Essays in Mechanism Design

著者	TAMURA SHOHEI
学位授与機関	Tohoku University
学位授与番号	11301甲第17294号
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Shohei Tamura

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Graduate School of Economics and Management
Tohoku University
Sendai, Miyagi, Japan

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Abstract

This thesis consists of three essays in mechanism design theory.

In Chapter 2, we study the design of incentive-compatible award mechanisms called *impartial nomination rules*. A group of agents has to choose one or more prize-winners from among themselves by aggregating each agent's disinterested opinion about who most deserves the prize except himself. A *nomination rule* determines the set of winners based on the opinions (nominations) represented by the agents, and is said to be *impartial* if it is designed so that one's winning is independent of one's own message, leaving no chance for anyone to influence his own winning by misrepresenting his disinterested opinion. Holzman and Moulin (2013) show that, if only one prize-winner can be selected, no impartial nomination rule simultaneously satisfies two compelling axioms called *positive unanimity* and *negative unanimity*, respectively. They also show that any single-valued impartial nomination rule satisfying an axiom called *anonymous ballots* is a constant selection. In this chapter, we show that if selecting multiple-winners is possible, there exists an impartial nomination rule satisfying the two unanimity axioms by proposing one which we call *plurality with runners-up*. On the other hand, we show that a multi-valued impartial nomination rule satisfying anonymous ballots is not necessarily constant, but, in general, violates positive unanimity.

In Chapter 3, we further investigate the design of impartial nomination rules in the same setting with the previous chapter having the following question: which multi-valued impartial rules are "superior" to others from various points of view?

To address this question, we introduce three axioms called *anonymity*, *symmetry*, and *monotonicity*, respectively, then establish a characterization result regarding the class of impartial rules satisfying these three axioms. We show that the plurality with runners-up presented in the previous chapter is the only *minimal* impartial nomination rule satisfying the three axioms, i.e., no other impartial rule satisfies the three axioms while giving a smaller (in the sense of set-inclusion) set of winners for every profile of nominations. Thus, subject to the three axioms, the impartial nomination rule that selects winners most strictly is the plurality with runners-up.

In Chapter 4, we turn our attention to a classical problem of allocating a single indivisible good to one of n agents when monetary transfer is possible. Each agent has a *valuation* for the indivisible good, and we study the design of *mechanisms* which determine who receives the indivisible good and how much money each agent receives or pays, based on the valuations reported by the agents. To prevent any agent from strategically misreporting his valuation, a celebrated incentive-compatibility axiom called *strategy-proofness* is imposed. Ando et al. (2008) show that there is no strategy-proof mechanism satisfying two desirable axioms called *symmetry* and *budget balance*, respectively, under an additional axiom of either "equal compensation," "normal compensation," or "individual rationality." In this chapter, we show that each of the last three axioms above is redundant for impossibility by proving that there is no strategy-proof mechanism satisfying symmetry and budget balance. For robustness of our result, we prove the result with a quite weak domain assumption: the sets of agents' possible valuations contains at least $n + 1$ common valuations.

Contents

Chapter 1

Introduction

Consider a situation in which a social planner wishes to allocate some goods among a group of agents (or implement some collective decision regarding themselves) in such a way that the allocation (decision) is "desirable." To know which allocation (decision) is desirable, he need to know information about the agents' preferences or opinions over all possible outcomes, which are all "private information," held only by the agents themselves. The problem here is that simply asking each agent to reveal his information would not work; someone might strategically misrepresent it to realize a more preferable outcome. In this situation, how can he achieve his objective which he has as a social planner?

Mechanism design, a field of study in economics, addresses this kind of problem (more precisely, the problem of *information asymmetry* in social and economic situations), investigating the possibility of designing institutions that make it possible for a social planner to achieve his objective through agents' strategic behavior.[1] Formally, the objective of the social planner is represented by a *social choice correspondence*, a correspondence which chooses a subset of *alternatives* for each profile of the agents'

[1]The study has a wide variety of applications, including the designs of voting rules, rules for public goods provision, auction procedures, matching markets, and so on.

private information called *types*. It is interpreted that all alternatives chosen by the correspondence are desirable according to the types of the agents. Indeed, the correspondence is regarded to satisfy one or more mathematical properties called *axioms*, each representing its normative view on which alternatives are desirable or should be chosen. Next, a *mechanism* is a pair of two elements: each agent's *message space* and an *outcome function*. Especially, the latter is a function which specifies a single alternative for each profile of messages that are chosen by the agents from their own message space. The point here is that, given the profile of the agents' *true* types, the mechanism generates a "game" situation in which each agent strategically chooses his message knowing his own type and forecasting the other agents' messages, which, combined with his own message, determine the outcome through the outcome function. Now, we say that a mechanism *implements* a social choice correspondence if, for *every* profile of types, the following two sets coincide: (i) the set of alternatives chosen by the social choice correspondence; (ii) the set of all alternatives given by the outcome function at every profile of messages that is an "equilibrium" of the generated game situation under some equilibrium concept(s). The goal of the study is to clarify which social choice correspondences are implementable by designing a mechanism.

There are two ways in which a particular social choice correspondence is implemented, namely, *direct implementation* and *indirect implementation*. Direct implementation treats the social choice correspondence itself as a mechanism by viewing the set of each agent's possible types as his message space and a single-valued selection of the social choice correspondence as the outcome function. The game situation generated by this direct mechanism is therefore a *type revelation game*, where each agent strategically chooses which type to inform, knowing his true one. This leads us to an idea of *truthful implementation*: a direct mechanism is said to be truthfully implementable if the profile of messages where each agent reports his true type is always an equilibrium under some equilibrium concept(s). Dominant strategy equilibrium and Bayesian Nash equilibrium are two central equilibrium concepts in such an idea of implementation, both of which are represented in the form of axioms: the former is represented by an axiom called *strategy-proofness*, and the latter is by one called *Bayesian incentive-compatibility*. On the other hand, indirect implementation does not require the social choice correspondence and the mechanism to be the same, allowing us to freely design both message spaces and an outcome function, including the construction of "dynamic" mechanisms. This way of implementation has been studied under various equilibrium concepts, including Nash equilibrium, subgame-perfect Nash equilibrium, iterated deletion of strictly dominated messages, and so on.

This thesis is a collection of three essays in mechanism design, all focusing on the design of direct implementation mechanisms. Chapter 2 and 3 study the design of incentive-compatible award mechanisms called *impartial nomination rules*, while Chapter 4 studies a classical mechanism design problem of allocating a single indivisible good among a number of agents with monetary transfer.

In Chapter 2, we consider a situation in which a group of agents has to choose one or more prize-winners from among themselves, by aggregating each agent's disinterested opinion about who most deserves the prize except himself. It is assumed that, while actually having such a disinterested opinion, each agent selfishly cares about the outcome of the selection: he cares about whether he himself wins the prize, but not about the winning of anyone else or the number of prize-winners.[2] In this situation, a concern is that someone might strategically misrepresent his disinterested opinion if his vote can be pivotal to his own winning as is in the plurality selection. To overcome this concern, we study the design of incentive-compatible award mechanisms called *impartial nomination rules*. Formally, a *nomination rule* is a correspondence which assigns a subset of agents, which is interpreted as the set

[2]The last part of this assumption, though it may sound strong for some occasion, is essential in our design of incentive-compatible award mechanisms. Indeed, in Section 2.5 of this chapter, we demonstrate that relaxing this assumption causes a difficulty in designing such mechanisms.

of prize-winners, for each profile of opinions represented by the agents in the form of nomination. The rule then satisfies *impartiality* if each agent's winning is determined independently of his nomination, meaning that no one has any chance to influence his own winning by strategically misrepresenting his disinterested opinion. Our goal is to find reasonable impartial nomination rules satisfying desirable axioms. Holzman and Moulin (2013) show, however, that, if we choose only one prize-winner, no impartial nomination rule simultaneously satisfies two axioms called *positive unanimity* and *negative unanimity*, respectively, where the former says that an agent should be selected as the (unique) winner if he is nominated by everyone else, while the latter says that an agent should not win if not nominated by anybody. Also, they show that any single-valued impartial nomination rule satisfying *anonymous ballots*, an axiom requiring that the determination of winners depend only on the number of nominations each agent receives so that everyone can cast his nomination anonymously, is a *constant selection*: the winner is always a predetermined agent. In this chapter, we examine whether these difficulties hold or can be escaped when we consider multi-valued nomination rules. First, we show that if selecting multiple-winners is possible, there exists an impartial nomination rule satisfying the above two unanimity axioms by proposing a simple variant of the plurality correspondence called *plurality with runners-up* (Theorem 2.1). Under this nomination rule, not only do the plurality winners always win, but also each "runner-up" wins if there is only one plurality winner, the runner-up nominates the plurality winner, and the difference of the numbers of nominations they obtain is one. On the other hand, we show that, though a multi-valued impartial nomination rule satisfying anonymous ballots is not necessarily constant, there is no impartial nomination rule satisfying anonymous ballots and positive unanimity (Theorem 2.2).

In Chapter 3, we further investigate the design of impartial nomination rules in the same setting with the previous chapter. What we learn from Theorem 2.1 is that, by considering multi-valued nomination rules, we can escape the impossibility result regarding the satisfaction of the two unanimity axioms which Holzman and Moulin (2013) prove in the single-valued case. This, nevertheless, leaves a question: which multi-valued impartial nomination rules are the best? Indeed, the plurality with runners-up presented in Theorem 2.1 is just an example of an impartial rule satisfying the two unanimity axioms, and no other axiom is considered except anonymous ballots. In this chapter, to tackle with this question, we first introduce three new axioms, namely, *anonymity*, *symmetry*, and *monotonicity*, respectively, then establish a characterization result regarding the class of multi-valued impartial rules satisfying these three axioms. Anonymity says that any exchange of nominations between two agents should not affect the winning of any other agent, so that they have the same influence on the winnings of the others. Symmetry says that the indexes of agents should be irrelevant to their winnings, so that they have an equal chance to win. Monotonicity imposes a certain consistency requirement: any subset of winners should be included in a new set of winners when each member in the subset obtains an additional nomination from others. The result is that the plurality with runners-up is the only *minimal* impartial nomination rule satisfying the three axioms (Theorem 3.1). We define an impartial rule satisfying the three axioms as minimal if there is no other impartial rule that satisfies the three axioms while giving a smaller (in the sense of set-inclusion) set of winners for every profile of nominations. Therefore, subject to the three axioms, the impartial nomination rule that can select winners most strictly is the plurality with runners-up.

In Chapter 4, we consider a situation in which a single indivisible good is allocated to one of n agents when monetary transfer among them is possible. It is assumed that each agent has a *valuation* for the indivisible good, thus having "quasilinear" preferences over all possible pairs of his consumption of the indivisible good and the amount of money he receives or pays. We study *mechanisms* which determine who receives the indivisible good and how much money each agent receives or pays, based on the valuations reported by the agents. For incentive compatibility of mechanisms, we impose them *strategy-proofness*, requiring that truth-telling of the valuation be a weakly dominant strategy for each agent. We also impose them two axioms called *symmetry* and *budget balance*, respectively, where the former says that any agents having the same valuation should achieve the same level of utility, and the latter requires the sum of monetary transfers to be always zero. In this setting, Ando et al. (2008) show intermediate impossibility results that there is no strategy-proof mechanism satisfying symmetry and budget balance, under an additional axiom of either "equal compensation," "normal compensation," or "individual rationality," without showing the independence of the last three axioms. In this chapter, we establish a stronger impossibility result: there is no strategy-proof mechanism satisfying symmetry and budget balance (Theorem 4.1). This result is robust. Indeed, we prove it with a quite weak domain assumption: the sets of agents' possible valuations includes at least $n + 1$ common valuations. As this assumption is easily satisfied in many economic environments, our result strongly concludes that it is impossible to construct any strategy-proof mechanism satisfying the two axioms.

Chapter 2

Impartial Nomination Correspondences*

2.1 Introduction

We study the prize award problem where a group of peers must choose one (or some) of them to receive a prize when everyone cares only about one's own winning. We consider *nomination rules* that determine who should get the prize on the basis of each agent's nomination, i.e., an opinion about who should get it if not herself. Our goal is to find a reasonable nomination rule that satisfies the desirable axioms. This framework is first proposed by Holzman and Moulin (2013), who study the consequences of nomination *functions*, i.e., nomination rules defined as single-valued functions. In this chapter, to deal with a situation in which the realization of several winners is possible, we generally consider nomination *correspondences*, i.e., nomination rules defined as multi-valued functions.

*This chapter is based on Tamura, S., and Ohseto, S. (2014). Impartial Nomination Correspondences. *Social Choice and Welfare*, 43, 47-54. doi:10.1007/s00355-013-0772-9.

We want to implement the award impartially in the sense that a collective outcome reflects no one's self-interest. A necessary condition for this purpose is to elicit everyone's disinterested opinion about who should get the prize. However, it is difficult to elicit an agent's disinterested opinion because she may misreport it in order to realize an outcome in her favor. To prevent such a strategic behavior, we impose the axiom of *impartiality* on nomination rules as an incentive compatibility constraint. A nomination rule is impartial if one's opinion never affects one's own winning.[3] It is natural to think that under the impartial nomination rule, each agent reports her own disinterested opinion truthfully.

Together with impartiality, we also consider the axioms of positive unanimity, negative unanimity, and anonymous ballots for nomination rules. Positive unanimity requires that if an agent is nominated by everyone else, then she should be the unique winner. Negative unanimity requires that if an agent is not nominated by anybody, then she should not win. Anonymous ballots requires each agent's nomination to be treated equally. Although these three axioms are natural for nomination rules, it follows from Holzman and Moulin (2013) that these are incompatible with impartiality. In their article, Holzman and Moulin establish two impossibility results for nomination functions. The first result is that there is no nomination function that satisfies impartiality, positive unanimity, and negative unanimity.[4] The second is that any impartial nomination function that satisfies anonymous ballots is constant, and thus, there is no nomination function that satisfies impartiality, anonymous ballots, and positive unanimity. In this chapter, we verify whether or not these two impossibilities hold even with nomination correspondences.

