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Solution of Two Linear Equation Singularly Perturbed of First Order PDE on Argument

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Abstract

Two systems of linear singularly perturbed problems in first order partial derivatives were considered. This system depends on a small parameter, and the small parameter exists in the left and right hands of it. Also, we prove the uniqueness of this solution is uniform in the space $\Psi = (0, X] \times (0, T]$, and asymptotic approximation of any rank was constructed.

Key words : Partial Differential Equation, Singularly Perturbed Problems, Asymptotic Solution Method,

المستخلص

تم اختيار نظامين من مسائل الحرجة المفردة الخطية للمشتقات جزئية من الرتبة الأولى. علما بان هذا النظام يعتمد على معلمه صغيرة و هذه المعلمة موجوده في طرفي النظام المعادلة . وأيضا برهنا على وحدانية هذه الحلول ومنتظمة ضمن المجال Ψ وثم برهنة متقاربة اية رتبة.

الكلمات المفتاحية : مسائل الحرجة المفردة، معادلات تفاضلية جزئية ، طرق الحل المتقاربة .

1- Introduction

Many researchers were studied singularly perturbed of partial differential equation and ordinary differential equation relying on a small parameter.

Where (Butuzov and Nedelko,1999) used elliptic equation with different power of ε and they proofed that the existence and local uniqueness, also the asymptotic of solution periodic in both variables was obtained. Also (Butuzov, 1977) studied elliptic equation with two parameter ϵ and μ in the space $\Omega = \{0 < x < a \times 0 < y < b\}$.

(Butuzov and Mamonov,1982) studied singularly perturbed elliptic PDE in critical case with BVP is depending on ε in the right and left hands of this equation and using asymptotic series of rank n and proved that the solution was unique and uniform in $O(\varepsilon^{n+1})$.while(Butuzov,1977) studied mixed singularly perturbed hyperbolic equation in the space $\Omega = \{0 \le x \le l \times 0 < t \le T\}$ with boundry conditions depending on $\varepsilon > 0$, with series of rank n.

(Butuzov and Buchnev,1989) studied system of singularly perturbed parabolic equation in 2-diminal depending on argument of $\boldsymbol{\mu} > \mathbf{0}$ in the space $\Omega = \{(\mathbf{l}, \mathbf{m}, \mathbf{i}): \mathbf{0} < \mathbf{l} < \mathbf{1}, \mathbf{0} < \mathbf{m} < \mathbf{1}, \mathbf{0} < \mathbf{i} \le \mathbf{T}\}$, with boundary value problem helped his found the solution of $\boldsymbol{\varepsilon}^{\mathbf{0}}$ and $\boldsymbol{\varepsilon}^{\mathbf{1}}$.

(Butuzov and Nefedov, 2002) considered the initial and BVP for a singularly perturbed system of two parabolic equations degenerating into a system from a finite equation and a first-order differential equation. they proved a theorem on the passage to the limit as the small parameter tends to zero from the solution of the original problem to the solution of a degenerate system in the case when the finite equation of a degenerate system has intersecting roots. While the researchers (Butuzov, 1997)and (Butuzov and Nikitin,1990) studied system of singularly perturbed parabolic equation depending on ε^4 in first equation and ε^2 in second equation with the space $\Omega = \{0 \le x \le 1 \times 0 < t \le T\}$.

(Butuzov and Levashova,2012) studied system of second order PDE depending on ε^4 in first equation and ε^2 in second equation in 1-dimensional case, this system belongs to the space $(u, v, x, t) \in I_u \times I_v \times [0,1] \times (0, \varepsilon_0]$.

(Butuzov and Kostin,2009), have studied a uniquely irritated arrangement of two second-arrange differential conditions (one quick and one moderate), the presence of an answer is proved and its asymptotics are got for the case in which the worsen condition has two meeting roots. While ,(Abood,2011) and (Abood and Ali,2012) specialist acquired an asymptotic extension, containing normal limit corner works in ε was built, to arrangement of moment arrange fractional differential condition.

(Abood and Ali,2012) developed asymptotic arrangement of a halfway differential condition with little parameter and they have demonstrated the arrangement is exceptional and uniform in area Ω , and asymptotic guess is $O(\varepsilon^2)$.

2-Setting of Main Problem

We consider the following systems of first order PDE

$$\left.\begin{aligned}\varepsilon\frac{\partial u}{\partial t}+b_1(x)\frac{\partial u}{\partial x}&=a_{11}(x,t)u+a_{12}(x,t)v+f_0(x,t,\varepsilon)+\varepsilon f_1(x,t,\varepsilon)\\ \frac{\partial v}{\partial t}+\varepsilon b_2(x)\frac{\partial v}{\partial x}&=a_{21}(x,t)u+a_{22}(x,t)v+g_0(x,t,\varepsilon)+\varepsilon g_1(x,t,\varepsilon)\end{aligned}\right\} \quad (2\text{-}1)$$

At the area $\Psi=(0,X]\times(0,T]$, with boundary conditions

$$u|_{t=0}=0,\quad v|_{t=0}=0,\quad u|_{x=0}=0,\ v|_{x=0}=0\ . \quad (2\text{-}2)$$

Where ε is small parameter, this system depended on the variables coefficient of t in first equation and the coefficient variables x in second equation. We consider the simple solution of system are enduring at space $\overline{\Psi}=[0,X]\times[0,T]$.

Requirements were made the construction:

1- $f_0(0,0,\varepsilon),\varepsilon f_1(0,0,\varepsilon)=0$ and $g_0(0,0,\varepsilon),\varepsilon g_1(0,0,\varepsilon)=0$

that the BVP be continuously conformable and verified (2-1) at (0, 0) and the first order PDF conformable stipulation at (0,0).

