Logical Investigations of Various Games

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Contents of Thesis

Abstract

This thesis consists of two parts. The first part treats eigen-distribution on game trees and the second part is to investigate some mathematical results which are used to prove several game properties in the context of reverse mathematics.

Game tree is a simple model of computation to analyze some difficult problems. We study the query complexity of algorithms on them. The uniform binary tree has been extensively investigated in many literatures. Compared with previous research in this area, we concentrate on multi-branching trees and eigen-distribution over independent distributions and correlated distributions for them.

In Chapter 4, we mainly show that for multi-branching trees, the uniqueness of eigen-distribution holds for the set of all alpha-beta pruning algorithms, although the uniqueness does not hold for the set of directional algorithms. In Chapter 5, we treat eigen-distribution over independent distributions (ID) for multi-branching trees. For any ID d, we define a directional algorithm DIR_d, and show it is optimal among all the depth-first algorithms with respect to d. Then we extend the Liu-Tanaka theorem for uniform binary trees to balanced multi-branching case, which has been left open since it was posed in [3].

Part II is devoted to investigating mathematics in second order arithmetic, including arguments of fixed point theorems and continuous games in the context of reverse mathematics. Since we know that fixed point theroems are used to prove the existence of Nash equilibria, we first investigate the logical strength of some fixed point theorems in the context of reverse mathematics. Then we move on to the study on reverse mathematics and continuous game, which is a mathematical generalization of finite games.

In Chapter 8, we investigate the logical strength of two types of fixed point theorems in the context of reverse mathematics. One is concerned with extensions of Banach contraction principle. We mainly show that Caristi fixed point theorem is equivalent to ACA_0 over RCA_0. The other is dedicated to topological fixed point theorems like Brouwer fixed point theorem. We introduce the variants of Fan-Browder fixed point theorem and Kakutani fixed point theorem, say FBFP and KFP respectively. Then we show that FBFP is equivalent to WKL_0 and KFP is equivalent to ACA_0, over RCA_0. We also study some applications of Fan-Browder fixed point theorem to game systems.

In Chapter 9, we investigate reverse mathematics and continuous games. We present a new version of Stone-Weierstrass theorem, which can define a product measure for a given mixed profile. Working in RCA_0, we show that the following is equivalent to ACA_0: any sequence of probability measures on a compact space has a weak convergent subsequence. Using this result, we show that the existence of mixed Nash equilibria for any continuous game is provable in ACA_0.

Part I: Game Trees

The main objective of the first part of my thesis is to investigate the query complexity of multi-branching trees, which is a continuation of study by Liu and Tanaka [3] on uniform binary AND-OR trees. We first extend their study to a multi-branching case, then investigate the optimal depth-first algorithms and equilibria of independent distributions on multi-branching trees.

By *balanced multi-branching*, we mean that all the nonterminal nodes at the same level have the same number of children and all paths from the root to the leaves are of the same length. It should be noted that the balancedness makes no restriction on the number of children for nodes at different levels. In this paper, we concentrate on T^h_n, n-branching trees of height h. We here notice that the argument for the uniform binary trees T^h_2 cannot be generalized to T^h_n $(n > 2)$ directly, since T^h_n inevitably corresponds to a non-uniform binary tree.

An AND-OR tree (OR-AND tree, respectively) is a tree whose root is labeled AND (OR), and sequentially the internal nodes are level-by-level labeled by OR-node and AND-node (AND-node and OR-node) alternatively. Each probed leaf is assigned with Boolean value 0 or 1, via an assignment. By evaluating a tree, we are trying to compute the Boolean value of the root. We start from probing the leaves. Each leaf returns its value. The computation stops when we get enough information to evaluate the root value of the tree. The cost of computation is the number of the leaves that are queried during this computation, regardless of the remaining unqueried leaves.