With regard to the former, we obtain a positive result. After showing that with

[3]The concept of impartiality is first proposed by de Clippel et al. (2008). They consider impartiality for rules that divide a surplus among partners on the basis of each partner's opinion about the relative contributions of the other partners to the surplus.

[4]This is also true for *randomized nomination rules*, i.e., nomination rules that determine the winner by a lottery. See Theorem 4 in Holzman and Moulin (2013).

three agents, impartiality is incompatible with positive unanimity (Proposition 2.1), we show that with four or more agents, there exists a nomination correspondence, named *plurality with runners-up*, that satisfies impartiality, positive unanimity, and negative unanimity (Theorem 2.1). As its name suggests, plurality with runners-up is a modification of the plurality correspondence. Under plurality with runners-up, each runner-up also wins if and only if she nominates the unique highest nominated agent who wins by only one point.

With regard to the latter, we obtain a negative result. We show that with four or more agents, there is no nomination correspondence that satisfies impartiality, anonymous ballots, and positive unanimity (Theorem 2.2). Although impartiality and anonymous ballots do not imply constancy in the class of correspondences, an impossibility result holds if these are combined with positive unanimity.

The study of peer ratings is closely related to our study of the prize award problem. In peer ratings, each agent reports a strict ranking of all the other agents, and a social welfare function determines the social ranking on the basis of each agent's report.[5] This framework is first proposed by Ando et al. (2003), who consider the axioms of the Pareto principle and the independence of irrelevant alternatives (Arrow, 1963) for social welfare functions. The former requires that if two agents are ranked in the same way by all the other agents, a social welfare function should weakly follow the ranking. The latter requires that the social ranking between two agents should depend only on the reported rankings between them. Ando et al. (2003) show that the only social welfare function satisfying the Pareto principle and the independence of irrelevant alternatives is the indifference rule, which always ranks all agents indifferently. Considering that the latter axiom is too strong, Ohseto (2007) focuses on the Pareto principle and characterizes the Borda rule as the unique scoring rule satisfying the Pareto principle. Our situation of the prize award problem can be

[5]Since such rankings are ordinal, we call the problem "ordinal" peer ratings. For the problem of "cardinal" peer ratings, see Ng and Sun (2003) and Ohseto (2012).

regarded as a special case of peer ratings if we focus on the information about the top-ranked agent in each report, and select one (or some) of the top-ranked agents in the social ranking. Although there exists such a similarity, the axioms considered in this chapter are different from the axioms considered in their articles.

This chapter is organized as follows. In Section 2.2, we introduce notation and definitions. In Section 2.3, we propose the nomination rule called plurality with runners-up, and examine its properties. In Section 2.4, we establish an impossibility result about impartiality and anonymous ballots. In Section 2.5, we discuss some interpretation issues of impartiality for correspondences.

2.2 Notation and Definitions

Let $N = \{1, \ldots, n\}$ ($n \geq 3$) be the set of *agents*. Each agent $i \in N$ reports her *nomination* $x_i \in N \setminus \{i\}$. A list $x = (x_i)_{i \in N}$ is called a *nomination profile*. Let N_{-}^{N} denote the set of all nomination profiles. When we focus on $i, \ldots, j$'s nominations at profile x, we write $(x_i, \ldots, x_j, x_{N \setminus \{i, \ldots, j\}})$ for x. For simplicity of notation, we often use (x_i, x_{-i}) instead of $(x_i, x_{N \setminus \{i\}})$. Let 2^N be the power set of N. A *nomination rule* φ is a correspondence $\varphi : N_{-}^{N} \to 2^N \setminus \{\emptyset\}$ that assigns a non-empty subset of agents to each nomination profile. We sometimes call φ a nomination correspondence. In particular, if $|\varphi(x)| = 1$ for all $x \in N_{-}^{N}$, we call φ a nomination function. Given $x \in N_{-}^{N}$, let $s_i(x) = |\{j \in N : x_j = i\}|$ and $s(x) = (s_i(x))_{i \in N}$ denote agent i's *score* and the profile of scores at x, respectively. Given $x \in N_{-}^{N}$, let $F_x = \{i \in N : s_i(x) \geq s_j(x) \text{ for all } j \in N\}$ denote the set of agents who obtain the highest score at x.

We introduce four main axioms. Impartiality requires that one's nomination should not affect one's own winning. Positive unanimity requires that if an agent is nominated by everyone else, then she should be the unique winner. Negative unanimity requires that if an agent is not nominated by anybody, then she should not win. Anonymous ballots requires each agent's nomination to be treated equally.

Impartiality: for all $x \in N_-^N$, all $i \in N$, and all $x_i' \in N \setminus \{i\}$,

$$i \in \varphi(x) \Leftrightarrow i \in \varphi(x_i', x_{-i}).$$

Positive unanimity: for all $x \in N_-^N$ and all $i \in N$,

$$s_i(x) = n - 1 \Rightarrow \varphi(x) = \{i\}.$$

Negative unanimity: for all $x \in N_-^N$ and all $i \in N$,

$$s_i(x) = 0 \Rightarrow i \notin \varphi(x).$$

Anonymous ballots: for all $x, x' \in N_-^N$,

$$s(x) = s(x') \Rightarrow \varphi(x) = \varphi(x').$$

We give two examples of nomination rules. The most natural nomination rule is the plurality correspondence: $\varphi(x) = F_x$ for all $x \in N_-^N$. It is easy to check that the plurality correspondence satisfies positive unanimity, negative unanimity, and anonymous ballots, but not impartiality.[6] The constant rules: there exists $B \subset N$ such that $\varphi(x) = B$ for all $x \in N_-^N$, satisfy impartiality and anonymous ballots, but not positive unanimity or negative unanimity.

[6]To check non-impartiality, consider $x \in N_-^N$, $i, j \in N$, and $x_i' \in N \setminus \{i\}$ such that $i, j \in \varphi(x)$ and $x_i \neq x_i' = j$. Then, $i \notin F_{(x_i', x_{-i})}$, and thus, $i \notin \varphi(x_i', x_{-i})$.

2.3 Impartial Nomination Correspondences

In this section, we consider nomination rules that satisfy impartiality, positive unanimity, and negative unanimity. It follows from Holzman and Moulin (2013) that there is no nomination function that satisfies impartiality, positive unanimity, and negative unanimity. This brings us to the question, is there any nomination correspondence that satisfies impartiality, positive unanimity, and negative unanimity?

The following proposition states that with three agents, impartiality is incompatible with positive unanimity. We introduce the following notation and definitions. Let x^c be the cyclic profile such that $x_i^c = i + 1$ for $i = 1, \ldots, n-1$ and $x_n^c = 1$. Given $i, j \in N$, we write x_i^j if i nominates j.

Proposition 2.1. *If $n = 3$, there is no nomination rule φ that satisfies impartiality and positive unanimity.*

Proof. Assume that φ satisfies impartiality and positive unanimity. Consider (x_1^3, x_2^c, x_3^c), (x_1^c, x_2^1, x_3^c), and (x_1^c, x_2^c, x_3^2). By positive unanimity, $1 \notin \varphi(x_1^3, x_2^c, x_3^c)$, $2 \notin \varphi(x_1^c, x_2^1, x_3^c)$, and $3 \notin \varphi(x_1^c, x_2^c, x_3^2)$. Then, impartiality implies that $1, 2, 3 \notin \varphi(x^c)$, which contradicts the fact that $\varphi(x^c)$ is non-empty. $\square$

We focus on the case of $n \geq 4$. As a positive result, we find a natural nomination rule that satisfies impartiality, positive unanimity, and negative unanimity among four or more agents. We call it *plurality with runners-up*. As its name suggests, it is a modification of the plurality correspondence. The difference is that, under plurality with runners-up, each runner-up also wins if and only if she nominates the unique highest nominated agent who wins by only one point. To define plurality with runners-up, we introduce additional notation and definitions. Given $x \in N_-^N$, let $S_x = \{i \in N \setminus F_x : s_i(x) \geq s_j(x) \text{ for all } j \in N \setminus F_x\}$ denote the set of agents who obtain the second highest score at x. Given $x \in N_-^N$ and distinct $i, j \in N$ with $s_i(x) \geq s_j(x)$, let $d_x(i, j) = s_i(x) - s_j(x)$ denote the difference of scores between i

and j at x.

Definition 2.1 (*Plurality with runners-up*)**.** Let φ be such that, for all $x \in N_-^N$,
(i) if there exists $i \in N$ such that $F_x = \{i\}$ and $d_x(i, j) = 1$ for some $j \in S_x$, then $\varphi(x) = F_x \cup \{j \in S_x : x_j = i\}$;
(ii) otherwise, $\varphi(x) = F_x$.

We show that with four or more agents, plurality with runners-up satisfies impartiality, positive unanimity, and negative unanimity.

Theorem 2.1. *If $n \geq 4$, then plurality with runners-up φ satisfies impartiality, positive unanimity, and negative unanimity.*

Proof. To check impartiality, suppose, for all $x \in N_-^N$ and all $i \in N$, that $i \in \varphi(x)$. There are two cases to consider. The first case is that $i \in F_x$. Let $x_i' \neq x_i$. If $x_i' = j$ for some $j \in F_x$, then $i \in S_{(x_i', x_{-i})}$, $F_{(x_i', x_{-i})} = \{j\}$, and $d_{(x_i', x_{-i})}(j, i) = 1$. Hence, $i \in \varphi(x_i', x_{-i})$. If not, then $i \in F_{(x_i', x_{-i})}$ since $d_x(i, j) \geq 1$ for all $j \in N \setminus F_x$. Therefore, $i \in \varphi(x_i', x_{-i})$. The second case is that $i \in S_x$. In this case, there exists $j \in N$ such that $F_x = \{j\}$, $d_x(j, i) = 1$, and $x_i = j$. Let $x_i' \neq x_i$. If $x_i' = k$ for some $k \in S_x$, then $i \in S_{(x_i', x_{-i})}$, $F_{(x_i', x_{-i})} = \{k\}$, and $d_{(x_i', x_{-i})}(k, i) = 1$. Hence, $i \in \varphi(x_i', x_{-i})$. If not, then $i \in F_{(x_i', x_{-i})}$ since $F_x = \{j\}$, $d_x(j, i) = 1$, $x_i = j$, and $d_x(i, k) \geq 1$ for all $k \in N \setminus (F_x \cup S_x)$. Therefore, $i \in \varphi(x_i', x_{-i})$.

To check positive unanimity, suppose, for all $x \in N_-^N$ and all $i \in N$, that $s_i(x) = n - 1$. Since $n \geq 4$, we have $F_x = \{i\}$ and $d_x(i, j) \geq 2$ for all $j \in N \setminus \{i\}$. Therefore, $\varphi(x) = \{i\}$. To check negative unanimity, suppose, for all $x \in N_-^N$ and all $i \in N$, that $s_i(x) = 0$. Then, $i \notin F_x$ and $d_x(j, i) \geq 2$ for all $j \in F_x$. Hence, $i \notin \varphi(x)$. $\square$

2.4 An Impossibility Result

In Section 2.3, we proposed the nomination rule called plurality with runners-up, and showed that with four or more agents, it satisfies impartiality, positive unanimity, and negative unanimity. Note that with four or more agents, plurality with runners-up does not satisfy anonymous ballots. Indeed, consider $x \in N_{-}^{N}$ and distinct $i, j, k, l \in N$ such that $F_x = \{i\}$, $j \in S_x$, $d_x(i, j) = 1$, $x_j = i$, and $x_k = l$. Then, $j \in \varphi(x)$. Let $x'_j = l$ and $x'_k = i$. Then, $j \notin \varphi(x'_j, x'_k, x_{N\setminus\{j,k\}})$, while $s(x'_j, x'_k, x_{N\setminus\{j,k\}}) = s(x)$.

In this section, we consider nomination rules that satisfy impartiality and anonymous ballots. It follows from Holzman and Moulin (2013) that the only nomination functions satisfying impartiality and anonymous ballots are the constant rules. It is easy to show, however, that in the class of correspondences, impartiality and anonymous ballots do not imply constancy. Let φ be such that, for all $x \in N_{-}^{N}$, $\varphi(x) = \{i \in N : s_i(x) \geq 1\}$. Under this nomination rule, each agent wins if and only if she gets support from at least one other agent. This rule is impartial: no one can affect one's own score, and it satisfies anonymous ballots: the determination of $\varphi(x)$ depends only on the profile of scores. Observe that it also satisfies negative unanimity, but does not satisfy positive unanimity.

Is there any nomination rule that satisfies impartiality, anonymous ballots, and positive unanimity? The following theorem states that the answer is negative. We show that with four or more agents, there is no nomination rule that satisfies impartiality, anonymous ballots, and positive unanimity.

We again use the notation x^c and x_i^j introduced in Section 2.3. In addition, given $B \subset N$ and $j \in N$, we write x_B^j if x_i^j for all $i \in B$.

