Locating Two Public Bads in an Interval

Peter Zhang*

Abstract

Public bads are facilities that are necessary for the whole society, but unfortunately, entail negative externalities for the social-economic welfare in the surrounding areas. The main objective of the present thesis is to find optimal locations of two public bads in a region given the preferences of two agents located there. Based on lexmin preference as the joint preference of each of the agents for locating pairs of public bads, the present thesis has defined and proved eleven lemmas determining the implications of strategy-proofness, anonymity and unanimity properties on the decision rule. As an extension to the traditional approach, this thesis has determined complete characterization of a set of strategy-proof, anonymous and unanimous rules, and shows that combination of these properties allows for inner solutions for optimal locations of two public bads in an interval given the preferences of the agents in the region.

1 Introduction

Public bads are facilities that are necessary for the whole society, but unfortunately, entail negative externalities for the social-economic welfare in the surrounding areas. For example, nuclear power plants or disposals of household are typical public bads. It is crucial to identify the most suitable location for a public bad in the related planning process.
The main objectives of the present thesis are to find optimal locations of two public bads in a region given the preferences of two agents in the region, to examine the implications of certain properties like strategy-proofness, anonymity and unanimity on the decision rule, and to determine if any combination of these properties allow for any inner solution. As a basic assumption, the considered region is modelled as an interval from 0 to 1 and each agent has an unique point in the interval which is denoted by the dip of the agent. Since negative effects of a public bad on an agent decreases with increasing distance between them, each agent has its single-dipped-preferences and its dip is thus the worst location of any of the two bads for the concerned agent. Preference of an agent to any other point increases as it becomes more distant from its dip. For locating two public bads, it is necessary to consider joint preference for each agent.
In this paper, I consider lexmin preference as joint preference. The preference concerning two pairs of location is determined by the distance to the nearer public bad and, in case of a tie that means two locations with the same distance to the nearer public

*Peter Zhang received a bachelor degree in Econometrics & Operations Research at Maastricht University in 2016. He is currently studying towards an MSc Finance degree at Imperial College London. Contact: peter.zhang@student.maastrichtuniversity.nl

bad, the preference of an agent is determined by its distance to the farer public bad. Furthermore, implications of three properties on the social choice rule are considered, namely strategy-proofness, anonymity and unanimity. Strategy-proofness ensures that unilateral misreporting is not beneficial for the agent, while unanimity implies that if all agents have the same top ranked alternatives, the social choice rule will select them. Anonymity means the outcome of the rule is invariant with respect to the permutation of the agents. Hence, the main tasks of the present thesis are to examine implications of the combination of the concerned three properties on the decision rule and to determine optimal locations of the concerned public bads. The implications of these properties will be analysed in detail and formally proved. The present paper makes use of the proving approach illustrated by Moulin (1980). In addition, it will be examined under which conditions these properties allow for an inner solution. This methodological approach extends the traditional approach for social choice rules, such as locating one public bad on a line with a strategy-proof and Pareto optimal rule (Barbera, 2012) or determining the location of one public bad on some particular region in two dimensional plane (Öztürk et al., 2014), thereby taking into account finitely many agents. Lastly, another main objective is to define the full characterisation of strategy-proof, anonymous and unanimous rules that select inner point. In Gibbard (1973) it is shown that given three or more alternatives, there is no nondictatorial, strategy-proof and Pareto optimal decision rule when the full preference domain is considered. However, in the present thesis only single-dipped preferences is considered and unanimity instead of Pareto optimality is taken into account.

The present thesis consists of five sections. Following the introduction, Section 2 describes the model setting. The internal solutions are addressed by Section 3. Section 4 presents the complete characterization of a set of strategy-proof, anonymous and unanimous rules for finding inner solutions for optimal locating of the concerned public bads. Finally, Section 5 presents the key findings and concluding statements.

2 Model

As mentioned in Section 1, the region concerned is modelled by $A := [0, 1]$. Let $N = \{i, j\}$ be the set of the agents. It is assumed that the two public bads are symmetric which implies that (a, b) and (b, a) are the same alternatives. Thus, the set of alternatives is $\mathcal{A} := \{(a, b) \in A \times A : 0 \leq a \leq b \leq 1\}$. For all $k \in N$, let $x(k)$ be the dip of agent k. According to the marginal single-dip preference, for any two points, $a \in A$ and $b \in A$, a is at least as good as b if $|a - x(i)| \geq |b - x(i)|$. If the inequality is strict, then a is strictly preferred to b. Denote $R_{x(i)}$ as the joint weak preference. According to the joint lexmin preference, for all $(a_1, b_1), (a_2, b_2) \in \mathcal{A}$, (a_1, b_1) is at least as good as (a_2, b_2) at $R_{x(i)}$, or $(a_1, b_1)R_{x(i)}(a_2, b_2)$, if

$$\min\{|a_1 - x(i)|, |b_1 - x(i)|\} > \min\{|a_2 - x(i)|, |b_2 - x(i)|\} \text{ , or}$$

$$\min\{|a_1 - x(i)|, |b_1 - x(i)|\} = \min\{|a_2 - x(i)|, |b_2 - x(i)|\} \text{ and}$$
$$\max\{|a_1 - x(i)|, |b_1 - x(i)|\} \geq \max\{|a_2 - x(i)|, |b_2 - x(i)|\}.$$

Note that the joint preferences of the agents are uniquely identified through the dips. Hence, the domain of the preference profile is $A \times A$, which is denoted by R.

Profile of the preference is defined as follows: $\mathbf{x}$ assigns a preference $x(k)$ to all agent $k \in N$. For any dip $x(i)$, let $\tau_1(R_{x(i)})$ denote the set of all top ranked alternatives according to $R_{x(i)}$.

In this setting, a decision rule ϕ assigns an outcome $\phi(\mathbf{x}) \in \mathcal{A}$ to every preference profile $\mathbf{x}$. Correspondingly, the precise way of expressing the three properties on ϕ is introduced below.

