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Fuzzy Euclidean Normed Spaces for Data Mining Applications

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Abstract: The aim of this paper is to introduce some special fuzzy norms on $\mathbb{K}^n$ and to obtain, in this way, fuzzy Euclidean normed spaces. In order to introduce this concept we have proved that the cartesian product of a finite family of fuzzy normed linear spaces is a fuzzy normed linear space. Thus any fuzzy norm on $\mathbb{K}$ generates a fuzzy norm on $\mathbb{K}^n$. Finally, we prove that each fuzzy Euclidean normed space is complete. Fuzzy Euclidean normed spaces can be proven to be a suitable tool for data mining. The method is based on embedding the data in fuzzy Euclidean normed spaces and to carry out data analysis in these spaces.
Keywords: fuzzy norm, fuzzy Euclidean normed spaces, data mining.

1 Introduction

Data mining and information retrieval are two important components of the same problem: discovering new and relevant information and knowledge, through investigation of a large amount of data, through extracting the information and knowledge out of a very large databases or data warehouse.

In information retrieval the user knows what he is looking for, but sometimes it is very difficult to express this thing. The use of fuzzy sets in representing the knowledge proves to be successful on many occasions, allowing the user to express his expectations in a language close to the natural one. On the other hand, many times the matching between the requests of the user and the existing data in the databases is only an approximate one, thus, the use of fuzzy sets and the degrees of membership proves to be not only useful but also necessary.

In data mining, the user looks for new knowledge. The aim is to divide the data into homogeneous categories, in data classes. The use of the of fuzzy sets brings about flexibility both in representing knowledge and in interpreting the results as well.

The measures of similarity are the most used, at all levels in the data mining and information retrieval. The notion of similarity, or more general of the measures of comparison is the central point for all applications in the real world. The measure of similarity aims at quantifying the degree to which two objects are similar or dissimilar, offering a numeric value for this comparison.

Machine learning techniques use similarity measures. Machine learning represents an important method of extracting the knowledge out of very large databases. A study concerning fuzzy learning methods was realized by E. Hüllermeier [4].

In the last twenty years, the World Wide Web has become a major source of data and information for all domains. Web mining is the process of discovering useful knowledge and information through investigating the web structure and its content. Different web mining tasks and advanced artificial intelligence methods for information retrieval and web mining are discussed by I. Dzitac & I. Moisil [3].

Clustering and classification are both important tasks in data mining. Since clustering means the grouping of similar objects, we need some suitable measures on data sets. In order to determinate the similarity or dissimilarity between any pair of objects, the most used measures are distance measures.

If each data point is view as a n-dimensional vector $x = (x_1, x_2, \cdots, x_n) \in \mathbb{R}^n$, where each component x_i is the value of an attribute of the data, the distance between the two data instances can be calculated using Euclidean distance, Manhattan distance, Minkowski distance, Max distance, etc.

What will we do if the distance between the vectors x and y can not be precisely measured and thus we are not able to assign it, with certainty, the value $t \in \mathbb{R}_+$. There are probably different approaches enabling to handle somehow this situation. One of them, fuzzy approach, consists in using on $\mathbb{R}^n$ some fuzzy metrics, i.e. mappings $M : \mathbb{R}^n \times \mathbb{R}^n \times [0, \infty) \to [0, 1]$, where $M(x, y, t) = \alpha$ indicates the truth value of the statement "the distance between x and y is smaller than the real number t" and which belongs to $[0, 1]$. It will be better that such fuzzy metrics to come from fuzzy norms on $\mathbb{R}^n$, namely $M(x, y, t) = N(x - y, t)$, where $N : \mathbb{R}^n \times [0, \infty) \to [0, 1]$.

The aim of this paper is to introduce some special fuzzy norms on $\mathbb{K}^n$ and to obtain, in this way, fuzzy Euclidean normed spaces. In order to introduce this concept we have proved that the cartesian product of a finite family of fuzzy normed linear spaces is a fuzzy normed linear space. Thus any fuzzy norm on $\mathbb{K}$ generates a fuzzy norm on $\mathbb{K}^n$. Finally, we prove that each fuzzy Euclidean normed space is complete. Fuzzy Euclidean normed spaces can be proven to be a suitable tool for data mining. The method is based on embedding the data in fuzzy Euclidean normed spaces and to carry out data analysis in these spaces.