Theorem 2.2. *If $n \geq 4$, there is no nomination rule φ that satisfies impartiality, anonymous ballots, and positive unanimity.*

Proof. Assume that φ satisfies impartiality, anonymous ballots, and positive unanimity. First, consider $(x_1^n, x_{n-1}^1, x_n^c, x_{N\setminus\{1,n-1,n\}}^1)$. By positive unanimity,

$$\varphi(x_1^n, x_{n-1}^1, x_n^c, x_{N\setminus\{1,n-1,n\}}^1) = \{1\}.$$

Then,

$$n \notin \varphi(x_1^n, x_{n-1}^1, x_n^c, x_{N\setminus\{1,n-1,n\}}^1).$$

Change agent n's nomination from x_n^c to x_n^{n-1}. Then, impartiality gives

$$n \notin \varphi(x_1^n, x_{n-1}^1, x_n^{n-1}, x_{N\setminus\{1,n-1,n\}}^1).$$

Since

$$s(x_1^n, x_{n-1}^1, x_n^{n-1}, x_{N\setminus\{1,n-1,n\}}^1)$$

and

$$s(x_1^{n-1}, x_{n-1}^c, x_n^c, x_{N\setminus\{1,n-1,n\}}^1)$$

are equal, anonymous ballots implies that

$$n \notin \varphi(x_1^{n-1}, x_{n-1}^c, x_n^c, x_{N\setminus\{1,n-1,n\}}^1). \tag{2.1}$$

Second, assume, for all $r = 2, \ldots, n-2$, that

$$n \notin \varphi(x_1^{n-(r-1)}, x_{n-r}^1, x_{n-(r-1)}^c, \ldots, x_{n-1}^c, x_n^c, x_{N\setminus\{1,n-r,n-(r-1),\ldots,n-1,n\}}^1). \tag{2.2}$$

Change agent n's nomination from x_n^c to x_n^{n-r}. Then, impartiality gives

$$n \notin \varphi(x_1^{n-(r-1)}, x_{n-r}^1, x_{n-(r-1)}^c, \ldots, x_{n-1}^c, x_n^{n-r}, x_{N\setminus\{1,n-r,n-(r-1),\ldots,n-1,n\}}^1).$$

Since

$$s(x_1^{n-(r-1)}, x_{n-r}^1, x_{n-(r-1)}^c, \ldots, x_{n-1}^c, x_n^{n-r}, x_{N\setminus\{1,n-r,n-(r-1),\ldots,n-1,n\}}^1)$$

and

$$s(x_1^{n-r}, x_{n-r}^c, x_{n-(r-1)}^c, \ldots, x_{n-1}^c, x_n^c, x^1_{N\setminus\{1,n-r,n-(r-1),\ldots,n-1,n\}})$$

are equal, anonymous ballots implies that

$$n \notin \varphi(x_1^{n-r}, x_{n-r}^c, x_{n-(r-1)}^c, \ldots, x_{n-1}^c, x_n^c, x^1_{N\setminus\{1,n-r,n-(r-1),\ldots,n-1,n\}}). \tag{2.3}$$

By (2.1), (2.2), and (2.3), we obtain $n \notin \varphi(x_1^2, x_2^c, \ldots, x_n^c) = \varphi(x^c)$. By repeating the same argument $n-1$ times and shifting the indexes of agents cyclically, we conclude that $i \notin \varphi(x^c)$ for all $i \in N$, which contradicts the fact that $\varphi(x^c)$ is non-empty. $\square$

Finally, we check the independence of the three axioms in Theorem 2.2. The plurality correspondence satisfies all the axioms except impartiality. Plurality with runners-up satisfies all the axioms except anonymous ballots. The nomination rule introduced in this section satisfies all the axioms except positive unanimity. Thus, the three axioms are mutually independent.

2.5 Discussion

Our formulation of impartiality for correspondences implicitly assumes that each agent is indifferent to the number of winners. Indeed, under plurality with runners-up, at the cyclic profile x^c, a winner has a chance to reduce the number of winners from n to 2 while maintaining her own position. It seems that, in practice, she is not indifferent to these numbers, and we should formulate impartiality for correspondences so that a change of a winner's nomination does not affect her own winning, nor the number of winners.

However, under this stronger formulation, the impossibility result of Holzman and Moulin (2013) extends also to correspondences. To see this, given φ, let $\lambda_i(x) \in \{0, 1\}$ be such that $\lambda_i(x) = 1$ if $i \in \varphi(x)$ and 0 if $i \notin \varphi(x)$. Then, the stronger formulation of impartiality is as follows: $\lambda_i(x)|\varphi(x)| = \lambda_i(x'_i, x_{-i})|\varphi(x'_i, x_{-i})|$ for all $x \in N_-^N$, all $i \in N$, and all $x'_i \in N \setminus \{i\}$. If there exists φ that satisfies this property and two unanimity axioms, then there must be the randomized nomination rule $\psi : N_-^N \to [0,1]^N$ that has the form $\psi_i(x) = \frac{\lambda_i(x)}{|\varphi(x)|}$ for all $x \in N_-^N$ and all $i \in N$, and that satisfies impartiality, positive unanimity, and negative unanimity for randomized nomination rules. This contradicts Theorem 4 in Holzman and Moulin (2013).

The above observation shows that we have to interpret correspondences such that any number of identical prizes may be awarded, and an agent is indifferent to the number of winners. We can minimize this difficulty, however, by adopting a refinement of plurality with runners-up φ, having the property that the prize is always awarded to one or two winners. The definition of this subcorrespondence φ' is as follows.[7] Fix an order on N. For any $x \in N_-^N$, the agent i who is the first member of F_x always wins, and there are two special cases in which there is one additional winner $j \neq i$: (i) if $|F_x| > 1$ and j is the second member of F_x with $x_j = i$, then $\varphi'(x) = \{i, j\}$; (ii) if $F_x = \{i\}$ and j is the first member of S_x with $d_x(i, j) = 1$, $x_j = i$, and j precedes i, then $\varphi'(x) = \{i, j\}$. Since φ' is a subcorrespondence of φ, it inherits the properties of positive and negative unanimity. To see that φ' is impartial, we check for the case in which $i \in \varphi'(x)$ is the first member of F_x. The other cases are checked similarly. Assume, without loss of generality, that the order is associated with the indexes of agents. Let $x'_i \neq x_i$. If $x'_i = j$ for some $j \in F_x \setminus \{i\}$, then $F_{(x'_i, x_{-i})} = \{j\}$, $d_{(x'_i, x_{-i})}(j, i) = 1$, $i < j$, and $i < k$ for all $k \in S_{(x'_i, x_{-i})} \setminus \{i\}$. Therefore, $i \in \varphi'(x'_i, x_{-i})$. If $x'_i = j$ for some $j \in S_x$ with $d_x(i, j) = 1$ and $j < i$, then $j < i < k$ for all $k \in F_{(x'_i, x_{-i})} \setminus \{i, j\}$, and thus, $i \in \varphi'(x'_i, x_{-i})$. Any other x'_i makes her be the first member of $F_{(x'_i, x_{-i})}$, and thus, $i \in \varphi'(x'_i, x_{-i})$.

As mentioned above, this subcorrespondence φ' has the good property that its winning sets are relatively small. On the other hand, the original plurality with

[7] The author is greatly indebted to an anonymous referee of *Social Choice and Welfare*, the journal in which Tamura and Ohseto (2014) is published, for suggesting this example.

runners-up φ is symmetric in the sense that each agent is treated equally as a candidate. To analyze these nomination rules, focusing on each of these good properties, would be an interesting topic for future research.

Chapter 3

Characterizing Minimal Impartial Rules for Awarding Prizes**

3.1 Introduction

Suppose that a foundation is considering awarding a prize to one or more members of a group of experts whose activities advance the public interest. The foundation's leader wishes to select members who most deserve the prize, but he cannot do so by himself because he lacks the expertise needed to evaluate their merits. Given that situation, this chapter considers the design of award rules that base the selection of winners on experts' views. In particular, we study *nomination rules* that ask each expert to nominate one other expert for the prize; the set of winners is then determined based on the profile of nominations. The challenge of this approach is that conflicts of interest might be created among selfish experts. In particular, a person caring only about her own winning might corrupt her nomination when there

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is a chance that she can influence her own likelihood of taking the prize. We are thus interested in nomination rules that create no such conflict of interest among selfish experts, and study those satisfying an axiom called *impartiality*. A nomination rule is impartial if it determines each person's winning independently of her nomination; a selfish person thus has no chance to influence her own winning when the rule satisfies impartiality.

The aim of this chapter is to identify reasonable impartial nomination rules among those satisfying three additional axioms: *anonymity*, *symmetry*, and *monotonicity*. Anonymity requires that an exchange of nominations between two people do not affect the winning of any other person. Symmetry requires the determination of the set of winners to be independent of the indexes of people. Monotonicity requires that any subset of winners be included in the new set of winners when each member in the subset obtains an additional nomination from another person.

Now, consider the nomination rule under which all people are always chosen as the winners. Although satisfying the three axioms and being impartial, we cannot describe such a nomination rule as reasonable. By always selecting too many winners, without examining their qualifications, it might degrade the prestige of the prize, which the foundation aims to maintain. It might also undermine the social practice of competition. These arguments confirm that it is desirable for a nomination rule to select winners as strictly as possible, leading us to the question of which nomination rules are optimal in this sense subject to all the four axioms.

In this chapter, we obtain an explicit answer to this question by exploring *minimal* nomination rules among those satisfying the four axioms. We define a nomination rule satisfying the four axioms as "minimal" if one cannot make a further refinement to the nomination rule while still preserving the four axioms, i.e., if there is no other nomination rule that satisfies the four axioms while assigning to every profile of nominations a set of winners that is smaller, compared by inclusion, than that assigned by the nomination rule under consideration. The result will thus characterize the set of all minimal nomination rules satisfying the four axioms. We show that *plurality with runners-up* (Tamura and Ohseto, 2014) *is the only minimal nomination rule satisfying impartiality, anonymity, symmetry, and monotonicity* (Theorem 3.1). Plurality with runners-up is a natural variant of the plurality correspondence. Indeed, the set of winners is always that of plurality winners except when there is a sole plurality winner who defeats the runners-up by only one point; in this case, a runner-up who nominates the sole plurality winner also wins.

This chapter is the first, to our knowledge, to establish a characterization result in the context of impartial nomination rules. Holzman and Moulin (2013) begin this area of study with "single-valued" nomination rules and propose interesting impartial rules called the *partition methods*. Instead of characterizing these partition methods, they establish two impossibility results regarding single-valued impartial nomination rules; one of these states that no such rule simultaneously satisfies two desirable axioms which they call "positive unanimity" and "negative unanimity."[8] Tamura and Ohseto (2014) then allow rules to be "multi-valued," as is done in this chapter, focusing on discussing whether Holzman and Moulin's impossibility results hold in a more general class of multi-valued nomination rules. By constructing the "plurality with runners-up" correspondence, they show that there exists an impartial rule meeting positive and negative unanimity when at least four people are involved.

In the closely related context of "impartial division rules," a characterization result has already been established. de Clippel et al. (2008) study the problem of dividing a surplus among a group of partners when each partner represents her subjective opinion about the relative contributions of the others to the surplus. A division rule determines the division of the surplus on the basis of the profile of opinions, and impartiality requires that the share of the surplus each person receives be independent of her own opinion. For situations of four or more partners, the

[8] Positive unanimity says that a person should be the (unique) winner if she is nominated by everyone else. Negative unanimity says that a person should not win if she is not nominated by anybody.

authors propose an infinite family of impartial rules that aggregate the opinions of the partners in a highly natural way. They then characterize that family by employing several reasonable axioms. A clear difference exists between de Clippel et al.'s result and ours: they characterize the whole class of rules meeting their axioms, whereas we characterize only the minimal rules satisfying our axioms. Nevertheless, this difference does not degrade the importance of our result; as explained above, investigating minimal nomination rules is itself meaningful in our context.

The rest of the chapter is organized as follows. In Section 3.2, we introduce the model and the axioms. In Section 3.3, we state and prove the result. In Section 3.4, we offer concluding remarks.

3.2 Model and Axioms

Let $N = \{1, \dots, n\}$ ($n \geq 3$) be the set of people. For each $i \in N$, let $x_i \in N \setminus \{i\}$ denote i's *nomination*. If $x_i = j$, it means that i nominates j. A list $x = (x_i)_{i \in N}$ is called a *nomination profile*. Let N_-^N denote the set of all nomination profiles. For each $x \in N_-^N$ and each $i_1, \dots, i_m \in N$, where $m = 1, \dots, n$, we sometimes write x for $(x_{\{i_1,\dots,i_m\}}, x_{N \setminus \{i_1,\dots,i_m\}})$ to distinguish the nominations of $i_1, \dots, i_m$ from those of the others in x. For simplicity of notation, we often use (x_i, x_{-i}) instead of $(x_{\{i\}}, x_{N \setminus \{i\}})$. A *nomination rule* is a correspondence $\varphi : N_-^N \to 2^N \setminus \{\emptyset\}$ that assigns a non-empty subset of people, which we mention as the set of winners, to each nomination profile.

We next introduce four axioms that we impose on nomination rules. First, as our central axiom, *impartiality* requires that one's nomination never influences one's own winning.

Impartiality: for all $x \in N_-^N$, all $i \in N$, and all $x'_i \in N \setminus \{i\}$,

$$i \in \varphi(x_i, x_{-i}) \Leftrightarrow i \in \varphi(x'_i, x_{-i}).$$

Second, we consider *anonymity* which ensures people to be treated equally as "voters." Suppose that two people, say, j, k, exchange their nominations each other. Anonymity says that this exchange should not affect the winning of any other person, i, so that j and k have the same influence on i's winning.

Anonymity: for all $x \in N_-^N$, all $i \in N$, all $j, k \in N \setminus \{i\}$, all $x'_j \in N \setminus \{j\}$, and all $x'_k \in N \setminus \{k\}$,

$$\text{if } \; x'_j = x_k \neq j, \;\text{ and }\; x'_k = x_j \neq k,$$

$$\text{then } \; i \in \varphi(x_{\{j,k\}}, x_{N\setminus\{j,k\}}) \Leftrightarrow i \in \varphi(x'_{\{j,k\}}, x_{N\setminus\{j,k\}}).$$

We should mention that there exists another anonymity axiom called *anonymous ballots*, whose meaning is slightly different from that of anonymity above. The axiom requires a rule to depend only on the number of nominations each person obtains,[9] meaning that the names of the voters are completely irrelevant to the determination of the winners and thus each voter can submit his nomination anonymously without signing his own name (on the other hand, anonymity says nothing about whether nominations be collected anonymously; its definition is only for guaranteeing equality between any two voters). It is shown, however, that the axiom is incompatible with impartiality,[10] whereas anonymity is compatible with it, as shown in the next section.