$b_1(x),b_2(x),\ a_{11}(x,t),\ a_{12}(x,t)\ ,a_{21}(x,t),\ a_{22}(x,t)$ are smoothness. Also $f_0(x,t,\varepsilon),\varepsilon f_1(x,t,\varepsilon),g_0(x,t,\varepsilon)\varepsilon g_1(x,t,\varepsilon)$ guarantee the presence it easy solve of the equations (2-1) and (2-2).

2- functions $b_1(x),b_2(x),\ a_{11}(x,t),\ a_{12}(x,t)\ ,a_{21}(x,t),\ a_{22}(x,t),f_0,\ f_1\ ,\ g_0,\ g_1$ are continuous of rank $(n+2)$, this condition is powerful to use building the asymptotic of arbitrarily of rank n Moreover, let us assume $b_1(x),b_2(x)$ is positive, i.e. $b_1(x),b_2(x)>0$.

Now if we put $\varepsilon=0$ in equation (2-1) we get,

$$(0).\frac{\partial u}{\partial t}+b_1(x)\frac{\partial u}{\partial x}=a_{11}(x,t)u+a_{12}(x,t)v+f_0(x,t,0)+(0).f_1(x,t,\varepsilon)$$

$$b_1(x)\frac{\partial u}{\partial x}=a_{11}(x,t)u+a_{12}(x,t)v+f_0(x,t,0)$$

$$\frac{\partial u}{\partial x}=\frac{a_{11}(x,t)u+a_{12}(x,t)v+f_0(x,t,0)}{b_1(x)}\quad,\ b_1(x)>0. \quad (2\text{-}3)$$

$$\frac{\partial v}{\partial t}+(0).b_2(x)\frac{\partial v}{\partial x}=a_{21}(x,t)u+a_{22}(x,t)v+g_0(x,t,0)+(0).g_1(x,t,\varepsilon)$$

$$\frac{\partial v}{\partial t}=a_{21}(x,t)u+a_{22}(x,t)v+g_0(x,t,0) \quad (2\text{-}4)$$

when use the degenerate system ($\varepsilon=0$) and $u|_{x=0}=0\ ,v|_{t=0}=0$ we get

$$\frac{\partial u}{\partial x}=\frac{a_{11}(0,0)u+a_{12}(0,0)v+f_0(0,0,0)}{b_1(0)}\quad,\ b_1(x)>0.$$

$$\frac{\partial u}{\partial x}=\frac{a_{11}(0,0)u+a_{12}(0,0)v}{b_1(0)} \quad (2\text{-}5)$$

and

$$\frac{\partial v}{\partial t}=a_{21}(0,0)u+a_{22}(0,0)v+g_0(0,0,0)$$

$$\frac{\partial v}{\partial t}=a_{21}(0,0)u+a_{22}(0,0)v \quad (2\text{-}6)$$

3-Building of Asymptotics

The framing of asymptotics extension to solve system equation $(2-1)$ and $(2-2)$ like double series depending on power ε as:

$$\left.\begin{array}{l} u(x,t,\varepsilon) = \sum_{k=0}^{\infty} \varepsilon^k \left[\bar{u}_k(x,t) + \Pi_k u(x,\tau) + Q_k u(\xi,t)\right] \\ v(x,t,\varepsilon) = \sum_{k=0}^{\infty} \varepsilon^k \left[\bar{v}_k(x,t) + \Pi_k v(x,\tau) + Q_k v(\xi,t)\right] \end{array}\right\} \quad (3\text{-}1)$$

The regular of first part ($\bar{u}_k(x,t)$ and $\bar{v}_k(x,t)$) of above series and the boundary layer of second part this series are $\Pi_k u(x,\tau), Q_k u(\xi,t), \Pi_k v(x,\tau), Q_k v(\xi,t)$. Also variables $\tau = \frac{t}{\varepsilon}$ and $\xi = \frac{x}{\varepsilon}$.

the coefficients of (3-1) is up determining of order n, b_2 is constant in problem (2-1). In general let the coefficient of $\frac{\partial v}{\partial x}$ in second equation equal to one. And we have $y = \int_0^x b_2^{-1}(\vartheta)d\vartheta$ by alteration of mutable, we get the coefficient of $\frac{\partial v}{\partial y}$ is ε.

That is when $b_2 = 1$ and $b(y) = \frac{b_1(x(y))}{b_2(x(y))} = b_1$ we have the following equation

$\varepsilon \frac{\partial u}{\partial t} + \frac{b_1(x(y))}{b_2(x(y))}\frac{\partial u}{\partial x} = a_{11}(x,t)u + a_{12}(x,t)v + f_0(x,t,\varepsilon) + \varepsilon f_1(x,t,\varepsilon)$, $b_1 = b(x) > 0$ and $b_2 = 1$.

We extension f_0, f_1, g_0 and g_1 in series in powers of ε as the following

$f_0 = f_{00}(x,t) + \varepsilon f_{01}(x,t) + \varepsilon^2 f_{02}(x,t) + \cdots$,

$f_1 = \varepsilon f_{10}(x,t) + \varepsilon^2 f_{11}(x,t) + \varepsilon^3 f_{12}(x,t) + \cdots$,

$g_0 = g_{00}(x,t) + \varepsilon g_{01}(x,t) + \varepsilon^2 g_{02}(x,t) + \cdots$, and

$g_1 = \varepsilon g_{10}(x,t) + \varepsilon^2 g_{11}(x,t) + \varepsilon^3 g_{12}(x,t) + \cdots$.