In studying fuzzy topological vector spaces, A.K. Katsaras [5] first introduced the notion of fuzzy norm on a linear space. Since then many mathematicians have introduced several notions of fuzzy norm from different points of view. Our definition looks similar, but it is more general, to the definitions introduced, almost in the same time, by T. Bag & S.K. Samanta (see [1], [2]) and R. Saadati & S.M. Vaezpour (see [7]). In 2006, R. Saadati & J.H. Park introduced the notion of intuitionistic fuzzy Euclidean normed space (see [8], [9]).

2 Preliminaries

Definition 1. [10] A binary operation $* : [0, 1] \times [0, 1] \to [0, 1]$ is called triangular norm (t-norm) if it satisfies the following conditions:

1. $a * b = b * a, (\forall) a, b \in [0, 1]$;
2. $a * 1 = a, (\forall) a \in [0, 1]$;
3. $(a * b) * c = a * (b * c), (\forall) a, b, c \in [0, 1]$;
4. If $a \le c$ and $b \le d$, with $a, b, c, d \in [0, 1]$, then $a * b \le c * d$.

Example 2. Three basic examples of continuous t-norms are $\wedge, \cdot, *_L$, which are defined by $a \wedge b = \min\{a, b\}$, $a \cdot b = ab$ (usual multiplication in $[0, 1]$) and $a *_L b = \max\{a + b - 1, 0\}$ (the Lukasiewicz t-norm).

Remark 3. $* \le \wedge$, i.e. $\wedge$ is stronger that any other t-norms.

Indeed, $a * b \le a * 1 = a$, $a * b \le 1 * b = b$. Thus $a * b \le a \wedge b$.

Definition 4. Let $*, *\prime$ be two t-norms. We say that $*\prime$ dominates $*$ and we denote $*\prime \gg *$ if

$$(x_1 * \prime x_2) * (y_1 * \prime y_2) \le (x_1 * y_1) * \prime (x_2 * y_2), (\forall) x_1, x_2, y_1, y_2 \in [0, 1].$$

Proposition 5. For any t-norm $*$ we have $\wedge \gg *$.

Proof: Let $x_1, x_2, y_1, y_2 \in [0,1]$.
Case 1. $x_1 \leq x_2, y_1 \leq y_2$. Then $x_1 * y_1 \leq x_2 * y_2$. Thus $(x_1 * y_1) \wedge (x_2 * y_2) = x_1 * y_1$. On the other hand $(x_1 \wedge x_2) * (y_1 \wedge y_2) = x_1 * y_1$. Therefore $(x_1 \wedge x_2) * (y_1 \wedge y_2) = (x_1 * y_1) \wedge (x_2 * y_2)$.
Case 2. $x_1 \leq x_2, y_2 \leq y_1$. As $x_1 \leq x_2$, we have $x_1 * y_2 \leq x_2 * y_2$. As $y_2 \leq y_1$, we have $x_1 * y_2 \leq x_1 * y_1$. Thus $x_1 * y_2 \leq (x_1 * y_1) \wedge (x_2 * y_2)$. Hence

$$(x_1 \wedge x_2) * (y_1 \wedge y_2) = x_1 * y_2 \leq (x_1 * y_1) \wedge (x_2 * y_2).$$

Case 3. $x_2 \leq x_1, y_1 \leq y_2$ and Case 4. $x_2 \leq x_1, y_2 \leq y_1$ are similar to previous cases. □

Definition 6. [6] Let X be a vector space over a field $\mathbb{K}$ (where $\mathbb{K}$ is $\mathbb{R}$ or $\mathbb{C}$) and $*$ be a continuous t-norm. A fuzzy set N in $X \times [0,\infty)$ is called a fuzzy norm on X if it satisfies:

(N1) $N(x,0) = 0, (\forall) x \in X$;

(N2) $[N(x,t) = 1, (\forall) t > 0]$ if and only if $x = 0$;

(N3) $N(\lambda x, t) = N\left(x, \frac{t}{|\lambda|}\right), (\forall) x \in X, (\forall) t \geq 0, (\forall) \lambda \in \mathbb{K}^*$;

(N4) $N(x+y, t+s) \geq N(x,t) * N(y,s), (\forall) x, y \in X, (\forall) t, s \geq 0$;

(N5) $(\forall) x \in X$, $N(x,\cdot)$ is left continuous and $\lim_{t\to\infty} N(x,t) = 1$.