Third, we consider *symmetry* which ensures people to be treated symmetrically as "candidates."[11] Let $\pi : N \to N$ be a permutation of N. The set of all such

[9] Thus, anonymous ballots is stronger than anonymity.

[10] Holzman and Moulin (2013) show that the only single-valued nomination rules satisfying impartiality and anonymous ballots are those that constantly choose a particular person. As for multi-valued rules, Tamura and Ohseto (2014) show that, although these two axioms do not imply the constancy described above, they are still incompatible when *positive unanimity*, an axiom which we mentioned in the introduction, is also required. Finally, the plurality with runners-up, which we introduce and characterize in the next section, does not satisfy anonymous ballots; see Tamura and Ohseto (2014).

[11] Holzman and Moulin (2013) define symmetry for *randomized single-valued nomination rules*,

permutations is denoted by Π_N. Given $\pi \in \Pi_N$ and $x \in N_-^N$, let x^π denote the nomination profile such that $x_i^\pi = \pi(x_{\pi^{-1}(i)})$ for all $i \in N$. Note that $x_{\pi(i)}^\pi = \pi(x_i)$ for any $i \in N$, which describes how π transforms x into x^π: if i nominates j in x, $\pi(i)$ nominates $\pi(j)$ in x^π. Symmetry says that if i wins (loses) in x, then $\pi(i)$ should win (lose) in x^π.

Symmetry: for all $\pi \in \Pi_N$, all $x \in N_-^N$, and all $i \in N$,

$$i \in \varphi(x) \Leftrightarrow \pi(i) \in \varphi(x^\pi).$$

Finally, we consider *monotonicity* which imposes a certain consistency requirement on nomination rules. Monotonicity says that any subset of winners should be included in the new set of winners when each member in the subset obtains an additional nomination from another person.

Monotonicity: for all $x \in N_-^N$, all $i_1, \ldots, i_m \in N$, all $j_1, \ldots, j_m \in N$, all $x'_{j_1} \in N \setminus \{j_1\}, \ldots$, and all $x'_{j_m} \in N \setminus \{j_m\}$,

$$\begin{aligned} \text{if } & \{i_1, \ldots, i_m\} \subset \varphi(x), \\ & x_{j_1}, \ldots, x_{j_m} \notin \{i_1, \ldots, i_m\}, \text{ and} \\ & x'_{j_1} = i_1, \ldots, x'_{j_m} = i_m, \\ \text{then } & \{i_1, \ldots, i_m\} \subset \varphi(x'_{\{j_1,\ldots,j_m\}}, x_{N\setminus\{j_1,\ldots,j_m\}}). \end{aligned}$$

Let Φ denote the set of all nomination rules satisfying impartiality, anonymity, symmetry, and monotonicity. We say that a nomination rule $\varphi \in \Phi$ is *minimal* if there is no $\varphi' \in \Phi$ such that $\varphi' \neq \varphi$ and $\varphi'(x) \subset \varphi(x)$ for all $x \in N_-^N$.

i.e., functions that assign every person's winning probability to each nomination profile.

3.3 Characterization Result

In this section, we show that *plurality with runners-up* (Tamura and Ohseto, 2014) is the only minimal nomination rule that belongs to Φ. To introduce the definition of plurality with runners-up, we give some additional notations. Given $x \in N_-^N$ and $i \in N$, let $s_i(x) = |\{j \in N \setminus \{i\} : x_j = i\}|$ denote i's *score* in x. Given $x \in N_-^N$, let $s_F(x) = \max_{i \in N} s_i(x)$ and $F_x = \{i \in N : s_i(x) = s_F(x)\}$ denote the (first) highest score and the set of people obtaining that score in x, respectively. Similarly, let $s_S(x) = \max_{i \in N \setminus F_x} s_i(x)$ and $S_x = \{i \in N : s_i(x) = s_S(x)\}$ denote the second highest score and the set of people obtaining that score in x, respectively.

Definition 3.1 (Tamura and Ohseto, 2014)**.** A nomination rule φ^* is plurality with runners-up if, for all $x \in N_-^N$,
(a) if $|F_x| = 1$ and $s_F(x) - s_S(x) = 1$, then $\varphi^*(x) = F_x \cup \{i \in S_x : x_i \in F_x\}$;
(b) else, $\varphi^*(x) = F_x$.

In words, plurality with runners-up is the nomination rule under which, not only do the plurality winners always win, but also a runner-up wins if she nominates the sole plurality winner who defeats the runner-up by only one point.

Before we state and prove the result, it should be noted that the plurality with runners-up is not the unique nomination rule that belongs to Φ. Indeed, for instance, the *indifference rule*, defined by $\varphi^{ind}(x) = N$ for all $x \in N_-^N$, also satisfies impartiality, anonymity, symmetry, and monotonicity. But we have $\varphi^*(x) \subset \varphi^{ind}(x)$ for all $x \in N_-^N$, which is consistent with the claim that φ^* is the only minimal nomination rule that belongs to Φ.

We show that the plurality with runners-up is the only minimal nomination rule satisfying impartiality, anonymity, symmetry, and monotonicity.

Theorem 3.1. *Plurality with runners-up is the only minimal nomination rule satisfying impartiality, anonymity, symmetry, and monotonicity.*

Proof. Let φ^* be the plurality with runners-up as in Definition 3.1. First of all, we have to verify that φ^* satisfies each of the four axioms.

Impartiality We propose an alternative verification that would be more intuitive than the one established in Tamura and Ohseto (2014). For all $x \in N_-^N$ and all $i \in N$, we have $i \in \varphi^*(x) \Leftrightarrow i \in F_{x_{-i}}$, where $F_{x_{-i}}$ denotes the set of people obtaining the first highest score in x provided that i's nomination is not counted.[12] Hence, since $F_{x_{-i}}$ is independent of i's nomination, φ^* satisfies impartiality.

Anonymity Let $x \in N_-^N$, $i \in N$, $j, k \in N \setminus \{i\}$, $x'_j \in N \setminus \{j\}$, and $x'_k \in N \setminus \{k\}$. Suppose that $i \in \varphi^*(x)$, $x'_j = x_k \neq j$, and $x'_k = x_j \neq k$. Let $x' = (x'_{\{j,k\}}, x_{N\setminus\{j,k\}})$. If $i \in F_x$, we have $i \in F_{x'}$. Hence, $i \in \varphi^*(x')$. If $i \in S_x$, then we have $|F_x| = 1$, $s_F(x) - s_S(x) = 1$, and $x_i \in F_x$. Therefore, $i \in S_{x'}$, $|F_{x'}| = 1$, $s_F(x') - s_S(x') = 1$, and $x'_i = x_i \in F_{x'}$. Hence, we obtain $i \in \varphi^*(x')$.

Symmetry Let $\pi \in \Pi_N$, $x \in N_-^N$, and $i \in N$. Suppose that $i \in \varphi^*(x)$. Note that $s_{\pi(j)}(x^\pi) = s_j(x)$ for any $j \in N$. Therefore, if $i \in F_x$, we have $\pi(i) \in F_{x^\pi}$. Hence, $\pi(i) \in \varphi^*(x^\pi)$. If $i \in S_x$, then we have $|F_x| = 1$, $s_F(x) - s_S(x) = 1$, and $x_i \in F_x$. Therefore, $\pi(i) \in S_{x^\pi}$, $|F_{x^\pi}| = 1$, $s_F(x^\pi) - s_S(x^\pi) = 1$, and $x^\pi_{\pi(i)} = \pi(x_i) \in F_{x^\pi}$. Hence, we obtain $\pi(i) \in \varphi^*(x^\pi)$.

Monotonicity Let $x \in N_-^N$, $i_1, \ldots, i_m \in N$, $j_1, \ldots, j_m \in N$, $x'_{j_1} \in N \setminus \{j_1\}, \ldots,$ and $x'_{j_m} \in N \setminus \{j_m\}$. Suppose that $\{i_1, \ldots, i_m\} \subset \varphi^*(x)$, $x_{j_1}, \ldots, x_{j_m} \notin \{i_1, \ldots, i_m\}$, and $x'_{j_1} = i_1, \ldots, x'_{j_m} = i_m$. Let $x' = (x'_{\{j_1,\ldots,j_m\}}, x_{N\setminus\{j_1,\ldots,j_m\}})$. Now, we distinguish the following three cases: (i) $|F_x| > 1$; (ii) $|F_x| = 1$ and $m = 1$; (iii) $|F_x| = 1$ and $m > 1$. If $|F_x| > 1$, then $\{i_1, \ldots, i_m\} \subset F_x$. Therefore, $\{i_1, \ldots, i_m\} \subset F_{x'}$. Hence, we obtain $\{i_1, \ldots, i_m\} \subset \varphi^*(x')$. If $|F_x| = 1$ and $m = 1$, then either $i_1 \in F_x$ or $i_1 \in S_x$ with $s_F(x) - s_S(x) = 1$. Therefore, in either of the two cases, we have

[12] A formal definition of $F_{x_{-i}}$ will be as follows. For each $x \in N_-^N$, each $i \in N$, and each $j \in N$, define $s_j(x_{-i}) = |\{k \in N \setminus \{i, j\} : x_k = j\}|$, and define $F_{x_{-i}} = \{j \in N : s_j(x_{-i}) = \max_{k \in N} s_k(x_{-i})\}$. Note that $s_i(x_{-i}) = s_i(x)$ for any $x \in N_-^N$ and any $i \in N$.

$i_1 \in F_{x'}$. Hence, we obtain $\{i_1\} \subset \varphi^*(x')$. If $|F_x| = 1$ and $m > 1$, then we have $s_F(x) - s_S(x) = 1$. Without loss of generality, assume that $i_2, \ldots, i_m \in S_x$. Then, $x_{i_2}, \ldots, x_{i_m} \in F_x$. Now, if $i_1 \in F_x$, then, we have $F_{x'} = \{i_1\}$, $i_2, \ldots, i_m \in S_{x'}$, and $s_F(x') - s_S(x') = 1$. Moreover, since $x_{j_1}, \ldots, x_{j_m} \notin \{i_1\}$ and $x_{i_2} = \ldots = x_{i_m} = i_1$, we have $i_2, \ldots, i_m \notin \{j_1, \ldots, j_m\}$, which implies that $x'_{i_2} = \ldots = x'_{i_m} = i_1$. Hence, we obtain $\{i_1, \ldots, i_m\} \subset \varphi^*(x')$. If $i_1 \in S_x$, then, since $s_F(x) - s_S(x) = 1$, we have $\{i_1, \ldots, i_m\} \subset F_{x'}$. Therefore, we obtain $\{i_1, \ldots, i_m\} \subset \varphi^*(x')$.

We now turn to the proof of the unique minimality of φ^*. Note that the claim is true if and only if, for any $\varphi \in \Phi$, we have $\varphi^*(x) \subset \varphi(x)$ for all $x \in N_-^N$.[13] Here, we prove the latter in two steps: for any $\varphi \in \Phi$, we first show that $F_x \subset \varphi(x)$ for all $x \in N_-^N$; then show that $\{i \in S_x : x_i \in F_x\} \subset \varphi(x)$ whenever $|F_x| = 1$ and $s_F(x) - s_S(x) = 1$.

Step 1. $F_x \subset \varphi(x)$ for all $x \in N_-^N$.

We show this by induction on $s_F(x)$, the first highest score in $x \in N_-^N$. First, let $x \in N_-^N$ be such that $s_F(x) = 1$. Then, $F_x = N$. Now, suppose that, for the sake of contradiction, we have $i \notin \varphi(x)$ for some $i \in F_x$. Without loss of generality, assume that $1 \notin \varphi(x)$ and that $x_1 = 2$. Consider the profile $x^c \in N_-^N$ such that $x_j^c = j + 1$ for all $j = 1, \ldots, n-1$, and $x_n^c = 1$. Note that $s_j(x^c) = s_j(x) = 1$ for all $j \in N$, and $x_1^c = x_1$. Now, we argue that $1 \notin \varphi(x^c)$ and prove this by iteratively changing each component of x into that of x^c with applying anonymity in each time. First, consider changing x_2 into x_2^c. If $x_2 = x_2^c$, then it immediately follows that $1 \notin \varphi(x^c_{\{1,2\}}, x_{N\setminus\{1,2\}})$. If $x_2 \neq x_2^c$, then, since $s_3(x) = 1$, there uniquely exists $k \in N \setminus \{1, 2, 3\}$ such that $x_k = x_2^c = 3$ ($k \neq 1$ follows from $x_1 = 2 \neq 3$). Now, if $x_2 \neq k$, then, since $x_k \neq 2$, the pairwise exchange of nominations between 2 and k is possible. Therefore, anonymity implies that $1 \notin \varphi(x^c_{\{1,2\}}, x_2, x_{N\setminus\{1,2,k\}})$. If $x_2 = k$, then consider an exchange between 2 and 3. Since $x_2 \neq 3$ and $x_3 \neq 2$, this

[13] If part is straightforward, and only if part follows from the finiteness of Φ.

exchange is possible. Therefore, anonymity implies that $1 \notin \varphi(x^c_{\{1\}}, x'_{\{2,3\}}, x_{N\setminus\{1,2,3\}})$, where $x'_2 = x_3$ and $x'_3 = x_2$. Now, since $s_k(x^c_{\{1\}}, x'_{\{2,3\}}, x_{N\setminus\{1,2,3\}}) = 1$ and $x'_3 = x_2 = k$, we have $x'_2 \neq k$, and this is the case we already proved above. Therefore, $1 \notin \varphi(x^c_{\{1,2\}}, x'_{\{2,3\}}, x_{N\setminus\{1,2,3,k\}})$. By repeating the same argument, one can finally obtain $1 \notin \varphi(x^c)$.

