From Sequences to Matrices, the Lagrange Identity, and Generalizations of Hadamard Matrices

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From Sequences to Matrices, the Lagrange Identity, and Generalizations of Hadamard Matrices

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by

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Say, "Behold (observe) all that is in the heavens and on the earth".

But neither signs nor warners benefit those who do not believe

(QS. Yunus: 101).

In memory of Mohammad Barmawi (1932 –2014).

To my husband and my parents

with eternal appreciation.

Contents

Introduction

An Hadamard matrix is a square $\{\pm 1\}$-matrix whose rows are mutually orthogonal. Precisely, if H is a square matrix of order n with all entries are in $\{\pm 1\}$ and $HH^T = nI$, then we say that H is an Hadamard matrix of order n. The set of Hadamard matrices of order n is denoted by $H(n)$. It is known that if $H(n)$ is nonempty, then n is divisible by 4. However, the converse remains as a conjecture, i.e., there exist Hadamard matrices of order $4t$ for every positive integer t. The conjecture is called the Hadamard conjecture and it is an important open question among the development of the theory of Hadamard matrices. The development of Hadamard matrices extends to nonbinary alphabets and higher dimensional arrays, and some properties for multilevel applications in signal processing, coding and cryptography [15]. The applications of Hadamard matrices are briefly presented in [15, 20]. Seberry, et al. in [20] give engineering and statistical applications of Hadamard matrices, especially in communications systems and digital image processing.

Many approaches have been introduced in order to prove the Hadamard conjecture. One of the method is by constructing some complementary sequences. There exist many classes of complementary sequences such as Golay sequences, T-sequences, and base sequences. Base sequences, and their special cases such as normal and near-normal sequences, play an important role in the construction of Hadamard matrices [10, 19]. For instance, Kharaghani and Tayfeh-Rezaie in [11] found an Hadamard matrix of order 428 by using base sequences of length 71 and 36. It is shown in [24] that if $(\boldsymbol{a}, \boldsymbol{b})$ is a pair of Golay sequences of order n and $A = circ(\boldsymbol{a})$ and $B = circ(\boldsymbol{b})$, then

$$\begin{bmatrix} A & B \\ -B^t & A^t \end{bmatrix} \in H(2n)$$

where $circ(\boldsymbol{a})$ is the circulant matrix with the first row $\boldsymbol{a}$. Moreover, it is importnat to note that we can obtain Hadamard matrices from T-sequences. We will discuss more detail of complementary sequences in Chapter 2.

Formally, a quadruple of (± 1)-sequences $(\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}, \boldsymbol{d})$ of length m, m, n, n, respectively, is called *base sequences* if it is a set of complementary sequences. We denote by $BS(m, n)$ the set of base sequences of length m, m, n, n. If $(\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}, \boldsymbol{d}) \in BS(m, n)$, then it is complementary with weight $2(m + n)$. In [23], Yang gave a method of constructing some complementary sequences from base sequences. More precisely, Yang's theorem says that if $BS(m+1, m) \neq \emptyset$ and $BS(n+1, n) \neq \emptyset$, then there exists a set of four complementary (± 1)-sequences of length $(2m + 1)(2n + 1)$. Furthermore, Yang constructs some paired ternary sequences from base sequences in his other papers (see [22]). An important key to prove his theorems was by using the Lagrange identity for the ring of Laurent polynomials. As our main results, we describe our approach to generalize Yang's result in Chapter 3. We focus on constructing matrices, rather than sequences, to obtain the complementary sequences from base sequences. In order to construct the matrices, we introduce some operations on sequences and matrices. This work is due to increase the posiibility on finding new T-sequences, which leads to finding new Hadamard matrices.

Since the Lagrange identity is crucial for the Yang multiplication theorem, we are interested in the interpretation of the Lagrange identity. Note that Đoković and Zhao [14] observed some connection between Yang's method and the octonion algebra. It is known that the octonion algebra can be obtained from an arbitrary ring by using the Cayley-Dickson process recursively. Thus, we will observe the connection between the Lagrange identity and the Cayley-Dickson process from an arbitrary ring. The detail of the interpretation of the Lagrange identity will be presented in Chapter 1.

On the other hand, we consider not only the existence of Hadamard matrices, but also the existence of weighing matrices. Actually, Hadamard matrices are known as a special family of weighing matrices. A weighing matrix W of order n with weight w is a square $\{0, \pm 1\}$-matrix such that $WW^T = wI$. Usually, the set of weighing matrices of order n with weight w is denoted by $W(n, w)$. Indeed, $W(n, n)$ is equal to $H(n)$. Some applications of weighing matrices are given in [13]. Recently, Best, Kharaghani, and Ramp [2] introduced the theory of unit weighing matrices, which is a generalization of the family of weighing matrices. Let $T = \{z \in \mathbb{C} : |z| = 1\}$. A square matrix $W = [w_{ij}]$ of order n with $w_{ij} \in T_0$ where $T_0 = T \cup \{0\}$ is called a unit weighing matrix of order n with weight w if $WW^* = wI_n$ where W^* denote the transpose conjugate of W. Indeed, an Hadamard matrix is a special case of a unit weighing matrix. The set of unit weighing matrices of order n with weight w is denoted by $UW(n, w)$. A unit weighing matrix is called a unit Hadamard matrix if $w = n$. The study on weighing matrices is extensively presented in [2, 3]. Best, Kharaghani, and Ramp in [2, Theorem 5] showed that every unit weighing matrix is equivalent to a standard form. This theorem plays an important role in establishing the classification of unit weighing matrices. However, the proof in the original paper was incorrect. Therefore, we will give a corrected proof of the standardization in Chapter 4. We also give applications of standard form in this chapter, based on the results in [2].

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1 The Lagrange Identity and Its Interpretation

In this chapter, we will give a brief introduction of the Lagrange identity. We will introduce the Lagrange identity as follows. Let $a, b, c, d, e, v, g, h \in R$ where R is a commutaive ring with identity and an involutive automorphism $*$. Set

$$\begin{aligned} q &= av^* + cg - b^*e + dh, \\ r &= bv^* + dg^* + a^*e - ch^*, \\ s &= ag^* - cv - bh - d^*e, \\ t &= bg - dv + ah^* + c^*e. \end{aligned} \tag{1.1}$$

Then

$$qq^* + rr^* + ss^* + tt^* = (aa^* + bb^* + cc^* + dd^*)(ee^* + ff^* + gg^* + hh^*).$$

When $*$ is just the identity map, the identity is known as Euler's identity. the Lagrange identity is crucial for the Yang multiplication theorem, and we will give the detail of the role of the Lagrange identity in Chapter 3.

1.1 Interpretation of the Lagrange Identity

Đoković and Zhao showed the relation between the Lagrange identity and an octonion algebra in [14]. They investigated a possibility of new the Lagrange identity for polynomials. However, as a result, all such identities are equivalent to each other. In the following sections, we will show that the Lagrange identity can be interpreted as a norm and a determinant. Also, we investigate a possible relation with the function RDet. We will give the detail information of RDet later.

1.1.1 Cayley-Dickson Process

Let R be a (not necessarily commutative but associative) ring, and let $\bar{\cdot} : R \to R$ be an involutive anti-automorphism of R, i.e.

$$\overline{\overline{a}} = a, \quad \overline{a+b} = \overline{a} + \overline{b}, \quad \overline{ab} = \overline{b}\,\overline{a} \quad (a, b \in R).$$

Assume

$$ab - ba + \overline{ab - ba} = 0 \quad (a, b \in R), \tag{1.2}$$

$$\bar{a}a = a\bar{a} \quad (a \in R), \tag{1.3}$$

$$\bar{a}ab = b\bar{a}a \quad (a, b \in R). \tag{1.4}$$

By using the Cayley-Dickson process, we will produce a non-associative algebra from R. Define

$$R_1 = R \oplus R$$

with the following multiplication:

$$(a, b)(c, d) = (ac - \bar{d}b, da + b\bar{c}). \tag{1.5}$$

We define $\tau : R_1 \to R_1$ by

$$(a, b)^\tau = (\bar{a}, -b). \tag{1.6}$$

It is easy to see that τ is involutive. Indeed, $((a, b)^\tau)^\tau = (\bar{a}, -b)^\tau = (a, b)$.

Lemma 1.1. *τ is an anti-automorphism.*

Proof. Let (a,b), $(c,d) \in R_1$. Then

$$\begin{aligned}((a,b)+(c,d))^\tau &= (a+c,b+d)^\tau \\ &= (\overline{a+c}, -b-d) \\ &= (\bar{a}, -b) + (\bar{c}, -d) \\ &= (a,b)^\tau + (c,d)^\tau.\end{aligned}$$

Also,

$$((a,b)(c,d))^\tau = (ac - \bar{d}b, da + b\bar{c})^\tau = (\overline{ac - \bar{d}b}, -da - b\bar{c}),$$

$$(c,d)^\tau (a,b)^\tau = (\bar{c}, -d)(\bar{a}, -b) = (\bar{c}\bar{a} - \bar{b}d, (-b)\bar{c} + (-d)a),$$

and hence

$$((a,b)(c,d))^\tau = (c,d)^\tau (a,b)^\tau.$$

Thus, the result follows. □

Lemma 1.2. *Let $\alpha, \beta, \gamma \in R_1$. Then*

(i) $\alpha\beta - \beta\alpha + (\alpha\beta - \beta\alpha)^\tau = 0$,

(ii) $\alpha^\tau \alpha = \alpha\alpha^\tau$,

(iii) $\alpha^\tau \alpha\beta = \beta\alpha^\tau \alpha$,

(iv) $(\alpha\beta)\gamma = \alpha(\beta\gamma)$ *holds only if R is commutative.*

Proof. Let $\alpha = (a,b)$, $\beta = (c,d)$, $\gamma = (e,f)$ $(a,b,c,d,e,f \in R)$.

(i) Since

$$\begin{aligned}\alpha\beta - \beta\alpha &= (a,b)(c,d) - (c,d)(a,b) \\ &= (ac - \bar{d}b, da + b\bar{c}) - (ca - \bar{b}d, bc + d\bar{a}) \\ &= (ac - ca + \bar{b}d - \bar{d}b, da - d\bar{a} + b\bar{c} - bc),\end{aligned}$$

and

$$
\begin{aligned}
(\alpha\beta - \beta\alpha)^{\tau} &= (\overline{ac - ca} + \overline{\bar{b}d - \bar{d}b}, -(da - d\bar{a} + b\bar{c} - bc)) \\
&= (\overline{ac - ca} + \bar{d}b - \bar{b}d, -(da - d\bar{a} + b\bar{c} - bc))
\end{aligned}
$$

we have

$$
\begin{aligned}
\alpha\beta - \beta\alpha + (\alpha\beta - \beta\alpha)^{\tau} &= (ac - ca + \overline{ac - ca}, 0) \\
&= 0 && \text{(by (1.2))}.
\end{aligned}
$$

(ii)

$$
\begin{aligned}
\alpha\alpha^{\tau} &= (a\bar{a} + \bar{b}b, 0) \\
&= (\bar{a}a + \bar{b}b, 0) && \text{(by (1.3))} \\
&= (\bar{a}a + \bar{b}b, b\bar{a} + (-b)\bar{a}) \\
&= (\bar{a}, -b)(a, b) \\
&= \alpha^{\tau}\alpha.
\end{aligned}
$$

(iii)

$$
\begin{aligned}
\alpha^{\tau}\alpha\beta &= (\bar{a}a + \bar{b}b, 0)(c, d) \\
&= ((\bar{a}a + \bar{b}b)c, d(\bar{a}a + \bar{b}b)) \\
&= (c(\bar{a}a + \bar{b}b), d(\bar{a}a + \bar{b}b)) && \text{(by (1.4))} \\
&= (c(\bar{a}a + \bar{b}b), d\overline{(\bar{a}a + \bar{b}b)}) \\
&= (c, d)(\bar{a}a + \bar{b}b, 0) \\
&= \beta\alpha^{\tau}\alpha.
\end{aligned}
$$

(iv) Since R is commutative and

$$\begin{aligned}(\alpha\beta)\gamma &= ((a,b)(c,d))(e,f)\\ &= (ac-\bar{d}b, da+b\bar{c})(e,f)\\ &= (ace-\bar{d}be-\bar{f}da-\bar{f}b\bar{c}, fac-f\bar{d}b+da\bar{e}-b\bar{c}\bar{e}),\end{aligned}$$

$$\begin{aligned}\alpha(\beta\gamma) &= (a,b)((c,d)(e,f))\\ &= (a,b)(ce-f\bar{f}d, fc-d\bar{e})\\ &= (ace-a\bar{f}d-\bar{f}\bar{c}b-\bar{d}eb, fca+d\bar{e}a+b\bar{c}\bar{e}-bf\bar{d}),\end{aligned}$$

the result follows. □

Now, we will define the norm in R_1. Define $N : R_1 \to R$ by

$$N((a,b)) = \bar{a}a + \bar{b}b \quad (a,b \in R). \tag{1.7}$$

Then

$$(a,b)^\tau(a,b) = (N(a,b),0) \quad (a,b \in R). \tag{1.8}$$

Lemma 1.3.

$$N(\alpha\beta) = N(\alpha)N(\beta) \quad (\alpha,\beta \in R_1).$$

Proof. Let $\alpha = (a,b)$, $\beta = (c,d)$, where $a,b,c,d \in R$. By (1.2), we have

$$c(\bar{b}da) - (\bar{b}da)c + \overline{c(\bar{b}da) - (\bar{b}da)c} = 0. \tag{1.9}$$

Thus

$$\begin{aligned}N(\alpha\beta) &= N((a,b)(c,d))\\ &= N(ac-\bar{d}b, da+b\bar{c})\\ &= \overline{(ac-\bar{d}b)}(ac-\bar{d}b) + \overline{(da+b\bar{c})}(da+b\bar{c})\end{aligned}$$

$$
\begin{aligned}
&= (\bar{c}\bar{a} - \bar{b}d)(ac - \bar{d}b) + (\bar{a}\bar{d} + c\bar{b})(da + b\bar{c}) \\
&= \bar{c}\bar{a}ac - \bar{c}\bar{a}\bar{d}b - \bar{b}dac + \bar{b}d\bar{d}b \\
&\quad + \bar{a}\bar{d}da + \bar{a}\bar{d}b\bar{c} + c\bar{b}da + c\bar{b}b\bar{c} \\
&= \bar{a}a\bar{c}c + \bar{a}a\bar{d}d + \bar{b}b\bar{c}c + \bar{b}b\bar{d}d \\
&\quad + c(\bar{b}da) - (\bar{b}da)c + \overline{(c(\bar{b}da) - (\bar{b}da)c)} && \text{(by (1.4))} \\
&= (\bar{a}a + \bar{b}b)(\bar{c}c + \bar{d}d) && \text{(by (1.2))} \\
&= N(a,b)N(c,d) \\
&= N(\alpha)N(\beta).
\end{aligned}
$$

□

Next, we will make a non-associative ring from R_1 by using the Cayley-Dickson process. Define

$$R_2 = R_1 \oplus R_1$$

with the following multiplication

$$(\alpha,\beta)(\gamma,\delta) = (\alpha\gamma - \delta^{\tau}\beta, \delta\alpha + \beta\gamma^{\tau}) \quad (\alpha,\beta,\gamma,\delta \in R_1). \tag{1.10}$$

Hence, if

$$\alpha = (a,b), \quad \beta = (c,d), \quad \gamma = (e,f), \quad \delta = (g,h), \quad (a,b,c,d,e,f,g,h \in R)$$

then (1.10) becomes

$$\begin{aligned}
(\alpha, \beta)(\gamma, \delta) &= (\alpha\gamma - \delta^\tau\beta, \delta\alpha + \beta\gamma^\tau) \\
&= ((a,b)(e,f) - (g,h)^\tau(c,d), (g,h)(a,b) + (c,d)(e,f)^\tau) \\
&= ((a,b)(e,f) - (\bar{g}, -h)(c,d), (g,h)(a,b) + (c,d)(\bar{e}, -f)) \\
&= ((ae - \bar{f}b, fa + b\bar{e}) - (\bar{g}c + \bar{d}h, d\bar{g} - h\bar{c}), \\
&\qquad (ga - \bar{b}h, bg + h\bar{a}) + (c\bar{e} + \bar{f}d, -fc + de)) \\
&= ((ae - \bar{f}b - \bar{g}c - \bar{d}h, fa + b\bar{e} - d\bar{g} + h\bar{c}), \\
&\qquad (ga - \bar{b}h + c\bar{e} + \bar{f}d, bg + h\bar{a} - fc + de)). \qquad (1.11)
\end{aligned}$$

Also, we define $* : R_2 \to R_2$ by

$$(\alpha, \beta) \mapsto (\alpha^\tau, -\beta). \quad (\alpha, \beta \in R_1).$$

Then, we can check that $*$ is an involutive anti-automorphism of R_2. Moreover, Lemma 1.2 (i)–(iv) holds in R_2. We will now define the norm in R_2 as follows. Define the norm $N_1 : R_2 \to R_1$ by

$$N_1((\alpha, \beta)) = \alpha^\tau\alpha + \beta^\tau\beta \quad (\alpha, \beta \in R_1). \qquad (1.12)$$

Lemma 1.4. *If $\alpha, \beta \in R_1$, then*

$$N_1(\alpha, \beta) = (N(\alpha) + N(\beta), 0).$$

Proof.

$$N_1(\alpha, \beta) = \alpha^\tau\alpha + \beta^\tau\beta = (N(\alpha), 0) + (N(\beta), 0) = (N(\alpha) + N(\beta), 0).$$

□

Corollary 1.5.

$$N_1(AB) = N_1(A)N_1(B) \quad (A, B \in R_2).$$

Proof. It follows by replacing R and $\bar{\cdot}$ with R_1 and τ, respectively, in the proof of Lemma

1.3. □

We recall the Lagrange identity for the ring R as follows.

Theorem 1.6. *Given $a, b, c, d, f, e, g, h \in R$, define*

$$q = ae - \bar{f}b - \bar{g}c - \bar{d}h,$$
$$r = fa + b\bar{e} - d\bar{g} + h\bar{c},$$
$$s = ga - \bar{b}h + c\bar{e} + \bar{f}d,$$
$$t = bg + h\bar{a} - fc + de.$$

Then

$$q\bar{q} + r\bar{r} + s\bar{s} + t\bar{t} = (a\bar{a} + b\bar{b} + c\bar{c} + d\bar{d})(e\bar{e} + f\bar{f} + g\bar{g} + h\bar{h}).$$

Proof. Straightforward calculation. □

Theorem 1.7. *Corollary 1.5 is equivalent to Theorem 1.6.*

Proof. Let $A = ((a, b), (c, d))$, $B = ((e, f), (g, h)) \in R_2$. Then

$$\begin{aligned} N_1(AB) &= N_1((q, r), (s, t)) && \text{(by (1.11))} \\ &= (q, r)^\tau (q, r) + (s, t)^\tau (s, t) && \text{(by (1.12))} \\ &= (q\bar{q} + r\bar{r}, 0) + (s\bar{s} + t\bar{t}, 0) && \text{(by (1.7), (1.8))} \\ &= (q\bar{q} + r\bar{r} + s\bar{s} + t\bar{t}, 0). \end{aligned}$$

On the other hand, we have

$$\begin{aligned} N_1(A) = N_1((a, b), (c, d)) &= (N(a, b) + N(c, d), 0) && \text{(by Lemma 1.4)} \\ &= (a\bar{a} + b\bar{b} + c\bar{c} + d\bar{d}, 0), && \text{(by (1.7))} \\ N_1(B) = N_1((e, f), (g, h)) &= (N(e, f) + N(g, h), 0) && \text{(by Lemma 1.4)} \\ &= (e\bar{e} + f\bar{f} + g\bar{g} + h\bar{h}, 0). && \text{(by (1.7))} \end{aligned}$$

by (1.12). Therefore, we have

$$\begin{aligned}(q\bar{q}+r\bar{r}+s\bar{s}+t\bar{t},0) &= N_1(AB)\\ &= N_1(A)N_1(B) && \text{(by Lemma 1.3)}\\ &= (a\bar{a}+b\bar{b}+c\bar{c}+d\bar{d},0)(e\bar{e}+f\bar{f}+g\bar{g}+h\bar{h},0)\\ &= ((a\bar{a}+b\bar{b}+c\bar{c}+d\bar{d})(e\bar{e}+f\bar{f}+g\bar{g}+h\bar{h}),0).\end{aligned}$$

Hence, the result holds. □

Theorem 1.7 means that the Lagrange identity can be interpreted as the norm N_1.

1.1.2 Matrix Representation of Cayley-Dickson Process

Recall the definition of R_1 and τ in Subsection 1.1.1. Define

$$\tilde{R} = \left\{ \begin{bmatrix} \alpha & \beta \\ -\beta^\tau & \alpha^\tau \end{bmatrix} \mid \alpha, \beta \in R_1 \right\}.$$

The addition in $\tilde{R}$ is the standard matrix addition and the multiplication in $\tilde{R}$ is defined by

$$\begin{aligned}\begin{bmatrix} \alpha & \beta \\ -\beta^\tau & \alpha^\tau \end{bmatrix} \star \begin{bmatrix} \gamma & \delta \\ -\delta^\tau & \gamma^\tau \end{bmatrix} &= \begin{bmatrix} \alpha\gamma - \delta^\tau\beta & \delta\alpha + \beta\gamma^\tau \\ -\gamma\beta^\tau - \alpha^\tau\delta^\tau & -\beta^\tau\delta + \gamma^\tau\alpha^\tau \end{bmatrix} \\ &= \begin{bmatrix} \alpha\gamma - \delta^\tau\beta & \delta\alpha + \beta\gamma^\tau \\ -(\delta\alpha + \beta\gamma^\tau)^\tau & (\alpha\gamma - \delta^\tau\beta)^\tau \end{bmatrix}\end{aligned} \tag{1.13}$$

for $\alpha, \beta, \gamma, \delta \in R_1$. The operation $\star$ gives the product (9) from the Cayley-Dickson process in the first row of R_1 and a form of the conjugate product in the second row.

Lemma 1.8. *Define $\sigma : \widetilde{R} \to \widetilde{R}$ by*

$$\left(\begin{bmatrix} \alpha & \beta \\ -\beta^\tau & \alpha^\tau \end{bmatrix} \right)^\sigma = \begin{bmatrix} \alpha^\tau & -\beta \\ \beta^\tau & \alpha \end{bmatrix} \qquad (\alpha, \beta \in R_1).$$

Then σ is an involutive anti-automorphism of $\tilde{R}$.

Proof. Let $\alpha, \beta, \gamma, \delta \in R_1$. Then

$$\left(\begin{bmatrix} \alpha & \beta \\ -\beta^\tau & \alpha^\tau \end{bmatrix} \star \begin{bmatrix} \gamma & \delta \\ -\delta^\tau & \gamma^\tau \end{bmatrix}\right)^\sigma = \left(\begin{bmatrix} \alpha\gamma - \delta^\tau\beta & \delta\alpha + \beta\gamma^\tau \\ -(\delta\alpha + \beta\gamma^\tau)^\tau & (\alpha\gamma - \delta^\tau\beta)^\tau \end{bmatrix}\right)^\sigma.$$

On the other hand,

$$\begin{aligned}\left(\begin{bmatrix} \gamma & \delta \\ -\delta^\tau & \gamma^\tau \end{bmatrix}\right)^\sigma \star \left(\begin{bmatrix} \alpha & \beta \\ -\beta^\tau & \alpha^\tau \end{bmatrix}\right)^\sigma &= \begin{bmatrix} \gamma^\tau & -\delta \\ \delta^\tau & \gamma \end{bmatrix} \star \begin{bmatrix} \alpha^\tau & -\beta \\ \beta^\tau & \alpha \end{bmatrix} \\ &= \begin{bmatrix} \gamma^\tau\alpha^\tau - \beta^\tau\delta & -\beta\gamma^\tau - \delta\alpha \\ \alpha^\tau\delta^\tau + \gamma\beta^\tau & -\delta^\tau\beta + \alpha\gamma \end{bmatrix} \\ &= \begin{bmatrix} (-\delta^\tau\beta + \alpha\gamma)^\tau & -(\beta\gamma^\tau + \delta\alpha) \\ (\beta\gamma^\tau + \delta\alpha)^\tau & -\delta^\tau\beta + \alpha\gamma \end{bmatrix} \\ &= \left(\begin{bmatrix} -\delta^\tau\beta + \alpha\gamma & \beta\gamma^\tau + \delta\alpha \\ -(\beta\gamma^\tau + \delta\alpha)^\tau & (-\delta^\tau\beta + \alpha\gamma)^\tau \end{bmatrix}\right)^\sigma.\end{aligned}$$

So, σ is an anti-automorphism of $\tilde{R}$. Also,

$$\left(\begin{bmatrix} \alpha^\tau & -\beta \\ \beta^\tau & \alpha \end{bmatrix}\right)^\sigma = \begin{bmatrix} \alpha & \beta \\ -\beta^\tau & \alpha^\tau \end{bmatrix}.$$

Hence, σ is involutive. □

We define $d : M_2(R_1) \to R_1$ by

$$d\left(\begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix}\right) = \alpha\delta - \beta\gamma.$$

So, for $\alpha, \beta \in R_1$,

$$\begin{aligned} d\left(\begin{bmatrix} \alpha & \beta \\ -\beta^\tau & \alpha^\tau \end{bmatrix}\right) &= \alpha\alpha^\tau + \beta\beta^\tau = \alpha^\tau\alpha + \beta^\tau\beta \qquad \text{(by Lemma 1.2 (ii))} \\ &= N_1((\alpha, \beta)). \end{aligned} \tag{1.14}$$

Theorem 1.9. *Let $\alpha, \beta, \gamma, \delta \in R_1$ and*

$$A = \begin{bmatrix} \alpha & \beta \\ -\beta^\tau & \alpha^\tau \end{bmatrix}, \quad B = \begin{bmatrix} \gamma & \delta \\ -\delta^\tau & \gamma^\tau \end{bmatrix} \in \tilde{R}.$$

Then $d(A \star B) = d(A)d(B)$.

Proof. Since

$$\begin{aligned}
N(\alpha\gamma - \delta^\tau\beta) + N(\delta\alpha + \beta\gamma^\tau) &= \alpha\gamma\gamma^\tau\alpha^\tau - \alpha\gamma\beta^\tau\delta - \delta^\tau\beta\gamma^\tau\alpha^\tau + \delta^\tau\beta\beta^\tau\delta^\tau \\
&\quad + \delta\alpha\alpha^\tau\delta^\tau + \delta\alpha\gamma\beta^\tau + \beta\gamma^\tau\alpha^\tau\delta^\tau + \beta\gamma^\tau\gamma\beta^\tau \\
&= \alpha\gamma\gamma^\tau\alpha^\tau + \delta^\tau\beta\beta^\tau\delta^\tau + \delta\alpha\alpha^\tau\delta^\tau + \beta\gamma^\tau\gamma\beta^\tau \\
&+ \delta\alpha\gamma\beta^\tau - \alpha\gamma\beta^\tau\delta + \beta\gamma^\tau\alpha^\tau\delta^\tau - \delta^\tau\beta\gamma^\tau\alpha^\tau \\
&= \alpha\alpha^\tau\gamma\gamma^\tau + \alpha\alpha^\tau\delta\delta^\tau + \beta\beta^\tau\gamma\gamma^\tau + \beta\beta^\tau\delta\delta^\tau + \\
&\quad + (\delta(\alpha\gamma\beta^\tau) - (\alpha\gamma\beta^\tau)\delta) + (\delta(\alpha\gamma\beta^\tau) - (\alpha\gamma\beta^\tau)\delta)^\tau \\
&= (\alpha\alpha^\tau + \beta\beta^\tau)(\gamma\gamma^\tau + \delta\delta^\tau) && \text{(by Lemma 1.2)} \\
&= (N(\alpha) + N(\beta))(N(\gamma) + N(\delta)), && (*)
\end{aligned}$$

we have

$$\begin{aligned}
d(A \star B) &= d\left(\begin{bmatrix} \alpha\gamma - \delta^\tau\beta & \delta\alpha + \beta\gamma^\tau \\ -(\delta\alpha + \beta\gamma^\tau)^\tau & (\alpha\gamma - \delta^\tau\beta)^\tau \end{bmatrix}\right) \\
&= N_1(\alpha\gamma - \delta^\tau\beta, \delta\alpha + \beta\gamma^\tau) && \text{(by (1.14))} \\
&= (N(\alpha\gamma - \delta^\tau\beta) + N(\delta\alpha + \beta\gamma^\tau), 0) && \text{(by Lemma 1.4)} \\
&= ((N(\alpha) + N(\beta))(N(\gamma) + N(\delta)), 0) \\
&= (N(\alpha) + N(\beta), 0)(N(\gamma) + N(\delta), 0) \\
&= N_1((\alpha, \beta))N_1((\gamma, \delta)) && \text{(by Lemma 1.4)} \\
&= d(A) \star d(B).
\end{aligned}$$

□

Lemma 1.10. *$\tilde{R}$ is isomorphic to R_2.*

Proof. Define $\pi : R_2 \to \tilde{R}$ by

$$(\alpha, \beta) \mapsto \begin{bmatrix} \alpha & \beta \\ -\beta^\tau & \alpha^\tau \end{bmatrix} \qquad (\alpha, \beta \in R_1). \tag{1.15}$$

Then π is a homomorphism since for every $\alpha, \beta, \gamma, \delta \in R_1$,

$$\begin{aligned} \pi((\alpha,\beta)(\gamma,\delta)) &= \pi(\alpha\gamma - \delta^\tau\beta, \delta\alpha + \beta\gamma^\tau) \\ &= \begin{bmatrix} \alpha\gamma - \delta^\tau\beta & \delta\alpha + \beta\gamma^\tau \\ -\gamma\beta^\tau - \alpha^\tau\delta^\tau & -\beta^\tau\delta + \gamma^\tau\alpha^\tau \end{bmatrix} \\ &= \begin{bmatrix} \alpha & \beta \\ -\beta^\tau & \alpha^\tau \end{bmatrix} \star \begin{bmatrix} \gamma & \delta \\ -\delta^\tau & \gamma^\tau \end{bmatrix} \\ &= \pi(\alpha,\beta)\pi(\gamma,\delta). \end{aligned}$$

Also, π is bijective. Thus, π is an isomorphism. □

Lemma 1.11. $d\,|_{\tilde{R}}\,\pi = N_1$.