Now we will Substitute double series (3-1) in (2-1) and (2-2) and equate the similarity of ε^i, $i = 0,1,2,\ldots,n,\ldots$ on all the parts of equation (2-1) and (2-2),that mean we getting regular part and boundary layer parts of order n.

3-1 Regular Terms $\bar{u}_0$ and $\bar{v}_0$

When ε^0we get the regular part of u_0 and v_o of the asymptotics and

$$\left.\begin{array}{l} b(x)\frac{\partial \bar{u}_0}{\partial x} = a_{11}(x,t)\bar{u}_0 + a_{12}(x,t)\bar{v}_0 + f_0(x,t,\varepsilon) \\ \frac{\partial \bar{u}_0}{\partial x} = \frac{(a_{11}(x,t)\bar{u}_0 + a_{12}(x,t)\bar{v}_0 + f_0(x,t,\varepsilon))}{b(x)}, \quad b(x) \neq 0, \\ \frac{\partial v_0}{\partial t} = a_{21}(x,t)\bar{u}_0 + a_{22}(x,t)\bar{v}_0 + g_0(x,t,\varepsilon) \\ \bar{u}_0(0,t) = 0, \quad \bar{v}_0(x,0) = 0 \end{array}\right\} \quad (3\text{-}2)$$

From system in the equation (3-2) lead to the system of a integral equations

$$\bar{u}_0(x,\varepsilon\tau) = \int_0^x exp\left(\int_\sigma^x b^{-1}(p)a_{11}(p,\varepsilon\tau)dp\right)b^{-1}(\theta)[a_{12}(\theta,\varepsilon\tau)\bar{v}_0(\theta,\varepsilon\tau) + f_{00}(\theta,\varepsilon\tau)]d\theta \quad (3\text{-}3)$$

$$\bar{v}_0(x,\varepsilon\tau) = \int_0^t exp\left(\int_\sigma^t a_{22}(x,p)dp\right)[a_{21}(x,s)\bar{u}_0(x,s) + g_{00}(x,s)]ds \quad (3\text{-}4)$$

When $t = \varepsilon\tau$ finally when we substitute (3-3) in (3-4),and then we get the first regular part of $\bar{u}_0(x,t)$:

$$\bar{u}_0(x,t) = \int_0^x \int_0^t K(x,t,\theta,s)\bar{u}_0(\theta,s)\, d\theta ds + \Omega(x,t)$$

at mean the functions $K(x,t,\theta,s)$, $\Omega(x,t)$ are known. By using integral equation we can solve it and the furthermore, the communicated as far as resolving $З(x,t,\theta,s)$ will replaced from nucleus $K(x,\varepsilon\tau,\theta,s)$ we get

$\bar{u}_0 = \Omega(x,t) + \int_0^x \int_0^t З(x,t,\theta,s)\Omega(x,t)\, d\theta ds$

Therefore the $\bar{v}_0(x,t)$ can be determined by solving a integral equation then we have the $\bar{v}_0(x,t)$ as

$\bar{v}_0 = \Omega(x,t) + \int_0^x \int_0^t D(x,t,\theta,s)\emptyset(\theta,s)\, d\theta ds$

3-2 Boundary Functions $\mathbf{Q_0u}$ and $\mathbf{Q_0v}$

For $Q_0u(\xi,t)$, since $\xi = \frac{x}{\varepsilon} = x = \varepsilon\,\xi$ we have the equation

$$\varepsilon \frac{\partial Q_0u}{\partial t} + b_1(\varepsilon\,\xi)\frac{\partial Q_0u}{\partial(\varepsilon\,\xi)} = a_{11}(\varepsilon\,\xi,t)Q_0u + a_{12}(\varepsilon\,\xi,t)Q_0 + f_0(\varepsilon\,\xi,t,\varepsilon) + \varepsilon f_1(\varepsilon\,\xi,t,\varepsilon)$$

That we get for ε^0 to find the boundary-layer part

$b(0)\frac{\partial Q_0u}{\partial\xi} = 0 \Rightarrow \frac{\partial Q_0u}{\partial\xi} = 0$, by the boundary functions at ∞, $Q_0u(\infty,t) = 0$. That mean the solution of $Q_0u \equiv 0$.

While the Q_0v we obtain the problem

$$\frac{\partial \mathrm{Q}_0v}{\partial t} + \varepsilon b_2(\varepsilon\,\xi)\frac{\partial \mathrm{Q}_0v}{\partial(\varepsilon\,\xi)} = a_{21}(\varepsilon\,\xi,t)Q_0u + a_{22}(\varepsilon\,\xi,t)\mathrm{Q}_0v + g_0(\varepsilon\,\xi,t,\varepsilon) + \varepsilon g_1(\varepsilon\,\xi,t,\varepsilon)$$

$$\frac{\partial \mathrm{Q}_0v}{\partial t} + \varepsilon b_2(\varepsilon\,\xi)\frac{\partial \mathrm{Q}_0v}{\partial(\varepsilon\,\xi)} = a_{21}(0,t)(0) + a_{22}(0,t)\mathrm{Q}_0v + g_0(0,t,0) + \varepsilon g_1(0,t,\varepsilon)$$

$$\frac{\partial \mathrm{Q}_0v}{\partial t} + \frac{\partial \mathrm{Q}_0v}{\partial\,\xi} = a_{22}(0,t)\mathrm{Q}_0v + g_0(0,t,0), \xi \ge 0,\ \ 0 \le t \le T$$

and since $\overline{v}_0(0{,}0) = 0$ and $\frac{\partial\overline{v}_0}{\partial t}(0{,}0) = 0$, we get from equation (3-2) that mean $g_0(0{,}0) = 0$ that lead to the final equation of Q_0v has the form