Call rule ϕ

- *Strategy-proof* if no agent can gain by unilaterally misreporting his preference. In other words, for all $k \in N$, $\phi(\mathbf{x})R_{x(i)}\phi(\mathbf{x}')$ for any $\mathbf{x}, \mathbf{x}' \in R$ such that $\mathbf{x}(N \setminus \{k\}) = \mathbf{x}'(N \setminus \{k\})$

- *Unanimous* if for any preference profile $(x(i), x(j))$ with $\tau_1(R_{x(i)}) \cap \tau_1(R_{x(j)}) \neq \emptyset$, then $\phi(x(i), x(j)) \in \tau_1(R_{x(i)}) \cap \tau_1(R_{x(j)})$.

- *Anonymous* if for every permutation $\pi : N \to N$ and for every preference profile $\mathbf{x} \in R$,
$$\phi(\pi(\mathbf{x})) = \phi(\mathbf{x}).$$

In the next section, an in-depth analysis on internal solutions is presented.

3 Internal solutions

In this section, it is shown how a strategy-proof, anonymous and unanimous rule selects an inner solution. In order to define and prove the lemmas, the following cases are considered:

- $R_1 = \{\mathbf{x} \in R : x(i) \leq \frac{1}{2}, x(j) \leq \frac{1}{2}\}$,

- $R_2 = \{\mathbf{x} \in R : x(i) \geq \frac{1}{2}, x(j) \geq \frac{1}{2}$ and $x(k) > \frac{1}{2}$ for at least one $k \in N\}$,

- $R_3 = \{\mathbf{x} \in R : x(i) < \frac{1}{2}, x(j) > \frac{1}{2}$ and $|x(j) - x(i)| < \frac{1}{2}\}$ and

- $R_4 = \{\mathbf{x} \in R : x(i) < \frac{1}{2}, x(j) > \frac{1}{2}$ and $|x(j) - x(i)| \geq \frac{1}{2}\}$.

Without loss of generality, assume that for all $\mathbf{x} \in R$, $x(i) \leq x(j)$. This is possible as $\phi(\mathbf{x})$ is anonymous. Let $\phi(\mathbf{x})$ be also a strategy-proof, anonymous and unanimous rule. Note that the partitions are mutually exclusive and collectively exhaustive, i.e. $R = R_1 \cup R_2 \cup R_3 \cup R_4$ and $R_i \cap R_j = \emptyset$ for all $i \neq j$.

Lemma 1. *For any profile* $\mathbf{x} \in R_1$, $\phi(\mathbf{x}) \in \{(0,0), (0,1), (1,1)\}$

Proof. As $x(i), x(j) \leq \frac{1}{2}$, the rule will only choose $(1,1)$ if either $x(i) < \frac{1}{2}$ or $x(j) < \frac{1}{2}$. Do also note that if $x(i) = x(j) = \frac{1}{2}$, then the decision rule chooses one element of the set $\{(0,0), (0,1), (1,1)\}$. □

Lemma 2. *For any profile* $\mathbf{x} \in R_2$, $\phi(\mathbf{x}) = (0,0)$.

Proof. Since $x(k) > \frac{1}{2}$ for at least one $k \in N$, by unanimity the rule chooses $(0,0)$. □

Lemma 3. *For any profile* $\mathbf{x} \in R_3$, $\phi(\mathbf{x}) = (0,0)$ *or* $(1,1)$ *or* $(0,1)$.

Proof. Suppose $\phi(\mathbf{x}) = (a, b)$ where $0 < a \leq b < 1$. Consider a or $b \in (2x(j) - 1, 1)$, then agent j can misreport his dip below $\frac{1}{2}$ and get better off with $(1, 1)$ by Lemma 1, leading to a violation of strategy-proofness. Similarly, if a or $b \in (0, 2x(i))$, then strategy-proofness does not hold because of Lemma 2. Hence, $a, b \notin (2x(j) - 1, 1)$ and $a, b \notin (0, 2x(i))$.
Now consider $2x(i) \leq a, b \leq 2x(j) - 1$ which is shown in Figure 1. This implies $2x(i) \leq 2x(j) - 1$ leading to $x(j) - x(i) \geq \frac{1}{2}$. This contradicts the assumption that $\mathbf{x} \in R_3$. Hence, $\phi(\mathbf{x}) \neq (a, b)$. Similarly, $\phi(\mathbf{x}) \notin \{(0, a), (b, 1)\}$.

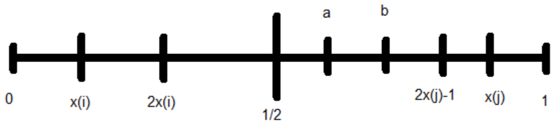

Figure 1: $2x(i) \leq a, b \leq 2x(j) - 1$

□

Lemma 4. *For any two profiles $\mathbf{x}, \mathbf{x}' \in R_3$, $\phi(\mathbf{x}) = \phi(\mathbf{x}')$.*

Proof. From Lemma 3, $\phi(\mathbf{x})$ can be $(0, 0), (0, 1)$ or $(1, 1)$. Suppose $\phi(\mathbf{x}) = (0, 0)$. This outcome is the worst from the three alternatives for agent i whose dip is $x(i) < \frac{1}{2}$. Then, for any $x'(i)$, $\phi(x'(i), x(j)) = (0, 0)$ by strategy-proofness. According to $x(j)$ and $x'(j)$, $(0, 0)$ is the top ranked alternative. Then, $\phi(x'(i), x'(j)) = (0, 0)$ for similar reason by strategy-proofness.
Now assume that $\phi(\mathbf{x}) = (0, 1)$. $\phi(x'(i), x(j)) = (1, 1)$ is not possible as agent i would deviate his dip from $x(i)$ to $x'(i)$ and be better off. Also, $\phi(x'(i), x(j)) = (0, 0)$ cannot be an outcome as the agent i would misreport his dip from $x'(i)$ to $x(i)$ in order to be better off. Thus, $\phi(x'(i), x(j)) = (0, 1)$. Now the outcome for two different dips of agent j are compared, namely $x(j)$ and $x'(j)$. If $\phi(x'(i), x'(j)) = (1, 1)$, deviation $x'(j) \to x(j)$ is more preferred for agent j. Else if $\phi(x'(i), x'(j)) = (0, 0)$, deviation $x(j) \to x'(j)$ is beneficial for agent j. Hence, $\phi(x'(i), x'(j)) = (0, 1)$.
Finally, it can also be shown that if $\phi(\mathbf{x}) = (1, 1)$, then $\phi(x'(i), x'(j)) = (1, 1)$ for any profile $(x'(i), x'(j))$ for the similar reason as above. □