Proof. Let $\alpha, \beta \in R_1$. Since the image of π is $\tilde{R}$, we have

$$\begin{aligned} d\,|_{\tilde{R}}\,\pi((\alpha,\beta)) &= d\,|_{\tilde{R}} \left(\begin{bmatrix} \alpha & \beta \\ -\beta^\tau & \alpha^\tau \end{bmatrix} \right) && \text{(by (1.15))} \\ &= \alpha\alpha^\tau + \beta\beta^\tau && \text{(by (1.14))} \\ &= N_1((\alpha,\beta)). && \text{(by (1.12))} \end{aligned}$$

□

Note that if $\alpha, \beta, \gamma, \delta \in R_1$ and $A = \pi(\alpha,\beta)$ and $B = \pi(\gamma,\delta) \in \widetilde{R}$, then we have

$$A \star B = \begin{bmatrix} \alpha\gamma - \delta^\tau\beta & \delta\alpha + \beta\gamma^\tau \\ -(\delta\alpha + \beta\gamma^\tau)^\tau & (\alpha\gamma - \delta^\tau\beta)^\tau \end{bmatrix} \in \widetilde{R}.$$

Thus

$$\begin{aligned}
d(A \star B) &= d\mid_{\tilde{R}} (A \star B) && \text{(since } A \star B \in \widetilde{R}\text{)} \\
&= d\mid_{\tilde{R}} \pi(\alpha\gamma - \delta^\tau\beta, \delta\alpha + \beta\gamma^\tau) \\
&= N_1(\alpha\gamma - \delta^\tau\beta, \delta\alpha + \beta\gamma^\tau) && \text{(by Lemma 1.11)} \\
&= N_1((\alpha, \beta)(\gamma, \delta)) && \text{(by (1.10))} \\
&= N_1((\alpha, \beta))N_1((\gamma, \delta)) && \text{(by Corollary 1.5)} \\
&= d\mid_{\tilde{R}} \pi((\alpha, \beta)) d\mid_{\tilde{R}} \pi((\gamma, \delta)) && \text{(by Lemma 1.11)} \\
&= d(A)d(B).
\end{aligned}$$

Hence, we also gave an interpretation of the Lagrange identity as the determinant.

1.2 RDet

In Section 1.1, we showed that the Lagrange identity can be interpreted as the norm and the determinant. Also, we see that the multiplicative property holds for N_1 and d where they are defined in Section 1.1, i.e. $N_1(AB) = N_1(A)N_1(B)$ for every $C, D \in R_2$ and $d(C \star D) = d(C)d(D)$ for arbitrary C, D in $\tilde{R}$. We are considering the function $\psi : M_2(R_1) \to M_4(R)$ and a new function $\mathrm{RDet} : M_4(R) \to R$ such that $N_1^2 = \mathrm{RDet}$ and RDet satisfies multiplicative property, i.e. $\mathrm{RDet}(AB) = \mathrm{RDet}(A)\,\mathrm{RDet}(B)$ for every $A, B \in M_4(R)$. When R_1 is the ring of quaternions, RDet is known as the *Study determinant*. The Study determinant was introduced by Eduard Study in 1920 [21]. His method was to produce a square $\mathbb{C}$-matrix M of order $2n$ and compute the determinant of M (see [1], [9] for more details). We investigate his method and observe the possibility to connect it with the Lagrange identity.

Throughout this subsection, we recall R and R_1 in Subsection 1.1.1. Denote the identity in R_1 by

$$1 = (1, 0), \quad j = (0, 1) \in R_1.$$

Then $R = R \oplus 0 \subset R_1$. Also, since for every $a, b \in R$, we have $a = (a, 0), b = (b, 0) \in R_1$ and

$$a + b = (a, 0) + (b, 0) = (a + b, 0) \in R \oplus 0 = R,$$
$$ab = (a, 0)(b, 0) = (ab, 0) \in R \oplus 0 = R.$$

Therefore, R is a subring of R_1 implies $M_n(R)$ is a subring of $M_n(R_1)$.

Lemma 1.12. *If $z \in R_1$, then*

$$zj = jz^{\tau} \tag{1.16}$$

if and only if $z \in R \subset R_1$.

Proof. Note that R is a subring of R_1. If $z \in R$, then $z = (z, 0) \in R_1$ and $z^{\tau} = (\bar{z}, 0) \in R_1$. Therefore

$$zj = (z, 0)(0, 1) = (0, z) = (0, 1)(\bar{z}, 0) = jz^{\tau}.$$

Let $z = (a, b) \in R_1$. Then we have $zj = (a, b)(0, 1) = (-b, a)$ and $jz^{\tau} = (0, 1)(\bar{a}, -b) = (b, a)$. So, if $zj = jz^{\tau}$, then $b = -b = 0$. Thus, $z = (a, 0) = a \in R$. Hence, the proof is complete. □

In the other word, Lemma 1.12 shows that $zj = j\bar{z}$ for every $z \in R$. Denote $\overline{A} = (\overline{a_{ij}}) \in M_n(R)$ for every $A = (a_{ij}) \in M_n(R))$. Then $jA = (ja_{ij}) = (\overline{a_{ij}}j) = \overline{A}j$. Note that for every $(a, b) \in R_1$,

$$(a, b) = (a, 0) + (0, b) = (a, 0) + j(\bar{b}, 0).$$

So, for every $z \in R_1$, there exist $z_1, z_2 \in R$ such that $z = z_1 + jz_2$. Consequently, for every $Z \in M_n(R_1)$, there exist $Z_1, Z_2 \in M_n(R)$ such that $Z = Z_1 + jZ_2$.

Proposition 1.13. Define $\psi : M_n(R_1) \to M_{2n}(R)$ by

$$\psi(A + jB) = \begin{pmatrix} A & -\overline{B} \\ B & \overline{A} \end{pmatrix} \qquad (A, B \in M_n(R))$$

Then ψ is an injective homomorphism and

$$\psi(M_n(R_1)) = \{N \in M_{2n}(R) \mid JN = \overline{N}J\}.$$

Proof. Since for every $A_1, A_2, B_1, B_2 \in M_n(R)$,

$$\begin{aligned}
\psi((A_1 + jB_1)(A_2 + jB_2)) &= \psi(A_1A_2 + A_1jB_2 + jB_1A_2 + jB_1jB_2) \\
&= \psi((A_1A_2 + j\overline{A_1}B_2 + jB_1A_2 - \overline{B_1}B_2)) \qquad \text{(by Lemma 1.12)} \\
&= \psi(A_1A_2 - \overline{B_1}B_2 + j(\overline{A_1}B_2 + B_1A_2)) \\
&= \begin{pmatrix} A_1A_2 - \overline{B_1}B_2 & -A_1\overline{B_2} + \overline{B_1A_2} \\ B_1A_2 + \overline{A_1}B_2 & -B_1\overline{B_2} + \overline{A_1A_2} \end{pmatrix} \\
&= \begin{pmatrix} A_1 & -\overline{B_1} \\ B_1 & \overline{A_1} \end{pmatrix} \begin{pmatrix} A_2 & -\overline{B_2} \\ B_2 & \overline{A_2} \end{pmatrix} \\
&= \psi(A_1 + jB_1)\psi(A_2 + jB_2),
\end{aligned}$$

ψ is a homomorphism. Also, $\mathrm{Ker}(\psi) = \{0\}$, thus ψ is injective.

Now, suppose

$$N = \begin{pmatrix} A & C \\ B & D \end{pmatrix} \in M_{2n}(R_1)$$

where $A, B, C, D \in M_n(R)$. Then we have

$$JN = \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix} \begin{pmatrix} A & C \\ B & D \end{pmatrix} = \begin{pmatrix} -B & -D \\ A & C \end{pmatrix}$$

and

$$\overline{N}J = \begin{pmatrix} \overline{A} & \overline{C} \\ \overline{B} & \overline{D} \end{pmatrix} \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix} = \begin{pmatrix} \overline{C} & -\overline{A} \\ \overline{D} & -\overline{B} \end{pmatrix}.$$

Therefore, $JN = \overline{N}J$ if and only if $C = -\overline{B}$ and $D = \overline{A}$. Hence

$$\psi(M_n(R_1) = \left\{ \begin{pmatrix} A & -\overline{B} \\ B & \overline{A} \end{pmatrix} \mid A, B \in M_n(R) \right\} = \{N \in M_{2n}(R) \mid JN = \overline{N}J\}.$$

□

Definition 1.14. Define $\det_R : M_n(R) \to R$ by

$$\det_R((a_{ij})) = \sum_{\sigma \in S_n} a_{1\sigma(1)} \cdots a_{n\sigma(n)}$$

and $\mathrm{RDet} : M_n(R_1) \to R$ by

$$\mathrm{RDet} = \det_R \psi.$$

We note again that RDet is just the Study determinant when R is $\mathbb{C}$.

Lemma 1.15. *The multiplicative property holds for* RDet.

Proof. Since ψ is a homomorphism by Lemma 1.13, the result follows. □

Lemma 1.16. *Extend*

$$\tau : R_1 \to R_1$$

to

$$\tau : M_n(R_1) \to M_n(R_1)$$

by

$$A + jB \mapsto \overline{A^t} - jB^t \qquad (A, B \in M_n(R)).$$

Then

(i) τ *is an anti-automorphism of* $M_n(R_1)$,

(ii) $\psi(M^\tau) = \overline{\psi(M)}^t \quad (M \in M_n(R_1))$,

(iii) $\mathrm{RDet}(M) = \overline{\mathrm{RDet}(M)} \quad (M \in M_n(R_1))$.

Proof. For every $A_1, A_2, B_1, B_2 \in M_n(R)$,

$$\begin{aligned}
((A_1 + jB_1)(A_2 + jB_2))^\tau &= (A_1A_2 + A_1jB_2 + jB_1A_2 + jB_1jB_2)^\tau \\
&= (A_1A_2 + j\overline{A_1}B_2 + jB_1A_2 - \overline{B_1}B_2)^\tau \qquad \text{(by Lemma 1.12)} \\
&= (A_1A_2 - \overline{B_1}B_2 + j(\overline{A_1}B_2 + B_1A_2))^\tau \\
&= \overline{(A_1A_2 - \overline{B_1}B_2)}^t - j(\overline{A_1}B_2 + B_1A_2)^t,
\end{aligned}$$

and

$$\begin{aligned}(A_1+jB_1)^\tau(A_2+jB_2)^\tau &= (\overline{A_2}^t - jB_2^t)(\overline{A_1} - jB_1^t)\\ &= \overline{A_2}^t\overline{A_1}^t - \overline{A_2}^t jB_1^t - jB_2^t\overline{A_1}^t + jB_2^t jB_1^t\\ &= \overline{A_2}^t\overline{A_2}^t - \overline{B_2}^t B_1^t - j(A_2^t B_1^t + B_2^t\overline{A_1}^t).\end{aligned}$$

Thus, (i) holds. Now, let $M = A + jB \in M_n(R_1)$ $(A, B \in M_n(R))$. Then

$$\psi(M^\tau) = \psi(\overline{A^t} - jB^t) = \begin{bmatrix}\overline{A^t} & \overline{B}^t\\ -B^t & A^t\end{bmatrix} = \overline{\psi(M)}^t.$$

Hence, we have (ii). Finally, let $M = A + jB \in M_n(R_1)$ $(A, B \in M_n(R))$. Notice that

$$\psi(M^\tau) = \psi(\overline{A}^t - jB^t) = \begin{bmatrix}\overline{A}^t & \overline{B}^t\\ -B^t & A^t\end{bmatrix}$$

and

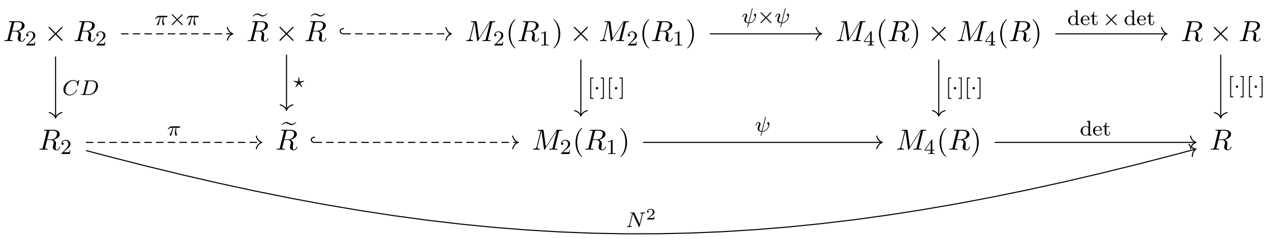
$$(\psi(M^\tau))^t = \begin{bmatrix}\overline{A} & -B\\ \overline{B} & A\end{bmatrix}.$$

Therefore, we have

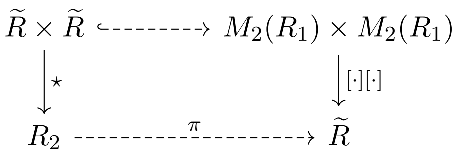
$$\begin{aligned}\det{}_R\psi(M) &= \det{}_R\left(\begin{bmatrix}A & -\overline{B}\\ B & \overline{A}\end{bmatrix}\right) = \det{}_R\left(\begin{bmatrix}0 & I_n\\ -I_n & 0\end{bmatrix}\begin{bmatrix}\overline{A} & -B\\ \overline{B} & A\end{bmatrix}\begin{bmatrix}0 & -I_n\\ I_n & 0\end{bmatrix}\right)\\ &= \det{}_R\left(\begin{bmatrix}\overline{A} & -B\\ \overline{B} & A\end{bmatrix}\right) = \det{}_R(\psi(M^\tau)^t) = \det{}_R\left(\overline{\psi(M)}\right) = \overline{\det{}_R(\psi(M))},\end{aligned}$$

and hence we have (iii). □

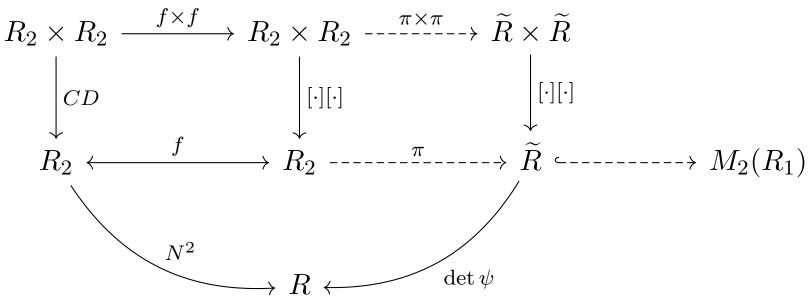
Now, consider the following diagram:



where "CD" denote the Cayley-Dickson product and $N^2(\alpha) = (N(\alpha))^2$ for every $\alpha \in R_2$. Note that only



is not commute. Therefore, we are considering $f : R_2 \to R_2$ such that



commute, i.e. $N^2(\alpha) = \det\psi\pi f(\alpha)$ for every $\alpha \in R_2$. Actually, we see that $f|_{R_1}$ is the identity map. By using the fact in the diagram, we have the following open problem.

Open Problem 1.17. *Does a bijection $f : R_2 \to R_2$ satisfying $N^2(\alpha) = \mathrm{RDet}(f(\alpha))$ exist?*

2 Complementary Sequences

Throughout this chapter, let $\mathcal{R}$ be a commutative ring with multiplicative identity 1, and $* : \mathcal{R} \to \mathcal{R}$ be a ring automorphism satisfying $(a^*)^* = a$ for every $a \in \mathcal{R}$. We first remark in the following lemma that there exists a subring W of $M_2(\mathcal{R})$ which contains a subring isomorphic to $\mathcal{R}$.

Lemma 2.1. *Define*

$$W = \left\{ \begin{pmatrix} a & b \\ -b^* & a^* \end{pmatrix} \middle| \, a, b \in \mathcal{R} \right\} \subset M_2(\mathcal{R}).$$

Then W is a ring which contains a subring isomorphic to $\mathcal{R}$. Moreover, W is commutative if and only if $*$ is the identity map.

Proof. We first show that W is a ring. Since $\mathcal{R}$ is a ring, it is closed under addition and multiplication. Therefore, W is also closed under matrix addition. For $a_1, a_2, b_1, b_2 \in \mathcal{R}$,

$$\begin{aligned} \begin{pmatrix} a_1 & b_1 \\ -b_1^* & a_1^* \end{pmatrix} \begin{pmatrix} a_2 & b_2 \\ -b_2^* & a_2^* \end{pmatrix} &= \begin{pmatrix} a_1a_2 - b_1b_2^* & a_1b_2 + b_1a_2^* \\ -b_1^*a_2 - a_1^*b_2^* & -b_1^*b_2 + a_1^*a_2^* \end{pmatrix} \\ &= \begin{pmatrix} a_1a_2 - b_1b_2^* & a_1b_2 + b_1a_2^* \\ -(a_1b_2 + b_1a_2^*)^* & (a_1a_2 - b_1b_2^*)^* \end{pmatrix} \in W. \end{aligned}$$

Thus, W is closed under matrix multiplication. From all of these facts, W is a subring of $M_2(\mathcal{R})$. Let

$$W' = \left\{ \begin{pmatrix} a & 0 \\ 0 & a^* \end{pmatrix} \middle| \, a \in \mathcal{R} \right\}.$$

Define a map $\phi : \mathcal{R} \to W'$ by $\phi(a) = \begin{pmatrix} a & 0 \\ 0 & a^* \end{pmatrix}$. For every $a, b \in \mathcal{R}$,

$$\phi(a+b) = \begin{pmatrix} a+b & 0 \\ 0 & (a+b)^* \end{pmatrix} = \begin{pmatrix} a+b & 0 \\ 0 & a^*+b^* \end{pmatrix} = \begin{pmatrix} a & 0 \\ 0 & a^* \end{pmatrix} + \begin{pmatrix} b & 0 \\ 0 & b^* \end{pmatrix} = \phi(a) + \phi(b),$$

$$\phi(ab) = \begin{pmatrix} ab & 0 \\ 0 & (ab)^* \end{pmatrix} = \begin{pmatrix} a & 0 \\ 0 & a^* \end{pmatrix} \begin{pmatrix} b & 0 \\ 0 & b^* \end{pmatrix} = \phi(a)\phi(b).$$

Thus, ϕ is a homomorphism and W' is a subring of W with an identity element 1_W. Since $\mathrm{Ker}(\phi) = \{0\}$ and $\mathrm{Im}(\phi) = W'$, W' is isomorphic to $\mathcal{R}$.

Now we will prove the last statement. If $*$ is the identity map, then

$$\begin{aligned}
\begin{pmatrix} a_1 & b_1 \\ -b_1^* & a_1^* \end{pmatrix} \begin{pmatrix} a_2 & b_2 \\ -b_2^* & a_2^* \end{pmatrix} &= \begin{pmatrix} a_1 & b_1 \\ -b_1 & a_1 \end{pmatrix} \begin{pmatrix} a_2 & b_2 \\ -b_2 & a_2 \end{pmatrix} \\
&= \begin{pmatrix} a_1a_2 - b_1b_2 & a_1b_2 + b_1a_2 \\ -b_1a_2 - a_1b_2 & -b_1b_2 + a_1a_2 \end{pmatrix} \\
&= \begin{pmatrix} a_2a_1 - b_2b_1 & a_2b_1 + b_2a_1 \\ -b_2a_1 - a_2b_1 & -b_2b_1 + a_2a_1 \end{pmatrix} \\
&= \begin{pmatrix} a_2 & b_2 \\ -b_2 & a_2 \end{pmatrix} \begin{pmatrix} a_1 & b_1 \\ -b_1 & a_1 \end{pmatrix} \\
&= \begin{pmatrix} a_2 & b_2 \\ -b_2^* & a_2^* \end{pmatrix} \begin{pmatrix} a_1 & b_1 \\ -b_1^* & a_1^* \end{pmatrix}.
\end{aligned}$$

Thus, W is commutative. Conversely, assume that W is commutative. Since for $a \in \mathcal{R}$,

$$\begin{pmatrix} a & 0 \\ 0 & a^* \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \in W,$$

we have

$$\begin{pmatrix} 0 & a \\ -a^* & 0 \end{pmatrix} = \begin{pmatrix} a & 0 \\ 0 & a^* \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} a & 0 \\ 0 & a^* \end{pmatrix} = \begin{pmatrix} 0 & a^* \\ -a & 0 \end{pmatrix}.$$

Thus, we have $a^* = a$ and therefore $*$ is the identity map. □

2.1 Polynomial Representation of Complementary Sequences

Definition 2.2 (Non-periodic autocorrelation)**.** Let $\boldsymbol{a} = (a_0, \dots, a_{l-1}) \in \mathcal{R}^l$. We define *the non-periodic autocorrelation* $N_{\boldsymbol{a}}$ *of* $\boldsymbol{a}$ by

$$N_{\boldsymbol{a}}(j) = \begin{cases} \sum_{i=0}^{l-j-1} a_i a_{i+j}^* & \text{if } 0 \leq j < l, \\ 0 & \text{otherwise.} \end{cases}$$

Definition 2.3 (Complementary sequences)**.** We say that a set of sequences $\{\boldsymbol{a}_1, \dots, \boldsymbol{a}_n\}$ is *complementary with weight* w if

$$\sum_{i=1}^{n} N_{\boldsymbol{a}_i}(j) = \begin{cases} w & \text{if } j = 0, \\ 0 & \text{otherwise.} \end{cases}$$

Moreover, the sequences $\boldsymbol{a}_1, \dots, \boldsymbol{a}_n$ are called complementary sequences.

We may note that the term *weight* is usually used to describe the number of nonzero components of an arbitrary sequence. The weight of any set of sequences means the total weight of all sequences in that set, provided that $aa^* = 1$ for all nonzero components a. The following sequences are complementary sequences with weight 14:

$$(1, -1, 1, 1, 1, -1, -1), \quad (1, 1, 0, 1, 0, -1, 1), \quad (0, 0, 1, 0, 1, 0, 0).$$

Complementary sequences do not necessarily have the same length. For example,

$$\boldsymbol{a} = (1, -1, -1, -1, 1), \quad \boldsymbol{b} = (1, 1, -1, 1, -1), \quad \boldsymbol{c} = (1, 1, 1, 1), \quad \boldsymbol{d} = (1, -1, -1, 1)$$

have different lengths, but they are complementary sequences with weight 18. Remark that this quadruple of sequences $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{c}$, $\boldsymbol{d}$ are known as near normal sequences, one of the class of complementary sequences.

Now, we will introduce a representation of complementary sequences by using polynomials. This representation becomes important in characterizing complementary sequences.

Definition 2.4. Let $\boldsymbol{a} = (a_0, \ldots, a_{l-1}) \in \mathcal{R}^l$. We define the *Hall polynomial* $f_{\boldsymbol{a}} = f_{\boldsymbol{a}}(x) \in \mathcal{R}[x]$ of $\boldsymbol{a}$ by

$$f_{\boldsymbol{a}}(x) = \sum_{i=0}^{l-1} a_i x^i.$$

Lemma 2.5. *Let* $* : \mathcal{R}[x^{\pm 1}] \to \mathcal{R}[x^{\pm 1}]$ *be an extension map of* $*$ *defined by*

$$\left(\sum_{i \in \mathbb{Z}} a_i x^i\right)^* = \sum_{i \in \mathbb{Z}} a_i^* x^{-i}.$$

Then $*$ *is a ring automorphism satisfying* $(f^*)^* = f$ *for every* $f \in \mathcal{R}[x^{\pm 1}]$.

Proof. We have

$$\begin{aligned}
\left(\sum_i a_i x^i + \sum_i b_i x^i\right)^* &= \left(\sum_i (a_i + b_i) x^i\right)^* \\
&= \sum_i (a_i + b_i)^* x^{-i} \\
&= \sum_i a_i^* x^{-i} + \sum_i b_i^* x^{-i} \\
&= \left(\sum_i a_i x^i\right)^* + \left(\sum_i b_i x^i\right)^*,
\end{aligned}$$

and

$$\begin{aligned}
\left(\sum_i a_i x^i \cdot \sum_j b_j x^j\right)^* &= \left(\sum_t \sum_{i+j=t} a_i b_j x^t\right)^* \\
&= \sum_t \left(\sum_{i+j=t} a_i b_j\right)^* x^{-t} \\
&= \sum_t \sum_{i+j=t} a_i^* b_j^* x^{-t} \\
&= \left(\sum_i a_i^* x^{-i}\right)\left(\sum_j b_j^* x^{-j}\right) \\
&= \left(\sum_i a_i x^i\right)^* \left(\sum_j b_j x^j\right)^*.
\end{aligned}$$

Since $(f^*)^* = f$ for $f \in \mathcal{R}[x^{\pm 1}]$, $*$ is bijective. □

Let $\boldsymbol{a} = (a_0, \ldots, a_{l-1}) \in \mathcal{R}^l$ and $\boldsymbol{b} \in \mathcal{R}^n$. We define a sequence $\boldsymbol{a} \otimes \boldsymbol{b}$ by

$$\boldsymbol{a} \otimes \boldsymbol{b} = (a_0 \boldsymbol{b}, \ldots, a_{l-1}\boldsymbol{b}) \in \mathcal{R}^{ln}.$$

Also, define $\boldsymbol{a}^* \in \mathcal{R}^l$ by $\boldsymbol{a}^* = (a_{l-1}^*, \ldots, a_0^*)$. The following lemmas are the basic properties of the operations in sequences.

Lemma 2.6. *Let $\boldsymbol{a} \in \mathcal{R}^m$ and $\boldsymbol{b} \in \mathcal{R}^n$. Then*

(i) $(\boldsymbol{a} + \boldsymbol{b})^* = \boldsymbol{a}^* + \boldsymbol{b}^*$ *if* $m = n$,

(ii) $(\boldsymbol{a} \otimes \boldsymbol{b})^* = \boldsymbol{a}^* \otimes \boldsymbol{b}^*$,

(iii) $\boldsymbol{a}^{**} = \boldsymbol{a}$.

Proof. (i), (ii) and (iii) are immediate. □

Lemma 2.7. *Let $\boldsymbol{a}, \boldsymbol{b} \in \mathcal{R}^l$. Then*

(i) $f_{\boldsymbol{a} \pm \boldsymbol{b}} = f_{\boldsymbol{a}} \pm f_{\boldsymbol{b}}$,

(ii) $f_{k\boldsymbol{a}} = k f_{\boldsymbol{a}}$ *for* $k \in \mathcal{R}$,

(iii) $f_{\boldsymbol{a} \pm \boldsymbol{b}}^* = f_{\boldsymbol{a}}^* \pm f_{\boldsymbol{b}}^*$,

(iv) $f^*_{k\boldsymbol{a}} = k^* f^*_{\boldsymbol{a}}$ *for* $k \in \mathcal{R}$,

(v) $f_{\boldsymbol{a}\otimes\boldsymbol{c}} = f_{\boldsymbol{a}}(x^n) f_{\boldsymbol{c}}(x)$ *for* $\boldsymbol{c} \in \mathcal{R}^n$,

(vi) $f^*_{\boldsymbol{a}}(x) = x^{1-l} f_{\boldsymbol{a}^*}(x)$,

(vii) $f_{\boldsymbol{a}} f^*_{\boldsymbol{b}} = f^*_{\boldsymbol{a}^*} f_{\boldsymbol{b}^*}$.

Proof. (i) and (ii) are immediate. (iii) and (iv) follow from (i) and (ii), respectively. We prove (v). For $\boldsymbol{c} = (c_0, \ldots, c_{n-1})$,

$$f_{\boldsymbol{a}\otimes\boldsymbol{c}} = \sum_{i=0}^{l-1}\sum_{j=0}^{n-1} a_i c_j x^{ni+j} = \sum_{i=0}^{l-1} a_i x^{ni} \sum_{j=0}^{n-1} c_j x^j = f_{\boldsymbol{a}}(x^n) f_{\boldsymbol{c}}(x).$$

(vi) If $\boldsymbol{a} = (a_0, \ldots, a_{l-1})$,

$$f^*_{\boldsymbol{a}}(x) = \sum_{i=0}^{l-1} a_i^* x^{-i} = \sum_{i=0}^{l-1} a^*_{l-1-i} x^{1-l+i} = x^{1-l} \sum_{i=0}^{l-1} a^*_{l-1-i} x^i = x^{1-l} f_{\boldsymbol{a}^*}(x).$$

(vii) Note that by (vi) and Lemma 2.6 (iii), $f_{\boldsymbol{a}}(x) = (f^*_{\boldsymbol{a}}(x))^* = (x^{1-l} f_{\boldsymbol{a}^*}(x))^* = x^{l-1} f^*_{\boldsymbol{a}^*}(x)$. Thus by (vi) again, $f_{\boldsymbol{a}} f^*_{\boldsymbol{b}} = x^{l-1} f^*_{\boldsymbol{a}^*} x^{1-l} f_{\boldsymbol{b}^*} = f^*_{\boldsymbol{a}^*} f_{\boldsymbol{b}^*}$. □

Lemma 2.8. *Let* $\boldsymbol{a}, \boldsymbol{b} \in R^l$. *Then* $f_{\boldsymbol{a}+\boldsymbol{b}} f^*_{\boldsymbol{a}+\boldsymbol{b}} + f_{\boldsymbol{a}-\boldsymbol{b}} f^*_{\boldsymbol{a}-\boldsymbol{b}} = 2(f_{\boldsymbol{a}} f^*_{\boldsymbol{a}} + f_{\boldsymbol{b}} f^*_{\boldsymbol{b}})$.