$$\frac{\partial \mathrm{Q}_0v}{\partial t} + \frac{\partial \mathrm{Q}_0v}{\partial\,\xi} = a_{22}(0,t)\mathrm{Q}_0v$$

$$Q_0v(0,t) = -\overline{v}_0(0,t), \quad Q_0v(\xi,0) = 0 \qquad \text{(3-5)}$$

This the $Q_0v(0,t) = -\overline{v}_0(0,t)$, $Q_0v(\xi,0) = 0$ satisfy the requirement of first rank at the $(0,0)$. And has solve like :

$$Q_0v(\xi,0) = \begin{cases} -\overline{v}_0(0,t-\xi)\exp\left(\int_0^{\xi}[a_{22}(0,\theta+t-\xi)]d\theta\right), & 0 \le \xi \le t \le T, \\ zero, & \xi \ge t \end{cases} \qquad \text{(3-6)}$$

3-3 The Boundary Functions $\prod_0 \boldsymbol{u}(\boldsymbol{x},\boldsymbol{\tau})$ and $\prod_0 \boldsymbol{v}(\boldsymbol{x},\boldsymbol{\tau})$

For $\prod_0 \mathrm{v}(\mathrm{x},\tau)$, since $\tau = \frac{t}{\varepsilon} = t = \varepsilon\tau$ we have the equation

$$\frac{\partial\prod_0 v}{\partial(\varepsilon\tau)} + \varepsilon b_2(x)\frac{\partial\prod_0 v}{\partial x} = a_{21}(x,\varepsilon\tau)\prod_0 u + a_{22}(x,\varepsilon\tau)\prod_0 v + g_0(x,\varepsilon\tau,\varepsilon) + \varepsilon g_1(x,\varepsilon\tau,\varepsilon)$$

$\frac{\partial\prod_0 v}{\partial\tau} = 0$, by the boundary functions at ∞, $\prod_0 v(x,\infty) = 0$. That mean the solution of $\prod_0 v \equiv 0$

While the $\prod_0 u$ we obtain the problem

$$\varepsilon\frac{\partial\prod_0 u}{\partial(\varepsilon\tau)} + b_1(x)\frac{\partial\prod_0 u}{\partial x} = a_{11}(x,\varepsilon\tau)\prod_0 u + a_{12}(x,\varepsilon\tau)\prod_0 v + f_0(x,\varepsilon\tau,\varepsilon) + \varepsilon f_1(x,\varepsilon\tau,\varepsilon)$$

$$\frac{\partial\prod_0 u}{\partial\tau} + b_1(x)\frac{\partial\prod_0 u}{\partial x} = a_{11}(x,0)\prod_0 u + a_{12}(x,\varepsilon\tau).(0) + f_0(x,0{,}0) + \varepsilon f_1(x,0{,}0)$$

and since $\overline{u}_0(0{,}0) = 0$ and $\frac{\partial u_0}{\partial x}(0{,}0) = 0$, we get from equation(3-2) that mean $f_0(0{,}0) = 0$ that lead the final equation of $\prod_0 u$ has the form

$$\frac{\partial\Pi_0 u}{\partial\tau} + b(x)\frac{\partial\Pi_0 u}{\partial x} = a_{11}(x,0)\Pi_0 u, \quad \tau \ge 0,\ \ x \in [0,X]$$

$$\Pi_0 u(0,\tau) = 0, \Pi_0 u(x,0) = -\overline{u}_0(x,0)$$

$$\Pi_0 u(x,\tau) = \begin{cases} -\overline{u}_0(\varphi^{-1}(\varphi(x)-\tau),0)\exp\left(\int_0^{\tau} a_{11}(\varphi^{-1}(\varphi(x)-\tau+s),0)ds\right), & 0 \le \tau \le \varphi(x), \\ zero, & \tau \ge \varphi(x) \end{cases}$$

where $\varphi(x) = \int_{\sigma}^{x} b^{-1}(\theta)d\theta$ and $\varphi^{-1}(y) =$ is inverse of the function $y = \varphi(x)$.

3-4 Regular Terms $\overline{\boldsymbol{u}}_1$ and $\overline{\boldsymbol{v}}_1$

When ε^1 we get the regular part of $\overline{u}_1$ and $\overline{v}_1$ of the asymptotics and by using the boundary conditions on $\overline{u}_1(x,t)$ and $\overline{v}_1(x,t)$ such as

$$\overline{u}_1(0,t) = -Q_1u(0,t), \quad \overline{v}_1(x,0) = -\prod_1 v(x,0) \qquad \text{(3-7)}$$

Then we getting the system equations for $\overline{u}_1$ and $\overline{v}_1$ as

$$\left.\begin{aligned}
&b(x)\frac{\partial \bar{u}_1}{\partial x}+\frac{\partial \bar{u}_0}{\partial t}=a_{11}(x,t)\bar{u}_1+a_{12}(x,t)\bar{v}_1+f_{01}(x,t)+f_{10}(x,t)\\
&b(x)\frac{\partial \bar{u}_1}{\partial x}=a_{11}(x,t)\bar{u}_1+a_{12}(x,t)\bar{v}_1+L\\
&\frac{\partial \bar{v}_1}{\partial t}+\frac{\partial v_0}{\partial x}=a_{21}(x,t)\bar{u}_1+a_{22}(x,t)\bar{v}_1+g_{01}(x,t,\varepsilon)+g_{10}(x,t,\varepsilon)\\
&\frac{\partial \bar{v}_1}{\partial t}=a_{21}(x,t)\bar{u}_1+a_{22}(x,t)\bar{v}_1+N
\end{aligned}\right\} \quad (3\text{-}8)$$

Where $L = f_{01}(x,t,\varepsilon)+f_{10}(x,t)-\frac{\partial \bar{u}_0}{\partial t}$ and $N = g_{01}(x,t,\varepsilon)+g_{10}(x,t,\varepsilon)-\frac{\partial v_0}{\partial x}$

By reducing the system(3-7) and (3-8) to integral equations we can find the solution of System (3-7),(3-8)) in same as the way in equation(3-2).

