \frac{\partial}{\partial x^i}\right) \to (x^i, \zeta_i, \xi^i, p_i), \tag{3.568}$$

and relates elements of $\mathcal{C}^\infty_1(\mathcal{M})$ with elements of $\Gamma(TM \oplus T^*M)$,

$$X^i\zeta_i + \alpha_i\xi^i \mapsto j^*(X^i\zeta_i + \alpha_i\xi^i) = X^i\partial_i + \alpha_i dx^i = X + \alpha \in \Gamma(TM \oplus T^*M), \tag{3.569}$$

where $X^i, \alpha_i \in \mathcal{C}^\infty(M)$. The classical master equation, $\{\Theta_R, \Theta_R\} = 0$, translates to the two conditions $[\Pi, \Pi]_S = 0$ and $[\Pi, R]_S = 0$. Therefore, Π is a Poisson tensor and R is a trivector which is d_Π-closed. Finally, the Poisson-Courant algebroid operations are reconstructed via pullback and derived brackets. The fiber metric on $TM \oplus T^*M$ is the pullback of the graded Poisson bracket,

$$\langle X + \alpha, Y + \beta\rangle = j^*\{j_*(X + \alpha), j_*(Y + \beta)\}, \tag{3.570}$$

where $X + \alpha, Y + \beta \in \Gamma(TM \oplus T^*M)$. The anchor map is reconstructed via

$$\rho(X + \alpha)f = j^*\{\{\Theta_R, j_*(X + \alpha)\}, f\}, \tag{3.571}$$

where $X + \alpha \in \Gamma(TM \oplus T^*M)$ and $f \in \mathcal{C}^\infty(M)$. Finally, the R-twisted Dorfman bracket is reconstructed by

$$[X + \alpha, Y + \beta]^{\Pi}_{\mathrm{D},R} = j^*\{\{\Theta_R, j_*(X + \alpha)\}, j_*(Y + \beta)\}, \tag{3.572}$$

where $X+\alpha, Y+\beta \in \Gamma(TM \oplus T^*M)$. The R-twisted Courant bracket is then given by antisymmetrization of the Dorfman bracket,

$$[X+\alpha, Y+\beta]^{\Pi}_{C,R} = \\ \frac{1}{2}(j^*\{\{\Theta_R, j_*(X+\alpha)\}, j_*(Y+\beta)\} - j^*\{\{\Theta_R, j_*(Y+\beta)\}, j_*(X+\alpha)\}). \quad (3.573)$$

We can summarize the result of this section in the following theorem.

Theorem 3.6.1 (Poisson-Courant algebroid) *The QP-manifold* $(\mathcal{M}, \Theta_R, \omega)$ *induces an R-twisted Poisson-Courant algebroid structure* $(TM \oplus T^*M \to (M, \Pi), \langle -,- \rangle, \rho, [-,-]^{\Pi}_{D,R})$.

We found the correct graded symplectic manifold description of the Poisson-Courant algebroid. In the following, begin by the investigation of its properties.

3.6.2 Poisson-Courant algebroid cohomology versus standard Courant algebroid cohomology

In this section, we derive the Poisson-Courant algebroid cohomology as well as the standard Courant algebroid cohomology on special subspaces.

The Poisson-Courant algebroid as well as the standard Courant algebroid are defined on the same graded manifold $\mathcal{M}$. Let us decompose the space of smooth functions on $\mathcal{M}$ by degree,

$$\mathcal{C}^{\infty}(\mathcal{M}) = \bigoplus_{k=0}^{\infty} \mathcal{C}^{\infty}_k(\mathcal{M}). \quad (3.574)$$

The homological vector fields of the Poisson-Courant algebroid and standard Courant algebroid are given by

$$Q_R = \{\Theta_R, -\} \quad (3.575)$$
$$= -\partial_m(\Theta_R)\frac{\vec{\partial}}{\partial p_m} + \Pi^{ij}\zeta_i\frac{\vec{\partial}}{\partial x^j} + \Pi^{ij}p_j\frac{\vec{\partial}}{\partial \xi^i} - \partial_k\Pi^{ij}\xi^k\zeta_i\frac{\vec{\partial}}{\partial \xi^j} - \frac{1}{2}\frac{\partial \Pi^{ij}}{\partial x^k}\zeta_i\zeta_j\frac{\vec{\partial}}{\partial \zeta_k} + \frac{1}{2}R^{ijk}\zeta_i\zeta_j\frac{\vec{\partial}}{\partial \xi^k},$$
$$Q_H = \{\Theta_H, -\} \quad (3.576)$$
$$= -\partial_m(\Theta_H)\frac{\vec{\partial}}{\partial p_m} + \xi^i\frac{\vec{\partial}}{\partial x^i} + p_i\frac{\vec{\partial}}{\partial \zeta_i} + \frac{1}{2}H_{ijk}\xi^i\xi^j\frac{\vec{\partial}}{\partial \zeta_k}.$$

If $[\Pi,\Pi]_S = [\Pi, R]_S = 0$, then the classical master equation is satisfied and the operator Q_R is nilpotent. Furthermore, if $dH = 0$, then $Q_H^2 = 0$. Therefore, we get the following two complexes

$$0 \to \mathcal{C}^{\infty}_0(\mathcal{M}) \xrightarrow{Q_H, Q_R} \mathcal{C}^{\infty}_1(\mathcal{M}) \xrightarrow{Q_H, Q_R} \mathcal{C}^{\infty}_2(\mathcal{M}) \xrightarrow{Q_H, Q_R} \cdots . \quad (3.577)$$

Then, we can define the associated Courant algebroid cohomologies,

$$H^k_{\mathrm{PCA}}(M, Q_R) = \frac{\ker(Q_R : \mathcal{C}^\infty_k(\mathcal{M}) \to \mathcal{C}^\infty_{k+1}(\mathcal{M}))}{\mathrm{im}(Q_R : \mathcal{C}^\infty_{k-1}(\mathcal{M}) \to \mathcal{C}^\infty_k(\mathcal{M}))}, \tag{3.578}$$

$$H^k_{\mathrm{SCA}}(M, Q_H) = \frac{\ker(Q_H : \mathcal{C}^\infty_k(\mathcal{M}) \to \mathcal{C}^\infty_{k+1}(\mathcal{M}))}{\mathrm{im}(Q_H : \mathcal{C}^\infty_{k-1}(\mathcal{M}) \to \mathcal{C}^\infty_k(\mathcal{M}))}. \tag{3.579}$$

Instead of analyzing the cohomologies in full generality, we will show how they are related to well-known cohomologies on certain hyperspaces.
Let us start with the standard Courant algebroid cohomology on the subspace $T[1]M \subset \mathcal{M}$, locally described by coordinates (x^i, ξ^i) of degrees $(0, 1)$. The space of smooth functions on this subspace can be associated to the space of polyforms, $j^*(\mathcal{C}^\infty(T[1]M)) = \Omega^\bullet(M)$. The pushforward of a general section $\gamma \in \Omega^k(M)$ can be written by

$$j_*(\gamma) = \frac{1}{k!}\gamma_{i_1\cdots i_k}\xi^{i_1}\cdots\xi^{i_k}.$$

The homological vector field of the standard Courant algebroid acts as the de Rham differential on this subspace,

$$(j^* \circ Q_H \circ j_*)\gamma = d\gamma. \tag{3.580}$$

We conclude, that the restriction of the standard Courant algebroid cohomology to functions on $\mathcal{C}^\infty(T[1]M)$ is equivalent to the de Rham cohomology over M,

$$H^k_{\mathrm{SCA}}(M, Q_H)|_{\mathcal{C}^\infty(T[1]M)} \cong H^k_{\text{de Rham}}(M, d). \tag{3.581}$$

Let us now investigate a special restriction of the Poisson-Courant algebroid cohomology. We consider the subspace $T^*[1]M \subset \mathcal{M}$, locally parameterized by coordinates (x^i, ζ_i). The decomposition of the space of smooth functions on this subspace by degree is isomorphic to the space of polyvectors, $j^*(\mathcal{C}^\infty(T^*[1]M)) = \mathfrak{X}^\bullet(M)$. So we can write the pushforward of a general section $V \in \mathfrak{X}^k(M)$ as

$$j_*(V) = \frac{1}{k!}V^{i_1\cdots i_k}\zeta_{i_1}\cdots\zeta_{i_k}. \tag{3.582}$$

On this subspace, we find that the Poisson-Courant algebroid homological vector field acts as the Lichnerowicz-Poisson differential,

$$(j^* \circ Q_H \circ j_*)V = d_\Pi V. \tag{3.583}$$

We conclude, that the Poisson-Courant algebroid cohomology restricted to the functions on the subspace $\mathcal{C}^\infty(T^*[1]M)$ is equivalent to the Lichnerowicz-Poisson cohomology of polyvector fields over M,

$$H^k_{\mathrm{PCA}}(M, Q_H)|_{\mathcal{C}^\infty(T^*[1]M)} \cong H^k_{\mathrm{LP}}(M, d_\Pi). \tag{3.584}$$

We are lead to the following theorem.

Theorem 3.6.2 *We have the following isomorphisms between cohomologies,*

$$H^k_{SCA}(M, Q_H)|_{\mathcal{C}^\infty(T[1]M)} \cong H^k_{de\ Rham}(M, d),$$
$$H^k_{PCA}(M, Q_H)|_{\mathcal{C}^\infty(T^*[1]M)} \cong H^k_{LP}(M, d_\Pi).$$

3.6.3 Flux duality I: QP-manifolds

In this section, we derive the duality transformation between the Poisson-Courant algebroid with R-flux and standard Courant algebroid with H-flux on the level of QP-manifolds. After that, we investigate the symmetry of both constructions.

We recognize that the R-twisted Poisson-Courant algebroid and H-twisted standard Courant algebroid are defined with respect to the same graded manifold $\mathcal{M}$ and graded symplectic structure ω. The only difference is the choice of the Hamiltonian function. Having this in mind, we will proof that both algebroids are related by a symplectomorphism if the Poisson structure is invertible.

Theorem 3.6.3 (Flux duality) *Let $(\mathcal{M}, \Theta_H, \omega)$ and $(\mathcal{M}, \Theta_R, \omega)$ be QP-manifolds that induce the H-twisted standard Courant algebroid and R-twisted Poisson-Courant algebroid, respectively. Furthermore, let us assume, that the Poisson structure Π is invertible.*
Then, both algebroids are related by the symplectomorphism $T : \Theta_H \mapsto \Theta_R$,

$$T : \Theta_H \mapsto \exp\left(-\frac{1}{2}\Pi^{ij}\zeta_i\zeta_j\right)\exp\left(\frac{1}{2}\Pi^{-1}_{ij}\xi^i\xi^j\right)\exp\left(-\frac{1}{2}\Pi^{ij}\zeta_i\zeta_j\right)\Theta_H = \Theta_R, \tag{3.585}$$

where $R = \Pi^\sharp H$.[1]

The transformation behavior of the local coordinates is given by

$$T : (x^i, \xi^i, \zeta_i, p_i) \mapsto \left(x^i, \Pi^{ji}\zeta_j, \Pi^{-1}_{ij}\xi^j, p_i - \frac{\partial \Pi^{jk}}{\partial x^i}\Pi^{-1}_{kl}\zeta_j\xi^l\right). \tag{3.586}$$

[1] In local coordinates, this is $R^{ijk} = \Pi^{im}\Pi^{jn}\Pi^{kl}H_{mnl}$.

We conclude, that the base manifold itself is not transformed. However, the space of 1-forms is bijectively mapped to the space of 1-vectors, and vice versa. A section of the generalized tangent bundle transforms as

$$j_*(X+\alpha) = X^i\zeta_i + \alpha_i\xi^i \mapsto X^i\Pi^{-1}_{ij}\xi^j - \alpha_i\Pi^{ij}\zeta_j = j_*(-\Pi^\sharp(\alpha) + (\Pi^{-1})^\flat(X)). \tag{3.587}$$

Here, $(\Pi^{-1})^\flat : TM \to T^*M$ is the musical isomorphism, locally described by $(\Pi^{-1})^\flat(X) = \Pi^{-1}_{ij}X^i dx^j$, where $\Pi^{-1} \in \wedge^2 T^*M$. The map T is invertible and its inverse is given by

$$T^{-1} = \exp\left(\frac{1}{2}\Pi^{ij}\zeta_i\zeta_j\right)\exp\left(-\frac{1}{2}\Pi^{-1}_{ij}\xi^i\xi^j\right)\exp\left(\frac{1}{2}\Pi^{ij}\zeta_i\zeta_j\right). \tag{3.588}$$

By further comparison of the associated Hamiltonian functions one recognizes the substantial symmetries,

$$\begin{aligned} \Pi^{ij}\zeta_i p_j - \frac{1}{2}\frac{\partial\Pi^{jk}}{\partial x^i}\xi^i\zeta_j\zeta_k \quad &\Leftrightarrow \quad p_i\xi^i, \\ \frac{1}{3!}R^{ijk}\zeta_i\zeta_j\zeta_k \quad &\Leftrightarrow \quad \frac{1}{3!}H_{ijk}\xi^i\xi^j\xi^k, \\ d_\Pi R = [\Pi, R]_S = 0 \quad &\Leftrightarrow \quad dH = 0, \\ [\Pi,\Pi]_S = 0 \quad &\Leftrightarrow \quad d^2 = 0. \end{aligned} \tag{3.589}$$

The first line generates the Poisson-Lichnerowicz and de Rham differentials, respectively. The Poisson-Courant algebroid as a contravariant model naturally provides 3-vector flux freedom, whereas the standard Courant algebroid serves as a model for 3-form flux. The 3-vector flux is a cohomology class in Poisson cohomology, whereas the 3-form flux represents a class in de Rham cohomology. The last line expresses the nilpotency of the both differentials, respectively. Finally, the β-twist of the Poisson-Courant algebroid naturally induces R-flux freedom by $R = d_\Pi\beta = [\Pi,\beta]_S \sim R$, whereas the B-twist of the standard Courant algebroid induces H-flux, $dB \sim H$. We conclude, that the Poisson-Courant algebroid realizing contravariant geometry is the most natural way to encode 3-vector freedom in a symmetric manner compared to the standard Courant algebroid with H-flux. However, the existence of a Poisson tensor is inevitable.

3.6.4 Flux duality II: Cohomologies

In this section, we show that the symplectomorphism T induces an isomorphism between the standard Courant algebroid and Poisson-Courant algebroid cohomologies in full generality.

In the former section, we showed that if Π is an invertible Poisson tensor, then $T : \Theta_H \mapsto \Theta_R$ is a symplectomorphism and invertible. We find

$$Q_H f = \{\Theta_H, f\} = \{T^{-1}(\Theta_R), f\} = T^{-1}\{\Theta_R, Tf\} = T^{-1}(Q_R(Tf)). \tag{3.590}$$

Therefore, for a Q_H-cocycle,

$$Q_H f = 0 \Leftrightarrow Q_R(Tf) = 0, \tag{3.591}$$

and for a Q_H-coboundary,

$$Q_H f = g \Leftrightarrow Tg = Q_R(Tf). \tag{3.592}$$

In other words, if f is a Q_H-cocycle, then Tf is a Q_R-cocycle and if f is a Q_H-coboundary, then Tf is a Q_R-coboundary. So we find that the flux duality map of the complexes $T : \mathcal{C}^\bullet(\mathcal{M}) \to \mathcal{C}^\bullet(\mathcal{M})$ lifts to a isomorphism of general Courant algebroid cohomologies,

$$T : H^k_{\text{SCA}}(M, Q_H) \cong H^k_{PCA}(M, Q_R). \tag{3.593}$$

This is a generalization of the famous theorem, that the map $\Pi^\sharp : \Omega^k(M) \to \wedge^k TM$ induces a homomorphism between de Rham cohomology and Lichnerowicz-Poisson cohomology,

$$\Pi^\sharp : H^k_{\text{de Rham}}(M, d) \to H^k_{\text{LP}}(M, d_\Pi), \tag{3.594}$$

which condenses to an isomorphism, if Π^{-1} is symplectic,

$$\Pi^{-1} \text{ is symplectic} \Rightarrow \Pi^\sharp : H^k_{\text{de Rham}}(M, d) \cong H^k_{\text{LP}}(M, d_\Pi). \tag{3.595}$$

From the perspective of the isomorphism between the Poisson and standard Courant algebroids, the map T gives an isomorphism of subspaces $\mathcal{C}^\infty(T[1]M) \cong \mathcal{C}^\infty(T^*[1]M)$ and therefore an isomorphism of restricted Poisson and standard Courant algebroid cohomologies to the respective polyform and polyvector subspaces.

Theorem 3.6.4 *Flux duality T lifts to an isomorphism of Courant algebroid cohomologies,*

$$T : H^k_{SCA}(M, Q_H) \cong H^k_{PCA}(M, Q_R).$$

As a result of the last two sections, we are lead to the conclusion, that the natural cohomology theory for the contravariant 3-vector flux is given by Poisson cohomology. Then, the formulation of 3-form and 3-vector theories is totally symmetric.

3.6.5 Embedding in double field theory

In this section, we shall show how the Poisson-Courant algebroid can be seen as a reduction of the double field theory Hamiltonian, i.e., a special solution to the section condition.
We start with the untwisted double field theory Hamiltonian,

$$\widehat{\Theta}_0 = p_i(\xi^i + \widetilde{\zeta}^i) + \widetilde{p}^i(\zeta_i + \widetilde{\xi}_i). \tag{3.596}$$

Let us recall the untwisted Poisson-Courant algebroid depending on coordinates $(y^i, \rho^i, \eta_i, q_i)$ of degrees $(0, 1, 1, 2)$,

$$\Theta_R = \Pi^{ij}\eta_i q_j - \frac{1}{2}\frac{\partial \Pi^{ij}}{\partial y^k}\rho^k \eta_i \eta_j. \tag{3.597}$$

The following reduction of $\widehat{\Theta}_0$,

$$\widetilde{\xi}^i = \rho^i, \quad \widetilde{\zeta}_i = \eta_i, \quad \widetilde{p}^i = \Pi^{ij}(y)q_j + \frac{1}{2}\frac{\partial \Pi^{ik}}{\partial y^j}\rho^j \eta_k, \tag{3.598}$$

under the projection to the winding frame by $(p_i = \xi^i = \zeta_i = 0)$ leads to the Poisson-Courant algebroid Hamiltonian Θ_R. The base manifold coordinates are related by

$$\widetilde{x}_i = \frac{1}{2}\int \Pi^{-1}_{ji}(y)dy^j. \tag{3.599}$$

From this perspective, we can conclude, that the Poisson-Courant algebroid solves the section condition as a Courant algebroid on the winding frame. However, for the case of a non-constant Poisson tensor, the contribution $(-\frac{1}{2}\frac{\partial \Pi^{ij}}{\partial y^k}\rho^k \eta_i \eta_j)$ corresponds to the torsion of the associated linear contravariant connection. Therefore, the Poisson-Courant algebroid lives in the winding space, which is deformed by a non-constant Poisson tensor inducing a non-trivial connection.
This can be made precise, when investigating the reduction of the graded symplectic structure $\widehat{\omega}$, which does not reduce to the ordinary graded symplectic structure of the Poisson-Courant algebroid,

$$\omega = -\delta y^i \wedge \delta q_i + \delta \rho^i \wedge \delta \eta_i, \tag{3.600}$$

but receives a deformation parameterized by the non-constant Poisson tensor Π. In the constant case, the deformation vanishes and the graded symplectic structure reduces correctly.
Let us summarize our findings in the following theorem.

Theorem 3.6.5 *The Poisson-Courant algebroid is a reduction of the untwisted double field theory pre-QP-manifold to the winding frame, if the Poisson tensor* Π *is constant. If the Poisson tensor is non-constant, the winding space is deformed through the emergence of a linear contravariant connection with torsion.*

We conclude, that the Poisson-Courant algebroid serves as a model of the double field theory winding space, which is deformed by the existence of a global Poisson tensor. Motivated by this result, we construct physical models of topological membranes and string sigma models in Poisson-Courant algebroid backgrounds as well as associated Poisson and R-twisted current algebras.

3.6.6 Poisson-Courant sigma model

In this section, we derive the topological sigma model associated to the Poisson-Courant algebroid using the AKZS procedure for BV models. We show that for non-zero worldvolume boundary, the boundary theory realizes a topological closed string model with R-flux Wess-Zumino-Witten term. By the introduction of a kinetic term we find the sigma model of a closed string traveling in R-space with Poisson-Courant algebroid target space structure.

We consider a membrane embedded into a QP-manifold target space with Poisson-Courant algebroid structure. The membrane worldvolume is denoted by X and has 3 dimensions. We promote it to the superworldvolume $\chi = T[1]X$, which locally is described by coordinates (σ^μ, θ^μ) of degrees $(0, 1)$ and the index μ runs from 1 to 3. The coordinates σ^μ locally parameterize X, whereas the Grassmann odd coordinates θ^μ parameterize the graded tangent bundle fiber.

The target space structure is induced by the QP-manifold $(\mathcal{M} = T^*[2]T[1]M, \Theta_R, \omega)$ as described in the former sections. Having fixed both, the superworldvolume and the target space QP-manifold, we now construct the AKSZ topological sigma model on the mapping space $\mathrm{Map}(\chi, \mathcal{M})$. For the details on the differential geometry on the mapping space and how the associated objects are defined locally, we refer to the sections B.3, B.4 and B.5 in the appendix. The graded symplectic structure on the mapping space is given by

$$\Omega = \int_\chi \mu_\chi(-\delta \boldsymbol{x}^i \wedge \delta \boldsymbol{p}_i + \delta \boldsymbol{\xi}^i \wedge \delta \boldsymbol{\zeta}_i), \tag{3.601}$$

locally expressed by elements of $\mathrm{Map}(\chi, \mathcal{M})$. These elements are images of the local coordinates on $\mathcal{M}$ under the pullback along the evaluation map ev,

$$ev^* : (x^i, p_i, \xi^i, \zeta_i) \mapsto (\boldsymbol{x}^i, \boldsymbol{p}^i, \boldsymbol{\xi}^i, \boldsymbol{\zeta}_i). \tag{3.602}$$

The fields $(\boldsymbol{x}^i, \boldsymbol{p}^i, \boldsymbol{\xi}^i, \boldsymbol{\zeta}_i)$ are functions on χ and therefore superfields, whose components contain physical components, ghosts and antifields. The interaction term of the BV action on the mapping space is given by

$$S_1 = \int_\chi \mu_\chi \left(\Pi^{ij}(\boldsymbol{x})\boldsymbol{\zeta}_i \boldsymbol{p}_j - \frac{1}{2}\frac{\partial \Pi^{ij}}{\partial x^k}(\boldsymbol{x})\boldsymbol{\xi}^k \boldsymbol{\zeta}_i \boldsymbol{\zeta}_j + \frac{1}{3!} R^{ijk}(\boldsymbol{x})\boldsymbol{\zeta}_i \boldsymbol{\zeta}_j \boldsymbol{\zeta}_k \right). \tag{3.603}$$

Here, the measure is given by $\mu_\chi = d\sigma^1 d\sigma^2 d\sigma^3 d\theta^3 d\theta^2 d\theta^1$. Taking the Liouville 1-form $\vartheta = -p_i \delta x^i - \zeta_i \delta \xi^i$, the associated Liouville 1-form on the mapping space is given

$$\boldsymbol{\vartheta} = \int_\chi \mu_\chi (\boldsymbol{p}_i \delta \boldsymbol{x}^i + \boldsymbol{\zeta}_i \delta \boldsymbol{\xi}^i), \tag{3.604}$$

so that the kinetic term of the BV action becomes

$$S_0 = \int_\chi \mu_\chi (-\boldsymbol{p}_i \boldsymbol{d} \boldsymbol{x}^i - \boldsymbol{\zeta}_i \boldsymbol{d} \boldsymbol{\xi}^i), \tag{3.605}$$

where $\boldsymbol{d} = \theta^\mu \partial_\mu$ is the superdifferential and ∂_μ denotes the derivative with respect to σ^μ. Summing up, we find the full BV action of a *Poisson-Courant sigma model* on the mapping space

$$\begin{aligned} S &= S_0 + S_1 \\ &= \int_\chi \mu_\chi \left(-\boldsymbol{p}_i \boldsymbol{d}\boldsymbol{x}^i - \boldsymbol{\zeta}_i \boldsymbol{d}\boldsymbol{\xi}^i + \Pi^{ij}(\boldsymbol{x})\boldsymbol{\zeta}_i \boldsymbol{p}_j - \frac{1}{2}\frac{\partial \Pi^{ij}}{\partial x^k}(\boldsymbol{x})\boldsymbol{\xi}^k \boldsymbol{\zeta}_i \boldsymbol{\zeta}_j + \frac{1}{3!} R^{ijk}(\boldsymbol{x})\boldsymbol{\zeta}_i \boldsymbol{\zeta}_j \boldsymbol{\zeta}_k \right). \end{aligned} \tag{3.606}$$

Definition 3.6.6 (Poisson-Courant sigma model) *The* 3*-tuple* $(Map(\chi, \mathcal{M}), S, \Omega)$*, where* $\chi = T[1]X$ *and* $\dim(X) = 3$*,* $\mathcal{M} = T^*[2]T[1]M$ *and*

$$\begin{aligned} S &= \int_\chi \mu_\chi \left(-\boldsymbol{p}_i \boldsymbol{d}\boldsymbol{x}^i - \boldsymbol{\zeta}_i \boldsymbol{d}\boldsymbol{\xi}^i + \Pi^{ij}(\boldsymbol{x})\boldsymbol{\zeta}_i \boldsymbol{p}_j - \frac{1}{2}\frac{\partial \Pi^{ij}}{\partial x^k}(\boldsymbol{x})\boldsymbol{\xi}^k \boldsymbol{\zeta}_i \boldsymbol{\zeta}_j + \frac{1}{3!} R^{ijk}(\boldsymbol{x})\boldsymbol{\zeta}_i \boldsymbol{\zeta}_j \boldsymbol{\zeta}_k \right), \\ \Omega &= \int_\chi \mu_\chi (-\delta \boldsymbol{x}^i \wedge \delta \boldsymbol{p}_i + \delta \boldsymbol{\xi}^i \wedge \delta \boldsymbol{\zeta}_i), \end{aligned}$$

is called the Poisson-Courant sigma model. It is a model of a topological membrane traveling in a target space with Poisson-Courant algebroid structure.

This sigma model can be referred to as contravariant Courant sigma model, since it realizes contravariant geometry.

Let us now derive the sigma model action of a closed string traveling in R-flux background induced from the Poisson-Courant algebroid structure on the target space. This can be derived from the membrane Poisson-Courant algebroid sigma model. The variation of (3.606) is given by

$$\begin{aligned}\delta S &= \int_{\chi} \mu_{\chi} \Big(-\delta \boldsymbol{p}_i \boldsymbol{d}\boldsymbol{x}^i - \boldsymbol{p}_i \boldsymbol{d}\delta\boldsymbol{x}^i - \delta\boldsymbol{\zeta}_i \boldsymbol{d}\boldsymbol{\xi}^i - \boldsymbol{\zeta}_i \boldsymbol{d}\delta\boldsymbol{\xi}^i \\ &\qquad + \delta \left(\Pi^{ij}(\boldsymbol{x})\boldsymbol{\zeta}_i \boldsymbol{p}_j - \frac{1}{2}\frac{\partial \Pi^{ij}}{\partial x^k}(\boldsymbol{x})\boldsymbol{\xi}^k \boldsymbol{\zeta}_i\boldsymbol{\zeta}_j + \frac{1}{3!}R^{ijk}(\boldsymbol{x})\boldsymbol{\zeta}_i\boldsymbol{\zeta}_j\boldsymbol{\zeta}_k \right)\Big) \\ &= \int_{\partial\chi} \mu_{\partial\chi}(\boldsymbol{\zeta}_i\delta\boldsymbol{\xi}^i - \boldsymbol{p}_i\delta\boldsymbol{x}^i) + \int_{\chi}\mu_{\chi}\Big(-\delta\boldsymbol{p}_i\boldsymbol{d}\boldsymbol{x}^i + \boldsymbol{d}\boldsymbol{p}_i\delta\boldsymbol{x}^i - \delta\boldsymbol{\zeta}_i\boldsymbol{d}\boldsymbol{\xi}^i - \boldsymbol{d}\boldsymbol{\zeta}_i\delta\boldsymbol{\xi}^i \\ &\qquad + \delta(\Pi^{ij}(\boldsymbol{x})\boldsymbol{\zeta}_i\boldsymbol{p}_j - \frac{1}{2}\frac{\partial \Pi^{ij}}{\partial x^k}(\boldsymbol{x})\boldsymbol{\xi}^k\boldsymbol{\zeta}_i\boldsymbol{\zeta}_j + \frac{1}{3!}R^{ijk}(\boldsymbol{x})\boldsymbol{\zeta}_i\boldsymbol{\zeta}_j\boldsymbol{\zeta}_k)\Big)\,. \end{aligned} \tag{3.607}$$

The measure $\mu_{\partial\chi}$ denotes the boundary measure on $\partial\chi$. The boundary variation term has to vanish. Therefore, the boundary of the membrane inherits the structure of the Poisson-Courant algebroid restricted to the Lagrangian submanifold $\mathcal{L}$, which is the zero locus of the Liouville 1-form ϑ.

The classical master equation on $(\mathrm{Map}(\chi,\mathcal{M}), S, \Omega)$ is given by

$$\begin{aligned}\{S,S\}_{\mathrm{BV}} &= \\ &\int_{\partial\chi}\mu_{\partial\chi}\left(-\boldsymbol{p}_i\boldsymbol{d}\boldsymbol{x}^i - \boldsymbol{\zeta}_i\boldsymbol{d}\boldsymbol{\xi}^i + \Pi^{ij}(\boldsymbol{x})\boldsymbol{\zeta}_i\boldsymbol{p}_j - \frac{1}{2}\frac{\partial\Pi^{ij}}{\partial x^k}(\boldsymbol{x})\boldsymbol{\xi}^k\boldsymbol{\zeta}_i\boldsymbol{\zeta}_j + \frac{1}{3!}R^{ijk}(\boldsymbol{x})\boldsymbol{\zeta}_i\boldsymbol{\zeta}_j\boldsymbol{\zeta}_k\right),\end{aligned} \tag{3.608}$$

which vanishes upon employment of the boundary conditions $\boldsymbol{p}_i|_{\partial\chi} = \boldsymbol{\zeta}_i|_{\partial\chi} = 0$. From the perspective of the target space QP-manifold, this choice corresponds to $p_i = \zeta_i = 0$, so that sections of the generalized tangent bundle get reduced to the cotangent subspace, $TM \oplus T^*M \to T^*M$, which is a *Dirac structure* of the Poisson-Courant algebroid.

In the next step, we consider the introduction of a boundary term by twist via $-\frac{1}{2}B_{ij}\xi^i\xi^j$, where $B \in \Omega^2(M)$. Since $\{-\frac{1}{2}B_{ij}\xi^i\xi^j, -\frac{1}{2}B_{ij}\xi^i\xi^j\} = 0$, we find the twisted homological function

$$S_B = S + \frac{1}{2}\int_{\partial\chi}\mu_{\partial\chi}B_{ij}(\boldsymbol{x})\boldsymbol{\xi}^i\boldsymbol{\xi}^j. \tag{3.609}$$

Investigating the boundary variation of S_B, we find, that the boundary conditions have been twisted,

$$\delta S_B = \int_{\partial\chi} \mu_{\partial\chi} \left(\boldsymbol{\zeta}_i \delta \boldsymbol{\xi}^i - \boldsymbol{p}_i \delta \boldsymbol{x}^i + \frac{1}{2} \frac{\partial B_{ij}}{\partial \boldsymbol{x}^k}(\boldsymbol{x}) \boldsymbol{\xi}^i \boldsymbol{\xi}^j \delta \boldsymbol{x}^k - B_{ij}(\boldsymbol{x}) \boldsymbol{\xi}^j \delta \boldsymbol{\xi}^i \right)$$
$$+ \int_{\chi} \mu_{\chi} \Big(-\delta \boldsymbol{p}_i \boldsymbol{d} \boldsymbol{x}^i + \boldsymbol{d} \boldsymbol{p}_i \delta \boldsymbol{x}^i - \delta \boldsymbol{\zeta}_i \boldsymbol{p} \boldsymbol{\xi}^i - \boldsymbol{d} \boldsymbol{\zeta}_i \delta \boldsymbol{\xi}^i$$
$$+ \delta \left(\Pi^{ij}(\boldsymbol{x}) \boldsymbol{\zeta}_i \boldsymbol{p}_j - \frac{1}{2} \frac{\partial \Pi^{ij}}{\partial x^k}(\boldsymbol{x}) \boldsymbol{\xi}^k \boldsymbol{\zeta}_i \boldsymbol{\zeta}_j + \frac{1}{3!} R^{ijk}(\boldsymbol{x}) \boldsymbol{\zeta}_i \boldsymbol{\zeta}_j \boldsymbol{\zeta}_k \right) \Big). \tag{3.610}$$

The vanishing of the boundary variation leads to the boundary conditions

$$\boldsymbol{p}_i|_{\partial\chi} = \frac{1}{2} \frac{\partial B_{jk}}{\partial \boldsymbol{x}^i}(\boldsymbol{x}) \boldsymbol{\xi}^j \boldsymbol{\xi}^k|_{\partial\chi}, \quad \boldsymbol{\zeta}_i|_{\partial\chi} = B_{ij}(\boldsymbol{x}) \boldsymbol{\xi}^j|_{\partial\chi}. \tag{3.611}$$

Upon solving the Liouville part via the boundary conditions, the twisted master equation gives

$$\{S_B, S_B\}_{\mathrm{BV}} = \left[\Pi^{ij}(\boldsymbol{x}) \boldsymbol{\zeta}_i \boldsymbol{p}_j - \frac{1}{2} \frac{\partial \Pi^{ij}}{\partial x^k}(\boldsymbol{x}) \boldsymbol{\xi}^k \boldsymbol{\zeta}_i \boldsymbol{\zeta}_j + \frac{1}{3!} R^{ijk}(\boldsymbol{x}) \boldsymbol{\zeta}_i \boldsymbol{\zeta}_j \boldsymbol{\zeta}_k \right]_{\partial\chi} = 0. \tag{3.612}$$

From the perspective of the target space, above equation is equivalent to the condition that the homological function Θ vanishes on the Lagrangian submanifold specified by the boundary conditions,

$$\Theta|_{\mathcal{L}_B} = \left[\Pi^{ij} \zeta_i p_j - \frac{1}{2} \frac{\partial \Pi^{ij}}{\partial x^k} \xi^k \zeta_i \zeta_j + \frac{1}{3!} R^{ijk} \zeta_i \zeta_j \zeta_k \right]_{\mathcal{L}_B} = 0, \tag{3.613}$$

where

$$\mathcal{L}_B = \left\{ (x^i, \xi^i, \zeta_i, p_i) \in \mathcal{M} \;\middle|\; p_i = \frac{1}{2} \frac{\partial B_{jk}}{\partial x^i} \xi^j \xi^k \text{ and } \zeta_i = B_{ij} \xi^j \right\}. \tag{3.614}$$

This condition is satisfied if $[B, B]_\Pi = B^\flat R$, where the Koszul bracket has been extended over the space of polyforms via the Leibniz rule.[2] The 2-form Koszul commutator is twisted by a 3-vector.