Proof.

$$\begin{aligned} f_{\boldsymbol{a}+\boldsymbol{b}} f^*_{\boldsymbol{a}+\boldsymbol{b}} + f_{\boldsymbol{a}-\boldsymbol{b}} f^*_{\boldsymbol{a}-\boldsymbol{b}} &= (f_{\boldsymbol{a}} + f_{\boldsymbol{b}})(f^*_{\boldsymbol{a}} + f^*_{\boldsymbol{b}}) + (f_{\boldsymbol{a}} - f_{\boldsymbol{b}})(f^*_{\boldsymbol{a}} - f^*_{\boldsymbol{b}}) \quad \text{(by Lemma 2.7 (ii)–(v))} \\ &= 2(f_{\boldsymbol{a}} f^*_{\boldsymbol{a}} + f_{\boldsymbol{b}} f^*_{\boldsymbol{b}}). \end{aligned}$$

□

Lemma 2.9. *Let* $\boldsymbol{a} \in \mathcal{R}^l$. *Then*

$$f_{\boldsymbol{a}}(x) f^*_{\boldsymbol{a}}(x) = N_{\boldsymbol{a}}(0) + \sum_{k=1}^{l-1} \left(N_{\boldsymbol{a}}(k) x^{-k} + N_{\boldsymbol{a}}(k)^* x^k \right).$$

Proof. For $\boldsymbol{a} = (a_0, \ldots, a_{l-1})$,

$$\begin{aligned}
f_{\boldsymbol{a}}(x) f_{\boldsymbol{a}}^*(x) &= \sum_{i=0}^{l-1} a_i x^i \sum_{j=0}^{l-1} a_j^* x^{-j} \\
&= \sum_{i,j=0}^{l-1} a_i x^i a_j^* x^{-j} \\
&= \sum_{i,j=0}^{l-1} a_i a_j^* x^{i-j} \\
&= \sum_{i=0}^{l-1} a_i a_i^* + \sum_{0 \le i < j \le l-1} a_i a_j^* x^{i-j} + \sum_{0 \le j < i \le l-1} a_i a_j^* x^{i-j} \\
&= \sum_{i=0}^{l-1} a_i a_i^* + \sum_{k=1}^{l-1} \left(\sum_{i=0}^{l-1-k} a_i a_{i+k}^* \right) x^{-k} + \sum_{k=1}^{l-1} \left(\sum_{j=0}^{l-1-k} a_{j+k} a_j^* \right) x^k \\
&= N_{\boldsymbol{a}}(0) + \sum_{k=1}^{l-1} N_{\boldsymbol{a}}(k) x^{-k} + \sum_{k=1}^{l-1} \left(\sum_{j=0}^{l-1-k} a_j a_{j+k}^* \right)^* x^k \\
&= N_{\boldsymbol{a}}(0) + \sum_{k=1}^{l-1} \left(N_{\boldsymbol{a}}(k) x^{-k} + N_{\boldsymbol{a}}(k)^* x^k \right).
\end{aligned}$$

□

Lemma 2.10. *Let $\boldsymbol{a}_1, \ldots, \boldsymbol{a}_n$ be arbitrary sequences with entries in $\mathcal{R}$. Then $\boldsymbol{a}_1, \ldots, \boldsymbol{a}_n$ are complementary with weight w if and only if*

$$\sum_{i=1}^{n} f_{\boldsymbol{a}_i}(x) f_{\boldsymbol{a}_i}^*(x) = w. \tag{2.1}$$

Proof. From Lemma 2.9, we have

$$\begin{aligned}
\sum_{i=1}^{n} f_{\boldsymbol{a}_i}(x) f_{\boldsymbol{a}_i}^*(x) &= \sum_{i=1}^{n} \left(N_{\boldsymbol{a}_i}(0) + \sum_{k=1}^{l-1} \left(N_{\boldsymbol{a}_i}(k) x^{-k} + N_{\boldsymbol{a}_i}(k)^* x^k \right) \right) \\
&= \sum_{i=1}^{n} N_{\boldsymbol{a}_i}(0) + \sum_{k=1}^{l-1} \sum_{i=1}^{n} N_{\boldsymbol{a}_i}(k) x^{-k} + \sum_{k=1}^{l-1} \sum_{i=1}^{n} N_{\boldsymbol{a}_i}(k)^* x^k.
\end{aligned}$$

The equation (2.1) means that w is the constant term of $\sum_{i=1}^{n} f_{\boldsymbol{a}_i}(x) f_{\boldsymbol{a}_i}^*(x)$, from which the result follows. □

Lemma 2.10 means that it suffices to compute $\sum_{i=1}^{n} f_{\boldsymbol{a}_i}(x) f_{\boldsymbol{a}_i}^*(x)$ for sequences $\boldsymbol{a}_1, \ldots, \boldsymbol{a}_n$ to determine whether they are complementary or not. We will use this fact to investigate general constructions of complementary sequences. This lemma is crucial for our result in Chapter 3.

2.2 Matrix Representation of Complementary Sequences

In this section, we define $A^* = [a^*_{ji}]_{i,j=0}^{n-1}$ for an arbitrary square matrix $A = [a_{ij}]_{i,j=0}^{n-1}$ of order n whose entries are in $\mathcal{R}$. Practically, rather than computing the summation of polynomials directly, we may associate $\mathbf{M} \in \mathcal{R}^{k\times k}[x^{\pm 1}]$ to $f_{\boldsymbol{a}_1}(x), \dots, f_{\boldsymbol{a}_n}(x)$ for arbitrary sequences $\boldsymbol{a}_1, \dots, \boldsymbol{a}_n \in \mathcal{R}^l$. Furthermore, the following results are due to the communication with R. Craigen. In what follows, we shall omit the reference for x in polynomials if there is no confusion.

Definition 2.11. Let $\boldsymbol{a}, \boldsymbol{b} \in \mathcal{R}^l$. Define

$$\mathbf{M}_{\boldsymbol{a},\boldsymbol{b}} = \mathbf{M}_{\boldsymbol{a},\boldsymbol{b}}(x) = \begin{pmatrix} f_{\boldsymbol{a}} & f_{\boldsymbol{b}} \\ -f_{\boldsymbol{b}^*} & f_{\boldsymbol{a}^*} \end{pmatrix}. \tag{2.2}$$

Lemma 2.12. *Let* $\boldsymbol{a}, \boldsymbol{b} \in \mathcal{R}^l$ *and* $\boldsymbol{c}, \boldsymbol{d} \in \mathcal{R}^n$. *Then*

(i) $\boldsymbol{M}_{\boldsymbol{a},b}\, \boldsymbol{M}^*_{\boldsymbol{a},b} = (f_{\boldsymbol{a}} f^*_{\boldsymbol{a}} + f_{\boldsymbol{b}} f^*_{\boldsymbol{b}}) I$,

(ii) $\boldsymbol{M}_{\boldsymbol{a},-\boldsymbol{b}}(x^n)\, \boldsymbol{M}_{\boldsymbol{c},\boldsymbol{d}}(x) = \boldsymbol{M}_{\boldsymbol{a}\otimes\boldsymbol{c}+\boldsymbol{b}\otimes\boldsymbol{d}^*,\boldsymbol{a}\otimes\boldsymbol{d}-\boldsymbol{b}\otimes\boldsymbol{c}^*}(x)$,

(iii) $\boldsymbol{M}_{\boldsymbol{a}\otimes\boldsymbol{c}+\boldsymbol{b}\otimes\boldsymbol{d}^*,\boldsymbol{a}\otimes\boldsymbol{d}-\boldsymbol{b}\otimes\boldsymbol{c}^*}\, \boldsymbol{M}^*_{\boldsymbol{a}\otimes\boldsymbol{c}+\boldsymbol{b}\otimes\boldsymbol{d}^*,\boldsymbol{a}\otimes\boldsymbol{d}-\boldsymbol{b}\otimes\boldsymbol{c}^*} = (f_{\boldsymbol{a}} f^*_{\boldsymbol{a}} + f_{\boldsymbol{b}} f^*_{\boldsymbol{b}})(f_{\boldsymbol{c}} f^*_{\boldsymbol{c}} + f_{\boldsymbol{d}} f^*_{\boldsymbol{d}}) I$.

Proof. (i)

$$\begin{aligned}
\mathbf{M}_{\boldsymbol{a},\boldsymbol{b}}\, \mathbf{M}^*_{\boldsymbol{a},\boldsymbol{b}} &= \begin{pmatrix} f_{\boldsymbol{a}} & f_{\boldsymbol{b}} \\ -f_{\boldsymbol{b}^*} & f_{\boldsymbol{a}^*} \end{pmatrix} \begin{pmatrix} f^*_{\boldsymbol{a}} & -f^*_{\boldsymbol{b}^*} \\ f^*_{\boldsymbol{b}} & f^*_{\boldsymbol{a}^*} \end{pmatrix} \\
&= \begin{pmatrix} f_{\boldsymbol{a}} f^*_{\boldsymbol{a}} + f_{\boldsymbol{b}} f^*_{\boldsymbol{b}} & -f_{\boldsymbol{a}} f^*_{\boldsymbol{b}^*} + f_{\boldsymbol{b}} f^*_{\boldsymbol{a}^*} \\ -f_{\boldsymbol{b}^*} f^*_{\boldsymbol{a}} + f_{\boldsymbol{a}^*} f^*_{\boldsymbol{b}} & f_{\boldsymbol{b}^*} f^*_{\boldsymbol{b}^*} + f_{\boldsymbol{a}^*} f^*_{\boldsymbol{a}^*} \end{pmatrix} \\
&= \begin{pmatrix} f_{\boldsymbol{a}} f^*_{\boldsymbol{a}} + f_{\boldsymbol{b}} f^*_{\boldsymbol{b}} & -f^*_{\boldsymbol{a}^*} f_{\boldsymbol{b}} + f_{\boldsymbol{b}} f^*_{\boldsymbol{a}^*} \\ -f_{\boldsymbol{b}^*} f^*_{\boldsymbol{a}} + f^*_{\boldsymbol{a}} f_{\boldsymbol{b}^*} & f_{\boldsymbol{a}} f^*_{\boldsymbol{a}} + f_{\boldsymbol{b}} f^*_{\boldsymbol{b}} \end{pmatrix} \qquad \text{(by Lemma 2.7 (vii))} \\
&= (f_{\boldsymbol{a}} f^*_{\boldsymbol{a}} + f_{\boldsymbol{b}} f^*_{\boldsymbol{b}}) I.
\end{aligned}$$

(ii) By Lemma 2.7 (v),

$$
\begin{aligned}
&\mathbf{M}_{\boldsymbol{a},-\boldsymbol{b}}(x^n)\,\mathbf{M}_{\boldsymbol{c},\boldsymbol{d}}(x)\\
&= \begin{pmatrix} f_{\boldsymbol{a}}(x^n) & -f_{\boldsymbol{b}}(x^n) \\ f_{\boldsymbol{b}^*}(x^n) & f_{\boldsymbol{a}^*}(x^n) \end{pmatrix} \begin{pmatrix} f_{\boldsymbol{c}}(x) & f_{\boldsymbol{d}}(x) \\ -f_{\boldsymbol{d}^*}(x) & f_{\boldsymbol{c}^*}(x) \end{pmatrix}\\
&= \begin{pmatrix} f_{\boldsymbol{a}}(x^n)f_{\boldsymbol{c}}(x) + f_{\boldsymbol{b}}(x^n)f_{\boldsymbol{d}^*}(x) & f_{\boldsymbol{a}}(x^n)f_{\boldsymbol{d}}(x) - f_{\boldsymbol{b}}(x^n)f_{\boldsymbol{c}^*}(x) \\ f_{\boldsymbol{b}^*}(x^n)f_{\boldsymbol{c}}(x) - f_{\boldsymbol{a}^*}(x^n)f_{\boldsymbol{d}^*}(x) & f_{\boldsymbol{b}^*}(x^n)f_{\boldsymbol{d}}(x) + f_{\boldsymbol{a}^*}(x^n)f_{\boldsymbol{c}^*}(x) \end{pmatrix}\\
&= \begin{pmatrix} f_{\boldsymbol{a}\otimes\boldsymbol{c}}(x) + f_{\boldsymbol{b}\otimes\boldsymbol{d}^*}(x) & f_{\boldsymbol{a}\otimes\boldsymbol{d}}(x) - f_{\boldsymbol{b}\otimes\boldsymbol{c}^*}(x) \\ f_{\boldsymbol{b}^*\otimes\boldsymbol{c}}(x) - f_{\boldsymbol{a}^*\otimes\boldsymbol{d}^*}(x) & f_{\boldsymbol{b}^*\otimes\boldsymbol{d}}(x) + f_{\boldsymbol{a}^*\otimes\boldsymbol{c}^*}(x) \end{pmatrix} && \text{(by Lemma 2.7 (v))}\\
&= \begin{pmatrix} f_{\boldsymbol{a}\otimes\boldsymbol{c}+\boldsymbol{b}\otimes\boldsymbol{d}^*}(x) & f_{\boldsymbol{a}\otimes\boldsymbol{d}-\boldsymbol{b}\otimes\boldsymbol{c}^*}(x) \\ -f_{\boldsymbol{a}^*\otimes\boldsymbol{d}^*-\boldsymbol{b}^*\otimes\boldsymbol{c}}(x) & f_{\boldsymbol{a}^*\otimes\boldsymbol{c}^*+\boldsymbol{b}^*\otimes\boldsymbol{d}}(x) \end{pmatrix} && \text{(by Lemma 2.7 (i))}\\
&= \begin{pmatrix} f_{\boldsymbol{a}\otimes\boldsymbol{c}+\boldsymbol{b}\otimes\boldsymbol{d}^*}(x) & f_{\boldsymbol{a}\otimes\boldsymbol{d}-\boldsymbol{b}\otimes\boldsymbol{c}^*}(x) \\ -f_{(\boldsymbol{a}\otimes\boldsymbol{d}-\boldsymbol{b}\otimes\boldsymbol{c}^*)^*}(x) & f_{(\boldsymbol{a}\otimes\boldsymbol{c}+\boldsymbol{b}\otimes\boldsymbol{d}^*)^*}(x) \end{pmatrix} && \text{(by Lemma 2.6 (ii))}\\
&= \mathbf{M}_{\boldsymbol{a}\otimes\boldsymbol{c}+\boldsymbol{b}\otimes\boldsymbol{d}^*,\boldsymbol{a}\otimes\boldsymbol{d}-\boldsymbol{b}\otimes\boldsymbol{c}^*}(x).
\end{aligned}
$$

(iii) Immediate from (i) and (ii).

□

Lemma 2.13. *Let $\boldsymbol{a},\boldsymbol{b} \in R^l$. Then*

$$\boldsymbol{M}_{\boldsymbol{a},\boldsymbol{b}}(x)\,\boldsymbol{M}_{\boldsymbol{a},\boldsymbol{b}}(x)^* = \boldsymbol{M}_{\boldsymbol{a},\boldsymbol{b}^*}(x)\,\boldsymbol{M}_{\boldsymbol{a},\boldsymbol{b}^*}(x)^*.$$

Proof.

$$
\begin{aligned}
\mathbf{M}_{\boldsymbol{a},\boldsymbol{b}^*}(x)\,\mathbf{M}_{\boldsymbol{a},\boldsymbol{b}^*}(x)^* &= (f_{\boldsymbol{a}}f_{\boldsymbol{a}}^* + f_{\boldsymbol{b}^*}f_{\boldsymbol{b}^*}^*)I && \text{(by Lemma 2.12 (i))}\\
&= (f_{\boldsymbol{a}}f_{\boldsymbol{a}}^* + f_{\boldsymbol{b}}f_{\boldsymbol{b}}^*)I && \text{(by Lemma 2.7 (vii))}\\
&= \mathbf{M}_{\boldsymbol{a},\boldsymbol{b}}(x)\,\mathbf{M}_{\boldsymbol{a},\boldsymbol{b}}(x)^*. && \text{(by Lemma 2.12 (i))}
\end{aligned}
$$

□

By using this matrix representation, we can characterize a pair of complementary sequences by using the following corollary.

Corollary 2.14. *Let* $\boldsymbol{a}, \boldsymbol{b} \in R^l$*. Then* $\boldsymbol{a}$ *and* $\boldsymbol{b}$ *are complementary sequences with weight* w *if and only if*

$$\boldsymbol{M}_{\boldsymbol{a},\boldsymbol{b}}\, \boldsymbol{M}^*_{\boldsymbol{a},\boldsymbol{b}} = wI.$$

Proof. Immediate from Lemma 2.10 and Lemma 2.12 (i). □

Moreover, we may also characterize a triple of complementary sequences by using the following matrix. For $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c} \in \mathcal{R}^l$, define

$$T_{\boldsymbol{a},\boldsymbol{b},\boldsymbol{c}} := \begin{bmatrix} 0 & -f_{\boldsymbol{a}} & -f_{\boldsymbol{b}} & -f_{\boldsymbol{c}} \\ f_{\boldsymbol{a}} & 0 & f^*_{\boldsymbol{c}} & -f^*_{\boldsymbol{b}} \\ f_{\boldsymbol{b}} & -f^*_{\boldsymbol{c}} & 0 & f^*_{\boldsymbol{a}} \\ f_{\boldsymbol{c}} & f^*_{\boldsymbol{b}} & -f_{\boldsymbol{a}} & 0 \end{bmatrix}.$$

Then by a direct calculation, we obtain

$$T_{\boldsymbol{a},\boldsymbol{b},\boldsymbol{c}} T^*_{\boldsymbol{a},\boldsymbol{b},\boldsymbol{c}} = (f_{\boldsymbol{a}} f^*_{\boldsymbol{a}} + f_{\boldsymbol{b}} f^*_{\boldsymbol{b}} + f_{\boldsymbol{c}} f^*_{\boldsymbol{c}})I. \tag{2.3}$$

Corollary 2.15. *Let* $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c} \in R^l$*. Then* $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}$ *are complementary sequences with weight* w *if and only if*

$$T_{\boldsymbol{a},\boldsymbol{b},\boldsymbol{c}} T^*_{\boldsymbol{a},\boldsymbol{b},\boldsymbol{c}} = wI.$$

Proof. Immediate by Lemma 2.10 and (2.3). □

2.3 Some Classes of Complementary Sequences

We will introduce some classes of complementary sequences in this section. First, we will give some examples of ternary complementary sequences, i.e. sequences with entries in $\{0, \pm 1\}$. The following results are well known among the study of complementary sequences.

2.3.1 Ternary Complementary Sequences

Definition 2.16 (Golay sequences)**.** A pair of sequences $(\boldsymbol{a}; \boldsymbol{b}) \in \{\pm 1\}^n$ is called *Golay sequences of length* n if they are complementary sequences. Denote by $GS(n)$ the set of Golay sequences of length n. A positive integer n is called a *Golay number* if $GS(n)$ is nonempty.

Example 2.17. The pairs

$$((1,1,1,-1);(1,1,-1,1)), \quad ((1,1,-1,1,-1,1,-1,-1,1,1);(1,1,-1,1,1,1,1,1,-1,-1))$$

are, respectively, Golay sequences of length 4 and 10.

Lemma 2.18. *Let* $\boldsymbol{a}, \boldsymbol{b} \in \{\pm 1\}^n$*. Then* $(\boldsymbol{a}; \boldsymbol{b})$ *are Golay sequences if and only if*

$$f_{\boldsymbol{a}} f_{\boldsymbol{a}}^* + f_{\boldsymbol{b}} f_{\boldsymbol{b}}^* = 2n.$$

Proof. Immediate from Lemma 2.10. □

Theorem 2.19. *If* $(\boldsymbol{a}; \boldsymbol{b}) \in GS(m)$ *and* $(\boldsymbol{c}; \boldsymbol{d}) \in GS(n)$ *then*

$$\left(\frac{1}{2}[(\boldsymbol{a}+\boldsymbol{b}) \otimes \boldsymbol{c} + (\boldsymbol{a}-\boldsymbol{b}) \otimes \boldsymbol{d}^*]; \frac{1}{2}[(\boldsymbol{a}+\boldsymbol{b}) \otimes \boldsymbol{d} - (\boldsymbol{a}-\boldsymbol{b}) \otimes \boldsymbol{c}^*]\right) \in GS(mn).$$

Proof. Let $\boldsymbol{a}' = \frac{\boldsymbol{a}+\boldsymbol{b}}{2}$ and $\boldsymbol{b}' = \frac{\boldsymbol{a}-\boldsymbol{b}}{2}$. Then by Lemma 2.8, we have

$$f_{\boldsymbol{a}'} f_{\boldsymbol{a}'}^* + f_{\boldsymbol{b}'} f_{\boldsymbol{b}'}^* = \frac{1}{2}(f_{\boldsymbol{a}} f_{\boldsymbol{a}}^* + f_{\boldsymbol{b}} f_{\boldsymbol{b}}^*) = m.$$

Also by Lemma 2.18,

$$f_{\boldsymbol{c}} f_{\boldsymbol{c}}^* + f_{\boldsymbol{d}} f_{\boldsymbol{d}}^* = 2n.$$

Recall $\mathbf{M}$ in (2.2). Then

$$\begin{aligned}
&\mathbf{M}_{\boldsymbol{a}'\otimes\boldsymbol{c}+\boldsymbol{b}'\otimes\boldsymbol{d}^*,\boldsymbol{a}'\otimes\boldsymbol{d}-\boldsymbol{b}'\otimes\boldsymbol{c}^*}(x)\cdot\mathbf{M}^*_{\boldsymbol{a}'\otimes\boldsymbol{c}+\boldsymbol{b}'\otimes\boldsymbol{d}^*,\boldsymbol{a}'\otimes\boldsymbol{d}-\boldsymbol{b}'\otimes\boldsymbol{c}^*}(x)\\
&=(f_{\boldsymbol{a}'}f^*_{\boldsymbol{a}'}+f_{\boldsymbol{b}'}f^*_{\boldsymbol{b}'})(f_{\boldsymbol{c}}f^*_{\boldsymbol{c}}+f_{\boldsymbol{d}}f^*_{\boldsymbol{d}})I \qquad \text{(by Lemma 2.12 (iii))}\\
&=2mnI.
\end{aligned}$$

From the fact that $\boldsymbol{a}',\boldsymbol{b}'$ are disjoint $\{0,\pm1\}$-sequences, and $\boldsymbol{c},\boldsymbol{d}$ are $\{\pm1\}$-sequences, $\boldsymbol{a}'\otimes\boldsymbol{c}+\boldsymbol{b}'\otimes\boldsymbol{d}^*$ and $\boldsymbol{a}'\otimes\boldsymbol{d}-\boldsymbol{b}'\otimes\boldsymbol{c}^*$ are $\{\pm1\}$-sequences. Therefore, by Lemma 2.18,

$$(\boldsymbol{a}'\otimes\boldsymbol{c}+\boldsymbol{b}'\otimes\boldsymbol{d}^*;\boldsymbol{a}'\otimes\boldsymbol{d}-\boldsymbol{b}'\otimes\boldsymbol{c}^*)\in GS(mn).$$

□

Lemma 2.20. *Let N be an even positive integer. If $(a_1,\dots,a_N)\in\{\pm1\}^N$ and $\sum_{i=1}^N a_j=0$, then $\prod_{j=1}^N a_j=(-1)^{N/2}$.*

Proof. Since $\sum_{i=1}^N a_j=0$, the total number of -1 in $a_1,\dots,a_N$ is $N/2$. Thus, the result follows. □

Lemma 2.21 ([7])**.** *If g is an even Golay number, then there exist complementary $(0,\pm1)$-sequences $\boldsymbol{u},\boldsymbol{v}$ of length g and weight $g/2$ such that $\boldsymbol{u}$, $\boldsymbol{v}$, $\boldsymbol{u}^*$ and $\boldsymbol{v}^*$ are all disjoint.*

Proof. Let $(\boldsymbol{z};\boldsymbol{y})\in GS(g)$, $\boldsymbol{z}=(z_0,\dots,z_{g-1})$ and $\boldsymbol{y}=(y_0,\dots,y_{g-1})$. For $0<i\le g-1$, let

$$a_k=\begin{cases} z_{k-1}z_{k+g-i-1} & \text{if } 1\le k\le i,\\ y_{k-1-i}y_{k-2i+g-1} & \text{if } i<k\le 2i.\end{cases}$$

By Lemma 2.18,

$$\begin{aligned}
2g - N_{\boldsymbol{z}}(0) - N_{\boldsymbol{y}}(0) &= f_{\boldsymbol{z}} f_{\boldsymbol{z}}^* + f_{\boldsymbol{y}} f_{\boldsymbol{y}}^* - N_{\boldsymbol{z}}(0) - N_{\boldsymbol{y}}(0) \\
&= \sum_{i=1}^{g-1} \Big((N_{\boldsymbol{z}}(i) + N_{\boldsymbol{y}}(i)) x^{-i} + (N_{\boldsymbol{z}}(i)^* + N_{\boldsymbol{y}}(i)^*) x^{i} \Big) \qquad \text{(by Definition 2.9)} \\
&= \sum_{i=1}^{g-1} (N_{\boldsymbol{z}}(i) + N_{\boldsymbol{y}}(i))(x^{-i} + x^{i}) \\
&= \sum_{i=1}^{g-1} (N_{\boldsymbol{z}}(g-i) + N_{\boldsymbol{y}}(g-i))(x^{-(g-i)} + x^{g-i}) \\
&= \sum_{i=1}^{g-1} \left(\sum_{k=0}^{i-1} z_k z_{k+g-i} + \sum_{k=0}^{i-1} y_k y_{k+g-i} \right) (x^{-(g-i)} + x^{g-i}) \\
&= \sum_{i=1}^{g-1} \left(\sum_{k=1}^{i} a_k + \sum_{k=i+1}^{2i} a_k \right) (x^{-(g-i)} + x^{g-i}) \\
&= \sum_{i=1}^{g-1} \sum_{k=1}^{2i} a_k (x^{-(g-i)} + x^{g-i}).
\end{aligned}$$

Thus $\sum_{k=1}^{2i} a_k = 0$. Therefore, by applying Lemma 2.20,

$$(-1)^i = \prod_{k=1}^{2i} a_k = \prod_{k=1}^{i} z_{k-1} z_{k+g-i-1} \prod_{k=1}^{i} y_{k-1} y_{k+g-i-1} = \prod_{k=1}^{i} z_{k-1} z_{k+g-i-1} y_{k-1} y_{k+g-i-1}. \tag{2.4}$$

Now, we will prove

$$z_{j-1} z_{g-j} y_{j-1} y_{g-j} = -1 \tag{2.5}$$

by induction on j for $1 \leq j \leq \frac{g}{2}$. Setting $i = 1$ in (2.4), we obtain $z_0 z_{g-1} y_0 y_{g-1} = -1$. Thus, (2.5) holds for $j = 1$. Assume (2.5) holds for all $k < j$, where $j \leq \frac{g}{2}$. Then

$$\prod_{k=1}^{j-1} z_{k-1} z_{g-k} y_{k-1} y_{g-k} = (-1)^{j-1}. \tag{2.6}$$

By setting $i = j$ in (2.4), we have

$$\begin{aligned}(-1)^j &= \prod_{k=1}^{j} z_{k-1}z_{k+g-j-1}y_{k-1}y_{k+g-j-1} \\ &= \prod_{k=1}^{j} z_{k-1}y_{k-1} \prod_{k=1}^{j} z_{g-(j+1-k)}y_{g-(j+1-k)} \\ &= \prod_{k=1}^{j} z_{k-1}y_{k-1} \prod_{k=1}^{j} z_{g-k}y_{g-k} \\ &= \prod_{k=1}^{j} z_{k-1}z_{g-k}y_{k-1}y_{g-k}\end{aligned}$$

and (2.5) follows by comparing this with (2.6).