3-5 Boundary Functions Q_1u and $\prod_1 v$.

For $Q_1u\left(\xi,\tau=\frac{t}{\varepsilon}\right)$ getting the :

$$b(0)\frac{\partial Q_1u}{\partial \xi}=a_{12}(0,t)Q_0v$$

since $Q_1u(\infty,\varepsilon\tau)=0$. And $\int_t^{\xi} Q_0v(\theta,t)d\theta$ also from section 3-2 we get $Q_0v(\xi,t)=0$ satisfy for $\xi\ge t$, we get the solution of $Q_1u(\xi,t)$ as

$$Q_1u(\xi,t)=\begin{cases} b^{-1}(0)a_{12}(0,t)\int_t^{\xi} Q_0v(\theta,t)d\theta & 0\le\xi\le t\le T,\\ zero, & \xi\ge t\end{cases}$$

For $\Pi_1 v(x,\tau)$. We can using $\Pi_1 v(x,\tau)$ so we will get the solution of $Q_1u(\xi,t)$ that is :

$$\Pi_1 v(x,\tau)=a_{21}(x,0)\int_{\infty}^{\tau}\Pi_0 u(x,s)ds\begin{cases} a_{21}(x,0)\int_{B(x)}^{\tau}\Pi_0 u(x,s)ds & 0\le\tau\le\varphi(x),\\ zero, & \tau\ge\varphi(x)\end{cases} \quad (3\text{-}9)$$

The function $\Pi_1 v(x,\tau)$ with partial derivatives was continuous of the first order also.

3-6 Boundary Function Q_1v and $\prod_1$u

For Q_1v we have the equation

$$\frac{\partial Q_1v}{\partial t}+\frac{\partial Q_1v}{\partial \xi}=a_{21}(0,t)Q_1v+a_{21}(0,t)Q_1u(\xi,t)+\frac{\partial a_{22}}{\partial x}(0,t)\xi Q_0v(\xi,t) \quad (3\text{-}10)$$

Here $a_{21}(0,t)Q_1u(\xi,t)+\frac{\partial a_{22}}{\partial x}(0,t)\xi Q_0v(\xi,t)$ with boundary conditions

$$Q_1v(0,t)=-\bar{v}_0(0,t),\ \ Q_1v(\xi,0)=0 \quad (3\text{-}11)$$

where $a_{21}(0,t)Q_1u(\xi,t)+\frac{\partial a_{22}}{\partial x}(0,t)\xi Q_0v(\xi,t)$ be smooth. if we put $x=0$ and $t=0$ and the condition $\bar{v}_1(0{,}0)=-\Pi_1 v(0{,}0)=0$ (see (3-7) and (3-9)), we get subsequently the boundary of equation (3-12) are continuous convenient $(0,0)$.

$$-\frac{\partial \bar{v}_1}{\partial t}(0{,}0)=a_{21}(0,t)Q_1u(0{,}0)+\frac{\partial a_{22}}{\partial x}(0{,}0)\times 0\times Q_0v(0{,}0)=0 \quad (3\text{-}12)$$

It follows from the expression the right hand of problem $(3-12)$ for $0\le t\le \xi$, becauce $Q_0v(\xi,t)$ and $Q_1u(\xi,t)$have the same property, that mean $\frac{\partial \bar{v}_1}{\partial t}(0{,}0)=0$, hence, equation(3-12) are holds. We use the second equation in(3-8) and since, $\bar{v}_0(x,0)=0$, it follows that $\frac{\partial \bar{v}_0}{\partial x}(0{,}0)=0$.

Hence $N(0,0)=0$, because $g_0(0{,}0,\varepsilon),\varepsilon g_1(0{,}0,\varepsilon)=0$ by requirement 1 , Further, we have already shown that $\bar{v}_1(0{,}0)=0$. Finally, (3-7) implies that $\bar{u}_1(0{,}0)=-Q_1u(0{,}0)$, and since $Q_1u(0{,}0)=0$, it follows that $\bar{u}_1(0{,}0)=0$. Thus, the $a_{21}(x,t)\bar{u}_1+a_{22}(x,t)\bar{v}_1+N$ in problem (3-8) is fade in $(0,0)$, so $\frac{\partial \bar{v}_1}{\partial t}(0{,}0)=0$.