In the special case, where $B = \Pi^{-1}$, we find

$$H = dB = B^\flat R. \tag{3.615}$$

Finally, we derive the string sigma model with R-flux on the boundary of the twisted Poisson-Courant sigma model. This is done by integrating out the auxiliary superfield $\boldsymbol{p}_i$. We start from (3.609) and use the equation of motion for $\boldsymbol{p}_i$ leading to

$$\boldsymbol{\zeta}_i = -\Pi^{-1}_{ij}(\boldsymbol{x}) \boldsymbol{d} \boldsymbol{x}^j, \tag{3.616}$$

[2] Locally, we can write $B^\flat R = \frac{1}{3!} B_{im} B_{jn} B_{kl} R^{mnl} dx^i \wedge dx^j \wedge dx^k$

under the assumption that Π is non-degenerate. The integral over χ is given by

$$\int_\chi \mu_\chi \left(\Pi^{-1}_{ij} \boldsymbol{dx}^j \boldsymbol{d\xi}^i - \frac{1}{2}\frac{\partial \Pi^{-1}_{ij}}{\partial x^k} \boldsymbol{\xi}^k \boldsymbol{dx}^i \boldsymbol{dx}^j - \frac{1}{3!} R^{ijk} \Pi^{-1}_{il} \Pi^{-1}_{jm} \Pi^{-1}_{kn} \boldsymbol{dx}^l \boldsymbol{dx}^m \boldsymbol{dx}^n \right). \tag{3.617}$$

Under the assumption that Π^{-1} is symplectic, we can apply Stokes theorem to obtain the boundary string sigma model with R-flux,

$$S = \int_{\partial\chi} \mu_{\partial\chi} \left(-\Pi^{-1}_{ij} \boldsymbol{\xi}^i \boldsymbol{dx}^j + \frac{1}{2} B_{ij} \boldsymbol{\xi}^i \boldsymbol{\xi}^j \right) - \frac{1}{3!} \int_\chi \mu_\chi R^{ijk} \Pi^{-1}_{il} \Pi^{-1}_{jm} \Pi^{-1}_{kn} \boldsymbol{dx}^l \boldsymbol{dx}^m \boldsymbol{dx}^n. \tag{3.618}$$

This is a Poisson sigma model with Wess-Zumino term $H = \Pi^\sharp R$.

Above action is written using superfields that contain ghosts and antifields. In order to obtain the ghost-free part of (3.618), we expand all the fields in supercoordinates and project out the ghost contribution by integration over the Berezin measure. The fields are expanded as

$$\boldsymbol{x}^i(\sigma, \theta) = \boldsymbol{x}^{(0),i}(\sigma) + \boldsymbol{x}^{(1),i}_\mu(\sigma)\theta^\mu + \frac{1}{2}\boldsymbol{x}^{(2),i}_{\mu\nu}(\sigma)\theta^\mu\theta^\nu, \tag{3.619}$$

$$\boldsymbol{\xi}^i(\sigma, \theta) = \boldsymbol{\xi}^{(0),i}(\sigma) + \boldsymbol{\xi}^{(1),i}_\mu(\sigma)\theta^\mu + \frac{1}{2}\boldsymbol{\xi}^{(2),i}_{\mu\nu}(\sigma)\theta^\mu\theta^\nu, \tag{3.620}$$

$$\boldsymbol{\zeta}_i(\sigma, \theta) = \boldsymbol{\zeta}^{(0)}_i(\sigma) + \boldsymbol{\zeta}^{(1)}_{i,\mu}(\sigma)\theta^\mu + \frac{1}{2}\boldsymbol{\zeta}^{(2)}_{i,\mu\nu}(\sigma)\theta^\mu\theta^\nu. \tag{3.621}$$

The physical component is the ghost-number zero component. If we insert the expansions and take the (σ^1, θ^1)-boundary, the relevant part is given by

$$\begin{aligned} S_{B,\text{ghost-free}} = &\\ &\int_{\partial\chi} d\sigma^1 d\sigma^2 d\theta^2 d\theta^1 \left(-\Pi^{-1}_{ij} \boldsymbol{\xi}^{(1),i}_\mu \theta^\mu\theta^\nu \partial_\nu \boldsymbol{x}^j + \frac{1}{2} B_{ij} \boldsymbol{\xi}^{(1),i}_\mu \theta^\mu \boldsymbol{\xi}^{(1),j}_\nu \theta^\nu \right) \\ &- \int_\chi d\sigma^1 d\sigma^2 d\sigma^3 d\theta^3 d\theta^2 d\theta^1 \left(\frac{1}{3!} R^{ijk} \Pi^{-1}_{il} \Pi^{-1}_{jm} \Pi^{-1}_{kn} \theta^\mu\theta^\nu\theta^\rho \partial_\mu \boldsymbol{x}^l \partial_\nu \boldsymbol{x}^m \partial_\rho \boldsymbol{x}^n \right) \\ = &\int_{\partial X} \left(-\Pi^{-1}_{ij} \xi^i \wedge dx^j + \frac{1}{2} B_{ij} \xi^i \wedge \xi^j \right) - \frac{1}{3!} \int_X R^{ijk} \Pi^{-1}_{il} \Pi^{-1}_{jm} \Pi^{-1}_{kn} dx^l \wedge dx^m \wedge dx^n, \end{aligned} \tag{3.622}$$

where we denoted $x^i = \boldsymbol{x}^{(0),i}$ and defined $\xi^i = \boldsymbol{\xi}^{(1),i}_\mu d\sigma^\mu$. Finally, we obtain the action of a string sigma model of a closed string traveling in R-flux background of the Poisson-Courant algebroid target space when adding the kinetic term,

$$\begin{aligned} S = &\int_{\partial X} \left(-\Pi^{-1}_{ij} \xi^i \wedge dx^j + \frac{1}{2} B_{ij} \xi^i \wedge \xi^j + \frac{1}{2} G_{ij} dx^i \wedge \star dx^j \right) \\ &- \frac{1}{3!} \int_X R^{ijk} \Pi^{-1}_{il} \Pi^{-1}_{jm} \Pi^{-1}_{kn} dx^l \wedge dx^m \wedge dx^n. \end{aligned} \tag{3.623}$$

Starting from the Poisson-Courant sigma model we found a closed string sigma model by a twist of the boundary conditions as the boundary theory of a topological membrane model with Poisson-Courant algebroid structure.

3.6.7 Flux duality III: Courant sigma models

In this section, we discuss how flux duality on the target space transforms the membrane Courant sigma models. We find that from the viewpoint of the membrane sigma model, the realization of the boundary theory as a Poisson sigma model with H-flux Wess-Zumino term or contravariant Poisson sigma model with R-flux Wess-Zumino term is crucially related to the choice of boundary conditions. In order to come to an understanding, we compare the derivation of both sigma models.

In both cases, the starting point is the graded manifold $\mathcal{M} = T^*[2]T[1]M$ with graded symplectic structure

$$\omega = -\delta x^i \wedge \delta p_i + \delta\xi^i \wedge \delta\zeta_i. \tag{3.624}$$

However, we can observe two differences. The first difference is given by the choice of Hamiltonian functions,

$$\Theta_H = p_i\xi^i + \frac{1}{3!}H_{ijk}\xi^i\xi^j\xi^k, \tag{3.625}$$

$$\Theta_R = \Pi^{ij}\zeta_i p_j - \frac{1}{2}\frac{\partial\Pi^{ij}}{\partial x^k}\xi^k\zeta_i\zeta_j + \frac{1}{3!}R^{ijk}\zeta_i\zeta_j\zeta_k, \tag{3.626}$$

which are related by the canonical transformation $T : \Theta_H \mapsto \Theta_R$. The second difference, which is crucial for the derivation of the appropriate boundary sigma model, is the choice of Liouville 1-form ϑ,

$$\vartheta_H = -p_i\delta x^i - \xi^i\delta\zeta_i, \tag{3.627}$$

$$\vartheta_R = -p_i\delta x^i - \zeta_i\delta\xi^i, \tag{3.628}$$

where we introduce the subscript H or R for convenience. The resulting Courant sigma models are then given by

$$S_H = \int_\chi \mu_\chi \left(-\boldsymbol{p}_i\boldsymbol{d}\boldsymbol{x}^i - \boldsymbol{\xi}^i\boldsymbol{d}\boldsymbol{\zeta}_i + \boldsymbol{p}_i\boldsymbol{\xi}^i + \frac{1}{3!}H_{ijk}(\boldsymbol{x})\boldsymbol{\xi}^i\boldsymbol{\xi}^j\boldsymbol{\xi}^k\right), \tag{3.629}$$

$$S_R = \int_\chi \mu \left(-\boldsymbol{p}_i\boldsymbol{d}\boldsymbol{x}^i - \boldsymbol{\zeta}_i\boldsymbol{d}\boldsymbol{\xi}^i + \Pi^{ij}(\boldsymbol{x})\boldsymbol{\zeta}_i\boldsymbol{p}_j - \frac{1}{2}\frac{\partial\Pi^{ij}}{\partial x^k}(\boldsymbol{x})\boldsymbol{\xi}^k\boldsymbol{\zeta}_i\boldsymbol{\zeta}_j + \frac{1}{3!}R^{ijk}(\boldsymbol{x})\boldsymbol{\zeta}_i\boldsymbol{\zeta}_j\boldsymbol{\zeta}_k\right). \tag{3.630}$$

The Poisson sigma model with H-flux Wess-Zumino term is realized in the zero locus of ϑ_H, whereas the contravariant Poisson sigma model with R-flux Wess-Zumino term is realized in the zero locus of ϑ_R. Recognizing that the duality transforms $T : \vartheta_H \mapsto \vartheta_R$, we come to the conclusion that T not only relates the Hamiltonian functions if Π is nondegenerate and $R = \Pi^\sharp H$, but also transforms the Lagrangian submanifolds related to both zero loci. Since the boundary theory of the membrane sigma model is mapped into the Lagrangian submanifold specified by the respective zero loci, we arrive at the following theorem.

Theorem 3.6.7 *From the perspective of the membrane theory, the Poisson sigma model with H-flux Wess-Zumino term and the contravariant Poisson sigma model with R-flux Wess-Zumino term are dual to each other by the transformation T in the sense that they are realized by dual boundary conditions.*

3.6.8 Contravariant Poisson algebra

In this section, we derive the Poisson algebra of physical canonical coordinates associated to the target space QP-manifold of the Poisson-Courant algebroid with R-flux.

The Poisson-Courant sigma model is a topological membrane model, which realizes R-flux freedom by a Wess-Zumino term. The associated boundary theory is the contravariant Poisson sigma model of a topological string. Let the worldsheet of the string be $\Sigma = S^1 \times \mathbb{R}$, where $\mathbb{R}$ denotes the time direction. Then, the Poisson algebra will be constructed on the mapping space from $\mathring{\chi} = T[1]S^1$ to the target superspace $\mathcal{M}$, $\mathrm{Map}(\mathring{\chi}, \mathcal{M})$. Let $\mathring{\chi}$ be locally parameterized by the coordinates (σ, θ) of degrees $(0, 1)$. The supermapping space $\mathrm{Map}(\mathring{\chi}, \mathcal{M})$ will in the end be reduced to the space of maps from the string to the cotangent bundle, $T^*LM = \mathrm{Map}(S^1, T^*M)$, locally parameterized by coordinates $x^i(\sigma)$ on M and $\xi^i(\sigma)$ on the cotangent fiber. Thus, we are lead to the Poisson algebra on T^*LM.

Please note that we use the following graded symplectic structure,

$$\omega = \delta x^i \wedge \delta p_i + \delta\xi^i \wedge \delta\zeta_i. \tag{3.631}$$

To get started, we choose the Lagrangian submanifold $\mathcal{L}$ on which $p_i = \zeta_i = 0$. The derived bracket does not close on $\mathcal{L}$,

$$\{\{x^i, \Theta_R\}, x^j\} = 0, \tag{3.632}$$

$$\{\{x^i, \Theta_R\}, \xi^j\} = -\Pi^{ij}, \tag{3.633}$$

$$\{\{\xi^i, \Theta_R\}, \xi^j\} = \frac{\partial \Pi^{ij}}{\partial x^k}\xi^k - R^{ijk}\zeta_k. \tag{3.634}$$

The Poisson bracket on the transgressed Lagrangian submanifold $\widehat{\mathcal{L}} \subset \mathrm{Map}(\mathring{\chi}, \mathcal{M})$ is then given by

$$\{\boldsymbol{x}^i(z), \boldsymbol{x}^j(z')\}_{\mathrm{PB},\widehat{\mathcal{L}}} = 0, \tag{3.635}$$

$$\{\boldsymbol{x}^i(z), \boldsymbol{\xi}^j(z')\}_{\mathrm{PB},\widehat{\mathcal{L}}} = -\Pi^{ij}(\boldsymbol{x}(z))\delta(z-z'), \tag{3.636}$$

$$\{\boldsymbol{\xi}^i(z), \boldsymbol{\xi}^j(z')\}_{\mathrm{PB},\widehat{\mathcal{L}}} = \frac{\partial \Pi^{ij}}{\partial \boldsymbol{x}^k}(\boldsymbol{x}(z))\boldsymbol{\xi}^k\delta(z-z'), \tag{3.637}$$

where $z = (\sigma, \theta)$ and $\delta(z-z') = \delta(\sigma-\sigma')\delta(\theta-\theta')$. One recognizes that the R-flux contribution has been projected out on $\widehat{\mathcal{L}}$. In order to generate the R-flux Poisson algebra, we twist by the Liouville 1-form on $\mathcal{L}$, $\alpha \equiv \iota_{\check{\boldsymbol{d}}}\mu_* \mathrm{ev}^* \vartheta_{\mathcal{L}}$, where

$$\vartheta_{\mathcal{L}} = \Pi^{-1}_{ij}\xi^i\delta x^j - \cdots, \tag{3.638}$$

where $\cdots$ contain terms without ξ^i, so that

$$\alpha \equiv -\int_{\mathring{\chi}} \mu_{\mathring{\chi}} \Pi^{-1}_{ij}\boldsymbol{\xi}^i \boldsymbol{d}\boldsymbol{x}^j + \cdots . \tag{3.639}$$

The twist transforms the coordinate $\boldsymbol{\zeta}_i$ via

$$\exp(\alpha)\boldsymbol{\zeta}_i = \boldsymbol{\zeta}_i - \Pi^{-1}_{ij}\boldsymbol{d}\boldsymbol{x}^j, \tag{3.640}$$

so that the Lagrangian submanifold is twisted $\widehat{\mathcal{L}} \to \widehat{\mathcal{L}}_\alpha$, on which $\boldsymbol{\zeta}_i - \Pi^{-1}_{ij}\boldsymbol{d}\boldsymbol{x}^j = 0$. We therefore find the following Poisson algebra with respect to $\mathcal{L}_\alpha$,

$$\{\boldsymbol{x}^i(z), \boldsymbol{x}^j(z')\}_{\mathrm{PB},\widehat{\mathcal{L}}_\alpha} = 0, \tag{3.641}$$

$$\{\boldsymbol{x}^i(z), \boldsymbol{\xi}^j(z')\}_{\mathrm{PB},\widehat{\mathcal{L}}_\alpha} = -\Pi^{ij}(\boldsymbol{x}(z))\delta(z-z'), \tag{3.642}$$

$$\{\boldsymbol{\xi}^i(z), \boldsymbol{\xi}^j(z')\}_{\mathrm{PB},\widehat{\mathcal{L}}_\alpha} = \left(\frac{\partial \Pi^{ij}}{\partial \boldsymbol{x}^k}(\boldsymbol{x}(z))\boldsymbol{\xi}^k - R^{ijk}(\boldsymbol{x}(z))\Pi^{-1}_{kl}\boldsymbol{d}\boldsymbol{x}^l\right)\delta(z-z'). \tag{3.643}$$

In order to derive the physical Poisson brackets, we expand the superfields in the Grassmann variables and project out the ghost-degree zero component. The superfield expansions are given by

$$\boldsymbol{x}^i(\sigma,\theta) = \boldsymbol{x}^{(0),i}(\sigma) + \boldsymbol{x}^{(1),i}(\sigma)\theta, \tag{3.644}$$

$$\boldsymbol{\xi}^i(\sigma,\theta) = \boldsymbol{\xi}^{(0),i}(\sigma) + \boldsymbol{\xi}^{(1),i}(\sigma)\theta, \tag{3.645}$$

$$\boldsymbol{\zeta}_i(\sigma,\theta) = \boldsymbol{\zeta}_i^{(0)}(\sigma) + \boldsymbol{\zeta}_i^{(1)}(\sigma)\theta, \tag{3.646}$$

where we denote the physical components by

$$x^i = \boldsymbol{x}^{(0),i}, \qquad \xi^i = \boldsymbol{\xi}^{(1),i}, \qquad \zeta_i = \boldsymbol{\zeta}_i^{(1)}. \tag{3.647}$$

After reduction to the ghost-degree zero component, the Poisson brackets on the space of physical canonical quantities are then given by

$$\{x^i(\sigma), x^j(\sigma')\}_{\mathrm{PB},\widehat{\mathcal{L}}_\alpha} = 0, \tag{3.648}$$

$$\{x^i(\sigma), \xi^j(\sigma')\}_{\mathrm{PB},\widehat{\mathcal{L}}_\alpha} = -\Pi^{ij}(x(\sigma))\delta(\sigma-\sigma'), \tag{3.649}$$

$$\{\xi^i(\sigma), \xi^j(\sigma')\}_{\mathrm{PB},\widehat{\mathcal{L}}_\alpha} = -\left(-\frac{\partial \Pi^{ij}}{\partial x^k}(x(\sigma))\xi^k + R^{ijk}(x(\sigma))\Pi^{-1}_{kl}\partial_\sigma x^l\right)\delta(\sigma-\sigma'). \tag{3.650}$$

We can easily read off the symplectic form that induces the resulting Poisson algebra on the space of physical canonical quantities,

$$\omega = -\int_{S^1} d\sigma \Pi^{-1}_{ij}\delta x^i \wedge \delta\xi^j + \frac{1}{2}\int_{S^1}\left(-\frac{\partial \Pi^{ij}}{\partial x^k}\xi^k + R^{ijk}\Pi^{-1}_{kl}\partial_\sigma x^l\right)\Pi^{-1}_{im}\delta x^m \wedge \Pi^{-1}_{jn}\delta x^n. \tag{3.651}$$

This symplectic form induces the contravariant Poisson algebra with R-flux on the space of physical canonical quantities (x^i, ξ^i). It is the phase space symplectic structure of a closed string traveling in R-space on a Poisson-Courant algebroid background.

3.6.9 Contravariant current algebra

In this section, we discuss the construction of the contravariant R-flux current algebra on the Poisson algebra derived above, based on the Poisson-Courant algebroid with R-flux.

The relevant subspaces of the space of smooth functions on $\mathcal{M}$ are given by

$$\mathcal{C}^\infty_0(\mathcal{M}) \cong \mathcal{C}^\infty(M), \quad \mathcal{C}^\infty_1(\mathcal{M}) \cong TM \oplus T^*M. \tag{3.652}$$

As in the standard Courant algebroid case, we assign to each element of a subspace an object, that will be transgressed to a current on the mapping space,

$$j_{[(0),f]} = f, \quad j_{[(1),X+\alpha]} = X^i\zeta_i + \alpha_i\xi^i, \tag{3.653}$$

where $f, X^i, \alpha_i \in \mathcal{C}^\infty(M)$. The derived brackets between these elements are given by

$$\{\{j_{[(0),f]}, \Theta_R\}, j_{[(0),g]}\} = 0, \tag{3.654}$$

$$\{\{j_{[(1),X+\alpha]}, \Theta_R\}, j_{[(0),g]}\} = -\Pi^\sharp(\alpha)(g), \tag{3.655}$$

$$\{\{j_{[(1),X+\alpha]}, \Theta_R\}, j_{[(1),Y+\beta]}\} = -j_{[(1),[X+\alpha,Y+\beta]^\Pi_{\mathrm{D},R}]}. \tag{3.656}$$

In the next step, we compute the transgression to the mapping space $\mathrm{Map}(\mathring{\chi}, \mathcal{M})$, where $\mathring{\chi} = T[1]S^1$. The supergeometric currents on the superloop space are given by

$$\boldsymbol{J}_{[(0),f]}(\varepsilon_{(1)}) = \mu_* \varepsilon_{(1)} \mathrm{ev}^* j_{[(0),f]} = \int_{\mathring{\chi}} \mu_{\mathring{\chi}} \varepsilon_{(1)} f(\boldsymbol{x}), \tag{3.657}$$

$$\boldsymbol{J}_{[(1),X+\alpha]}(\varepsilon_{(0)}) = \mu_* \varepsilon_{(1)} \mathrm{ev}^* j_{[(0),X+\alpha]} = \int_{\mathring{\chi}} \mu_{\mathring{\chi}} \varepsilon_{(1)} (X^i(\boldsymbol{x}) \boldsymbol{\zeta}_i + \alpha_i(\boldsymbol{x}) \boldsymbol{\xi}^i), \tag{3.658}$$

where we introduced $\varepsilon_{(1)}$ and $\varepsilon_{(0)}$ of degrees 1 and zero. On the transgression of the Lagrangian submanifold defined by $p_i = \zeta_i = 0$, some part of the currents gets projected out. Therefore, we twist $\mathcal{L}$ by the Liouville 1-form as in the previous section leading to the following supercurrents on $\widehat{\mathcal{L}}_\alpha$,

$$\boldsymbol{J}_{[(0),f]}(\varepsilon_{(1)})|_{\widehat{\mathcal{L}}_\alpha} = \mu_* \varepsilon_{(1)} \mathrm{ev}^* j_{[(0),f]}|_{\widehat{\mathcal{L}}_\alpha} = \int_{\mathring{\chi}} \mu_{\mathring{\chi}} \varepsilon_{(1)} f(\boldsymbol{x}), \tag{3.659}$$

$$\boldsymbol{J}_{[(1),X+\alpha]}(\varepsilon_{(0)})|_{\widehat{\mathcal{L}}_\alpha} = \mu_* \varepsilon_{(1)} \mathrm{ev}^* j_{[(0),X+\alpha]}|_{\widehat{\mathcal{L}}_\alpha} = \int_{\mathring{\chi}} \mu_{\mathring{\chi}} \varepsilon_{(1)} (X^i(\boldsymbol{x}) \Pi^{-1}_{ij} \boldsymbol{d}\boldsymbol{x}^j + \alpha_i(\boldsymbol{x}) \boldsymbol{\xi}^i). \tag{3.660}$$

The Poisson algebra of supercurrents on the transgressed twisted Lagrangian submanifold $\widehat{\mathcal{L}}_\alpha$ is then given by

$$\{\boldsymbol{J}_{[(0),f]}(\epsilon), \boldsymbol{J}_{[(0),g]}(\varepsilon')\}_{\mathrm{PB},\widehat{\mathcal{L}}_\alpha} = 0, \tag{3.661}$$

$$\{\boldsymbol{J}_{[(1),X+\alpha]}(\epsilon), \boldsymbol{J}_{[(0),g]}(\varepsilon')\}_{\mathrm{PB},\widehat{\mathcal{L}}_\alpha} = \Pi^\sharp(\alpha) \boldsymbol{J}_{[(0),g]}(\varepsilon\varepsilon'), \tag{3.662}$$

$$\begin{aligned} \{\boldsymbol{J}_{[(1),X+\alpha]}(\epsilon), \boldsymbol{J}_{[(1),Y+\beta]}(\varepsilon')\}_{\mathrm{PB},\widehat{\mathcal{L}}_\alpha} &= \boldsymbol{J}_{[(1),[X+\alpha,Y+\beta]^\Pi_{\mathrm{D},R}]}(\varepsilon\varepsilon') \\ &\quad + \int_{\mathring{\chi}} \mu_{\mathring{\chi}} \boldsymbol{d}\varepsilon_{(0)} \varepsilon'_{(0)} \langle X+\alpha, Y+\beta\rangle(\boldsymbol{x}). \end{aligned} \tag{3.663}$$

Finally, we find the physical contravariant R-flux current algebra by restriction to the ghost-number zero component,

$$\{J_{[(0),f]}(\sigma), J_{[(0),g]}(\sigma')\}_{\mathrm{PB},\widehat{\mathcal{L}}_\alpha} = 0, \tag{3.664}$$

$$\{J_{[(1),X+\alpha]}(\sigma), J_{[(0),g]}(\sigma')\}_{\mathrm{PB},\widehat{\mathcal{L}}_\alpha} = -\Pi^\sharp(\alpha) J_{[(0),g]}(\sigma)\delta(\sigma-\sigma'), \tag{3.665}$$

$$\begin{aligned} \{J_{[(1),X+\alpha]}(\sigma), J_{[(1),Y+\beta]}(\sigma')\}_{\mathrm{PB},\widehat{\mathcal{L}}_\alpha} &= -J_{[(1),[X+\alpha,Y+\beta]^\Pi_{\mathrm{D},R}]}(\sigma)\delta(\sigma-\sigma') \\ &\quad + \langle X+\alpha, Y+\beta\rangle(\sigma')\partial_\sigma \delta(\sigma-\sigma'), \end{aligned} \tag{3.666}$$

where the generalized physical currents are given by

$$J_{[(0),f]}(\sigma) = f(x(\sigma)), \tag{3.667}$$

$$J_{[(1),X+\alpha]}(\sigma) = X^i(x(\sigma))\Pi^{-1}_{ij}\partial_\sigma x^j(\sigma) + \alpha_i(x(\sigma))\xi^i(\sigma). \tag{3.668}$$

We successfully derived the current algebra on the cotangent bundle of the loop space $T^*LM = \mathrm{Map}(S^1, T^*M)$ of a closed string traveling in R-space associated with the Poisson-Courant algebroid. Two currents close on a third, which is computed by the R-twisted Courant bracket of the Poisson-Courant algebroid. The anomalous term is given by the fiber product on the generalized tangent bundle and vanishes on a Dirac structure of the Poisson-Courant algebroid.
This ends the analysis of the Poisson-Courant algebroid and its relevance as a non-geometric background of toroidally compactified string theory. In the remainder of the first part of this thesis, we investigate general duality structures in string theory and M-theory.

3.7 Generalized geometries

In this section, we investigate the local symmetry L_∞-algebras of higher gerbe structures associated with the various generalized geometries, which underly the duality structures of string theory and M-theory in terms of T-duality and U-duality. This section is based on calculations done associated with [4], which is still work in progress. Exceptional generalized structures and B_n generalized geometry have also been studied from the standpoint of Leibniz algebroids in [78] and from the standpoint of dg-manifolds in [79] to which we will compare our construction.
Section 3.7.1 concerns ordinary generalized geometry appearing as the geometry of toroidal closed string compactifications with H-flux and should be considered as a warm-up. Then, in section 3.7.2 we discuss the structure associated with the tensor product of the generalized tangent bundle with a line bundle. If the line bundle is exchanged by an adjoint bundle of a non-abelian Lie algebra, then this case is related to T-duality in heterotic string theory, where there is an additional 1-form gauge field present. In 3.7.3 and 3.7.4, we investigate the higher gerbe structures associated with exceptional generalized tangent bundles appearing, when considering U-duality symmetry in M-theory. The section 3.7.3 concerns exceptional tangent bundles, which can accommodate M2-brane wrapping modes. Then, in section 3.7.3,

exceptional tangent bundles will be discussed, which can accommodate both M2- and M5-brane wrapping modes.

In section 3.3.7, we described abelian n-gerbes in terms of Čech-Deligne cocycles. By inspecting the structure of the total cohomology, in which the respective cocycle is located, one recognizes that the local symmetries of the n-gerbe is encoded in the following L_∞-algebra,

$$\mathcal{C}^\infty(M) \xrightarrow{d} \Omega^1(M) \xrightarrow{d} \Omega^2(M) \xrightarrow{d} \cdots \xrightarrow{d} \Omega^{n-1}(M) \xrightarrow{d} \Gamma(TM \oplus \wedge^n T^*M), \tag{3.669}$$

where M is the smooth manifold over which the n-gerbe is defined. The complex encodes the symmetries-of-symmetries of the associated n-gerbe. Comparing to theorem 3.3.33 in section 3.3.3, we recognize that this constitutes a Lie $(n+1)$-algebra, which is associated with Courant algebroids of degree n.

3.7.1 Generalized geometry

As we described in the preliminary sections, at the heart of generalized geometry lies the exact Courant algebroid, which captures the transformation behavior of the metric and B-field under T-duality. Such a Courant algebroid has a description as QP-manifold of degree 2. The associated semistrict Lie 2-algebra structure is defined on the complex

$$\mathcal{C}^\infty(M) \xrightarrow{d} \Gamma(TM \oplus T^*M) \tag{3.670}$$

and captures the local symmetries of the 1-gerbe with curvature $dB = H$.

3.7.2 B_n-generalized geometry

B_n-generalized geometry was introduced in [78]. It turns out that it is an version of generalized geometry related to T-duality in string theory with Yang-Mills gauge field, e.g. heterotic string theory or type I string theory [120]. We shortly describe the underlying construction. The notion B_n comes from the special orthogonal group $B_n = SO(n+1, n)$, which is reflected by signature of the fiber metric, we define in the following.

Let M be a smooth manifold of dimension n. Let us consider the vector bundle $E = TM \oplus T^*M \oplus 1 \to M$ and endow it with the fiber metric

$$\langle X + \alpha + f, Y + \beta + g\rangle = \iota_X\beta + \iota_Y\alpha + fg, \tag{3.671}$$

where $X, Y \in TM$, $\alpha, \beta \in T^*M$ and $f, g \in \mathcal{C}^\infty(M)$. Furthermore, we define the Dorfman bracket by

$$[X+\alpha+f, Y+\beta+g]_{\mathrm{D}} = [X,Y]_{\mathrm{Lie}} + X(g) - Y(f) + L_X\beta - \iota_Y\alpha + gdf. \tag{3.672}$$

Finally, we have the bundle morphism $\rho : E \to TM$, defined by projection to TM,

$$\rho(X+\alpha+f) = X. \tag{3.673}$$

The 4-tuple $(E, \langle -,-\rangle, \rho, [-,-]_{\mathrm{D}})$ defines a transitive Courant algebroid. The adjoint bundle of the vector bundle E is given by

$$\wedge^2 E = \wedge^2 TM \oplus TM \oplus \mathrm{End}(TM) \oplus T^*M \oplus \wedge^2 T^*M. \tag{3.674}$$

The Lie algebra of endomorphisms of E, that preserve ρ is given by

$$\mathrm{End}(TM) \oplus T^*M \oplus \wedge^2 T^*M, \tag{3.675}$$

where $\mathrm{End}(TM)$ induces diffeomorphisms and T^*M as well as $\wedge^2 T^*M$ induce 1-form and 2-form twists, respectively. The infinitesimal actions by the twists are given by

$$A \triangleright (X+\alpha+f) = \iota_X A - Af, \tag{3.676}$$

$$B \triangleright (X+\alpha+f) = -\iota_X B, \tag{3.677}$$

where $A \in \Omega^1(M)$ and $B \in \Omega^2(M)$. The the Courant algebroid $(E, \langle -,-\rangle, \rho, [-,-]_{\mathrm{D}})$ is invariant under diffeomorphisms, and twists by closed 1-forms and closed 2-forms.

In the next step, we reconstruct the Courant algebroid associated with B_n-generalized geometry using symplectic NQ-manifolds. Let M be a smooth manifold. Furthermore, let $\mathcal{M} = T^*[2](T[1]M \oplus \mathbb{R}[1])$ be a graded manifold over M. We associate local coordinates $(x^i, \xi^i, \tau, \zeta_i, p_i, \overline{\tau})$ of degrees $(0,1,1,1,2,1)$ to $\mathcal{M}$. The coordinates (x^i, ξ^i) parameterize $T[1]M$, while τ parameterizes $\mathbb{R}[1]$. Then, we associate conjugate coordinates $(p_i, \zeta_i, \overline{\tau})$ to (x^i, ξ^i, τ). For now, we simplify the structure by identifying $\tau = \overline{\tau} \equiv \theta$.

Then, we equip $\mathcal{M}$ with the graded symplectic structure

$$\omega = -\delta x^i \wedge \delta p_i + \delta\xi^i \wedge \delta\zeta_i + \frac{1}{2}\delta\theta \wedge \delta\theta. \tag{3.678}$$

Finally, we define the Hamiltonian function by

$$\Theta = \xi^i p_i. \tag{3.679}$$

The associated homological vector field is given by

$$Q = \xi^i \frac{\overrightarrow{\partial}}{\partial x^i} + p_i \frac{\overrightarrow{\partial}}{\partial \zeta_i}. \tag{3.680}$$

The sections of the bundle $E = TM \oplus T^*M \oplus 1$ are given by the degree 1 subspace of $\mathcal{C}^\infty(\mathcal{M})$. We can define the injection map

$$\begin{aligned} j : (TM \oplus T^*M \oplus 1) \oplus TM &\to \mathcal{M}, \\ \left(x^i, \partial_i, dx^i, 1, \frac{\partial}{\partial x^i} \right) &\mapsto (x^i, \zeta_i, \xi^i, \theta, p_i), \end{aligned} \tag{3.681}$$

so that a section is given by the pullback of elements of $\mathcal{C}_1^\infty(\mathcal{M})$ along j,

$$j^*(X^i \zeta_i + \alpha_i \xi^i + f\theta) = X^i \partial_i + \alpha_i dx^i + f \in \Gamma(E). \tag{3.682}$$

The fiber metric is reconstructed by the pullback of the symplectic structure,

$$\begin{aligned} \langle X + \alpha + f, Y + \beta + g \rangle &= j^*\{j_*(X + \alpha + f), j_*(Y + \beta + g)\} \\ &= \iota_X \beta + \iota_Y \alpha + fg. \end{aligned} \tag{3.683}$$

The anchor map $\rho : E \to TM$, is reconstructed via derived bracket,

$$\begin{aligned} \rho(X + \alpha + f)(g) &= j^*\{Qj_*(X + \alpha + f), j_*(g)\} \\ &= X(f). \end{aligned} \tag{3.684}$$

Finally, the Dorfman bracket is induced by

$$\begin{aligned} [X + \alpha + f, Y + \beta + g]_{\mathrm{D}} &= j * \{Qj_*(X + \alpha + f), j_*(Y + \beta + g)\}, \\ &= [X, Y]_{\mathrm{Lie}} + L_X \beta - \iota_Y \alpha + X(g) - Y(f) + gdf. \end{aligned} \tag{3.685}$$

The classical master equation, $Q^2 = 0$, is obviously trivially solved. We are lead to the following theorem.