The Lemmas 1, 2, 3 show that for any profile in R_1, R_2 and R_3, a strategy-proof, anonymous and unanimous rule cannot choose inner point. In addition, Lemma 4 shows that, such a rule is constant over all $\mathbf{x} \in R_3$

Lemma 5. *If $\mathbf{x} \in R_4$ and $\phi(\mathbf{x}) = (0, a)$, then there exists a $\mathbf{x}' \in R_4$ such that $\phi(\mathbf{x}') = (0, a)$, $x'(i) > 0$, $x'(j) < 1$ and $0 < a < 1$*

Proof. Suppose $x(i) = 0$ and $x(j) = 1$. Consider deviation from $x(i)$ to $x'(i)$ where $x'(i) < \frac{a}{2}$. Then, by strategy proofness, $\phi(x'(i), x(j)) = (0, a)$ as any other outcome would incentivise agent i to deviate. Now suppose $\phi(x'(i), x'(j)) = (\alpha, \beta)$. In addition, consider deviation from $x(j)$ to $x'(j)$ where $\frac{3+a}{4} < x'(j)$. If $\beta < a$, then it is beneficial to misreport for agent j. Now, consider deviation from $x'(j)$ to $x(j)$. If $\beta > a$, then it is more preferred to misreport for agent j. Hence, $\beta = a$. By lexmin preference, $\alpha = 0$. □

Corollary 1. *If $\mathbf{x} \in R_4$ and $\phi(\mathbf{x}) = (b, 1)$, then there exists a $\mathbf{x}' \in R_4$ such that $\phi(\mathbf{x}') = (b, 1)$, $x'(i) > 0$, $x'(j) < 1$ and $0 < b < 1$*

Proof. This is similar as the proof for Lemma 5. □

Lemma 6. *For any profile* $\mathbf{x} \in R_4$, $\phi(\mathbf{x}) \neq (0, a)$ *where* $a \in (0, 1)$.

Proof. Suppose $\phi(\mathbf{x}) = (0, a)$. Choose $x'(i)$ such that $x(i) < x'(i) < \frac{1}{2}$ and $x(j) - x'(i) < \frac{1}{2}$. Existence of such $x'(i)$ and $x(j)$ is guaranteed through Lemma 5. Then by Lemma 3, $\phi(x'(i), x(j))$ can be $(0, 0), (0, 1)$ or $(1, 1)$. This leads to the following three cases that need to be examined:

Case 1 First, assume $\phi(x'(i), x(j)) = (0, 1)$, then agent i misreports his dip from $x(i)$ to $x'(i)$. He benefits through deviating which violates strategy-proofness.

Case 2 Similarly, if $\phi(x'(i), x(j)) = (1, 1)$, then agent i deviates his dip from $x(i)$ to $x'(i)$ violating strategy-proofness.

Case 3 Finally, suppose $\phi(x'(i), x(j)) = (0, 0)$. Then, for $\epsilon > 0$ choose a $x''(i)$ such that $x''(i) = \frac{a}{2} - \epsilon$ and $x'(j)$ such that $|x'(j) - x''(i)| < \frac{1}{2}$. Hence, the profile $(x'(j), x''(i)) \in R_3$.
Then, by the Lemma 4, $\phi(x''(i), x'(j)) = (0, 0)$. Now consider the profile $(x''(i), x(j))$, then $\phi(x''(i), x(j))$ must be equal to j's top ranked alternative $(0, 0)$, otherwise deviation $x(j) \to x'(j)$ is beneficial for agent j.

But then, agent i can deviate from $x''(i)$ to $x(i)$ violating strategy-proofness. Hence, $\phi(x'(i), x(j)) \neq (0, 0)$.

□

Lemma 7. *For any profile* $\mathbf{x} \in R_4$, $\phi(\mathbf{x}) \neq (b, 1)$ *where* $b \in (0, 1)$.

Proof. The proof is similar to Lemma 6. □

Lemmas 6 and 7 show that if $\mathbf{x} \in R_4$, inner solutions $(0, a)$ and $(b, 1)$ where $0 < a, b < 1$ cannot be chosen. However, for any profile in R_4, the specific type of inner solution (b, c), where $0 < b, c < 1$, can be selected. This is shown in Lemma 8.

Lemma 8. *If for a profile* $\mathbf{x} \in R_4$ *and* $\phi(\mathbf{x}) = (a, b)$, *then* $2x(i) < a, b < 2x(j) - 1$.

Proof. Note that, if $a < 2x(i)$ (or $b > 2x(j) - 1$), then i (or j) can deviate to $\frac{1}{2}$ and be better off. So suppose $a = 2x(i)$, $x'(i) > x(i)$ and $|x(j) - x'(i)| < \frac{1}{2}$. This is depicted in Figure 2. Consider deviation from $x(i)$ to $x'(i)$. Then, $\phi(x'(i), x(j))$ can take the value $(0, 0), (0, 1)$ or $(1, 1)$ by Lemma 3. More specifically, the following three cases have to be addressed:

Case 1 Suppose $\phi(x'(i), x(j)) = (1, 1)$ which is agent i's top ranked alternative. Thus, he would deviate from $x(i)$ to $x'(i)$ violating strategy-proofness.