Now, let

$$\boldsymbol{u} = \frac{\boldsymbol{z}+\boldsymbol{y}+\boldsymbol{z}^*-\boldsymbol{y}^*}{4}, \quad \boldsymbol{v} = \frac{\boldsymbol{z}+\boldsymbol{y}-\boldsymbol{z}^*+\boldsymbol{y}^*}{4}.$$

Since $\boldsymbol{z}$ and $\boldsymbol{y}$ are constrained by (2.5), the following table shows that $\boldsymbol{u}$, $\boldsymbol{v}$, $\boldsymbol{u}^*$, $\boldsymbol{v}^*$ are disjoint $(0, \pm 1)$-sequences.

z_{j-1}	y_{j-1}	z_{g-j}	y_{g-j}	u_{j-1}	v_{j-1}	u_{g-j}	v_{g-j}
1	1	1	−1	1	0	0	0
1	1	−1	1	0	1	0	0
−1	−1	1	−1	0	−1	0	0
−1	−1	−1	1	−1	0	0	0
1	−1	1	1	0	0	1	0
−1	1	1	1	0	0	0	1
1	−1	−1	−1	0	0	0	−1
−1	1	−1	−1	0	0	−1	0

Moreover,

$$\begin{aligned} f_{\boldsymbol{u}} f_{\boldsymbol{u}}^* + f_{\boldsymbol{v}} f_{\boldsymbol{v}}^* &= \frac{1}{16}(f_{(\boldsymbol{z}+\boldsymbol{y})+(\boldsymbol{z}^*-\boldsymbol{y}^*)} f^*_{(\boldsymbol{z}+\boldsymbol{y})+(\boldsymbol{z}^*-\boldsymbol{y}^*)} + f_{(\boldsymbol{z}+\boldsymbol{y})-(\boldsymbol{z}^*-\boldsymbol{y}^*)} f^*_{(\boldsymbol{z}+\boldsymbol{y})-(\boldsymbol{z}^*-\boldsymbol{y}^*)}) \\ &= \frac{1}{8}(f_{\boldsymbol{z}+\boldsymbol{y}} f^*_{\boldsymbol{z}+\boldsymbol{y}} + f_{\boldsymbol{z}^*-\boldsymbol{y}^*} f^*_{\boldsymbol{z}^*-\boldsymbol{y}^*}) && \text{(by Lemma 2.8)} \\ &= \frac{1}{8}(f_{\boldsymbol{z}+\boldsymbol{y}} f^*_{\boldsymbol{z}+\boldsymbol{y}} + f_{\boldsymbol{z}-\boldsymbol{y}} f^*_{\boldsymbol{z}-\boldsymbol{y}}) && \text{(by Lemma 2.7)} \\ &= \frac{1}{4}(f_{\boldsymbol{z}} f^*_{\boldsymbol{z}} + f_{\boldsymbol{y}} f^*_{\boldsymbol{y}}) && \text{(by Lemma 2.8)} \\ &= \frac{g}{2}. \end{aligned}$$

Therefore $\boldsymbol{u}$ and $\boldsymbol{v}$ are complementary with weight $\frac{g}{2}$ by Lemma 2.10. □

Definition 2.22 (Base sequences)**.** A quadruple of (± 1)-sequences $(\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}, \boldsymbol{d})$ of length m, m, n, n, respectively, is called *base sequences* if it is a set of complementary sequences. We denote by $BS(m, n)$ the set of base sequences of length m, m, n, n. If $(\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}, \boldsymbol{d}) \in BS(m, n)$, then it is complementary with weight $2(m + n)$.

Definition 2.23 (Paired ternary sequences)**.** We call a quadruple of complementary $(0, \pm 1)$-sequences with weight $2l$, $(\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}, \boldsymbol{d})$ of length l a *paired ternary sequences* if

(i) the pairs $\{\boldsymbol{a}, \boldsymbol{b}\}$ and $\{\boldsymbol{c}, \boldsymbol{d}\}$ are each conjoint,

(ii) the pair $\{\boldsymbol{a}, \boldsymbol{c}\}$ is disjoint.

Denote by $PT(l)$ the set of paired ternary sequences of length l.

Definition 2.24 (T-sequences)**.** A T-sequences of length n is a quadraple of complementary $\{0, \pm 1\}$-sequences $(\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}, \boldsymbol{d})$, with weight n such that $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}, \boldsymbol{d}$ are mutually disjoint sequences, i.e. every pair in $\{\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}, \boldsymbol{d}\}$ is disjoint. Denote by $TS(n)$, the set of T-sequences of length n.

Lemma 2.25. *Let m and n be positive integers.*

(i) *For every positive integer n, $PT(n)$ is nonempty if and only if $TS(n)$ is nonempty,*

(ii) *$BS(m,n) \neq \emptyset$ implies $PT(m+n) \neq \emptyset$ and $TS(m+n) \neq \emptyset$.*

Proof. First, we will prove (i). Let $(\boldsymbol{a},\boldsymbol{b},\boldsymbol{c},\boldsymbol{d}) \in TS(n)$. Set $\boldsymbol{q} = \boldsymbol{a}+\boldsymbol{b}$, $\boldsymbol{r} = \boldsymbol{a}-\boldsymbol{b}$, $\boldsymbol{s} = \boldsymbol{c}+\boldsymbol{d}$, and $\boldsymbol{t} = \boldsymbol{c}-\boldsymbol{d}$. Then by Lemma 2.8, we have

$$(f_{\boldsymbol{q}}f_{\boldsymbol{q}}^* + f_{\boldsymbol{r}}f_{\boldsymbol{r}}^* + f_{\boldsymbol{s}}f_{\boldsymbol{s}}^* + f_{\boldsymbol{t}}f_{\boldsymbol{t}}^*)(x) = 2(f_{\boldsymbol{a}}f_{\boldsymbol{a}}^* + f_{\boldsymbol{b}}f_{\boldsymbol{b}}^* + f_{\boldsymbol{c}}f_{\boldsymbol{c}}^* + f_{\boldsymbol{d}}f_{\boldsymbol{d}}^*)(x) = 2n.$$

Since $\boldsymbol{a},\boldsymbol{b},\boldsymbol{c},\boldsymbol{d}$ are mutually disjoint sequences, the pairs $(\boldsymbol{q},\boldsymbol{r})$ and $(\boldsymbol{s},\boldsymbol{t})$ are each conjoint, and it can be checked that $\boldsymbol{q}$ and $\boldsymbol{s}$ are disjoint. Therefore, $(\boldsymbol{q},\boldsymbol{r};\boldsymbol{s},\boldsymbol{t}) \in PT(n)$. Conversely, let $(\boldsymbol{q},\boldsymbol{r};\boldsymbol{s},\boldsymbol{t}) \in PT(n)$. Set $\boldsymbol{a} = \frac{1}{2}(\boldsymbol{q}+\boldsymbol{r})$, $\boldsymbol{b} = \frac{1}{2}(\boldsymbol{q}-\boldsymbol{r})$, $\boldsymbol{c} = \frac{1}{2}(\boldsymbol{s}+\boldsymbol{t})$, $\boldsymbol{d} = \frac{1}{2}(\boldsymbol{s}-\boldsymbol{t})$. Again by Lemma 2.8,

$$(f_{\boldsymbol{q}}f_{\boldsymbol{q}}^* + f_{\boldsymbol{r}}f_{\boldsymbol{r}}^* + f_{\boldsymbol{s}}f_{\boldsymbol{s}}^* + f_{\boldsymbol{t}}f_{\boldsymbol{t}}^*)(x) = (f_{\boldsymbol{a}}f_{\boldsymbol{a}}^* + f_{\boldsymbol{b}}f_{\boldsymbol{b}}^* + f_{\boldsymbol{c}}f_{\boldsymbol{c}}^* + f_{\boldsymbol{d}}f_{\boldsymbol{d}}^*)(x) = n.$$

Since $\boldsymbol{q}$ and $\boldsymbol{r}$ are conjoint and $\{0,\pm1\}$-sequences, $\text{supp}(\boldsymbol{a})\cap\text{supp}(\boldsymbol{b}) = \text{supp}(\boldsymbol{q}+\boldsymbol{r})\cap\text{supp}(\boldsymbol{q}-\boldsymbol{r}) = \emptyset$. By a similar argument, $\text{supp}(\boldsymbol{c})\cap\text{supp}(\boldsymbol{d}) = \text{supp}(\boldsymbol{s}+\boldsymbol{t})\cap\text{supp}(\boldsymbol{s}-\boldsymbol{t}) = \emptyset$. Moreover, since $\text{supp}(\boldsymbol{x})\cap\text{supp}(\boldsymbol{y}) = \emptyset$ for every $\boldsymbol{x} \in \{\boldsymbol{q},\boldsymbol{r}\}$ and $\boldsymbol{y} \in \{\boldsymbol{s},\boldsymbol{t}\}$, we have $\text{supp}(\boldsymbol{u})\cap\text{supp}(\boldsymbol{v}) = \emptyset$ for every $\boldsymbol{u} \in \{\boldsymbol{a},\boldsymbol{b}\}$ and $\boldsymbol{v} \in \{\boldsymbol{c},\boldsymbol{d}\}$. Therefore, $\boldsymbol{a},\boldsymbol{b},\boldsymbol{c},\boldsymbol{d}$ are mutually disjoint sequences. Hence, $(\boldsymbol{a},\boldsymbol{b},\boldsymbol{c},\boldsymbol{d}) \in TS(n)$.

Next, we will prove (ii). Let $(\boldsymbol{a},\boldsymbol{b},\boldsymbol{c},\boldsymbol{d}) \in BS(m,n)$. Set $\boldsymbol{q} = (\boldsymbol{a},0_n), \boldsymbol{r} = (\boldsymbol{b},0_n), \boldsymbol{s} = (0_m,\boldsymbol{c}), \boldsymbol{t} = (0_m,\boldsymbol{d})$. Clearly, the pairs $(\boldsymbol{q};\boldsymbol{r})$ and $(\boldsymbol{s},\boldsymbol{t})$ are each conjoint, and the pair $(\boldsymbol{q};\boldsymbol{s})$ is disjoint. Also, we have

$$(f_{\boldsymbol{q}}f_{\boldsymbol{q}}^* + f_{\boldsymbol{r}}f_{\boldsymbol{r}}^* + f_{\boldsymbol{s}}f_{\boldsymbol{s}}^* + f_{\boldsymbol{t}}f_{\boldsymbol{t}}^*)(x) = (f_{\boldsymbol{a}}f_{\boldsymbol{a}}^* + f_{\boldsymbol{b}}f_{\boldsymbol{b}}^* + f_{\boldsymbol{c}}f_{\boldsymbol{c}}^* + f_{\boldsymbol{d}}f_{\boldsymbol{d}}^*)(x) = 2(m+n).$$

Hence, $(\boldsymbol{q},\boldsymbol{r},\boldsymbol{s},\boldsymbol{t}) \in PT(m+n)$, and also $TS(m+n)$ by (i). □

2.3.2 Complementary Sequences with Entries in $\{0,\pm i,\pm1\}$

Recently, we were considering to extend the entries of complementary sequences to an arbitrary commutative ring $\mathcal{R}$. Specifically, the entries will be in $\mathcal{T}\cup\{0\}$ where $\mathcal{T}$ is a multiplicatively closed subset of $\mathcal{R}$ satisfying $-1 \in \mathcal{T}$. For example, if we take $\mathcal{R}$ to be the ring of integers, then $\mathcal{T} = \{\pm1\}$. Moreover, if $\mathcal{R} = \mathbb{C}$, then we may take $\mathcal{T} = \{\pm i,\pm1\}$. The existence of complementary sequences whose entries are in $\{\pm i, \pm 1\}$ has been studied in [7] and [17]. In this subsection, we will give some examples of complementary sequences with entries in $\{0, \pm i, \pm 1\}$. All of the following results are due to [7] and [17].

Definition 2.26 (Complex Golay Sequence)**.** Two $(\pm i, \pm 1)$-sequences of length l are called a complex Golay pair (or complex Golay sequences), if they together have zero autocorrelation. Let $CGS(l)$ be the set of complex Golay sequences of length l. A positive integer l is called a *complex Golay number* if $CGS(l) \neq \emptyset$.

Lemma 2.27. *Let* $\boldsymbol{a}, \boldsymbol{b} \in \{\pm i, \pm 1\}^l$*. Then* $f_{\boldsymbol{a}} f_{\boldsymbol{a}}^* + f_{\boldsymbol{b}} f_{\boldsymbol{b}}^* = 2l$ *if and only if* $(\boldsymbol{a}; \boldsymbol{b})$ *is a complex Golay pair.*

Proof. Immediate from Lemma 2.10. □

Theorem 2.28. *If* $(\boldsymbol{a}; \boldsymbol{b}) \in CGS(m)$ *and* $(C; D) \in CGS(n)$*. Then,*

(i) $((\boldsymbol{a} \otimes \boldsymbol{c}; \boldsymbol{b} \otimes \boldsymbol{d}^*); (\boldsymbol{a} \otimes \boldsymbol{d}; -\boldsymbol{b} \otimes \boldsymbol{c}^*)) \in CGS(2mn)$,

(ii) *If* $\boldsymbol{a}, \boldsymbol{b} \in \{\pm 1\}^m$*, then*

$$\left(\frac{1}{2}[(\boldsymbol{a} + \boldsymbol{b}) \otimes \boldsymbol{c} + (\boldsymbol{a} - \boldsymbol{b}) \otimes \boldsymbol{d}^*]; \frac{1}{2}[(\boldsymbol{a} + \boldsymbol{b}) \otimes \boldsymbol{d} - (\boldsymbol{a} - \boldsymbol{b}) \otimes s\boldsymbol{c}^*] \right) \in CGS(mn).$$

Proof. First, we prove part (i). We have

$$\begin{aligned}
&\mathbf{M}_{\boldsymbol{a}\otimes\boldsymbol{c}+\boldsymbol{b}\otimes\boldsymbol{d}^*,\boldsymbol{a}\otimes\boldsymbol{d}-\boldsymbol{b}\otimes\boldsymbol{c}^*}(x) \cdot \mathbf{M}_{\boldsymbol{a}\otimes\boldsymbol{c}+\boldsymbol{b}\otimes\boldsymbol{d}^*,\boldsymbol{a}\otimes\boldsymbol{d}-\boldsymbol{b}\otimes\boldsymbol{c}^*}(x)^* \\
&= (f_{\boldsymbol{a}} f_{\boldsymbol{a}}^* + f_{\boldsymbol{b}} f_{\boldsymbol{b}}^*)(f_{\boldsymbol{c}} f_{\boldsymbol{c}}^* + f_{\boldsymbol{d}} f_{\boldsymbol{d}}^*) I \qquad \text{(by Lemma 2.12 (iii))} \\
&= 4mnI.
\end{aligned}$$

Next, we prove (ii). Let

$$\boldsymbol{a}' = \frac{\boldsymbol{a} + \boldsymbol{b}}{2} \quad \text{and} \quad \boldsymbol{b}' = \frac{\boldsymbol{a} - \boldsymbol{b}}{2}.$$

Since $\boldsymbol{a}$ and $\boldsymbol{b}$ are $\{\pm 1\}$-sequences, $\boldsymbol{a}'$ and $\boldsymbol{b}'$ are disjoint, by Lemma 2.8 and Lemma 2.12 (i).

By Lemma 2.8, we have $f_{\boldsymbol{a}'}f^*_{\boldsymbol{a}'} + f_{\boldsymbol{b}'}f^*_{\boldsymbol{b}'} = \frac{1}{2}(f_{\boldsymbol{a}}f^*_{\boldsymbol{a}} + f_{\boldsymbol{b}}f^*_{\boldsymbol{b}}) = m$. Thus,

$$\begin{aligned}
&\mathbf{M}_{\boldsymbol{a}'\otimes\boldsymbol{c}+\boldsymbol{b}'\otimes\boldsymbol{d}^*,\boldsymbol{a}'\otimes\boldsymbol{d}-\boldsymbol{b}'\otimes\boldsymbol{c}^*}(x)\cdot\mathbf{M}^*_{\boldsymbol{a}'\otimes\boldsymbol{c}+\boldsymbol{b}'\otimes\boldsymbol{d}^*,\boldsymbol{a}'\otimes\boldsymbol{d}-\boldsymbol{b}'\otimes\boldsymbol{c}^*}(x)\\
&= (f_{\boldsymbol{a}'}f^*_{\boldsymbol{a}'} + f_{\boldsymbol{b}'}f^*_{\boldsymbol{b}'})(f_{\boldsymbol{c}}f^*_{\boldsymbol{c}} + f_{\boldsymbol{d}}f^*_{\boldsymbol{d}})I \qquad \text{(by Lemma 2.12 (iii))}\\
&= 2mnI.
\end{aligned}$$

Therefore, the result holds by Lemma 2.27. □

Theorem 2.29 ([7])**.** *Let g_1 and g_2 be complex Golay numbers and g be an even Golay number. Then gg_1g_2 is a complex Golay number.*

Proof. Let $(\boldsymbol{a};\boldsymbol{b}) \in CGS(g_1)$ and $(\boldsymbol{c};\boldsymbol{d}) \in CGS(g_2)$. Let $(\boldsymbol{y},\boldsymbol{z}) \in GS(g)$. By Lemma 2.21, there exist complementary $(0,\pm1)$-sequences $\boldsymbol{u},\boldsymbol{v}$ of length g and weight $g/2$ such that $\boldsymbol{u}$, $\boldsymbol{v}$, $\boldsymbol{u}^*$ and $\boldsymbol{v}^*$ are all disjoint. Define

$$\boldsymbol{s} = \boldsymbol{a}\otimes\boldsymbol{u} + \boldsymbol{b}\otimes\boldsymbol{v}^*, \quad \boldsymbol{t} = \boldsymbol{a}\otimes\boldsymbol{v} - \boldsymbol{b}\otimes\boldsymbol{u}^*.$$

Since $\boldsymbol{a},\boldsymbol{b}$ are $(\pm i,\pm1)$-sequences and $\boldsymbol{u},\boldsymbol{v},\boldsymbol{u}^*,\boldsymbol{v}^*$ are all disjoint $(0,\pm1)$-sequences, S,T are disjoint $(0,\pm i,\pm1)$-sequences. Now

$$\begin{aligned}
\mathbf{M}_{\boldsymbol{s},\boldsymbol{t}}(x)\,\mathbf{M}_{\boldsymbol{s},\boldsymbol{t}}(x)^* &= (f_{\boldsymbol{a}}f^*_{\boldsymbol{a}} + f_{\boldsymbol{b}}f^*_{\boldsymbol{b}})(f_{\boldsymbol{u}}f^*_{\boldsymbol{u}} + f_{\boldsymbol{v}}f^*_{\boldsymbol{v}})I \qquad \text{(by Lemma 2.12 (iii))}\\
&= 2g_1\cdot\frac{g}{2}\\
&= gg_1.
\end{aligned}$$

Moreover,

$$\begin{aligned}
&\mathbf{M}_{\boldsymbol{c}\otimes\boldsymbol{s}+\boldsymbol{d}^*\otimes\boldsymbol{t},\boldsymbol{c}\otimes\boldsymbol{t}^*-\boldsymbol{d}^*\otimes\boldsymbol{s}^*}(x)\,\mathbf{M}_{\boldsymbol{c}\otimes\boldsymbol{s}+\boldsymbol{d}^*\otimes\boldsymbol{t},\boldsymbol{c}\otimes\boldsymbol{t}^*-\boldsymbol{d}^*\otimes\boldsymbol{s}^*}(x)^*\\
&\qquad= (f_{\boldsymbol{c}}f^*_{\boldsymbol{c}} + f_{\boldsymbol{d}}f^*_{\boldsymbol{d}})(f_{\boldsymbol{s}}f^*_{\boldsymbol{s}} + f_{\boldsymbol{t}}f^*_{\boldsymbol{t}})I \qquad \text{(by Lemma 2.12 (iii))}\\
&\qquad= 2g_2\cdot gg_1\\
&\qquad= 2gg_1g_2.
\end{aligned}$$

Also, the length and the weight of $\boldsymbol{c} \otimes \boldsymbol{s} + \boldsymbol{d}^* \otimes \boldsymbol{t}$ and $\boldsymbol{c} \otimes \boldsymbol{t}^* - \boldsymbol{d}^* \otimes \boldsymbol{s}^*$ are the same. Thus, they are $(\pm i, \pm 1)$-sequences and hence, they are complex Golay pair by Lemma 2.27. □

2.4 Constructions of Hadamard Matrices from Complementary Sequences

We will give two constructions of Hadamard matrices that is obtained by using Golay sequences and T-sequences. Denote by $\mathrm{circ}(\boldsymbol{a})$ the circulant matrix with the first row $\boldsymbol{a}$.

Theorem 2.30 ([24])**.** *Let $(\boldsymbol{a}; \boldsymbol{b}) \in GS(s)$. Let $A = \mathrm{circ}(\boldsymbol{a})$ and $B = \mathrm{circ}(\boldsymbol{b})$. Then*

$$\begin{bmatrix} A & B \\ -B^T & A^T \end{bmatrix}$$

is an Hadamard matrix of order $2s$.

To give a construction of an Hadamard matrix from T-sequences, we need the Goethals-Seidel array.

Theorem 2.31 (Goethals-Seidel array)**.** *Suppose A, B, C, D are circulant $\{\pm 1\}$-matrices of order n such that*

$$AA^T + BB^T + CC^T + DD^T = 4nI.$$

Let R be the back-diagonal matrix, i.e.

$$R = \begin{bmatrix} & & 1 \\ & \iddots & \\ 1 & & \end{bmatrix}.$$

Then

$$H = \begin{bmatrix} A & BR & CR & DR \\ -BR & A & -D^T R & C^T R \\ -CR & D^T R & A & -B^T R \\ -DR & -C^T R & B^T R & A \end{bmatrix} \tag{2.7}$$

is an Hadamard matrix of order $4n$. *Moreover, if* A, B, C, D *are symmetric, then*

$$A^2 + B^2 + C^2 + D^2 = 4nI$$

and

$$H = \begin{bmatrix} A & B & C & D \\ -B & A & -D & C \\ -C & D & A & -B \\ -D & -C & B & A \end{bmatrix}. \tag{2.8}$$

The construction (2.8) is known as a Williamson matrix. Therefore, Williamson matrices are special cases of Goethals-Seidel arrays. For further information about Goethals-Seidel array and Williamson matrices, we refer the reader to [19] and [25]. By using Goethals-Seidel array, we can actually construct an Hadamard matrix from T-sequences.

Theorem 2.32. *Let* $(\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}, \boldsymbol{d}) \in TS(n)$. *Let* A', B', C', D' *be* $\text{circ}(\boldsymbol{a}), \text{circ}(\boldsymbol{b}), \text{circ}(\boldsymbol{c}), \text{circ}(\boldsymbol{d})$, *respectively. Set*

$$A = A' + B' + C' + D', \qquad B = -A' + B' + C' - D',$$
$$C = -A' - B' + C' + D', \qquad D = -A' + B' - C' + D'.$$

Then

$$H = \begin{bmatrix} A & BR & CR & DR \\ -BR & A & -D^TR & C^TR \\ -CR & D^TR & A & -B^TR \\ -DR & -C^TR & B^TR & A \end{bmatrix}$$

is an Hadamard matrix of order $4n$.

Proof. Since $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}, \boldsymbol{d}$ are T-sequences, $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}, \boldsymbol{d}$ are mutually disjoint sequences. This implies that A', B', C', D' are mutually disjoint circulant matrices. Therefore, A, B, C, D are circulant $\{\pm 1\}$-matrices. By Theorem 2.31, H is an Hadamard matrix of order $4n$. □

3 Yang Multiplication Theorem

Yang multiplication theorem, basically, is known as a method to find paired ternary sequences from base sequences. As shown in Lemma 2.25, the existence of paired ternary sequences will lead to the existence of T-sequences and the existence of T-sequences implies to the existence of Hadamard matrices (see Theorem 2.32). Therefore, we try to increase the possibility of finding new T-sequences by generalizing Yang multiplication theorem.

3.1 A Generalization of Yang Multiplication Theorem

In this section, we introduce one result of Yang, who gave a method of constructing some complementary sequences from base sequences. More specifically,

Theorem 3.1 ([23, Theorem 4])**.** *If $BS(m+1,m) \neq \emptyset$ and $BS(n+1,n) \neq \emptyset$, then there exists a set of four complementary (± 1)-sequences of length $(2m+1)(2n+1)$.*

In order to prove Theorem 3.1, Yang used the Lagrange identity for polynomial rings. Let $\mathcal{R}$ be a commutative ring with identity and an involutive automorphism $*$. Moreover, let $\mathcal{R}[x^{\pm 1}]$ be the ring of Laurent polynomials over $\mathcal{R}$ and $* : \mathcal{R}[x^{\pm 1}] \to \mathcal{R}[x^{\pm 1}]$ be the extension of the involutive automorphism $*$ of $\mathcal{R}$ defined by $x \mapsto x^{-1}$. Then, $*$ is an involutive automorphism of $\mathcal{R}[x^{\pm 1}]$.

Definition 3.2. Let $\boldsymbol{a} = (a_0, \ldots, a_{l-1}) \in \mathcal{R}^l$. We define the *Hall polynomial* $f_{\boldsymbol{a}}(x) \in \mathcal{R}[x^{\pm 1}]$ of $\boldsymbol{a}$ by

$$f_{\boldsymbol{a}}(x) = \sum_{i=0}^{l-1} a_i x^i.$$

Hall polynomials have been used not only by Yang, but also others. See [8] and references therein.

By using the notation in Definition 3.2, we can generalize Theorem 3.1 as follows.

Theorem 3.3. *Let $\mathcal{T}$ be a multiplicatively closed subset of $\mathcal{R}$ satisfying $-1 \in \mathcal{T} = \mathcal{T}^*$. Let*

$$\boldsymbol{a}, \boldsymbol{b} \in \mathcal{T}^{m+1},$$
$$\boldsymbol{c}, \boldsymbol{d} \in \mathcal{T}^{m},$$
$$\boldsymbol{v}, \boldsymbol{g} \in \mathcal{T}^{n+1},$$
$$\boldsymbol{h}, \boldsymbol{e} \in \mathcal{T}^{n}$$

satisfy

$$(f_{\boldsymbol{a}} f_{\boldsymbol{a}}^* + f_{\boldsymbol{b}} f_{\boldsymbol{b}}^* + f_{\boldsymbol{c}} f_{\boldsymbol{c}}^* + f_{\boldsymbol{d}} f_{\boldsymbol{d}}^*)(x) = 2(2m+1),$$
$$(f_{\boldsymbol{v}} f_{\boldsymbol{v}}^* + f_{\boldsymbol{g}} f_{\boldsymbol{g}}^* + f_{\boldsymbol{h}} f_{\boldsymbol{h}}^* + f_{\boldsymbol{e}} f_{\boldsymbol{e}}^*)(x) = 2(2n+1).$$

Then there exist $\boldsymbol{q}, \boldsymbol{r}, \boldsymbol{s}, \boldsymbol{t} \in \mathcal{T}^{(2m+1)(2n+1)}$ such that

$$(f_{\boldsymbol{q}} f_{\boldsymbol{q}}^* + f_{\boldsymbol{r}} f_{\boldsymbol{r}}^* + f_{\boldsymbol{s}} f_{\boldsymbol{s}}^* + f_{\boldsymbol{t}} f_{\boldsymbol{t}}^*)(x) = 4(2m+1)(2n+1).$$

Theorem 3.1 follows from Theorem 3.3 by setting $\mathcal{T} = \{\pm 1\} \subset \mathbb{Z}$, $(\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}, \boldsymbol{d}) \in BS(m+1, m)$, and $(\boldsymbol{v}, \boldsymbol{g}, \boldsymbol{h}, \boldsymbol{e}) \in BS(n+1, n)$. The proof of Theorem 3.1 in [23] is by establishing the identity

$$\begin{aligned}
&(f_{\boldsymbol{q}} f_{\boldsymbol{q}}^* + f_{\boldsymbol{r}} f_{\boldsymbol{r}}^* + f_{\boldsymbol{s}} f_{\boldsymbol{s}}^* + f_{\boldsymbol{t}} f_{\boldsymbol{t}}^*)(x) \\
&= (f_{\boldsymbol{a}} f_{\boldsymbol{a}}^* + f_{\boldsymbol{b}} f_{\boldsymbol{b}}^* + f_{\boldsymbol{c}} f_{\boldsymbol{c}}^* + f_{\boldsymbol{d}} f_{\boldsymbol{d}}^*)(x^2)(f_{\boldsymbol{v}} f_{\boldsymbol{v}}^* + f_{\boldsymbol{g}} f_{\boldsymbol{g}}^* + f_{\boldsymbol{h}} f_{\boldsymbol{h}}^* + f_{\boldsymbol{e}} f_{\boldsymbol{e}}^*)(x^{2(2m+1)}),
\end{aligned} \tag{3.1}$$

after defining the sequences $\boldsymbol{q}$, $\boldsymbol{r}$, $\boldsymbol{s}$, $\boldsymbol{t}$ appropriately such that, in particular,

$$\begin{aligned}
f_{\boldsymbol{q}}(x) = {} & f_{\boldsymbol{a}}(x^2) f_{\boldsymbol{v}^*}(x^{2(2m+1)}) + x f_{\boldsymbol{c}}(x^2) f_{\boldsymbol{g}}(x^{2(2m+1)}) \\
& - x^{2(2m+1)} f_{\boldsymbol{b}^*}(x^2) f_{\boldsymbol{e}}(x^{2(2m+1)}) + x^{2(2m+1)+1} f_{\boldsymbol{d}}(x^2) f_{\boldsymbol{h}}(x^{2(2m+1)}).
\end{aligned}$$

A key to the proof is the Lagrange identity: given a, b, c, d, e, v, g, h in a commutative ring with identity and an involutive automorphism $*$, set

$$\begin{aligned} q &= av^* + cg - b^*e + dh, \\ r &= bv^* + dg^* + a^*e - ch^*, \\ s &= ag^* - cv - bh - d^*e, \\ t &= bg - dv + ah^* + c^*e. \end{aligned} \tag{3.2}$$

Then

$$qq^* + rr^* + ss^* + tt^* = (aa^* + bb^* + cc^* + dd^*)(ee^* + vv^* + gg^* + hh^*). \tag{3.3}$$

The derivation of (3.1) from (3.3) is not so immediate since one has to define a, b, c, d, v, g, h, e as

$$\begin{aligned} &f_{\boldsymbol{a}}(x^2), f_{\boldsymbol{b}}(x^2), x f_{\boldsymbol{c}}(x^2), x f_{\boldsymbol{d}}(x^2), \\ &x^{-n(2m+1)} f_{\boldsymbol{v}}(x^{2(2m+1)}), x^{-n(2m+1)} f_{\boldsymbol{g}}(x^{2(2m+1)}), \\ &x^{(1-n)(2m+1)} f_{\boldsymbol{h}}(x^{2(2m+1)}), x^{2m+(1-n)(2m+1)} f_{\boldsymbol{e}}(x^{2(2m+1)}), \end{aligned}$$

rather than

$$f_{\boldsymbol{a}}(x^2), f_{\boldsymbol{b}}(x^2), f_{\boldsymbol{c}}(x^2), f_{\boldsymbol{d}}(x^2), f_{\boldsymbol{v}}(x^{2(2m+1)}), f_{\boldsymbol{g}}(x^{2(2m+1)}), f_{\boldsymbol{h}}(x^{2(2m+1)}), f_{\boldsymbol{e}}(x^{2(2m+1)}),$$

respectively. We note that Đoković and Zhao [14] observed some connection between Yang's method and the octonion algebra.