Theorem 3.7.1 *The QP-manifold $(\mathcal{M}, Q, \omega)$ induces the Courant algebroid associated to B_n-generalized geometry.*

Now, let us investigate the 1-form and 2-form twists from the parabolic subalgebra of the adjoint bundle. These are easily reconstructed by

$$\exp(\delta_A) = \exp(A_i \xi^i \theta), \tag{3.686}$$

$$\exp(\delta_B) = \exp\left(\frac{1}{2} B_{ij} \xi^i \xi^j \right), \tag{3.687}$$

where $A \in \Omega^1(M)$ and $B \in \Omega^2(M)$. The twist is defined as usual via exponential adjoint action on the degree 1 subspace of $\mathcal{C}^\infty(M)$. The non-parabolic part of the adjoint bundle can also be represented in a similar way,

$$\exp(\delta_\Xi) = \exp(\Xi^i \zeta_i \theta), \tag{3.688}$$

$$\exp(\delta_\beta) = \exp\left(\frac{1}{2}\beta^{ij}\zeta_i\zeta_j\right), \tag{3.689}$$

where $\Xi \in \mathfrak{X}^1(M)$ and $\beta \in \mathfrak{X}^2(M)$. This leads to the first order transformation behavior

$$\Xi \triangleright (X + \alpha + f) = \iota_\Xi \alpha - f\Xi, \tag{3.690}$$

$$\beta \triangleright (X + \alpha + f) = \iota_\alpha \beta. \tag{3.691}$$

We summarize our findings in the following theorem.

Theorem 3.7.2 *The adjoint bundle of $E = TM \oplus T^*M \oplus 1$ is represented by twists of the QP-manifold $(\mathcal{M}, Q, \omega)$.*

Let us investigate the degree decomposition of $\mathcal{C}^\infty(\mathcal{M})$ in order to derive the associated Lie 2-algebra. The important subspaces are of degree 0 and 1,

$$\mathcal{C}^\infty_0(\mathcal{M}) \cong \mathcal{C}^\infty(M), \tag{3.692}$$

$$\mathcal{C}^\infty_1(\mathcal{M}) \cong \Gamma(TM \oplus T^*M \oplus 1). \tag{3.693}$$

The associated Lie 2-algebra is then given by the complex

$$\mathcal{C}^\infty(M) \xrightarrow{\mu_1} \Gamma(TM \oplus T^*M \oplus 1), \tag{3.694}$$

where μ_1 is induced by Q and gives the de Rham differential, $\mu_1 = d$. The map μ_2 is reconstructed by the derived bracket and gives

$$\mu_2(X + \alpha + f, Y + \beta + g) = [X + \alpha + f, Y + \beta + g]_{\mathrm{C}}, \tag{3.695}$$

$$\mu_2(X + \alpha + f, h) = \rho(X + \alpha + f)(h) = X(h), \tag{3.696}$$

where $[X + \alpha + f, Y + \beta + g]_{\mathrm{C}}$ is the antisymmetrization of the Dorfman bracket, $X + \alpha + f, Y + \beta + g \in \Gamma(E)$ and $h \in \mathcal{C}^\infty(M)$. Finally, μ_3 defined by

$$\begin{aligned} \mu_3(X + \alpha + f, &Y + \beta + g, Z + \gamma + h) = \\ &\frac{1}{3!}(\{\{Q(X + \alpha + f), Y + \beta + g\}, Z + \gamma + h\} \pm \text{perm.}), \end{aligned} \tag{3.697}$$

gives the Jacobiator of an ordinary Courant algebroid, since the functions do not contribute for this choice of homological vector field. However, for twisted homological functions, there might be a non-trivial contribution. We will investigate twisted homological functions below. First, we summarize our findings in the following theorem.

Theorem 3.7.3 *The QP-manifold $(\mathcal{M}, Q, \omega)$ associated with B_n-generalized geometry induces the structure of a Lie 2-algebra on the complex*

$$\mathcal{C}^\infty(M) \xrightarrow{d} \Gamma(TM \oplus T^*M \oplus 1),$$

with multilinear products defined above.

This Lie 2-algebra encodes the local gauge structure of a metric, 2-form gauge potential and Yang-Mills $U(1)$-gauge field. It corresponds to the symmetry L_∞-algebra of a 0-1-gerbe.
Let us investigate the twisted versions. Introducing curvatures of the $U(1)$-gauge field A and 2-form B-field by $F \in \Omega^2(M)$ and $H \in \Omega^3(M)$, we can define the twisted Hamiltonian function

$$\Theta' = \xi^i p_i + \frac{1}{2} F_{ij} \xi^i \xi^j \theta + \frac{1}{3!} H_{ijk} \xi^i \xi^j \xi^k. \tag{3.698}$$

The classical master equation, $\{\Theta', \Theta'\} = 0$, is equivalent to

$$dF = 0, \tag{3.699}$$

$$dH + F \wedge F = 0. \tag{3.700}$$

The anchor map is not twisted and remains $\rho(X + \alpha + f)(g) = X(g)$. However, the Dorfman bracket is twisted to

$$\begin{aligned}[X + \alpha + f, Y + \beta + g]_{\mathrm{D}} = [X, Y]_{\mathrm{Lie}} + L_X\beta - \iota_Y\alpha + X(g) - Y(f) + gdf \\ + \iota_X\iota_Y F + \iota_X\iota_Y H + g\iota_X F - f\iota_Y F.\end{aligned} \tag{3.701}$$

Here, H and F are the curvatures of the associated 0-1-gerbe and equations (3.699) and (3.700) the consistency conditions on the two cocycles. This is an example of a transitive Courant algebroid of the form [121]

$$E = TM \oplus T^*M \oplus \mathcal{G}, \tag{3.702}$$

where $\mathcal{G} = \ker(\rho)/\ker(\rho)^{\perp}$ is bundle of Lie algebras [120]. Special transitive Courant algebroids are heterotic Courant algebroids [122], which under the existence of a splitting yield a decomposition of the bundle by

$$E = TM \oplus T^*M \oplus \mathfrak{g}_P, \tag{3.703}$$

with bundle map $\rho(X + \alpha + f) = X$ and fiber metric

$$\langle X + \alpha + f, Y + \beta + g \rangle = \iota_X \beta + \iota_Y \alpha + \langle f, g \rangle_{\mathfrak{g}_P}, \tag{3.704}$$

where $\mathfrak{g}_P$ is the adjoint bundle of a G-principal bundle P and $\langle -, - \rangle_{\mathfrak{g}_P}$ is a metric on $\mathfrak{g}_P$. A natural choice would be the Killing metric. Requiring consistency then gives us a relation between the curvature of the 1-gerbe and the first Pontriyagin class of the G-principal bundle. We found the construction in the case of an abelian structure group. The L_∞-algebra, that we derived, encodes the local structure of a 0-1 gerbe with 3-form curvature and abelian Yang-Mills field strength F. It corresponds to generalized geometry with a $U(1)$-gauge field. The lift to the non-abelian construction is work in progress.

3.7.3 Exceptional generalized geometry with M2-branes

Now, let us step into the realm of U-duality in M-theory. The exceptional tangent bundle, which incorporates the modes of M2-branes, is given by

$$E = TM \oplus \wedge^2 T^*M, \tag{3.705}$$

where M is a smooth manifold. This includes the cases E_2, E_3 and E_4. With anchor map, Dorfman bracket and fiber metric it gains the structure of a Courant algebroid of degree 2, $(E, \langle -, - \rangle, \rho, [-, -]_{\mathrm{D}})$. As explained in the previous sections, there is an associated semistrict Lie 3-algebra structure on the complex

$$\mathcal{C}^\infty(M) \xrightarrow{d} \Omega^1(M) \xrightarrow{d} \Gamma(TM \oplus \wedge^2 T^*M). \tag{3.706}$$

We can define a Vinogradov Lie 3-algebroid structure on $\mathcal{M} = T^*[3]T[1]M$ with local coordinates $(x^i, \xi^i, \zeta_i, p_i)$ of degrees $(0, 1, 2, 3)$. The untwisted Hamiltonian is given by

$$\Theta = \xi^i p_i. \tag{3.707}$$

The twisted Hamiltonian is given by

$$\Theta' = \xi^i p_i + \frac{1}{4!} F_{4,ijkl} \xi^i \xi^j \xi^k \xi^l, \tag{3.708}$$

where $F_4 \in \Omega^4(M)$. The classical master equation requires F_4 to be closed, $dF_4 = 0$. There is a natural 3-form twist on this bundle by $A_3 \in \Omega^3(M)$,

$$\exp(\delta_{A_3}) = \exp\left(\frac{1}{3!} A_{3,ijk} \xi^i \xi^j \xi^k\right). \tag{3.709}$$

The parabolic subalgebra of the adjoint bundle acting on E consists of diffeomorphisms and 3-form twists. The associated untwisted Courant bracket,

$$[X+\sigma, Y+\gamma]_{\mathrm{C}} = \frac{1}{2}(j^*\{Qj_*(X+\sigma), j_*(Y+\gamma)\} - j^*\{Qj_*(Y+\gamma), j_*(X+\sigma)\}), \tag{3.710}$$

is invariant under the diffeomorphisms and closed 3-form twists. The resulting Lie 3-algebra encodes the local symmetry structure of a 2-gerbe with curvature $F_4 = dA_3$.

3.7.4 Exceptional generalized geometry with M2- and M5-branes

In this section, we compute the L_∞-algebra of local symmetries associated with the exceptional generalized geometry on exceptional tangent bundles, which accommodate M2-brane as well as M5-brane modes. We start with exceptional tangent bundles, which transform irreducibly under the exceptional group E_5 and E_6. It arises as the gauge group of the Kaluza-Klein compactification of 11-dimensional supergravity on a 5- or 6-torus. In the remainder of this section, we comment on the issues associated with E_7, when trying to fit it into the framework of QP-manifolds.

Let us shortly state the Leibniz algebroid structure of exceptional generalized geometry with E_6-adjoint bundle according to [78]. The adjoint bundle decomposes as

$$E_{\mathrm{adj}} = \wedge^6 TM \oplus \wedge^3 TM \oplus \mathrm{End}(TM) \oplus \wedge^3 T^*M \oplus \wedge^6 T^*M. \tag{3.711}$$

We see that the parabolic subalgebra incorporates 3-form and 6-form twists, which are the potentials associated to M2- and M5-branes. The associated bundle transforms in the **27** of E_{adj}, which decomposes under $GL(6)$ as

$$E = (TM \oplus \wedge^2 T^*M \oplus \wedge^5 T^*M) \otimes (\wedge^6 T^*M)^{-\frac{1}{3}}, \tag{3.712}$$

where the determinant line bundle arises to ensure the triviality of the determinant bundle of E. As described in [78], if M is orientable, then a choice of volume form reduces the bundle to

$$E = TM \oplus \wedge^2 T^* M \oplus \wedge^5 T^* M. \tag{3.713}$$

The Dorfman bracket on E is given by

$$[X + \sigma + \overline{\sigma}, Y + \gamma + \overline{\gamma}]_{\mathrm{D}} = [X, Y] + L_X \gamma - \iota_Y d\sigma + L_X \overline{\gamma} - \iota_Y d\overline{\sigma} + d\sigma \wedge \gamma, \tag{3.714}$$

where $X, Y \in TM$, $\sigma, \gamma \in \Omega^2(M)$ and $\overline{\sigma}, \overline{\gamma} \in \Omega^5(M)$. The anchor map is defined as projection to TM,

$$\rho(X + \sigma + \overline{\sigma}) = X. \tag{3.715}$$

The 3-form and 6-form twists from the parabolic subalgebra of the adjoint bundle act infinitesimally by

$$A_3 \triangleright (X + \sigma + \overline{\sigma}) = \iota_X A_3 - A_3 \wedge \sigma, \tag{3.716}$$

$$A_6 \triangleright (X + \sigma + \overline{\sigma}) = -\iota_X A_6. \tag{3.717}$$

The Dorfman bracket is invariant under diffeomorphism as well as closed 3-form and closed 6-form twists.

Let us now construct the graded symplectic manifold, which derives this structure. Let M be a smooth manifold. The graded manifold $\mathcal{M}$ is defined as $\mathcal{M} = T^*[6](T[1]M \oplus \mathbb{R}[3])$, locally parameterized by coordinates $(x^i, \xi^i, \tau, p_i, \zeta_i, \overline{\tau})$ of degrees $(0, 1, 3, 6, 5, 3)$. We again combine $\tau = \overline{\tau} \equiv \theta$ and define the graded symplectic form as

$$\omega = -\delta x^i \wedge \delta p_i + \delta \xi^i \wedge \delta \zeta_i + \frac{1}{2} \delta\theta \wedge \delta\theta. \tag{3.718}$$

We start with the untwisted Hamiltonian, which we define by

$$\Theta = \xi^i p_i, \tag{3.719}$$

yielding the Q-structure

$$Q = \xi^i \frac{\overrightarrow{\partial}}{\partial x^i} + p_i \frac{\overrightarrow{\partial}}{\partial \zeta_i}. \tag{3.720}$$

In the next step, we define the injection map j by

$$\begin{aligned} &j : (TM \oplus \wedge^2 T^* M \oplus \wedge^5 T^* M) \oplus TM \to \mathcal{M}, \\ &(x^i, \partial_i, dx^i \wedge dx^j, dx^i \wedge dx^j \wedge dx^k \wedge dx^l \wedge dx^m, \frac{\partial}{\partial x^i}) \mapsto (x^i, \zeta_i, \xi^i \xi^j \theta, \xi^i \xi^j \xi^k \xi^l \xi^m, p_i). \end{aligned} \tag{3.721}$$

A section on the bundle $X+\sigma+\overline{\sigma}\in\Gamma(E)$ is then given by the pullback,

$$j^*\left(X^i\zeta_i+\frac{1}{2}\sigma_{ij}\xi^i\xi^j\theta+\frac{1}{5!}\overline{\sigma}_{ijklm}\xi^i\xi^j\xi^k\xi^l\xi^m\right)=X+\sigma+\overline{\sigma}. \tag{3.722}$$

We can induce a fiber metric $\langle -,-\rangle : E\otimes E\to\Omega^1(M)\oplus\Omega^4(M)$ by pullback of the graded symplectic structure,

$$\begin{aligned}\langle X+\sigma+\overline{\sigma},Y+\gamma+\overline{\gamma}\rangle &= j^*\{j_*(X+\sigma+\overline{\sigma}),j_*(Y+\gamma+\overline{\gamma})\}\\ &=\iota_X(\gamma+\overline{\gamma})+\iota_Y(\sigma+\overline{\sigma})+\sigma\wedge\gamma.\end{aligned} \tag{3.723}$$

The anchor map is induced by the derived bracket,

$$\begin{aligned}\rho(X+\sigma+\overline{\sigma})(f) &= j^*\{Qj_*(X+\sigma+\overline{\sigma}),j_*(f)\}\\ &= X(f).\end{aligned} \tag{3.724}$$

The Dorfman bracket is reconstructed by the following derived bracket,

$$\begin{aligned}[X+\sigma+\overline{\sigma},Y+\gamma+\overline{\gamma}]_{\mathrm{D}} &= j^*\{Qj_*(X+\sigma+\overline{\sigma}),j_*(Y+\gamma+\overline{\gamma})\}\\ &=[X,Y]_{\mathrm{Lie}}+L_X\gamma-\iota_Y d\sigma+L_X\overline{\gamma}-\iota_Y d\overline{\gamma}+d\sigma\wedge\gamma.\end{aligned} \tag{3.725}$$

The classical master equation, $Q^2=0$, is trivially solved. The parabolic subalgebra of the adjoint bundle contains 3-form and 6-form twist. On the QP-manifold their action is represented by twist,

$$\exp(\delta_{A_3})=\exp\left(\frac{1}{3!}A_{3,ijk}\xi^i\xi^j\xi^k\theta\right), \tag{3.726}$$

$$\exp(\delta_{A_6})=\exp\left(-\frac{1}{6!}A_{6,ijklmn}\xi^i\xi^j\xi^k\xi^l\xi^m\xi^n\theta\right). \tag{3.727}$$

Via exponential adjoint action on the space of sections, we find the first order transformation behavior, that matches the action of the parabolic subalgebra of the adjoint bundle. We note, that the implementation of the non-parabolic twists, which are associated to M-theoretically non-geometric fluxes, is not obvious in this case, in contrast to the B_n-generalized geometry case.

The Hamiltonian function can be twisted by the curvatures $F_4\in\Omega^4(M)$ and $F_7\in\Omega^7(M)$ of the higher gauge fields A_3 and A_6. It is given by

$$\Theta'=\xi^i p_i+\frac{1}{4!}F_{4,ijkl}\xi^i\xi^j\xi^k\xi^l\theta+\frac{1}{7!}F_{7,ijklmno}\xi^i\xi^j\xi^k\xi^l\xi^m\xi^n\xi^o. \tag{3.728}$$

The classical master equation, $\{\Theta', \Theta'\} = 0$, induces the relations between the curvatures,

$$dF_4 = 0, \tag{3.729}$$

$$dF_7 + \frac{1}{2} F_4 \wedge F_4 = 0. \tag{3.730}$$

By twist of the Hamiltonian Θ by 3- and 6-form we find the local symmetries of the curvatures. Twist by A_3 induces $F_4 = dA_3$ and $F_7 = -\frac{1}{2} dA_3 \wedge A_3$, whereas twist by A_6 induces $F_7 = dA_6$. The fiber metric as well as the anchor map are invariant under the twists. However, the Dorfman bracket changes,

$$\begin{aligned} [X + \sigma + \overline{\sigma}, Y + \gamma + \overline{\gamma}]_{\mathrm{D}} &= j^* \{Q' j_*(X + \sigma + \overline{\sigma}), j_*(Y + \gamma + \overline{\gamma})\} \\ &= [X, Y]_{\mathrm{Lie}} + L_X \gamma - \iota_Y d\sigma + L_X \overline{\gamma} - \iota_Y d\overline{\gamma} + d\sigma \wedge \gamma \\ &\quad + \iota_X \iota_Y F_4 + \iota_X \iota_Y F_7 + \iota_X F_4 \wedge \gamma. \end{aligned} \tag{3.731}$$

Let us summarize our findings in the following theorem.

Theorem 3.7.4 *The QP-manifold $(\mathcal{M}, Q, \omega)$ and its twisted version $(\mathcal{M}, Q', \omega)$ induces the local structure of E_6-generalized geometry. The parabolic subalgebra of the adjoint bundle is representable via exponential adjoint action on the QP-manifold.*

Let us now compute the associated L_∞-algebra of local symmetries of the associated 2-5-gerbe. This will be an L_∞-algebra concentrated in degrees $\{0, 1, 2, 3, 4, 5\}$, or semistrict Lie 6-algebra. The relevant degree subspaces $\mathcal{C}^\infty(\mathcal{M})$ are given by

$$\mathcal{C}^\infty_0(\mathcal{M}) \cong \mathcal{C}^\infty(M), \tag{3.732}$$

$$\mathcal{C}^\infty_1(\mathcal{M}) \cong \Omega^1(M), \tag{3.733}$$

$$\mathcal{C}^\infty_2(\mathcal{M}) \cong \Omega^2(M), \tag{3.734}$$

$$\mathcal{C}^\infty_3(\mathcal{M}) \cong \Omega^3(M) \oplus \mathcal{C}^\infty(M), \tag{3.735}$$

$$\mathcal{C}^\infty_4(\mathcal{M}) \cong \Omega^4(M) \oplus \Omega^1(M), \tag{3.736}$$

$$\mathcal{C}^\infty_5(\mathcal{M}) \cong \mathfrak{X}^1(M) \oplus \Omega^5(M) \oplus \Omega^2(M). \tag{3.737}$$

The Lie 6-algebra structure is defined on the following complex,

$$\begin{aligned} L(\mathcal{M}) = \Big(&\mathcal{C}^\infty(M) \xrightarrow{\mu_1} \Omega^1(M) \xrightarrow{\mu_1} \Omega^2(M) \xrightarrow{\mu_1} \Omega^3(M) \oplus \mathcal{C}^\infty(M) \\ &\xrightarrow{\mu_1} \Omega^4(M) \oplus \Omega^1(M) \xrightarrow{\mu_1} \mathfrak{X}^1(M) \oplus \Omega^5(M) \oplus \Omega^2(M) \Big). \end{aligned} \tag{3.738}$$

The unary map is induced by the untwisted homological vector field Q via

$$\mu_1(\alpha) = j^* \circ Q \circ j_*(\alpha) = d\alpha, \tag{3.739}$$

where $\alpha \in L(\mathcal{M})$. We recognize, that it acts as de Rham differential d. In the case of the twisted homological vector field Q', μ_1 becomes the F_4-twisted de Rham differential $d + F_4\wedge$ on the lower form degree component in $\mathcal{C}_3^\infty(\mathcal{M})$ and $\mathcal{C}_4^\infty(\mathcal{M})$. As required by the homotopy Jacobi identities, the de Rham and the F_4-twisted de Rham differential are nilpotent since $dF_4 = 0$ by the classical master equation.

Let us give the results for some of the maps μ_2 in the case of the untwisted homological vector field. The action of the bundle $\Gamma(E)$ onto the lower subspaces is given by

$$\mu_2 : \Gamma(E) \oplus \mathcal{C}^\infty(M) \mapsto \mathcal{C}^\infty(M), \quad \mu_2(X + \sigma + \overline{\sigma}, f) \sim L_X f, \tag{3.740}$$

$$\mu_2 : \Gamma(E) \oplus \Omega^1(M) \mapsto \Omega^1(M), \quad \mu_2(X + \sigma + \overline{\sigma}, \omega_1) \sim L_X \omega_1, \tag{3.741}$$

$$\mu_2 : \Gamma(E) \oplus \Omega^2(M) \mapsto \Omega^2(M), \quad \mu_2(X + \sigma + \overline{\sigma}, \omega_2) \sim L_X \omega_2, \tag{3.742}$$

$$\begin{aligned} \mu_2 : \Gamma(E) \oplus \Omega^3(M) \oplus \mathcal{C}^\infty(M) &\mapsto \Omega^3(M) \oplus \mathcal{C}^\infty(M), \\ \mu_2(X + \sigma + \overline{\sigma}, \omega_3 + f) &\sim L_X \omega_3 + L_X f + d\sigma \wedge f, \end{aligned} \tag{3.743}$$

$$\begin{aligned} \mu_2 : \Gamma(E) \oplus \Omega^4(M) \oplus \Omega^3(M) &\mapsto \Omega^4(M) \oplus \Omega^3(M), \\ \mu_2(X + \sigma + \overline{\sigma}, \omega_4 + \omega_1) &\sim L_X \omega_4 + L_X \omega_1 - d\sigma \wedge \omega_1, \end{aligned} \tag{3.744}$$

and the action of the bundle onto itself is given by the exceptional Courant bracket,

$$\mu_2 : \Gamma(E) \wedge \Gamma(E) \mapsto \Gamma(E), \quad \mu_2(X + \sigma + \overline{\sigma}, Y + \gamma + \overline{\gamma}) = [X + \sigma + \overline{\sigma}, Y + \gamma + \overline{\gamma}]_{\mathrm{C}}, \tag{3.745}$$

where

$$\begin{aligned} [X + \sigma + \overline{\sigma}, Y + \gamma + \overline{\gamma}]_{\mathrm{C}} &= \frac{1}{2}([X + \sigma + \overline{\sigma}, Y + \gamma + \overline{\gamma}]_{\mathrm{D}} - [Y + \gamma + \overline{\gamma}, X + \sigma + \overline{\sigma}]_{\mathrm{D}}) \\ &= [X, Y]_{\mathrm{Lie}} + L_X(\gamma + \overline{\gamma}) - L_Y(\sigma + \overline{\sigma}) - \frac{1}{2} d(\iota_X(\gamma + \overline{\gamma}) \\ &\quad - \iota_Y(\sigma + \overline{\sigma})) + \frac{1}{2}(d\sigma \wedge \gamma - d\gamma \wedge \sigma). \end{aligned} \tag{3.746}$$

The resulting semistrict Lie 6-algebra encodes the local symmetries of the 2-5-gerbe with curvatures F_4 and F_7, which underlies exceptional generalized geometry that accommodates modes of M2-branes as well as M5-branes.

Comment on $E_{7(7)}$

The case of exceptional generalized geometry with $E_{7(7)}$-structure was investigated in [78] from the algebraic perspective. The generalized tangent bundle is given by

$$E = TM \oplus \wedge^2 T^*M \oplus \wedge^5 T^*M \oplus (\wedge^7 T^*M \otimes T^*M), \tag{3.747}$$

where M is a 7-dimensional smooth manifold. The $(\wedge^7 T^*M \otimes T^*M)$-part arises due to the duals of diffeomorphism vectors of KK-monopoles. The adjoint bundle is given by

$$E = \wedge^6 TM \oplus \wedge^3 TM \oplus \mathrm{End}(TM) \oplus \wedge^3 T^*M \oplus \wedge^6 T^*M. \tag{3.748}$$

The 3- and 6-form twists act as follows on the sections of the bundle,

$$A_3 \rhd (X + \sigma + \overline{\sigma} + u) = \iota_X A_3 - A_3 \wedge \sigma + A_3 \diamond \overline{\sigma}, \tag{3.749}$$

$$A_6 \rhd (X + \sigma + \overline{\sigma} + u) = -\iota_X A_6 + A_6 \diamond \sigma. \tag{3.750}$$

The diamond operation $\diamond$ is defined by $\diamond : \wedge^k T^*M \otimes \wedge^{8-k} T^*M \to \wedge^7 T^*M \otimes T^*M$ so that

$$(\alpha \diamond \beta)(X) = \iota_X \alpha \wedge \beta. \tag{3.751}$$

The untwisted Dorfman bracket is given by

$$\begin{aligned}[X + \sigma + \overline{\sigma} + u, Y + \gamma + \overline{\gamma}]_{\mathrm{D}} = [X, Y]_{\mathrm{Lie}} &+ L_X \gamma - \iota_Y d\sigma + L_X \overline{\gamma} - \iota_Y d\overline{\sigma} \\ &+ d\sigma \wedge \gamma + L_X v - d\sigma \diamond \overline{\gamma} + d\overline{\sigma} \diamond \gamma.\end{aligned} \tag{3.752}$$

It can be twisted by the field strengths $F_4 \in \Omega^4(M)$ and $F_7 \in \Omega^7(M)$ associated with both gauge potentials, giving the twisted Dorfman bracket

$$\begin{aligned}[X + \sigma + \overline{\sigma} + u, Y + \gamma + \overline{\gamma}]_{\mathrm{D}} = [X, Y]_{\mathrm{Lie}} &+ L_X \gamma - \iota_Y d\sigma + L_X \overline{\gamma} - \iota_Y d\overline{\sigma} \\ &+ d\sigma \wedge \gamma + L_X v - d\sigma \diamond \overline{\gamma} + d\overline{\sigma} \diamond \gamma + \iota_X \iota_Y F_4 \\ &+ \iota_X \iota_Y F_7 + \iota_X F_4 \wedge \gamma - (\iota_X F_4) \diamond \overline{\gamma} + (\iota_X F_7) \diamond \gamma.\end{aligned} \tag{3.753}$$

The antisymmetrization of the Dorfman bracket is called the exceptional Courant bracket. Due to the $(\wedge^7 T^*M \otimes T^*M)$-component in the bundle structure and the resulting diamond product, a supergeometric reconstruction is problematic. A generalization of the approach becomes necessary as we will discuss below.

3.8 Summary

Let us summarize the main results of the first chapter of this thesis. We started by recalling the graded symplectic manifold setup of the standard Courant algebroid with H-flux and encountered insufficiencies of the ansatz when naïvely trying to insert other fluxes by hand.

After investigation of the twists that are available we constructed the β-twisted Courant algebroid as an example for a non-geometric background with non-trivial Q- and R-fluxes. We analyzed its consistency conditions, which implied the flux Bianchi identities, and found that the associated cohomology is the total cohomology of the Poisson-de Rham double complex. The β-twisted Courant algebroid encodes the local symmetry structure of a non-geometric background with Q- and R-flux sourced by β-potential.

Motivated by this result, we extended the symplectic NQ-manifold by a generalized frame bundle, that brought us into the position to generate metric f-flux. After that, we constructed the f-twisted Courant algebroid of a nilmanifold background.

In the next step, we combined our knowledge about the H-twisted, β-twisted and f-twisted Courant algebroids to construct the fully twisted Courant algebroid, which encodes the local symmetry structure of a background, which is sources by geometric H-, F- as well as non-geometric Q- and R-fluxes in a consistent manner. The consistency condition of the twisted Courant algebroid naturally induces the flux Bianchi identities among all fluxes. The reduction of the twisted Courant algebroid along its twisted anchor for integer fluxes yields the general form of the non-abelian gauge algebra of gauged supergravities. The classical master equation is induces the closure condition of the gauge algebra.

Then, we lifted the whole construction to the double space of double field theory. We prepared the underlying graded symplectic manifold that induces local symmetries of double field theory. The associated graded symplectic manifold does not trivially solve the strong constraint, but it turned out to be necessary to consider half-rank projections of the double field theory QP-manifold. These half-rank projections are associated to Courant algebroids on T-duality frames. We conducted an investigation of possible twists of the underlying setup and extended it by the introduction of a double generalized frame bundle. This brought us into the position to induce geometric flux contributions not only on the supergravity frame, but also on the winding frame. After application of a succession of twists, we derived the local expressions of all H-, F-, Q- and R-fluxes in terms of their potentials B, β and vielbein in double space. We concluded, that any half-rank projection of the twisted setup that solves the section condition leads to a twisted Courant algebroid on the respective T-duality frame. On each T-duality frame, there lives a twisted Courant algebroid that realizes the background fluxes in a consistent manner and encodes the local symmetry structure of that background. We computed two examples in terms of the winding Courant algebroid with fluxes and the non-geometric Courant algebroid in the supergravity frame. Finally, we constructed a presentation of T-duality in terms of maps between graded symplectic manifolds and could reconstruct well-known examples like the T-duality chain on 3-torus backgrounds. In our definition, T-duality is a discrete map between Hamiltonian functions that encode the properties of a respective T-dual background.

In the next step, we analyzed the Poisson-Courant algebroid as a model for non-geometric R-flux backgrounds. Our first result was the reconstruction of the Poisson-Courant algebroid with R-flux in the supergeometric setting. Then, we investigated the Poisson-Courant algebroid cohomology and standard Courant algebroid cohomology on distinct subspaces. We found an isomorphism between the Poisson-Courant algebroid cohomology and Poisson-Lichnerowicz cohomology and between standard Courant algebroid cohomology and de Rham cohomology under certain conditions. Then, we constructed a duality transformation, called flux duality, between the QP-manifolds of the H-twisted standard Courant algebroid and R-twisted Poisson-Courant algebroid and lifted it to an isomorphism of Courant algebroid cohomologies. We found the Poisson-Courant algebroid as winding frame reduction from double field theory, if the Poisson tensor is constant. For non-constant Poisson tensor, the winding frame is deformed by a linear contravariant connection. Then, we computed the Poisson-Courant sigma model of a membrane traveling in R-twisted Poisson-Courant algebroid target space by using the AKSZ procedure. After that, we derived string sigma model with R-flux on the boundary of the membrane, which is equivalent to the Poisson sigma model with Wess-Zumino term. We analyzed the flux duality on the level of Courant sigma models and came to the conclusion, that the Poisson-Courant sigma model and the standard Courant sigma model are equivalent theories, that realize dual boundary conditions. Then, we computed the Poisson algebra and the current algebra associated with the Poisson-Courant algebroid with R-flux. We found an Alekseev-Strobl type generalized current algebra, which realizes Poisson-Courant algebroid symmetries and is anomaly-free on a Dirac structure of the Poisson-Courant algebroid generalized tangent bundle.

In the final section, we reconsidered various generalized geometries associated with T-duality geometries of closed string compactifications and heterotic compactifications as well as U-duality emerging from toroidal compactifications of 11-dimensional supergravity. We investigated the local symmetry L_∞-algebras associated to the higher gerbes appearing in the formulation of these (exceptional) generalized geometries and B_n-geometries using graded symplectic manifold techniques.

Chapter 4

Higher gauge theory and multiple M5-branes

We believe that we know something about the things themselves when we speak of trees, colors, snow, and flowers; and yet we possess nothing but metaphors for things — metaphors which correspond in no way to the original entities.
– Friedrich Nietzsche, *Über Wahrheit und Lüge im außermoralischen Sinn* (1873)

4.1 Introduction

This chapter constitutes the second part of this thesis. We started by investigating the intriguing mysteries surrounding T-duality and non-geometric backgrounds in string theory. Then, we took a step back and investigated the local symmetry L_∞-algebras of higher gerbe structures of generalized geometry with $U(1)$-gauge field and exceptional tangent bundles in M-theory. The exceptional tangent bundles encode the symmetry structures of toroidally compactified 11-dimensional supergravity and therefore the wrapping modes of M2- and M5-branes as well as KK6-modes.
For this chapter, we will stay in the realm of M-theory and devote ourselves to the thorough investigation of the system of multiple M5-branes as a higher gauge theory, from the viewpoint of higher categorification. The close relation between higher categorification, L_∞-algebras and higher gerbe structures will become very clear in the preliminary sections. This chapter is based on the published paper [5].
In contrast to the system of multiple M2-branes, the analysis of systems of multiple M5-branes is still a field which is not very well understood. What is known is that M5-branes

interact by M2-branes, which extend between them. Then, the 1-dimensional boundaries of the M2-branes turn out to be soliton solutions of a 6-dimensional superconformal $\mathcal{N} = (2,0)$ theory [123, 124]. Since these solutions are charged under a self-dual 2-form gauge field, they are also called *self-dual strings* [125]. For the case of a single M5-brane, the dynamics of the self-dual strings is governed by a so-called principal 2-bundle with with abelian gerbe structure. The dynamics for multiple M5-branes is still unclear. However, it is believed that the governing structure is a non-abelianization of the principal 2-bundle, or a non-abelian gerbe.