Now the solution of the equation(3-11) and (3-12)are satisfied. We can written this solution as

$$Q_1v(\xi,t)=\begin{cases}-\overline{v}_1(0,t-\xi)exp\left(\int_0^\xi a_{22}(0,\theta+t-\xi)d\theta\right)+C, & 0\le\xi\le t\le T\\ zero, & \xi\ge t\end{cases} \quad (3\text{-}13)$$

where

$$C=\int_0^\xi exp\left(\int_\sigma^\xi a_{22}(0,p+t-\xi)dp\right)\left(a_{21}(\theta,\theta+t-\xi)Q_1u(\theta,\theta+t-\xi)+\frac{\partial a_{22}}{\partial x}(\theta,\theta+t-\xi)\xi Q_0v(\theta,\theta+t-\xi)\right)d\theta$$

While the boundary layer $\Pi_1u(x,\tau)$ can be acquainting as the solution of the equation

$$\frac{\partial\Pi_1u}{\partial\tau}+b(x)\frac{\partial\Pi_1u}{\partial x}\approx a_{11}(x,0)\Pi_1u+\gamma_1(x,\tau),$$
$$\Pi_1u(x,0)=-\bar{u}_1(x,0),\quad \Pi_1u(0,\tau)=0,$$

Where is smooth of function

$\gamma_1(x,\tau)=\frac{\partial a_{22}}{\partial t}(x,0)\tau\Pi_0u(0,\tau)+a_{12}(x,0)\Pi_1v(x,\tau)$. Similarity as of the part of Q_1v we can show the PDE of first order harmonious requirement to the equation was hold. Similarly of Q_1v *we* see that $\Pi_1u(x,\tau)=0$ for $\tau\ge\varphi(x)$.

4-Construction of the Asymptoties of an Arbitrary Order

In this section we will find the solution of the regular part and boundary layer of order n. And now suppose $i = 0,1,\dots,n-1$ we get

$\Pi_iv(x,\tau)=0,\quad \Pi_iu(x,\tau)=0$ for $\tau\ge\varphi(x)$,

$Q_iu(\xi,t)=0,\ Q_iv(\xi,t)=0$ for $\xi\ge t$.

from section (3) these qualifications are holds of $n=j$ where $j=1,2$. Also us they these conditions we can enable to find a solution of regular and boundary layer with rank n.

4-1 Boundary Functions Q_nu and Π_nv.

For Q_nu we have the equation

$$b(0)\frac{\partial Q_nu}{\partial\xi}=\beta_n(\xi,t), \quad (4\text{-}1)$$

where the function

$$\beta_n(\xi,t)=\sum_{k=0}^{n-1}(k!)^{-1}\xi^k\left(\frac{\partial a_{11}}{\partial x^k}(0,t)Q_{n-1-k}u+\frac{\partial a_{12}}{\partial x^k}(0,t)Q_{n-1-k}v\right)-\left\{\frac{\partial Q_{n-2}u}{\partial t}+\sum_{k=1}^{n}(k!)^{-1}b^k(0)\xi^k\frac{\partial Q_{n-2}u}{\partial\xi}\right\}$$

is known

now $\forall\,\xi\le t\Rightarrow\beta_n(\xi,t)=0$ through supposition . We appreciate perform the $\beta_n(\xi,t)$ in the space $t\ge\xi$ like

$$\beta_n(\xi,t)=(t-\xi)\alpha_n(\xi,t) \quad (4\text{-}2)$$

where the function $\alpha_n(\xi,t)$ is smoothness . Integrate (3-14) and using the conditions $Q_nu(\infty,t)=0$, we get the solution of $Q_nu(\xi,t)$ as: $Q_nu(\xi,t)=$

$$\begin{cases}b^{-1}(0)\int_t^\xi\beta_n(\theta,t)d\theta & 0\le\xi\le t\le T,\\ zero, & \xi\ge t\end{cases}$$

that the formulation together with represent problem (4-2), that mean $Q_nu(\xi,t))$ is smoothness everywhere, include the characters line $\xi=t$.

While the function $\Pi_nv(x,\tau)$ can be shown as the solution of the problem as :

$$\frac{\partial\Pi_nv}{\partial\tau}=\rho_n(x,\tau)$$

and the condition $\Pi_nv(x,\infty)=0$,are satisfy where $\rho_n(x,\tau)$was function wellknown fade at $\tau \geq \varphi(x)$ and when $\tau = \varphi(x)$ make jump. Therefore by integrate we get

$$\Pi_n v(x,\tau) = \begin{cases} \int_{\varphi(x)}^{\tau} \rho_n(x,s)\, ds & 0 \leq \tau \leq \varphi(x), \\ zero, & \tau \geq \varphi(x) \end{cases} \quad (4\text{-}3)$$

this the function $\Pi_n v$ is smoothness in everywhere.

4-2 Regular Terms $\bar{u}_n$ and $\bar{v}_n$.

For the regular terms in order n we get the following equation

$$\left.\begin{aligned} & b(x)\frac{\partial \bar{u}_n}{\partial x} + \frac{\partial \bar{u}_{n-1}}{\partial t} = a_{11}(x,t)\bar{u}_n + a_{12}(x,t)\bar{v}_n + f_{0n}(x,t,\varepsilon) + f_{1n-1}(x,t) \\ & b(x)\frac{\partial \bar{u}_n}{\partial x} = a_{11}(x,t)\bar{u}_n + a_{12}(x,t)\bar{v}_n + L_n \\ & \frac{\partial \bar{v}_n}{\partial t} + \frac{\partial \bar{v}_{n-1}}{\partial x} = a_{21}(x,t)\bar{u}_1 + a_{22}(x,t)\bar{v}_1 + g_{01}(x,t,\varepsilon) + g_{1n-1}(x,t,\varepsilon) \\ & \frac{\partial \bar{v}_n}{\partial t} = a_{21}(x,t)\bar{u}_n + a_{22}(x,t)\bar{v}_n + N_n \end{aligned}\right\} \quad (4\text{-}4)$$

Where $L_n = f_{0n}(x,t,\varepsilon) + f_{1n-1}(x,t) - \frac{\partial \bar{u}_{n-1}}{\partial t}$ and

$N_n = g_{0n}(x,t,\varepsilon) + g_{1n-1}(x,t,\varepsilon) - \frac{\partial \bar{v}_{n-1}}{\partial x}$

$$\bar{u}_n(0,t) = -Q_n u(0,t), \quad \bar{v}_n(x,0) = -\Pi_n v(x,0) \quad (4\text{-}5)$$

This equation is the same as equation(3-7) and (3-11) and by reducing this system to integral equations We can find the solution it.