In this section, we will give a more straightforward proof of Theorem 3.3. Our approach is by constructing a matrix Q from the eight sequences $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}, \boldsymbol{d}, \boldsymbol{v}, \boldsymbol{g}, \boldsymbol{h}, \boldsymbol{e}$ and produce a Laurent polynomial of single variable for each sequence and a Laurent polynomial of two variables for the matrix Q, such that

$$\psi_Q(x, y) = \psi_{\boldsymbol{a}}(x)\psi_{\boldsymbol{v}}(y) + \psi_{\boldsymbol{c}}(x)\psi_{\boldsymbol{g}}(y) + \psi_{\boldsymbol{b}}(x)\psi_{\boldsymbol{e}}(y) + \psi_{\boldsymbol{d}}(x)\psi_{\boldsymbol{h}}(y).$$

This gives an interpretation of the Lagrange identity in term of sequences and matrices, i.e.

there exist matrices Q, R, S, T such that

$$\begin{aligned}&(\psi_Q\psi_Q^* + \psi_R\psi_R^* + \psi_S\psi_S^* + \psi_T\psi_T^*)(x, y)\\&= (\psi_{\boldsymbol{a}}\psi_{\boldsymbol{a}}^* + \psi_{\boldsymbol{b}}\psi_{\boldsymbol{b}}^* + \psi_{\boldsymbol{c}}\psi_{\boldsymbol{c}}^* + \psi_{\boldsymbol{d}}\psi_{\boldsymbol{d}}^*)(x)(\psi_{\boldsymbol{e}}\psi_{\boldsymbol{e}}^* + \psi_{\boldsymbol{v}}\psi_{\boldsymbol{v}}^* + \psi_{\boldsymbol{g}}\psi_{\boldsymbol{g}}^* + \psi_{\boldsymbol{h}}\psi_{\boldsymbol{h}}^*)(y).\end{aligned}$$

Then (3.1) follows immediately by noticing $\psi_Q(x, x^{(2m+1)}) = \psi_{\boldsymbol{q}}(x)$ and $(\psi_{\boldsymbol{a}}\psi_{\boldsymbol{a}}^*)(x) = (f_{\boldsymbol{a}}f_{\boldsymbol{a}}^*)(x^2)$ for a sequence $\boldsymbol{a}$.

We fix a multiplicatively closed subset $\mathcal{T}$ of $\mathcal{R}$ satisfying $-1 \in \mathcal{T} = \mathcal{T}^*$. Also, we denote $\mathcal{T}_0 = \mathcal{T} \cup \{0\}$.

Definition 3.4. Let $\boldsymbol{a} = (a_0, \dots a_{l-1}) \in \mathcal{R}^l$. We define a Laurent polynomial $\psi_{\boldsymbol{a}}(x) \in \mathcal{R}[x^{\pm 1}]$ by

$$\psi_{\boldsymbol{a}}(x) = x^{1-l} f_{\boldsymbol{a}}(x^2).$$

For sequence $\boldsymbol{a} = (a_0, \dots, a_{l-1}) \in \mathcal{R}^l$ of length l we define $\boldsymbol{a}^* \in \mathcal{R}^l$ by $(a_{l-1}^*, \dots, a_0^*)$. It follows immediately that $\boldsymbol{a}^{**} = \boldsymbol{a}$ for every $\boldsymbol{a} \in \mathcal{R}^l$.

Lemma 3.5. *Let l be a positive integer and $\boldsymbol{a} \in \mathcal{R}^l$. Then*

$$\psi_{\boldsymbol{a}^*}(x) = \psi_{\boldsymbol{a}}^*(x).$$

Proof.

$$\begin{aligned}\psi_{\boldsymbol{a}^*}(x) &= x^{1-l} f_{\boldsymbol{a}^*}(x^2)\\&= x^{1-l+2(l-1)} f_{\boldsymbol{a}}^*(x^2) && \text{(by Lemma 2.7 (vi))}\\&= x^{l-1} f_{\boldsymbol{a}}^*(x^2)\\&= \psi_{\boldsymbol{a}}^*(x).\end{aligned}$$

□

Lemma 3.6. *For any sequence $\boldsymbol{a}$,*

$$f_{\boldsymbol{a}}(x^2)f^*_{\boldsymbol{a}}(x^2) = \psi_{\boldsymbol{a}}(x)\psi^*_{\boldsymbol{a}}(x).$$

Proof. Immediate from Definition 3.4. □

Corollary 3.7. *Let $\boldsymbol{a}_1, \dots, \boldsymbol{a}_n$ be arbitrary sequences with entries in $\mathcal{R}$. Then $\boldsymbol{a}_1, \dots, \boldsymbol{a}_n$ are complementary sequences with weight w if and only if*

$$\sum_{i=1}^{n}(\psi_{\boldsymbol{a}_i}\psi^*_{\boldsymbol{a}_i})(x) = w.$$

Proof. The proof is immediate from Lemma 2.10 and Lemma 3.6. □

Definition 3.8. Let $\boldsymbol{a} = (a_0, \dots, a_{l-1}) \in \mathcal{R}^l$. Define

$$\boldsymbol{a}/0 = (a_0, 0, a_1, \dots, 0, a_{l-1}) \in \mathcal{R}^{2l-1}, \quad 0/\boldsymbol{a} = (0, a_0, 0, \dots, a_{l-1}, 0) \in \mathcal{R}^{2l+1}.$$

Lemma 3.9. *Let $\boldsymbol{a} \in \mathcal{R}^l$ and let p be an odd positive integer. Then*

$$\psi_{\boldsymbol{a}/0}(x) = \psi_{0/\boldsymbol{a}}(x) = \psi_{\boldsymbol{a}}(x^2).$$

Proof. By Definition 3.4 and Definition 3.8, we have

$$\begin{aligned}
\psi_{\boldsymbol{a}/0}(x) &= x^{1-(2l-1)}f_{\boldsymbol{a}/0}(x^2) = x^{2-2l}f_{\boldsymbol{a}}(x^4) = \psi_{\boldsymbol{a}}(x^2),\\
\psi_{0/\boldsymbol{a}}(x) &= x^{1-(2l+1)}f_{0/\boldsymbol{a}}(x^2) = x^{-2l}x^2 f_{\boldsymbol{a}}(x^4) = \psi_{\boldsymbol{a}}(x^2).
\end{aligned}$$

□

Let $\mathcal{R}[x^{\pm 1}, y^{\pm 1}]$ be the ring of Laurent polynomials in two variables x, y. We define an involutive ring automorphism $* : \mathcal{R}[x^{\pm 1}, y^{\pm 1}] \to \mathcal{R}[x^{\pm 1}, y^{\pm 1}]$ by $x \mapsto x^{-1}$, $y \mapsto y^{-1}$ and $a \mapsto a^*$ for $a \in \mathcal{R}$. For the remainder of this section, we denote the row vectors of a matrix A by $\boldsymbol{a}_0, \boldsymbol{a}_1, \dots$ and those of a matrix B by $\boldsymbol{b}_0, \boldsymbol{b}_1, \dots$.

Definition 3.10. For $A \in \mathcal{R}^{n\times m}$, we define

$$\mathrm{seq}(A) = (\boldsymbol{a}_0 \mid \boldsymbol{a}_1 \mid \cdots \mid \boldsymbol{a}_{n-1}) \in \mathcal{R}^{mn},$$

where $\mid$ denotes concatenation, and

$$\psi_A(x,y) = \sum_{i=0}^{n-1} y^{2i+1-n}\psi_{\boldsymbol{a}_i}(x).$$

Lemma 3.11. *Let $A, B \in \mathcal{R}^{n\times m}$. Then*

$$\psi_{A\pm B}(x,y) = \psi_A(x,y) \pm \psi_B(x,y).$$

Proof. Notice that for every $\boldsymbol{a}, \boldsymbol{b} \in \mathcal{R}^l$, $\psi_{\boldsymbol{a}+\boldsymbol{b}}(x) = \psi_{\boldsymbol{a}}(x) + \psi_{\boldsymbol{b}}(x)$ by Definition 3.4 and Lemma 2.7 (i). Thus, the result follows. □

Note that we may regard $\mathcal{R}^n$ as $\mathcal{R}^{1\times n}$. So, for every $\boldsymbol{a} \in \mathcal{R}^n$, we have $\boldsymbol{a}^t \in \mathcal{R}^{n\times 1}$ where t denotes the transpose of a matrix.

Lemma 3.12. *Let $\boldsymbol{a} \in \mathcal{R}^n$ and $\boldsymbol{b} \in \mathcal{R}^m$. Then*

$$\psi_{\boldsymbol{a}^t\boldsymbol{b}}(x,y) = \psi_{\boldsymbol{a}}(y)\psi_{\boldsymbol{b}}(x).$$

Proof. Let $\boldsymbol{a} = (a_0, \ldots, a_{n-1})$. Then the ith row of the matrix $\boldsymbol{a}^t\boldsymbol{b}$ is $a_i\boldsymbol{b}$. Thus

$$\begin{aligned}
\psi_{\boldsymbol{a}^t\boldsymbol{b}}(x,y) &= \sum_{i=0}^{n-1} y^{2i+1-n}\psi_{a_i\boldsymbol{b}}(x) \\
&= \sum_{i=0}^{n-1} y^{2i+1-n} a_i \psi_{\boldsymbol{b}}(x) \\
&= \psi_{\boldsymbol{b}}(x) y^{1-n} \sum_{i=0}^{n-1} a_i y^{2i} \\
&= \psi_{\boldsymbol{b}}(x) y^{1-n} f_{\boldsymbol{a}}(y^2) \\
&= \psi_{\boldsymbol{b}}(x)\psi_{\boldsymbol{a}}(y).
\end{aligned}$$

□

Lemma 3.13. *Let*

$$\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}, \boldsymbol{d} \in \mathcal{R}^m, \quad \boldsymbol{v}, \boldsymbol{g}, \boldsymbol{h}, \boldsymbol{e} \in \mathcal{R}^n.$$

Set

$$\begin{aligned}
Q &= \boldsymbol{v}^{*t}\boldsymbol{a} + \boldsymbol{g}^t\boldsymbol{c} - \boldsymbol{e}^t\boldsymbol{b}^* + \boldsymbol{h}^t\boldsymbol{d}, \\
R &= \boldsymbol{v}^{*t}\boldsymbol{b} + \boldsymbol{g}^{*t}\boldsymbol{d} + \boldsymbol{e}^t\boldsymbol{a}^* - \boldsymbol{h}^{*t}\boldsymbol{c}, \\
S &= \boldsymbol{g}^{*t}\boldsymbol{a} - \boldsymbol{v}^t\boldsymbol{c} - \boldsymbol{h}^t\boldsymbol{b} - \boldsymbol{e}^t\boldsymbol{d}^*, \\
T &= \boldsymbol{g}^t\boldsymbol{b} - \boldsymbol{v}^t\boldsymbol{d} + \boldsymbol{h}^{*t}\boldsymbol{a} + \boldsymbol{e}^t\boldsymbol{c}^*.
\end{aligned}$$

Then

$$\begin{aligned}
&(\psi_Q\psi_Q^* + \psi_R\psi_R^* + \psi_S\psi_S^* + \psi_T\psi_T^*)(x, y) \\
&= (\psi_{\boldsymbol{a}}\psi_{\boldsymbol{a}}^* + \psi_{\boldsymbol{b}}\psi_{\boldsymbol{b}}^* + \psi_{\boldsymbol{c}}\psi_{\boldsymbol{c}}^* + \psi_{\boldsymbol{d}}\psi_{\boldsymbol{d}}^*)(x)(\psi_{\boldsymbol{e}}\psi_{\boldsymbol{e}}^* + \psi_{\boldsymbol{v}}\psi_{\boldsymbol{v}}^* + \psi_{\boldsymbol{g}}\psi_{\boldsymbol{g}}^* + \psi_{\boldsymbol{h}}\psi_{\boldsymbol{h}}^*)(y).
\end{aligned}$$

Proof. By Lemma 3.11, we have

$$\begin{aligned}
\psi_Q(x, y) &= (\psi_{\boldsymbol{v}^{*t}\boldsymbol{a}} + \psi_{\boldsymbol{g}^t\boldsymbol{c}} - \psi_{\boldsymbol{e}^t\boldsymbol{b}^*} + \psi_{\boldsymbol{h}^t\boldsymbol{d}})(x, y), \\
\psi_R(x, y) &= (\psi_{\boldsymbol{v}^{*t}\boldsymbol{b}} + \psi_{\boldsymbol{g}^{*t}\boldsymbol{d}} + \psi_{\boldsymbol{e}^t\boldsymbol{a}^*} - \psi_{\boldsymbol{h}^{*t}\boldsymbol{c}})(x, y), \\
\psi_S(x, y) &= (\psi_{\boldsymbol{g}^{*t}\boldsymbol{a}} - \psi_{\boldsymbol{v}^t\boldsymbol{c}} - \psi_{\boldsymbol{h}^t\boldsymbol{b}} - \psi_{\boldsymbol{e}^t\boldsymbol{d}^*})(x, y), \\
\psi_T(x, y) &= (\psi_{\boldsymbol{g}^t\boldsymbol{b}} - \psi_{\boldsymbol{v}^t\boldsymbol{d}} + \psi_{\boldsymbol{h}^{*t}\boldsymbol{a}} + \psi_{\boldsymbol{e}^t\boldsymbol{c}^*})(x, y).
\end{aligned}$$

Also, by Lemma 3.5 and Lemma 3.12, we have

$$\begin{aligned}
\psi_Q(x, y) &= \psi_{\boldsymbol{a}}(x)\psi_{\boldsymbol{v}}^*(y) + \psi_{\boldsymbol{c}}(x)\psi_{\boldsymbol{g}}(y) - \psi_{\boldsymbol{b}}^*(x)\psi_{\boldsymbol{e}}(y) + \psi_{\boldsymbol{d}}(x)\psi_{\boldsymbol{h}}(y), \\
\psi_R(x, y) &= \psi_{\boldsymbol{b}}(x)\psi_{\boldsymbol{v}}^*(y) + \psi_{\boldsymbol{d}}(x)\psi_{\boldsymbol{g}}^*(y) + \psi_{\boldsymbol{a}}(x)\psi_{\boldsymbol{e}}^*(y) - \psi_{\boldsymbol{c}}(x)\psi_{\boldsymbol{h}}^*(y), \\
\psi_S(x, y) &= \psi_{\boldsymbol{a}}(x)\psi_{\boldsymbol{g}}^*(y) - \psi_{\boldsymbol{c}}(x)\psi_{\boldsymbol{v}}(y) - \psi_{\boldsymbol{b}}(x)\psi_{\boldsymbol{h}}(y) - \psi_{\boldsymbol{d}}^*(x)\psi_{\boldsymbol{e}}(y), \\
\psi_T(x, y) &= \psi_{\boldsymbol{b}}(x)\psi_{\boldsymbol{g}}(y) - \psi_{\boldsymbol{d}}(x)\psi_{\boldsymbol{v}}(y) + \psi_{\boldsymbol{a}}(x)\psi_{\boldsymbol{h}}^*(y) + \psi_{\boldsymbol{c}}^*(x)\psi_{\boldsymbol{e}}(y).
\end{aligned}$$

Thus, by applying the Lagrange identity (3.3), the result follows. □

Lemma 3.14. *Let p and p' be odd positive integers,*

$$\boldsymbol{a}, \boldsymbol{b} \in \mathcal{T}^{m+p},$$
$$\boldsymbol{c}, \boldsymbol{d} \in \mathcal{T}^{m},$$
$$\boldsymbol{v}, \boldsymbol{g} \in \mathcal{T}^{n+p'},$$
$$\boldsymbol{h}, \boldsymbol{e} \in \mathcal{T}^{n}.$$

Set

$$\boldsymbol{a}' = \boldsymbol{a}/0, \quad \boldsymbol{b}' = \boldsymbol{b}/0, \quad \boldsymbol{c}' = 0/\widetilde{\boldsymbol{c}}, \quad \boldsymbol{d}' = 0/\widetilde{\boldsymbol{d}},$$
$$\boldsymbol{v}' = \boldsymbol{v}/0, \quad \boldsymbol{g}' = \boldsymbol{g}/0, \quad \boldsymbol{h}' = 0/\widetilde{\boldsymbol{h}}, \quad \boldsymbol{e}' = 0/\widetilde{\boldsymbol{e}},$$

where

$$\widetilde{\boldsymbol{c}} = (0_{(p-1)/2}, \boldsymbol{c}, 0_{(p-1)/2}),$$
$$\widetilde{\boldsymbol{d}} = (0_{(p-1)/2}, \boldsymbol{d}, 0_{(p-1)/2}),$$
$$\widetilde{\boldsymbol{h}} = (0_{(p'-1)/2}, \boldsymbol{h}, 0_{(p'-1)/2}),$$
$$\widetilde{\boldsymbol{e}} = (0_{(p'-1)/2}, \boldsymbol{e}, 0_{(p'-1)/2}).$$

Write

$$Q = \boldsymbol{v}'^{*t}\boldsymbol{a}' + \boldsymbol{g}'^{t}\boldsymbol{c}' - \boldsymbol{e}^{t}\boldsymbol{b}'^{*} + \boldsymbol{h}'^{t}\boldsymbol{d}', \tag{3.4}$$
$$R = \boldsymbol{v}'^{*t}\boldsymbol{b}' + \boldsymbol{g}'^{*t}\boldsymbol{d}' + \boldsymbol{e}'^{t}\boldsymbol{a}'^{*} - \boldsymbol{h}'^{*t}\boldsymbol{c}', \tag{3.5}$$
$$S = \boldsymbol{g}'^{*t}\boldsymbol{a}' - \boldsymbol{v}'^{t}\boldsymbol{c}' - \boldsymbol{h}'^{t}\boldsymbol{b}' - \boldsymbol{e}'^{t}\boldsymbol{d}'^{*}, \tag{3.6}$$
$$T = \boldsymbol{g}'^{t}\boldsymbol{b}' - \boldsymbol{v}'^{t}\boldsymbol{d}' + \boldsymbol{h}'^{*t}\boldsymbol{a}' + \boldsymbol{e}'^{t}\boldsymbol{c}'^{*}. \tag{3.7}$$

Then

$$\begin{aligned}&(\psi_Q\psi_Q^* + \psi_R\psi_R^* + \psi_S\psi_S^* + \psi_T\psi_T^*)(x,y)\\&= (\psi_{\boldsymbol{a}}\psi_{\boldsymbol{a}}^* + \psi_{\boldsymbol{b}}\psi_{\boldsymbol{b}}^* + \psi_{\boldsymbol{c}}\psi_{\boldsymbol{c}}^* + \psi_{\boldsymbol{d}}\psi_{\boldsymbol{d}}^*)(x^2)(\psi_{\boldsymbol{e}}\psi_{\boldsymbol{e}}^* + \psi_{\boldsymbol{v}}\psi_{\boldsymbol{v}}^* + \psi_{\boldsymbol{g}}\psi_{\boldsymbol{g}}^* + \psi_{\boldsymbol{h}}\psi_{\boldsymbol{h}}^*)(y^2).\end{aligned}$$

Moreover, $Q, R, S, T \in \mathcal{T}_0^{(2(n+p')-1)\times(2(m+p)-1)}$ *if* $p, p' > 1$, *and* $Q, R, S, T \in \mathcal{T}^{(2n+1)\times(2m+1)}$ *if* $p = p' = 1$.

Proof. Notice that $\boldsymbol{a}', \boldsymbol{b}', \boldsymbol{c}', \boldsymbol{d}' \in \mathcal{T}_0^{2(m+p)-1}$ and $\boldsymbol{v}', \boldsymbol{g}', \boldsymbol{h}', \boldsymbol{e}' \in \mathcal{T}_0^{2(n+p')-1}$. It can be checked that $\psi_{\widetilde{\boldsymbol{s}}}(x) = \psi_{\boldsymbol{s}}(x)$ for every $\boldsymbol{s} \in \{\boldsymbol{c}, \boldsymbol{d}, \boldsymbol{h}, \boldsymbol{e}\}$. Thus, by Lemma 3.9 and Lemma 3.13, we have

$$\begin{aligned}&(\psi_Q\psi_Q^* + \psi_R\psi_R^* + \psi_S\psi_S^* + \psi_T\psi_T^*)(x,y)\\&= (\psi_{\boldsymbol{a}'}\psi_{\boldsymbol{a}'}^* + \psi_{\boldsymbol{b}'}\psi_{\boldsymbol{b}'}^* + \psi_{\boldsymbol{c}'}\psi_{\boldsymbol{c}'}^* + \psi_{\boldsymbol{d}'}\psi_{\boldsymbol{d}'}^*)(x)(\psi_{\boldsymbol{e}'}\psi_{\boldsymbol{e}'}^* + \psi_{\boldsymbol{v}'}\psi_{\boldsymbol{v}'}^* + \psi_{\boldsymbol{g}'}\psi_{\boldsymbol{g}'}^* + \psi_{\boldsymbol{h}'}\psi_{\boldsymbol{h}'}^*)(y)\\&= (\psi_{\boldsymbol{a}}\psi_{\boldsymbol{a}}^* + \psi_{\boldsymbol{b}}\psi_{\boldsymbol{b}}^* + \psi_{\boldsymbol{c}}\psi_{\boldsymbol{c}}^* + \psi_{\boldsymbol{d}}\psi_{\boldsymbol{d}}^*)(x^2)(\psi_{\boldsymbol{e}}\psi_{\boldsymbol{e}}^* + \psi_{\boldsymbol{v}}\psi_{\boldsymbol{v}}^* + \psi_{\boldsymbol{g}}\psi_{\boldsymbol{g}}^* + \psi_{\boldsymbol{h}}\psi_{\boldsymbol{h}}^*)(y^2).\end{aligned}$$

The last statement is clear by Definition 3.8. □

Lemma 3.15. *If* $A \in \mathcal{R}^{n\times m}$, *then*

$$\psi_{\mathrm{seq}(A)}(x) = \psi_A(x, x^m).$$

Proof.

$$\begin{aligned}\psi_{\mathrm{seq}(A)}(x) &= x^{1-nm} f_{\mathrm{seq}(A)}(x^2)\\&= x^{1-nm}\sum_{i=0}^{n-1} x^{2im} f_{\boldsymbol{a}_i}(x^2)\\&= x^{1-nm}\sum_{i=0}^{n-1} x^{2im+m-1}\psi_{\boldsymbol{a}_i}(x)\\&= \sum_{i=0}^{n-1} x^{m(2i+1-n)}\psi_{\boldsymbol{a}_i}(x)\\&= \psi_A(x, x^m).\end{aligned}$$

□

The assumption that p and p' be odd in Lemma 3.13 is insignificant, if we allow 0 in sequences. Indeed, if $a \in \mathcal{R}^{m+p}$ with p even, then $a' = (a \mid 0) \in \mathcal{R}^{m+p+1}$ and $\psi_{\boldsymbol{a}}\psi_{\boldsymbol{a}}^* = \psi_{\boldsymbol{a}'}\psi_{\boldsymbol{a}'}^*$.

Theorem 3.16. *Let m, n be positive integers, p, p' be odd positive integers, and*

$$\begin{aligned} &\boldsymbol{a}, \boldsymbol{b} \in \mathcal{T}^{m+p}, \\ &\boldsymbol{c}, \boldsymbol{d} \in \mathcal{T}^{m}, \\ &\boldsymbol{v}, \boldsymbol{g} \in \mathcal{T}^{n+p'}, \\ &\boldsymbol{h}, \boldsymbol{e} \in \mathcal{T}^{n} \end{aligned}$$

satisfy

$$\begin{aligned} (f_{\boldsymbol{a}}f_{\boldsymbol{a}}^* + f_{\boldsymbol{b}}f_{\boldsymbol{b}}^* + f_{\boldsymbol{c}}f_{\boldsymbol{c}}^* + f_{\boldsymbol{d}}f_{\boldsymbol{d}}^*)(x) &= 2(2m+p), \\ (f_{\boldsymbol{v}}f_{\boldsymbol{v}}^* + f_{\boldsymbol{g}}f_{\boldsymbol{g}}^* + f_{\boldsymbol{h}}f_{\boldsymbol{h}}^* + f_{\boldsymbol{e}}f_{\boldsymbol{e}}^*)(x) &= 2(2n+p'). \end{aligned}$$

Then there exist $\boldsymbol{q}, \boldsymbol{r}, \boldsymbol{s}, \boldsymbol{t} \in \mathcal{T}_0^{(2(n+p')-1)(2(m+p)-1)}$ such that

$$(f_{\boldsymbol{q}}f_{\boldsymbol{q}}^* + f_{\boldsymbol{r}}f_{\boldsymbol{r}}^* + f_{\boldsymbol{s}}f_{\boldsymbol{s}}^* + f_{\boldsymbol{t}}f_{\boldsymbol{t}}^*)(x) = 4(2m+p)(2n+p').$$

Moreover, $\boldsymbol{q}, \boldsymbol{r}, \boldsymbol{s}, \boldsymbol{t} \in \mathcal{T}^{(2n+1)(2m+1)}$ if $p = p' = 1$.

Proof. Define Q, R, S, T as in (3.4), (3.5), (3.6), (3.7) in Lemma 3.14, respectively. Write

$$\boldsymbol{q} = \operatorname{seq}(Q), \quad \boldsymbol{r} = \operatorname{seq}(R), \quad \boldsymbol{s} = \operatorname{seq}(S), \quad \boldsymbol{t} = \operatorname{seq}(T).$$

By Lemma 3.14, $Q, R, S, T \in \mathcal{T}^{(2n+1)\times(2m+1)}$ if $p = p' = 1$. Thus, the last statement holds immediately by Definition 3.10. Applying Lemma 3.6, Lemma 3.14 and Lemma 3.15, we have

$$\begin{aligned}
&(f_{\boldsymbol{q}}f_{\boldsymbol{q}}^* + f_{\boldsymbol{r}}f_{\boldsymbol{r}}^* + f_{\boldsymbol{s}}f_{\boldsymbol{s}}^* + f_{\boldsymbol{t}}f_{\boldsymbol{t}}^*)(x^2) \\
&= (\psi_{\boldsymbol{q}}\psi_{\boldsymbol{q}}^* + \psi_{\boldsymbol{r}}\psi_{\boldsymbol{r}}^* + \psi_{\boldsymbol{s}}\psi_{\boldsymbol{s}}^* + \psi_{\boldsymbol{t}}\psi_{\boldsymbol{t}}^*)(x) \\
&= (\psi_Q\psi_Q^* + \psi_R\psi_R^* + \psi_S\psi_S^* + \psi_T\psi_T^*)(x, x^{2(m+p)-1}) \\
&= (\psi_{\boldsymbol{a}}\psi_{\boldsymbol{a}}^* + \psi_{\boldsymbol{b}}\psi_{\boldsymbol{b}}^* + \psi_{\boldsymbol{c}}\psi_{\boldsymbol{c}}^* + \psi_{\boldsymbol{d}}\psi_{\boldsymbol{d}}^*)(x^2)(\psi_{\boldsymbol{e}}\psi_{\boldsymbol{e}}^* + \psi_{\boldsymbol{v}}\psi_{\boldsymbol{v}}^* + \psi_{\boldsymbol{g}}\psi_{\boldsymbol{g}}^* + \psi_{\boldsymbol{h}}\psi_{\boldsymbol{h}}^*)(x^{2(2(m+p)-1)}) \\
&= (f_{\boldsymbol{a}}f_{\boldsymbol{a}}^* + f_{\boldsymbol{b}}f_{\boldsymbol{b}}^* + f_{\boldsymbol{c}}f_{\boldsymbol{c}}^* + f_{\boldsymbol{d}}f_{\boldsymbol{d}}^*)(x^2)(f_{\boldsymbol{e}}f_{\boldsymbol{e}}^* + f_{\boldsymbol{v}}f_{\boldsymbol{v}}^* + f_{\boldsymbol{g}}f_{\boldsymbol{g}}^* + f_{\boldsymbol{h}}f_{\boldsymbol{h}}^*)(x^{2(2(m+p)-1))}) \\
&= 4(2m+p)(2n+p').
\end{aligned}$$

Hence the proof is complete. □

Finally, we see that Theorem 3.3 is a special case of Theorem 3.16 with $p = p' = 1$. We already showed that Theorem 3.1 follows from Theorem 3.3. Hence, this method also can be used for proving Theorem 3.1.

3.2 Some Constructions of Paired Ternary Sequences

Throughout this section, we will use the same notions from Section 3.1. Also, we introduce another version of Yang multiplication theorem: if $BS(m+p,m)$ is nonempty, then $PT(7(2m+p))$ is nonempty. Also, he showed that if $BS(m+p,m)$ and $GS(s)$ are nonempty, then $PT((2m+p)(2s+1))$ is also nonempty. More specifically,

Theorem 3.17 (Yang [22])**.** *Let* $(\boldsymbol{a},\boldsymbol{b},\boldsymbol{c},\boldsymbol{d}) \in BS(m+p,m)$ *and* $(\boldsymbol{v};\boldsymbol{g}) \in GS(s)$. *Then there exist* $\boldsymbol{q},\boldsymbol{r},\boldsymbol{s},\boldsymbol{t} \in \{0,\pm 1\}^{(2s+1)(2m+p)}$ *such that* $(\boldsymbol{q},\boldsymbol{r},\boldsymbol{s},\boldsymbol{t}) \in PT((2s+1)(2m+p))$.