The governing structure of the self-dual strings should incorporate a notion of parallel transport of 1-dimensional objects. In ordinary Yang-Mills theory there is a notion of Wilson line, or holonomy of point-objects, or ordinary parallel transport. Exactly the generalization of ordinary parallel transport and the consistent definition of associated higher principal bundles is provided by the young mathematical field of *higher gauge theory*. Since it highly relies on methods of *higher categorification*, we provide a compact introduction to category theory and the derivation of 2-form higher gauge theories [126, 127, 128]. Using the viewpoint of higher categorification, a notion of parallel transport of 1-dimensional objects has been defined in [126]. It turned out that the underlying higher gauge structure is given by a so-called *differential crossed module*. This defines a non-abelian gerbe with 3-form curvature H, which is believed to be the appropriate framework for the non-abelianization of the $\mathcal{N} = (2,0)$ theory governing the dynamics of multiple M5-branes [129, 130, 131].

The resulting 2-form higher gauge theory associated with the differential crossed module incorporates the usual Yang-Mills 1-form gauge field A with 2-form curvature F and a 2-form gauge field B with 3-form curvature H. However, it turns out that it is a generic feature of higher gauge theories, that the consistent definition of a higher Wilson volume, or higher parallel transport, requires all lower form curvatures but the highest form one to vanish. For the 2-form higher gauge theory, it turns out that the covariant gauge transformation of H requires $F = 0$. This is called *fake curvature condition*. This requirement highly restricts the dynamics of possible Lagrangian theories. If we want to construct a Lagrangian with non-abelian gauge symmetry of 2-form gauge fields, then the fake curvature condition restricts the construction to a BF-type topological field theory leading to an essentially free theory [126].

In this chapter, we propose a method, called **off-shell covariantization**, to circumvent the fake curvature condition towards a non-topologically interacting theory with non-abelian gerbe structure. The method makes use of the prescription to construct higher gauge theories from QP-manifold structures due to [132]. Off-shell covariantization introduces auxiliary gauge freedom to the initial higher gauge theory of the non-abelian gerbe and constrains the resulting system such that the residual gauge transformations lead to a non-abelian gerbe structure, which does not inherit the fake curvature issue. We perform a detailed calculation, which leads to a theory closely related to a system of multiple M5-branes compactified on a circle [133, 134, 135]. For the investigation of off-shell gauge symmetries in the Poisson sigma model see [136].

This chapter is structured as follows. In section 4.2, we provide an introduction to M-branes, especially the recent development regarding M5-branes. Section 4.3 concerns the introduction to category theory and higher gauge theory. Furthermore, we provide a description of the method to generate higher gauge theories from graded symplectic manifolds. We refer to the preliminary section *graded manifolds* of the first part of this thesis for a thorough introduction to the underlying mathematical framework. Section 4.4 constitutes the main analysis. We describe the method of off-shell covariantization and apply it successfully to a 2-form higher gauge theory. The result is related to a system of multiple M5-branes compactified on a circle. In section 4.5 we provide a final summary.

4.2 Physical preliminaries: M-branes

This section provides an introduction to M-theory branes and their relation to higher gauge theory. Excellent reviews on M-branes are [137, 138]. The relation of M-branes with higher gauge theory is pointed out and reviewed in [139].

The stable objects in M-theory are M2-branes and their magnetic duals, the M5-branes. They are half-BPS states of 11-dimensional supergravity. The M2-brane is electrically charged under the 3-form potential C_3, whereas the M5-brane is charged under its dual, C_6. The dual of the field strength F_4 is given by

$$F_7 = \star F_4 - \frac{1}{2} C_3 \wedge F_4. \tag{4.1}$$

The magnetic charge is defined by

$$Q_M = \int_{S^4} F_4, \tag{4.2}$$

where the 4-sphere encloses the M5-brane. The electric charge

$$Q_E = \int_{S^4} F_7, \tag{4.3}$$

is assigned to the M2-brane.

Under dimensional reduction of 11-dimensional supergravity, the M5-brane yields the D4- and NS5-branes in type II supergravity. The M2-brane corresponds to fundamental strings in type IIA supergravity and the 3-form potential C_3 becomes the Kalb-Ramond B-field. As such we can see the membrane as the M-theoretic analogue to the string in superstring theory. All branes from type IIA string theory can be seen as reductions from branes of M-theory.

The actions of a single M2-brane [140] and of a single M5-brane [141] have already been studied in the old days. However, progress on systems of multiple M2- and M5-branes was made recently. The research of the system of multiple M2-branes is substantially more developed than the research on the system of multiple M5-branes. In the following, we will shortly state the recent development on M2-brane actions and then turn to the many difficulties faced when trying to understand M5-brane systems.

The system of 2 coincident M2-branes is described by the so-called BLG model [142, 143]. It is a superconformal gauge theory with $SU(2) \times SU(2)$ Chern-Simons-matter term in 3 dimensions and is based on a 3-algebra structure. This structure should not be confused with a Lie 3-algebra in the L_∞-algebra sense. The BLG model is conformally invariant, has 16 supersymmetries and $SO(8)$ R-symmetry. Furthermore, it can explain the $N^{\frac{3}{2}}$ scaling of the entropy of N coincident M2-branes.

The ABJM model [144] is a Chern-Simons theory with matter coupling, which generalizes the BLG model. It is supposed to describe the worldvolume theory for N coincident flat M2-branes with $\mathbb{Z}_k$-orbifold singularity and reduces to the BLG model for $N = 2$. In contrast to the BLG model, supersymmetry is reduced to $\mathcal{N} = 6$ due to orbifolding.

Now let us turn to the M5-brane. The worldvolume theory is supposed to be a 6-dimensional superconformal field theory. It breaks the Lorentz symmetry via $SO(1,10) \to SO(1,5) \times SO(5)$ and has $SO(5)$ R-symmetry as well as $\mathcal{N} = (2,0)$ supersymmetry. This theory can also be called $(2,0)$-theory for short. Breaking of translation invariance and supersymmetry leads to 5 scalars and fermions that together with a self-dual 3-form field strength form a supermultiplet with 16 supersymmetries. The bosonic field content of an M5-brane theory is given by a 2-form gauge potential of a self-dual 3-form curvature and scalars, that parameterize the fluctations of the M5-brane in the transverse directions. The system forms a self-dual $U(1)$-gerbe in the free or abelian case, which is associated with the dynamics of a single M5-brane. The non-abelianization of the theory is believed to yield a theory of multiple interacting M5-branes. Since it is superconformal, it does not contain dimensionful parameters. However, the fact that the theory contains a self-dual gauge field raises many difficulties in writing down a Lagrangian [145, 146, 147, 148].

The problem of constructing the action of a self-dual $U(1)$-gerbe with 3-form curvature H in 6 dimensions can be observed [131], when investigating the natural action

$$S = \int_{M_6} H \wedge \star H, \tag{4.4}$$

which is zero due to the self-duality of H. Therefore, it is believed, that the system of a self-dual gerbe is a non-Lagrangian theory and that self-dual gauge theories cannot be formulated in a manifestly Lorentz-invariant manner without auxiliary fields. However, the Bianchi identity and the self-duality equation of the curvature is classically as well as on quantum level conformally invariant.

The dynamics of multiple M5-branes is still unknown. However, it is believed, that the non-abelian generalization of the $(2,0)$-theory can be associated with the dynamics of such a system. The resulting theory would go beyond ordinary Yang-Mills theory. A further hint towards the necessity of a generalization is that the entropy of N coincident M5-branes scales as N^3, which cannot be achieved by ordinary Yang-Mills theory scaling as N^2.

Two M5-branes interact by open M2-branes, whose boundaries are in the M5-brane worldvolume. The boundaries become closed strings. In the zero distance limit, the closed strings in the M5-brane worldvolume become charged under the self-dual 2-form gauge field B yielding so-called *self-dual strings*. Self-dual strings are tension-less and arise as soliton solutions of the $(2,0)$-theory. As described before, in the case of a single M5-brane, the gauge sector of the theory is governed by a $U(1)$-gerbe. Although the gauge sector of multiple M5-branes is expected to be governed by a non-abelianization of that gerbe structure, the resulting mathematical structure is unclear. Non-abelian gerbes are associated with categorified principal bundles. Categorified principal bundles encode a notion of parallel transport of higher-dimensional objects, in this case, the parallel transport of self-dual strings. As we will describe below, a higher parallel transport requires the existence of higher form gauge fields.

In the case of the self-dual string, it would be a self-dual 2-form gauge field. In general, gauge theories of higher gauge fields that are constructed via categorification of objects in ordinary gauge theory are called higher gauge theories [126, 127, 128].

The setup of a D4-brane on which a fundamental string ends lifts to an M-theory configuration of an M2-brane ending on an M5-brane. Open strings can end on D-branes. In the case of a single D-brane, the endpoints of open strings induce an abelian 1-form gauge field on the worldvolume of the D-brane. In the effective description, this leads to a gauge theory on the D-brane worldvolume. In the case of a stack of N D-branes, the gauge theory becomes non-abelian with gauge group $U(N)$. When an open M2-brane ends on a M5-brane, then the boundary in the M5-brane worldvolume becomes a self-dual string and is subject to an abelian higher gauge theory.

Let us discuss the setup of a stack of D2-branes ending on a D4-brane in flat Minkowski space. Let the D4-branes be extended in directions x^1, x^2, x^3, x^5 and the D2-branes in directions x^5 and x^9 while ending on the D4-brane in x^9-direction. The setup is summarized in the following table.

Brane	0	1	2	3	4	5	6	7	8	9
D4	×	×	×	×		×				
D2	×					×				⊥

From the perspective of the D4-brane this yields the *Bogomol'nyi monopole equation*,

$$F_{\mu\nu} = \varepsilon_{\mu\nu\rho} \nabla_\rho \Phi, \tag{4.5}$$

where $\mu, \nu, \rho = 1, 2, 3$ and F is the curvature of a $U(N)$-principal bundle. The scalar field Φ is in the adjoint representation and describes the D4-position in x^9-direction. The situation is equivalently described from the perspective of the D2-branes by the *Nahm equation*,

$$\nabla_{x^9} X^\mu = \frac{1}{2} \varepsilon^{\mu\nu\rho} [X^\nu, X^\rho], \tag{4.6}$$

where the scalar fields X^μ describe the D2-position in the x^1, x^2, x^3-directions. The equivalence between both equations is given by the *Nahm transform*. A solution of the Nahm equation is given by

$$X^\mu = \frac{1}{x^9} T^\mu, \quad T^\mu = \varepsilon^{\mu\nu\rho} [T^\nu, T^\rho], \tag{4.7}$$

which has the form of a fuzzy funnel. As one approaches the D4-brane, the fuzzy funnel becomes a fuzzy 2-sphere with coordinates T^μ in $SU(2)$-representation.

Let us now describe the analogous setup with a single and multiple M2-branes ending on a M5-brane, investigated in [149] and [125]. Let the spatial part of the M5-brane worldvolume be extended in the directions $x^1, \dots, x^5$, with the self-dual string being extended in x^5-direction. The M2-branes are taken to be extended in x^5- and x^{10}-directions so that they end at $x^{10} = 0$ on the M5-brane. The setup is summarized in the following table.

Brane	0	1	2	3	4	5	6	7	8	9	10
M5	×	×	×	×	×	×					
M2	×					×					⊥

Investigation of this system from the perspective of the worldvolume theory of a single M2-brane ending on an M5-brane has been conducted in [125] and lead to the *self-dual string equation*,

$$H_{\mu\nu\rho}(r) = \varepsilon_{\mu\nu\rho\sigma}\partial_\sigma \Phi(r), \tag{4.8}$$

where the Greek indices run over $1, \dots, 4$. Here, H is the self-dual 2-cuvature of the abelian 1-gerbe, and r is the radial coordinate given by

$$r^2 = (x^1)^2 + (x^2)^2 + (x^3)^2 + (x^4)^2 \tag{4.9}$$

parameterizes the space transverse to the self-dual string inside the M5-brane worldvolume. The scalar Φ is given by one of the coordinates transverse to the M5-brane. The solution, called the *self-dual string soliton*, is half-BPS and is given by

$$\Phi = \Phi_0 + \frac{2Q}{r^2}, \tag{4.10}$$

where Q is the charge of the self-dual string. The non-abelianization of the self-dual string equation towards a theory of multiple M5-branes is one of the goals which to achieve by higher gauge theory.

In [149], the authors propose an analogue equation to the Nahm equation relying on a 3-algebra structure. This equation, called the *Basu-Harvey equation*, is given by

$$\frac{dX^\mu}{dx^{10}} = \frac{1}{3!}\varepsilon^{\mu\nu\rho\sigma}[X^\nu, X^\rho, X^\sigma], \tag{4.11}$$

where the Greek indices run over $1, \dots, 4$ and denote the directions transverse to the M2-branes. It is supposed to describe the M5-brane from the perspective of the worldvolume theory of multiple M2-branes. The fields take values in a 3-Lie algebra. It has been shown that the solutions of the Basu-Harvey equation form fuzzy 3-spheres,

$$X^\mu(x^{10}) = \frac{T^\mu}{\sqrt{2x^{10}}}, \quad T^\mu = \varepsilon^{\mu\nu\rho\sigma}[T^\nu, T^\rho, T^\sigma]. \tag{4.12}$$

With smaller distance to the M5-brane, the sphere radia diverge. Since this equation is a proposal, a consistent derivation is missing. It is believed that these equations are related to the quantization of a 3-sphere, which remains an open problem until today. Contrary to the relation of the Bogomol'nyi equation with the Nahm equation by the Nahm transform in the case of D-branes, there is no known relation between the solutions of the self-dual string equation and Basu-Harvey equation, yet.

In [150, 151] it was shown that in the case of the BLG model for a stack of infinite M2-branes with infinite-dimensional 3-Lie algebra, a description of a single M5-brane in a large constant C-field background emerges [152]. It introduces a non-abelian self-dual gauge theory based on a Nambu-Poisson structure. The Nambu bracket associated with the Nambu-Poisson structure is associated with the 3-Lie algebra of the BLG model and defined on a 3-dimensional manifold, which is treated as the internal space of the M2-brane. It turns out, that it can also be associated with a subspace of the emerging M5-brane worldvolume. See also [153] for the construction of an action with non-abelian 2-form in 6 dimensions as a proposal for the system of multiple M5-branes in flat space.

In [133, 134, 133] the theory of an S^1-compactified system of M5-branes has been considered. The model does not contain matter coupling and is free of supersymmetry. It turns out that in the zero radius limit, $R \to 0$, the M5-branes become D4-branes by double-dimensional reduction and the system reduces to super Yang-Mills theory in 5 dimensions. Furthermore, this model possesses the structure of a non-abelian gerbe with self-dual 2-form potential in 6 dimensions.

Since our construction will yield a 2-form higher gauge theory, which is closely related to this system, we shortly describe this theory according to [134]. Let $M = \mathbb{R}^5 \times S^1$ be the worldvolume of the M5-brane locally parameterized by coordinates x^M, where $M = 0, \ldots, 5$. Let R be the radius of the circle. The S^1-direction is parameterized by x^5. The local coordinates of the component $\mathbb{R}^5$ are denoted by x^μ. The 2-form gauge field B_{MN} and its 1-form gauge parameter μ_M take values in $\mathrm{Lie}(U(N))$ in a flat background of N coincident M5-branes. The 0-form gauge parameter ε is defined as the zero mode of μ,

$$\varepsilon = \int_{S^1} dx^5 \mu_5 = 2\pi R \mu_5^{(0)}, \tag{4.13}$$

so that $\varepsilon \in \Omega^0(\mathbb{R}^5)$. The zero-mode of $B_{\mu 5}$ is identified with the 1-form gauge potential

$A \in \Omega^1(\mathbb{R}^5)$,

$$A_\mu = \int_{S^1} dx^5 B_{\mu 5} = 2\pi R B^{(0)}_{\mu 5}. \tag{4.14}$$

The covariant derivative and associated field strength is given by

$$D_\mu = \partial_\mu + A_\mu, \quad F_{\mu\nu} = [D_\mu, D_\nu]. \tag{4.15}$$

The non-abelian gauge transformations of the 2-form gauge potential are defined by [134]

$$\delta B_{\mu 5} = [D_\mu, \mu_5] - \partial_5 \mu_\mu + [B^{(\mathrm{KK})}_{\mu 5}, \varepsilon], \tag{4.16}$$

$$\delta B_{\mu\nu} = [D_\mu, \mu_\nu] - [D_\nu, \lambda_\mu] + [B_{\mu\nu}, \varepsilon] - [F_{\mu\nu}, \partial_5^{-1} \mu_5^{(KK)}]. \tag{4.17}$$

The 3-form field strength is defined by

$$H_{\mu\nu 5} = \frac{1}{2\pi R} F_{\mu\nu} + \partial_5 B_{\mu\nu} + 2[D_{[\mu}, B^{(\mathrm{KK})}_{\nu]5}], \tag{4.18}$$

$$H_{\mu\nu\rho} = 3[D_{[\mu}, B^{(\mathrm{KK})}_{\nu\rho]}] + 3[F_{[\mu\nu}, \partial_5^{-1} B^{(\mathrm{KK})}_{\rho]5}]. \tag{4.19}$$

The decomposition in KK-modes and zero-modes is given by

$$H^{(\mathrm{KK})}_{\mu\nu 5} = \partial_5 B_{\mu\nu} + 2[D_{[\mu}, B^{(\mathrm{KK})}_{\nu]5}], \tag{4.20}$$

$$H^{(0)}_{\mu\nu 5} = \frac{1}{2\pi R} F_{\mu\nu}, \tag{4.21}$$

The action is defined by

$$S = -\frac{1}{2\pi R} \int_{\mathbb{R}^5} d^5x \frac{1}{4} \mathrm{Tr}(F_{\mu\nu} F^{\mu\nu}) - \int_{\mathbb{R}^5 \times S^1} d^6x \mathrm{Tr}(\widetilde{H}^{\mu\nu 5}_{(\mathrm{KK})} (H^{(\mathrm{KK})}_{\mu\nu 5} - \widetilde{H}^{(\mathrm{KK})}_{\mu\nu 5})), \tag{4.22}$$

where

$$\widetilde{H}_{\mu\nu 5} = -\frac{1}{6} \epsilon_{\mu\nu\lambda\sigma\rho} H^{\lambda\sigma\rho}. \tag{4.23}$$

The Yang-Mills equation is equivalent to the self-duality equation on the zero-modes of H. The variation of the action then implies self-duality of the KK-modes of H. The self-duality of the zero-modes is identical to the condition

$$H^{(0)}_{\mu\nu\rho} = \frac{1}{2} \epsilon_{\mu\nu\rho\lambda\sigma} F^{\lambda\sigma}. \tag{4.24}$$

Higher gauge theory leads to a consistent definition of a Wilson surface and non-abelian 2-form gauge theory governed by the structure of a differential crossed module. The categorified principal bundle is given by a principal 2-bundle with structure Lie 2-group. However, the consistency implies that the 2-form curvature of the non-abelian gerbe is zero reducing the system to be free or the theory to a topological theory of BF-type. This constraint is called fake curvature condition. In [135], a modification of the crossed module was proposed in order to have the 2-form curvature transforming covariantly without fake curvature constraint.

4.3 Mathematical preliminaries

4.3.1 Category theory and higher gauge theory

In this section, we provide an introduction to the realm of higher gauge theory from the perspective of category theory. We will define fundamental structures, that will be used in the main text. An excellent introduction, which underlies the structure of this section, is given by [126].

Higher gauge theory generalizes ordinary 1-form gauge theory on G-principal bundles, where G is the structure Lie group, to categorified principal bundles encoding higher form gauge theory taking values in categorified gauge algebras. Crucial for the understanding of the underlying structures is the method of categorification, which is one of the main topics of this section. In this sense, categorification is an intuitive approach to arrive at generalized gauge structures. It yields so-called n-bundles with structure n-groups as underlying gauge structure of the associated higher form gauge fields. Higher gauge theory is the answer to the question of how to consistently define parallel transport of higher-dimensional objects analogous to the ordinary parallel transport in 1-form gauge theory, governed by the so-called Wilson line. One main goal is therefore to define a Wilson surface, Wilson volume and higher analogues. In this exposition, we will restrict ourselves to the case of parallel transport of strings and associated 2-form gauge structures. Yielding a notion of parallel transport of 1-dimensional objects, i.e. strings, it is reasonable to use the structures of higher gauge theory to investigate string theory and M-theory. However, obviously it also serves as the appropriate context to analyze the various branes that appear in string theory and their M-theoretic colleagues, the M2- and M5-branes, from the perspective of higher parallel transport. For a rigorous account on categorification [154], higher gauge theory [127], 2-bundles [128], 2-connections [155], Lie 2-algebras [156] and 2-groups [157] we refer to the respective reference.

We start with the definition of the fundamentals of category theory: category, object, morphism and functor. After that, we develop a notion of holonomy from the category point of view, as it is found in ordinary gauge theory as Wilson line. Then, we generalize this notion using higher categories leading to parallel transport of 1-dimensional objects. Finally, examples are in order.

Definition 4.3.1 ((Small) Category) *A category C consists of two sets, the object-set C_0 and the morphism-set C_1. Furthermore, it contains a source map $s : C_1 \to C_0$ and a target map $t : C_1 \to C_0$. It is equipped with a binary operation $(-,-) : C_1 \times C_1 \to C_1$, $(f,g) = fg$, for pairs of morphisms $f,g \in C_1$, for which $t(f) = s(g)$. For example, a morphism $f : x \to y$ has the source $s(f) = x$ and target $t(f) = y$. The maps have the following properties: Compatibility of $\circ$ with the source and target maps,*

$$s(g \circ f) = s(f), \quad t(g \circ f) = t(g). \tag{4.25}$$

Associativity of $\circ$,

$$f \circ (g \circ h) = (f \circ g) \circ h. \tag{4.26}$$

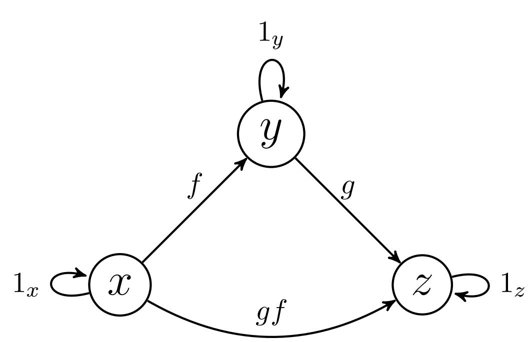
Existence of an identity morphism $1_x : x \to x$ for every object $x \in C_0$, such that for any morphisms f, g

$$1_x \circ g = g, \quad f \circ 1_x = f, \tag{4.27}$$

where $t(g) = x$ and $s(f) = x$.

The map $(-,-)$ is called *composition*. All categories, we introduce, are small categories, compared to large categories, where C_0 and C_1 are not ordinary sets but proper classes. Morphisms are also called and denoted by arrows. Let us describe an introductory example.

Example 4.3.1 (A simple category) Let C be a category with 3 objects, $C_0 = \{x, y, z\}$, and 2 morphisms besides the identity morphisms, $C_1 = \{f : x \to y, g : y \to z, 1_x, 1_y, 1_z\}$. Since the target of f is equal to the source of g, $t(f) = s(g)$, we can compute the composition arrow $gf : x \to z$. We can depict this setup as follows.



Let us give some easy examples of categories to get accommodated to this notion.

Example 4.3.2 (Category of sets) The category of sets is denoted by **Set**. The objects of **Set** are given by sets and the morphisms of **Set** are given by functions between sets. The composition is given by concatenation of functions.

Example 4.3.3 (Category of groups) The category of groups is denoted by **Grp**. Its objets are given by groups and its morphisms are given by group homomorphisms. Its composition is given by concatenation of group homomorphisms.

Example 4.3.4 (Category of topological spaces) The category of topological spaces is denoted by **Top**. The objects of **Top** are given by topological spaces and the morphisms of **Top** are given by continuous maps between topological spaces. Its composition is given by concatenation of continuous maps.

The first main goal of this section is the understanding of ordinary gauge theory from the category point of view. For this, we have to introduce a notion of a categorified analogue of a group. Such a *groupoid* is a category with one extra feature.

Definition 4.3.2 (Groupoid) *A groupoid is a category, where every morphism is invertible.*

The simplest example of a groupoid is a category with a single object, where every morphism is invertible. In this case, all morphisms start and end at the same object and therefore can be concatenated with each other. Since furthermore all morphisms are invertible, we conclude that the set of morphisms with composition becomes an ordinary group.
Having understood the meaning of category and groupoid, we now go on and define a map between two categories, the *functor*.

Definition 4.3.3 (Functor) *Let C and D be two categories. A map $F : C \to D$, that associates to each object $x \in C_0$ and each morphism $(f : a \to b) \in C_1$ an object $F(x) \in D_0$ and morphism $(F(f) : F(a) \to F(b)) \in D_1$ such that it respects the composition*

$$F(f \circ g) = F(f) \circ F(g), \tag{4.28}$$

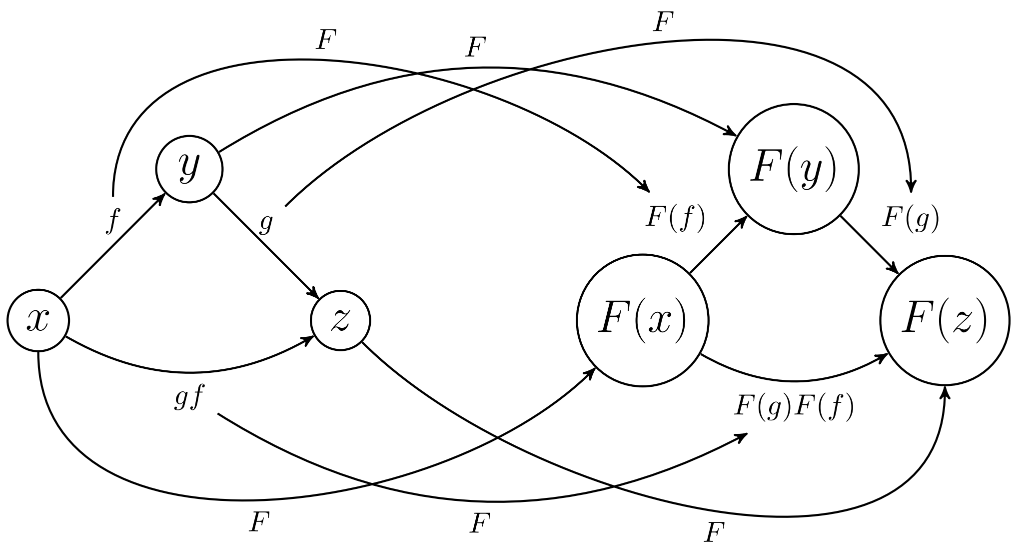
for any morphisms $f, g \in C_1$, and the identity morphisms

$$F(1_x) = 1_{F(x)}, \tag{4.29}$$

for any object $x \in C_0$. The map F is called functor.

The functor is a means to transport structure from one category to another.

Example 4.3.5 (A simple functor) Let us consider our category C with 3 objects again. If there exists a category D and a functor $F : C \to D$, then the image of the C_0 under F is subset of D_0 and the image of C_1 under F is subset of D_1. The identities and compositions are preserved.



From now, we develop a notion *holonomy*. For this, we need a category, which encodes paths on a manifold, on the one hand, and a functor, that assigns to each path a group element, on the other. This functor will become the *holonomy functor* and is comparable with a Wilson line in ordinary gauge theory. The so-called *path groupoid* is what we are looking for. In the following, M denotes a smooth manifold, which we associate with the spacetime manifold.

Definition 4.3.4 (Path groupoid) *Let M be a smooth manifold. The path groupoid $P_1(M)$ of M is defined as follows. The objects of $P_1(M)$ are given by the points $x \in M$. The morphisms of $P_1(M)$ are given by* thin *homotopy classes of* lazy *paths in M.*
A thin homotopy is a homotopy, which sweeps out a zero area surface, and a path $\gamma : [0,1] \to M$ is lazy, if γ is smooth and constant in a neighborhood around the two points $t = \{0,1\}$. The composition is defined as follows. If $\delta : [0,1] \to M$ and $\gamma : [0,1] \to M$ are lazy paths in

M, then the composition $\gamma\delta : [0,1] \to M$ is given by

$$(\gamma\delta)(t) = \begin{cases} \delta(2t) & for\ \ 0 \leq t \leq 0.5 \\ \gamma(2t-1) & for\ \ 0.5 \leq t \leq 1 \end{cases} . \tag{4.30}$$

The inverse of a path $\gamma : [0,1] \to M$ is defined by

$$\gamma^{-1}(t) = \gamma(1-t). \tag{4.31}$$

One can prove, that the path groupoid is indeed a groupoid.
In ordinary gauge theory with structure group G, there is a notion of holonomy. Let γ be a path that a particle travels through spacetime. The Wilson line assigns to this path an element of a group G,

$$\mathrm{hol}(\gamma) = \mathrm{Pexp}\left(\int_\gamma A\right), \tag{4.32}$$

where A is the 1-form gauge field talking values in Lie(G). The operation Pexp denotes the path-ordered exponential. This group element encodes the transformation behavior of the particle, when it travels along γ.
The map hol can be defined as a functor between the path groupoid $P_1(M)$ and the Lie group G interpreted as a groupoid $\mathcal{G}$ with one object,

$$\begin{aligned} \mathrm{hol} : P_1(M) &\to \mathcal{G}, \\ \mathrm{hol}(\gamma) &= \mathrm{Pexp}\left(\int_\gamma A\right). \end{aligned} \tag{4.33}$$

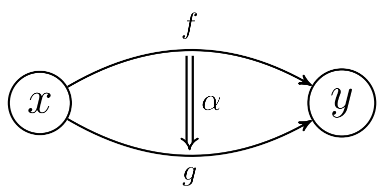
Since $\mathcal{G}$ only has a single object, which we denote by $\star$ for convenience, all objects $x \in M$ of the path groupoid get mapped to $\star$. However, every homotopy class of lazy paths $[\gamma]$ gets mapped to a group element $\mathrm{Pexp}\left(\int_\gamma A\right)$. It turns out, that hol preserves the composition and identity and therefore is a functor. We state the following theorem of equivalence of the gauge theoretical notion and the notion developed using category theory.

Theorem 4.3.5 ([126]) *Let M be a smooth manifold and G a Lie group. Then, connections on the trivial principal G-bundle over M are one-to-one with $\mathfrak{g}$-valued 1-forms on M are one-to-one with* smooth *functors hol : $P_1(M) \to G$.*

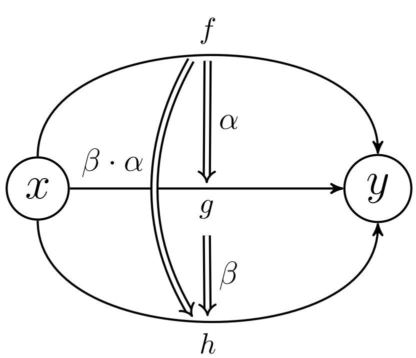
The functor hol is smooth, if the composition of paths gives compositions of holonomies and it depends smoothly on the paths.

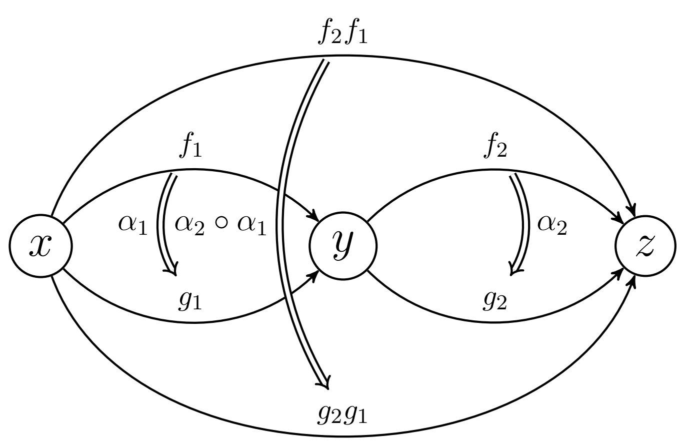
We conclude, that we now have a categorified notion of an ordinary connection. From now, we will step into the realm of higher categorification in order to arrive at a notion of parallel transport of 1-dimensional objects, or strings, and associated 2-bundles. For this, we need a "higher" category. In this case, a 2-category is sufficient, but for even higher parallel transport general n-categories are needed.

Definition 4.3.6 (2-category) *A 2-category C consists of a set of objects, C_0. Furthermore, for each pair of objects x, y it consists of a category $C(x, y)$, whose objects are morphisms from x to y and whose morphisms are 2-morphisms $\alpha : f \Rightarrow g$, where $(f, g) : x \to y$. Therefore, 2-morphisms are also called morphisms between morphisms. The target of α is g and the source of α is f.*



On the level of morphisms, we have the usual composition $(-,-)$. However, on the level of 2-morphisms, there are two different compositions available: vertical composition $\cdot$ and horizontal composition $\circ$.
For three morphisms $(f, g, h) : x \to y$ and two 2-morphisms $\alpha : f \Rightarrow g$ and $\beta : g \Rightarrow h$, one can compute the vertical composite $\beta \cdot \alpha : f \Rightarrow h$. For four morphisms $f_1, g_1 : x \to y$ and $f_2, g_2 : y \to z$ and two 2-morphisms $\alpha_1 : f_1 \Rightarrow g_1$ and $\alpha_2 : f_2 \Rightarrow g_2$, we one can compute the horizontal composite $\alpha_2 \circ \alpha_1 : (f_2 \circ f_1) \Rightarrow (g_2 \circ g_1)$.



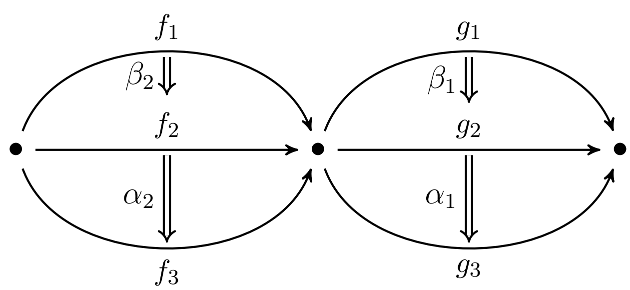


Then, the following conditions have to hold:

- *Associativity of the composition of morphisms and existence of an identity morphism* 1_x *for any* $x \in C_0$.
- *Associativity of the vertical composition of* 2*-morphisms and existence of a vertical identity* 2*-morphism* 1_f *for any morphism* f.
- *Associativity of the horizontal composition of* 2*-morphisms and existence a horizontal identity* 2*-morphism* 1_{1_x} *for any object* x.
- *The interchange law,*

$$(\alpha_1 \cdot \beta_1) \circ (\alpha_2 \cdot \beta_2) = (\alpha_1 \circ \alpha_2) \cdot (\beta_1 \circ \beta_2). \tag{4.34}$$

The interchange law guarantees the equivalence of two ways to contract the following diagram: first vertically, then horizontally, or vice versa.