An algorithm tells us how to proceed to evaluate a tree. The performance of algorithms makes a significant effect on the cost of computation. Among all these algorithms, alpha-beta pruning algorithm is known as one of the classical and effective algorithms [17] [6]. Knuth and Moore [2] conducted a detailed study on the alpha-beta pruning algorithm, which we briefly explain as follows. While evaluating an AND-node, if some child returns value 0, then the value of the AND-node is regarded as 0 without searching other children of this AND-node (which is known as an alpha-cut). On the other hand, when evaluating an OR-node, if some child returns value 1, then the value of the OR-node is recognized as 1 without searching other children of this OR-node (which is known as a beta-cut). An algorithm is directional if it queries the leaves in a fixed order, independent from the query history [5]. If an algorithm proceeds depending on its query history, then we say it is non-directional. In this study, we denote $\mathcal{A}_D$ the set of all alpha-beta pruning algorithms, and $\mathcal{A}_{dir}$ the set of all directional algorithms.

A randomized algorithm is a distribution over a family of deterministic algorithms. For a randomized algorithm, cost is computed as the expected cost over the corresponding family of deterministic algorithms. Yao's principle [18] indicates the relation between randomized complexity and distributional complexity as follows,

$$\underbrace{\min_{A_R}\max_{\omega} cost(A_R,\omega)}_{\text{Randomized complexity}} = \underbrace{\max_{d}\min_{A_D} cost(A_D,d)}_{\text{Distributional complexity}}.$$

where A_R ranges over randomized algorithms, ω over assignments for leaves, d over distributions on assignments and A_D over deterministic algorithms. This result provides a new perspective to analyze randomized complexity. Saks and Wigderson [11] showed that for any n-branching tree, the randomized complexity is

$$\Theta((\frac{n-1+\sqrt{n^2+14n+1}}{4})^h),$$

where h is the height of tree.

Recently, several works have been done for uniform binary trees. Based on Saks and Wigderson [11], Liu and Tanaka [3] proposed the concept of eigen-distribution (a distribution which achieves the distributional complexity) on assignments. They claimed that an eigen-distribution among the independent distributions (ID) is actually independently and identically distributed (IID). Recently, Suzuki and Niida [16] investigated the case where the probability of root is constrained for uniform binary trees.

Liu and Tanaka [3] also introduced a reverse assigning technique to formulate sets of assignments for T_2^h, namely 1-set and 0-set, in the case that assignments to leaves are correlated distributed (CD). They showed that E^1-distribution (a distribution on 1-set such that all deterministic algorithms have the same cost) is a unique eigen-distribution (the Liu-Tanaka Theorem). Suzuki and Nakamura [15] furthermore studied certain subsets of alpha-beta pruning algorithms on T_2^h and proved that the eigen-distribution with respect to a "closed" subset of alpha-beta pruning algorithms is unique, but for a set of directional algorithms, the uniqueness does not hold.

Compared with previous research in this area, we concentrate on multi-branching trees and eigen-distribution under ID and CD for them.

Chapter 4 contains our results on eigen-distribution for balanced multi-branching trees under CD. At first, we introduce some important notions on n-branching game trees.

Definition 1 (Transposition of assignment)**.** For T_n^h, suppose that u is an internal node, and ω an assignment. The i-th u-transposition of ω, denote $\mathrm{tr}_i^u(\omega)$, is defined by $\mathrm{tr}_i^u(\omega)(v) = \omega(\mathrm{tr}_i^u(v))$, where v is a leaf of T_n^h.

Definition 2 (Transposition of algorithm)**.** For T_n^h, suppose that u is an internal node, and A an algorithm in $\mathcal{A}_D$. For each assignment ω and the query history $(\alpha^1,\cdots,\alpha^m)$ of $(A,\mathrm{tr}_i^u(\omega))$, the i-th u-transposition of $\mathbb{A}$, denote $\mathrm{tr}_i^u(A)$, has the query history $(\beta^1,\cdots,\beta^m)$ such that $\beta^j = \mathrm{tr}_i^u(\alpha^j)$

for each $j \leq m$.