Now, consider $\pi \in \Pi_N$ such that $\pi(j) = x^c_j$ for all $j \in N$. Then, $(x^c)^\pi = x^c$. Therefore, symmetry implies that $[1 \notin \varphi(x^c) \Rightarrow 2 \notin \varphi(x^c)], [2 \notin \varphi(x^c) \Rightarrow 3 \notin \varphi(x^c)], \ldots$, and $[n-1 \notin \varphi(x^c) \Rightarrow n \notin \varphi(x^c)]$. Thus, we have $\varphi(x^c) = \emptyset$, a contradiction.

Next, for all $r = 2, \ldots, n-1$, assume that $F_x \subset \varphi(x)$ whenever $x \in N^N_-$ is such that $s_F(x) = r - 1$ (induction hypothesis). Let $x \in N^N_-$ be such that $s_F(x) = r$. We show that $F_x \subset \varphi(x)$. Suppose that $|F_x| = m$, where $1 \leq m \leq n/2$, and let $F_x = \{i_1, \ldots, i_m\}$. Let $H_x = \{h \in N : s_h(x) = 0\}$ denote the set of people not nominated by anybody in x. Note that, since $s_F(x) = r \geq 2$, we have $|H_x| \geq m$. Let $h_1, \ldots, h_m \in H_x$. On the other hand, since $s_F(x) = r \geq 2$, there exist distinct $k_1, l_1, \ldots, k_m, l_m$ such that $x_{k_1} = x_{l_1} = i_1, \ldots, x_{k_m} = x_{l_m} = i_m$. Now, let $j_1, \ldots, j_m$ be as follows: $j_1 = k_1$ if $k_1 \neq h_1$ and $j_1 = l_1$ if $k_1 = h_1$; $\ldots$; $j_m = k_m$ if $k_m \neq h_m$ and $j_m = l_m$ if $k_m = h_m$. Then, consider $x' \in N^N_-$ such that $x'_{j_1} = h_1, \ldots, x'_{j_m} = h_m$, and $x'_i = x_i$ for all $i \in N \setminus \{j_1, \ldots, j_m\}$. Then, $\{i_1, \ldots, i_m\} \subset F_{x'}$ and $s_F(x') = r - 1$. Therefore, induction hypothesis implies that $\{i_1, \ldots, i_m\} \subset \varphi(x')$. Hence, by monotonicity, we obtain $F_x = \{i_1, \ldots, i_m\} \subset \varphi(x)$.

Step 2. $\{i \in S_x : x_i \in F_x\} \subset \varphi(x)$ if $|F_x| = 1$ and $s_F(x) - s_S(x) = 1$.

Let $x \in N^N_-$ be such that $|F_x| = 1$ and $s_F(x) - s_S(x) = 1$. Let $i \in S_x$ with $x_i \in F_x$. Note that, since $|F_x| = 1$, we have $s_F(x) \geq 2$. Hence, there exists $h \in N \setminus (F_x \cup S_x)$ such that $s_h(x) = 0$. Let $x'_i = h$. Then, since $|F_x| = 1$, $i \in S_x$, $s_F(x) - s_S(x) = 1$, and $x_i \in F_x$, we have $i \in F_{(x'_i, x_{-i})}$. Therefore, by Step 1, we have $i \in F_{(x'_i, x_{-i})} \subset \varphi(x'_i, x_{-i})$. Hence, by impartiality, we obtain $i \in \varphi(x)$. $\square$

We check that the four axioms are needed to establish the statement. We show that, if we drop each of the four axioms, then there exists another nomination rule φ that satisfies all the other axioms and that $\varphi^*(x) \not\subset \varphi(x)$ for some $x \in N_-^N$. We omit all easy verifications.

Example 3.1 (Dropping impartiality)**.** The plurality correspondence, defined by $\varphi(x) = F_x$ for all $x \in N_-^N$, satisfies anonymity, symmetry, and monotonicity, but not impartiality.

Example 3.2 (Dropping anonymity)**.** Consider the following subcorrespondence φ of the plurality with runners-up: for all $x \in N_-^N$,

Case A. if $|F_x| > 1,$ and

(i) if $s_F(x) > 1,$ then $\varphi(x) = F_x;$

(ii) if $s_F(x) = 1$ and $|\{i \in F_x : \exists i' \in N \setminus \{i\},\ x_{i'} = i \text{ and } x_i = i'\}| = n,$

then $\varphi(x) = F_x;$

(iii) if $s_F(x) = 1$ and $|\{i \in F_x : \exists i' \in N \setminus \{i\},\ x_{i'} = i \text{ and } x_i = i'\}| < n,$

then $\varphi(x) = F_x^*,$

where $F_x^* = \{i \in F_x : \exists j, k \in N \setminus \{i\},\ x_j = i \text{ and } x_k = j\};$

Case B. if $|F_x| = 1,\ s_F(x) - s_S(x) = 1,$ and

(i) if $s_F(x) > 2,$ then $\varphi(x) = F_x \cup \{i \in S_x : x_i \in F_x\};$

(ii) if $s_F(x) = 2$ and $|\{i \in S_x : \exists i' \in N \setminus \{i\},\ x_{i'} = i \text{ and } x_i = i'\}| \geq n - 3,$

then $\varphi(x) = F_x \cup \{i \in S_x : x_i \in F_x\};$

(iii) if $s_F(x) = 2$ and $|\{i \in S_x : \exists i' \in N \setminus \{i\},\ x_{i'} = i \text{ and } x_i = i'\}| < n - 3,$

then $\varphi(x) = F_x \cup \{i \in S_x^* : x_i \in F_x\}$,

where $S_x^* = \{i \in S_x : \exists j, k \in N \setminus \{i\},\ x_j = i \text{ and } x_k = j\}$;

Case C. if $|F_x| = 1$ and $s_F(x) - s_S(x) > 1,$ then $\varphi(x) = F_x$.

This subcorrespondence satisfies impartiality, symmetry, and monotonicity, but not anonymity if $n \geq 5$.

Example 3.3 (Dropping symmetry)**.** We introduce a subcorrespondence φ of the plurality with runners-up mentioned in Tamura and Ohseto (2014). Fix an order on N. For any $x \in N_-^N$, the person i being the first member of F_x always wins, and there are two special cases in which there is one additional winner $j \neq i$: (i) if $|F_x| > 1$, $i \in F_x$, and j is the second member of F_x with $x_j = i$, then $\varphi(x) = \{i, j\}$; (ii) if $|F_x| = 1$, $i \in F_x$, and j is the first member of S_x with $s_i(x) - s_j(x) = 1$, $x_j = i$, and j precedes i, then $\varphi(x) = \{i, j\}$. This subcorrespondence satisfies impartiality, anonymity, and monotonicity, but not symmetry.

Example 3.4 (Dropping monotonicity)**.** Let φ be such that, for all $x \in N_-^N$,
(a) if $s_F(x) = n - 1$, then $\varphi(x) = \{i \in N : s_i(x) = 1\}$;
(b) else, $\varphi(x) = \{i \in N : s_i(x) \geq 1\}$.
This nomination rule satisfies impartiality, anonymity, and symmetry, but not monotonicity.

Remark 3.1. We have seen in Example 3.2 that anonymity is necessary for the result whenever $n \geq 5$. This is, however, no longer true if $n \leq 4$ since, in this case, we can establish the result without using anonymity. To see this, it suffices to show that, if $n \leq 4$, symmetry implies that $\varphi(x) = F_x = N$ whenever $s_F(x) = 1$. Let $x \in N_-^N$ be such that $s_F(x) = 1$. First, consider the case that $n = 3$. Then, there always exist $i_1, i_2, i_3 \in N$ such that $x_{i_1} = i_2$, $x_{i_2} = i_3$, and $x_{i_3} = i_1$. Consider $\pi \in \Pi_N$ such that $\pi(i) = x_i$. Then, since $x^\pi = x$, symmetry implies that $\varphi(x) = \{i_1, i_2, i_3\}$.

Second, consider the case that $n = 4$. In this case, we distinguish the following two subcases: (i) $\exists i_1, i_2, i_3, i_4 \in N$ such that $x_{i_1} = i_2$, $x_{i_2} = i_3$, $x_{i_3} = i_4$, and $x_{i_4} = i_1$; (ii) $\exists i_1, i_2, j_1, j_2 \in N$ such that $x_{i_1} = i_2$, $x_{i_2} = i_1$, $x_{j_1} = j_2$, and $x_{j_2} = j_1$. If (i), then, by the same argument with $n = 3$, one can obtain $\varphi(x) = \{i_1, i_2, i_3, i_4\}$. If (ii), consider $\pi \in \Pi_N$ such that $\pi(i_1) = i_2$, $\pi(i_2) = i_1$, $\pi(j_1) = j_2$, and $\pi(j_2) = j_1$. Then, we have $x^\pi = x$, and symmetry implies that we have $\{i_1, i_2\} \subset \varphi(x)$ or $\{j_1, j_2\} \subset \varphi(x)$ (or both). On the other hand, consider $\pi' \in \Pi_N$ such that $\pi'(i_1) = j_1$, $\pi'(i_2) = j_2$, $\pi'(j_1) = i_1$, and $\pi'(j_2) = i_2$. Then, we have $x^{\pi'} = x$, and symmetry implies that we have $\{i_1, j_1\} \subset \varphi(x)$ or $\{i_2, j_2\} \subset \varphi(x)$ (or both). Therefore, we obtain $\varphi(x) = \{i_1, i_2, j_1, j_2\}$.

Remark 3.2. Example 3.4 violates positive unanimity, while the other examples satisfy it. Given this observation, one may wonder if one can establish Theorem 3.1 using positive unanimity instead of monotonicity, that is, whether one can prove that plurality with runners-up is a subcorrespondence of any other nomination rule satisfying impartiality, anonymity, symmetry, and positive unanimity. In regard to this, we can say that the answer is no: if $n \geq 5$,[14] there is a (non-monotonic) nomination rule satisfying the above four axioms while not including the plurality with runners-up. To see this, consider the following nomination rule φ: for any $x \in N^N_-$,

(a) if $s_F(x) \leq n - 3$, then $\varphi(x) = \{i \in N : s_i(x) \geq 1\}$;
(b) if $s_F(x) = n - 2$, then $\varphi(x) = \{i \in N : s_i(x) \geq 1 \text{ and } x_i \in F_x\}$;
(c) if $s_F(x) = n - 1$, then $\varphi(x) = F_x$.

One can easily check that, if $n \geq 5$, then this rule satisfies impartiality, anonymity, symmetry, and positive unanimity (but not monotonicity) and $\varphi^*(x) \not\subset \varphi(x)$ when $s_F(x) = n - 2$. Note that the rule also satisfies negative unanimity, an axiom which

[14] If $n = 4$, one can easily show that plurality with runners-up is a subcorrespondence of any other nomination rule satisfying impartiality, symmetry, and positive unanimity. Finally, if $n = 3$, there is no nomination rule satisfying impartiality and positive unanimity, as shown in Tamura and Ohseto (2014).

we also mentioned in the introduction.

3.4 Concluding Remarks

We showed that plurality with runners-up is the only minimal nomination rule satisfying impartiality, anonymity, symmetry, and monotonicity. It would be fair to say that our three axioms, as well as impartiality, are desirable properties in practical situations. Therefore, our result suggests that plurality with runners-up is a reasonable impartial rule to use in such situations. Moreover, as we have seen in the proof of Theorem 3.1, the rule becomes simple enough for practical use if it is represented as follows: a person wins if and only if she is one of the plurality winners when her nomination is not counted.

Given our result, one may wonder if one can establish a complete characterization of the plurality with runners-up, that is, whether one can find a set of axioms that deduces the rule. In regard to this, we were able to show that, if $n = 4$, then plurality with runners-up is the unique nomination rule satisfying impartiality, symmetry, positive unanimity, and negative unanimity. However, we have not seen any such characterization in the case of $n \geq 5$, and leave this question for future research.

Chapter 4

Strategy-Proofness versus Symmetry in Economies with an Indivisible Good and Money[***]

4.1 Introduction

We consider the problem of allocating a single indivisible good among n agents when monetary transfers are allowed. We assume that each agent has a valuation of the indivisible good (i.e., quasilinear preferences), and the indivisible good must be received by one agent. This model can handle a situation where some agents have positive valuations and others have negative valuations. In addition, it can be interpreted as an auction model when agents' valuations are restricted to be non-negative, and as a task assignment model when agents' valuations are restricted to

[***]This chapter is based on Kato, M., Ohseto, S., and Tamura, S. (2015). Strategy-Proofness versus Symmetry in Economies with an Indivisible Good and Money. *International Journal of Game Theory*, 44, 195-207. doi:10.1007/s00182-014-0425-y.

be non-positive. We study the possibility of constructing desirable mechanisms that determine who receives the indivisible good and how to make monetary transfers on the basis of agents' valuations.

We introduce three main axioms. Strategy-proofness requires that truthful revelation of a valuation should be a weakly dominant strategy for each agent (Gibbard, 1973; Satterthwaite, 1975).[15] Symmetry requires that if the valuations of two agents are the same, they should receive indifferent consumption bundles.[16] Budget balance requires that the total amount of net monetary transfers should be equal to zero. In this chapter, we investigate whether or not there exists a strategy-proof, symmetric, and budget balanced mechanism in economies with an indivisible good and money.