Having understood the notion of a 2-category, we can define the 2-groupoid.

Definition 4.3.7 (2-groupoid) *A* 2*-groupoid is a* 2*-category, where every morphism is invertible and every* 2*-morphism is vertically and horizontally invertible.*

Finally, we can define the path 2-groupoid. A 2-*functor* between the path 2-groupoid of a smooth manifold M to a 2-groupoid will be the main ingredient to define the notion of parallel transport of 1-dimensional objects and the associated 2-form higher gauge theory.

Definition 4.3.8 (Path 2-groupoid) *Let* M *be a smooth manifold. The path* 2*-groupoid of* M, $P_2(M)$, *is a* 2*-groupoid, where the objects are the points* $x \in M$, *the morphisms are thin homotopy classes of lazy paths in* M, *and the* 2*-morphisms are* thin homotopy classes of lazy surfaces *in* M.

A homotopy $\Sigma : [0,1]^2 \to M$ between two paths $(\gamma, \delta) : x \to y$ is a lazy surface if each of the paths in the 1-parameter family $\gamma_s(t) = \Sigma(s,t)$ is lazy and $\gamma_s = \gamma_0$ in a neighborhood of $t = 0$ and $\gamma_s = \gamma_1$ in a neighborhood of $t = 1$.
A homotopy between lazy surfaces Σ and Ξ is a smooth map $H : [0,1]^3 \to M$ such that $H(0,s,t) = \Sigma(s,t)$ and $H(1,s,t) = \Xi(s,t)$. This homotopy is thin if H does not sweep out a volume.

Now, we understand the generalization of the path groupoid to the stringy picture. The higher holonomy functor, which we define below involves a generalization of a Lie group to a Lie 2-group, or *crossed module of Lie groups*. In other words, the gauge fields in the resulting higher gauge theory take values in the Lie 2-algebra associated with a Lie 2-group. Let us describe what this means. We start by defining the 2-group.

Definition 4.3.9 (2-group) *A 2-group is a 2-groupoid with one object.*

Since the definition of a 2-group is still quite abstract, we now define the structure of a crossed module and discuss their equivalence.

Definition 4.3.10 (Crossed module) *Let G and H be a pair of groups, that are endowed with two group homomorphisms $t : H \to G$ and $\alpha : G \to Aut(H)$, where $Aut(H)$ is the automorphism group over H. Let the two group homomorphisms satisfy the relations*

$$\alpha(t(h))(h') = hh'h, \tag{4.35}$$

$$t(\alpha(g)(h)) = gt(h)g^{-1}, \tag{4.36}$$

where $g \in G$ and $h, h' \in H$. The 4-tuple (G, H, t, α) is called a crossed module.

This structure also turns out to be the underlying structure of higher-dimensional Yang-Mills theory, where the Lie group is exchanged by a Lie 2-group [158].

Theorem 4.3.11 (Equivalence of crossed module and 2-group, [157]) *Let (G, H, t, α) be a crossed module. We can uniquely construct a 2-group $\mathcal{G}$ as follows.*
Let G be the group of morphisms in $\mathcal{G}$. To any set $(g, h) \in G \times H$ with $g' = t(h)g$ we assign a 2-morphism $\alpha : g \Rightarrow g'$. The vertical and horizontal composition of 2-morphism is then

given by

$$(g,h)\cdot(g',h') = (g',hh'), \tag{4.37}$$

$$(g,h)\circ(g',h') = (gg',h\alpha(g)h'). \tag{4.38}$$

Now, the other way. Let $\mathcal{G}$ be a 2-group. We can uniquely construct a crossed module (G,H,t,α) as follows.
Let the group of morphisms in $\mathcal{G}$ be the group G. Let the group of 2-morphisms α with $s(\alpha)=1_\bullet$ be the group H. Let target map on 2-morphisms in H be the group homomorphism $t:H\to G$. Finally, let the action $\alpha:G\to Aut(H)$ be given by

$$\alpha(g)H = 1_g\circ h\circ 1_{g^{-1}}. \tag{4.39}$$

Let us remark, that the groupoids discussed here are so-called strict groupoids. We will omit a detailed discussion on generalizations of these structures.

Definition 4.3.12 (Lie crossed module) *A Lie crossed module is a crossed module (G,H,t,α), where H and G are Lie groups.*

A Lie crossed module is also referred to as Lie 2-group.
The infinitesimal object associated with a smooth crossed module is given by the differential crossed module. It can be easily derived by Lie derivation.

Definition 4.3.13 (Differential crossed module) *Let $\mathfrak{g}$ and $\mathfrak{h}$ be a pair of Lie algebras. Furthermore, let the pair of Lie algebras be endowed with two Lie algebra homomorphisms $\underline{t}:\mathfrak{h}\to\mathfrak{g}$ and $\underline{\alpha}:\mathfrak{g}\to Der(\mathfrak{h})$, where $Der(\mathfrak{h})$ is the Lie algebra of Lie derivations over $\mathfrak{h}$. The homomorphisms obey*

$$\underline{t}(\underline{\alpha}(g)(h)) = [g,\underline{t}(h)], \tag{4.40}$$

$$\underline{\alpha}(\underline{t}(h))(h') = [h,h'], \tag{4.41}$$

for all $g\in\mathfrak{g}$ and $h\in\mathfrak{h}$. The 4-tuple $(\mathfrak{g},\mathfrak{h},\underline{t},\underline{\alpha})$ is called a differential crossed module.

The property of $\underline{t}$ and $\underline{\alpha}$ to be homomorphisms leads to the additional conditions

$$\underline{t}([h,h']) = [\underline{t}(h),\underline{t}(h')], \tag{4.42}$$

$$\underline{\alpha}([g,g']) = [\underline{\alpha}(g),\underline{\alpha}(g')], \tag{4.43}$$

$$\underline{\alpha}(g)([h,h']) = [\underline{\alpha}(g)(h),h'] + [h,\underline{\alpha}(g)(h')], \tag{4.44}$$

where $g, g' \in \mathfrak{g}$ and $h, h' \in \mathfrak{h}$. The last equation is the derivation property. A differential crossed module can be identified with a strict Lie 2-algebra, as we showed in the L_∞-algebra section.

Having generalized the path groupoid and the group to the stringy picture, we now have to introduce a generalized version of the functor itself, the 2-*functor*. The 2-functor between the path 2-groupoid and a Lie 2-group will then lead to the correct notion of parallel transport of 1-dimensional objects. The resulting structure is called a 2-*connection*, in contrast to the ordinary connection in 1-form gauge theory.

Definition 4.3.14 (2-functor) *A 2-functor $F : C \to D$ between two 2-categories C and D is a map of each object $x \in C$ to $F(x) \in D$, each morphism $(f : x \to y) \in C$ to $(F(f) : F(x) \to F(y)) \in D$ and each 2-morphism $(\alpha : f \Rightarrow g) \in C$ to $(F(\alpha) : F(f) \Rightarrow F(g)) \in D$. It has the following properties:*

It preserves the composition of morphisms, vertical and horizontal composition of 2-morphisms as well as all identities,

$$F(f \circ g) = F(f) \circ F(g), \quad F(\alpha \cdot \beta) = F(\alpha) \cdot F(\beta), \quad F(\alpha \circ \beta) = F(\alpha) \circ F(\beta), \\ F(1_f) = 1_{F(f)}, \quad F(1_x) = 1_{F(x)}. \tag{4.45}$$

Let us now state the most important theorem of this section, the equivalence between holonomy 2-functors and 2-form higher gauge theories with differential crossed module gauge structure.

Theorem 4.3.15 ([128, 159, 160]) *Let M be a smooth manifold and $\mathcal{G}$ be a Lie 2-group. Then, 2-connections on the trivial principal $\mathcal{G}$-2-bundle over M are one-to-one to pairs $(A, B) \in \Omega^1(M; \mathfrak{g}) \times \Omega^2(M; \mathfrak{h})$ such that*

$$\underline{t}(B) = dA + A \wedge A, \tag{4.46}$$

are one-to-one to smooth *2-functors $hol : P_2(M) \to \mathcal{G}$.*

Roughly speaking, a smooth 2-functor depends smoothly on the parameter of smoothly parameterized families of lazy paths as well as lazy surfaces.

We now understand, that the differential crossed module is the underlying structure of parallel transport of 1-dimensional objects. The holonomy of a surface requires the existence of both a 1-form and a 2-form. This leads to 2-form higher gauge theory with differential crossed module gauge structure. The equation that is required for consistency when computing the surface holonomy,

$$\underline{t}(B) = dA + A \wedge A, \tag{4.47}$$

is called the **fake curvature condition**, $F = \underline{t}(B)$. The so-called fake curvature $\mathcal{F}$ can be defined by

$$\mathcal{F} = F - \underline{t}(B). \tag{4.48}$$

Then, for consistency, this curvature has to vanish, $\mathcal{F} = 0$, in order to yield well-defined higher parallel transport. In the main section of this part we will develop a method to circumvent the fake curvature condition by using more intricate higher gauge algebras. This ultimately leads to a higher gauge theory, that can be associated to a multiple M5-brane system compactified on a circle.

A gauge transformation between two 2-connections (A, B) and (A', B') is parameterized by a smooth function $g : M \to G$ and a 1-form $a \in \Omega^1(M; \mathfrak{h})$ taking values in $\mathfrak{h}$,

$$A' = gAg^{-1} + gdg^{-1} + \underline{t}(a), \tag{4.49}$$

$$B' = \alpha(g)(B) + \underline{\alpha}(A') \wedge a + da - a \wedge a. \tag{4.50}$$

The 2-form curvature F transforms as

$$F' = gFg^{-1} + \underline{t}(da) + [\underline{t}(a), A'] - \underline{t}(a) \wedge \underline{t}(a). \tag{4.51}$$

The 2-*curvature* H of the 2-connection is defined by

$$H = dB + \underline{\alpha}(A) \wedge B. \tag{4.52}$$

It describes the holonomy of the 2-connection over an infinitesimal 2-sphere. A 2-connection is called flat, if its curvature and 2-curvature vanish.

It turns out, that a $U(1)$-gerbe is a 2-connection, where the Lie crossed module is given by (G, H, t, α) with G trivial, H abelian and t and α trivial maps. The curvature of the $U(1)$-gerbe is given by the 2-curvature of the 2-connection, $H = dB_i$ on U_i. On double-overlaps U_{ij} we have

$$B_i - B_j = da_{ij}, \tag{4.53}$$

and on triple-overlaps U_{ijk} we have

$$a_{ij} + a_{jk} - a_{ik} = h_{ijk} dh^{-1}_{ijk}. \tag{4.54}$$

Also compare this construction to the abelian gerbes that arise as Čech-Deligne cocycles in 3.3.7.

For general differential crossed modules the connection of a so-called non-abelian gerbe arises, where

$$a_{ij} + \rho(g_{ij})a_{jk} - h_{ijk}a_{ik}h_{ijk}^{-1} = h_{ijk}\underline{\alpha}(A_i)h_{ijk}^{-1} + h_{ijk}dh_{ijk}^{-1}, \tag{4.55}$$

on triple-overlaps U_{ijk}. We summarize, the 2-form higher gauge theory on a $\mathcal{G}$-2-bundle is described by a connection of a non-abelian gerbe. As a remark, general non-abelian Čech cocycles are encoded in n-functors from the so-called Čech n-groupoid to an n-groupoid. For generalizations of higher gauge theory, where even the underlying manifold is categorified to a so-called 2-space, we refer to [106].

4.3.2 Higher gauge theory from QP-manifolds

In this section, we show how to generate ordinary and higher gauge theories with gauge structure induced by arbitrary QP-manifolds. The mathematical fundament is given by supergeometry and can be applied in any dimension. The method is due to [132]. It relies on the AKSZ method to generate a BV-BRST formalism on the mapping space between two graded manifolds used in the first part of this thesis.

The gauge theory shall be constructed on a d-dimensional worldvolume manifold X. For this, we promote the worldvolume to the superworldvolume $\chi = T[1]X$. The local parameterization of χ is given by coordinates σ^μ of degree zero on X and θ^μ of degree 1 on the fiber. The Grassmann odd coordinates will be identified with the generators of the cotangent bundle $T^*X \to X$. The index μ runs over $\mu = 1, \ldots, d$.

The gauge theory defined on the worldvolume inherits the structure of a QP-manifold by a morphism of differential graded algebras, or *dga-morphism*, as follows. Let the QP-manifold be denoted by $(\mathcal{M}, \Theta, \omega)$. Then, we can define a map $a : \chi \to \mathcal{M}$ between the graded manifolds χ and $\mathcal{M}$ as follows. Let us fix a coordinate z of arbitrary degree $|z|$. The image of the pullback of z along a is a superfield $\boldsymbol{z} \in \mathcal{C}^\infty(\chi)$ such that $|\boldsymbol{z}| = |z|$. We can expand the superfield in Grassmann odd coordinates,

$$a^*(z) = \boldsymbol{z}(\sigma, \theta) = \sum_{j=0}^{d} \boldsymbol{z}^{(j)}(\sigma, \theta) = \sum_{j=0}^{d} \frac{1}{j!}\theta^{\mu_1} \cdots \theta^{\mu_j} z^{(j)}_{\mu_1 \cdots \mu_j}(\sigma). \tag{4.56}$$

The result is a superfield, which includes the gauge fields as well as the ghosts and antifields of the induced BV formalism.

We are interested in the physical field, which is given by the ghost-number zero component. Let us explain how the ghost-number is counted. For this, we first introduce the form-degree $|-|_{\text{Form}}$ and assign $|\theta^\mu|_{\text{Form}} = 1$ and $|\sigma^\mu|_{\text{Form}} = 0$ to the local coordinates of the superworldvolume χ. Then, the ghost-number $|-|_{\text{Ghost}}$ of a superfield component $z^{(j)}_{\mu_1\cdots\mu_j}$ is defined as the difference $|z^{(j)}_{\mu_1\cdots\mu_j}|_{\text{Ghost}} = |\boldsymbol{z}| - |\boldsymbol{z}^{(j)}|_{\text{Form}}$. Therefore, the ghost number of the j-component is given by $|z^{(j)}_{\mu_1\cdots\mu_j}|_{\text{Ghost}} = |\boldsymbol{z}| - j$. The ghost-number zero component is a physical field. For our superfield $\boldsymbol{z}$, it means that the $|\boldsymbol{z}|$-th component is physical, $z^{(|\boldsymbol{z}|)}_{\mu_1\cdots\mu_{|\boldsymbol{z}|}}$. Furthermore, a field, for which $|-|_{\text{Ghost}} > 0$, is called a ghost, whereas a field, for which $|-|_{\text{Ghost}} < 0$, is called an antifield.

The ghost-number zero component of a superfield will be associated with a gauge field. More explicitly, $z^{(|\boldsymbol{z}|)}_{\mu_1\cdots\mu_{|\boldsymbol{z}|}}$ will be associated with the components of a $|\boldsymbol{z}|$-form gauge field on the worldvolume X in the following way. We define a second map $\widetilde{a} : \chi \to \mathcal{M}$, which associates the corresponding gauge field on X to the ghost-number zero component via pullback. In local coordinates, for the coordinate z of degree $|z|$ on $\mathcal{M}$ we find by pullback along $\widetilde{a}$,

$$\widetilde{a}^*(z) = \frac{1}{|\boldsymbol{z}|!} d\sigma^{\mu_1} \wedge \cdots \wedge d\sigma^{\mu_{|\boldsymbol{z}|}} z^{(|\boldsymbol{z}|)}_{\mu_1\cdots\mu_{|\boldsymbol{z}|}}, \tag{4.57}$$

so that $\widetilde{a}^*(z) \in \Omega^{|z|}(X)$.

The ghost-number 1 component of the superfield $\boldsymbol{z}$ is the $(|\boldsymbol{z}|-1)$-form gauge parameter $z^{(|\boldsymbol{z}|-1)}_{\mu_1\cdots\mu_{|\boldsymbol{z}|-1}}$ associated with the $|\boldsymbol{z}|$-form gauge field $z^{(|\boldsymbol{z}|)}_{\mu_1\cdots\mu_{|\boldsymbol{z}|}}$.

The super field strength associated to the $|\boldsymbol{z}|$-form gauge field $z^{(|\boldsymbol{z}|)}_{\mu_1\cdots\mu_{|\boldsymbol{z}|}}$ is given by

$$\boldsymbol{F}_z = \boldsymbol{d} \circ a^*(z) - a^* \circ Q(z), \tag{4.58}$$

where $\boldsymbol{d} = \theta^\mu \partial_\mu$ is the superdifferential as usual and $Q = \{\Theta, -\}$ is the homological vector field associated with the Hamiltonian function Θ on the QP-manifold $\mathcal{M}$. Since the superdifferential contributes one form-degree, one recognizes that the physical component in $\boldsymbol{F}_z$ is at order $1 + |\boldsymbol{z}^{(|\boldsymbol{z}|)}|_{\text{Form}} = 1 + |\boldsymbol{z}|$,

$$\boldsymbol{F}_z|_{1+|\boldsymbol{z}|} = [\boldsymbol{d} \circ a^*(z) - a^* \circ Q(z)]_{1+|\boldsymbol{z}|}. \tag{4.59}$$

By exchanging the map a with $\widetilde{a}$, we can define the physical field strength,

$$F_z = d \circ \widetilde{a}^*(z) - \widetilde{a}^* \circ Q(z), \tag{4.60}$$

where the superdifferential collapsed to the standard de Rham differential d.
Finally, let us define the infinitesimal gauge transformations. The information on this transformation is encoded in the ghost-number 1 component of the super field strength,

$$\delta \boldsymbol{z} = \boldsymbol{F}_z|_{|\boldsymbol{z}|} = [\boldsymbol{d} \circ a^*(z) - a^* \circ Q(z)]_{|\boldsymbol{z}|}. \tag{4.61}$$

In BV language, this is a BRST transformation with Grassmann odd ghost gauge parameter. In order to extract this component with the ghost-number 1 gauge parameter $z^{(|\boldsymbol{z}|-1)}_{\mu_1\cdots\mu_{|\boldsymbol{z}|-1}}$, we define a map of degree (-1), $\widetilde{a}_{-1} : \chi \to \mathcal{M}$. The pullback of the local coordinate z on $\mathcal{M}$ along $\widetilde{a}_{-1}$ extracts the $(|\boldsymbol{z}|-1)$-form gauge parameter,

$$\widetilde{a}^*_{-1}(z) = \frac{1}{(k-1)!} d\sigma^{\mu_1} \wedge \cdots \wedge d\sigma^{\mu_{|\boldsymbol{z}|-1}} z^{(|\boldsymbol{z}|-1)}_{\mu_1\cdots\mu_{|\boldsymbol{z}|-1}}. \tag{4.62}$$

Using this map, we can define the gauge transformation of the physical gauge field as

$$\delta \widetilde{a}^*(z) = d \circ \widetilde{a}^*_{-1}(z) - \widetilde{a}^*_{-1} \circ Q(z). \tag{4.63}$$

It turns out that the physical field strength obeys the Bianchi identity since the homological vector field Q is nilpotent, $Q^2 = 0$,

$$dF_z = -F \circ Q(z) + \widetilde{a}^* \circ Q^2(z) = -F \circ Q(z), \tag{4.64}$$

where F denotes the map

$$\begin{aligned} F &: \mathcal{M} \to \Omega^{|\boldsymbol{z}|+1}(X), \\ F &: z \mapsto F_z. \end{aligned} \tag{4.65}$$

Let us summarize the method. We choose a worldvolume manifold X and promote it to a graded manifold $\chi = T[1]M$. Then, we choose a QP-manifold $(\mathcal{M}, \Theta, \omega)$, which should serve as the underlying structure of our (higher) gauge theory. Finally, to each coordinate on $\mathcal{M}$ we associate a (higher) gauge field with (higher) field strength, gauge transformation and get a consistent Bianchi identity for free, if Θ obeys the classical master equation on $\mathcal{M}$.
Let us work out a simple example to get accommodated with the formalism. We show how to recover ordinary gauge theory, if the underlying QP-manifold induces the structure of a Lie algebra.

Example 4.3.6 (Ordinary gauge theory) In order to induce an ordinary gauge theory with Lie algebra structure, we start from the QP-structure inducing an ordinary Lie algebra.

Let $\mathcal{M} = T^*[n]V[1]$ be a graded manifold, where V is a finite-dimensional vector space. Let $\mathcal{M}$ be locally parameterized by coordinates (v^a, v_a) of degrees $(1, n-1)$ and be endowed with the following graded symplectic structure,

$$\omega = (-1)^n \delta v^a \wedge \delta v_a. \tag{4.66}$$

Finally, let the Hamiltonian function be defined by

$$\Theta = \frac{1}{2} f_{ab}^c v^a v^b v_c, \tag{4.67}$$

where f_{ab}^c is constant.

The local coordinates on $\mathcal{M}$ will correspond to gauge fields of the associated BRST-BV formalism. We assign the physical 1-form gauge field $A^a \in \Omega^1(M)$ to the coordinate v^a of degree 1,

$$\widetilde{a}^*(v^a) = A^a(\sigma). \tag{4.68}$$

The gauge parameter of ghost-degree 1 is given by a function ε^a,

$$\widetilde{a}^*_{-1}(v^a) = \varepsilon^a(\sigma). \tag{4.69}$$

The associated 2-form field strength is given by

$$\begin{aligned} F^a = F_{v^a} &= d \circ \widetilde{a}^*(v^a) - \widetilde{a}^* \circ Q(v^a) \\ &= dA^a - \frac{1}{2} f_{bc}^a A^b A^c, \end{aligned} \tag{4.70}$$

where $Q = \{\Theta, -\}$. The gauge transformation of A^a is given by

$$\begin{aligned} \delta A^a &= d \circ \widetilde{a}^*_{-1}(v^a) - \widetilde{a}^*_{-1} \circ Q(v^a) \\ &= d\varepsilon^a - f_{bc}^a A^b \varepsilon^c. \end{aligned} \tag{4.71}$$

The resulting gauge structure is an ordinary gauge theory with internal structure induced by the QP-manifold equivalent to the Lie algebra $(V, [-,-])$.

For an application of the formalism to higher bundles associated with non-abelian superconformal models in 6 spacetime dimensions see [161]. For an account on supergravity in string theory from the perspective dual to supergeometry, using higher Cartan geometry we recommend [162]. The picture in terms of so-called ∞-bundles with connections was established in [163].

4.4 Higher gauge theories of multiple M5-branes

We clarified above, that there are strong hints that the dynamics of multiple M5-branes is governed by a non-abelian version of higher gauge theory. However, the construction of local actions with manifest Lorentz symmetry with non-trivially interacting self-dual gauge fields poses problems. We described the ansatz of [126] to develop a notion of higher parallel transport from the viewpoint of category theory, which leads to a 2-form higher gauge theory governed by a differential crossed module structure. The theory describes the parallel transport of 1-dimensional objects sweeping out a Wilson surface, the higher analogue of a Wilson line in ordinary gauge theory. However, it turns out that the covariant gauge transformation of the 3-form field strength H requires the lower 2-form field strength F to vanish. This is called **fake curvature condition**. The resulting theory becomes topological and dynamically highly restricted.

This section is devoted to the construction of a topologically non-trivial interacting covariantly transforming non-abelian higher gauge theory, which can be related to a system of multiple M5-branes compactified on a circle along the lines of [133, 134, 135]. It is based on the published article [5].

The crucial step to the success of our construction is our newly proposed method to generate off-shell covariant (higher) gauge theories induced by graded symplectic manifolds, called **off-shell covariantization**. The starting point of our journey is a Lie 2-algebra gauge theory, which inherits the fake curvature issue. The underlying gauge Lie 2-algebra, will be extended by additional structures, and auxiliary gauge fields will be added in a covariant manner. By restricting the gauge transformations of the auxiliary gauge fields to a hypersurface, the residual gauge transformations become covariant while circumventing the fake curvature condition $F = 0$ (off-shell). This leads to an off-shell covariant higher gauge theory.

This section is structured as follows. In section 4.4.1, we describe our method of off-shell covariantization. The graded symplectic manifold setup, which we use to construct the higher gauge theory, is introduced in section 4.4.2. In Section 4.4.3, we define the graded symplectic manifold, which induces a semistrict Lie 2-algebra as the basic structure of our higher gauge theory. Section 4.4.4 concerns the derivation of the higher gauge theory with underlying semistrict Lie 2-algebra gauge structure and discussion of the fake curvature condition. In section 4.4.5, we explicitly construct the off-shell covariantized higher gauge theory, which circumvents the fake curvature condition and relate it to the system of multiple M5-branes compactified on a circle along the lines of [133, 134, 135].

4.4.1 Off-shell covariantization

Off-shell covariantization is a method to covariantize the gauge transformation behavior of the highest form field strength while keeping the lower form field strengths non-vanishing. It is a way to circumvent the fake curvature condition. The modified field strengths after the off-shell covariantization procedure can then be used to construct quadratic actions, which are invariant under the modified gauge transformations.

We start by fixing a QP-manifold $(\mathcal{M}, \Theta, \omega)$, which serves as the underlying gauge structure of the higher gauge theory. In general, the QP-manifold induces a the structure of a *symplectic* L_∞*-algebroid*. All gauge fields, field strengths, gauge parameters and gauge transformations are derived using the local structure of the QP-manifold. The Hamiltonian function on the QP-manifold directly induces the gauge structure of the resulting theory. In general, the resulting gauge theory is on-shell covariant, which means that it inherits the fake curvature issue. In such a case, our off-shell covariantization procedure can be applied.

The off-shell covariantization procedure makes use of the auxiliary gauge fields and field strengths associated with the conjugate coordinates on the QP-manifold as follows. We choose a QP-manifold of the structure $\mathcal{M} = T^*[n]\mathcal{X}$, where $\mathcal{X}$ is another graded manifold. Then, we use the local coordinates on $\mathcal{X}$ to induce the higher gauge fields and associated field strengths. This gauge structure in general inherits the fake curvature issue. In the next step, the gauge structure associated with the $T^*[n]$-fiber coordinates is used to generate auxiliary gauge fields with auxiliary field strengths. This auxiliary gauge freedom enters the gauge transformation of the ordinary higher gauge theory. Thus, by *constraining* the auxiliary gauge fields, the auxiliary gauge freedom can be used to deform the gauge transformations and circumvent the fake curvature condition.

The underlying gauge structure is governed by a symplectic L_∞-algebroid. It turns out that in general the successful circumvention of the fake curvature issue by constraining the auxiliary gauge fields leads to a reduction of the underlying gauge structure of the symplectic L_∞-algebroid to a subalgebroid.

We can summarize the off-shell covariantization procedure in the following recipe.

Off-shell covariantization	
1.	Fix the QP-structure $(\mathcal{M}, \Theta, \omega)$.
2.	Compute associated symplectic L_∞-algebroid.
3.	Derive full gauge structure.
4.	Covariantize using the auxiliary gauge structure and by reduction of the full gauge structure.
5.	Check that field strengths transform off-shell covariantly under the residual gauge symmetry.

4.4.2 Graded symplectic manifold setup and classification

In this section, we describe the graded symplectic manifold setup and fix the QP-manifold, which we use as starting point of the off-shell covariantization procedure.

Let V and W be finite-dimensional vector spaces. We consider the family of graded manifolds $\mathcal{M}_n = T^*[n]\mathcal{X}$, where $\mathcal{X} = W[1] \oplus V[2]$ and $n \in \mathbb{N}$. Let $W[1]$ be locally parameterized by coordinates ξ^a of degree 1 and $V[2]$ be locally parameterized by coordinates ζ^A of degree 2. Then, we take conjugate coordinates with respect to the $T^*[n]$-fiber to be $(\bar{\xi}_a, \bar{\zeta}_A)$ of degrees $(n-1, n-2)$. In a nutshell, $\mathcal{M}_n$ is parameterized by local coordinates $(\xi^a, \zeta^A, \bar{\xi}_a, \bar{\zeta}_A)$ of degrees $(1, 2, n-1, n-2)$.

Let $\mathcal{M}_n$ be endowed with the following graded symplectic structure,

$$\omega = (-1)^n \delta\xi^a \wedge \delta\bar{\xi}_a + \delta\zeta^A \wedge \delta\bar{\zeta}_A. \tag{4.72}$$

In the next step, we perform a classification of available Hamiltonian functions for each $n \in \mathbb{N}$. We can expand any Hamiltonian function Θ in conjugate coordinates $(\bar{\xi}_a, \bar{\zeta}_A)$,

$$\Theta = \sum_i \Theta^{(i)}, \tag{4.73}$$

where $\Theta^{(i)}$ is i-th order in $(\bar{\xi}_a, \bar{\zeta}_A)$. Recall that the Hamiltonian function on a degree n QP-manifold is of degree $n+1$. The following table summarizes all available Hamiltonian functions for any $n \in \mathbb{N}$ by degree counting.

Available Hamiltonian functions	
$n \in \mathbb{N}$	Hamiltonian function Θ
$n \geq 6$	$\Theta = \Theta^{(0)} + \Theta^{(1)}$
$n = 4, 5$	$\Theta = \Theta^{(0)} + \Theta^{(1)} + \Theta^{(2)}$
$n = 3$	$\Theta = \Theta^{(0)} + \Theta^{(1)} + \Theta^{(2)} + \Theta^{(3)} + \Theta^{(4)}$
$n = 2$	Courant algebroid

For the case $n = 2$, the degrees of the local coordinates $(\xi^a, \zeta^A, \bar{\xi}_a, \bar{\zeta}_A)$ are given by $(1, 2, 1, 0)$. Therefore, the Hamiltonian function can contain arbitrary functions in $\bar{\zeta}_A$. We find

$$T^*[2](W[1] \oplus V[2]) \cong W[1] \oplus V[2] \oplus W^*[1] \oplus V^* \cong T^*[2](E[1]), \tag{4.74}$$

which induces a Courant algebroid on the vector bundle $E \to V^*$ with fiber W.

4.4.3 Symplectic NQ-manifold of a semistrict Lie 2-algebra

In this section, we show that the QP-manifold $(\mathcal{M}_n, \Theta, \omega)$ contains a semistrict Lie 2-algebra as substructure. This Lie 2-algebra serves as the underlying structure of the 2-form higher gauge theory. However, it inherits the fake curvature issue, as we will show below.
The Hamiltonian function, which is first order in the conjugate coordinates, is given by

$$\Theta^{(1)} = t^a_A \zeta^A \bar{\xi}_a + \frac{(-1)^n}{2} f^c_{ab} \xi^a \xi^b \bar{\xi}_c + \alpha^B_{aA} \xi^a \zeta^A \bar{\zeta}_B + \frac{(-1)^n}{3!} T^A_{abc} \xi^a \xi^b \xi^c \bar{\zeta}_A, \tag{4.75}$$

where t^a_A, f^c_{ab}, α^B_{aA} and T^A_{abc} are constant. Let e_a denote the generators of the vector space V and E_A the generators of the vector space W. We can define the following injection map j,

$$\begin{aligned} j &: W \oplus V \to \mathcal{M}, \\ j &: (e_a, E_A, 0, 0) \mapsto (\bar{\xi}_a, \bar{\zeta}_A, \xi^a, \zeta^A). \end{aligned} \tag{4.76}$$

Then, the operations of the semistrict Lie 2-algebra are recovered via

$$[g^a_1 e_a, g^b_2 e_b]_{\text{Lie}} = (-1)^{n+1} j^* \{\{\Theta^{(1)}, j_*(g^a_1 e_a)\}, j_*(g^b_2 e_b)\} = f^c_{ab} g^a_1 g^b_2 e_c, \tag{4.77}$$

$$\underline{t}(h^A E_A) = j^* \{\Theta^{(1)}, j_*(h^A E_A)\} = t^a_A h^A e_a, \tag{4.78}$$

$$\underline{\alpha}(g^a e_a)(h^A E_A) = (-1)^n j^* \{\{\Theta^{(1)}, j_*(g^a e_a)\}, j_*(h^A E_A)\} = \alpha^B_{aA} g^a h^A E_B, \tag{4.79}$$

$$\text{Jac}(g^a_1 e_a, g^b_2 e_b, g^c_3 e_c) = -j^* \{\{\{\Theta^{(1)}, j_*(g^a_1 e_a)\}, j_*(g^b_2 e_b)\}, j_*(g^c_3 e_c)\} = T^A_{abc} g^a g^b g^c E_A. \tag{4.80}$$

Here, Jac denotes the Jacobiator. The classical master equation, $\{\Theta^{(1)}, \Theta^{(1)}\} = 0$, is equivalent to the following algebraic equations,

$$\begin{gathered} \frac{1}{2} f^d_{e[a} f^e_{bc]} - \frac{1}{3!} t^d_A T^A_{abc} = 0, \quad t^c_A f^a_{cb} - t^a_B \alpha^B_{bA} = 0, \\ \frac{1}{2} \alpha^B_{cA} f^c_{ab} + \alpha^B_{[a|C|} \alpha^C_{b]A} + \frac{1}{2} t^c_A T^B_{cab} = 0, \quad \frac{3}{2} f^e_{[ab} T^A_{cd]e} + \alpha^A_{[a|B|} T^B_{bcd]} = 0, \quad \alpha^C_{a(A} t^a_{B)} = 0. \end{gathered} \tag{4.81}$$

The first equation is the Jacobi identity, which is broken by a Jacobiator. The operations together with the algebraic equations form the structure of a semistrict Lie 2-algebra. We summarize our findings in the following theorem.

Theorem 4.4.1 *The QP-structure* $(\mathcal{M}_n, \Theta^{(1)}, \omega)$ *induces the structure of a semistrict Lie 2-algebra.*

In the case, where $T^A_{abc} = 0$, the structure collapses to a strict Lie 2-algebra, which is categorically equivalent to a differential crossed module.
The gauge structure induced by $(\mathcal{M}_n, \Theta^{(1)}, \omega)$ serves as the basis for the 2-form higher gauge theory. However, it naturally is subject to the fake curvature issue. This issue can be tackled by enlarging the structure of the Lie 2-algebra by introduction of additional contributions to the Hamiltonian function different from $\Theta^{(1)}$.