Definition 3 (Equivalent assignment class, closedness)**.** For T_n^h, any assignments ω, ω', we denote $\omega \approx \omega'$ if $\omega' = \mathrm{tr}_i^u(\omega)$ for some u and i. An assignment ω is equivalent to ω' if there exists a sequence of assignments $\langle \omega_i \rangle_{i=1,\cdots,s}$ such that $\omega \approx \omega_1 \approx \cdots \approx \omega_s \approx \omega'$ for some $s \in \mathbb{N}$. Then we denote $[\![\omega]\!]$ as the equivalent assignment class of ω.

- A set Ω of assignments is closed if $\Omega = \bigcup_{\omega \in \Omega} [\![\omega]\!]$.
- Given $\mathcal{A} \subseteq \mathcal{A}_D$, $\mathcal{A}$ is closed (under transposition) if for any $A \in \mathcal{A}$, each internal node u and $i < n$, $\mathrm{tr}_i^u(A) \in \mathcal{A}$.

Then, we give some technical lemmas to prove that the average cost on 1-set is larger than that on other closed sets. Based on these results, we show the following results.

Theorem 1. *Assume an AND-OR tree T_n^h, d is a probability distribution on the assignments, $\mathcal{A}$ is a closed subset of $\mathcal{A}_D$. Then the following two conditions are equivalent.*

a) d is an eigen-distribution w.r.t. $\mathcal{A}$.

b) d is an E^1-distribution w.r.t. $\mathcal{A}$.

But the uniqueness of eigen-distribution does not hold when we only consider directional algorithms.

Theorem 2. *For any tree T_n^h ($h \geq 2$), there are uncountably many eigen-distributions w.r.t. $\mathcal{A}_{dir}$.*

In Chapter 5, we treat eigen-distribution over independent distributions (ID) for multi-branching trees. For any ID d, we define a directional algorithm DIR_d, and show it is optimal among all the depth-first algorithms with respect to d. Applying this result, we show that, for any ID d, there exists an IID d' such that the expected cost with d is not larger than that with d' following DIR_d. Then we extend the Liu-Tanaka's results for uniform binary trees to balanced multi-branching case, which has been left open since it was posed in [3].

Theorem 3. *For any balanced multi-branching AND-OR tree T, suppose that $d \in$ ID and $p_\lambda(d) \neq 0$ or 1. If d achieves the distributional complexity, then, $d \in$ IID.*

The results represented in Chapters 4 have been published in [7] and [8], and the results in Chapter 5 appeared in [9].

Part II: Reverse Mathematics

Part II is devoed to investigating mathematics in second order arithmetic, especially arguments of functional analysis concerned with fixed point theorems and continuous game in the context of reverse mathematics. Reverse mathematics is a research program in the foundations of mathematics. It was founded by Harvey Friedman and Stephen Simpson in the 1970's and developed in many publications. A basic goal of reverse mathematics is to classify mathematical theorems according to the set existence axioms used in their proofs [14].

In our study, we mainly consider three kinds of subsystems of second order arithmetic, which are RCA_0, ACA_0 and WKL_0. RCA_0 is the formal system of recursive comprehension that guarantees the existence of recursively definable sets. This is the weakest system we will discuss, but it is still strong enough to explore some basic theorems. For example, the intermediate value theorem and Banach fixed point theorem are provable in RCA_0. The axioms of WKL_0 are those of RCA_0 and Weak König's Lemma, which asserts that each infinite tree of finite sequences of 0's and 1's has an infinite path. Many theorems, such as Hahn-Banach theorem for separable Banach space are equivalent to WKL_0 over RCA_0. The system ACA_0 stronger than WKL_0 consists of RCA_0 plus the arithmetic comprehension axioms, and it proves many theorems related to convergence. For details of the definitions of these three subsystems, see [14].

Though fruitful reverse mathmatical results have been achieved in various fields, game theory is so far one of the undeveloped lands. In the second part of my thesis, we mainly investigate some mathematical results which are used to prove several game properties in the context of reverse mathematics.