The compatibility between strategy-proofness and Pareto efficiency has been one of the central issues in the mechanism design literature. Many authors have established interesting results in quasilinear environments, where Pareto efficiency can be decomposed into assignment-efficiency (i.e., an agent with the highest valuation receives the indivisible good in our model) and budget balance. It follows from Holmström (1979) that Groves mechanisms (Groves, 1973) are the only strategy-proof and assignment-efficient mechanisms. It also follows as a corollary of Holmström's characterization that there is no strategy-proof and Pareto efficient mechanism.[17]

One research direction after Holmström's characterization is the selection of Groves mechanisms. Fairness axioms such as envy-freeness (Foley, 1967) or egalitarian-equivalence (Pazner and Schmeidler, 1978) are often employed. Pápai (2003), Ohseto (2006), and Svensson (2009) characterize envy-free Groves mechanisms. Ohseto (2004) and Yengin (2012) characterize egalitarian-equivalent Groves mechanisms. Chew and Serizawa (2007) characterize the Vickrey mechanism (Vickrey, 1961) as

[15] See Sprumont (1995) and Barberà (2001, 2012) for excellent surveys of the literature on strategy-proofness.

[16] Symmetry is also called equal treatment of equals.

[17] Ohseto (2000) and Schummer (2000) formally prove this statement in related models.

the unique individually rational Groves mechanism.[18] Porter et al. (2004), Cavallo (2006), and Moulin (2009) construct some Groves mechanisms whose budget imbalance is smaller than that of the Vickrey mechanism.

Another research direction is to drop the requirement of assignment-efficiency, and find strategy-proof and budget balanced mechanisms that satisfy other desirable axioms. Schummer (2000) characterizes strategy-proof and budget balanced mechanisms in the two-agent case and Ohseto (1999) characterizes strategy-proof and budget balanced mechanisms that satisfy individual rationality, equal compensation, and demand monotonicity. However, their mechanisms do not satisfy symmetry. Symmetry is important in this context since it is considered as a weak axiom of fairness. Indeed, it is implied by envy-freeness or egalitarian-equivalence. Our problem is translated to the problem of whether or not strategy-proofness is compatible with weak axioms of fairness (symmetry) and Pareto efficiency (budget balance). Ando et al. (2008) show intermediate results that there is no strategy-proof, symmetric, and budget balanced mechanism under an additional axiom of either (i) equal compensation, (ii) normal compensation, or (iii) individual rationality.

In this chapter, we show that in general there is no strategy-proof, symmetric, and budget balanced mechanism. More precisely, we show that there is no strategy-proof, symmetric, and budget balanced mechanism on the following restricted domains: the set of agent's possible valuations includes at least $n + 1$ common valuations (Theorem 4.1). This domain condition is very weak, and it is satisfied with standard domains such as (i) the set of agent's possible valuations consisting of non-negative real numbers (as in the auction models), and (ii) the set of agent's possible valuations consisting of non-positive real numbers (as in the task assignment models). Moreover, if we represent the value of the indivisible good by the minimum unit of currency, the set of agent's possible valuations consisting of non-negative integers is an interesting

[18]This result also follows as a corollary of Holmström (1979). See Saitoh and Serizawa (2008), Sakai (2008, 2013a), Ashlagi and Serizawa (2012), and Sprumont (2013) for alternative characterizations of the Vickrey mechanism.

domain that satisfies the above domain condition.

It is a fundamental fact that the non-existence of strategy-proof, symmetric, and budget balanced mechanism on a smaller domain implies the same result on a larger domain.[19] The consequences of Theorem 4.1 follow immediately: (i) in the model where each agent may have non-quasilinear preferences in addition to all quasilinear preferences, and/or (ii) in the model where there is a set of n heterogeneous indivisible goods (i.e., all indivisible goods are distinct), each agent has an arbitrary valuation vector of the indivisible goods (i.e., agent's possible valuations are unrestricted), and each agent receives exactly one unit of the indivisible goods. In the former model, Saitoh and Serizawa (2008) and Sakai (2008, 2013a) provide some characterizations on arbitrary domains of non-quasilinear preferences that include the set of all quasilinear preferences.[20] Theorem 4.1 implies that there is no strategy-proof, symmetric, and budget balanced mechanism on arbitrary domains that include the set of all quasilinear preferences. In the latter model, Miyagawa (2001) and Svensson and Larsson (2002) characterize strategy-proof and budget balanced mechanisms that satisfy some auxiliary axioms. Consider a situation where there is one indivisible good which is valuable to the agents and the remaining $n-1$ indivisible goods are valueless to the agents. Since agent's possible valuations are unrestricted, the mechanisms must be defined on the domain including the situation mentioned above. Theorem 4.1 implies that there is no strategy-proof, symmetric, and budget balanced mechanism in the model of allocating n heterogeneous indivisible goods.

The rest of this chapter is organized as follows. Section 4.2 contains notation and definitions. Section 4.3 establishes the main impossibility result. Section 4.4 states

[19] The definitions of strategy-proofness and budget balance can be extended to larger domains straightforwardly. The definition of symmetry can be extended to larger domains in the weak sense that it requires the same condition on the set of all valuation profiles and it requires nothing on the outside of that set. See Hashimoto and Saitoh (2010, Definition 4) for this type of extension in the context of public good economies.

[20] Saitoh and Serizawa (2008, Theorems 2 and 3) also provide characterizations on some domains of non-quasilinear preferences that do not include any quasilinear preferences.

some concluding remarks.

4.2 Notation and Definitions

Let $N = \{1, ..., n\}$ $(n \geq 2)$ be the set of agents. We consider economies with a single indivisible good and a transferable good. The indivisible good must be assigned to one agent. We prohibit the agent from disposing of the indivisible good even if it is undesirable to him. The transferable good, often regarded as money, is used for compensation. Agent i's consumption space is the set of consumption bundles $(s_i, t_i) \in \{0, 1\} \times \mathbb{R}$, where s_i denotes his consumption of the indivisible good and t_i denotes the net monetary transfer he receives (if $t_i > 0$) or he pays (if $t_i < 0$). The set of *feasible allocations* is $Z = \{z = (z_1, ..., z_n) = ((s_1, t_1), ..., (s_n, t_n)) \in [\{0, 1\} \times \mathbb{R}]^n : \sum_{i \in N} s_i = 1 \text{ and } \sum_{i \in N} t_i \leq 0\}$.

Each agent i has a *valuation* $v_i \in \mathbb{R}$ of the indivisible good, and his preference can be represented by a quasilinear utility function $U((s_i, t_i); v_i) = v_i s_i + t_i$. Let $V_i \subset \mathbb{R}$ be the set of agent i's possible valuations of the indivisible good. We do not assume a priori that V_i consists of non-negative real numbers as in the auction models, nor non-positive real numbers as in the task assignment models. Let V be the Cartesian product of V_i, and an element $v = (v_1, ..., v_n) \in V$ is called a *valuation profile.* Given a coalition $C \subset N$, let (v'_C, v_{-C}) denote the valuation profile whose i-th component is v'_i if $i \in C$ and v_i if $i \notin C$. For simplicity of notation, we often use (v'_i, v_{-i}) instead of $(v'_{\{i\}}, v_{-\{i\}})$.

A *mechanism* is a function $f : V \to Z$, which associates a feasible allocation with each valuation profile. Given a mechanism f and $v \in V$, we write $f(v) = ((s_1(v), t_1(v)), ..., (s_n(v), t_n(v)))$, and $f_i(v) = (s_i(v), t_i(v))$ for all $i \in N$. Let $C(v) = \{i \in N : s_i(v) = 1\}$ and $NC(v) = \{i \in N : s_i(v) = 0\}$ denote the consumer and the non-consumers of the indivisible good at $v \in V$, respectively. Since we consider economies with a single indivisible good and money, $C(v)$ is a singleton for all $v \in V$.

We introduce three main axioms. Strategy-proofness requires that truth-telling should be a weakly dominant strategy for each agent. Symmetry requires that if the valuations of two agents are the same, they should receive indifferent consumption bundles. Budget balance requires that the total amount of net monetary transfers should be equal to zero.

Strategy-proofness: for all $v \in V$, all $i \in N$, and all $v'_i \in V_i$,

$$U(f_i(v); v_i) \geq U(f_i(v'_i, v_{-i}); v_i).$$

Symmetry: for all $v \in V$ and all $i, j \in N$,

$$v_i = v_j \Rightarrow U(f_i(v); v_i) = U(f_j(v); v_i).$$

Budget balance: for all $v \in V$,

$$\sum_{i \in N} t_i(v) = 0.$$

A direct result of symmetry is that the non-consumers with the same valuation receive the same consumption bundle, i.e., for all $v \in V$ and all $i, j \in NC(v)$, $[v_i = v_j] \Rightarrow [f_i(v) = f_j(v)]$. Therefore, it is convenient to introduce the following notation. Let $f^x_{NC}(v) = (s^x_{NC}(v), t^x_{NC}(v))$ denote the consumption bundle of the non-consumers whose valuation is $x \in \mathbb{R}$ at $v \in V$. Let $f_C(v) = (s_C(v), t_C(v))$ denote the consumption bundle of the consumer at $v \in V$.

We present two useful lemmas. Lemma 4.1 shows that under a strategy-proof mechanism, (i) an increase of the consumer's valuation does not change his consumption bundle, and (ii) a decrease of a non-consumer's valuation does not change his consumption bundle.

Lemma 4.1. *Let f be a strategy-proof mechanism.*

(i) *For all* $v \in V$*, all* $i \in N$*, and all* $v'_i \in V_i$*, if* $i \in C(v)$ *and* $v_i < v'_i$*, then* $f_i(v) = f_i(v'_i, v_{-i})$.

(ii) *For all* $v \in V$*, all* $i \in N$*, and all* $v'_i \in V_i$*, if* $i \in NC(v)$ *and* $v_i > v'_i$*, then* $f_i(v) = f_i(v'_i, v_{-i})$.

Proof. (i) Suppose that $i \in NC(v'_i, v_{-i})$. By strategy-proofness, $U(f_i(v); v_i) = v_i + t_i(v) \geq t_i(v'_i, v_{-i}) = U(f_i(v'_i, v_{-i}); v_i)$ and $U(f_i(v'_i, v_{-i}); v'_i) = t_i(v'_i, v_{-i}) \geq v'_i + t_i(v) = U(f_i(v); v'_i)$. These inequalities imply that $v'_i \leq v_i$, a contradiction. Hence, $i \in C(v'_i, v_{-i})$. By strategy-proofness, $t_i(v) = t_i(v'_i, v_{-i})$. Therefore, $f_i(v) = f_i(v'_i, v_{-i})$.

(ii) Suppose that $i \in C(v'_i, v_{-i})$. By strategy-proofness, $U(f_i(v); v_i) = t_i(v) \geq v_i + t_i(v'_i, v_{-i}) = U(f_i(v'_i, v_{-i}); v_i)$ and $U(f_i(v'_i, v_{-i}); v'_i) = v'_i + t_i(v'_i, v_{-i}) \geq t_i(v) = U(f_i(v); v'_i)$. These inequalities imply that $v_i \leq v'_i$, a contradiction. Hence, $i \in NC(v'_i, v_{-i})$. By strategy-proofness, $t_i(v) = t_i(v'_i, v_{-i})$. Therefore, $f_i(v) = f_i(v'_i, v_{-i})$. □

Lemma 4.2 describes how a symmetric and budget balanced mechanism specifies a feasible allocation when all agents have the same valuation a. It specifies the consumption bundles of the consumer and the non-consumers, but it does not specify who consumes the indivisible good.

Lemma 4.2. *Let* f *be a symmetric and budget balanced mechanism. For all* $a \in \mathbb{R}$*, let* $v = (a, ..., a) \in V$*. Then,* $f_i(v) = (1, -(n-1)a/n)$ *for* $i \in C(v)$ *and* $f_j(v) = (0, a/n)$ *for all* $j \in NC(v)$*.*

Proof. By symmetry, $U(f_i(v); a) = a + t_C(v) = t^a_{NC}(v) = U(f_j(v); a)$ for $i \in C(v)$ and all $j \in NC(v)$. By budget balance, $t_C(v) + (n-1)t^a_{NC}(v) = 0$. By solving these equations, $t_C(v) = -(n-1)a/n$ and $t^a_{NC}(v) = a/n$. □

4.3 The Impossibility Result

In this section, we establish the impossibility result: there is no strategy-proof, symmetric, and budget balanced mechanism if the set of possible valuations includes at least $n + 1$ common valuations across the agents.

Theorem 4.1. *Let V be such that the intersection $\cap_{i \in N} V_i$ contains at least $n + 1$ valuations. Then, there is no strategy-proof, symmetric, and budget balanced mechanism.*

We briefly explain how to prove Theorem 4.1. We consider a unanimous valuation profile $v = (a, ..., a)$. Suppose that a non-consumer at the valuation profile v decreases his valuation to b. Then, we obtain a new valuation profile. Similarly, suppose that a non-consumer at the new valuation profile decreases his valuation to b. We repeat this process $n-1$ times, and then we obtain the valuation profile where one agent's valuation is a and the other $n-1$ agents' valuation is b. By induction, the following three successive lemmas (Lemmas 4.3-4.5) specify (i) who is the consumer of the indivisible good, and (ii) the consumption bundles of the consumer and the non-consumers at this valuation profile. Using these results, we show that the consumption bundles of agents at $n + 1$ unanimous valuation profiles are incompatible under strategy-proofness, symmetry, and budget balance.

First, consider a valuation profile $v = (a, ..., a)$, where all agents have the same valuation, as in Lemma 4.2. After a non-consumer, say h_1, at the valuation profile v decreases his valuation to b, we obtain a new valuation profile. Lemma 4.3 shows that the consumer is in $N \setminus \{h_1\}$, and the consumption bundles of the consumer and the non-consumers remain unchanged.