It is already known from Lemma 2.25 that $PT(l)$ is equivalent to $TS(l)$ and the existence of T-sequences implies to the existence of Hadamard matrices (see Theorem 2.32). Therefore, we interested to investigate some constructions by Yang that lead to the existence of some paired ternary sequences.

The proof of Theorem 3.17 is by establishing the identity

$$(f_{\boldsymbol{q}}f_{\boldsymbol{q}}^*+f_{\boldsymbol{r}}f_{\boldsymbol{r}}^*+f_{\boldsymbol{s}}f_{\boldsymbol{s}}^*+f_{\boldsymbol{t}}f_{\boldsymbol{t}}^*)(x)=(f_{\boldsymbol{a}}f_{\boldsymbol{a}}^*+f_{\boldsymbol{b}}f_{\boldsymbol{b}}^*+f_{\boldsymbol{c}}f_{\boldsymbol{c}}^*+f_{\boldsymbol{d}}f_{\boldsymbol{d}}^*)(x)(f_{\boldsymbol{v}}f_{\boldsymbol{v}}^*+f_{\boldsymbol{g}}f_{\boldsymbol{g}}^*+f_{\boldsymbol{e}}f_{\boldsymbol{e}}^*)(x^{2(2m+p)}) \tag{3.8}$$

after defining the sequences $\boldsymbol{q}, \boldsymbol{r}, \boldsymbol{s}, \boldsymbol{t}$ appropriately. As written in the original proof by Yang, we need to use the following identity: let $a, b, c, d, e, v, g, w \in \mathcal{R}$ and $ww^* = 1$. Set

$$\begin{aligned} q &= av^* + cg - b^*e, \\ r &= bv^* + dg^* + a^*e, \\ s &= w(ag^* - cv - d^*e), \\ t &= w(bg - dv + c^*e). \end{aligned} \tag{3.9}$$

Then

$$qq^* + rr^* + ss^* + tt^* = (aa^* + bb^* + cc^* + dd^*)(vv^* + gg^* + ee^*). \tag{3.10}$$

We can see easily that his identity is just a derivation of the Lagrange identity (see (1.1)) with $h = 0$. Also, the derivation of (3.8) from (3.10) is not so immediate since one has to define a, b, c, d, v, g as

$$\begin{gathered} f_{\boldsymbol{a}}(x), f_{\boldsymbol{b}}(x), x^{m+p}f_{\boldsymbol{c}}(x), x^{m+p}f_{\boldsymbol{d}}(x), \\ x^{(1-2s)(2m+p)}f_{\boldsymbol{v}}(x^{2(2m+p)}), x^{(1-2s)(2m+p)}f_{\boldsymbol{g}}(x^{2(2m+p)}), \end{gathered}$$

rather than

$$f_{\boldsymbol{a}}(x), f_{\boldsymbol{b}}(x), f_{\boldsymbol{c}}(x), f_{\boldsymbol{d}}(x), f_{\boldsymbol{v}}(x^{2(2m+p)}), f_{\boldsymbol{g}}(x^{2(2m+p)}).$$

In this section, we give some results to approach a generalization of Theorem 3.17.

Lemma 3.18. *Let k, k', l be positive integers and $\boldsymbol{a} \in \mathcal{R}^l$. Then*

$$\psi_{(0_k,\boldsymbol{a},0_{k'})}(x) = x^{k-k'}\psi_{\boldsymbol{a}}(x).$$

Proof.

$$\begin{aligned}\psi_{(0_k,\boldsymbol{a},0_{k'})}(x) &= x^{1-l-k-k'} f_{(0_k,\boldsymbol{a},0_{k'})}(x^2)\\ &= x^{1-l-k-k'} f_{(0_k,\boldsymbol{a})}(x^2)\\ &= x^{1-l+k-k'} f_{\boldsymbol{a}}(x^2)\\ &= x^{k-k'} \cdot x^{1-l} f_{\boldsymbol{a}}(x^2)\\ &= x^{k-k'} \psi_{\boldsymbol{a}}(x).\end{aligned}$$

□

Lemma 3.19. *Let* $\boldsymbol{a}_j, \boldsymbol{b}_j, \boldsymbol{c}_j, \boldsymbol{d}_j \in R^m$ *for* $j = 1, 2$ *and* $\boldsymbol{v}_i, \boldsymbol{g}_i, \boldsymbol{h}_i, \boldsymbol{e}_i \in R^n$ *for every* $i = 1, 2, 3, 4$. *Set*

$$\begin{aligned}Q &= \boldsymbol{v}_1^t \boldsymbol{a}_1 + \boldsymbol{g}_1^t \boldsymbol{c}_1 + \boldsymbol{h}_1^t \boldsymbol{d}_1 - \boldsymbol{e}_1^t \boldsymbol{b}_2,\\ R &= \boldsymbol{v}_2^t \boldsymbol{b}_1 + \boldsymbol{g}_2^t \boldsymbol{d}_1 - \boldsymbol{h}_2^t \boldsymbol{c}_1 + \boldsymbol{e}_2^t \boldsymbol{a}_2,\\ S &= \boldsymbol{g}_3^t \boldsymbol{a}_1 - \boldsymbol{v}_3^t \boldsymbol{c}_1 - \boldsymbol{h}_3^t \boldsymbol{b}_1 - \boldsymbol{e}_3^t \boldsymbol{d}_2,\\ T &= \boldsymbol{g}_4^t \boldsymbol{b}_1 - \boldsymbol{v}_4^t \boldsymbol{d}_1 + \boldsymbol{h}_4^t \boldsymbol{a}_1 + \boldsymbol{e}_4^t \boldsymbol{c}_2.\end{aligned}$$

If

(i) $\mathrm{supp}(x_i) = \mathrm{supp}(y_i)$ *for* $(x, y) \in \{(\boldsymbol{a}, \boldsymbol{b}), (\boldsymbol{c}, \boldsymbol{d})\}$ *and* $i = 1, 2$,

(ii) $\mathrm{supp}(\boldsymbol{a}_i) \cap \mathrm{supp}(\boldsymbol{c}_i) = \emptyset$ *for* $i = 1, 2$,

(iii) $\mathrm{supp}(x_i) = \mathrm{supp}(x_j)$ *for* $x \in \{\boldsymbol{v}, \boldsymbol{g}, \boldsymbol{h}, \boldsymbol{e}\}$ *and* $(i, j) \in \{(1, 2), (3, 4)\}$,

(iv) $\mathrm{supp}(z_i) \cap \mathrm{supp}(\boldsymbol{e}_j) = \emptyset$ *for* $z \in \{\boldsymbol{v}, \boldsymbol{g}, \boldsymbol{h}\}$ *and* $i, j = 1, 3$,

(v) $\mathrm{supp}(\boldsymbol{v}_1) \cap \mathrm{supp}(x_3) = \emptyset$ *and* $\mathrm{supp}(\boldsymbol{v}_3) \cap \mathrm{supp}(x_1) = \emptyset$ *for every* $x \in \{\boldsymbol{g}, \boldsymbol{h}\}$,

(vi) $\mathrm{supp}(\boldsymbol{g}_i) \cap \mathrm{supp}(\boldsymbol{h}_i) = \emptyset$ *for* $i = 1, 3$,

then the pairs (Q, R) *and* (S, T) *are each conjoint and the pair* (Q, S) *is disjoint.*

Proof. By (iv), we have

$$\operatorname{supp}(Q) = \operatorname{supp}(\boldsymbol{v}_1^t\boldsymbol{a}_1 + \boldsymbol{g}_1^t\boldsymbol{c}_1 + \boldsymbol{h}_1^t\boldsymbol{d}_1) \cup \operatorname{supp}(\boldsymbol{e}_1^t\boldsymbol{b}_2), \tag{3.11}$$

$$\operatorname{supp}(R) = \operatorname{supp}(\boldsymbol{v}_2^t\boldsymbol{b}_1 + \boldsymbol{g}_2^t\boldsymbol{d}_1 - \boldsymbol{h}_2^t\boldsymbol{c}_1) \cup \operatorname{supp}(\boldsymbol{e}_2^t\boldsymbol{a}_2), \tag{3.12}$$

$$\operatorname{supp}(S) = \operatorname{supp}(\boldsymbol{g}_3^t\boldsymbol{a}_1 - \boldsymbol{v}_3^t\boldsymbol{c}_1 - \boldsymbol{h}_3 t\boldsymbol{b}_1) \cup \operatorname{supp}(\boldsymbol{e}_3^t\boldsymbol{d}_2), \tag{3.13}$$

$$\operatorname{supp}(T) = \operatorname{supp}(\boldsymbol{g}_4^t\boldsymbol{b}_1 - \boldsymbol{v}_4^t\boldsymbol{d}_1 + \boldsymbol{h}_4^t\boldsymbol{a}_1) \cup \operatorname{supp}(\boldsymbol{e}_4^t\boldsymbol{c}_2). \tag{3.14}$$

Also,

$$\operatorname{supp}(\boldsymbol{v}_1^t\boldsymbol{a}_1) \cap \operatorname{supp}(\boldsymbol{g}_1^t\boldsymbol{c}_1) = \emptyset \qquad \text{(by (ii))}, \tag{3.15}$$

$$\operatorname{supp}(\boldsymbol{v}_1^t\boldsymbol{a}_1) \cap \operatorname{supp}(\boldsymbol{h}_1^t\boldsymbol{d}_1) = \emptyset \qquad \text{(by (i) and (ii))}, \tag{3.16}$$

$$\operatorname{supp}(\boldsymbol{g}_1^t\boldsymbol{c}_1) \cap \operatorname{supp}(\boldsymbol{h}_1^t\boldsymbol{d}_1) = \emptyset \qquad \text{(by (vi))}. \tag{3.17}$$

Hence

$$\operatorname{supp}(Q) = \operatorname{supp}(\boldsymbol{v}_1^t\boldsymbol{a}_1) \cup \operatorname{supp}(\boldsymbol{g}_1^t\boldsymbol{c}_1) \cup \operatorname{supp}(\boldsymbol{h}_1^t\boldsymbol{d}_1) \cup \operatorname{supp}(\boldsymbol{e}_1^t\boldsymbol{b}_2). \tag{3.18}$$

We have

$$\operatorname{supp}(\boldsymbol{v}_2^t\boldsymbol{b}_1) \cap \operatorname{supp}(\boldsymbol{g}_2^t\boldsymbol{d}_1) = \emptyset \qquad \text{(by (i),(ii))},$$

$$\operatorname{supp}(\boldsymbol{v}_2^t\boldsymbol{b}_1) \cap \operatorname{supp}(\boldsymbol{h}_2^t\boldsymbol{c}_1) = \emptyset \qquad \text{(by (i), (ii))},$$

$$\operatorname{supp}(\boldsymbol{g}_2^t\boldsymbol{d}_1) \cap \operatorname{supp}(\boldsymbol{h}_2^t\boldsymbol{c}_1) = \emptyset \qquad \text{(by (iii), (vi))}.$$

Therefore, by (ii), (iii), (3.11) and (3.12), we obtain

$$\begin{aligned}\operatorname{supp}(R) &= \operatorname{supp}(\boldsymbol{v}_2^t\boldsymbol{b}_1) \cup \operatorname{supp}(\boldsymbol{g}_2^t\boldsymbol{d}_1) \cup \operatorname{supp}(\boldsymbol{h}_2^t\boldsymbol{c}_1) \cup \operatorname{supp}(\boldsymbol{e}_2^t\boldsymbol{a}_2)\\ &= \operatorname{supp}(\boldsymbol{v}_1^t\boldsymbol{a}_1) \cup \operatorname{supp}(\boldsymbol{g}_1^t\boldsymbol{c}_1) \cup \operatorname{supp}(\boldsymbol{h}_1^t\boldsymbol{d}_1) \cup \operatorname{supp}(\boldsymbol{e}_1^t\boldsymbol{b}_2)\\ &= \operatorname{supp}(Q),\end{aligned}$$

and hence, the pair Q, R is conjoint. Note that since

$$\operatorname{supp}(\boldsymbol{g}_3^t\boldsymbol{a}_1)\cap\operatorname{supp}(\boldsymbol{v}_3^t\boldsymbol{c}_1)=\emptyset \qquad \text{(by (ii))}, \tag{3.19}$$

$$\operatorname{supp}(\boldsymbol{g}_3^t\boldsymbol{a}_1)\cap\operatorname{supp}(\boldsymbol{h}_3^t\boldsymbol{b}_1)=\emptyset \qquad \text{(by (vi))}, \tag{3.20}$$

$$\operatorname{supp}(\boldsymbol{v}_3^t\boldsymbol{c}_1)\cap\operatorname{supp}(\boldsymbol{h}_3^t\boldsymbol{b}_1)=\emptyset \qquad \text{(by (i), (ii))}, \tag{3.21}$$

by (3.13), we have

$$\operatorname{supp}(S)=\operatorname{supp}(\boldsymbol{g}_3^t\boldsymbol{a}_1)\cup\operatorname{supp}(\boldsymbol{v}_3^t\boldsymbol{c}_1)\cup\operatorname{supp}(\boldsymbol{h}_3^t\boldsymbol{b}_1)\cup\operatorname{supp}(\boldsymbol{e}_3^t\boldsymbol{d}_2). \tag{3.22}$$

Also, by (i), (ii) and (iii), we have

$$\operatorname{supp}(\boldsymbol{g}_4^t\boldsymbol{b}_1)\cap\operatorname{supp}(\boldsymbol{v}_4^t\boldsymbol{d}_1)=\emptyset \qquad \text{(by (3.19))},$$

$$\operatorname{supp}(\boldsymbol{g}_4^t\boldsymbol{b}_1)\cap\operatorname{supp}(\boldsymbol{h}_4^t\boldsymbol{a}_1)=\emptyset \qquad \text{(by (3.20))},$$

$$\operatorname{supp}(\boldsymbol{v}_4^t\boldsymbol{d}_1)\cap\operatorname{supp}(\boldsymbol{h}_4^t\boldsymbol{a}_1)=\emptyset \qquad \text{(by (3.21))}.$$

Therefore, by (i), (iii), (3.13) and (3.14),

$$\begin{aligned}\operatorname{supp}(T)&=\operatorname{supp}(\boldsymbol{g}_4^t\boldsymbol{b}_1)\cup\operatorname{supp}(\boldsymbol{v}_4^t\boldsymbol{d}_1)\cup\operatorname{supp}(\boldsymbol{h}_4^t\boldsymbol{a}_1)\cup\operatorname{supp}(\boldsymbol{e}_4^t\boldsymbol{c}_2)\\&=\operatorname{supp}(\boldsymbol{g}_3^t\boldsymbol{a}_1)\cup\operatorname{supp}(\boldsymbol{v}_3^t\boldsymbol{c}_1)\cup\operatorname{supp}(\boldsymbol{h}_3^t\boldsymbol{b}_1)\cup\operatorname{supp}(\boldsymbol{e}_3^t\boldsymbol{d}_2)\\&=\operatorname{supp}(S).\end{aligned}$$

Therefore, the pair S and T is also conjoint. Finally, by (i), (ii), (iv), (v)

$$\operatorname{supp}(A)\cap\operatorname{supp}(B)=\emptyset$$

for every $A\in\{\boldsymbol{v}_1^t\boldsymbol{a}_1,\boldsymbol{g}_1^t\boldsymbol{c}_1,\boldsymbol{h}_1^t\boldsymbol{d}_1,\boldsymbol{e}_1^t\boldsymbol{b}_2\}$ and $B\in\{\boldsymbol{g}_3^t\boldsymbol{a}_1,\boldsymbol{v}_3^t\boldsymbol{c}_1,\boldsymbol{h}_3^t\boldsymbol{b}_1,\boldsymbol{e}_3^t\boldsymbol{d}_2\}$. Hence, by (3.18) and (3.22),

$$\begin{aligned}\operatorname{supp}(Q) \cap \operatorname{supp}(S) &= (\operatorname{supp}(\boldsymbol{v}_1^t \boldsymbol{a}_1) \cup \operatorname{supp}(\boldsymbol{g}_1^t \boldsymbol{c}_1) \cup \operatorname{supp}(\boldsymbol{h}_1^t \boldsymbol{d}_1) \cup \operatorname{supp}(\boldsymbol{e}_1^t \boldsymbol{b}_2)) \\ &\quad \cap (\operatorname{supp}(\boldsymbol{g}_3^t \boldsymbol{a}_1) \cup \operatorname{supp}(\boldsymbol{v}_3^t \boldsymbol{c}_1) \cup \operatorname{supp}(\boldsymbol{h}_3^t \boldsymbol{b}_1) \cup \operatorname{supp}(\boldsymbol{e}_3^t \boldsymbol{d}_2)) \\ &= \emptyset.\end{aligned}$$

So, the pair Q, S is disjoint and the proof is complete. □

Theorem 3.20. *Let m, n, p be positive integers and*

$$\begin{aligned}\boldsymbol{a}, \boldsymbol{b} &\in \mathcal{T}^{m+p}, \\ \boldsymbol{c}, \boldsymbol{d} &\in \mathcal{T}^{m}, \\ \boldsymbol{v}, \boldsymbol{g}, \boldsymbol{h}, \boldsymbol{e} &\in \mathcal{T}_0^{n}, \\ \boldsymbol{e} &\in \mathcal{T}_0^{2n+1}.\end{aligned}$$

Let $\boldsymbol{a}_i, \boldsymbol{b}_i, \boldsymbol{c}_i, \boldsymbol{d}_i \in \mathcal{T}_0^{2m+p}$ for $i = 1, 2$ and $\boldsymbol{v}_j, \boldsymbol{g}_j, \boldsymbol{h}_j \in \mathcal{T}_0^{2n+1}$ for $j = 1, 2, 3, 4$ satisfy the following conditions.

$$\begin{aligned}\psi_{\boldsymbol{a}_1}(x) &= x^{-m}\psi_{\boldsymbol{a}}(x), & \psi_{\boldsymbol{a}_2}(x) &= x^{-m}\psi_{\boldsymbol{a}}^*(x), \\ \psi_{\boldsymbol{b}_1}(x) &= x^{-m}\psi_{\boldsymbol{b}}(x), & \psi_{\boldsymbol{b}_2}(x) &= x^{-m}\psi_{\boldsymbol{b}}^*(x), \\ \psi_{\boldsymbol{c}_1}(x) &= x^{m+p}\psi_{\boldsymbol{c}}(x), & \psi_{\boldsymbol{c}_2}(x) &= x^{m+p}\psi_{\boldsymbol{c}}^*(x), \\ \psi_{\boldsymbol{d}_1}(x) &= x^{m+p}\psi_{\boldsymbol{d}}(x), & \psi_{\boldsymbol{d}_2}(x) &= x^{m+p}\psi_{\boldsymbol{d}}^*(x),\end{aligned}$$

$$\psi_{\boldsymbol{v}_1}(y) = \psi_{\boldsymbol{v}_2}(y) = y^{-2}\psi_{\boldsymbol{v}}^*(y^2), \quad \psi_{\boldsymbol{v}_3}(y) = \psi_{\boldsymbol{v}_4}(y) = \psi_{\boldsymbol{v}}(y^2),$$

$$\begin{aligned}\psi_{\boldsymbol{g}_1}(y) &= y^{-2}\psi_{\boldsymbol{g}}(y^2), & \psi_{\boldsymbol{h}_1}(y) &= y^{-2}\psi_{\boldsymbol{h}}(y^2), \\ \psi_{\boldsymbol{g}_2}(y) &= y^{-2}\psi_{\boldsymbol{g}}^*(y^2), & \psi_{\boldsymbol{h}_2}(y) &= y^{-2}\psi_{\boldsymbol{h}}^*(y^2), \\ \psi_{\boldsymbol{g}_3}(y) &= \psi_{\boldsymbol{g}}^*(y^2), & \psi_{\boldsymbol{h}_3}(y) &= \psi_{\boldsymbol{h}}(y^2), \\ \psi_{\boldsymbol{g}_4}(y) &= \psi_{\boldsymbol{g}}(y^2), & \psi_{\boldsymbol{h}_4}(y) &= \psi_{\boldsymbol{h}}^*(y^2).\end{aligned}$$

Set

$$\begin{aligned} Q &= \boldsymbol{v}_1^t \boldsymbol{a}_1 + \boldsymbol{g}_1^t \boldsymbol{c}_1 + \boldsymbol{h}_1^t \boldsymbol{d}_1 - \boldsymbol{e}^t \boldsymbol{b}_2, \\ R &= \boldsymbol{v}_2^t \boldsymbol{b}_1 + \boldsymbol{g}_2^t \boldsymbol{d}_1 - \boldsymbol{h}_2^t \boldsymbol{c}_1 + \boldsymbol{e}^t \boldsymbol{a}_2, \\ S &= \boldsymbol{g}_3^t \boldsymbol{a}_1 - \boldsymbol{v}_3^t \boldsymbol{c}_1 - \boldsymbol{h}_3^t \boldsymbol{b}_1 - \boldsymbol{e}^t \boldsymbol{d}_2, \\ T &= \boldsymbol{g}_4^t \boldsymbol{b}_1 - \boldsymbol{v}_4^t \boldsymbol{d}_1 + \boldsymbol{h}_4^t \boldsymbol{a}_1 + \boldsymbol{e}^t \boldsymbol{c}_2. \end{aligned} \tag{3.23}$$

Then

$$\begin{aligned} &(\psi_{\boldsymbol{q}}\psi_{\boldsymbol{q}}^* + \psi_{\boldsymbol{r}}\psi_{\boldsymbol{r}}^* + \psi_{\boldsymbol{s}}\psi_{\boldsymbol{s}}^* + \psi_{\boldsymbol{t}}\psi_{\boldsymbol{t}}^*)(x) \\ &= (\psi_{\boldsymbol{a}}\psi_{\boldsymbol{a}}^* + \psi_{\boldsymbol{b}}\psi_{\boldsymbol{b}}^* + \psi_{\boldsymbol{c}}\psi_{\boldsymbol{c}}^* + \psi_{\boldsymbol{d}}\psi_{\boldsymbol{d}}^*)(x)((\psi_{\boldsymbol{v}}\psi_{\boldsymbol{v}}^* + \psi_{\boldsymbol{g}}\psi_{\boldsymbol{g}}^* + \psi_{\boldsymbol{h}}\psi_{\boldsymbol{h}}^*)(x^{2(2m+p)}) + (\psi_{\boldsymbol{e}}\psi_{\boldsymbol{e}}^*)(x^{2m+p})) \end{aligned}$$

where $\boldsymbol{q} = \mathrm{seq}(Q)$, $\boldsymbol{r} = \mathrm{seq}(R)$, $\boldsymbol{s} = \mathrm{seq}(S)$, $\boldsymbol{t} = \mathrm{seq}(T)$.

Proof. By the constructions, we have

$$\begin{aligned} y^2\psi_Q(x,y) &= y^2(\psi_{\boldsymbol{v}_1}(y)\psi_{\boldsymbol{a}_1}(x) + \psi_{\boldsymbol{g}_1}(y)\psi_{\boldsymbol{c}_1}(x) + \psi_{\boldsymbol{h}_1}(y)\psi_{\boldsymbol{d}_1}(x) - \psi_{\boldsymbol{e}}(y)\psi_{\boldsymbol{b}_2}(x)) \\ &= \psi_{\boldsymbol{v}}^*(y^2)x^{-m}\psi_{\boldsymbol{a}}(x) + \psi_{\boldsymbol{g}}(y^2)x^{m+p}\psi_{\boldsymbol{c}}(x) + \psi_{\boldsymbol{h}}(y^2)x^{m+p}\psi_{\boldsymbol{d}}(x) - y^2\psi_{\boldsymbol{e}}(y)x^{-m}\psi_{\boldsymbol{b}}^*(x), \end{aligned} \tag{3.24}$$

$$\begin{aligned} y^2\psi_R(x,y) &= (\psi_{\boldsymbol{v}_2}(y)\psi_{\boldsymbol{b}_1}(x) + \psi_{\boldsymbol{g}_2}(y)\psi_{\boldsymbol{d}_1}(x) - \psi_{\boldsymbol{h}_2}(y)\psi_{\boldsymbol{c}_1}(x) - \psi_{\boldsymbol{e}}(y)\psi_{\boldsymbol{a}_2}(x)) \\ &= \psi_{\boldsymbol{v}}^*(y^2)x^{-m}\psi_{\boldsymbol{b}}(x) + \psi_{\boldsymbol{g}}^*(y^2)x^{m+p}\psi_{\boldsymbol{d}}(x) - \psi_{\boldsymbol{h}}^*(y^2)x^{m+p}\psi_{\boldsymbol{c}}(x) - y^2\psi_{\boldsymbol{e}}(y)x^{-m}\psi_{\boldsymbol{a}}^*(x), \end{aligned} \tag{3.25}$$

$$\begin{aligned} \psi_S(x,y) &= \psi_{\boldsymbol{g}_3}(y)\psi_{\boldsymbol{a}_1}(x) - \psi_{\boldsymbol{v}_3}(y)\psi_{\boldsymbol{c}_1}(x) - \psi_{\boldsymbol{h}_3}(y)\psi_{\boldsymbol{b}_1}(x) + \psi_{\boldsymbol{e}}(y)\psi_{\boldsymbol{d}_2}(x) \\ &= \psi_{\boldsymbol{g}}^*(y^2)x^{-m}\psi_{\boldsymbol{a}}(x) - \psi_{\boldsymbol{v}}(y^2)x^{m+p}\psi_{\boldsymbol{c}}(x) - \psi_{\boldsymbol{h}}(y^2)x^{-m}\psi_{\boldsymbol{b}}(x) + \psi_{\boldsymbol{e}}(y)x^{m+p}\psi_{\boldsymbol{d}}^*(x), \end{aligned} \tag{3.26}$$

$$\begin{aligned} \psi_T(x,y) &= \psi_{\boldsymbol{g}_4}(y)\psi_{\boldsymbol{b}_1}(x) - \psi_{\boldsymbol{v}_4}(y)\psi_{\boldsymbol{d}_1}(x) - \psi_{\boldsymbol{h}_4}(y)\psi_{\boldsymbol{a}_1}(x) + \psi_{\boldsymbol{e}}(y)\psi_{\boldsymbol{c}_2}(x) \\ &= \psi_{\boldsymbol{g}}(y^2)x^{-m}\psi_{\boldsymbol{b}}(x) - \psi_{\boldsymbol{v}}(y^2)x^{m+p}\psi_{\boldsymbol{d}}(x) - \psi_{\boldsymbol{h}}^*(y^2)x^{-m}\psi_{\boldsymbol{a}}(x) + \psi_{\boldsymbol{e}}(y)x^{m+p}\psi_{\boldsymbol{c}}^*(x). \end{aligned} \tag{3.27}$$

Set

$$\begin{aligned}
a &= x^{-m}\psi_{\boldsymbol{a}}(x),\\
b &= x^{-m}\psi_{\boldsymbol{b}}(x),\\
c &= x^{m+p}\psi_{\boldsymbol{c}}(x),\\
d &= x^{m+p}\psi_{\boldsymbol{d}}(x),\\
v &= \psi_{\boldsymbol{v}}(x^{4m+2p}),\\
g &= \psi_{\boldsymbol{g}}(x^{4m+2p}),\\
h &= \psi_{\boldsymbol{h}}(x^{4m+2p}),\\
e &= x^{2(m+p)}\psi_{\boldsymbol{e}}(x^{2m+p}).
\end{aligned}$$

