The minimal generating set of the presentation ideal of Backelin semigroup ring

Abstract

Let K be a field, and let $n_1, \ldots, n_r$ be positive integers. The Ideal of relations of the semigroup ring $K[x_1, \ldots, x_r]$ generated by $n_1, \ldots, n_r$ is $P(n_1, \ldots, n_r) = \{f(x_1, \ldots, x_r) : f(x_1, \ldots, x_r) \in K[x_1, \ldots, x_r], f(t^{n_1}, \ldots, t^{n_r}) = 0\}$ with t is transcendental over K.

In 1970, Herzog showed that the least upper bound on the number of generators of $P(n_1, \ldots, n_r)$ for $r = 3$ is 3. In 1975, Bresinsky showed that the lowest upper bound on the number of generators of $P(n_1, \ldots, n_r)$, can be arbitrarily large if $r \geq 4$. More recently, Herzog and Stamate provided a closed form for the number of generators for the semigroup ring in Bresinskys example, and showed that in this case, the number of generators for the semigroup ring is arbitrarly large but even. Progress have been made in finding a closed form for the number of generators of ideal of relations for various semigroup rings with four or more generators. All established work on this subject produced examples where this number is always an **even** number.
However, in 2017, Stamate considered a semigroup suggested by Backelin, which has the following structure

$$\big\langle r(3n+2)+3,\ r(3n+2)+6,\ r(3n+2)+3n+4,\ r(3n+2)+3n+5 \big\rangle$$

where $n \geq 2$ and r $\geq 3n$+2. Stamate reports that, computations using Singular and GAP (Groups, Algorithms, Programming - a System for Computational Discrete Algebra) indicate that the number of generators for this semigroup is $3n$+4, which can be an **odd** number. In 2018 we showed that the conjecture holds for the special case where $n = 2$. In this work, we prove that the result holds for any n.