4.4.4 Higher gauge theory of a semistrict Lie 2-algebra

In this section, we compute the associated 2-form higher gauge theory for the semistrict Lie 2-algebra. Let X be a smooth manifold. Then, let $\chi = T[1]X$ be the graded manifold, on which the gauge theory shall be defined. Let χ be locally parameterized by coordinates (σ^μ, θ^μ) of degrees $(0, 1)$. Then, we associate 1-form and 2-form gauge fields to the basis coordinates of degree 1 and 2 of $\mathcal{M}_n$,

$$\widetilde{a}^*(\xi^a) = A^a = A^a_\mu d\sigma^\mu, \tag{4.82}$$

$$\widetilde{a}^*(\zeta^A) = B^A = \frac{1}{2} B^A_{\mu\nu} d\sigma^\mu \wedge d\sigma^\nu. \tag{4.83}$$

We find the associated field strengths to be

$$F^a = F_{\xi^a} = d \circ \widetilde{a}^*(\xi^a) - \widetilde{a}^*(\{\Theta^{(1)}, \xi^a\}) = dA^a - \frac{1}{2} f^a_{bc} A^b \wedge A^c - (-1)^n t^a_A B^A, \tag{4.84}$$

$$H^A = F_{\zeta^A} = d \circ \widetilde{a}^*(\zeta^A) - \widetilde{a}^*(\{\Theta^{(1)}, \zeta^A\}) = dB^A + \alpha^A_{aC} A^a \wedge B^C + \frac{(-1)^n}{3!} T^A_{abc} A^a \wedge A^b \wedge A^c. \tag{4.85}$$

The ghost-number 1 gauge parameters are given by

$$\varepsilon^a = \widetilde{a}^*_{-1}(\xi^a), \quad \mu^A = \widetilde{a}^*_{-1}(\zeta^A), \tag{4.86}$$

where ε^a is a function and μ^A is a 1-form. Then, the gauge transformations are easily computed,

$$\delta A^a = d\varepsilon^a - f^a_{bc} A^b \varepsilon^c + t^a_A \mu^A, \tag{4.87}$$

$$\delta B^A = d\mu^A + \alpha^A_{aB} A^a \wedge \mu^B - \alpha^A_{aC} \varepsilon^a B^C + \frac{1}{2} T^A_{abc} A^a \wedge A^b \varepsilon^c. \tag{4.88}$$

Finally, we can compute the gauge transformations of the field strengths,

$$\delta F^a = f^a_{bc} F^b \varepsilon^c, \tag{4.89}$$

$$\delta H^A = \alpha^A_{aB} H^B \varepsilon^a - \alpha^A_{aB} F^a \wedge \mu^B + T^A_{abc} A^a \wedge F^c \varepsilon^b. \tag{4.90}$$

From the transformation behavior of the 3-form field strength, we conclude that for the 3-form field strength to transform covariantly, the 2-form field strength has to vanish, $F^a = 0$. Then, $\delta H \sim H$ and trivially $\delta F \sim F$. This is the fake curvature condition. Without setting $F^a = 0$ it is impossible to introduce the associated kinetic terms of both field strengths when constructing the Lagrangian of the resulting higher gauge theory.

4.4.5 Higher gauge theory of multiple M5-branes

In this section, we will apply the procedure of off-shell covariantization to the gauge structure induced by the QP-manifold $(\mathcal{M}_4, \Theta, \omega)$ for $n = 4$. It turns out that the additional structure provided by the term $\Theta^{(2)}$ is crucial for a successful off-shell covariantization. We will comment on the choice of $n = 4$ after showing the success of our proposed method.

Underlying QP-manifold

Let us shortly summarize the setup. We consider the QP-manifold $(\mathcal{M}_4, \Theta, \omega)$, where $\mathcal{M}_4 = T^*[4](W[1] \oplus V[2])$ is locally parameterized by coordinates $(\xi^a, \zeta^A, \overline{\xi}_a, \overline{\zeta}_A)$ of degrees $(1, 2, 3, 2)$. The most general Hamiltonian function is given by

$$\Theta = \Theta^{(0)} + \Theta^{(1)} + \Theta^{(2)}, \tag{4.91}$$

where

$$\Theta^{(0)} = \frac{1}{5!} m_{abcde} \xi^a \xi^b \xi^c \xi^d \xi^e + \frac{1}{3!} m_{abcA} \xi^a \xi^b \xi^c \zeta^A + \frac{1}{2} m_{aAB} \xi^a \zeta^A \zeta^B, \tag{4.92}$$

$$\Theta^{(1)} = \frac{1}{2} f^c_{ab} \xi^a \xi^b \overline{\xi}_c + t^a_A \zeta^A \overline{\xi}_a + \alpha^B_{aA} \xi^a \zeta^A \overline{\zeta}_B + \frac{1}{3!} T^A_{abc} \xi^a \xi^b \xi^c \overline{\zeta}_A, \tag{4.93}$$

$$\Theta^{(2)} = s^{aA} \overline{\xi}_a \overline{\zeta}_A + \frac{1}{2} n^{AB}_a \xi^a \overline{\zeta}_A \overline{\zeta}_B. \tag{4.94}$$

Here, all coefficients are constant. The classical master equation, $\{\Theta, \Theta\} = 0$, can be decomposed by order in conjugate coordinates,

$$\{\Theta^{(0)}, \Theta^{(0)}\} = 0, \tag{4.95}$$

$$\{\Theta^{(0)}, \Theta^{(1)}\} + \{\Theta^{(1)}, \Theta^{(0)}\} = 0, \tag{4.96}$$

$$\{\Theta^{(1)}, \Theta^{(1)}\} + \{\Theta^{(0)}, \Theta^{(2)}\} + \{\Theta^{(2)}, \Theta^{(0)}\} = 0, \tag{4.97}$$

$$\{\Theta^{(1)}, \Theta^{(2)}\} + \{\Theta^{(2)}, \Theta^{(1)}\} = 0, \tag{4.98}$$

$$\{\Theta^{(2)}, \Theta^{(2)}\} = 0. \tag{4.99}$$

Induced higher gauge theory

Here, we discuss the higher gauge theory induced by the QP-manifold $(\mathcal{M}_4, \Theta, \omega)$. We consider a higher gauge theory on the graded manifold $\chi = T[1]X$, where X is a smooth manifold of any dimension. The graded manifold χ is locally parameterized by coordinates (σ^μ, θ^μ) of degrees $(0, 1)$. We assign 1-form gauge field A^a and 2-form gauge field B^A to the coordinates of the base of $\mathcal{M}_4$,

$$A^a = \widetilde{a}^*(\xi^a) = A^a_\mu d\sigma^\mu, \tag{4.100}$$

$$B^A = \widetilde{a}^*(\zeta^A) = \frac{1}{2} B^A_{\mu\nu} d\sigma^\mu \wedge d\sigma^\nu. \tag{4.101}$$

The associated field strengths are given by

$$\begin{aligned} F^a = F_{\xi^a} &= d \circ \widetilde{a}^*(\xi^a) - \widetilde{a}^*(\{\Theta, \xi^a\}) \\ &= d \circ \widetilde{a}^*(\xi^a) - \widetilde{a}^*(\{\Theta^{(1)}, \xi^a\}) - \widetilde{a}^*(\{\Theta^{(2)}, \xi^a\}), \end{aligned} \tag{4.102}$$

$$\begin{aligned} H^A = F_{\zeta^A} &= d \circ \widetilde{a}^*(\zeta^A) - \widetilde{a}^*(\{\Theta, \zeta^A\}) \\ &= d \circ \widetilde{a}^*(\zeta^A) - \widetilde{a}^*(\{\Theta^{(1)}, \zeta^A\}) - \widetilde{a}^*(\{\Theta^{(2)}, \zeta^A\}). \end{aligned} \tag{4.103}$$

We recognize, that the term $\Theta^{(0)}$ does not contribute to the expressions for the field strengths. Furthermore, as we discussed above, the term $\Theta^{(1)}$ induces the gauge structure of a semistrict Lie 2-algebra, which inherits the fake curvature issue. In the following, we will see how we can use the additional structure provided by $\Theta^{(2)}$ together with the auxiliary gauge fields to circumvent the fake curvature condition. Note that without associating additional gauge fields to the conjugate coordinates, the terms $\widetilde{a}^*(\{\Theta^{(2)}, \zeta^A\})$ and $\widetilde{a}^*(\{\Theta^{(2)}, \xi^a\})$ are pulled back to zero. So we are effectively still dealing with usual semistrict Lie 2-algebra higher gauge theory.

Analysis of the classical master equation

Since the term $\Theta^{(0)}$ does not deform the expressions of the field strengths, we will omit it in the following. The decomposition of the classical master equation simplifies to

$$\{\Theta^{(1)},\Theta^{(1)}\}=0,\quad \{\Theta^{(1)},\Theta^{(2)}\}=0,\quad \{\Theta^{(2)},\Theta^{(2)}\}=0. \tag{4.104}$$

The first equation induces the structure of a semistrict Lie 2-algebra. However, this structure is deformed by the second equation. The third equation then is an intrinsic condition on the deformation term $\Theta^{(2)}$. Inserting the Hamiltonian function, we find the following full set of algebraic relations,

$$\begin{gathered}\frac{1}{2}f^d_{e[a}f^e_{bc]}-\frac{1}{3!}t^d_AT^A_{abc}=0,\quad t^c_Af^a_{cb}-t^a_B\alpha^B_{bA}=0,\\ \frac{1}{2}\alpha^B_{cA}f^c_{ab}+\alpha^B_{[a|C|}\alpha^C_{b]A}+\frac{1}{2}t^c_AT^B_{cab}=0,\quad \frac{3}{2}f^e_{[ab}T^A_{cd]e}+\alpha^A_{[a|B|}T^B_{bcd]}=0,\\ \alpha^C_{a(A}t^a_{B)}=0,\quad s^{a(A}n^{BC)}_a=0,\quad s^{cA}f^b_{ca}+\alpha^A_{aB}s^{bB}-t^b_Bn^{AB}_a=0,\\ \frac{1}{2}s^{c(A}T^{B)}_{abc}+\frac{1}{4}n^{AB}_cf^c_{ab}+\alpha^{(A}_{[a|C|}n^{B)C}_{b]}=0,\quad s^{a(A}\alpha^{B)}_{aC}+\frac{1}{2}t^a_Cn^{AB}_a=0,\quad t^{[a}_As^{b]A}=0.\end{gathered} \tag{4.105}$$

The resulting structure is a 4-term L_∞-algebra, which is trivial in degree 3. This is the underlying structure of the full higher gauge theory including auxiliary fields associated with the conjugate coordinates, which we will introduce below. Let us for convenience decompose the space of smooth functions on the graded manifold $\mathcal{M}$ by degree,

$$\mathcal{C}^\infty(\mathcal{M})=\bigoplus_i\mathcal{C}^\infty_i(\mathcal{M}). \tag{4.106}$$

It is sufficient to analyze the decomposition until degree 3, since this is the degree of the highest degree local coordinate on $\mathcal{M}$. Let k be the underlying field. The spaces are given by

$$\mathcal{C}^\infty_0(\mathcal{M})=\emptyset, \tag{4.107}$$

$$\mathcal{C}^\infty_1(\mathcal{M})=\{X_a\xi^a|X_a\in k\} \tag{4.108}$$

$$\mathcal{C}^\infty_2(\mathcal{M})=\left\{\frac{1}{2}X_{ab}\xi^a\xi^b+Y_A\zeta^A+Z^A\bar{\zeta}_A\;\middle|\;X_{ab},Y_A,Z^A\in k\right\}, \tag{4.109}$$

$$\mathcal{C}^\infty_3(\mathcal{M})=\left\{\frac{1}{3!}X_{abc}\xi^a\xi^b\xi^c+Y_{Aa}\zeta^A\xi^a+Z^A_a\bar{\zeta}_A\xi^a+R^a\bar{\xi}_a\;\middle|\;X_{abc},Y_{Aa},Z^A_a,R^a\in k\right\}. \tag{4.110}$$

The differential crossed module lives in the subalgebra

$$C^\infty_2(\mathcal{N})\xrightarrow{t}C^\infty_3(\mathcal{N}), \tag{4.111}$$

over the subspace $\mathcal{N} \subset \mathcal{M}$ parameterized by $(\bar{\xi}_a, \bar{\zeta}_A)$. In other words, we extended the higher gauge structure to a 4-term L_∞-algebra containing the differential crossed module. The additional structure will be used below, to circumvent the fake curvature issue.

Introduction of auxiliary gauge fields

Let us now introduce auxiliary gauge fields associated with the conjugate coordinates $(\bar{\xi}_a, \bar{\zeta}_A)$ in $\mathcal{M}_4$ in order to make use of the additional structure provided by $\Theta^{(2)}$. Since $(\bar{\xi}_a, \bar{\zeta}_A)$ are of degrees $(3, 2)$, we get additional 3-form and 2-form auxiliary gauge fields,

$$\widetilde{a}^*(\bar{\xi}_a) = C_a = \frac{1}{3!} C_{a,\mu\nu\rho} d\sigma^\mu \wedge d\sigma^\nu \wedge d\sigma^\rho, \tag{4.112}$$

$$\widetilde{a}^*(\bar{\zeta}_A) = D_A = \frac{1}{2} D_{A,\mu\nu} d\sigma^\mu \wedge d\sigma^\nu. \tag{4.113}$$

The associated 4-form and 3-form auxiliary field strengths are given by

$$F_a^{(C)} = dC_a - f^c_{ab} A^b \wedge C_c - \alpha^A_{aB} B^B \wedge D_A - \frac{1}{2} T^A_{abc} A^b \wedge A^c \wedge D_A - \frac{1}{2} n_a^{AB} D_A \wedge D_B, \tag{4.114}$$

$$F_A^{(D)} = dD_A - t^a_A C_a - \alpha^B_{aA} A^a \wedge D_B. \tag{4.115}$$

The auxiliary gauge fields also enter the 2-form and 3-form field strengths F^a and H^A. The deformed expressions are given by

$$F^a = dA^a - \frac{1}{2} f^a_{bc} A^b \wedge A^c - t^a_A B^A - s^{aA} D_A, \tag{4.116}$$

$$H^A = dB^A + \alpha^A_{aB} A^a \wedge B^B + \frac{1}{3!} T^A_{abc} A^a \wedge A^b \wedge A^c + s^{bA} C_b + n_a^{AB} A^a \wedge D_B. \tag{4.117}$$

Assigning the ghost-number 1 gauge parameters to the auxiliary gauge fields,

$$\varepsilon'_a = \widetilde{a}^*_{-1}(\bar{\xi}_a), \quad \mu'_A = \widetilde{a}^*_{-1}(\bar{\zeta}_A). \tag{4.118}$$

The full gauge transformations are computed to be

$$\delta A^a = d\varepsilon^a - f^a_{bc} A^b \varepsilon^c - t^a_A \mu^A - s^{aA} \mu'_A, \tag{4.119}$$

$$\begin{aligned} \delta B^A = {} & d\mu^A + \alpha^A_{aB}(A^a \wedge \mu^B + \varepsilon^a B^B) + \frac{1}{2} T^A_{abc} A^a \wedge A^b \varepsilon^c + s^{bA} \varepsilon'_b \\ & + n_a^{AB}(A^a \wedge \mu'_B + \epsilon^a \wedge D_B), \end{aligned} \tag{4.120}$$

$$\begin{aligned} \delta C_a = {} & d\varepsilon'_a - f^c_{ab}(A^b \wedge \varepsilon'_c + \varepsilon^b \wedge C_c) - \alpha^A_{aB}(B^B \wedge \mu'_A + \mu^B \wedge D_A) \\ & - \frac{1}{2} T^A_{abc}(2A^b \wedge D_A \varepsilon^c + A^b \wedge A^c \wedge \mu'_A) - n_a^{AB} D_A \wedge \mu'_B, \end{aligned} \tag{4.121}$$

$$\delta D_A = d\mu'_A - t^a_A \varepsilon'_a - \alpha^B_{aA}(A^a \wedge \mu'_B + \varepsilon^a D_B). \tag{4.122}$$

Using above equations, we can compute the gauge transformations of the 2-form and 3-form field strengths,

$$\delta F^a = f^a_{bc} F^b \varepsilon^c, \tag{4.123}$$
$$\delta H^A = \alpha^A_{aB} H^B \varepsilon^a + n^{AB}_a F^{(D)}_B \varepsilon^a + T^A_{abc} A^a \wedge F^c \varepsilon^b - \alpha^A_{aB} F^a \wedge \mu^B - n^{AB}_a F^a \wedge \mu'_B. \tag{4.124}$$

We recognize that now there is more space to covariantize the gauge transformation behavior of the 3-form field strength without having to impose $F^a = 0$. For this, we restrict the auxiliary gauge fields and compute the restricted gauge transformations using a special subspace of solutions of the classical master equation on $\mathcal{M}_4$.

Off-shell covariantization procedure

We will be using a special subset of solutions to the classical master equation for the off-shell covariantization procedure. Based on this subset of solutions, the 3-form field strength H^A can be covariantized by constraining the auxiliary gauge fields without setting the 2-form field strength F^a to zero.

Let us in the following describe this special subset of solutions. For simplicity, we assume $T^A_{abc} = 0$. We start by introducing the symmetric matrix $G^{ab} = s^{aB} t^b_B$, which in general is not invertible. In the next step, we assume that W is a metric vector space with metric g_{ab}, which we use to introduce the new constant $s^A_b = s^{aA} g_{ab}$. Then, we assume $s^A_a t^a_B = \delta^A_B$ and introduce the projector $P^a_b = t^a_A s^A_b$. We find $G^{ab} = P^a_c g^{cb}$ and conclude that the metric g is in general not invertible in the image of P.

It turns out that the space of solutions under above assumptions is equivalent to the space of solutions of the differential crossed module inducing Hamiltonian function $\Theta^{(1)}$,

$$\exp(\delta_{s^{aA} s^B_a})(\Theta^{(1)}) = \Theta^{(1)} + \Theta^{(2)}, \tag{4.125}$$
$$\exp(\delta_{s^{aA} s^B_a})(\{\Theta^{(1)}, \Theta^{(1)}\}) = \{\Theta^{(1)} + \Theta^{(2)}, \Theta^{(1)} + \Theta^{(2)}\}, \tag{4.126}$$

so that

$$\{\Theta^{(1)}, \Theta^{(1)}\} = 0 \Rightarrow \{\Theta^{(1)} + \Theta^{(2)}, \Theta^{(1)} + \Theta^{(2)}\} = 0. \tag{4.127}$$

Please find the detailed calculation in section A.1 of appendix A. On this subspace of solutions, we will perform the procedure off-shell covariantization of the higher gauge theory.

In the next step, we impose constraints on the auxiliary gauge fields C_a and D_A. We choose the ansatz given by

$$C_a = -K_{abc}F^b \wedge A^c, \quad D_A = 0, \tag{4.128}$$

leading to the field strengths

$$F^a = dA^a - \frac{1}{2}f^a_{bc}A^b \wedge A^c - t^a_A B^A, \tag{4.129}$$

$$H^A = dB^A + \alpha^A_{aB}A^a \wedge B^B - s^{bA}K_{bcd}F^c \wedge A^d. \tag{4.130}$$

It turns out that the choice $K_{abc} = g_{ad}t^d_A\alpha^A_{bB}s^B_c$ leads to the desired off-shell covariant field strengths as we will show in the following. The resulting field strengths are given by

$$F^a = dA^a - \frac{1}{2}f^a_{bc}A^b \wedge A^c - t^a_A B^A, \tag{4.131}$$

$$H^A = dB^A + \alpha^A_{aB}A^a \wedge B^B - \alpha^A_{aB}s^B_c F^a \wedge A^c. \tag{4.132}$$

We have to show that the residual gauge transformations on the hypersurface defined by the constraints off-shell covariantize the higher gauge theory. This has been proven for a special choice of vector spaces W and V. Let V be the Lie algebra $\mathfrak{h}$ and let W be endowed with the structure of a semi-direct product, $W = \mathfrak{g} = K \ltimes \mathfrak{h}$, where K is a Lie algebra with a representation ρ on $\mathfrak{h}$. The Lie bracket on W is defined by

$$[(k,h),(k',h')] = ([k,k'], \rho(k)h' - \rho(k')h), \tag{4.133}$$

where $k, k' \in K$ and $h, h' \in \mathfrak{h}$. Furthermore, the maps $\underline{\alpha} : \mathfrak{g} \to \mathrm{Der}(\mathfrak{h})$, $\underline{t} : \mathfrak{h} \to \mathfrak{g}$ and $\underline{s} : \mathfrak{g} \to \mathfrak{h}$ are defined as

$$\underline{\alpha}((k,h))h' = \rho(k)h', \quad \underline{t}(h) = (0, Mh), \quad \underline{s}(k,h) = M^{-1}h, \tag{4.134}$$

where $k \in K$, $h, h' \in \mathfrak{h}$ and M is an invertible matrix.

In this setup, we can compute the higher gauge theory on the constraint surface with residual gauge transformations denoted by $\widehat{\delta}$,

$$F^a = dA^a - \frac{1}{2}f^a_{bc}A^b \wedge A^c - t^a_A B^A, \tag{4.135}$$

$$H^A = dB^A + \alpha^A_{aB}A^a \wedge B^B - \alpha^A_{aB}s^B_c F^a \wedge A^c, \tag{4.136}$$

$$\widehat{\delta}A^a = d\widehat{\varepsilon}^a - f^a_{bc}A^b\widehat{\varepsilon}^c - t^a_A\widehat{\mu}^A, \tag{4.137}$$

$$\widehat{\delta}B^A = d\widehat{\mu}^A + \alpha^A_{jB}(A^j \wedge \widehat{\mu}^B + \widehat{\varepsilon}^j B^B) - \alpha^A_{jB}s^B_c\widehat{\varepsilon}^c F^j, \tag{4.138}$$

$$\widehat{\delta}F^a = f^a_{bc}F^b(\widehat{\varepsilon}^c - (P\widehat{\varepsilon})^c), \tag{4.139}$$

$$\widehat{\delta}H^A = \alpha^A_{aB}H^B(\widehat{\varepsilon}^a - (P\widehat{\varepsilon})^a), \tag{4.140}$$

where the gauge parameters of the residual gauge transformations are denoted by $\widehat{\varepsilon}^a$ and $\widehat{\mu}^A$. Please find the detailed calculation in section A.2 of the appendix.

Finally, we analyze the closure of the residual gauge transformations. First, we pull out the ghost-number from the gauge transformations, giving

$$\tilde{\delta} A^a = d\tilde{\epsilon}^a - f^a_{bc} A^b \tilde{\epsilon}^c + t^a_A \tilde{\mu}^A, \tag{4.141}$$

$$\tilde{\delta} B^A = d\tilde{\mu}^A + \alpha^A_{jB}(A^j \wedge \tilde{\mu}^B - \tilde{\epsilon}^j B^B) + \alpha^A_{jB} s^B_c \tilde{\epsilon}^c F^j, \tag{4.142}$$

where we introduced the ghost-number zero gauge parameters $\widetilde{\varepsilon}^a$ and $\widetilde{\mu}^A$. Calculating the commutator of two gauge transformations, we find

$$[\widetilde{\delta}_1, \widetilde{\delta}_2] A^a = d\widetilde{\varepsilon}^a_3 - f^a_{bc} A^b \widetilde{\varepsilon}^c_3 + t^a_A \widetilde{\mu}^A_3, \tag{4.143}$$

$$[\widetilde{\delta}_1, \widetilde{\delta}_2] B^A = d\widetilde{\mu}^A_3 + \alpha^A_{jB}(A^j \wedge \widetilde{\mu}^B_3 - \widetilde{\varepsilon}^j_3 B^B) + \alpha^A_{jB} s^B_c \widetilde{\varepsilon}^c_3 F^j + \Lambda^A, \tag{4.144}$$

where

$$\Lambda^A = \alpha^A_{jB} f^j_{ke} s^B_c P^e_b (\widetilde{\varepsilon}^b_1 \widetilde{\varepsilon}^c_2 - \widetilde{\varepsilon}^b_2 \widetilde{\varepsilon}^c_1) F^k \tag{4.145}$$

and

$$\widetilde{\varepsilon}^a_3 = -f^a_{bc} \widetilde{\varepsilon}^b_1 \widetilde{\varepsilon}^c_2, \quad \widetilde{\mu}^A_3 = \alpha^A_{bB}(\widetilde{\varepsilon}^b_1 \widetilde{\mu}^B_2 - \widetilde{\varepsilon}^b_2 \widetilde{\mu}^B_1). \tag{4.146}$$

Closure of the gauge transformations requires $\Lambda^A = 0$, which could be solved by restricting the theory on-shell, $F^a = 0$, or by

$$\alpha^A_{jB} f^j_{ke} s^B_c P^e_b = \alpha^A_{jB} f^j_{ke} s^B_c t^e_D s^D_b = 0. \tag{4.147}$$

Our setup obeys equation (4.147). Therefore, we showed off-shell closure of the off-shell covariantized residual gauge transformations of the 2-form higher gauge theory. We successfully circumvented the fake curvature condition by making use of the auxiliary gauge freedom. Our result matches with the construction of a non-abelian gerbe by [135], which is related to the system of multiple M5-branes compactified on a circle [133].

Context of the result

We follow the presentation in [135], in order to relate our result to the S^1-compactified M5-brane system. Let $\mathbb{R}^5 \times S^1$ the 6-dimensional worldvolume of the M5-brane system. We can decompose the 1-form gauge fields A^a into K- and $\mathfrak{h}$-components, $A^a = (\widehat{A}^a, \widetilde{A}^a)$. We do the same for the 2-form curvature, $F^a = (\widehat{F}^a, \widetilde{F}^a)$. Then, under the decomposition we find

$$\widehat{F}^a = d\widehat{A}^a - \frac{1}{2} f^a_{bc} \widehat{A}^b \wedge \widehat{A}^c, \tag{4.148}$$

$$\widetilde{F}^a = d\widetilde{A}^a - \frac{1}{2} f^a_{bc} \widetilde{A}^b \wedge \widetilde{A}^c - MB^a. \tag{4.149}$$

Let $X = \mathbb{R}^5$ be the decompactified 5-dimensional part of the M5-brane worldvolume. Then, we can write down the gauge invariant action,

$$S = \frac{1}{4} \int_X \Big(\mathrm{Tr}(\star\widehat{F} \wedge \widehat{F}) + \mathrm{Tr}(\star\widetilde{F} \wedge \widetilde{F}) \Big) + \frac{1}{12} \int_X \mathrm{Tr}(\star H \wedge H). \tag{4.150}$$

Gauge fixing of $\widetilde{A} = 0$ gives $\widetilde{F} = -\underline{t}(B) = -MB$, leading to

$$S = \frac{1}{4} \int_X \Big(\mathrm{Tr}(\star\widehat{F} \wedge \widehat{F}) + \mathrm{Tr}(\star MB \wedge MB) \Big) + \frac{1}{12} \int_X \mathrm{Tr}(\star H \wedge H). \tag{4.151}$$

The result is that the field B acquires mass proportional to the eigenvalues of spectral decomposition of M according to an irreducible representation of $\mathfrak{h}$ via Stückelberg mechanism. It was furthermore noted in [135], that in the case of X being 5-dimensional one may impose

$$H_{\mu\nu\rho} = \frac{1}{2} \epsilon_{\mu\nu\rho\lambda\sigma} \widetilde{F}^{\lambda\sigma}, \tag{4.152}$$

leading to

$$H_{\mu\nu\rho} = \frac{m^{(\alpha)}}{2} \epsilon_{\mu\nu\rho\lambda\sigma} B^{(\alpha)\lambda\sigma} \tag{4.153}$$

at linear order. Here (α) labels the decomposition of the mass matrix M into irreducible representations. When taking $K = SU(N)$, identifying $\widetilde{A}$ and B with the Kaluza-Klein modes of the 2-form gauge field in 6 dimensions emerging from S^1-compactification and the 1-form gauge field A with the zero mode of $B_{\mu 5}$ in 6 dimensions, then this system can be interpreted as a generalization of a non-abelian higher gauge theory of N coincident M5-branes compactified on an S^1 with self-dual 2-form gauge field [133]. The action is given by (4.23) and the equation (4.152) arises from the self-duality of the zero-modes.

Why $n = 4$?

Let us take a look at the table, that summarizes all available Hamiltonian functions for any $n \in \mathbb{N}$. The term $\Theta^{(0)}$ does not deform the expressions of the field strengths and the term $\Theta^{(1)}$ inherits the fake curvature issue. To extend the underlying algebraic structure in a meaningful way, we need higher order terms $\Theta^{(i)}$ for $i \geq 2$. This renders the cases $n \geq 6$

useless. For $n = 4$, the off-shell covariantization procedure has been shown to be successful. However, we cannot exclude that off-shell covariantization can also be successful in the case $n = 5$. We can perform a simple analysis of the Hamiltonian function for this case. The crucial part is $\Theta^{(2)}$, which is given by

$$\Theta^{(2)} = \frac{1}{2} u^{AB} \bar{\zeta}_A \bar{\zeta}_B, \tag{4.154}$$

where u^{AB} is constant. Since in this case $|\bar{\zeta}_A| = 3$, we find a deformation of the 3-form curvature,

$$H^A = dB^A + \alpha^A_{aB} A^a \wedge B^B + \frac{1}{3!} T^A_{abc} A^a \wedge A^b \wedge A^c + u^{AB} D_B, \tag{4.155}$$

whereas the 2-from curvature F^a remains undeformed contrary to the case $n = 4$. Here, $D_B = \widetilde{a}^*(\bar{\zeta}_B)$ is the 3-form auxiliary gauge field. The additional algebraic conditions on u^{AB} are given by

$$t^a_A u^{AB} = 0, \quad \alpha^{[B}_{aA} u^{C]A} = 0. \tag{4.156}$$

Although, we cannot exclude that off-shell covariantization can be successful in this case, we conclude that compared to the case $n = 4$, the additional structures are highly restricted.

In the case $n = 3$, the local coordinates $(\xi^a, \zeta^A, \bar{\xi}_a, \bar{\zeta}_A)$ are of degrees $(1, 2, 2, 1)$ and the available Hamiltonian functions provide a rich structure. Since the conjugate variables attain the same degrees as the coordinates on $\mathcal{X}$, there are strong deformations of F and H. Therefore, a successful off-shell covariantization can be feasible.

The case $n = 2$ is special, since it leads to a Courant algebroid structure. The algebraic structures become continuous in the conjugate coordinate $\bar{\zeta}_A$, which attains degree 0.

4.5 Summary

Although the dynamics of single M2- and M5-branes has already been studied for decades, just recently there has been made progress in the study of the dynamics of multiple M2- and M5-branes. However, whereas we now have a very good understanding of the system of multiple M2-branes, the construction of action functionals that model the dynamics of systems of multiple M5-branes is still a field surrounded by many unresolved mysteries. Higher gauge theory from the perspective of higher categorification turns out to be the appropriate framework to tackle these difficult questions towards a correct description of multiple M5-branes.

In general, higher gauge theory suffers from the fake curvature condition, which highly restricts the dynamics of possible Lagrangian theories. It requires all lower form curvatures to vanish leading to an essentially free system or a BF-type topological theory, where the fake curvature condition is required by equation of motion (on-shell). We proposed a method to circumvent the fake curvature condition, thus leading to a topologically non-trivially interacting covariantly transforming higher gauge theory of non-abelian gerbes.
This method of **off-shell covariantization** makes use of the auxiliary gauge freedom. The auxiliary gauge fields are restricted to a hypersurface, such that the residual gauge transformations lead to a higher gauge theory, where the fake curvature condition is circumvented. We applied the procedure successfully for the QP-manifold $(T^*[4]W[1] \oplus V[2], \Theta^{(1)} + \Theta^{(2)}, \omega)$ and related the resulting system to the non-abelian gerbe theory of a system of multiple M5-branes compactified on a circle along the lines of [133, 134, 135].

Chapter 5

Discussion and outlook

You must learn all things, both the unshaken heart of persuasive truth, and the opinions of mortals in which there is no true warranty.

– Parmenides

String theory and M-theory are not only physically but also mathematically tremendously rich fields of research. In this thesis, we have come a long way through highly mathematical terrain to arrive at various physical results. Without recalling each and every result, let us shortly sketch the way we have come. Our journey started in 10 dimensions. We investigated various symmetry structures associated to T-duality in string theory, the geometry of non-geometric backgrounds and generalized flux algebras. Then, we climbed up to 11 dimensions and investigated the higher gerbe structures associated with U-duality symmetric exceptional tangent bundles in M-theory. Finally, we stepped over into the realm of branes in M-theory, where we constructed a higher gauge theory, which can be associated with a system of multiple M5-branes compactified on a circle, using our method of off-shell covariantization.

Dualities in string theory and M-theory

We found a way how to encode all geometric as well as non-geometric fluxes in one single twisted Courant algebroid, summarized in theorem 3.4.3. The Courant algebroid is induced by a QP-manifold, whose classical master equation is equivalent to the generalized flux Bianchi identities. Under the restriction to Q- and R-fluxes with β being a Poisson tensor, we discovered the total cohomology of the Poisson-de Rham double complex as associated twisted Courant algebroid cohomology. It is very intriguing that the underlying graded manifold encodes various cohomological structures depending on the flux background. The Courant bracket twisted by all H-, f-, Q- and R-fluxes encodes the deformed local symmetry structure of the associated abelian gerbe of the fully twisted flux background. The reduction of this Courant algebroid along the twisted anchor for integer fluxes yields the non-abelian gauge algebra of gauged supergravities. When considering T-duality in toroidal string theory compactifications, then also the dilaton transforms non-trivially. However, it is not clear how to encode this transformation in our ansatz. It might be possible to pull a conformal factor out of the metric, $g \mapsto e^{-\phi} g$, and the vielbein, $e^a{}_i \mapsto e^{-\frac{\phi}{2}} e^a{}_i$, and mimic in this way the dilaton transformation behavior in our ansatz.

When we lifted our construction to double field theory by doubling the underlying manifold to incorporate the winding space, we could recover the local expressions for all geometric as well as non-geometric fluxes in double space. The underlying graded manifold became a pre-QP-manifold, which under reduction to a physical subspace condenses to a twisted Courant algebroid on the respective T-duality hypersurface. The twisted Courant algebroid encodes all local fluxes, generalized flux Bianchi identities and local symmetry structures of the gerbe associated with the T-duality frame. We constructed a presentation of T-duality based on the pre-QP-manifold, which induces an isomorphism of twisted Courant algebroids realized on each T-dual subspace. The pre-QP-manifold encodes the local gauge structure of double field theory, which reduces to the local gauge structure of the physical subspace under reduction via strong constraint. It remains an open problem to extract the topological information of T-duality covariant formulations from the pre-QP-manifold setup.