Then from equations (3.24), (3.25), (3.26), (3.27), we obtain

$$\begin{aligned}
x^{4m+2p}\psi_{\boldsymbol{q}}(x) &= x^{4m+2p}\psi_Q(x, x^{2m+p})\\
&= \psi^*_{\boldsymbol{v}}(x^{4m+2p})x^{-m}\psi_{\boldsymbol{a}}(x) + \psi_{\boldsymbol{g}}(x^{4m+2p})x^{m+p}\psi_{\boldsymbol{c}}(x)\\
&\quad + \psi_{\boldsymbol{h}}(x^{4m+2p})x^{m+p}\psi_{\boldsymbol{d}}(x) - x^{4m+2p}\psi_{\boldsymbol{e}}(x^{2m+p})x^{-m}\psi^*_{\boldsymbol{b}}(x)\\
&= \psi^*_{\boldsymbol{v}}(x^{4m+2p})x^{-m}\psi_{\boldsymbol{a}}(x) + \psi_{\boldsymbol{g}}(x^{4m+2p})x^{m+p}\psi_{\boldsymbol{c}}(x)\\
&\quad + \psi_{\boldsymbol{h}}(x^{4m+2p})x^{m+p}\psi_{\boldsymbol{d}}(x) - x^{2m+2p}\psi_{\boldsymbol{e}}(x^{2m+p})x^{m}\psi^*_{\boldsymbol{b}}(x)\\
&= v^*a + gc + hd - eb^*,
\end{aligned}$$

$$\begin{aligned}
x^{4m+2p}\psi_{\boldsymbol{r}}(x) &= x^{4m+2p}\psi_R(x, x^{2m+p})\\
&= \psi^*_{\boldsymbol{v}}(x^{4m+2p})x^{-m}\psi_{\boldsymbol{b}}(x) + \psi^*_{\boldsymbol{g}}(x^{4m+2p})x^{m+p}\psi_{\boldsymbol{d}}(x)\\
&\quad - \psi^*_{\boldsymbol{h}}(x^{4m+2p})x^{m+p}\psi_{\boldsymbol{c}}(x) - x^{4m+2p}\psi_{\boldsymbol{e}}(x^{2m+p})x^{-m}\psi^*_{\boldsymbol{a}}(x)\\
&= \psi^*_{\boldsymbol{v}}(x^{4m+2p})x^{-m}\psi_{\boldsymbol{b}}(x) + \psi^*_{\boldsymbol{g}}(x^{4m+2p})x^{m+p}\psi_{\boldsymbol{d}}(x)\\
&\quad - \psi^*_{\boldsymbol{h}}(x^{4m+2p})x^{m+p}\psi_{\boldsymbol{c}}(x) - x^{2m+2p}\psi_{\boldsymbol{e}}(x^{2m+p})x^{m}\psi^*_{\boldsymbol{a}}(x)\\
&= v^*b + g^*d - h^*c - ea^*,
\end{aligned}$$

$$\begin{aligned}
\psi_{\boldsymbol{s}}(x) &= \psi_S(x, x^{2m+p}) \\
&= \psi_{\boldsymbol{g}}^*(x^{4m+2p})x^{-m}\psi_{\boldsymbol{a}}(x) - \psi_{\boldsymbol{v}}(x^{4m+2p})x^{m+p}\psi_{\boldsymbol{c}}(x) \\
&\quad - \psi_{\boldsymbol{h}}(x^{4m+2p})x^{-m}\psi_{\boldsymbol{b}}(x) + \psi_{\boldsymbol{e}}(x^{2m+p})x^{m+p}\psi_{\boldsymbol{d}}^*(x) \\
&= \psi_{\boldsymbol{g}}^*(x^{4m+2p})x^{-m}\psi_{\boldsymbol{a}}(x) - \psi_{\boldsymbol{v}}(x^{4m+2p})x^{m+p}\psi_{\boldsymbol{c}}(x) \\
&\quad - \psi_{\boldsymbol{h}}(x^{4m+2p})x^{-m}\psi_{\boldsymbol{b}}(x) + x^{2m+2p}\psi_{\boldsymbol{e}}(x^{2m+p})x^{-m-p}\psi_{\boldsymbol{d}}^*(x) \\
&= g^*a - vc - hb + ed^*,
\end{aligned}$$

$$\begin{aligned}
\psi_{\boldsymbol{t}}(x) &= \psi_T(x, x^{2m+p}) \\
&= \psi_{\boldsymbol{g}}(x^{4m+2p})x^{-m}\psi_{\boldsymbol{b}}(x) - \psi_{\boldsymbol{v}}(x^{4m+2p})x^{m+p}\psi_{\boldsymbol{d}}(x) \\
&\quad - \psi_{\boldsymbol{h}}^*(x^{4m+2p})x^{-m}\psi_{\boldsymbol{a}}(x) + \psi_{\boldsymbol{e}}(x^{2m+p})x^{m+p}\psi_{\boldsymbol{c}}^*(x) \\
&= \psi_{\boldsymbol{g}}(x^{4m+2p})x^{-m}\psi_{\boldsymbol{b}}(x) - \psi_{\boldsymbol{v}}(x^{4m+2p})x^{m+p}\psi_{\boldsymbol{d}}(x) \\
&\quad - \psi_{\boldsymbol{h}}^*(x^{4m+2p})x^{-m}\psi_{\boldsymbol{a}}(x) + x^{2m+2p}\psi_{\boldsymbol{e}}(x^{2m+p})x^{-m-p}\psi_{\boldsymbol{c}}^*(x) \\
&= gb - vd - h^*a + ec^*.
\end{aligned}$$

By the Lagrange identity,

$$\begin{aligned}
&(\psi_{\boldsymbol{q}}\psi_{\boldsymbol{q}}^* + \psi_{\boldsymbol{r}}\psi_{\boldsymbol{r}}^* + \psi_{\boldsymbol{s}}\psi_{\boldsymbol{s}}^* + \psi_{\boldsymbol{t}}\psi_{\boldsymbol{t}}^*)(x) = (aa^* + bb^* + cc^* + dd^*)(ee^* + vv^* + gg^* + hh^*) \\
&= (\psi_{\boldsymbol{a}}\psi_{\boldsymbol{a}}^* + \psi_{\boldsymbol{b}}\psi_{\boldsymbol{b}}^* + \psi_{\boldsymbol{c}}\psi_{\boldsymbol{c}}^* + \psi_{\boldsymbol{d}}\psi_{\boldsymbol{d}}^*)(x)((\psi_{\boldsymbol{v}}\psi_{\boldsymbol{v}}^* + \psi_{\boldsymbol{g}}\psi_{\boldsymbol{g}}^* + \psi_{\boldsymbol{h}}\psi_{\boldsymbol{h}}^*)(x^{2(2m+p)}) + (\psi_{\boldsymbol{e}}\psi_{\boldsymbol{e}}^*)(x^{2m+p})).
\end{aligned}$$

□

Corollary 3.21. *Let m, n, p be positive integers and*

$$\begin{aligned}
\boldsymbol{a}, \boldsymbol{b} &\in \mathcal{T}^{m+p}, \\
\boldsymbol{c}, \boldsymbol{d} &\in \mathcal{T}^{m}, \\
\boldsymbol{v}, \boldsymbol{g}, \boldsymbol{h} &\in \mathcal{T}_0^{n}, \\
\boldsymbol{e} &\in \mathcal{T}_0^{2n+1}.
\end{aligned}$$

Define

$$\begin{aligned} \boldsymbol{a}_1 &= (\boldsymbol{a}, 0_m), & \boldsymbol{a}_2 &= (\boldsymbol{a}^*, 0_m), \\ \boldsymbol{b}_1 &= (\boldsymbol{b}, 0_m), & \boldsymbol{b}_2 &= (\boldsymbol{b}^*, 0_m), \\ \boldsymbol{c}_1 &= (0_{m+p}, \boldsymbol{c}), & \boldsymbol{c}_2 &= (0_{m+p}, \boldsymbol{c}^*), \\ \boldsymbol{d}_1 &= (0_{m+p}, \boldsymbol{d}), & \boldsymbol{d}_2 &= (0_{m+p}, \boldsymbol{d}^*), \end{aligned}$$

and set

$$\begin{aligned} &\boldsymbol{v}_1 = \boldsymbol{v}_2 = (\boldsymbol{v}^*/0,0,0), \quad \boldsymbol{v}_3 = \boldsymbol{v}_4 = (0, \boldsymbol{v}/0,0), \\ &\boldsymbol{g}_1 = (\boldsymbol{g}/0,0,0), \qquad \boldsymbol{h}_1 = (\boldsymbol{h}/0,0,0), \\ &\boldsymbol{g}_2 = (\boldsymbol{g}^*/0,0,0), \qquad \boldsymbol{h}_2 = (\boldsymbol{h}^*/0,0,0), \\ &\boldsymbol{g}_3 = (0, \boldsymbol{g}^*/0,0), \qquad \boldsymbol{h}_3 = (0, \boldsymbol{h}/0,0), \\ &\boldsymbol{g}_4 = (0, \boldsymbol{g}/0,0), \qquad \boldsymbol{h}_4 = (0, \boldsymbol{h}^*/0,0). \end{aligned} \tag{3.28}$$

Set

$$\begin{aligned} Q &= \boldsymbol{v}_1^t \boldsymbol{a}_1 + \boldsymbol{g}_1^t \boldsymbol{c}_1 + \boldsymbol{h}_1^t \boldsymbol{d}_1 - \boldsymbol{e}^t \boldsymbol{b}_2, \\ R &= \boldsymbol{v}_2^t \boldsymbol{b}_1 + \boldsymbol{g}_2^t \boldsymbol{d}_1 - \boldsymbol{h}_2^t \boldsymbol{c}_1 + \boldsymbol{e}^t \boldsymbol{a}_2, \\ S &= \boldsymbol{g}_3^t \boldsymbol{a}_1 - \boldsymbol{v}_3^t \boldsymbol{c}_1 - \boldsymbol{h}_3^t \boldsymbol{b}_1 - \boldsymbol{e}^t \boldsymbol{d}_2, \\ T &= \boldsymbol{g}_4^t \boldsymbol{b}_1 - \boldsymbol{v}_4^t \boldsymbol{d}_1 + \boldsymbol{h}_4^t \boldsymbol{a}_1 + \boldsymbol{e}^t \boldsymbol{c}_2. \end{aligned} \tag{3.29}$$

Then

$$\begin{aligned} &(\psi_{\boldsymbol{q}}\psi_{\boldsymbol{q}}^* + \psi_{\boldsymbol{r}}\psi_{\boldsymbol{r}}^* + \psi_{\boldsymbol{s}}\psi_{\boldsymbol{s}}^* + \psi_{\boldsymbol{t}}\psi_{\boldsymbol{t}}^*)(x) \\ &= (\psi_{\boldsymbol{a}}\psi_{\boldsymbol{a}}^* + \psi_{\boldsymbol{b}}\psi_{\boldsymbol{b}}^* + \psi_{\boldsymbol{c}}\psi_{\boldsymbol{c}}^* + \psi_{\boldsymbol{d}}\psi_{\boldsymbol{d}}^*)(x)((\psi_{\boldsymbol{v}}\psi_{\boldsymbol{v}}^* + \psi_{\boldsymbol{g}}\psi_{\boldsymbol{g}}^* + \psi_{\boldsymbol{h}}\psi_{\boldsymbol{h}}^*)(x^{2(2m+p)}) + (\psi_{\boldsymbol{e}}\psi_{\boldsymbol{e}}^*)(x^{2m+p})) \end{aligned}$$

where $\boldsymbol{q} = \operatorname{seq}(Q)$, $\boldsymbol{r} = \operatorname{seq}(R)$, $\boldsymbol{s} = \operatorname{seq}(S)$, $\boldsymbol{t} = \operatorname{seq}(T)$.

Proof. Since

$$
\begin{aligned}
&\psi_{\boldsymbol{a}_1}(x) = x^{-m}\psi_{\boldsymbol{a}}(x),\\
&\psi_{\boldsymbol{b}_1}(x) = x^{-m}\psi_{\boldsymbol{b}}(x),\\
&\psi_{\boldsymbol{c}_1}(x) = x^{m+p}\psi_{\boldsymbol{c}}(x),\\
&\psi_{\boldsymbol{d}_1}(x) = x^{m+p}\psi_{\boldsymbol{d}}(x),\\
&\psi_{\boldsymbol{a}_2}(x) = x^{-m}\psi_{\boldsymbol{a}^*}(x) = x^{-m}\psi^*_{\boldsymbol{a}}(x),\\
&\psi_{\boldsymbol{b}_2}(x) = x^{-m}\psi_{\boldsymbol{b}^*}(x) = x^{-m}\psi^*_{\boldsymbol{b}}(x),\\
&\psi_{\boldsymbol{c}_2}(x) = x^{m+p}\psi_{\boldsymbol{c}^*}(x) = x^{m+p}\psi^*_{\boldsymbol{c}}(x),\\
&\psi_{\boldsymbol{d}_2}(x) = x^{m+p}\psi_{\boldsymbol{d}^*}(x) = x^{m+p}\psi^*_{\boldsymbol{d}}(x),\\
&\psi_{\boldsymbol{v}_1}(y) = \psi_{\boldsymbol{v}_2}(y) = y^{-2}\psi_{(\boldsymbol{v}^*/0)}(y) = y^{-2}\psi_{\boldsymbol{v}^*}(y^2) = y^{-2}\psi^*_{\boldsymbol{v}}(y^2),\\
&\psi_{\boldsymbol{v}_3}(y) = \psi_{\boldsymbol{v}_4}(y) = \psi_{(\boldsymbol{v}/0)}(y) = \psi_{\boldsymbol{v}}(y^2),\\
&\psi_{\boldsymbol{g}_1}(y) = y^{-2}\psi_{(0/\boldsymbol{g})}(y) = y^{-2}\psi_{\boldsymbol{g}}(y^2),\\
&\psi_{\boldsymbol{g}_2}(y) = y^{-2}\psi_{(0/\boldsymbol{g}^*)}(y) = y^{-2}\psi_{\boldsymbol{g}^*}(y^2) = y^{-2}\psi^*_{\boldsymbol{g}}(y^2),\\
&\psi_{\boldsymbol{g}_3}(y) = \psi_{(0/\boldsymbol{g}^*)}(y) = \psi^*_{\boldsymbol{g}}(y^2),\\
&\psi_{\boldsymbol{g}_4}(y) = \psi_{(0/\boldsymbol{g})}(y) = \psi_{\boldsymbol{g}}(y^2),\\
&\psi_{\boldsymbol{h}_1}(y) = y^{-2}\psi_{(0/\boldsymbol{h})}(y) = y^{-2}\psi_{\boldsymbol{h}}(y^2),\\
&\psi_{\boldsymbol{h}_2}(y) = y^{-2}\psi_{(0/\boldsymbol{h}^*)}(y) = y^{-2}\psi_{\boldsymbol{h}^*}(y^2) = y^{-2}\psi^*_{\boldsymbol{h}}(y^2),\\
&\psi_{\boldsymbol{h}_3}(y) = \psi_{(0/\boldsymbol{h})}(y) = \psi_{\boldsymbol{h}}(y^2),\\
&\psi_{\boldsymbol{h}_4}(y) = \psi_{(0/\boldsymbol{h}^*)}(y) = \psi^*_{\boldsymbol{h}}(y^2),
\end{aligned}
$$

the conditions in Theorem 3.20 are satisfied. Thus, by Theorem 3.20, the result follows. □

Lemma 3.22. *Under the conditions of Corollary 3.21, if*

a. $\mathrm{supp}(z_i) \cap \mathrm{supp}(\boldsymbol{e}) = \emptyset$ *for* $z \in \{\boldsymbol{v}, \boldsymbol{g}, \boldsymbol{h}\}$ *and* $i = 1, 3$,

b. $\mathrm{supp}(\boldsymbol{g}_i) \cap \mathrm{supp}(\boldsymbol{h}_i) = \emptyset$ *for* $i = 1, 3$,

then the pairs $(\boldsymbol{q}, \boldsymbol{r})$ *and* $(\boldsymbol{s}, \boldsymbol{t})$ *are each conjoint and the pair* $(\boldsymbol{q}, \boldsymbol{s})$ *is disjoint.*

Proof. Recall Lemma 3.19. By the constructions, the conditions (i), (ii), (iii), and (v) are already satisfied. Since (a.) and (b.) holds, the conditions (iv) and (vi) are also satisfied. Therefore, by Lemma 3.19, the pairs (Q, R) and (S, T) are each conjoint and the pair (Q, S) is disjoint. Hence, the pairs $(\boldsymbol{q}, \boldsymbol{r})$ and $(\boldsymbol{s}, \boldsymbol{t})$ are each conjoint and the pair $(\boldsymbol{q}, \boldsymbol{s})$ is disjoint. □

Lemma 3.23. *Let m, n, p be positive integers and set*

$$\boldsymbol{a}, \boldsymbol{b} \in \mathcal{T}^{m+p},$$
$$\boldsymbol{c}, \boldsymbol{d} \in \mathcal{T}^{m}.$$

Let

$$\boldsymbol{v} = (1, 1, -1), \qquad \boldsymbol{g} = (1, 0, 1), \qquad \boldsymbol{h} = (0, 1, 0), \qquad \boldsymbol{e} = (0_6, 1),$$

and apply Corollary 3.21. Then

$$(\psi_{\boldsymbol{q}}\psi_{\boldsymbol{q}}^* + \psi_{\boldsymbol{r}}\psi_{\boldsymbol{r}}^* + \psi_{\boldsymbol{s}}\psi_{\boldsymbol{s}}^* + \psi_{\boldsymbol{t}}\psi_{\boldsymbol{t}}^*)(x) = 7(\psi_{\boldsymbol{a}}\psi_{\boldsymbol{a}}^* + \psi_{\boldsymbol{b}}\psi_{\boldsymbol{b}}^* + \psi_{\boldsymbol{c}}\psi_{\boldsymbol{c}}^* + \psi_{\boldsymbol{d}}\psi_{\boldsymbol{d}}^*)(x). \tag{3.30}$$

Moreover, the pairs $(\boldsymbol{q}, \boldsymbol{r})$ and $(\boldsymbol{s}, \boldsymbol{t})$ are each conjoint and the pair $(\boldsymbol{q}, \boldsymbol{s})$ is disjoint.

Proof. By direct calculation, we have $(\psi_{\boldsymbol{v}}\psi_{\boldsymbol{v}}^* + \psi_{\boldsymbol{g}}\psi_{\boldsymbol{g}}^* + \psi_{\boldsymbol{h}}\psi_{\boldsymbol{h}}^*)(y) = 6$ and $\psi_{\boldsymbol{e}}\psi_{\boldsymbol{e}}^*(y) = 1$. Thus, we obtain (3.30) immediately by Corollary 3.21. Moreover, we have

$$\begin{aligned}
\boldsymbol{v}_1 = \boldsymbol{v}_2 &= (\ -1\ \ 0\ \ 1\ \ 0\ \ 1\ \ 0\ \ 0\ ),\\
\boldsymbol{g}_1 = \boldsymbol{g}_2 &= (\ 1\ \ 0\ \ 0\ \ 0\ \ 1\ \ 0\ \ 0\ ),\\
\boldsymbol{h}_1 = \boldsymbol{h}_2 &= (\ 0\ \ 0\ \ 1\ \ 0\ \ 0\ \ 0\ \ 0\ ),\\
\boldsymbol{v}_3 = \boldsymbol{v}_4 &= (\ 0\ \ 1\ \ 0\ \ 1\ \ 0\ \ -1\ \ 0\ ),\\
\boldsymbol{g}_3 = \boldsymbol{g}_4 &= (\ 0\ \ 1\ \ 0\ \ 0\ \ 0\ \ 1\ \ 0\ ),\\
\boldsymbol{h}_3 = \boldsymbol{h}_4 &= (\ 0\ \ 0\ \ 0\ \ 1\ \ 0\ \ 0\ \ 0\ ),\\
\boldsymbol{e} &= (\ 0\ \ 0\ \ 0\ \ 0\ \ 0\ \ 0\ \ 1\ ).
\end{aligned}$$

Since $\operatorname{supp}(z_i) \cap \operatorname{supp}(\boldsymbol{e}) = \emptyset$ for $z \in \{\boldsymbol{v}, \boldsymbol{g}, \boldsymbol{h}\}$ and $i = 1, 3$, and $\operatorname{supp}(\boldsymbol{g}_i) \cap \operatorname{supp}(\boldsymbol{h}_i) = \emptyset$ for $i = 1, 3$, by Lemma 3.22, the pairs $(\boldsymbol{q}, \boldsymbol{r})$ and $(\boldsymbol{s}, \boldsymbol{t})$ are each conjoint and the pair $(\boldsymbol{q}, \boldsymbol{s})$ is disjoint. □

Theorem 3.24. *If $BS(m+p,m)$ is nonempty, then $PT(7(2m+p))$ is nonempty.*

Proof. Let $(\boldsymbol{a},\boldsymbol{b},\boldsymbol{c},\boldsymbol{d}) \in BS(m+p,m)$. By applying Lemma 3.23, we obtain $\boldsymbol{q},\boldsymbol{r},\boldsymbol{s},\boldsymbol{t} \in \{0,\pm1\}^{7(2m+p)}$ such that

$$\begin{aligned}(\psi_{\boldsymbol{q}}\psi_{\boldsymbol{q}}^* + \psi_{\boldsymbol{r}}\psi_{\boldsymbol{r}}^* + \psi_{\boldsymbol{s}}\psi_{\boldsymbol{s}}^* + \psi_{\boldsymbol{t}}\psi_{\boldsymbol{t}}^*)(x) &= 7(\psi_{\boldsymbol{a}}\psi_{\boldsymbol{a}}^* + \psi_{\boldsymbol{b}}\psi_{\boldsymbol{b}}^* + \psi_{\boldsymbol{c}}\psi_{\boldsymbol{c}}^* + \psi_{\boldsymbol{d}}\psi_{\boldsymbol{d}}^*)(x) \\ &= 14(2m+p)\end{aligned}$$

with the pairs $(\boldsymbol{q},\boldsymbol{r})$ and $(\boldsymbol{s},\boldsymbol{t})$ are each conjoint and the pair $(\boldsymbol{q},\boldsymbol{s})$ is disjoint. Therefore, $(\boldsymbol{q},\boldsymbol{r},\boldsymbol{s},\boldsymbol{t}) \in PT(7(2m+p))$. □

Theorem 3.25. *If $BS(m+p,m)$ and $GS(s)$ are nonempty, then $PT((2s+1)(2m+p))$ is nonempty.*

Proof. Let $(\boldsymbol{a},\boldsymbol{b},\boldsymbol{c},\boldsymbol{d}) \in BS(m+p,m)$ and $(\boldsymbol{v},\boldsymbol{g}) \in GS(s)$. Define all sequences in Corollary 3.21 with $\boldsymbol{h} = 0_s$ and $\boldsymbol{e} = (0_{2s},1)$. Clearly, $\operatorname{supp}(z_i) \cap \operatorname{supp}(\boldsymbol{e}) = \emptyset$ for $z \in \{\boldsymbol{v},\boldsymbol{g},\boldsymbol{h}\}$ and $i = 1,3$, and $\operatorname{supp}(\boldsymbol{g}_i) \cap \operatorname{supp}(\boldsymbol{h}_i) = \emptyset$ for $i = 1,3$. Therefore, the pairs $(\boldsymbol{q},\boldsymbol{r})$ and $(\boldsymbol{s},\boldsymbol{t})$ are each conjoint and the pair $(\boldsymbol{q},\boldsymbol{s})$ is disjoint by Lemma 3.22. By Corollary 3.21,

$$\begin{aligned}&(\psi_{\boldsymbol{q}}\psi_{\boldsymbol{q}}^* + \psi_{\boldsymbol{r}}\psi_{\boldsymbol{r}}^* + \psi_{\boldsymbol{s}}\psi_{\boldsymbol{s}}^* + \psi_{\boldsymbol{t}}\psi_{\boldsymbol{t}}^*)(x) \\ &= (\psi_{\boldsymbol{a}}\psi_{\boldsymbol{a}}^* + \psi_{\boldsymbol{b}}\psi_{\boldsymbol{b}}^* + \psi_{\boldsymbol{c}}\psi_{\boldsymbol{c}}^* + \psi_{\boldsymbol{d}}\psi_{\boldsymbol{d}}^*)(x)((\psi_{\boldsymbol{v}}\psi_{\boldsymbol{v}}^* + \psi_{\boldsymbol{g}}\psi_{\boldsymbol{g}}^* + \psi_{\boldsymbol{h}}\psi_{\boldsymbol{h}}^*)(x^{2(2m+p)}) + (\psi_{\boldsymbol{e}}\psi_{\boldsymbol{e}}^*)(x^{2m+p})) \\ &= 2(2m+p)(2s+1).\end{aligned}$$

Therefore, $(\boldsymbol{q},\boldsymbol{r},\boldsymbol{s},\boldsymbol{t}) \in PT((2s+1)(2m+p))$. □

4 The Theorem of Standardization

Throughout this section, let R be a commutative ring with identity. We fix a subgroup T of the group of units of R, and set $T_0 = T \cup \{0\}$. The set of $m \times n$ matrices with entries in T_0 is denoted by $T_0^{m\times n}$. If $T = \{z \in \mathbb{C} : |z| = 1\}$, then $W \in T_0^{n\times n}$ is called *a unit weighing matrix of order n with weight w* provided that $WW^* = wI$ where W^* is the transpose conjugate of W. Unit weighing matrices are introduced by D. Best, H. Kharaghani, and H. Ramp in [2, 3]. Moreover, a unit weighing matrix is known as a unit Hadamard matrix if $w = n$ (see [4]). A unit weighing matrix in which every entry is in $\{0, \pm 1\}$ is called a *weighing matrix*. We refer the reader to [6] for an extensive discussion of weighing matrices, and to [13] for more information on applications of weighing matrices.

The study on the number of inequivalent unit weighing matrices was initiated in [2]. Also, observing the number of weighing matrices in standard form leads to an upper bound on the number of inequivalent unit weighing matrices [2]. In this thesis, we will introduce a standard form of an arbitrary matrix in $T_0^{m\times n}$ and show that every matrix in $T_0^{m\times n}$ is equivalent to a matrix in standard form.

We equip T_0 with a total ordering $\prec$ satisfying $\min(T_0) = 1$ and $\max(T_0) = 0$. Moreover, let $\boldsymbol{a} = (a_1, \dots, a_n)$ and $\boldsymbol{b} = (b_1, \dots, b_n)$ be arbitrary row vectors with entries in T_0. If k is the smallest index such that $a_k \neq b_k$, then we write $\boldsymbol{a} < \boldsymbol{b}$ provided $a_k \prec b_k$. We write $\boldsymbol{a} \leq \boldsymbol{b}$ if $\boldsymbol{a} < \boldsymbol{b}$ or $\boldsymbol{a} = \boldsymbol{b}$. If $\boldsymbol{a}_1, \dots, \boldsymbol{a}_m$ are row vectors of a matrix $A \in T_0^{m\times n}$ and $\boldsymbol{a}_1 < \cdots < \boldsymbol{a}_m$, then we say that the rows of A are in *lexicographical order*.

Definition 4.1. We say that a matrix in $T_0^{m\times n}$ is in *standard form* if the following conditions are satisfied:

(S1) The first non-zero entry in each row is 1.

(S2) The first non-zero entry in each column is 1.

(S3) The first row is ones followed by zeros.

(S4) The rows are in lexicographical order according to $\prec$.

The subset of $T_0^{m\times m}$ consisting of permutation matrices, nonsingular diagonal matrices and monomial matrices, are denoted respectively, by $\mathbb{P}_m, \mathbb{D}_m$ and $\mathbb{M}_m$. Then $\mathbb{M}_m = \mathbb{P}_m\mathbb{D}_m$.

Definition 4.2. For $A, B \in T_0^{m\times n}$, we say that A is *equivalent* to B if there exist monomial T_0-matrices M_1 and M_2 such that $M_1AM_2 = B$.

Now we will restate [2, Theorem 5] to the following theorem.

Theorem 4.3. *Every unit weighing matrices is equivalent to a matrix that is in a standard form.*

For convenience, we will restate the proof of Theorem 4.3 as the following algorithm.

Algorithm 4.4. Let W be an arbitrary unit weighing matrix.

(1) We multiply each ith row of W by r_i^{-1} where r_i is the first non-zero entry in ith row. Denote the obtained matrix by $W^{(1)}$.

(2) Let c_j be the first non-zero entry in jth column of $W^{(1)}$. Let $W^{(2)}$ obtained from $W^{(1)}$ by multiplying each jth column by c_j^{-1}.

(3) Permute the columns of $W^{(2)}$ so that the first row has w ones. Denote the resulting matrix by $W^{(3)}$.

(4) Let $W^{(4)}$ be a matrix obtained from $W^{(3)}$ by sorting the rows of $W^{(3)}$ lexicographically with the ordering $\prec$.

Then $W^{(4)}$ is in standard form.

However, we provide a counterexample to show that this algorithm does not produce a standard form.

Example 4.5. The matrix

$$W = \begin{bmatrix} 1 & -i & i & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & i & i \\ 1 & 0 & 0 & -1 & -i & i \\ 1 & 0 & 0 & -1 & i & -i \\ 0 & 1 & 1 & 0 & -i & -i \\ 1 & i & -i & 1 & 0 & 0 \end{bmatrix}$$

is a unit weighing matrix, where i is a 4th root of unity in $\mathbb{C}$. Also,we equip the set $\{0, \pm i, \pm 1\}$ with a total ordering $\prec$ defined by $1 \prec -1 \prec i \prec -i \prec 0$. Since the first nonzero entry in each row of W is one, $W^{(1)} = W$. Applying step (2), we obtain

$$W^{(2)} = \begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & i & -i & 0 & 1 & 1 \\ 1 & 0 & 0 & -1 & -1 & 1 \\ 1 & 0 & 0 & -1 & 1 & -1 \\ 0 & i & -i & 0 & -1 & -1 \\ 1 & -1 & -1 & 1 & 0 & 0 \end{bmatrix}.$$

Notice that the first row of $W^{(2)}$ is all ones followed by zeros. So, $W^{(3)} = W^{(2)}$. Finally, by applying the last step of the algorithm, we have

$$W^{(4)} = \begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & -1 & -1 & 1 & 0 & 0 \\ 1 & 0 & 0 & -1 & 1 & -1 \\ 1 & 0 & 0 & -1 & -1 & 1 \\ 0 & i & -i & 0 & 1 & 1 \\ 0 & i & -i & 0 & -1 & -1 \end{bmatrix}.$$

We see that $W^{(4)}$ is not in standard form. So, we conclude that the algorithm does not produce a matrix in standard form as claimed.

This counterexample shows that the additional steps are needed to complete the proof of Theorem 4.3. In the next section, we will prove a more general theorem than Theorem 4.3 by showing that every matrix in $T_0^{m\times n}$ is equivalent to a matrix that is in standard form.

4.1 Improvement of the Proof of the Standardization

In addition to the conditions (S1)–(S4) in Definition 4.1, we will consider the following condition:

(S3)′ The first nonzero row is ones followed by zeros.

Note that (S3)′ is weaker than (S3). The condition (S3)′ is crucial in the proof of Lemma 4.6, where we encounter a matrix whose first row consists entirely of zeros.