Since we know that fixed point theroems are used to prove the existence of Nash equilibria of games, we first investigate the logical strength of some fixed point theorems in the context of reverse mathematics. The logical strength of Brouwer fixed point theorem is investigated by Shioji and Tanaka [13]. They showed that it is equivalent over RCA_0 to WKL_0, which directly implies that the existence of Nash equilibria for finite game is provable in WKL_0.

Roughly speaking, we can divide fixed point theorems into three groups. The first group is a collection of fixed point theorems derived from the metric properties of the underlying spaces and self-maps, such as Banach fixed point theorem, say metric fixed point theorems. The second group consists of fixed point theorems derived from the topological properties such as Brouwer fixed point theorem, say topological fixed point theorem. The third consists of fixed point theorems for certain types of posets, say order-theoretic fixed point theorems. For this aspect, Sato and Yamazaki [12] have investigated reverse mathematics on order-theoretic fixed point theorems.

The purpose of Chapter 8 is to study reverse mathematical aspects of metric and topological fixed point theorems. We first concentrate on extensions of Banach contraction principle. Among theorems in this type, we mainly show that

Theorem 4 (RCA_0). *The following are pairwise equivalent.*

1. ACA_0.

2. *Caristi fixed point theorem: Let $\widehat{X}$ be a complete separable metric space, and let $f : \widehat{X} \to \widehat{X}$ and $g : \widehat{X} \to [0, +\infty)$ be continuous functions. Suppose that, for all $x \in \widehat{X}$,*

$$d\big(x, f(x)\big) \leq g(x) - g\big(f(x)\big).$$

Then f has a fixed point on $\widehat{X}$.

Then, we introduce the variants of Fan-Browder fixed point theorem and Kakutani fixed point theorem, say FBFP and KFP respectively. We show the following results.

Theorem 5. *The following assertions are pairwise equivalent over* $\mathsf{RCA_0}$.

1. $\mathsf{ACA_0}$.

2. KFP.

Theorem 6. *The following assertions are pairwise equivalent over* $\mathsf{RCA_0}$.

1. $\mathsf{WKL_0}$.

2. FBFP.

On the other hand, we also concentrate on reverse mathematics and continuous game, which is a mathematical generalization used in finite games. A continuous game is usually defined and studied as an infinite game (game with infinte strategies) in which the strategy sets are compact and the payoff functions are continuous. Glicksberg [1] presented a generalized version of Kakutani fixed point theorem and showed the existence of mixed Nash equilibrium for continuous games. Our proof is based on the Glicksberg's theorem proved by Ozdaglar [4], which approximated the continuous game to the sequence of finite games.

In Chapter 9, we first investigate some basics of probability and present a new version of Stone-Weierstrass theorem, which can define a product measure. Working in $\mathsf{RCA_0}$, we show that any sequence of probability measures on a compact space that has a weak convergent subsequence is equivalent to $\mathsf{ACA_0}$. Then, we have the following result.

Theorem 7 ($\mathsf{RCA_0}$)**.** *The following assertions are pairwise equivalent.*

1. $\mathsf{ACA_0}$.

2. *Let $\langle G_i : i \in \mathbb{N} \rangle$ be a sequence of n-player finite games. Then, there exists a sequence of Nash equilibria $\langle \mu_i : i \in \mathbb{N} \rangle$ such that μ_i is a Nash equlibrium for G_i.*

3. *Let $\langle G_i : i \in \mathbb{N} \rangle$ be a sequence of 1-player finite games. Then, there exists a sequence of Nash equilibria $\langle \mu_i : i \in \mathbb{N} \rangle$ such that μ_i is a Nash equlibrium for G_i.*

Using these results, we show that the existence of mixed Nash equilibria for any continuous game is provable in $\mathsf{ACA_0}$.

The results in Chapter 8 appeared in [10].

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