The triple $(X, N, *)$ will be called fuzzy normed linear space (briefly FNL-space).

Remark 7. a) T. Bag and S.K. Samanta [1], [2] gave a similar definition for $* = \wedge$, but in order to obtain some important results they assume that the fuzzy norm satisfies also the following conditions:

(N6) $N(x,t) > 0, (\forall) t > 0 \Rightarrow x = 0$;

(N7) $(\forall) x \neq 0, N(x,\cdot)$ is a continuous function and strictly increasing on the subset $\{t : 0 < N(x,t) < 1\}$ of $\mathbb{R}$.

The results obtained by T. Bag and S.K. Samanta can be found in this more general settings.
b) R. Saadati and S.M. Vaezpour [7] suppose that

1. $N(x,t) > 0, (\forall) t > 0$;
2. $N(x,\cdot)$ is a continuous function, $(\forall) x \neq 0$.

Remark 8. $N(x,\cdot)$ is nondecreasing, $(\forall) x \in X$.

Theorem 2.1. [6] Let $(X, N, *)$ be a FNL-space. For $x \in X, r \in (0,1), t > 0$ we define the open ball

$$B(x,r,t) := \{y \in X \ : \ N(x-y,t) > 1-r\} .$$

Then

$$\mathcal{T}_N := \{T \subset X \ : \ x \in T \text{ iff } (\exists) t > 0, r \in (0,1) \ : \ B(x,r,t) \subseteq T\}$$

is a topology on X.

Moreover, if the t-norm $*$ satisfies $\sup_{x\in(0,1)} x * x = 1$, then $(X, \mathcal{T}_N)$ is Hausdorff.

Theorem 2.2. [6] Let $(X, N, \wedge)$ be a FNL-space. Let

$$p_\alpha(x) := \inf\{t > 0 \ : \ N(x,t) > \alpha\}, \alpha \in (0,1) .$$

Then $\mathcal{P} = \{p_\alpha\}_{\alpha\in(0,1)}$ is an ascending family of semi-norms on X.

Moreover, for $x \in X, s > 0, \alpha \in (0,1)$ we have: $p_\alpha(x) < s$ if and only if $N(x,s) > \alpha$.

3 Convergence in FNL-spaces

Definition 9. [2] Let $(X, N, *)$ be a FNL-space and (x_n) be a sequence in X. The sequence (x_n) is said to be convergent if $(\exists)x \in X$ such that $\lim_{n\to\infty} N(x_n - x, t) = 1$, $(\forall)t > 0$. In this case, x is called the limit of the sequence (x_n) and we denote $\lim_{n\to\infty} x_n = x$ or $x_n \to x$.

Definition 10. [2] Let $(X, N, *)$ be a FNL-space and (x_n) be a sequence in X. The sequence (x_n) is called Cauchy sequence if $\lim_{n\to\infty} N(x_{n+p} - x_n, t) = 1$, $(\forall)t > 0, (\forall)p \in \mathbb{N}^*$.

Remark 11. If $(X, N, *)$ is a FNL-space, then every convergent sequence is Cauchy sequence.

Definition 12. [2] Let $(X, N, *)$ be a FNL-space. $(X, N, *)$ is said to be complete if any Cauchy sequence in X is convergent to a point in X. A complete FNL-space will be called fuzzy Banach space.

Definition 13. Let $(X, N, *)$ be a FNL-space, $\alpha \in (0, 1)$ and (x_n) be a sequence in X. The sequence (x_n) is said to be α-convergent if exists $x \in X$ such that

$$(\forall)t > 0, (\exists)n_0 \in \mathbb{N} \, : \, N(x_n - x, t) > \alpha, (\forall)n \geq n_0 \, .$$

In this case, x is called the α-limit of the sequence (x_n) and we denote $x_n \stackrel{\alpha}{\to} x$.

Theorem 3.1. Let $(X, N, *)$ be a FNL-space and (x_n) be a sequence in X. The following sentences are equivalent:

1. (x_n) is convergent to x;
2. (x_n) is convergent to x in topology $\mathcal{T}_N$;
3. (x_n) is α-convergent to x, $(\forall)\alpha \in (0, 1)$;
4. $\lim_{n\to\infty} p_\alpha(x_n - x) = 0, (\forall)\alpha \in (0, 1)$.