Lemma 4.6. *Let*

$$A = \begin{bmatrix} A_1 & A_2 \end{bmatrix} \in T_0^{m\times(n_1+n_2)},$$

where $A_i \in T_0^{m\times n_i}$, $i = 1, 2$. *Then there exist* $P \in \mathbb{P}_m$ *and* $M \in \mathbb{M}_{n_2}$ *such that* PA_2M *satisfies* (S2) *and* (S3)′, *and* $\begin{bmatrix} PA_1 & PA_2M \end{bmatrix}$ *satisfies* (S4).

Proof. Without loss of generality, we may assume A_1 satisfies (S4). Then there exist row vectors $\boldsymbol{a}_1, \dots, \boldsymbol{a}_k$ of A_1 such that $\boldsymbol{a}_1 < \cdots < \boldsymbol{a}_k$, and positive integers $m_1, \dots, m_k$ such that

$$A_1 = \begin{bmatrix} \mathbf{1}_{m_1}^\top & & \\ & \ddots & \\ & & \mathbf{1}_{m_k}^\top \end{bmatrix} \begin{bmatrix} \boldsymbol{a}_1 \\ \vdots \\ \boldsymbol{a}_k \end{bmatrix},$$

where $\sum_{i=1}^k m_i = m$. Write

$$A_2 = \begin{bmatrix} B_1 \\ \vdots \\ B_k \end{bmatrix},$$

where $B_i \in T_0^{m_i\times n_2}$ for $i = 1, 2, \dots, k$. We may assume $B_1 \neq 0$, since otherwise the proof reduces to establishing the assertion for the matrix A with the first m_1 rows deleted. Let $\boldsymbol{b}$ be a row vector of B_1 with maximum number of nonzero components. Then there exists $M \in \mathbb{M}_{n_2}$

such that the vector $\boldsymbol{b}M$ constitutes ones followed by zeros. Moreover, for each $i \in \{1, \dots, k\}$, there exists $P_i \in \mathbb{P}_{m_i}$ such that the rows of $P_i B_i M$ are in lexicographic order. It follows that $\boldsymbol{b}M$ is the first row of $P_1 B_1 M$, that is also the first row of PA_2M. Set $P = \operatorname{diag}(P_1, \dots, P_k)$. Then PA_2M satisfies (S3). Since $PA_1 = A_1$, we see that $\begin{bmatrix} PA_1 & PA_2M \end{bmatrix}$ satisfies (S4).

With the above notation, we prove the assertion by induction on n_2. First we treat the case where $\boldsymbol{b}M = \mathbf{1}$. This in particular includes the case where $n_2 = 1$, the starting point of the induction. In this case, the first row of PA_2M is $\mathbf{1}$, hence PA_2M satisfies (S2). The other assertions have been proved already.

Next we consider the case where $\boldsymbol{b}M = \begin{bmatrix} \mathbf{1}_{n_2-n_2'} & \mathbf{0}_{n_2'} \end{bmatrix}$, with $0 < n_2' < n_2$. Define $A_1' \in T_0^{(m-1)\times(n_1+n_2-n_2')}$ and $A_2' \in T_0^{(m-1)\times n_2'}$ by setting $\begin{bmatrix} A_1' & A_2' \end{bmatrix}$ to be the matrix obtained from $\begin{bmatrix} A_1 & PA_2M \end{bmatrix}$ by deleting the first row. By inductive hypothesis, there exist $P' \in \mathbb{P}_{m-1}$ and $M' \in \mathbb{M}_{n_2'}$ such that $P'A_2'M'$ satisfies (S2) and (S3)$'$, and $\begin{bmatrix} P'A_1' & P'A_2'M' \end{bmatrix}$ satisfies (S4). By our choice of $\boldsymbol{b}$, the row vector $\boldsymbol{b}M$ is lexicographically the smallest member among the rows of P_1B_1M, and the same is true among the rows of the matrix P_1B_1M'', where

$$M'' = M \begin{bmatrix} I_{n_2-n_2'} & 0 \\ 0 & M' \end{bmatrix}.$$

It follows that the matrix

$$\begin{bmatrix} 1 & 0 \\ 0 & P' \end{bmatrix} \begin{bmatrix} A_1 & PA_2M'' \end{bmatrix} = \begin{bmatrix} * & 0 \\ P'A_1' & P'A_2'M' \end{bmatrix}$$

satisfies (S4). Set

$$P'' = \begin{bmatrix} 1 & 0 \\ 0 & P' \end{bmatrix} P.$$

Since $P'A_2'M'$ satisfies (S2), while the first row of $P''A_2M''$ is the same as that of PA_2M which is $\begin{bmatrix} \mathbf{1}_{n_2-n_2'} & \mathbf{0}_{n_2'} \end{bmatrix}$, the matrix $P''A_2M''$ satisfies both (S2) and (S3)$'$. We have already shown that the matrix $\begin{bmatrix} P''A_1 & P''A_2M \end{bmatrix}$ satisfies (S4). □

Lemma 4.7. *Under the same assumption as in Lemma 4.6, there exist $M_1 \in \mathbb{M}_m$ and*

$M_2 \in \mathbb{M}_{n_2}$ *such that* $\begin{bmatrix} M_1A_1 & M_1A_2M_2 \end{bmatrix}$ *satisfies* (S1) *and* (S4), *and* $M_1A_2M_2$ *satisfies* (S2) *and* (S3)$'$.

Proof. We will prove the assertion by induction on m. Suppose $m = 1$. It is clear that every single row vector always satisfies (S4). Also, every single row vector satisfying (S3)$'$ necessarily satisfies (S2). Now, if $A_1 = 0$ or $n_1 = 0$, then there exists $M_2 \in \mathbb{M}_{n_2}$ such that A_2M_2 satisfies (S3)$'$ and hence (S1) is satisfied. If $A_1 \neq 0$, then there exist $a \in T$ and $M_2 \in \mathbb{M}_{n_2}$ such that aA_1 satisfies (S1) and aA_2M_2 satisfies (S3)$'$.

Assume the assertion is true up to $m-1$. First, we consider the case where $A_1 = 0$ or $n_1 = 0$. Without loss of generality, we may assume $A_2 \neq 0$. Furthermore, we may assume that the first row and the first column of A_2 are ones followed by zeros. Then there exists $P' \in \mathbb{P}_{n_2}$ such that

$$A_2P' = \left[\begin{array}{ccc|c} 1 & \mathbf{1} & 0 & 0 \\ \mathbf{1}^T & B_1 & B_2 & 0 \\ 0 & \multicolumn{2}{c|}{C_1} & C_2 \end{array}\right]$$

where $B_2 \in T_0^{m_1 \times t}$ has no zero column. By Lemma 4.6, there exist $P \in \mathbb{P}_{m_1}$ and $M \in \mathbb{M}_t$ such that PB_2M satisfies (S2) and (S3)$'$ and $\begin{bmatrix} PB_1 & PB_2M \end{bmatrix}$ satisfies (S4). Let

$$C_1' = C_1 \begin{bmatrix} I_{n_2-n_2'-t-1} & 0 \\ 0 & M \end{bmatrix}.$$

By inductive hypothesis, there exist $M_1' \in \mathbb{M}_{m-m_1-1}$, and $M_2' \in \mathbb{M}_{n_2'}$ such that $\begin{bmatrix} M_1'C_1' & M_1'C_2M_2' \end{bmatrix}$ satisfies (S1) and (S4), and $M_1'C_2M_2'$ satisfies (S2) and (S3)$'$. By setting

$$M_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & P & 0 \\ 0 & 0 & M_1' \end{bmatrix}, \quad M_2 = P' \begin{bmatrix} I_{n_2-n_2'-t} & 0 & 0 \\ 0 & M & 0 \\ 0 & 0 & M_2' \end{bmatrix},$$

the matrix $M_1A_2M_2$ satisfies (S1)–(S4).

Next we consider the case $A_1 \neq 0$. Without loss of generality, we may assume that the first nonzero column in A_1 is ones followed by zeros. Write

$$A_1 = \begin{bmatrix} 0_{m\times t} & \begin{matrix} \mathbf{1}^T & B_1 \\ 0 & D_1 \end{matrix} \end{bmatrix}$$

for some $t < n_1$, with $B_1 \in T_0^{m_1\times(n_1-t-1)}$ and $D_1 \in T_0^{m_2\times(n_1-t-1)}$ for some m_1, m_2 with $m_1 + m_2 = m$ and $m_2 < m$. Then there exists $P' \in \mathbb{P}_{n_2}$ such that

$$A_2P' = \begin{bmatrix} B_2 & 0_{m_1\times n_2'} \\ D_2 & C_2 \end{bmatrix}$$

for some $n_2' \geq 0$, where $B_2 \in T_0^{m_1\times(n_2-n_2')}$ has no zero column. By Lemma 4.6, there exist $P \in \mathbb{P}_{m_1}$ and $M \in \mathbb{M}_{n_2-n_2'}$ such that PB_2M satisfies (S2) and (S3)′ and $\begin{bmatrix} PB_1 & PB_2M \end{bmatrix}$ satisfies (S4). Let $C_1 = \begin{bmatrix} D_1 & D_2M \end{bmatrix}$. Then by inductive hypothesis, there exist $M_1' \in \mathbb{M}_{m_2}$ and $M_2' \in \mathbb{M}_{n_2'}$ such that $\begin{bmatrix} M_1'C_1 & M_1'C_2M_2' \end{bmatrix}$ satisfies (S1) and (S4), and $M_1'C_2M_2'$ satisfies (S2) and (S3)′. By setting

$$M_1 = \begin{bmatrix} P & 0 \\ 0 & M_1' \end{bmatrix}, \quad M_2 = P' \begin{bmatrix} M & 0 \\ 0 & M_2' \end{bmatrix},$$

the proof is complete. □

Theorem 4.8. *Every nonzero matrix in $T_0^{m\times n}$ is equivalent to a matrix that is in standard form.*

Proof. Let $W \in T_0^{m\times n}$. Setting $A_1 = \varnothing$ and $A_2 = W$ in Lemma 4.7, we see that W is equivalent to a matrix that is in standard form. □

As a consequence of Theorem 4.3, we see that by setting $T = \{z \in \mathbb{C} : |z| = 1\}$ in Theorem 4.8, we have every unit weighing matrices is equivalent to a matrix that is in a standard form. Therefore, we improve the proof of [2, Theorem 5].

4.2 Applications of the Standardization to the Existence of Unit Weighing Matrices

Note that the theorems and lemmas in this section are presented in [2] except Theorem 4.9. In this section, we assume $T = \{z \in \mathbb{C} \mid |z| = 1\}$.

Theorem 4.9. *$UW(n,1) = \mathbb{M}_n$.*

Proof. It is clear that every monomial matrix is in $UW(n,1)$. Conversely, let $A = [a_{ij}] \in UW(n,1)$. Since $AA^* = I_n$, we have

$$[AA^*]_{ij} = \sum_{k=1}^{n} a_{ik}\overline{a_{jk}} = \begin{cases} 1 & \text{for } i = j, \\ 0 & \text{otherwise.} \end{cases}$$

Therefore, there exists an index $k(i) \in \{1,\ldots,n\}$ for every $i \in \{1,\ldots,n\}$ such that $a_{i,k(i)} \in T$ and $a_{i,k'} = 0$ for $k' \in \{1,\ldots,n\} - k(i)$. Next, we will show that $k(i)$ is unique for every $i \in \{1,\ldots,n\}$. For fixed i, let $j \neq i$ and $j \in \{1,\ldots,n\}$ such that $k(i) = k(j)$. Then $a_{i,k(i)} = a_{j,k(j)}$ and

$$\sum_{k=1}^{n} a_{ik}\overline{a_{jk}} = a_{i,k(i)}\overline{a_{j,k(j)}} = a_{i,k(i)}\overline{a_{i,k(i)}} = 1.$$

This is a contradiction since $\sum_{i=k}^{n} a_{ik}\overline{a_{jk}} = 0$. Thus, for each $i \in \{1,\ldots,n\}$, there exists a unique $k(i)$ such that $a_{i,k(i)} \in T$ and $a_{i,k'} = 0$ for $k' \in \{1,\ldots,n\} - k(i)$. This implies $A \in \mathbb{M}_n$. □

Definition 4.10. Let $S \subset T$. Then S is said to have n-orthogonality, if there exist $a_1, b_1, \ldots, a_n, b_n \in S$ such that $\sum_{i=1}^{n} a_i\overline{b_i} = 0$.

For a matrix W of order n, define

$$Z_W(k,j) = |\{i \mid 1 \leq i \leq j,\ W_{ik} = 0\}| \quad (1 \leq j,k \leq n),$$

$$E_W(k) = \max\{i \mid W_{ik} = 0\} \quad (1 \leq k \leq n).$$

Lemma 4.11. *If W is a unit weighing matrix of order n and weight w, and $n > z^2 - z + 1$, where $z = n - w$, then the set $S = \{W_{ij} \mid 1 \leq i, j \leq n\} \setminus \{0\}$ has $(n-2z)$-orthogonality.*

Proof. If $z = 0$, then the result is clear. If $z = 1$, by the non-singularity of W, there exist two rows with distinct support. This implies that S has $(n-2)$-orthogonality.

Now assume $z > 1$. We may assume without loss of generality that

$$W_{1k} = 0 \quad (1 \leq j \leq z), \tag{4.1}$$

$$W_{i1} = 0 \quad (1 \leq i \leq z). \tag{4.2}$$

For $1 \leq k \leq z$, define

$$Z_k = \{i \mid 1 \leq i \leq n, \ W_{ik} = 0\}.$$

Then

$$|Z_k| = z, \tag{4.3}$$

$$1 \in Z_k \qquad \text{(by (4.1))}, \tag{4.4}$$

for $1 \leq k \leq z$, and

$$Z_1 = \{1, \ldots, z\} \qquad \text{(by (4.2))}. \tag{4.5}$$

This implies

$$\begin{aligned}
\left| \bigcup_{k=1}^{z} Z_k \right| &= 1 + \left| \bigcup_{k=1}^{z} (Z_k \setminus \{1\}) \right| && \text{(by (4.4))} \\
&\leq 1 + \sum_{k=1}^{z} (|Z_k| - 1) \\
&= 1 + \sum_{k=1}^{z} (z-1) && \text{(by (4.3))} \\
&= z^2 - z + 1 \\
&< n.
\end{aligned}$$

Thus, there exists

$$i \in \{1, \ldots, n\} \setminus \bigcup_{k=1}^{z} Z_k.$$

Observe $i \neq 1$ by (4.5). Since $i \notin Z_k$, we have $W_{ik} \neq 0$ for $1 \leq k \leq z$. This implies

$$|\{k \mid z+1 \leq k \leq n,\ W_{ik} = 0\}| = z. \tag{4.6}$$

Thus

$$\begin{aligned}
&|\{k \mid 1 \leq k \leq n,\ W_{1k} \neq 0,\ W_{ik} \neq 0\}| \\
&= |\{k \mid z+1 \leq k \leq n,\ \ W_{ik} \neq 0\}| && \text{(by (4.1))} \\
&= n - z - |\{k \mid z+1 \leq k \leq n,\ W_{ik} = 0\}| \\
&= n - 2z && \text{(by (4.6))}.
\end{aligned}$$

□

Lemma 4.12. *$UW(3,2)$ is an empty set.*

Proof. By Lemma 4.11, the existence of $UW(3,2)$ implies the existence of a set that has 1-orthogonality. But there is no set satisfying 1-orthogonality. Thus, $UW(3,2)$ is an empty set. □

Proposition 4.13. Let k be a positive integer. Then every matrix in $UW(2k,2)$ is equivalent to

$$\bigoplus_{i=1}^{k} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}. \tag{4.7}$$

Moreover, there is exactly one equivalence class of $UW(2k,2)$.

Proof. We will prove by induction on k. The case $k=1$ is clear. Without loss of generality, let $W \in UW(2k,2)$ be a matrix in a standard form. Then

$$W = \begin{bmatrix} 1 & 1 & \\ 1 & -1 & \\ & & W' \end{bmatrix}$$

where $W' \in UW(2(k-1),2)$. By inductive hypothesis, we have W' is equivalent to $\bigoplus_{i=1}^{k-1}\begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix}$. Therefore, W is equivalent to $\bigoplus_{i=1}^{k}\begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix}$.

Moreover, there is exactly one equivalence class of $UW(2k,2)$ since the number of equivalence classes of weighing matrices is bounded above by the number of the matrix that is in a standard form, and (4.7) is the only one standard form matrix in $UW(2k,2)$. □

Theorem 4.14. *$UW(n,2)$ is nonempty if and only if n is even.*

Proof. The "if" part is immediate from Proposition 4.13. Without loss of generality, assume that $W \in UW(n,2)$ is in a standard form. Suppose n is an odd positive integer. Then W is equivalent to

$$\begin{bmatrix}W_1 & \\ & W_2\end{bmatrix}$$

where $W_1 = \bigoplus_{i=1}^{(n-3)/2}\begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix}$ by Proposition 4.13. Hence, W_2 should be in $UW(3,2)$. But $UW(3,2)$ is an empty set by Lemma 4.12. Thus n should be an even positive integer. □

Lemma 4.15. *Let $W \in UW(n,3)$. Then W is equivalent to a matrix with the top left submatrix is either*

$$\begin{bmatrix}1 & 1 & 1\\ 1 & \alpha & \bar{\alpha}\\ 1 & \bar{\alpha} & \alpha\end{bmatrix} \quad \textit{or} \quad \begin{bmatrix}1 & 1 & 1 & 0\\ 1 & - & 0 & 1\\ 1 & 0 & - & -\\ 0 & 1 & - & 1\end{bmatrix}$$

where $\alpha + \overline{\alpha} + 1 = 0$.

Proof. For convenience, we denote by W_i the ith row of W. Without loss of generality, let W be in a standard form, i.e.

$$W_1 = \begin{pmatrix}1 & 1 & 1 & 0 & \cdots & 0\end{pmatrix}.$$

Thus, without loss of generality, W_2 is one of the following form:

$$(A)\begin{pmatrix}1 & a & b & 0 & \cdots & 0\end{pmatrix}, \quad (B)\begin{pmatrix}1 & a & 0 & 1 & 0 & \cdots & 0\end{pmatrix}.$$

Suppose $W_2 = (A)$. Then $a \in \{\alpha, \bar{\alpha}\}$ and $b = \bar{a}$ by comparing W_1 and W_2. Moreover, in this case, W_3, without loss of generality, is one of the following form.

(i) $\begin{pmatrix}1 & c & d & 0 & \cdots & 0\end{pmatrix}$,

(ii) $\begin{pmatrix}1 & c & 0 & 1 & \cdots & 0\end{pmatrix}$,

(iii) $\begin{pmatrix}1 & 0 & c & 1 & \cdots & 0\end{pmatrix}$.

If $W_3 =$ (ii), then $c = -1$ by comparing W_1 and W_3. But by comparing W_2 and W_3, we obtain $c = -a$. This is a contradiction. Similar conclusion is occured if we set $W_3 =$ (iii). For the case $W_3 =$ (i), we obtain $c = \bar{d} = \bar{a}$, the orthogonality condition is satisfied for each pair of rows. Thus, the top left submatrix of W is

$$\begin{bmatrix}1 & 1 & 1\\ 1 & a & \bar{a}\\ 1 & \bar{a} & a\end{bmatrix}.$$

Now, suppose $W_2 = (B)$. Then, $a = -1$. In this case, W_3 is equivalent to one of the following form:

(i) $\begin{pmatrix}1 & b & c & 0 & 0 & 0 & \cdots & 0\end{pmatrix}$,

(ii) $\begin{pmatrix}1 & b & 0 & c & 0 & 0 & \cdots & 0\end{pmatrix}$,

(iii) $\begin{pmatrix}1 & b & 0 & 0 & 1 & 0 & \cdots & 0\end{pmatrix}$,

(iv) $\begin{pmatrix}1 & 0 & b & c & 0 & 0 & \cdots & 0\end{pmatrix}$.

Thus, if $W_3 =$ (i), $b = 1$ by comparing W_3 and W_2, but $b \neq 1$ by comparing W_2 and W_3. If $W_3 =$ (ii), $b = -1$ by comparing W_3 and W_1, but $b \neq -1$ by comparing W_3 and W_2. If $W_3 =$ (iii), $b = -1$ by comparing W_3 and W_1, but $b \neq -1$ by comparing W_3 and W_2. If $W_3 =$ (iv), we have $b = c = -1$. Therefore, without loss of generality, $W_4 = \begin{pmatrix} 0 & 1 & d & f & 0 & 0 & \cdots & 0 \end{pmatrix}$ and hence $d = -f = -1$. So, we have the top left submatrix of W is

$$\begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & - & 0 & 1 \\ 1 & 0 & - & - \\ 0 & 1 & - & 1 \end{bmatrix}.$$

□

Theorem 4.16. *Let $W \in UW(n,3)$. Then there exist positive integers k and l such that W is equivalent to*

$$\left(\bigoplus_{i=1}^{k} \begin{bmatrix} 1 & 1 & 1 \\ 1 & \alpha & \bar{\alpha} \\ 1 & \bar{\alpha} & \alpha \end{bmatrix}\right) \bigoplus \left(\bigoplus_{i=1}^{l} \begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & - & 0 & 1 \\ 1 & 0 & - & - \\ 0 & 1 & - & 1 \end{bmatrix}\right)$$

where α is a primitive 3rd root of unity.

Proof. Note that $n \geq 3$. Without loss of generality, assume that W is in a standard form. We will prove by induction on n. The cases $n = 3$ and $n = 4$ are trivial. By Lemma 4.15, the top left submatrix of W is either

$$U_1 = \begin{bmatrix} 1 & 1 & 1 \\ 1 & \alpha & \bar{\alpha} \\ 1 & \bar{\alpha} & \alpha \end{bmatrix} \quad \text{or} \quad U_2 = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & - & 0 & 1 \\ 1 & 0 & - & - \\ 0 & 1 & - & 1 \end{bmatrix}.$$

Thus,

$$W = \begin{bmatrix} U_1 & \\ & W_1 \end{bmatrix} \quad \text{or} \quad W = \begin{bmatrix} U_2 & \\ & W_2 \end{bmatrix}$$

where $W_1 \in UW(n-3,3)$ and $W_2 \in UW(n-4,3)$. By inductive hypothesis, there exist positive integers k_1, l_1, k_2 and l_2 such that

$$W_1 = \left(\bigoplus_{i=1}^{k_1} \begin{bmatrix} 1 & 1 & 1 \\ 1 & \alpha & \bar{\alpha} \\ 1 & \bar{\alpha} & \alpha \end{bmatrix}\right) \bigoplus \left(\bigoplus_{i=1}^{l_1} \begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & - & 0 & 1 \\ 1 & 0 & - & - \\ 0 & 1 & - & 1 \end{bmatrix}\right)$$

and

$$W_2 = \left(\bigoplus_{i=1}^{k_2} \begin{bmatrix} 1 & 1 & 1 \\ 1 & a & \bar{a} \\ 1 & \bar{a} & a \end{bmatrix}\right) \bigoplus \left(\bigoplus_{i=1}^{l_2} \begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & - & 0 & 1 \\ 1 & 0 & - & - \\ 0 & 1 & - & 1 \end{bmatrix}\right).$$

The assertion follows immediately for $W = \begin{bmatrix} U_1 & \\ & W_1 \end{bmatrix}$. However, if $W = \begin{bmatrix} U_2 & \\ & W_2 \end{bmatrix}$, the result follows after appropriate row and column permutations. Hence, the proof is complete. □

Proposition 4.17. Let $n \neq 5$ be a positive integer. Then $n = 3k + 4l$ for some positive integers k and l.

Proof. Note that $n \equiv 0 \pmod 4$, $n \equiv 1 \pmod 4$, $n \equiv 2 \pmod 4$ or $n \equiv 3 \pmod 4$. The cases $n \equiv 0 \pmod 4$ and $n \equiv 3 \pmod 4$ are clear. Suppose $n = 4k+1$. We will prove this case by induction on k. For the case $k = 2$, clearly, $3 \mid 9$. Thus, we have $4k+1 = 4(k-1)+1+4 = 3k' + 4l' + 4$ for some k' and l' by inductive hypothesis. Thus, $4k + 1 = 3k' + 4(l' + 1)$. Now, we will prove the case $n = 4k+2$ by induction on k. The case $k = 1$ is clear since $3 \mid 6$. Thus, $4k+2 = 4(k-1)+2+4 = 3k'+4l'+4$ by inductive hypothesis. Therefore, $4k+2 = 3k'+4(l'+1)$. Thus, the proof is complete. □

Corollary 4.18. *$UW(n, 3)$ is non empty if and only if $n \neq 5$.*

Proof. Since there is no set with 1-orthogonality, $UW(5, 3)$ is an empty set by Lemma 4.11. Also, the "if" part is immediate by Proposition 4.17 and Theorem 4.16. □

Corollary 4.19. *There exists a $W(n,3)$ if and only if n is divisible by 4. Moreover, there is only one equivalence class of such matrices.*

Proof. Note that by Theorem 4.16, a matrix in $W(n,3)$ is equivalent to

$$\bigoplus_{i=1}^{k} \begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & - & 0 & 1 \\ 1 & 0 & - & - \\ 0 & 1 & - & 1 \end{bmatrix} \tag{4.8}$$

for some positive integer k and hence the first statement is satisfied. Moreover, since the number of inequivalent class of weighing matrices is bounded above by the number of the matrix that is in a standard form, and (4.8) is the only one standard form matrix of $W(4,3)$, there is only one class of inequivalent matrices in $W(n,3)$.

□

Bibliography

[1] H. Aslaksen, The Quaternionic Determinants, The Mathematical Intelligencer, vol. 18, 3 (1996), 57–65.

[2] D. Best, H. Kharaghani, H. Ramp, On unit matrices with small weight, Disc. Math. 313 (2013), 855–864.

[3] D. Best, H. Kharaghani, H. Ramp, Mutually unbiased weighing matrices, Des. Codes Cryptogr. 76 (2015), 237–256.

[4] D. Best, H. Kharaghani, Unbiased complex Hadamard matrices and bases, Cryptography and Communications 2, (2010), 199–209.

[5] G. Cohen, D. Rubie, J. Seberry, C. Koukouvinos, S. Kounias, and M. Yamada, A survey of base sequences, disjoint complementary sequences and $OD(4t; t, t, t, t)$, J. Combin. Math. Combin. Comput. 5 (1989), 69–103.

[6] R. Craigen, H. Kharaghani, Orthogonal designs in: Handbook of Comb. Des. (C.J. Colbourn and J.H. Dinitz., eds.), 2nd Ed., pp. 280–295, Chapman & Hall/CRC Press, Boca Raton, FL, 2007.

[7] R. Craigen, W. Holzmann, H. Kharaghani, Complex Golay sequences: structure and applications, Disc. Math. 252 (2000), 73–89.

[8] R. Craigen, W. Gibson, C. Koukouvinos, An update on primitive ternary complementary pairs, J. Combin. Theory Ser. A 114 (2007), 957–963.

[9] M.L. Curtis, Matrix Groups, New York: Springer-Verlag, 1979.

[10] H. Kharaghani, C. Koukouvinos, Complementary, base and Turyn sequences in: Handbook of Comb. Des. (C.J. Colbourn and J.H. Dinitz., eds.), 2nd Ed., pp. 317–321, Chapman & Hall/CRC Press, Boca Raton, FL, 2007.

[11] H. Kharaghani, B. Tayfeh-Rezaie, A Hadamard matrix of order 428, J. Combin. Designs 13 (2005), 435–440.

[12] N. Konno, H. Mitsuhashi, I. Sato, The quaternionic second weighted zeta function of a graph and the Study determinant, Linear Algebra and its Application 510 (2016), 92–109.

[13] C. Koukouvinos, J. Seberry, Weighing matrices and their applications, JSPI 62 (1997), 91–101.

[14] D. Ž. Đoković, K. Zhao, An octonion algebra originating in combinatorics, Proc. Amer. Math. Soc. 138 (2010), 4187–4195.

[15] K. J. Horadam, Hadamard matrices and their applications, Princeton University Press, Princeton and Oxford, 2007.

[16] C. Koukouvinos, J. Seberry, Addendum to Further results on base sequences, disjoint complementary sequences, $OD(4t; t, t, t, t)$, and the excess of Hadamard matrices, Cong. Numer. 82 (1991), 97–103.

[17] I. Livinskyi, Asymptotic existence of Hadamard matrices, M.Sc. thesis, University of Manitoba, `http://hdl.handle.net/1993/8915`, 2012.

[18] J. H. R. Ng, Quaternions and The Four Square Theorem, Summer VIGRE REU, 2008.

[19] J. Seberry, M. Yamada, Hadamard matrices, sequences and block designs, in Contemporary Design Theory: A Collection of Surveys, Eds. J.H. Dinitz and D.R. Stinson, J. Wiley, New York, 1992, pp. 431–560.

[20] J. Seberry, B J. Wysocki, T. A. Wysocki, On some applications of Hadamard matrices, Metrika, 62 (2005), 221–239.

[21] E. Study, Zur Theorie der linearen Gleichungen, Acta Math. 42 (1920), 1–61.

[22] C. H. Yang, Lagrange identity for polynomials and δ-codes of length $7t$ and $13t$, Proc. Amer. Math. Soc. 88 (1983), 746–750.

[23] C. H. Yang, On composition of four-symbol δ-codes and Hadamard matrices, Proc. Amer. Math. Soc. 107 (1989), 763–776.

[24] C. H. Yang, On Hadamard Matrices Constructible by Circulant Submatrices, Math. Comp. Vol 25, 113 (1971), 181–186.

[25] R. J. Turyn, An infinite class of Williamson matrices, J. Combin. Theory Ser. A, 12 (1972), 319–321.