4-3 The Boundary Function Of $Q_n v$ and $\Pi_n u$

For $Q_n v(\xi,t)$ we have the following problem

$$\frac{\partial Q_n v}{\partial t} + \frac{\partial Q_n v}{\partial \xi} = a_{22}(0,t)Q_n v + q_n(\xi,t) \quad (4\text{-}6)$$

$$Q_n v(0,t) = -\bar{v}_n(0,t), \quad Q_n v(\xi,0) = 0 \quad (4\text{-}7)$$

where

$q_n(\xi,t) = \sum_{k=0}^{n}(k!)^{-1}\frac{\partial^k a_{21}}{\partial x^k}(0,t)\xi^k Q_{n-k}u(\xi,t) + \sum_{k=1}^{n}(k!)^{-1}\frac{\partial^k a_{22}}{\partial x^k}(0,t)\xi^k Q_{n-k}v(\xi,t)$ and the function $q_n(\xi,t)$ is smoothness.

Since $\bar{v}_n(0,0) = -\Pi_n v(0,0)$ (see (4-6)) and $\Pi_n v(0,0) = 0$ (see (4-5)), that is $\bar{v}_n(0,0) = 0$. Therefore the boundary condition of equation(4-7) are continuous harmonious in $(0,0)$.Next parity is needful to boundary to verify equation(4-6) at the point (0, 0):

$$-\frac{\partial \bar{v}_n}{\partial t}(0,0) = q_n(0,0)\ .$$

The terms of the right-hand is equivalence to disappear , since $q_n(\xi,t) = 0$ when $t \in [\xi, 0]$. It was just offering the $\frac{\partial \bar{v}_n}{\partial t}(0,0) = 0$. Lets go to problem (4-4) of second part, so will offer that $(a_{21}(x,t)\bar{u}_n + a_{22}(x,t)\bar{v}_n + N_n)$ will disappear in point (0,0). This will hint that the $\frac{\partial \bar{v}_n}{\partial t}(0,0) = 0$. Belong equation (4-5) we get $\bar{u}_n(0,0) = -Q_n u(0,0)$, and cause $Q_n u(\xi,t) = 0$ at $t \in [\xi, 0]$, that mean the $Q_n u(0,0) = 0$, therefore $\bar{u}_n(0,0)$and $a_{21}(x,t)\bar{u}_n$ equation(4-4) fade at the point (0,0). The next terms (second term, $a_{22}(x,t)\bar{v}_n$) will disappear, because $\bar{v}_n(0,0) = 0$. By condition 1 we get $g_{0n}(x,t)$=$g_{1n-1}(x,t) = 0$ and (so $\bar{v}_{n-1}(x,0) = -\Pi_{n-1}v(x,0)$) that is

$$\frac{\partial \bar{v}_{n-1}}{\partial x}(0,0) = -\frac{\partial \Pi_{n-1}v}{\partial x}(0,0).$$

Anyway $\Pi_{n-1}v(x,\tau)$ for $\tau \geq \varphi(x)$; therefore $\frac{\partial \Pi_{n-1}v}{\partial x}(x,\tau) \approx 0$ for $\tau > \varphi(x)$ and, by continuity, also for $\tau = \varphi(x)$. In particular, $\frac{\partial \Pi_{n-1}v}{\partial x}(0,0) = 0$. Consequently, $\frac{\partial \bar{v}_{n-1}}{\partial x}(0,0) = 0$, and hence $N_n(0,0) = 0$.

Thus, the first order compatible condition was satisfied at point (0, 0). so the equation(4-6) and (4-7) have smooth solution in whole space ($\xi \geq 0$) and $t \in [0,T]$ it will write up in the same as to equation(3-13).

The $\Pi_n u(x,\tau)$ can be shown as the solution of the equation

$$\frac{\partial \Pi_n u}{\partial \tau} + b(x)\frac{\partial \Pi_n u}{\partial x} = a_{11}(x,0)\Pi_n u + \gamma_n(x,\tau),$$
$$\Pi_n u(x,0) = -\bar{u}_n(x,0), \quad \Pi_n u(0,\tau) = 0,$$

and $\gamma_n(x,\tau)$ is smoothness function which is known. For situation the function $Q_n v$, we can offer the $Q_n v$ has a smoothness solution, also $\Pi_n u(x,\tau) = 0$ for $\tau \geq \varphi(x)$.

5- Estimating of the Remainder Terms

We will indicate by $\chi_n(x,t,\varepsilon)$ and $\psi_n(x,t,\varepsilon)$ the n^{th} of equation $(3-1)$

Theorem. Next estimates, the solution of $u_n(x,t,\varepsilon)$ and $v_n(x,t,\varepsilon)$ in equation $(2-1)$, $(2-2)$ that hold for ε and orderly for space $\overline{\Psi} = [0,X] \times [0,T]$:

$u(x,t,\varepsilon) - \chi_n(x,t,\varepsilon) = O(\varepsilon^{n+1})$ and

$v(x,t,\varepsilon) - \psi_n(x,t,\varepsilon) = O(\varepsilon^{n+1})$

Proof. Suppose that $u = \chi_{n+1} + \varrho_1$ and $v = \psi_{n+1} + \varrho_2$. Replacing the $u - \chi_{n+1}$ and $v - \psi_{n+1}$ in equation (2-1) and $(2-2)$ to get equation of estimates of the ϱ_1 and ϱ_2:

$$\varepsilon\frac{\partial \varrho_1}{\partial t} + b_1(x)\frac{\partial \varrho_1}{\partial x} = a_{11}(x,t)\varrho_1 + a_{12}(x,t)\varrho_2 + h_0(x,t,\varepsilon) + \varepsilon h_1(x,t,\varepsilon)$$
$$\frac{\partial \varrho_2}{\partial t} + \varepsilon b_2(x)\frac{\partial \varrho_2}{\partial x} = a_{21}(x,t)\varrho_1 + a_{22}(x,t)\varrho_2 + z_0(x,t,\varepsilon) + \varepsilon z_1(x,t,\varepsilon)$$
$$\varrho_i|_{t=0} = \varrho_i|_{x=0} = 0; \qquad i = 1,2.$$

clearly, the terms of the right hand side above h_0 , h_1 and z_0, z_1 can be estimated as $h_i(x,t,\varepsilon) = z_i(x,t,\varepsilon) = O(\varepsilon^{n+1})$ during the space $\overline{\Psi} = [0,X] \times [0,T]$ was been uniform

Now we want to proven estimation hold to the set $u - \chi_{n+1}$ and $v - \psi_{n+1}$ was similarity. Let us see the equality

$$u - \chi_n = (u - \chi_{n+1}) + (\chi_{n+1} - \chi_n) = \varrho_1 + O(\varepsilon^{n+1}),$$
$$v - \psi_n = (v - \psi_{n+1}) + (\psi_{n+1} - \psi_n) = \varrho_2 + O(\varepsilon^{n+1})$$

this will infer the declaration of the thermos.

Let us make changing variable for $\varrho_1 = s_1 \exp(\text{Ɔ})$ and $\varrho_2 = s_2 \exp(\text{Ɔ})$, where $\text{Ɔ} = k(x+t)$ and the constant k is positive . Hence getting problems of $s_i, i = 1,2$

$$\left.\begin{aligned} \varepsilon\frac{\partial s_1}{\partial t} + b_1(x)\frac{\partial s}{\partial x} &= p_{11}s_1 + p_{12}s_2 + H_0 + \varepsilon H_1 \\ \frac{\partial s_2}{\partial t} + \varepsilon b_2(x)\frac{\partial s_2}{\partial x} &= p_{21}s_1 + p_{22}s_2 + Z_0 + \varepsilon Z_1 \end{aligned}\right\} \qquad (5\text{-}1)$$

where

$p_{22} = a_{22} - k(1+\varepsilon), \qquad p_{11} = a_{11} - \varpi$

$p_{12} = a_{12,} \qquad p_{21} = a_{21}$

$\varpi = k(b(x) + \varepsilon)$ and $H_i = h_i e^{-k(x+t)} = O(\varepsilon^{n+1}) = Z_i = z_i e^{-k(x+t)}$ we will take the k sufficient large, then the inequality

$$p_{11} + |p_{12}| \leq -1, \quad |p_{21}| + p_{22} \leq -1 \qquad (5\text{-}2)$$

hold.

suppose that $|s_1|$ has a maximum at a point $\omega_1(x_1, t_1)$ of $\overline{G}$ and $|s_2|$ has a maximum at point $\omega_2(x_2, t_2)$, also suppose that $|s_1(\omega_1)| \geq |s_2(\omega_2)|$, we will be consider the first problem of (5-1) at a point $\omega_1(x_1, t_1)$. (If the $|s_1(\omega_1)| \leq |s_2(\omega_2)|$, then we will consider the second problem of (5-1) at point $\omega_2(x_2, t_2)$.) We rewrite this problem as

$$\varepsilon \frac{\partial s_1}{\partial t} + b_1(x) \frac{\partial s_1}{\partial x} - p_{11} s_1 - p_{12} s_2 = H_0 + \varepsilon H_1 \qquad (5\text{-}3)$$

Now we suppose that the $s_1 < 0$ and have a min. for point $\omega_1(x_1, t_1)$. (in the same as way we will consider the positive maximum case) . Then $\frac{\partial s_1}{\partial t} \leq 0$ and $\frac{\partial s_1}{\partial x} \leq 0$ at the point $\omega_1(x_1, t_1)$ and by equation (5-2) we get ,

$$-p_{11}s_1 - p_{12}s_2 \leq -p_{11}s_1 + |p_{12}| \,.\, |s_2| \leq -(-p_{11} + |p_{12}|)s_1 \leq s_1$$

therefore the left hand side of equation(5-3) is negative at point $\omega_1(x_1, t_1)$ and isn't greater than $s_1(\omega_1)$, whilst $H_0 + \varepsilon H_1$ of order ε^{n+1}.thus, $|s_1(\omega_1)| = max_{\overline{G}}|s_1(x,t)| = O(\varepsilon^{n+1})$. Since $|s_1(\omega_1)| \geq |s_2(\omega_2)|$, it follows $|s_2(\omega_2)| = max_{\overline{G}}|s_2(x,t)| = O(\varepsilon^{n+1})$. Therefore $s_i(x,t) = O(\varepsilon^{n+1})$ is uniformly in the domain $\overline{\Psi} = [0, \mathrm{X}] \times [0, \mathrm{T}]$. Hence getting that $\varrho_1 = s_1 \exp(\mathrm{Ɔ}) = O(\varepsilon^{n+1})$ and $\varrho_2 = s_2 \exp(\mathrm{Ɔ}) = O(\varepsilon^{n+1})$ are uniformly in $\overline{\Psi} = [0, \mathrm{X}] \times [0, \mathrm{T}]$. And this the theorem was proved.

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