Proof: (2) $\Leftrightarrow$ (1)

$$x_n \to x \text{ in the topology } \mathcal{T}_M \Leftrightarrow$$

$$(\forall)r \in (0, 1), (\forall)t > 0, (\exists)n_0 \in \mathbb{N} \, : \, x_n \in B(x, r, t), (\forall)n \geq n_0 \Leftrightarrow$$

$$(\forall)r \in (0, 1), (\forall)t > 0, (\exists)n_0 \in \mathbb{N} \, : \, N(x_n - x, t) > 1 - r, (\forall)n \geq n_0 \Leftrightarrow$$

$$\lim_{n\to\infty} N(x_n - x, t) = 1 \, , \, (\forall)t > 0 \, .$$

(1) $\Leftrightarrow$ (3) It is obvious.
(4) $\Leftrightarrow$ (3)

$$\lim_{n\to\infty} p_\alpha(x_n - x) = 0 \Leftrightarrow (\forall)t > 0, (\exists)n_0 \in \mathbb{N} \, : \, p_\alpha(x_n - x) < t, (\forall)n \geq n_0$$

$$\Leftrightarrow (\forall)t > 0, (\exists)n_0 \in \mathbb{N} \, : \, N(x_n - x, t) > \alpha, (\forall)n \geq n_0 \Leftrightarrow x_n \stackrel{\alpha}{\to} x \, .$$

□

Theorem 3.2. Let $(X, N, *)$ be a FNL-space and (x_n) be a sequence in X. Then (x_n) is a Cauchy sequence if and only if $\lim_{n\to\infty} p_\alpha(x_{n+p} - x_n) = 0, (\forall)\alpha \in (0, 1), (\forall)p \geq 1$.

Proof:

$$\lim_{n\to\infty} p_\alpha(x_{n+p}-x_n)=0, (\forall)\alpha\in(0,1), (\forall)p\geq 1$$

$$\Leftrightarrow (\forall)t>0, (\exists)n_0\in\mathbb{N} \,:\, p_\alpha(x_{n+p}-x_n)<t, (\forall)n\geq n_0, (\forall)\alpha\in(0,1), (\forall)p\geq 1$$

$$\Leftrightarrow (\forall)t>0, (\exists)n_0\in\mathbb{N} \,:\, N(x_{n+p}-x_n,t)>\alpha, (\forall)n\geq n_0, (\forall)\alpha\in(0,1), (\forall)p\geq 1$$

$$\Leftrightarrow (\forall)t>0, \lim_{n\to\infty} N(x_{n+p}-x_n,t)\geq\alpha, (\forall)\alpha\in(0,1), (\forall)p\geq 1$$

$$\Leftrightarrow (\forall)t>0, \lim_{n\to\infty} N(x_{n+p}-x_n,t)=1, (\forall)p\geq 1 \Leftrightarrow (x_n) \text{ is a Cauchy sequence .}$$

□

Definition 14. Let $(X,N,*)$, $(X,N',*')$ be two FNL-space. The fuzzy norms N and N' are said to be equivalent if for any sequence (x_n) in X, we have $x_n\to x$ in $(X,N,*)$ if and only if $x_n\to x$ in $(X,N',*')$.

4 Fuzzy Euclidean normed spaces

In this section we will denote by $\mathbb{K}$ the field of real numbers $\mathbb{R}$ or the field of complex numbers $\mathbb{C}$.

Theorem 4.1. Let $(X_1,N_1,*),(X_2,N_2,*),\cdots,(X_n,N_n,*)$ be FNL-spaces. Let $*'$ be a continuous t-norm such that $*'\gg *$. Let $N: X_1\times X_2\times\cdots X_n\times[0,\infty)\to[0,1]$,

$$N(x_1,x_2,\cdots,x_n,t)=N_1(x_1,t)*'N_2(x_2,t)*'\cdots*'N_n(x_n,t)\ .$$

Then $(X_1\times X_2\times\cdots X_n,N,*)$ is a FNL-space.