Case 2 Suppose $\phi(x'(i), x(j)) = (0, 0)$. By Lemma 4, for any profile in R_3, the outcome is $(0, 0)$. Hence, the outcome for the profile $(x(i), x'(j))$ such that $x'(j) > \frac{1}{2}$ and $|x'(j) - x(i)| < \frac{1}{2}$ is also $(0, 0)$. This gives agent j an opportunity to deviate from $x(j)$ to $x'(j)$ and get his top ranked alternative violating strategy-proofness.

Case 3 Suppose $\phi(x'(i), x(j)) = (0, 1)$. Agent i prefers $(0, 1)$ to (a, b) because $|a - x(i)| = |x(i) - 0|$ and $|b - x(i)| < |1 - x(i)|$. So deviation from $x(i)$ to $x'(i)$ makes agent i better off. Hence, $a \neq 2x(i)$.

Similarly, it can be shown that $b \neq 2x(j) - 1$.

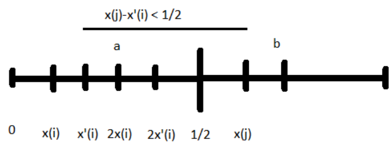

Figure 2: $2x(i) = a$

□

Lemma 9. *If for a profile $\mathbf{x} \in R_4$ and $\phi(\mathbf{x}) = (a, b)$, where $0 < a \leq b < 1$, then for any profile $\mathbf{x}' \in R_3$, $\phi(\mathbf{x}') = (0, 1)$*

Proof. Suppose $\phi(\mathbf{x}) = (a, b)$ for some $\mathbf{x} \in R_4$. Next, it is shown in the following two cases that $\phi(\mathbf{x}') \notin \{(0, 0), (1, 1)\}$ by contradiction.

Case 1 Suppose $\phi(\mathbf{x}') = (0, 0)$ where $\mathbf{x}' \in R_3$. For agent j, $(0, 0)$ is the top ranked alternative. Consider the deviation from $x(j)$ to $x'(j)$ such that $x'(j) > \frac{1}{2}$ and $|x'(j) - x(i)| < \frac{1}{2}$. So, the profile $(x(i), x'(j)) \in R_3$. Hence, by Lemma 4, $\phi(x(i), x'(j)) = (0, 0)$. Thus, deviation from $x(j)$ to $x'(j)$ is beneficial for agent j as he prefers $(0, 0)$ over (a, b).

Case 2 Suppose $\phi(\mathbf{x}') = (1, 1)$ where $\mathbf{x}' \in R_3$. For player i, $(1, 1)$ is the top ranked alternative. Consider the deviation from $x(i)$ to $x'(i)$ such that $x'(i) < \frac{1}{2}$ and $|x(j) - x'(i)| < \frac{1}{2}$. So, the profile $(x'(i), x(j)) \in R_3$. Hence, by Lemma 4, $\phi(x'(i), x(j)) = (1, 1)$. Thus, deviation from $x(i)$ to $x'(i)$ is better off for agent i as he prefers $(1, 1)$ over (a, b).

Hence, this shows that $\phi(\mathbf{x}') \notin \{(0, 0), (1, 1)\}$. As $\mathbf{x}' \in R_3$, by Lemma 3 $\phi(\mathbf{x}') \in \{(0, 0), (0, 1), (1, 1)\}$. Hence, it follows that $\phi(\mathbf{x}') = (0, 1)$. □

Lemma 9 states that the rule ϕ selects the alternative $(0, 1)$ for any profile $\mathbf{x}' \in R_3$, if an inner point is selected for some profile $\mathbf{x} \in R_4$.

Lemma 10. *If $\mathbf{x} \in R_4$ and $\phi(\mathbf{x}) = (0, 0)$, then there exists a $\mathbf{x}'$ such that $\phi(\mathbf{x}') = (0, 0)$ with $x'(i) > 0$, $x'(j) < 1$.*

Proof. Suppose $x(i) = 0$ and $x(j) = 1$. Consider deviation from $x(i)$ to $x'(i)$ where $0 < x'(i) < \frac{1}{2}$. Note that $(0, 0)$ is the worst alternative with respect to $x(i) = 0$. Then by strategy-proofness, $\phi(x'(i), x(j)) = (0, 0)$, otherwise agent i will deviate from $x(i)$ to $x'(i)$ and get better off. Now consider deviation of agent j, in other words $x(j) \to x'(j)$. Since $(0, 0)$ is the top ranked alternative for both $x(j)$ and $x'(j)$, the rule cannot choose anything else than $(0, 0)$ for the profile $(x'(i), x'(j))$. Hence, $\phi(x'(i), x'(j)) = (0, 0)$ □

Corollary 2. *If $\mathbf{x} \in R_4$ and $\phi(\mathbf{x}) = (1, 1)$, then there exists a $\mathbf{x}' \in R_4$ such that $\phi(\mathbf{x}') = (1, 1)$, $x'(i) > 0$, $x'(j) < 1$.*

Proof. This is similar as the proof for Lemma 10. □

Lemma 11. *If $\mathbf{x} \in R_4$ and $\phi(\mathbf{x}) = (a, b)$, then $\phi(\mathbf{x}') \in \{(a, b), (0, 1)\}$ for any $\mathbf{x}' \in R_4$.*

Proof. Suppose $\phi(\mathbf{x}) = (a, b)$ for some $\mathbf{x} \in R_4$. Following this, it is shown that $\phi(\mathbf{x}') \notin \{(0,0),(1,1)\}$ for any $\mathbf{x}' \in R_4$.

Proposition 1. *$\phi(\mathbf{x}') \notin \{(0,0),(1,1)\}$ for any $\mathbf{x}' \in R_4$.*

Proof of Proposition 1. This proposition is proven by contradiction as follows.