The Poisson-Courant algebroid is obviously the natural framework to introduce a 3-vector freedom to a Courant algebroid structure in a symmetric fashion compared to the standard Courant algebroid with H-flux. The standard Courant algebroid with H-flux as the underlying structure of generalized geometry captures the geometry of T-dual backgrounds with H-flux. Therefore, it is a natural step to analyze the features of the Poisson-Courant algebroid and what role it plays as a T-duality frame realizing R-flux in string theory. We found its supergeometric construction and a generalization of the well-known isomorphism between Lichnerowicz-Poisson cohomology and de Rham cohomology to an isomorphism of Courant algebroid cohomologies. More generally, we constructed a *flux duality* isomorphism between the QP-manifolds of the standard Courant algebroid and the Poisson-Courant algebroid. Using the AKSZ method, we constructed a topological membrane model with Poisson-Courant algebroid structure and found a string sigma model with R-flux on its boundary. This string sigma model is equivalent to a Poisson sigma model with Wess-Zumino term. The flux duality isomorphism shows the equivalence between the Poisson sigma model with H-flux Wess-Zumino term with the contravariant sigma model with R-flux Wess-Zumino term. Both models are realized by dual boundary conditions. We constructed the contravariant current algebra on the loop space with Poisson-Courant algebroid structure. It turned out to be the contravariant version of the Alekseev-Strobl current algebra with H-flux. Finally, we found that the Poisson-Courant algebroid can be interpreted as living in the winding space of double field theory. One major question that remains is *"What is the physical origin of the Poisson tensor* Π*?"*. However, if there arises a Poisson structure from a certain flux compactification of string theory, then the Poisson-Courant algebroid is the natural structure to analyze its behavior under T-duality due to the natural $O(D,D)$-structure.

In the final section, we started by recalling the gerbe structure associated with the Courant algebroid in generalized geometry. Then, we constructed the underlying graded manifold of B_n-generalized geometry, which is related to T-duality in heterotic string compactifications. The resulting structure is a heterotic Courant algebroid and we found the local symmetry Lie 2-algebra of an abelian 0-1-gerbe derived bracket construction. The gerbe contains a 2-form curvature of a Yang-Mills $U(1)$-gauge field and a 3-form curvature of a 2-form gauge field. The consistency of the underlying graded manifold requires the first Pontryagin class of the $U(1)$-principal bundle to vanish. Then, we explored the realm of exceptional generalized geometry starting with exceptional generalized tangent bundles, which can accommodate modes of M2-branes. We recovered the underlying Lie 3-algebra of local symmetries of the associated 2-gerbe via derived bracket construction. Finally, we generalized the construction to also accommodate modes of M5-branes and found the local symmetry Lie 6-algebra associated with a 2-5-gerbe. The consistency condition of the underlying graded manifold is equivalent to the relation between the 4- and 7-form curvatures from the equation of motion of 11-dimensional supergravity.

When trying to incorporate the degrees of freedom from the KK6-monopole to the generalized tangent bundle, the transformation under the adjoint of E_7 becomes highly intricate. Despite a lot of effort we did not yet discover a natural way to reconstruct the resulting bundle using supergeometric structures. The whole analysis points towards a non-trivial generalization of the underlying graded manifold structures. However, it is unclear *what* should be the starting point of the generalization program. If a good principle can be found from which to set off, the whole analysis has to be reconsidered in light of the new principle. This will inevitably lead to new and inspiring insights into the realm of underlying mathematical structures of U-duality in M-theory.

Higher gauge theory and multiple M5-branes

We proposed a new method to generate higher gauge theories that are free of the fake curvature issue. This method of *off-shell covariantization* makes use of the auxiliary gauge freedom to deform the gauge transformations of the various n-form field strengths. We applied the method to the setup of a 2-form higher gauge theory with differential crossed module structure. After introduction of the auxiliary gauge freedom and restriction of the auxiliary gauge fields to a hypersurface, we could derive a deformation of the differential crossed module structure, which leads to a 2-form higher gauge theory without fake curvature issue. It turned out that the resulting gauge theory is related to a system of multiple M5-branes compactified on a circle forming a non-abelian gerbe. The starting point was a QP-manifold of the form $\mathcal{M}_n = T^*[n](W[1] \oplus V[2])$. Our calculation was based on the case $n = 4$. However, as stated before, a successful off-shell covariantization might also be possible for other cases. It crucially depends on the freedom provided by the extension of the differential crossed module structure, since it is used to deform the initial gauge transformations. Furthermore, we could show a successful off-shell covariantization only after assuming a certain structure on the vector spaces W and V. It is unclear, if there might be other solutions. However, since the off-shell covariantization was proven by hand, finding other solutions can be a cumbersome task.

Outlook

Let us now elaborate on further open questions related to the analysis of this thesis.

Obviously, there is still a lot to explore regarding the underlying algebraic structures of exceptional generalized geometry. The analysis presented in this thesis is still work in progress. One main point is that the constructed algebras only capture the local symmetry structure of the underlying higher gerbes. There has to be made more effort to extract the topological information from this construction. A way to to this is by using generalizations of Chern-Weil theory. Furthermore, we apparently face problems when trying to construct E_7-bundles and higher using graded symplectic manifolds. Some generalization of the ansatz is necessary. However, it still remains unclear where to start. For lower E_n-bundles, we have the construction. The next step is to recover generalized metric and if possible a generalized Riemannian geometry that captures the U-duality symmetries.

In the generalized geometry case, we could reconstruct the geometric and non-geometric fluxes. It would be intriguing if we could do the same for exceptional generalized bundles. However, also here we face problems. It turns out that there is no problem in representing the parabolic subalgebra of the adjoint bundle as transformation of the underlying graded symplectic manifold. This would correspond to geometric orbits in the U-dual space. When trying to introduce non-geometric twists, not being represented by the parabolic subalgebra, there arise conceptual problems. One might tend to think that the problems regarding the representability of the non-parabolic subalgebra is related to the problems with the implementation of the KK6-monopole symmetry in E_7 and higher. Maybe if we can solve one of those problems, we get the solution of the other for free.

Finally, there is a closely related theory, which we did not touch at all in this thesis: *exceptional field theory*. Roughly spoken, it plays the role of double field theory in M-theory, now with E_n-symmetry instead of $O(D, D)$-symmetry. Double field theory turns out to be describable by the pre-QP-manifold generalization. So it is tempting to assume that a pre-QP-manifold generalization of the graded symplectic manifolds with E_n-structure might lead to a description of exceptional field theory. The reason, why we did not touch this terrain in this thesis is that it does not seem to work. Double field theory is based on generalized geometry, whose generalized tangent bundle $TM \oplus T^*M$ is highly symmetric. Due to this symmetry, a polarization of sections on $T\widehat{M} \oplus T^*\widehat{M}$ after introduction of the double space is possible leading to the tangent bundle structure of double field theory. However, E_n-bundles do not possess such type of symmetry and polarization is impossible. A generalization of the underlying graded symplectic manifold structure might lead out of the dilemma. If we can overcome all these conceptual problems, we find insight in the underlying mathematical structures of U-duality and U-duality symmetric field theories.

When studying the method of off-shell covariantization for the case $n = 4$ in more detail, one recognizes that one of the auxiliary field strengths has the same form degree as the 3-form field strength H. Recall, that the covariant gauge transformation of H induces the fake curvature issue. It is natural to consider a generalized variable that combines the degree 2 local coordinates in $\mathcal{M}_n$. This generalized variable induces a generalized 3-form field strength. What now happens during off-shell covariantization is that the generalized 3-form field strength splits into H and the auxiliary 3-form field strength in a special way, that leads to a covariant transformation behavior of H. This splitting can be analyzed on the level of the associated L_∞-algebra. When the mechanism of splitting can be understood from the viewpoint of graded manifolds and L_∞-algebras, then we could take it as the starting point to off-shell covariantize even more complicated n-form higher gauge theories. This would lead to new insight towards more intricate M-brane dynamics. A further generalization of the ansatz is to promote the structure constants to structure functions, leading to algebroid gauge theories [164]. Let us also note that the procedure of off-shell covariantization does not automatically lead to a self-dual field strength. Investigations towards an integration of the self-duality feature into the covariantization method would be one of the next goals. Finally, there might be a way to facilitate the restriction process of the auxiliary gauge fields as gauge fixing procedure.

Appendix A

Detail calculations

A.1 Special subset of solutions

Proposition A.1.1 *Let* $(\mathcal{M}_4 = T^*[4](W[1] \oplus V[2]), \Theta^{(1)} + \Theta^{(2)}, \omega)$ *the QP-manifold as defined in 4.4.5. Assume that* $T^A_{abc} = 0$*. Furthermore, assume that* g_{ab} *is a metric on* W *and* $s^A_a t^a_B = \delta^A_B$*, where* $s^A_a = g_{ab} s^{bA}$*. Then,*

$$\exp(\delta_{s^{aA} s^B_a})(\Theta^{(1)}) = \Theta^{(1)} + \Theta^{(2)}. \tag{A.1}$$

Proof Let us define $G^{ab} = t^a_A s^{bA}$, which is (ab)-symmetric. Furthermore, let us define the projector $P^a_b = t^a_A s^A_b$. From $\{\Theta^{(1)} + \Theta^{(2)}, \Theta^{(1)} + \Theta^{(2)}\} = 0$ we find

$$\alpha^A_{aB} = s^A_b t^c_B f^b_{ca}, \tag{A.2}$$

$$n^{AB}_a = 2 s^{c(A} s^{B)}_b f^b_{ca}, \tag{A.3}$$

$$0 = g^{dg} s^{(A}_g \{ s^{B)}_e (\delta^f_c - P^f_c) f^e_{f[a} f^c_{b]d} \}. \tag{A.4}$$

The Hamiltonian function $\Theta = \Theta^{(1)} + \Theta^{(1)}$ generating this special subset of solutions can be reached by canonical transformation of $\Theta^{(1)}$,

$$\exp(\delta_{s^{aA} s^B_a})(\Theta^{(1)}) = \Theta^{(1)} + \Theta^{(2)}. \tag{A.5}$$

This finishes the proof. ■

A.2 Residual gauge transformations on constraint surface

Proposition A.2.1 *Let* $(\mathcal{M}_4 = T^*[4](W[1] \oplus V[2]), \Theta^{(1)} + \Theta^{(2)}, \omega)$ *be the QP-manifold as defined in 4.4.5 under the assumptions of proposition A.1.1. Furthermore, let* $W = \mathfrak{g} = K \ltimes \mathfrak{h}$*,* *where K is a Lie algebra with a representation ρ on $\mathfrak{h}$, such that the Lie bracket is given by*

$$[(k,h),(k',h')] = ([k,k'],\rho(k)h' - \rho(k')h), \tag{A.6}$$

where $k, k' \in K$ and $h, h' \in \mathfrak{h}$. Furthermore, let $V = \mathfrak{h}$ and let the operations $\underline{\alpha} : \mathfrak{g} \to Der(\mathfrak{h})$, $\underline{t} : \mathfrak{h} \to \mathfrak{g}$ and $\underline{s} : \mathfrak{g} \to \mathfrak{h}$ of the associated L_∞-algebra be defined as

$$\underline{\alpha}((k,h))h' = \rho(k)h', \tag{A.7}$$

$$\underline{t}(h) = (0, Mh), \tag{A.8}$$

$$\underline{s}(k,h) = M^{-1}h, \tag{A.9}$$

where $k \in K$, $h, h' \in \mathfrak{h}$ and M is an invertible matrix. Then, the associated higher gauge theory collapses on the constraint surface

$$C_a = -K_{abc}F^b \wedge A^c, \tag{A.10}$$

$$D_A = 0, \tag{A.11}$$

where $K_{abc} = g_{ad}t^d_A\alpha^A_{bB}s^B_c$ to

$$F^a = dA^a - \frac{1}{2}f^a_{bc}A^b \wedge A^c - t^a_A B^A, \tag{A.12}$$

$$H^A = dB^A + \alpha^A_{aB}A^a \wedge B^B - \alpha^A_{aB}s^B_c F^a \wedge A^c, \tag{A.13}$$

$$\widehat{\delta}A^a = d\widehat{\varepsilon}^a - f^a_{bc}A^b\widehat{\varepsilon}^c - t^a_A\widehat{\mu}^A, \tag{A.14}$$

$$\widehat{\delta}B^A = d\widehat{\mu}^A + \alpha^A_{jB}(A^j \wedge \widehat{\mu}^B + \widehat{\varepsilon}^j B^B) - \alpha^A_{jB}s^B_c\widehat{\varepsilon}^c F^j, \tag{A.15}$$

$$\widehat{\delta}F^a = f^a_{bc}F^b(\widehat{\varepsilon}^c - (P\widehat{\varepsilon})^c), \tag{A.16}$$

$$\widehat{\delta}H^A = \alpha^A_{aB}H^B(\widehat{\varepsilon}^a - (P\widehat{\varepsilon})^a)., \tag{A.17}$$

where $\widehat{\delta}$ denotes the residual gauge transformations on the constraint hypersurface.

Proof The gauge transformation of the 3-form field strength H^A is given by

$$\delta H^A = \alpha^A_{aB}H^B\epsilon^a - \alpha^A_{aB}F^a \wedge \mu^B - n^{AB}_a F^a \wedge \mu'_B + n^{AB}_a F^{(D)}_B \epsilon^a \tag{A.18}$$

$$\equiv \alpha^A_{aB}H^B\epsilon^a - \triangle^A. \tag{A.19}$$

If we can show that $\triangle^A = 0$ on the constraint hypersurface, then we show off-shell covariantization. We take the index convention $g^a = (g^i, g^I) \in K \ltimes \mathfrak{h}$. We can decompose

$$\begin{aligned}\triangle^A = {} & s^A_a t^b_B f^a_{jb}F^j \wedge \mu^B + (s^{bA}s^B_a f^a_{jb} + s^{bB}s^A_a f^a_{jb})F^j \wedge \mu'_B \\ & - n^{AB}_j dD_B\epsilon^j - n^{AB}_j t^a_B C_a\epsilon^j - n^{AB}_j \alpha^C_{aB}A^a \wedge D_C\epsilon^j.\end{aligned} \tag{A.20}$$

We make use of the freedom of the conjugate auxiliary fields C_a and D_A and find

$$\triangle^A = s^A_a t^b_B f^a_{jb} F^j \wedge \mu^B + (s^{bA} s^B_a f^a_{jb} + s^{bB} s^A_a f^a_{jb}) F^j \wedge \mu'_B - n^{AB}_j t^a_A C_a \epsilon^j. \tag{A.21}$$

In the next step, we introduce the gauge parameters $\widehat{\varepsilon}^a$ and $\widehat{\mu}^A$, which are associated with the residual gauge symmetry. Then, we require, that the reduced gauge transformation of the 1-form gauge field is given by

$$\delta A^a = D_0 \varepsilon^a - t^a_A \mu^A - s^{aA} \mu'_A \equiv D_0 \widehat{\varepsilon}^a - t^a_A \widehat{\mu}^A, \tag{A.22}$$

where we introduced the covariant differential $D_0 \widehat{\varepsilon}^a \equiv d\widehat{\varepsilon}^a - f^a_{bc} A^b \widehat{\varepsilon}^c$. Applying $(1-P)$ to (A.22) Gives

$$(1-P) D_0 \epsilon = (1-P) D_0 \widehat{\varepsilon}. \tag{A.23}$$

Making use of the equation

$$f^a_{ba'} P^{a'}_d = P^a_c f^c_{ba'} P^{a'}_d, \tag{A.24}$$

we find

$$\varepsilon^a - \widehat{\varepsilon}^a = (PX)^a, \tag{A.25}$$

which tells us that X is a function of $\widehat{\varepsilon}$ or zero. We apply P to (A.22),

$$s^{aA} \mu'_A + t^a_A \mu^A = t^a_A \widehat{\mu}^A + P^a_b D_0 P^b_c X^c. \tag{A.26}$$

Then, the solution of $\delta D_A = 0$ is given by

$$P^b_a \epsilon'_b = P^b_a [d(s^A_b \mu'_A) - f^d_{bc} A^c (s^A_d \mu'_A)] \equiv P^b_a D_0 s^B_b \mu'_B. \tag{A.27}$$

Furthermore, by application of PD_0 to (A.27) we find

$$P^{aa'} D_0 P^b_{a'} \epsilon'_b = P^{ab} F^i f^{b'}_{ib} s^B_{b'} \mu'_B. \tag{A.28}$$

Let us now investigate the second constraint equation $\delta C_a = -g_{ad} t^d_A \alpha^A_{b'B} s^B_c \delta(F^{b'} \wedge A^c)$. It is sufficient to analyze the projected part of this condition. First, have a look at the left hand side. Using (A.28), we can rewrite the projected part of the gauge transformation of C_a as

$$P^{a'}_a \delta C_{a'} = P^{a'}_a \varepsilon^j f^c_{ja'} P^d_c C_d + P^{a'}_a D_0 \varepsilon'_{a'}. \tag{A.29}$$

Now let us investigate the right hand side of the second gauge fixing equation. Under the requirement, that the tensor part of the 1-form gauge field transforms homogeneously under the gauge transformation

$$s^A_b \delta A^b = s^A_b D_0 \varepsilon^d - s^A_b (t^b_B \mu^B + s^{bB} \mu'_B) = -s^A_b f^b_{ac} A^a \varepsilon^c + Y^A \tag{A.30}$$

we find

$$\delta(\alpha^A_{aB} s^B_c F^a \wedge A^c) = \varepsilon^k \alpha^A_{kB'} (\alpha^{B'}_{jB} F^j \wedge A^{c'} s^B_{c'}) + \alpha^A_{jB} F^j \wedge (-s^B_b D_0 P^b_c X^c - \widehat{\mu}^B). \tag{A.31}$$

By hitting (A.31) with t^a_B one derives

$$-\delta(f^a_{iB} F^i \wedge \tilde{A}^B) = \varepsilon^k f^a_{kc} (f^c_{jB} F^j \wedge \tilde{A}^B) - f^a_{jb} F^j \wedge t^b_B (-s^B_d D_0 P^d_c X^c - \widehat{\mu}^B), \tag{A.32}$$

where $\tilde{A}^B = (PA)^B$ denotes the projected part. We can now use (A.29) and (A.32) in order to derive the covariance condition coming from the second constraint equation, leading to

$$P^{aa'} D_0 \varepsilon'_{a'} = \varepsilon^j t^a_A t^c_B n^{AB}_j C_c + f^a_{jb} F^j \wedge t^b_B (-s^B_d D_0 P^d_c X^c - \widehat{\mu}^B). \tag{A.33}$$

Using the covariance condition following from the first gauge fixing constraint (A.28), we finally derive the condition on μ' in terms of $\widehat{\varepsilon}$ and $\widehat{\mu}$, which makes H^A covariant

$$P^{ab} F^i f^{b'}_{ib} s^B_{b'} \mu'_B = \varepsilon^j t^a_A t^c_B n^{AB}_j C_c + f^a_{jb} F^j \wedge t^b_B (-s^B_d D_0 P^d_c X^c - \widehat{\mu}^B). \tag{A.34}$$

Let us show this explicitly by investigating $\triangle^A$,

$$\triangle^A = s^A_a t^b_B f^a_{jb} F^j \wedge \mu^B + (s^{bA} s^B_a f^a_{jb} + s^{bB} s^A_a f^a_{jb}) F^j \wedge \mu'_B - n^{AB}_j t^a_A C_a \varepsilon^j \tag{A.35}$$

$$= s^A_a f^a_{jb} F^j \wedge t^b_B (\widehat{\mu}^B + s^B_d D_0 P^d_c X^c) + s^{bA} s^B_a f^a_{jb} F^j \wedge \mu'_B - n^{AB}_j t^b_B C_b \varepsilon^j. \tag{A.36}$$

It turns out that $\triangle^A = 0$ if $X^a = -\widehat{\varepsilon}^a$ on the constraint hypersurface. The last condition emerges from the orthogonal projection of the second constraint equationm

$$(1-P)^b_a \delta C_b = -(1-P)^b_a g_{bd} t^d_A \alpha^A_{b'B} s^B_c \delta(F^{b'} \wedge A^c) = 0. \tag{A.37}$$

This equation imposes $(1-P)^a_i \epsilon'_a = \epsilon'_i$. Inserting the residual gauge parameters into the gauge transformations, we find the residual gauge transformations. We showed that the gauge fixing procedure leads to off-shell covariantization. This finishes the proof. ■

Appendix B

Conventions and formulas

In this section, we provide a detailed introduction to the (graded) mathematics underlying this thesis.

B.1 Graded differential geometry

Here, we define the objects which we use for differential geometry on graded manifolds. Let $\mathcal{M}$ be a graded manifold and let the local coordinates on $\mathcal{M}$ be denoted as z^i. Let $f \in \mathcal{C}^\infty(\mathcal{M})$ be a smooth function on the graded space $\mathcal{M}$. Degrees of any objects defined on and over $\mathcal{M}$ are denoted by $|-|$. We define the de Rham differential δ on $\mathcal{M}$ via

$$\delta f = \delta z^i \frac{\vec{\partial} f}{\partial z^i}. \tag{B.1}$$

We assign degree 1 to the de Rham differential, $|\delta| = 1$, which is due to the fact that it raises the form degree by one. Furthermore, a vector field $X \in \mathfrak{X}^1(\mathcal{M})$ over $\mathcal{M}$ is defined by

$$X = X^i \partial_i, \tag{B.2}$$

where we denote $\partial_i = \frac{\partial}{\partial z^i}$. Then, we define the interior product with respect to a vector field X via

$$\iota_X = (-1)^{|X|} X^i \frac{\vec{\partial}}{\partial \delta z^i}, \tag{B.3}$$

where $|X|$ denotes the degree of X. The differential is defined such that

$$\frac{\vec{\partial}}{\partial \delta z^i} \delta z^j = \delta^i_j. \tag{B.4}$$

By degree counting, we easily find $|\delta z^a| = |z^a| + 1$ and $|\iota_X| = |X| - 1$.

In the next step, we compute the graded Lie commutator of vector fields over $\mathcal{M}$. Let $X, Y \in \mathfrak{X}(\mathcal{M})$ be vector fields over $\mathcal{M}$. Their graded Lie commutator is given by

$$[X, Y] = X^i \frac{\vec{\partial} Y^j}{\partial z^i} \frac{\vec{\partial}}{\partial z^j} - (-1)^{|X||Y|} Y^i \frac{\vec{\partial} X^j}{\partial z^i} \frac{\vec{\partial}}{\partial z^j}, \tag{B.5}$$

where the additional factor $(-1)^{|X||Y|}$ is due to the fact that the vector fields are graded. The action of a graded vector field X on a function $f \in \mathcal{C}^\infty(\mathcal{M})$ is given by

$$Xf = (-1)^{|X|} \iota_X \delta f = (-1)^{(|f|+1)|X|} \delta f(X), \tag{B.6}$$

where

$$\delta z^i \left(\frac{\vec{\partial}}{\partial z^j} \right) = \delta^i_j. \tag{B.7}$$

Finally, the Lie derivative along a vector field X is defined by

$$L_X = \iota_X \delta - (-1)^{(|X|-1)} \delta \iota_X = \iota_X \delta + (-1)^{|X|} \delta \iota_X, \tag{B.8}$$

and has degree $|L_X| = |\iota_X| + |\delta| = |X| - 1 + 1 = |X|$.

B.2 Graded symplectic geometry

Here, we will introduce the necessary tools to perform manipulations in graded symplectic geometry.

Let $(\mathcal{M}, \omega)$ be a graded manifold $\mathcal{M}$ with graded symplectic structure ω of degree n. First, let us introduce Darboux coordinates (ξ^i, ζ_i) such that $|\xi^i| + |\zeta_i| = n$. Then, the graded symplectic form can be expanded via

$$\omega = (-1)^{|\xi^i|(|\zeta_i|+1)} \delta \xi^i \wedge \delta \zeta_i = (-1)^{|\zeta_i|+1} \delta \zeta_i \wedge \delta \xi^i. \tag{B.9}$$

In the next step, we introduce the Liouville 1-form by the defining relation

$$\omega = -\delta \vartheta. \tag{B.10}$$

A choice for the Liouville 1-form is given by

$$\vartheta = (-1)^{|\zeta_i|} \zeta_i \delta \xi^i = -\delta \zeta_i \xi^i. \tag{B.11}$$

The Hamiltonian vector field X_f with respect to a function $f \in \mathcal{C}^\infty(\mathcal{M})$ is defined by the equation

$$\iota_{X_f}\omega = -\delta f, \tag{B.12}$$

and its degree is given by $|X_f| = |f| - n$. Let us expand the vector field X locally in Darboux coordinates by

$$X = X_i \frac{\overrightarrow{\partial}}{\partial \zeta_i} + Y^i \frac{\overrightarrow{\partial}}{\partial \xi^i}. \tag{B.13}$$

Then, we find the associated Hamiltonian vector field X_f as

$$X_f = \frac{f\overleftarrow{\partial}}{\partial \xi^i}\frac{\overrightarrow{\partial}}{\partial \zeta_i} - (-1)^{|\xi^i||\zeta_i|}\frac{f\overleftarrow{\partial}}{\partial \zeta_i}\frac{\overrightarrow{\partial}}{\partial \xi^i}. \tag{B.14}$$

Furthermore, we have the swapping rule

$$\frac{f\overleftarrow{\partial}}{\partial \xi^i} = (-1)^{(|f|-|\xi^i|)|\xi^i|}\frac{\overrightarrow{\partial} f}{\partial \xi^i}. \tag{B.15}$$

The graded Poisson bracket induced by the graded symplectic structure ω is defined by

$$\{f, g\} = X_f g = (-1)^{|f|-n}\iota_{X_f} dg = (-1)^{|f|-n+1}\iota_{X_f}\iota_{X_g}\omega. \tag{B.16}$$

One can verify the graded antisymmetry, graded Leibniz rule and graded Jacobi identity of the graded Poisson bracket,

$$\{f, g\} = -(-1)^{(|f|-n)(|g|-n)}\{g, f\}, \tag{B.17}$$

$$\{f, gh\} = \{f, g\}h + (-1)^{(|f|-n)|g|}g\{f, g\}, \tag{B.18}$$

$$\{f, \{g, h\}\} = \{\{f, g\}, h\} + (-1)^{(|f|-n)(|g|-n)}\{g, \{f, h\}\}. \tag{B.19}$$

Locally, the graded Poisson bracket is given by

$$\{\xi^i, \zeta_j\} = \delta^i_j, \tag{B.20}$$

$$\{\zeta_j, \xi^i\} = -(-1)^{|\xi^i||\zeta_i|}\delta^i_j, \tag{B.21}$$

for Darboux coordinates and

$$\{f, g\} = \frac{f\overleftarrow{\partial}}{\partial \xi^i}\frac{\overrightarrow{\partial} g}{\partial \zeta_i} - (-1)^{|\xi^i||\zeta_i|}\frac{f\overleftarrow{\partial}}{\partial \zeta_i}\frac{\overrightarrow{\partial} g}{\partial \xi^i}, \tag{B.22}$$

where f and g are smooth functions on $\mathcal{M}$ in Darboux coordinates.

B.3 Graded functional analysis I: Fields and derivatives

Here, we define the tools for the graded functional analysis on the supergeometric mapping space.

Let $\chi = T[1]X$ be a graded manifold, where X is a d-dimensional smooth manifold. Locally, χ is parameterized by coordinates (σ^μ, θ^μ) of degrees $(0, 1)$ and $\mu = 1, \ldots, d$. The local coordinates σ^μ parameterize X and the degree-shifted fiber is parameterized by local coordinates θ^μ, which are Grassmann odd. For convenience, we might write the local coordinates in a combined way, $z = (\sigma, \theta)$.

Let $\Psi = \Psi(\sigma, \theta)$ be a field of degree $|\Psi|$. We choose its expansion in Grassmann coordinates in the following way,

$$\Psi(\sigma, \theta) = \sum_{j=0}^{d} \frac{1}{j!} \psi_{\mu_1 \cdots \mu_j}(\sigma) \theta^{\mu_1} \cdots \theta^{\mu_j}. \tag{B.23}$$

Since $|\theta^\mu| = 1$, the field components have degree $|\psi_{\mu_1 \cdots \mu_j}| = |\Psi| - j$.

In the next step, we find the local form of the functional right-derivative.

Proposition B.3.1 *The form of the field expansion* (B.23) *and the requirement*

$$\frac{\vec{\delta}\,\Psi(\sigma, \theta)}{\delta \Psi(\sigma', \theta')} = \delta^d(\sigma' - \sigma) \delta^d(\theta' - \theta) = \delta^{d,d}(z' - z) \tag{B.24}$$

fixes the form of the functional right-derivative to be

$$\frac{\vec{\delta}}{\delta \Psi(\sigma, \theta)} = \sum_{j=0}^{d} \frac{(-1)^{d-j}}{j!(d-j)!} \theta^{\mu_1} \cdots \theta^{\mu_j} \epsilon_{\mu_1 \cdots \mu_j \mu_{j+1} \cdots \mu_d} \frac{\vec{\delta}}{\delta \psi_{\mu_{j+1} \cdots \mu_d}(\sigma)}, \tag{B.25}$$

where ϵ denotes the Levi-Civita symbol. The components of the functional right-derivative have degree $\left| \frac{\vec{\delta}}{\delta \psi_{\mu_{j+1} \cdots \mu_d}(\sigma)} \right| = -(|\Psi| - d + j)$.

Proof

$$\frac{\overrightarrow{\delta}\,\Psi(\sigma,\theta)}{\delta\Psi(\sigma',\theta')} = \sum_{j=0}^{d}\sum_{k=0}^{d} \frac{(-1)^{d-j}}{j!(d-j)!k!}\theta'^{\mu_1}\cdots\theta'^{\mu_j}\epsilon_{\mu_1\cdots\mu_j\mu_{j+1}\cdots\mu_d}\frac{\overrightarrow{\delta}}{\delta\psi_{\mu_{j+1}\cdots\mu_d}(\sigma')}\left[\psi_{\nu_1\cdots\nu_k}(\sigma)\theta^{\nu_1}\cdots\theta^{\nu_k}\right]$$

$$= \sum_{j,k=0,\, j=d-k}^{d} \frac{(-1)^{d-j}}{j!(d-j)!k!}\theta'^{\mu_1}\cdots\theta'^{\mu_j}\epsilon_{\mu_1\cdots\mu_j\mu_{j+1}\cdots\mu_d}\delta^{\mu_1}_{\nu_1}\cdots\delta^{\mu_d}_{\nu_k}(k!)\delta^d(\sigma-\sigma')\theta^{\nu_1}\cdots\theta^{\nu_k}$$

$$= \sum_{j=0}^{d}\frac{(-1)^{d-j}}{j!(d-j)!}\theta'^{\mu_1}\cdots\theta'^{\mu_j}\epsilon_{\mu_1\cdots\mu_j\mu_{j+1}\cdots\mu_d}\theta^{\mu_{j+1}}\cdots\theta^{\mu_d}\delta^d(\sigma-\sigma')$$

$$= \sum_{j=0}^{d}\frac{(-1)^{d-j}}{j!(d-j)!}\frac{1}{\binom{d}{j}}\epsilon_{\mu_1\cdots\mu_d}\,(\text{all distributions of } j\ \theta^{\mu_i} \text{ into } d-j\ \theta'^{\mu_k})\,\delta^d(\sigma-\sigma')$$

$$= \sum_{j=0}^{d}(-1)^{d-j}\,(\text{all distributions of } j\ \theta^{\mu_i} \text{ into } d-j\ \theta'^{\mu_k} \text{ (indices fixed in order 1 to } d))\,\delta^d(\sigma-\sigma')$$

$$= (-1)^d\prod_{\mu=1}^{d}(\theta^\mu-\theta'^\mu)\delta^d(\sigma-\sigma')$$

$$= (-1)^d\delta^d(\theta-\theta')\delta^d(\sigma-\sigma')$$

$$= \delta^d(\theta'-\theta)\delta^d(\sigma'-\sigma)$$

$$= \delta^{d,d}(z'-z) \tag{B.26}$$

We used

$$\frac{\overrightarrow{\delta}\,\psi(\sigma)}{\delta\psi(\sigma')} = \delta^d(\sigma-\sigma'). \tag{B.27}$$

This finishes the proof. ■

Note that the Grassmann odd d-dimensional δ-function is given by

$$\delta^d(\theta'-\theta) = \prod_{\mu}^{d}(\theta'^\mu-\theta^\mu). \tag{B.28}$$

Using an additional requirement, we can now fix the functional left-derivative.

Proposition B.3.2 *Let us require the following swapping rule between functional right- and left-derivatives,*

$$\frac{\overrightarrow{\delta}\,F}{\delta\Psi} = (-1)^{|F|(|\Psi|-d)}\frac{F\overleftarrow{\delta}}{\delta\Psi}, \tag{B.29}$$

where F is an arbitrary superfield. Then, we can fix the form of the left-derivative to be

$$\frac{\overleftarrow{\delta}}{\delta\Psi(\sigma,\theta)} = \sum_{j=0}^{d}\frac{1}{j!(d-j)!}(-1)^{|\Psi|+j(|\Psi|+d+1)}\frac{\overleftarrow{\delta}}{\delta\psi_{\mu_{j+1}\cdots\mu_d}}\theta^{\mu_1}\cdots\theta^{\mu_j}\epsilon_{\mu_1\cdots\mu_j\mu_{j+1}\cdots\mu_d}. \tag{B.30}$$

Proof The functional right-derivative can be manipulated as follows,

$$
\begin{aligned}
\frac{\overrightarrow{\delta} F}{\delta\Psi(\sigma,\theta)} &= \sum_{j=0}^{d} \frac{(-1)^{d-j}}{j!(d-j)!}\theta^{\mu_1}\cdots\theta^{\mu_j}\epsilon_{\mu_1\cdots\mu_j\mu_{j+1}\cdots\mu_d}\frac{\overrightarrow{\delta} F}{\delta\psi_{\mu_{j+1}\cdots\mu_d}(\sigma)} \\
&= \sum_{j=0}^{d} \frac{(-1)^{d-j+(|\Psi|-(d-j))((|\Psi|-(d-j))-|F|)}}{j!(d-j)!}\theta^{\mu_1}\cdots\theta^{\mu_j}\epsilon_{\mu_1\cdots\mu_j\mu_{j+1}\cdots\mu_d}\frac{F\overleftarrow{\delta}}{\delta\psi_{\mu_{j+1}\cdots\mu_d}(\sigma)} \\
&= \sum_{j=0}^{d} \frac{(-1)^{d-j+(|\Psi|-(d-j))((|\Psi|-(d-j))-|F|)+j(|F|+(|\Psi|-(d-j)))}}{j!(d-j)!}\frac{F\overleftarrow{\delta}}{\delta\psi_{\mu_{j+1}\cdots\mu_d}(\sigma)}\theta^{\mu_1}\cdots\theta^{\mu_j}\epsilon_{\mu_1\cdots\mu_j\mu_{j+1}\cdots\mu_d}.
\end{aligned}
\tag{B.31}
$$

The sign factor gives

$$
\begin{aligned}
&d-j+(|\Psi|-(d-j))((|\Psi|-(d-j))-|F|)+j(|F|+(|\Psi|-(d-j))) \bmod 2 \\
&\quad = d+j+|\Psi|+d+j+|\Psi||F|+d|F|+j|F|+j|F|+j|\Psi|+jd+j \bmod 2 \\
&\quad = |\Psi|+|\Psi||F|+d|F|+j|\Psi|+jd+j \bmod 2.
\end{aligned}
\tag{B.32}
$$

In the next step, pull out $(|F||\Psi|-|F|d)$ to give

$$
\begin{aligned}
\frac{\overrightarrow{\delta} F}{\delta\Psi(\sigma,\theta)} &= (-1)^{|F|(|\Psi|-d)}\sum_{j=0}^{d}\frac{(-1)^{|\Psi|+j(|\Psi|+d+1)}}{j!(d-j)!}\frac{F\overleftarrow{\delta}}{\delta\psi_{\mu_{j+1}\cdots\mu_d}(\sigma)}\theta^{\mu_1}\cdots\theta^{\mu_j}\epsilon_{\mu_1\cdots\mu_j\mu_{j+1}\cdots\mu_d} \\
&= (-1)^{|F|(|\Psi|-d)}\frac{F\overleftarrow{\delta}}{\delta\Psi(\sigma,\theta)}
\end{aligned}
\tag{B.33}
$$

while we fixing the form of the functional left-derivative to be

$$
\frac{F\overleftarrow{\delta}}{\delta\Psi(\sigma,\theta)} = \sum_{j=0}^{d}\frac{(-1)^{|\Psi|+j(|\Psi|+d+1)}}{j!(d-j)!}\frac{F\overleftarrow{\delta}}{\delta\psi_{\mu_{j+1}\cdots\mu_d}(\sigma)}\theta^{\mu_1}\cdots\theta^{\mu_j}\epsilon_{\mu_1\cdots\mu_j\mu_{j+1}\cdots\mu_d}. \tag{B.34}
$$

This finishes the proof. ■

Then, we can evaluate the action of the functional left- and right-derivatives along a function on the function itself.