Proof: (N1) $N(x_1,x_2,\cdots,x_n,0)=N_1(x_1,0)*'N_2(x_2,0)*'\cdots*'N_n(x_n,0)=0$.
(N2)

$$N(0,0,\cdots,0,t)=N_1(0,t)*'N_2(0,t)*'\cdots*'N_n(0,t)=1\ .$$

Conversely, if $N(x_1,x_2,\cdots,x_n,t)=1, (\forall)t>0$, we obtain that

$$N_1(x_1,t)*'N_2(x_2,t)*'\cdots*'N_n(x_n,t)=1, (\forall)t>0.$$

As $*'\leq\wedge$, we have

$$1\leq\min\{N_1(x_1,t),N_2(x_2,t),\cdots,N_n(x_n,t)\}, (\forall)t>0.$$

Thus $N_1(x_1,t)=1, N_2(x_2,t)=1,\cdots,N_n(x_n,t)=1, (\forall)t>0$. Hence $x_1=x_2=\cdots=x_n=0$.
(N3) For $\lambda\neq 0$, we have

$$N(\lambda x_1,\lambda x_2,\cdots,\lambda x_n,t)=N_1(\lambda x_1,t)*'N_2(\lambda x_2,t)*'\cdots*'N_n(\lambda x_n,t)=$$
$$=N_1\left(x_1,\frac{t}{|\lambda|}\right)*'N_2\left(x_2,\frac{t}{|\lambda|}\right)*'\cdots N_n\left(x_n,\frac{t}{|\lambda|}\right)=N\left(x_1,x_2,\cdots,x_n,\frac{t}{|\lambda|}\right).$$

(N4)

$$N(x_1+y_1,x_2+y_2,\cdots,x_n+y_n,t+s)=N_1(x_1+y_1,t+s)*'N_2(x_2+y_2,t+s)*'\cdots*'N_n(x_n+y_n,t+s)\geq$$
$$\geq(N_1(x_1,t)*N_1(y_1,s))*'(N_2(x_2,t)*N_2(y_2,s))*'\cdots*'(N_n(x_n,t)*N_n(y_n,s))\geq$$
$$\geq(N_1(x_1,t)*'N_2(x_2,t)*'\cdots*'N_n(x_n,t))*(N_1(y_1,s)*'N_2(y_2,s)*'\cdots*'N_n(y_n,s))=$$
$$=N(x_1,x_2,\cdots,x_n,t)*N(y_1,y_2,\cdots,y_n,s).$$

(N5) Let $x=(x_1,x_2,\cdots,x_n)\in X_1\times X_2\times\cdots X_n$. As $N_1(x_1,\cdot),N_2(x_2,\cdot),\cdots,N_n(x_n,\cdot)$ are left continuous and $*'$ is a continuous t-norm, we obtain that $N(x,\cdot)$ is left continuous. It is obvious that $\lim_{t\to\infty}N(x,t)=1$. □

Proposition 15. If $(X_1 \times X_2 \times \cdots X_n, N, *)$ is a FNL-space, then $(X_1, N_1, *)$, $(X_2, N_2, *)$, $\cdots$, $(X_n, N_n, *)$ are FNL-spaces, where $N_1(x_1,t) = N((x_1, 0, \cdots, 0), t)$, $N_2(x_2,t) = N((0, x_2, \cdots, 0), t)$, $\cdots$, $N_n(x_n,t) = N((0, 0, \cdots, x_n), t)$.

Proof: We will prove that N_1 is a fuzzy norm. Similarly it can be shown that $N_2, \cdots, N_n$ are fuzzy norm.
(N1) $N_1(x_1, 0) = N((x_1, 0, \cdots, 0), 0) = 0$;
(N2) $N_1(x_1,t) = 1, (\forall)t > 0 \Leftrightarrow N((x_1, 0, \cdots, 0), t) = 1, (\forall)t > 0 \Leftrightarrow (x_1, 0, \cdots, 0) = 0 \Leftrightarrow x_1 = 0$;
(N3)

$$N_1(\lambda x_1, t) = N((\lambda x_1, 0, \cdots, 0), t) = N(\lambda(x_1, 0, \cdots, 0), t) =$$
$$= N\left((x_1, 0, \cdots, 0), \frac{t}{|\lambda|}\right) = N_1\left(x_1, \frac{t}{|\lambda|}\right) \; ;$$