Lemma 4.3. *Let f be a strategy-proof, symmetric, and budget balanced mechanism. For all $a, b \in \mathbb{R}$ with $a > b$, let $v = (a, ..., a) \in V$, and $v' = (b, ..., b) \in V$. Then, for all $h_1 \in NC(v)$,*

(i) $C(v'_{h_1}, v_{-h_1}) \subset N \setminus \{h_1\}$,

(ii) $f_C(v'_{h_1}, v_{-h_1}) = (1, -(n-1)a/n)$, *and*

(iii) $f^a_{NC}(v'_{h_1}, v_{-h_1}) = f^b_{NC}(v'_{h_1}, v_{-h_1}) = (0, a/n)$.

Proof. Since $h_1 \in NC(v)$, by Lemma 4.2, $f_{h_1}(v) = (0, a/n)$. By Lemma 4.1 (ii), $f_{h_1}(v'_{h_1}, v_{-h_1}) = (0, a/n)$, and thus $t^b_{NC}(v'_{h_1}, v_{-h_1}) = a/n$. Note that the consumer's valuation is a. By symmetry, $a + t_C(v'_{h_1}, v_{-h_1}) = t^a_{NC}(v'_{h_1}, v_{-h_1})$.[21] By budget balance, $t_C(v'_{h_1}, v_{-h_1}) + (n-2)t^a_{NC}(v'_{h_1}, v_{-h_1}) + a/n = 0$. By solving these equations, $t_C(v'_{h_1}, v_{-h_1}) = -(n-1)a/n$ and $t^a_{NC}(v'_{h_1}, v_{-h_1}) = a/n$. $\square$

As in Lemma 4.2, let $v = (a, ..., a)$ be a valuation profile where all agents have the same valuation. Suppose that a non-consumer, say h_1, at the valuation profile v decreases his valuation to $v'_{h_1} = b$. Then, we obtain the valuation profile (v'_{h_1}, v_{-h_1}). Similarly, suppose that a non-consumer, say h_2, at the valuation profile (v'_{h_1}, v_{-h_1}) decreases his valuation to $v'_{h_2} = b$. This process is repeated k times, and then we obtain the new valuation profile $(v'_{\{h_1,...,h_k\}}, v_{-\{h_1,...,h_k\}})$. Lemma 4.4 assumes that the consumer is in $N \setminus \{h_1, ..., h_k\}$, and the consumption bundles of the consumer and the non-consumers remain unchanged (Induction hypothesis). Suppose that a non-consumer, say h_{k+1}, at the valuation profile $(v'_{\{h_1,...,h_k\}}, v_{-\{h_1,...,h_k\}})$ decreases his valuation to $v'_{h_{k+1}} = b$. Then, we obtain the new valuation profile $(v'_{\{h_1,...,h_{k+1}\}}, v_{-\{h_1,...,h_{k+1}\}})$. Lemma 4.4 shows that the consumer is in $N \setminus \{h_1, ..., h_{k+1}\}$, and the consumption bundles of the consumer and the non-consumers remain unchanged.

Lemma 4.4. *Let $n \geq 3$ and f be a strategy-proof, symmetric, and budget balanced mechanism. For all $a, b \in \mathbb{R}$ with $a > b$, let $v = (a, ..., a) \in V$ and $v' = (b, ..., b) \in V$. For all $h_1 \in NC(v)$, all $h_2 \in NC(v'_{h_1}, v_{-h_1}) \setminus \{h_1\}$,...,all $h_k \in NC(v'_{\{h_1,...,h_{k-1}\}}, v_{-\{h_1,...,h_{k-1}\}}) \setminus \{h_1, ..., h_{k-1}\}$, where $1 \leq k \leq n-2$, we assume*

[21] Ignore this equation in the two-agent case since there is no non-consumer whose valuation is a.

(i) $C(v'_{\{h_1,...,h_k\}}, v_{-\{h_1,...,h_k\}}) \subset N \setminus \{h_1, ..., h_k\}$,
(ii) $f_C(v'_{\{h_1,...,h_k\}}, v_{-\{h_1,...,h_k\}}) = (1, -(n-1)a/n)$, *and*
(iii) $f^a_{NC}(v'_{\{h_1,...,h_k\}}, v_{-\{h_1,...,h_k\}}) = f^b_{NC}(v'_{\{h_1,...,h_k\}}, v_{-\{h_1,...,h_k\}}) = (0, a/n)$.

Then, for all $h_{k+1} \in NC(v'_{\{h_1,...,h_k\}}, v_{-\{h_1,...,h_k\}}) \setminus \{h_1, ..., h_k\}$, *we have*
(i)' $C(v'_{\{h_1,...,h_{k+1}\}}, v_{-\{h_1,...,h_{k+1}\}}) \subset N \setminus \{h_1, ..., h_{k+1}\}$,
(ii)' $f_C(v'_{\{h_1,...,h_{k+1}\}}, v_{-\{h_1,...,h_{k+1}\}}) = (1, -(n-1)a/n)$, *and*
(iii)' $f^a_{NC}(v'_{\{h_1,...,h_{k+1}\}}, v_{-\{h_1,...,h_{k+1}\}}) = f^b_{NC}(v'_{\{h_1,...,h_{k+1}\}}, v_{-\{h_1,...,h_{k+1}\}}) = (0, a/n)$.

Proof. Suppose by way of contradiction that (i)' does not hold. To simplify the notation, let $H_{k+1} = \{h_1, ..., h_{k+1}\}$. Then, $C(v'_{H_{k+1}}, v_{-H_{k+1}}) \subset H_{k+1}$. By the premise (iii) of this lemma and Lemma 4.1 (ii), $f_{h_{k+1}}(v'_{H_{k+1}}, v_{-H_{k+1}}) = (0, a/n)$. Note that the consumer's valuation is b. By symmetry, $b + t_C(v'_{H_{k+1}}, v_{-H_{k+1}}) = t^b_{NC}(v'_{H_{k+1}}, v_{-H_{k+1}}) = a/n$. By solving this equation, $t_C(v'_{H_{k+1}}, v_{-H_{k+1}}) = (a - nb)/n$. Therefore,

$$f_C(v'_{H_{k+1}}, v_{-H_{k+1}}) = (1, (a-nb)/n).$$

Let $i_1 \in C(v'_{H_{k+1}}, v_{-H_{k+1}}) \subset \{h_1, ..., h_k\}$. Change agent i_1's valuation from b to a. By Lemma 4.1 (i),

$$f_{i_1}(v'_{H_{k+1}\setminus\{i_1\}}, v_{-\{H_{k+1}\setminus\{i_1\}\}}) = (1, (a-nb)/n).$$

Note that consumer i_1's valuation is a. By symmetry,
$t^a_{NC}(v'_{H_{k+1}\setminus\{i_1\}}, v_{-\{H_{k+1}\setminus\{i_1\}\}}) = a + t_C(v'_{H_{k+1}\setminus\{i_1\}}, v_{-\{H_{k+1}\setminus\{i_1\}\}}) = a + (a-nb)/n$. By budget balance, $t_C(v'_{H_{k+1}\setminus\{i_1\}}, v_{-\{H_{k+1}\setminus\{i_1\}\}}) + (n-k-1)t^a_{NC}(v'_{H_{k+1}\setminus\{i_1\}}, v_{-\{H_{k+1}\setminus\{i_1\}\}}) + kt^b_{NC}(v'_{H_{k+1}\setminus\{i_1\}}, v_{-\{H_{k+1}\setminus\{i_1\}\}}) = 0$. By solving these equations,
$t^a_{NC}(v'_{H_{k+1}\setminus\{i_1\}}, v_{-\{H_{k+1}\setminus\{i_1\}\}}) = \{(n+1)a - nb\}/n$ and $t^b_{NC}(v'_{H_{k+1}\setminus\{i_1\}}, v_{-\{H_{k+1}\setminus\{i_1\}\}}) = \{(-n^2 + nk + k)a + n(n-k)b\}/(nk)$. Therefore,

$$f^a_{NC}(v'_{H_{k+1}\setminus\{i_1\}}, v_{-\{H_{k+1}\setminus\{i_1\}\}}) = (0, \{(n+1)a - nb\}/n), \text{ and} \tag{4.1}$$

$$f^b_{NC}(v'_{H_{k+1}\setminus\{i_1\}}, v_{-\{H_{k+1}\setminus\{i_1\}\}}) = (0, \{(-n^2+nk+k)a + n(n-k)b\}/(nk)). \quad (4.2)$$

Note that the set $H_{k+1} \setminus \{i_1\}$ consists of k agents. Choose $k-1$ agents from the set in the following way. Let $j_1 \in NC(v) \cap \{H_{k+1} \setminus \{i_1\}\}$, $j_2 \in NC(v'_{j_1}, v_{-j_1}) \cap \{H_{k+1} \setminus \{i_1, j_1\}\}$, ..., and $j_{k-1} \in NC(v'_{\{j_1,...,j_{k-2}\}}, v_{-\{j_1,...,j_{k-2}\}}) \cap \{H_{k+1} \setminus \{i_1, j_1, ..., j_{k-2}\}\}$. To simplify the notation, let $J_{k-1} = \{j_1, ..., j_{k-1}\}$. Note that the set $H_{k+1} \setminus \{J_{k-1} \cup \{i_1\}\}$ is a singleton and let $i_2 \in H_{k+1} \setminus \{J_{k-1} \cup \{i_1\}\}$. We consider the following two cases.

Case 1. $i_2 \in NC(v'_{J_{k-1}}, v_{-J_{k-1}})$.

Change agent i_2's valuation from a to b. By the premise (i) and (iii) of this lemma,

$$f_{i_2}(v'_{J_{k-1}\cup\{i_2\}}, v_{-\{J_{k-1}\cup\{i_2\}\}}) = f^b_{NC}(v'_{J_{k-1}\cup\{i_2\}}, v_{-\{J_{k-1}\cup\{i_2\}\}}) = (0, a/n). \quad (4.3)$$

Note that the valuation profile $(v'_{H_{k+1}\setminus\{i_1\}}, v_{-\{H_{k+1}\setminus\{i_1\}\}})$ in (4.2) is identical with the valuation profile $(v'_{J_{k-1}\cup\{i_2\}}, v_{-\{J_{k-1}\cup\{i_2\}\}})$ in (4.3). Comparing (4.3) with (4.2), we have $a/n - \{(-n^2+nk+k)a + n(n-k)b\}/(nk) = (n-k)(a-b)/k > 0$, a contradiction.

Case 2. $i_2 \in C(v'_{J_{k-1}}, v_{-J_{k-1}})$.

Note that $H_{k+1} = J_{k-1} \cup \{i_1, i_2\}$. Choose some $i_3 \in N \setminus H_{k+1}$. Change agent i_3's valuation from a to b. By the premise (ii) and (iii) of this lemma,

$$f_C(v'_{J_{k-1}\cup\{i_3\}}, v_{-\{J_{k-1}\cup\{i_3\}\}}) = (1, -(n-1)a/n), \text{ and} \quad (4.4)$$

$$f^a_{NC}(v'_{J_{k-1}\cup\{i_3\}}, v_{-\{J_{k-1}\cup\{i_3\}\}}) = (0, a/n). \quad (4.5)$$

Consider the valuation profile $(v'_{H_{k+1}\setminus\{i_1\}}, v_{-\{H_{k+1}\setminus\{i_1\}\}})$ in (4.1). Remember that $i_1 \in C(v'_{H_{k+1}\setminus\{i_1\}}, v_{-\{H_{k+1}\setminus\{i_1\}\}})$. Change agent i_3's valuation from a to b. Note that $(v'_{\{H_{k+1}\setminus\{i_1\}\}\cup\{i_3\}}, v_{-\{\{H_{k+1}\setminus\{i_1\}\}\cup\{i_3\}\}}) = (v'_{J_{k-1}\cup\{i_2,i_3\}}, v_{-\{J_{k-1}\cup\{i_2,i_3\}\}}) \in V$. By (4.1) and Lemma 4.1 (ii),

$$f_{i_3}(v'_{J_{k-1}\cup\{i_2,i_3\}}, v_{-\{J_{k-1}\cup\{i_2,i_3\}\}}) = (0, \{(n+1)a - nb\}/n). \tag{4.6}$$

We consider the following two subcases.

Subcase 2A. $i_2 \in NC(v'_{J_{k-1}\cup\{i_2,i_3\}}, v_{-\{J_{k-1}\cup\{i_2,i_3\}\}})$.

By (4.6) and symmetry,

$t_{i_2}(v'_{J_{k-1}\cup\{i_2,i_3\}}, v_{-\{J_{k-1}\cup\{i_2,i_3\}\}}) = t_{i_3}(v'_{J_{k-1}\cup\{i_2,i_3\}}, v_{-\{J_{k-1}\cup\{i_2,i_3\}\}}) = \{(n+1)a - nb\}/n$. By (4.4), if $i_2 \in C(v'_{J_{k-1}\cup\{i_3\}}, v_{-\{J_{k-1}\cup\{i_3\}\}})$, agent i_2's utility is $U((1, -(n-1)a/n); a) = a/n$. By (4.5), if $i_2 \in NC(v'_{J_{k-1}\cup\{i_3\}}, v_{-\{J_{k-1}\cup\{i_3\}\}})$, agent i_2's utility is $U((0, a/n); a) = a/n$. Since $\{(n+1)a - nb\}/n - a/n = a - b > 0$, both cases contradict strategy-proofness.

Subcase 2B. $i_2 \in C(v'_{J_{k-1}\cup\{i_2,i_3\}}, v_{-\{J_{k-1}\cup\{i_2,i_3\}\}})$.