Case 1 Suppose $\phi(\mathbf{x}') = (0,0)$. Due to Lemma 10, it is possible to assume that $0 < x'(i) < \frac{1}{2}$ and $\frac{1}{2} < x'(j) < 1$. With regard to $x'(i)$, $(0,1)$ is preferred over $(0,0)$. Suppose $x'(j) = 1 - \epsilon$. Then choose $x''(i) = \frac{1}{2} - \frac{\epsilon}{2}$ such that $|x'(j) - x''(i)| = \frac{1}{2} - \frac{\epsilon}{2} < \frac{1}{2}$. Hence, profile $(x''(i), x'(j)) \in R_3$. Since $\phi(\mathbf{x}) = (a, b)$, by Lemma 9, $\phi(x''(i), x'(j)) = (0,1)$. This creates an opportunity for agent i to misreport and benefit. Hence, $\phi(\mathbf{x}') = (0,0)$ is not possible.

Case 2 Suppose $\phi(\mathbf{x}') = (1,1)$. This case is similar to the previous case, except agent j's deviation is considered and Corollary 2 in order to get the contradiction.

This concludes the proof of Proposition 1.

Now it is shown that $\phi(\mathbf{x}') \in \{(a,b),(0,1)\}$ for any $\mathbf{x}' \in R_4$. First the profile $(x'(i), x(j))$ is considered. Hence, the following cases are considered:

Case 1 If $(x'(i), x(j)) \in R_3$, then by Lemma 9 $\phi(x'(i), x(j)) = (0,1)$.

Case 2 If $(x'(i), x(j)) \in R_4$, then by Lemma 6, 7 and Proposition 1 $\phi(x'(i), x(j)) \in \{(c,d),(0,1)\}$ where $0 < c, d < 1$. Now we show that if $\phi(x'(i), x(j)) = (c,d)$, then $c = a$ and $d = b$.

Proposition 2. *$(c, d) = (a, b)$.*

Proof of Proposition 2. Suppose agent i moves. Further, assume $\phi(x'(i), x(j)) = (c,d)$ where $a \neq c$ and $b \neq d$. Now consider the following cases.

Case 2a Consider $x'(i) < x(i)$. This is shown in Figure 3. By strategy-proofness, the closest outcome with respect to $x'(i)$ is in $[0, a]$ but not in $(0, a)$. Hence, the closest outcome must be a. By lexmin preference, the furthest outcome also stays at b.

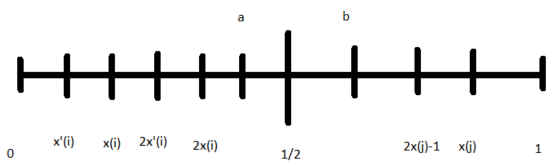

Figure 3: $x'(i) < x(i)$

Case 2b Now the case $x'(i) > x(i)$ is taken into account which is depicted in Figure 4. Consider deviation from $x(i)$ to $x'(i)$. Strategy-proofness implies that either c or d is in the interval $[0, a]$. Similarly, for the other way deviation, strategy-proofness implies that neither c nor d is in the interval $(2x'(i) - a, a)$. Due to Lemma 8, it follows that $2x'(i) < c$. As $\phi(x'(i), x(j)) = (c, d)$, where

$0 < c \leq d < 1$, it follows that $2x'(i) \leq a$. Then it follows that $c = a$. This in turn implies, because of lexmin preferences and strategy-proofness that $d = b$.

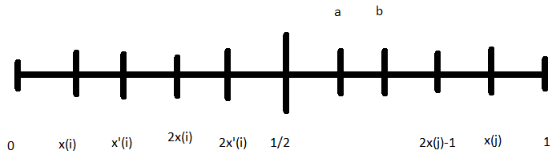

Figure 4: $x'(i) > x(i)$

Hence, this concludes the proof of Proposition 2.

Now agent j moves. From the previous cases it follows that $\phi(x'(i), x(j)) \in \{(0,1), (a,b)\}$. Hence, two cases are considered:

Case 1 $\phi(x'(i), x(j)) = (a, b)$.

Case 1a Consider $x(j) < x'(j)$. This is shown in Figure 5. By strategy-proofness, the closest outcome with respect to $x'(j)$ is in $[b, 1]$ but not in $(b, 1)$. Hence, the closest outcome must be b. By lexmin preference, the furthest outcome also stays at a.

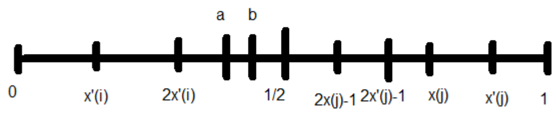

Figure 5: $x(j) < x'(j)$

Case 1b Now the case $x'(j) < x(j)$ is taken into account which is depicted in Figure 6. Suppose $\phi(\mathbf{x}') = (a_2, b_2)$ for some $(a_2, b_2) \in \mathcal{A}$. As $\mathbf{x}' \in R_4$ and $\phi(\mathbf{x}) = (a, b)$, Lemmas 6, 7 and Proposition 1 implies that either $(a_2, b_2) = (0, 1)$, or $0 < a_2 \leq b_2 < 1$. Now suppose that $0 < a_2 \leq b_2 < 1$. It is shown that $(a_2, b_2) = (a, b)$. Consider deviation from $x(j)$ to $x'(j)$. Strategy-proofness implies that either a_2 or b_2 is in the interval $[b, 1]$. Similarly, for the other way deviation, strategy-proofness implies that neither a_2 nor b_2 is in the interval $(b, 2x'(j) - b)$. Due to Lemma 8, it follows that $d < 2x'(j) - 1$. As $0 < a_2 \leq b_2 < 1$, $2x'(j) - 1 \geq b$. Then it follows that $b_2 = b$. This in turn implies, because of lexmin preferences and strategy-proofness that $a_2 = a$.