(N4)

$$N_1(x_1 + y_1, t + s) = N((x_1 + y_1, 0, \cdots, 0), t + s) = N((x_1, 0, \cdots, 0) + (y_1, 0, \cdots, 0), t + s) \geq$$
$$\geq N((x_1, 0, \cdots, 0), t) * N((y_1, 0, \cdots, 0), s) = N_1(x_1, t) * N_1(y_1, s) \; ;$$

(N5) It is obvious. □

Example 16. Let $N : \mathbb{K} \times [0, \infty) \to [0, 1]$, defined by

$$N(x,t) := \begin{cases} e^{-\frac{|x|}{t}}, & \text{if} \quad t > 0 \\ 0, & \text{if} \quad t = 0 \end{cases} \; .$$

Then $(\mathbb{K}, N, \wedge)$ is a FNL-space.

Proof: (N1) It is obvious.
(N2) $N(x,t) = 1, (\forall)t > 0 \Leftrightarrow e^{-\frac{|x|}{t}} = 1, (\forall)t > 0 \Leftrightarrow -\frac{|x|}{t} = 0, (\forall)t > 0 \Leftrightarrow x = 0$.
(N3) Let $x \in \mathbb{R}, t > 0, \lambda \in \mathbb{R}*$. Then

$$N(\lambda x, t) = e^{-\frac{|\lambda x|}{t}} = e^{-\frac{|x|}{t/|\lambda|}} = N\left(x, \frac{t}{|\lambda|}\right) \; .$$

(N4) Fix $x, y \in \mathbb{R}, t, s > 0$. We assume, without restricting the generality, that $e^{-\frac{|x|}{t}} \leq e^{-\frac{|y|}{s}}$. Thus $-\frac{|x|}{t} \leq -\frac{|y|}{s}$, i.e. $|x|s \geq |y|t$. We will show that $e^{-\frac{|x+y|}{t+s}} \geq e^{-\frac{|x|}{t}}$, namely $-\frac{|x+y|}{t+s} \geq -\frac{|x|}{t}$, i.e. $|x + y|t \leq |x|(t + s)$. But

$$|x + y|t \leq (|x| + |y|)t = |x|t + |y|t \leq |x|t + |x|s = |x|(t + s) \; .$$

Therefore

$$N(x + y, t + s) = e^{-\frac{|x+y|}{t+s}} \geq \min\left\{e^{-\frac{|x|}{t}}, e^{-\frac{|y|}{s}}\right\} = N(x,t) \wedge N(y, s) \; .$$

(N5) It is obvious. □

Lemma 17. *Let* $(\mathbb{K}, N, *)$ *be a FNL-space. Then there exists* $\alpha \in (0, 1)$ *such that* $p_\alpha(1) \neq 0$.

Proof: $p_\alpha(1) = \inf\{t > 0 : N(1,t) > \alpha\}$. We suppose that $p_\alpha(1) = 0, (\forall)\alpha \in (0, 1)$. Then $N(1,t) > \alpha, (\forall)\alpha \in (0, 1), (\forall)t > 0$. Thus $N(1,t) = 1, (\forall)t > 0$. Therefore $1 = 0$, contradiction. □

Proposition 18. A sequence (x_n) is convergent in a FNL-space $(\mathbb{K}, N, *)$ if and only if (x_n) is convergent in $(\mathbb{K}, |\cdot|)$.

Proof: A sequence (x_n) is convergent to x in $(\mathbb{K}, N, *) \Leftrightarrow \lim_{n\to\infty} p_\alpha(x_n - x) = 0, (\forall)\alpha \in (0,1)$ $\Leftrightarrow \lim_{n\to\infty} |x_n - x| p_\alpha(1) = 0, (\forall)\alpha \in (0,1) \Leftrightarrow \lim_{n\to\infty} |x_n - x| = 0 \Leftrightarrow (x_n)$ is convergent in $(\mathbb{K}, |\cdot|)$ □

Corollary 19. Any two fuzzy norm on $\mathbb{K}$ are equivalent.

Definition 20. The triplet $(\mathbb{K}^n, N, *)$ is called fuzzy Euclidean normed space (briefly FEN-space) if $*$ is a continuous t-norm and $N : \mathbb{K}^n \times [0,\infty) \to [0,1]$ is a fuzzy norm defined by

$$N(x_1, x_2, \cdots, x_n, t) = N_1(x_1, t) \wedge N_2(x_2, t) \wedge \cdots \wedge N_n(x_n, t) ,$$

where $N_1, N_2, \cdots, N_n$ are fuzzy norms on $\mathbb{K}$ ((N4) is satisfied with the t-norm $*$, for all fuzzy norms $N_1, N_2, \cdots, N_n$).