Proposition B.3.3 *The following two equations hold*

$$
\frac{\overrightarrow{\delta}\Psi(\sigma,\theta)}{\delta\Psi(\sigma',\theta')} = \delta^{d,d}(z'-z), \tag{B.35}
$$

$$
\frac{\Psi(\sigma,\theta)\overleftarrow{\delta}}{\delta\Psi(\sigma',\theta')} = (-1)^{|\Psi|(1+d)+d}\delta^{d,d}(z-z'). \tag{B.36}
$$

Proof The first equation was shown above. The second equation can be computed to give

$$\frac{\Psi(\sigma,\theta)\overleftarrow{\delta}}{\delta\Psi(\sigma',\theta')} = \sum_{j=0}^{d}\sum_{k=0}^{d}\frac{(-1)^{|\Psi|+j(|\Psi|+d+1)}}{j!(d-j)!k!}\psi_{\nu_1\cdots\nu_k}(\sigma)\theta^{\nu_1}\cdots\theta^{\nu_k}\frac{\overleftarrow{\delta}}{\delta\psi_{\mu_{j+1}\cdots\mu_d}(\sigma')}\theta'^{\mu_1}\cdots\theta'^{\mu_j}\epsilon_{\mu_1\cdots\mu_j\mu_{j+1}\cdots\mu_d}$$

$$= \sum_{j=0}^{d}\frac{(-1)^{|\Psi|+j(|\Psi|+d+1)+(|\Psi|-d+j)(d-j)}}{j!(d-j)!}\theta^{\mu_{j+1}}\cdots\theta^{\mu_d}\theta'^{\mu_1}\cdots\theta'^{\mu_j}\epsilon_{\mu_1\cdots\mu_d}\delta^d(\sigma-\sigma')$$

$$= \sum_{j=0}^{d}\frac{(-1)^{|\Psi|+jd+j|\Psi|+j+|\Psi|d+|\Psi|j+d+dj+dj+j+dj}}{j!(d-j)!}\theta'^{\mu_1}\cdots\theta'^{\mu_j}\theta^{\mu_{j+1}}\cdots\theta^{\mu_d}\epsilon_{\mu_1\cdots\mu_d}\delta^d(\sigma-\sigma')$$

$$= (-1)^{|\Psi|(d+1)+d}\prod_{\mu=1}^{d}(\theta^\mu-\theta'^\mu)\delta^d(\sigma-\sigma')$$

$$= (-1)^{|\Psi|(d+1)+d}\delta^{d,d}(z-z'), \tag{B.37}$$

where we used

$$\frac{\Psi(\sigma)\overleftarrow{\delta}}{\delta\Psi(\sigma')} = \delta^d(\sigma-\sigma'). \tag{B.38}$$

This finishes the proof. ■

Let us summarize the degrees of the functional right- and left-derivatives.

Proposition B.3.4 *The degrees of the functional right- and left-derivatives are given by*

$$\left|\frac{\overrightarrow{\delta}}{\delta\Psi(\sigma,\theta)}\right| = \left|\frac{\overleftarrow{\delta}}{\delta\Psi(\sigma,\theta)}\right| = d-|\Psi|. \tag{B.39}$$

Furthermore, for an arbitrary superfield F of degree $|F|$ we find the following degrees,

$$\left|\frac{\overrightarrow{\delta}F}{\delta\Psi(\sigma,\theta)}\right| = \left|\frac{F\overleftarrow{\delta}}{\delta\Psi(\sigma,\theta)}\right| = |F|+d-|\Psi|. \tag{B.40}$$

Proof Thorough inspection of the objects involved suffices,

$$\frac{\overrightarrow{\delta}}{\delta\Psi(\sigma,\theta)} = \sum_{j=0}^{d}\frac{(-1)^{d-j}}{j!(d-j)!}\theta^{\mu_1}\cdots\theta^{\mu_j}\epsilon_{\mu_1\cdots\mu_j\mu_{j+1}\cdots\mu_d}\frac{\overrightarrow{\delta}}{\delta\psi_{\mu_{j+1}\cdots\mu_d}(\sigma)}. \tag{B.41}$$

As $|\theta^\mu| = 1$ and $\left|\frac{\vec{\delta}}{\delta\phi_{\mu_{j+1}\cdots\mu_d}(\sigma)}\right| = -(|\Phi|-(d-j))$ and we have a product consisting of j θ^μ we find the whole degree to be the sum

$$j-(|\Psi|-(d-j)) \bmod 2 = d-|\Psi| \bmod 2.$$

The reasoning is of course also true for the right-functional derivative. In the case of an action onto a superfield F, the degree rises by $|F|$ to be $|F|+d-|\Phi|$. This finishes the proof. ■

In the remainder of this section, we show the graded Leibniz rules.

Proposition B.3.5 *The following left and right Leibniz rules hold for arbitrary superfields F and G,*

$$\frac{\overrightarrow{\delta}(FG)}{\delta\Psi} = \frac{\overrightarrow{\delta}F}{\delta\Psi}\cdot G + (-1)^{|F|(d-|\Psi|)} F\cdot\frac{\overrightarrow{\delta}G}{\delta\Psi} \tag{B.42}$$

$$\frac{(FG)\overleftarrow{\delta}}{\delta\Psi} = F\cdot\frac{G\overleftarrow{\delta}}{\delta\Psi} + (-1)^{|G|(d-|\Psi|)}\frac{F\overleftarrow{\delta}}{\delta\Psi}\cdot G. \tag{B.43}$$

Proof The first equation has to be shown using the definition of the functional right-derivative. The second equation can then be deduced from the first. The first equation gives

$$\begin{aligned}
\frac{\overrightarrow{\delta}(FG)}{\delta\Psi(\sigma,\theta)} &= \sum_{j=0}^{d}\frac{(-1)^{d-j}}{j!(d-j)!}\theta^{\mu_1}\cdots\theta^{\mu_j}\epsilon_{\mu_1\cdots\mu_j\mu_{j+1}\cdots\mu_d}\frac{\overrightarrow{\delta}(FG)}{\delta\psi_{\mu_{j+1}\cdots\mu_d}(\sigma)} \\
&= \sum_{j=0}^{d}\frac{(-1)^{d-j}}{j!(d-j)!}\theta^{\mu_1}\cdots\theta^{\mu_j}\epsilon_{\mu_1\cdots\mu_j\mu_{j+1}\cdots\mu_d}\frac{\overrightarrow{\delta}F}{\delta\psi_{\mu_{j+1}\cdots\mu_d}(\sigma)}\cdot G \\
&\quad + (-1)^{|F|(|\Psi|-(d-j))}\sum_{j=0}^{d}\frac{(-1)^{d-j}}{j!(d-j)!}\theta^{\mu_1}\cdots\theta^{\mu_j}\epsilon_{\mu_1\cdots\mu_j\mu_{j+1}\cdots\mu_d}F\cdot\frac{\overrightarrow{\delta}G}{\delta\psi_{\mu_{j+1}\cdots\mu_d}(\sigma)} \\
&= \sum_{j=0}^{d}\frac{(-1)^{d-j}}{j!(d-j)!}\theta^{\mu_1}\cdots\theta^{\mu_j}\epsilon_{\mu_1\cdots\mu_j\mu_{j+1}\cdots\mu_d}\frac{\overrightarrow{\delta}F}{\delta\psi_{\mu_{j+1}\cdots\mu_d}(\sigma)}\cdot G \\
&\quad + (-1)^{|F|(|\Psi|-(d-j))+j|F|}F\cdot\sum_{j=0}^{d}\frac{(-1)^{d-j}}{j!(d-j)!}\theta^{\mu_1}\cdots\theta^{\mu_j}\epsilon_{\mu_1\cdots\mu_j\mu_{j+1}\cdots\mu_d}\frac{\overrightarrow{\delta}G}{\delta\psi_{\mu_{j+1}\cdots\mu_d}(\sigma)} \\
&= \frac{\overrightarrow{\delta}F}{\delta\Psi(\sigma,\theta)}\cdot G + (-1)^{|F|(d-|\Psi|)}F\cdot\frac{\overrightarrow{\delta}G}{\delta\Psi(\sigma,\theta)}.
\end{aligned} \tag{B.44}$$

The second equation follows via swapping rule,

$$\begin{aligned}
\frac{\overrightarrow{\delta}(FG)}{\delta\Psi} &= \frac{\overrightarrow{\delta}F}{\delta\Psi}\cdot G + (-1)^{|F|(d-|\Psi|)}F\cdot\frac{\overrightarrow{\delta}G}{\delta\Psi} \\
(-1)^{(d-|\Psi|)((|F|+|G|)+1)}\frac{(FG)\overleftarrow{\delta}}{\delta\Psi} &= (-1)^{(d-|\Psi|)(|F|+1)}\frac{F\overleftarrow{\delta}}{\delta\Psi}\cdot G + (-1)^{|F|(d-|\Psi|)+(d-|\Psi|)(|G|+1)}F\cdot\frac{F\overleftarrow{\delta}}{\delta\Psi} \\
\frac{(FG)\overleftarrow{\delta}}{\delta\Psi} &= (-1)^{(d-|\Psi|)(|G|+|F|+1)+(d-|\Psi|)((|F|+|G|)+1)}F\cdot\frac{G\overleftarrow{\delta}}{\delta\Psi} \\
&\quad + (-1)^{(d-|\Psi|)(|F|+1)+(d-|\Psi|)((|F|+|G|)+1)}\frac{F\overleftarrow{\delta}}{\delta\Psi}\cdot G \\
\frac{(FG)\overleftarrow{\delta}}{\delta\Psi} &= F\cdot\frac{G\overleftarrow{\delta}}{\delta\Psi} + (-1)^{|G|(d-|\Psi|)}\frac{F\overleftarrow{\delta}}{\delta\Psi}\cdot G. \qquad \text{(B.45)}
\end{aligned}$$

This finishes the proof. ■

B.4 Graded functional analysis II: Graded symplectic geometry

In this section, we explain the differential and symplectic geometry performed via graded functional analysis on the supergeometric mapping space.

Let $\chi = T[1]X$ be a graded manifold, where X is a d-dimensional smooth manifold. Local coordinates on χ are given by (σ^μ, θ^μ) of degrees $(0,1)$. Furthermore, Let $(\mathcal{M}, \Theta, \omega)$ be a QP-manifold of degree n, such that $\mathrm{Map}(\chi, \mathcal{M})$ is a supergeometric mapping space, i.e., the space of embeddings of χ into $\mathcal{M}$. Then, let z^i be local coordinates of degrees $|z^i|$ on $\mathcal{M}$. The image of z^i under the pullback along the evaluation map shall be denoted by boldface, $\boldsymbol{z}^i = \mathrm{ev}^*(z^i)$. The element $\boldsymbol{z}^i = \boldsymbol{z}^i(\sigma,\theta)$ is a function on the supergeometric mapping space $\mathrm{Map}(\chi, \mathcal{M})$ and since the evaluation map is degree-preserving, we have $|\boldsymbol{z}^i| = |z^i|$. The local coordinates z^i on $\mathcal{M}$ become local basis superfields $\boldsymbol{z}^i$ on $\mathrm{Map}(\chi, \mathcal{M})$, whose differential geometry we now want to describe.

A vector field X is defined by

$$X = \int_\chi \mu_\chi (-1)^{d|X^i|} X^i(\boldsymbol{z}(\sigma,\theta))\frac{\overrightarrow{\delta}}{\delta\boldsymbol{z}^i(\sigma,\theta)}. \qquad \text{(B.46)}$$

Then, we can define the interior product with respect to the vector field X via

$$\iota_X = (-1)^{|X|}\int_\chi \mu_\chi (-1)^{d|X^i|} X^i(\boldsymbol{z}(\sigma,\theta))\frac{\overrightarrow{\delta}}{\delta(\delta\boldsymbol{z}^i)(\sigma,\theta)}. \qquad \text{(B.47)}$$

The interior product has total degree $|\iota_X| = |X| - 1$. The graded symplectic form Ω on the mapping space arises as the transgression of the graded symplectic form on the target space QP-manifold, $\Omega = \mu_* \mathrm{ev}^* \omega$. It has degree $|\Omega| = |\omega| - d = n - d$. Let the local expression of ω in Darboux coordinates (ξ^i, ζ_i) be given by

$$\omega = (-1)^{n|\xi^i|} \delta\xi^i \wedge \delta\zeta_i. \tag{B.48}$$

Then, the transgressed graded symplectic form shall be given by

$$\Omega = \int_\chi \mu_\chi (-1)^{n|\xi^i|} (\delta\boldsymbol{\xi}^i)(\sigma,\theta) \wedge (\delta\boldsymbol{\zeta}_i)(\sigma,\theta), \tag{B.49}$$

keeping the relative sign structure. Furthermore, we define the differential $\boldsymbol{\delta}$ on a function $f \in \mathrm{Map}(\chi, \mathcal{M})$ via

$$\boldsymbol{\delta} f = \int_\chi \mu_\chi (-1)^{d(|\boldsymbol{z}|+1)} (\delta\boldsymbol{z}^i)(\sigma,\theta) \frac{\overrightarrow{\delta} f}{\delta \boldsymbol{z}^i(\sigma,\theta)}. \tag{B.50}$$

The differential $\boldsymbol{\delta}$ has total degree 1, which arises from the form degree $|\delta| = 1$. Then, the Hamiltonian vector field X_f associated with a function $f \in \mathrm{Map}(\chi, \mathcal{M})$ is defined by

$$\iota_{X_f} \Omega = -\boldsymbol{\delta} f. \tag{B.51}$$

The degree of the Hamiltonian vector field is therefore $|X_f| = |f| - d + n$. Finally, the graded Poisson bracket on the supergeometric mapping space is defined by

$$\{f, g\}_\Omega = X_f g. \tag{B.52}$$

Proposition B.4.1 *We find the local expression of the graded Poisson bracket given by*

$$\{f,g\}_\Omega = (-1)^{d(n+1)+|\xi^i|} \int_\chi \left(\frac{f\overleftarrow{\delta}}{\delta\boldsymbol{\xi}^i} \mu_\chi \frac{\overrightarrow{\delta} g}{\delta \boldsymbol{\zeta}_i} + (-1)^{(1+n)(1+|\xi^i|)} \frac{f\overleftarrow{\delta}}{\delta\boldsymbol{\zeta}_i} \mu_\chi \frac{\overrightarrow{\delta} g}{\delta \boldsymbol{\xi}^i} \right), \tag{B.53}$$

which collapses for $d = n+1$ to

$$\{f,g\}_{BV} = \int_\chi (-1)^{d+|\xi^i|} \left(\frac{f\overleftarrow{\delta}}{\delta\boldsymbol{\xi}^i} \mu_\chi \frac{\overrightarrow{\delta} g}{\delta \boldsymbol{\zeta}_i} + (-1)^{d(1+|\xi^i|)} \frac{f\overleftarrow{\delta}}{\delta\boldsymbol{\zeta}_i} \mu_\chi \frac{\overrightarrow{\delta} g}{\delta \boldsymbol{\xi}^i} \right), \tag{B.54}$$

and for $d = n$ to

$$\{f,g\}_{PB} = \int_\chi (-1)^{|\xi^i|} \left(\frac{f\overleftarrow{\delta}}{\delta\boldsymbol{\xi}^i} \mu_\chi \frac{\overrightarrow{\delta} g}{\delta \boldsymbol{\zeta}_i} + (-1)^{(1+d)(1+|\xi^i|)} \frac{f\overleftarrow{\delta}}{\delta\boldsymbol{\zeta}_i} \mu_\chi \frac{\overrightarrow{\delta} g}{\delta \boldsymbol{\xi}^i} \right). \tag{B.55}$$

Proof Let us prove the form of the brackets. We start by writing the Hamiltonian vector field X_f in Darboux coordinates,

$$X_f = \int_\chi \mu_\chi \left[X_i(\boldsymbol{\xi}(\sigma,\theta), \boldsymbol{\zeta}(\sigma,\theta)) \frac{\vec{\delta}}{\delta \boldsymbol{\zeta}_i(\sigma,\theta)} + Y^i(\boldsymbol{\xi}(\sigma,\theta), \boldsymbol{\zeta}(\sigma,\theta)) \frac{\vec{\delta}}{\delta \boldsymbol{\xi}^i(\sigma,\theta)} \right]. \tag{B.56}$$

The interior product of the Hamiltonian vector field is then given by

$$\iota_{X_f} = (-1)^{|X_f|} \int_\chi \mu_\chi \left[X_i(\boldsymbol{\xi}(z), \boldsymbol{\zeta}(z)) \frac{\vec{\delta}}{\delta(\delta \boldsymbol{\zeta}_i)(z)} + Y^i(\boldsymbol{\xi}(z), \boldsymbol{\zeta}(z)) \frac{\vec{\delta}}{\delta(\delta \boldsymbol{\xi}^i)(z)} \right]. \tag{B.57}$$

Now we use the defining relation of the Hamiltonian vector field in order to fix the components,

$$\begin{aligned}
\iota_{X_f}\Omega &= (-1)^{|X_f|+n|\xi^i|} \left[\int_\chi \mu_\chi X_i(z) \frac{\vec{\delta}}{\delta(\delta \boldsymbol{\zeta}_i)(z)} \int_\chi \mu'_\chi \delta \boldsymbol{\xi}^j(z') \wedge \delta \boldsymbol{\zeta}_j(z') \right. \\
&\quad \left. + \int_\chi \mu_\chi Y^i(z) \frac{\vec{\delta}}{\delta(\delta \boldsymbol{\xi}^i)(z)} \int_\chi \mu'_\chi \delta \boldsymbol{\xi}^j(z') \wedge \delta \boldsymbol{\zeta}_j(z') \right] \\
&= (-1)^{|X_f|+n|\xi^i|} \left[(-1)^{(d-(1+|\zeta_i|))(-d+1+|\xi^i|)} \int_\chi \mu_\chi X_i(z) \int_\chi \mu'_\chi \delta \boldsymbol{\xi}^i(z') \delta^{d,d}(z-z') \right. \\
&\quad \left. + (-1)^{-d(d-(1+|\xi^i|))} \int_\chi \mu_\chi Y^i(z) \int_\chi \mu'_\chi \delta^{d,d}(z-z') \delta \boldsymbol{\zeta}_i(z') \right] \\
&= (-1)^{|X_f|+n|\xi^i|} \left[(-1)^{(d-(1+|\zeta_i|))(-d+1+|\xi^i|)+d+d(1+|\xi^i|)} \int_\chi \mu_\chi X_i(z) \delta \boldsymbol{\xi}^i(z) \right. \\
&\quad \left. + (-1)^{-d(d-(1+|\xi^i|))+d} \int_\chi \mu_\chi Y^i(z) \delta \boldsymbol{\zeta}_i(z) \right] \\
&= (-1)^{|X_f|+n|\xi^i|} \left[(-1)^{(d-(1+|\zeta_i|))(-d+1+|\xi^i|)+d+d(1+|\xi^i|)+|X_i|(1+|\xi^i|)} \int_\chi \mu_\chi \delta \boldsymbol{\xi}^i(z) X_i(z) \right. \\
&\quad \left. + (-1)^{-d(d-(1+|\xi^i|))+d+|Y^i|(1+|\zeta^i|)} \int_\chi \mu_\chi \delta \boldsymbol{\zeta}_i(z) Y^i(z) \right].
\end{aligned} \tag{B.58}$$

On the other hand,

$$-\boldsymbol{\delta} f = - \left[\int_\chi \mu_\chi (-1)^{d(|\xi^i|+1)} \delta \boldsymbol{\xi}^i(z) \frac{\vec{\delta} f}{\delta \boldsymbol{\xi}^i(z)} + \int_\chi \mu_\chi (-1)^{d(|\zeta_i|+1)} \delta \boldsymbol{\zeta}^i(z) \frac{\vec{\delta} f}{\delta \boldsymbol{\zeta}^i(z)} \right]. \tag{B.59}$$

Setting both hand sides equal, we find

$$\frac{\overrightarrow{\delta} f}{\delta \boldsymbol{\xi}^i(z)} = (-1)^{|\xi^i|+|\zeta_i|d+f|\xi^i|} X_i(z), \tag{B.60}$$

$$\frac{\overrightarrow{\delta} f}{\delta \boldsymbol{\zeta}_i(z)} = (-1)^{1+n|\xi^i|+n-d|\xi^i|+f|\zeta^i|} Y^i(z), \tag{B.61}$$

where we used $|X_f| = |f| + d - n$, $|X_i| = |f| + d - n + |\zeta_i|$ and $|Y^i| = f + d - n + |\xi^i|$. Using the swapping rule we can bring the Hamiltonian vector field into the form

$$X_f = (-1)^{d(n+1)+|\xi^i|} \int_\chi \left(\frac{f \overleftarrow{\delta}}{\delta \boldsymbol{\xi}^i} \mu_\chi \frac{\overrightarrow{\delta}}{\delta \boldsymbol{\zeta}_i} + (-1)^{(1+n)(1+|\xi^i|)} \frac{f \overleftarrow{\delta}}{\delta \boldsymbol{\zeta}_i} \mu_\chi \frac{\overrightarrow{\delta}}{\delta \boldsymbol{\xi}^i} \right). \tag{B.62}$$

Since $\{f, g\}_\Omega = X_f g$, finishes the proof. ■

B.5 Graded functional analysis III: Integration

In this section, we develop some formulas concerning integration and Stokes theorem in supergeometry.

The measure on χ is denoted by μ_χ and defined by

$$\mu_\chi = d\sigma^1 \wedge \cdots \wedge d\sigma^d d\theta^d \cdots d\theta^1. \tag{B.63}$$

Note the opposite order of the Berezin measure compared to the ordinary measure. Due to this choice of ordering, the Berezin integration over a Grassmann odd δ-function comes out conveniently,

$$\int_\chi \mu_\chi \delta^d(\theta - \theta') \Psi(\sigma, \theta) = \int_X d\sigma^1 \wedge \cdots \wedge d\sigma^d \Psi(\sigma, \theta'), \tag{B.64}$$

where $\delta^d(\theta - \theta') = \prod_{\mu=1}^d (\theta^\mu - \theta'^\mu)$.

Proposition B.5.1 *The following equation for the Grassmann odd δ-function holds,*

$$\int_\chi \mu_{\chi,\theta} \delta^d(\theta - \theta') \Psi(\sigma, \theta) = \Psi(\sigma, \theta'), \tag{B.65}$$

where we restrict to the Grassmann odd variable measure $\mu_{\chi,\theta} = d\theta^d \cdots d\theta^1$, for convenience.

Proof

$$
\begin{aligned}
\int_\chi \mu_{\chi,\theta}\delta^d(\theta-\theta')\Psi(\sigma,\theta) &= \int_\chi d\theta^d\cdots d\theta^1 \prod_{i=1}^d(\theta^\mu-\theta'^\mu)\sum_{k=0}^d \frac{1}{k!}\psi_{\nu_1\cdots\nu_k}(\sigma)\theta^{\nu_1}\cdots\theta^{\nu_k} \\
&= \int_\chi d\theta^d\cdots d\theta^1 \sum_{j=0}^d \frac{(-1)^j}{j!(d-j)!}\theta'^{\mu_1}\cdots\theta'^{\mu_j}\epsilon_{\mu_1\cdots\mu_d}\theta^{\mu_{j+1}}\cdots\theta^{\mu_d}\sum_{k=0}^d\frac{1}{k!}\psi_{\nu_1\cdots\nu_k}(\sigma)\theta^{\nu_1}\cdots\theta^{\nu_k} \\
&= \sum_{k=0}^d\sum_{j=0}^d \frac{(-1)^j}{k!j!(d-j)!}\int_\chi d\theta^d\cdots d\theta^1\theta'^{\mu_1}\cdots\theta'^{\mu_j}\epsilon_{\mu_1\cdots\mu_d}\theta^{\mu_{j+1}}\cdots\theta^{\mu_d}\theta^{\nu_1}\cdots\theta^{\nu_k}\psi_{\nu_1\cdots\nu_k}(\sigma)(-1)^{k(n-k)} \\
&= \sum_{k=0}^d \frac{(-1)^j}{(k!)^2(d-k)!}d\theta^d\cdots d\theta^1\theta'^{\mu_1}\cdots\theta'^{\mu_k}\epsilon_{\mu_1\cdots\mu_d}\theta^{\mu_{k+1}}\cdots\theta^{\mu_d}\theta^{\nu_1}\cdots\theta^{\nu_k}\psi_{\nu_1\cdots\nu_k}(\sigma)(-1)^{k(n-k)} \\
&= \sum_{k=0}^d \frac{(-1)^j}{(k!)^2(d-k)!}d\theta^d\cdots d\theta^1\theta^{\nu_1}\cdots\theta^{\nu_k}\theta^{\mu_{k+1}}\cdots\theta^{\mu_d}\psi_{\nu_1\cdots\nu_k}(\sigma)\theta'^{\mu_1}\cdots\theta'^{\mu_k}\epsilon_{\mu_1\cdots\mu_d} \\
&= \sum_{k=0}^d \frac{(-1)^j}{(k!)^2(d-k)!}d\theta^d\cdots d\theta^1\epsilon^{\nu_1\cdots\nu_k\mu_{k+1}\cdots\mu_d}\psi_{\nu_1\cdots\nu_k}(\sigma)\theta'^{\mu_1}\cdots\theta'^{\mu_k}\epsilon_{\mu_1\cdots\mu_d} \\
&= \sum_{k=0}^d \frac{1}{(k!)^2}\psi_{\nu_1\cdots\nu_k}(\sigma)\theta'^{\nu_1}\cdots\theta'^{\nu_k}\delta^{\nu_1\cdots\nu_k}_{\mu_1\cdots\mu_k} \\
&= \sum_{k=0}^d \frac{1}{k!}\psi_{\nu_1\cdots\nu_k}(\sigma)\theta'^{\nu_1}\cdots\theta'^{\nu_k} \\
&= \Psi(\sigma,\theta'),
\end{aligned}
\tag{B.66}
$$

where $\delta^{\nu_1\cdots\nu_k}_{\mu_1\cdots\mu_k} = k!\delta^{\nu_1}_{[\mu_1}\cdots\delta^{\nu_k}_{\mu_k]}$ denotes the generalized Kronecker delta. This finishes the proof. ■

We now collect some formulas regarding integration on manifolds with boundary.

Proposition B.5.2 *Let χ a (d,d)-dimensional manifold, bounded in every Grassmann even direction σ^μ, denoted by $\partial\chi_\mu$. Then, the following equation holds,*

$$
\int_\chi \mu_\chi \boldsymbol{d}\Psi = \sum_{i=1}^d \int_{\partial\chi_i}\mu_{\partial\chi_i}[\Psi]_i, \tag{B.67}
$$

where the boundary measure is given by $\mu_{\partial\chi_i} = d\sigma^1\wedge\cdots\wedge\widehat{d\sigma^\mu}\wedge\cdots\wedge d\sigma^d d\theta^d\cdots d\theta^1$ and the symbol $[\Psi]_i$ denotes Ψ taken on the boundary of σ^μ, setting $\theta^\mu=0$. Hat denotes omission. For simplicity, we take all surface normals to be of order 1.

Proof

$$\begin{aligned}\int_\chi \mu_\chi \boldsymbol{d}\Psi &= \int_\chi d\sigma^1\wedge\cdots\wedge d\sigma^d d\theta^d\cdots d\theta^1\theta^\nu\partial_\nu\Psi\\ &=\sum_{\mu=1}^d(-1)^{\mu-1}\int_\chi d\sigma^1\wedge\cdots\wedge d\sigma^d d\theta^d\cdots d\widehat{\theta^\mu}\cdots d\theta^1\partial_\mu\Psi_{\theta^\mu=0}\\ &=\sum_{\mu=1}^d\int_{\partial\chi_i} d\sigma^1\wedge\cdots\wedge d\widehat{\sigma}^\mu\wedge\cdots\wedge d\sigma^d d\theta^d\cdots d\widehat{\theta^\mu}\cdots d\theta^1[\Psi]_\mu\\ &=\sum_{\mu=1}^d\int_{\partial\chi_\mu}\mu_{\partial\chi_\mu}[\Psi]_\mu,\end{aligned}\tag{B.68}$$

where hat denotes omission. This finishes the proof. ■

Proposition B.5.3 *Let χ a (d,d)-dimensional dimensional manifold, bounded in every Grassmann even direction σ^μ, denoted by $\partial\chi_\mu$. Then, the following equation holds,*

$$\int_\chi \mu_\chi\Psi\cdot\boldsymbol{d}\Phi = (-1)^{|\Psi|}\sum_{\mu=0}^d\int_{\partial\chi_\mu}\mu_{\partial\chi_\mu}[\Psi\cdot\Phi]_\mu-(-1)^{|\Psi|}\int_\chi\mu_\chi\boldsymbol{d}\Psi\cdot\Phi.\tag{B.69}$$

Proof

$$\begin{aligned}&\int_\chi\mu_\chi\Psi\cdot\boldsymbol{d}\Phi=\int_\chi d\sigma^1\wedge\cdots\wedge d\sigma^d d\theta^d\cdots d\theta^1\Psi\theta^\nu\partial_\nu\Phi\\ &=(-1)^{|\Psi|}\sum_{\mu=1}^d(-1)^{\mu-1}\int_\chi d\sigma^1\wedge\cdots\wedge d\sigma^d d\theta^d\cdots d\widehat{\theta^\mu}\cdots d\theta^1(\Psi\partial_\mu\Phi)\\ &=(-1)^{|\Psi|}\sum_{\mu=1}^d\int_{\partial\chi_\mu}\mu_{\partial\chi_\mu}[\Psi\Phi]_\mu-(-1)^{|\Psi|}\sum_{\mu=1}^d(-1)^{\mu-1}\int_\chi d\sigma^1\wedge\cdots\wedge d\sigma^d d\theta^d\cdots d\widehat{\theta^\mu}\cdots d\theta^1\partial_\mu\Psi\cdot\Phi\\ &=(-1)^{|\Psi|}\sum_{\mu=1}^d\int_{\partial\chi_\mu}\mu_{\partial\chi_\mu}[\Psi\Phi]_\mu-(-1)^{|\Psi|}\int_\chi\mu_\chi\boldsymbol{d}\Psi\cdot\Phi.\end{aligned}\tag{B.70}$$

That finishes the proof. ■

Corollary B.5.4 *In the case, where χ does not have boundaries, $\partial_{\chi_\mu}=\emptyset$, then above proposition gives the usual partial integration rule for a degree 1 derivation,*

$$\int_\chi\mu_\chi\Psi\cdot\boldsymbol{d}\Phi=-(-1)^{|\Psi|}\int_\chi\mu_\chi\boldsymbol{d}\Psi\cdot\Phi.\tag{B.71}$$

Proposition B.5.5 *Let χ be a supermanifold with boundary $\partial\chi_1 \neq \emptyset$. Then, the following equation holds,*

$$\int_\chi \mu'_\chi \int_\chi \mu_\chi \boldsymbol{d}[\delta(z-z')\Psi(z)]\Phi(z') = \int_{\partial\chi_1} \mu_{\partial\chi_1} \Psi(z'_{\partial\chi_1})\Phi(z'_{\partial\chi_1}), \tag{B.72}$$

where we took the σ^1-boundary given by $\theta^1 = 0$ and evaluating the fields at $\sigma^1 = 0$ and $\sigma^1 \to \infty$, where we take the fields vanishing at infinity. Furthermore, $z_{\partial\chi_1} = (0, \sigma^2, \cdots, \sigma^d, 0, \theta^2, \cdots, \theta^d)$ and $\mu_{\partial\chi_1} = d\sigma^2 \wedge \cdots \wedge d\sigma^d d\theta^d \cdots d\theta^2$.

Proof

$$\begin{aligned}
\int_\chi \mu'_\chi \int_\chi \mu_\chi \boldsymbol{d}[\delta^{d,d}(z-z')A(z)]\Phi(z') &= \int_\chi \mu'_\chi \int_\chi \mu_\chi \boldsymbol{d}[\delta^{d,d}(z-z')\Psi(z)\Phi(z')] \\
&= \int_\chi \mu'_\chi \sum_{\mu=1}^{d} \int_{\partial\chi_\mu} \mu_{\partial\chi_\mu}[\delta^{d,d}(z-z')\Psi(z)\Phi(z')]_\mu \\
&= -\int_\chi \mu'_\chi \left[\delta(0-\sigma'^1)\delta(0-\theta'^1)\Psi(z'_{\partial\chi_1})\Phi(z')\right] \\
&= \int_{\partial\chi_1} \mu_{\partial\chi_1} \Psi(z'_{\partial\chi_1})\Phi(z'_{\partial\chi_1}).
\end{aligned} \tag{B.73}$$

In the third line we took the σ^1-boundary as described above. This finishes the proof. ■

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