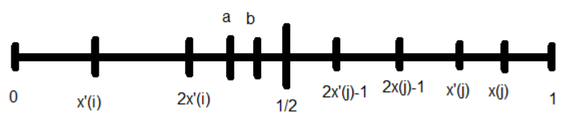

Figure 6: $x'(j) < x(j)$

So it is concluded that if $\phi(x'(i), x(j)) = (a, b)$, then $\phi(\mathbf{x}') \in \{(0,1), (a,b)\}$.

Case 2 $\phi(x'(i), x(j)) = (0,1)$. In this case, $\phi(\mathbf{x}') = (a_3, b_3)$, where either $(a_3, b_3) = (0,1)$ or $0 < a_3 \leq b_3 < 1$. This is due to Lemmas 6, 7 and Proposition 1. Consider deviation from $x(j)$ to $x'(j)$, then strategy-proofness implies that either a_3 or b_3 is in the interval $[2x(j) - 1, 1]$. Consider deviation from $x'(j)$ to $x(j)$, then strategy-proofness implies that neither a_3 nor b_3 is in the interval $(2x'(j) - 1, 1)$. Now the following cases are considered.

Case 2a Suppose $x'(j) < x(j)$ which is illustrated in Figure 7. Then $b_3 = 1$. This implies that $\phi(\mathbf{x}') = (0,1)$.

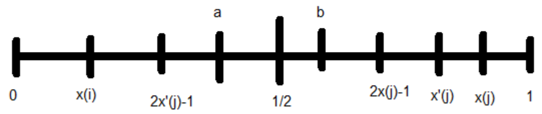

Figure 7: $x'(j) < x(j)$

Case 2b Assume $x'(j) > x(j)$ which is shown in Figure 8. Then $b_3 \in [2x(j) - 1, 2x'(j) - 1] \cup \{1\}$. Now consider the deviations in the following sequence: $(x(i), x(j)) \to (x(i), x'(j)) \to (x'(i), x'(j))$, where $x'(i) > x(i)$. Recall that $\phi(x(i), x(j)) = (a, b)$ and assume $\phi(x(i), x'(j)) = (a_4, b_4)$. Then b_4 must be equal to b. Otherwise deviation from $x'(j)$ to $x(j)$ is better off for agent j. By lexmin $a_4 = a$. Consider deviation from $x(i)$ to $x'(i)$, then strategy-proofness implies that either a_3 or b_3 is in the interval $[0, a]$. Furthermore, suppose deviation from $x'(i)$ to $x(i)$, then strategy-proofness implies that neither a_3 nor b_3 is in the interval $(2x'(i) - a, a)$. But, because of Lemma 8 it follows that, neither a_3 nor b_3 is in the interval $(0, 2x'(i))$. So if $2x'(i) > a$, it follows that $\phi(\mathbf{x}') = (0,1)$. Otherwise we have $\phi(\mathbf{x}') = (a,b)$. Note that $b \notin [2x(j) - 1, 2x'(j) - 1]$. Hence, $\phi(\mathbf{x}') = (0,1)$.

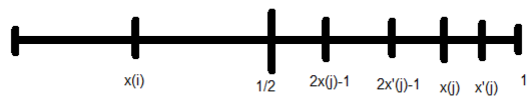

Figure 8: $x'(j) > x(j)$

So it is concluded that if $\phi(x'(i), x(j)) = (0,1)$, then $\phi(\mathbf{x}') = (0,1)$.

Combining these cases, concludes the proof of Lemma 11. □

Lemma 11 shows that if a strategy-proof, anonymous and unanimous rule selects an inner point for some profile in R_4, then that rule selects either the same inner point or $(0,1)$ for all other profile in R_4.

4 Generalized Example

This section determines and presents the complete characterization of a set of strategy-proof, anonymous and unanimous rules for finding inner solutions for optimal locating of the public bads.

Let $y_1(\mathbf{x}) := \min\{x(i), x(j)\}$ and $y_2(\mathbf{x}) := \max\{x(i), x(j)\}$. For any given $a, b \in (0, 1)$, define rule $h^{(a,b)}$ in the following way:

$$h^{(a,b)}(\mathbf{x}) = \begin{cases} (1,1) & \text{if } x(i) \leq \frac{1}{2} \text{ and } x(j) \leq \frac{1}{2} \\ (0,0) & \text{if } x(i) > \frac{1}{2} \text{ and } x(j) > \frac{1}{2} \\ & \text{or } x(i) = \frac{1}{2} \text{ and } x(j) > \frac{1}{2} \\ & \text{or } x(i) > \frac{1}{2} \text{ and } x(j) = \frac{1}{2} \\ (a,b) & \text{if } 2y_1(\mathbf{x}) < a \leq b < 2y_2(\mathbf{x}) - 1 \\ (0,1) & \text{otherwise.} \end{cases}$$

Theorem 1. *For any given $a, b \in (0, 1)$ such that $a \leq b$, rule $h^{(a,b)}$ is strategy-proof, anonymous and unanimous.*

Proof. Note that from the definition of $h^{(a,b)}$, it follows that the rule is anonymous. First of all, it is shown that this rule is unanimous.

Case 1. Suppose $x(i), x(j) \leq \frac{1}{2}$, which is shown in Figure 9. If $x(i) < \frac{1}{2}$ and $x(j) < \frac{1}{2}$, then agent i's and j's top ranked alternatives are $(1, 1)$. Thus, the set of common top ranked alternative is $\tau_1(R_{x(i)}) \cap \tau_1(R_{x(j)}) = (1, 1)$. If $x(i) = \frac{1}{2}$ and $x(j) = \frac{1}{2}$, then agent i's and j's top ranked alternatives are $(0, 0), (0, 1)$ and $(1, 1)$. This leads to $\tau_1(R_{x(i)}) \cap \tau_1(R_{x(j)}) = \{(0, 0), (0, 1), (1, 1)\}$. The rule selects $(1, 1)$. For the cases $x(i) < \frac{1}{2}$ and $x(j) = \frac{1}{2}$ and $x(i) = \frac{1}{2}$ and $x(j) < \frac{1}{2}$ resulting in $\tau_1(R_{x(i)}) \cap \tau_1(R_{x(j)}) = \{(0, 0), (0, 1), (1, 1)\}$, the rule chooses $(1, 1)$. Hence, in this case the rule does not violate unanimity.