By (4.6) and symmetry,

$b + t_C(v'_{J_{k-1}\cup\{i_2,i_3\}}, v_{-\{J_{k-1}\cup\{i_2,i_3\}\}}) = t_{i_3}(v'_{J_{k-1}\cup\{i_2,i_3\}}, v_{-\{J_{k-1}\cup\{i_2,i_3\}\}}) = \{(n+1)a - nb\}/n$. Hence, $f_{i_2}(v'_{J_{k-1}\cup\{i_2,i_3\}}, v_{-\{J_{k-1}\cup\{i_2,i_3\}\}}) = (1, \{(n+1)a - 2nb\}/n\})$. Change agent i_2's valuation from b to a. By Lemma 4.1 (i), $f_{i_2}(v'_{J_{k-1}\cup\{i_3\}}, v_{-\{J_{k-1}\cup\{i_3\}\}}) = (1, \{(n+1)a - 2nb\}/n\})$. Comparing this with (4.4), we have $\{(n+1)a - 2nb\}/n\} - \{-(n-1)a/n\} = 2(a-b) > 0$, a contradiction.

We have proved that the supposition is false, and (i)' holds. By Lemma 4.1 (ii), $f_{h_{k+1}}(v'_{H_{k+1}}, v_{-H_{k+1}}) = (0, a/n)$. Then, $t^b_{NC}(v'_{H_{k+1}}, v_{-H_{k+1}}) = a/n$. Note that the consumer's valuation is a. By symmetry, $a + t_C(v'_{H_{k+1}}, v_{-H_{k+1}}) = t^a_{NC}(v'_{H_{k+1}}, v_{-H_{k+1}})$. By budget balance,

$t_C(v'_{H_{k+1}}, v_{-H_{k+1}}) + (n-k-2)t^a_{NC}(v'_{H_{k+1}}, v_{-H_{k+1}}) + (k+1)t^b_{NC}(v'_{H_{k+1}}, v_{-H_{k+1}}) = 0$. By solving these equations, $t_C(v'_{H_{k+1}}, v_{-H_{k+1}}) = -(n-1)a/n$ and $t^a_{NC}(v'_{H_{k+1}}, v_{-H_{k+1}}) = a/n$. Therefore, (ii)' and (iii)' hold. $\square$

As in Lemma 4.2, let $v = (a, ..., a)$ be a valuation profile where all agents have the same valuation. Suppose that a non-consumer, say h_1, at the valuation profile v decreases his valuation to $v'_{h_1} = b$. Then, we obtain the valuation profile (v'_{h_1}, v_{-h_1}). Similarly, suppose that a non-consumer, say h_2, at the valuation profile (v'_{h_1}, v_{-h_1}) decreases his valuation to $v'_{h_2} = b$. This process is repeated $n-1$ times, and then we obtain the new valuation profile $(v'_{\{h_1,...,h_{n-1}\}}, v_{-\{h_1,...,h_{n-1}\}})$. Lemma 4.5 shows that the consumer is $i \in N \setminus \{h_1, ..., h_{n-1}\}$, and the consumption bundles of the consumer and the non-consumers remain unchanged.

Lemma 4.5. *Let f be a strategy-proof, symmetric, and budget balanced mechanism. For all $a, b \in \mathbb{R}$ with $a > b$, let $v = (a, ..., a) \in V$ and $v' = (b, ..., b) \in V$. Then, there exists some $i \in N$ such that $f_i(v_i, v'_{-i}) = (1, -(n-1)a/n)$, and $f_j(v_i, v'_{-i}) = (0, a/n)$ for all $j \neq i$.*

Proof. It follows directly from Lemma 4.3 in the two-agent case. Hence, we consider the case of $n \geq 3$. By Lemma 4.3, for all $h_1 \in NC(v)$, we have (i) $C(v'_{h_1}, v_{-h_1}) \subset N \setminus \{h_1\}$, (ii) $f_C(v'_{h_1}, v_{-h_1}) = (1, -(n-1)a/n)$, and
(iii) $f^a_{NC}(v'_{h_1}, v_{-h_1}) = f^b_{NC}(v'_{h_1}, v_{-h_1}) = (0, a/n)$. By applying Lemma 4.4 repeatedly for $k = 1, ..., n-2$, for all $h_1 \in NC(v)$, all $h_2 \in NC(v'_{h_1}, v_{-h_1}) \setminus \{h_1\}$, ..., all $h_{n-1} \in NC(v'_{H_{n-2}}, v_{-H_{n-2}}) \setminus H_{n-2}$, we have (i)* $C(v'_{H_{n-1}}, v_{-H_{n-1}}) \subset N \setminus H_{n-1}$, (ii)* $f_C(v'_{H_{n-1}}, v_{-H_{n-1}}) = (1, -(n-1)a/n)$, and
(iii)* $f^a_{NC}(v'_{H_{n-1}}, v_{-H_{n-1}}) = f^b_{NC}(v'_{H_{n-1}}, v_{-H_{n-1}}) = (0, a/n)$, where $H_{n-2} = \{h_1, ..., h_{n-2}\}$ and $H_{n-1} = \{h_1, ..., h_{n-1}\}$. Note that the set $N \setminus H_{n-1}$ is a singleton and let $i \in N \setminus H_{n-1}$. Since $(v'_{H_{n-1}}, v_{-H_{n-1}}) = (v_i, v'_{-i})$, (i)* and (ii)* imply $f_i(v_i, v'_{-i}) = (1, -(n-1)a/n)$, and (i)* and (iii)* imply $f_j(v_i, v'_{-i}) = (0, a/n)$ for all $j \neq i$. $\square$

Proof of Theorem 4.1. By the assumption on V, let $v^1, v^2, ..., v^{n+1}$ be such that $\{v^1, v^2, ..., v^{n+1}\} \subset \cap_{i \in N} V_i$, where $v^1 > v^2 > \cdots > v^{n+1}$. Let $v^{(m)} = (v^m, ..., v^m) \in V$ for all $m \in \{1, ..., n\}$ and $v^{(n+1)} = (v^{n+1}, ..., v^{n+1}) \in V$. We regard $v^{(m)}$ and $v^{(n+1)}$ as v and v' in Lemma 4.5, respectively, where $m \in \{1, ..., n\}$. By Lemma 4.5, for all $m \in \{1, ..., n\}$, there exists some $i_m \in N$ such that $f_{i_m}(v^{(m)}_{i_m}, v^{(n+1)}_{-i_m}) = (1, -(n-1)v^m/n)$, and $f_j(v^{(m)}_{i_m}, v^{(n+1)}_{-i_m}) = (0, v^m/n)$ for all $j \neq i_m$ (Note that the consumer of the indivisible good is dependent on $m \in \{1, ..., n\}$). Let us consider $n+1$ valuation profiles $v^{(n+1)}$ and $(v^{(m)}_{i_m}, v^{(n+1)}_{-i_m})$ for $m = 1, ..., n$. By Lemma 4.2, $f_i(v^{(n+1)}) = (1, -(n-1)v^{n+1}/n)$ for $i \in C(v^{(n+1)})$. Then, since there are only n agents, either (i) there is $m \in \{1, ..., n\}$ such that $i = i_m \in N$, $f_i(v^{(n+1)}) = (1, -(n-1)v^{n+1}/n)$, and $f_{i_m}(v^{(m)}_{i_m}, v^{(n+1)}_{-i_m}) = (1, -(n-1)v^m/n)$, or (ii) there are $m_1, m_2 \in \{1, ..., n\}$ $(m_1 \neq m_2)$ such that $i_{m_1} = i_{m_2} \in N$, $f_{i_{m_1}}(v^{(m_1)}_{i_{m_1}}, v^{(n+1)}_{-i_{m_1}}) = (1, -(n-1)v^{m_1}/n)$, and $f_{i_{m_2}}(v^{(m_2)}_{i_{m_2}}, v^{(n+1)}_{-i_{m_2}}) = (1, -(n-1)v^{m_2}/n)$. Both cases contradict strategy-proofness. $\square$

We show the independence of three axioms in Theorem 4.1. We introduce the following notation: for all $v \in V$, $v^{[1]}$ and $v^{[2]}$ denote the highest and the second highest valuations in v, respectively. For simplicity, we assume that the set of agent's possible valuations consists of all non-negative real numbers. The following examples show that strategy-proofness, symmetry, and budget balance are mutually independent.

Example 4.1. Let V be such that $V_i = [0, +\infty)$ for all $i \in N$. Let f be such that for all $v \in V$,

(1) $C(v) \subset \text{Argmax}_{i \in N}\ v_i$,

(2a) $t_i(v) = -v^{[1]}(n-1)/n$ for $i \in C(v)$, and

(2b) $t_j(v) = v^{[1]}/n$ for all $j \in NC(v)$.

Then, f satisfies symmetry and budget balance, but does not satisfy strategy-proofness.

Example 4.2. Let V be such that $V_i = [0, +\infty)$ for all $i \in N$. Fix $i \in N$ and let f be such that for all $v \in V$, $f_i(v) = (1, 0)$ and $f_j(v) = (0, 0)$ for all $j \neq i$. Then, f

satisfies strategy-proofness and budget balance, but does not satisfy symmetry.

Example 4.3. (Vickrey, 1961). Let V be such that $V_i = [0, +\infty)$ for all $i \in N$. Let f be such that for all $v \in V$,

(1) $C(v) \subset \mathrm{Argmax}_{i \in N}\, v_i$,

(2a) $t_i(v) = -v^{[2]}$ for $i \in C(v)$, and

(2b) $t_j(v) = 0$ for all $j \in NC(v)$.

Then, f satisfies strategy-proofness and symmetry, but does not satisfy budget balance.

We mention two interesting results related to Theorem 4.1.

Remark 4.1. A general result of Fujinaka and Sakai (2007, Corollary 3) implies that if V is such that the intersection $\cap_{i \in N} V_i$ contains at least $n+1$ valuations, then there is no Maskin monotonic, symmetric, and budget balanced mechanism.[22] The difference between this result and our result lies between Maskin monotonicity and strategy-proofness. While Maskin monotonicity and strategy-proofness are equivalent or very close in some environments (e.g., Muller and Satterthwaite, 1977; Dasgupta et al., 1979; Takamiya, 2001), they are independent in our model. Indeed, let f be such that for all $v \in V$, (i) if $v_1 \geq 1$, then $C(v) = \{1\}$ and $t_i(v) = 0$ for all $i \in N$, and (ii) if $v_1 < 1$, then $C(v) = \{2\}$ and $t_i(v) = 0$ for all $i \in N$. Then, f satisfies Maskin monotonicity, but does not satisfy strategy-proofness. Also, Example 4.3 above satisfies strategy-proofness, but does not satisfy Maskin monotonicity.

Remark 4.2. Ando et al. (2008) present a class of "sequential mechanisms" on a particular domain V, where the set of agent's possible valuations consists of exactly n common valuations. Their mechanisms satisfy strategy-proofness, weak symmetry, and budget balance, but do not satisfy symmetry.[23] It is then natural to inquire what

[22] See Fujinaka and Sakai (2007) for a general definition of Maskin monotonicity. In our model, a mechanism f is *Maskin monotonic* if for all $v, v' \in V$, if $v'_i \geq v_i$ for $i \in C(v)$ and $v'_j \leq v_j$ for all $j \in NC(v)$, then $f(v') = f(v)$.

[23] A mechanism f is *weakly symmetic* if for all $v \in V$, $[v_i = v_j$ for all $i, j \in N] \Rightarrow [U(f_i(v); v_i) = U(f_j(v); v_i)$ for all $i, j \in N]$.

happens if the domain condition in Theorem 4.1 is not satisfied. More precisely, one may wonder if there exists a strategy-proof, symmetric, and budget balanced mechanism under the domain condition that the set of agent's possible valuations includes at most n common valuations. We leave this interesting question for future research.

4.4 Concluding Remarks

We showed that there is no strategy-proof, symmetric, and budget balanced mechanism under the weak domain condition that the set of agent's possible valuations includes at least $n + 1$ common valuations. This domain condition is very weak, and it is satisfied with standard domains such as (i) the set of agent's possible valuations consisting of non-negative real numbers, and (ii) the set of agent's possible valuations consisting of non-positive real numbers. As explained in the introduction, Theorem 4.1 applies to (i) the model where agents may have non-quasilinear preferences and/or (ii) the unit-demand model of allocating n heterogeneous indivisible goods.

We mention three interesting topics for future research. First, it is interesting to characterize the entire class of strategy-proof and budget balanced mechanisms. Schummer (2000) characterizes the class in the two-agent case and Ohseto (1999) characterizes the class in the n-agent case by using some auxiliary axioms. Such characterizations enable us to understand how asymmetric those mechanisms are. However, as indicated by Example 4.2 in Ohseto (1999), it seems very complicated to characterize the entire class of strategy-proof and budget balanced mechanisms in the n-agent case without making use of any auxiliary axioms.

Second, Ohseto (2004, 2006), Saitoh and Serizawa (2008), and Ashlagi and Serizawa (2012) consider the problem of allocating k ($1 \leq k \leq n - 1$) homogeneous indivisible goods. By the same proof technique, we can examine the case of $n - 1$

homogeneous indivisible goods. Consider the valuation profile $(b, ..., b)$. Changing some consumer's valuation from b to a $(a > b)$ successively as in the proof of Theorem 4.1 leads to the same impossibility result. However, we have not proved whether there is no strategy-proof, symmetric, and budget balanced mechanism in the case of k $(2 \leq k \leq n-2)$ homogeneous indivisible goods.

Third, Sakai (2013b) and Sprumont (2013) study the model which allows an indivisible good to be unallocated. They mention the mechanism which never allocates the indivisible good and makes no monetary transfer among agents. Obviously, this mechanism satisfies strategy-proofness, symmetry, and budget balance, but is not appealing. Hence, it is natural to investigate whether there exist interesting mechanisms that satisfy strategy-proofness, symmetry, and budget balance in their model.

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