Remark 21. Theorem 4.1 and the fact that $\wedge \gg *$ assure the accuracy of the previous definition, meaning that N is fuzzy norm on $\mathbb{K}^n$ indeed.

Proposition 22. A sequence (x_k) is convergent in a FEN-space $(\mathbb{K}^n, N, *)$ if and only if (x_k) is convergent in $(\mathbb{K}^n, ||\cdot||)$, where $||\cdot||$ denotes the Euclidean norm on $\mathbb{K}^n$.

Proof:

$$(x_k) \text{ is convergent to } x \text{ in } (\mathbb{K}^n, N, *) \Leftrightarrow \lim_{k\to\infty} N(x_k - x, t) = 1, (\forall)t > 0$$

$$\Leftrightarrow \lim_{k\to\infty} N_1(x_k^1 - x^1, t) \wedge N_2(x_k^2 - x^2, t) \wedge \cdots \wedge N_n(x_k^n - x^n, t) = 1, (\forall)t > 0$$

$$\Leftrightarrow \lim_{k\to\infty} N_i(x_k^i - x^i, t) = 1, (\forall)t > 0, (\forall)i = \overline{1,n} \Leftrightarrow |x_k^i - x^i| \to 0, (\forall)i = \overline{1,n} \Leftrightarrow ||x_k - x|| \to 0 .$$

□

Theorem 4.2. Any FEN-space $(\mathbb{K}^n, N, *)$ is complete.

Proof: Let (x_k) be a Cauchy sequence in $(\mathbb{K}^n, N, *)$. Then

$$p_{\alpha,N}(x_{k+p} - x_k) = \inf\{t > 0 \ : \ N(x_{k+p} - x_k, t) > \alpha\} =$$

$$= \inf\{t > 0 \ : \ N_1(x_{k+p}^1 - x_k^1, t) \wedge N_2(x_{k+p}^2 - x_k^2, t) \wedge \cdots \wedge N_n(x_{k+p}^n - x_k^n, t) > \alpha\} =$$

$$= \inf\{t > 0 \ : \ N_1(x_{k+p}^1 - x_k^1, t) > \alpha, N_2(x_{k+p}^2 - x_k^2, t) > \alpha, \cdots, N_n(x_{k+p}^n - x_k^n, t) > \alpha\} \geq$$

$$\geq \inf\{t > 0 \ : \ N_i(x_{k+p}^i - x_k^i, t) > \alpha\}, (\forall)i = \overline{1,n} .$$

Thus

$$p_{\alpha,N}(x_{k+p} - x_k) \geq p_{\alpha,N_i}(x_{k+p}^i - x_k^i) = |x_{k+p}^i - x_k^i| p_{\alpha,N_i}(1), (\forall)i = \overline{1,n} .$$

By Lemma 4.4, applied to fuzzy norm N_i, we obtain that there exists $\alpha_i \in (0,1)$ such that $p_{\alpha_i,N_i}(1) \neq 0$. Therefore

$$p_{\alpha_i,N}(x_{k+p} - x_k) \geq |x_{k+p}^i - x_k^i| p_{\alpha_i,N_i}(1), (\forall)i = \overline{1,n} .$$

As (x_k) is a Cauchy sequence in $(\mathbb{K}^n, N, *)$, we obtain that (x_k^i) is a Cauchy sequence in $(\mathbb{K}, |\cdot|), (\forall)i = \overline{1,n}$. Thus (x_k^i) is convergent to $x^i, (\forall)i = \overline{1,n}$. Therefore (x_k) is convergent to $x = (x^1, x^2, \cdots, x^n)$ in $(\mathbb{K}^n, ||\cdot||)$ and previous proposition implies that (x_k) is convergent to x in $(\mathbb{K}^n, N, *)$. □

5 Conclusion

In this paper some special fuzzy norms on $\mathbb{K}^n$ is given in order to obtain, in this way, fuzzy Euclidean normed spaces. These spaces can be proven to be a suitable tool for data mining. The method is based on embedding the data in fuzzy Euclidean normed spaces and to carry out data analysis in these spaces.

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