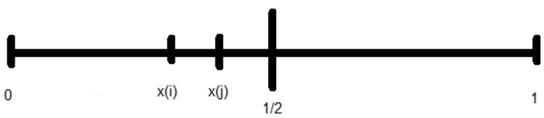

Figure 9: $x(i), x(j) \leq \frac{1}{2}$

Case 2. In this case, the following is possible: $x(i) > \frac{1}{2}$ and $x(j) > \frac{1}{2}$ or $x(i) = \frac{1}{2}$ and $x(j) > \frac{1}{2}$ or $x(i) > \frac{1}{2}$ and $x(j) = \frac{1}{2}$.

It is observed that the rule chooses an element of $\tau_1(R_{x(i)}) \cap \tau_1(R_{x(j)})$, that is $(0, 0)$. Thus, there is no violation of unanimity.

Case 3. Suppose the rule chooses (a, b) if $2y_1 < a \leq b < 2y_2 - 1$, or equivalently, $y_1 < \frac{a}{2} \leq \frac{b+1}{2} < y_2$.

Assume without loss of generality, $x(i) \leq x(j)$. Then $y_1 = x(i)$ and $y_2 = x(j)$. Furthermore, $x(i) < \frac{1}{2}$ and $x(j) > \frac{1}{2}$. Figure 10 depicts the setting.

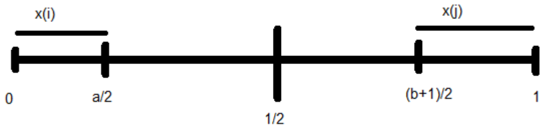

Figure 10: $x(i) < \frac{a}{2} \leq \frac{b+1}{2} < x(j)$

This leads to the result that agent i's and j's top ranked alternative are $(1,1)$ and $(0,0)$, respectively. The unanimity condition does not apply as $\tau_1(R_{x(i)}) \cap \tau_1(R_{x(j)}) = \emptyset$.

Case 4. In this case, the following is possible: $a < 2y_1 < 1$ and $2y_2 - 1 > b$ or $a \leq 2y_1 < 1$ and $0 \leq 2y_2 - 1 \leq b$ or $2y_1 \leq a$ and $0 < 2y_2 - 1 \leq b$.
Below the reader finds Figure 11 that depicts the situation.

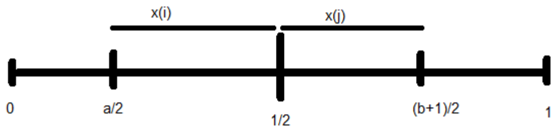

Figure 11: $a \leq 2y_1 < 1$ and $0 \leq 2y_2 - 1 \leq b$

Similar to Case 3, unanimity is not applicable.

Hence, as for all cases the rule chooses $\tau_1(R_{x(i)}) \cap \tau_1(R_{x(j)})$, where applicable. Thus, the rule satisfy the condition for the unanimity property.

Finally, it is shown that the rule is strategy-proof. First, Case 1 is considered. Since agent i and j are already getting their top alternative $(1,1)$, they have no incentive to misreport. Similarly, I come to the same conclusion for Case 2. Now consider case 3 and assume without loss of generality $x(i) < x(j)$, then $2x(i) < a \leq b < 2x(j) - 1$. Deviation from this profile can result in outcomes $(0,0)$, $(0,1)$ and $(1,1)$. Consider deviation of player i. Player i himself cannot ensure the outcome $(1,1)$ as $x(j) > \frac{1}{2}$. Moreover, agent i would not misreport his dip in order to achieve the other pairs of locations. The reason for that is as follows: Since $x(i) < \frac{a}{2}$,

$$\min\{|a - x(i)|, |b - x(i)|\} = a - x(i) > \frac{a}{2},$$

$$\min\{|0 - x(i)|, |1 - x(i)|\} = |0 - x(i)| < \frac{a}{2}$$

and

$$\min\{|0 - x(i)|, |0 - x(i)|\} = |0 - x(i)| < \frac{a}{2}.$$

Similarly, agent j cannot deviate and get better off.
Finally, I consider Case 4. Again, assume without loss of generality, $x(i) < x(j)$ which implies $y_1 = x(i)$ and $y_2 = x(j)$. There are three distinct situations to violate $2x(i) < a \leq b < 2x(j) - 1$:

(i) $x(i) \geq \frac{a}{2}$ and $x(j) > \frac{b+1}{2}$ In order to distinguish from Case 1, $x(i)$ must be smaller than $\frac{1}{2}$ implying $\frac{a}{2} \leq x(i) < \frac{1}{2}$. The setting is shown in Figure 12.

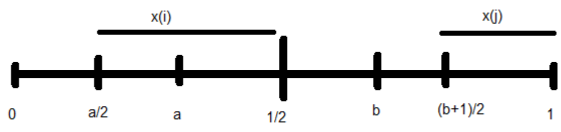

Figure 12: $\frac{a}{2} \leq x(i) < \frac{1}{2}$ and $\frac{b+1}{2} < x(j) \leq 1$

Deviation from this profile can result in outcomes $(0,0)$, (a,b) and $(1,1)$. Consider deviation of player j. Player j himself cannot ensure the outcome $(0,0)$ and (a,b) as $\frac{a}{2} \leq x(i) < \frac{1}{2}$. Moreover, agent j does not prefer $(1,1)$ over the current outcome since $x(j) > \frac{b+1}{2}$,

$$\min\{|0-x(j)|, |1-x(j)|\} = 1-x(j) = \min\{|1-x(j)|, |1-x(j)|\}$$

and

$$\max\{|0-x(j)|, |1-x(j)|\} = |0-x(j)| > \max\{|1-x(j)|, |1-x(j)|\} = 1-x(j).$$

For agent i, $(0,0)$ is worse than $(0,1)$ as $\min\{|0-x(i)|, |0-x(i)|\} = \min\{|0-x(i)|, |1-x(i)|\}$ and $\max\{|0-x(i)|, |0-x(i)|\} \leq \max\{|0-x(i)|, |1-x(i)|\}$. Now consider outcome (a,b). Since $\min\{|a-x(i)|, |b-x(i)|\} = a - x(i) \leq x(i) - 0 = \min\{|0-x(i)|, |1-x(i)|\}$ and $\max\{|a-x(i)|, |b-x(i)|\} = b - x(i) < 1 - x(i) = \max\{|0-x(i)|, |1-x(i)|\}$, for agent i (a,b) is worse than $(0,1)$ by lexmin preference. Agent i cannot get $(1,1)$ as $x(j)$ is above $\frac{1}{2}$.

(ii) $x(i) < \frac{a}{2}$ and $x(j) \leq \frac{b+1}{2}$ This case is illustrated by Figure 13.

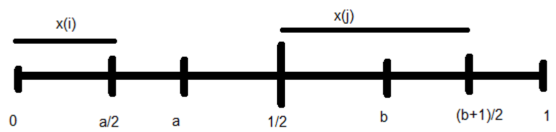

Figure 13: $x(i) < \frac{a}{2}$ and $x(j) \leq \frac{b+1}{2}$

I identify the two same possible outcomes to deviate for the agents. Now take into account the outcome $(0,0)$. For agent i, $(0,0)$ is worse than $(0,1)$. Further, agent j cannot get the outcome as he has no control over agent i's dip which is below $\frac{1}{2}$. Finally, consider outcome (a,b) which is not possible to attain for agent i as he has no control over agent j. Moreover, agent j would prefer $(0,1)$ over (a,b) because $x(j) \leq \frac{b+1}{2}$. Lastly, consider outcome $(1,1)$. Agent j is better off with $(0,1)$ compared to $(1,1)$. In addition, agent i cannot achieve $(1,1)$ as he has no control over agent j whose dip is above $\frac{1}{2}$.

(iii) $x(i) \geq \frac{a}{2}$ and $x(j) \leq \frac{b+1}{2}$
Similarly, I proceed with the analysis for Case (iii) leading to the result that no agent is better off by deviating.

□

Hence, the rule satisfy the condition for strategy-proofness. To conclude, the rule is strategy-proof, anonymous and unanimous.
Note that the rule $h^{(a,b)}(\mathbf{x})$ contains one internal solution, that is, for a given (a, b) if $2y_1(\mathbf{x}) < a \leq b < 2y_2(\mathbf{x}) - 1$.

5 Key Findings and Conclusions

The key findings of the present thesis are summarised with the following theorem:

Theorem 2. *Suppose ϕ be a rule. Then ϕ is strategy-proof, unanimous, anonymous and selects inner points if and only if $\phi = h^{(a,b)}$ for some $a, b \in (0, 1)$ such that $a \leq b$.*

Proof. The only if-case follows from Theorem 1. In addition, the if-case follows from Lemmas 1 to 11. □

Based on lexmin preference of the concerned agents for locating pairs of public bads, the present thesis has defined and proved eleven lemmas determining the implications of the properties, strategy-proofness, anonymity and unanimity on the decision rule. In addition, it has shown that the combination of these properties allows for inner solutions. In Figure 14, the preference domain is partitioned as follows: As noted in Section 2, R_1, R_2, R_3 and R_4 represent the different combinations of the locations of dips of the agents i and j, denoted as $x(i)$ and $x(j)$ respectively. The alternatives that the rule chooses for the specific partition are defined in the brackets. The following observations are made: In R_1, R_2 and R_3 the decision rule chooses no internal solution. However, an alternative where both public bads are inside the interval can be selected from the rule. Furthermore, if there exists a profile $\mathbf{x} \in R_4$ and $\phi(\mathbf{x}) = (a, b)$, where $0 < a, b < 1$, then for any profile $\mathbf{x}' \in R_3$, $\phi(\mathbf{x}') = (0, 1)$. The reader should also note that in R_4, the decision rule can also choose a boundary point. Hence, if there exists a profile $\mathbf{x} \in R_4$ and $\phi(\mathbf{x}) = (0, 1)$, then for any profile $\mathbf{x}' \in R_3$, $\phi(\mathbf{x}') \in \{(0, 0), (0, 1), (1, 1)\}$.

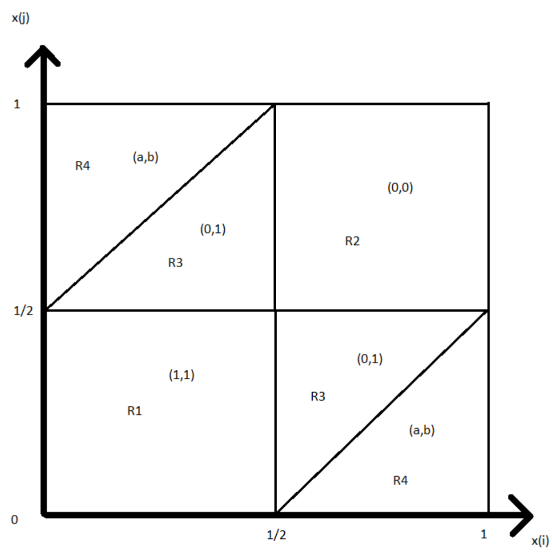

Figure 14: Partition of Preference Domain

As an extension to the traditional approach, the present thesis has determined the complete characterization of a set of strategy-proof, anonymous and unanimous rules for selecting inner solutions for optimal locations of two public bads in a region given the preferences of the concerned agents in the region. The thesis has thus fully achieved the predefined objectives.

6 References

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