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Some Geometric Properties of Multivalent Convex Function for Operator on Hilbert Space

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Abstract

By making use of the operator on Hilbert space, we introduce and study some properties of geometric of a subclass $W\text{Ӝ}_{\text{Ƥ}}(\acute{\upsilon}, \acute{\omega}, \acute{\eta}, \text{Ҭ})$ of multivalent convex functions with negative coefficients.

Also we obtain some geometric properties, such as, coefficient inequality, growth and distortion theorem, extreme points, convex set, closure theorem, radius of close-to-convexity, weighted mean and inclusive properties.

1. Introducti0n

In this paper, the aim is to mention the basic facts about geometric function theory, which this is obtained from mixing of geometry and analysis. Its origin started from the 19th century, but it continued and continually applicable till now. Geometric Function Theory is an important branch of complex analysis; It deals with the geometric properties of the analytic functions. In particular , we will concentrate on the important ideas in this theory. The fundamentals of this theory are explained in most text books on this subject. Also, we review and consider the basic ideas, principles, definitions and the general principles of complex analysis, which underline the geometric function theory of a complex variable rather than the basic lemmas which are needed in the proofs of our results. A full discussion of these principles can be found in standard text books, [1], [2].

The study of multivalent functions is one of the main branches of geometric function theory and plays a central role in complex analysis.

Let $W_{\text{Ƥ}}$ be the class of functi0ns $\mathfrak{f}$ of the f0rm:

$$\mathfrak{f}(\mathfrak{z}) = \mathfrak{z}^{\text{Ƥ}} + \sum_{\tilde{n}=1}^{\infty} a_{\tilde{n}+\text{Ƥ}} \mathfrak{z}^{\tilde{n}+\text{Ƥ}} \quad (\text{Ƥ} \in \mathbb{N} = \{1,2,\dots\}), \tag{1}$$

which are analytic and Ƥ- valent in the open unit disk $\mathring{U} = \{\mathfrak{z} \in \mathbb{C}: |\mathfrak{z}| < 1\}$. Let $\text{Ӝ}_{\text{Ƥ}}$ denote the subclass of $W_{\text{Ƥ}}$ consisting of functions of the f0rm:

$$\mathfrak{f}(\mathfrak{z}) = \mathfrak{z}^{\text{Ƥ}} - \sum_{\tilde{n}=1}^{\infty} a_{\tilde{n}+\text{Ƥ}} \mathfrak{z}^{\tilde{n}+\text{Ƥ}} \quad (a_{\tilde{n}+\text{Ƥ}} \geq 0, \text{Ƥ} \in \mathbb{N} = \{1,2,\dots\}), \tag{2}$$

Definition (1.1): A function $\mathfrak{f} \in W_{\text{Ƥ}}$ is said to be in the class $W\text{Ӝ}_{\text{Ƥ}}(\acute{\upsilon}, \acute{\omega}, \acute{\eta})$ if it satisfies

$$\left| \frac{\mathfrak{z}\mathfrak{f}''(\mathfrak{z}) + 2\mathfrak{f}'(\mathfrak{z}) - \text{Ƥ}(1+\text{Ƥ})\mathfrak{z}^{\text{Ƥ}-1}}{\acute{\upsilon}(\mathfrak{z}\mathfrak{f}''(\mathfrak{z}) + 2\mathfrak{f}'(\mathfrak{z}) - \text{Ƥ}\acute{\omega}) + \text{Ƥ}(\text{Ƥ} - \acute{\omega} + 1)} \right| < 0,$$

where $0 \leq ύ < 1, 0 \leq ώ < Ƥ, 0 < ή \leq 1$ dan $ʑ \in Ủ$.

Let H be a Hilbert space on the complex field. Let Ƭ be a linear operator on H. For a complex analytic function ƒ on the unit disk Ủ, we denoted ƒ(Ƭ), the operator on H defined by the usual Riesz- Dunford integral [3]

$$ƒ(Ƭ) = \frac{1}{2\pi i}\int_{c} ƒ(ʑ)(ʑI - Ƭ)^{-1}dʑ,$$

where I is the identity operator on H, c is a positively oriented simple closed rectifiable contour lying in Ủ and containing the spectrum κ(Ƭ) of Ƭ in its interi0r domain [4]. Also ƒ(Ƭ) can be defined by the series

$$ƒ(Ƭ) = \sum_{ñ=0}^{\infty}\frac{ƒ^{(ñ)}(0)}{ñ!}Ƭ^{ñ},$$

which converges in the norm topology [5].

Definition (1.2): Let H be a Hilbert space and Ƭ be an operator on H such that $Ƭ \neq \varphi$

and $\|Ƭ\| < 1$. Let ύ, ώ be real numbers such that $0 \leq ύ < 1, 0 \leq ώ < Ƥ, 0 < ή \leq 1$.

An analytic functi0n ƒ on the unit disk belong to the class $WЖ_Ƥ(ύ, ώ, ή, Ƭ)$ if it satisfy the inequality

$$\left\|Ƭƒ''(Ƭ) + 2ƒ'(Ƭ) - Ƥ(1+Ƥ)Ƭ^{Ƥ-1}\right\| < ή\|ύ(Ƭƒ''(Ƭ) + 2ƒ'(Ƭ) - Ƥώ) + Ƥ(Ƥ - ώ + 1)\|,$$

The operator on Hilbert space were consider recently by [6]-[12].

2. Main results:

The first theorem gives C0efficient inequality for a function ƒ to be in the class $WЖ_Ƥ(ύ, ώ, ή, Ƭ)$.

Theorem (2.1): Let $ƒ \in Ж_Ƥ$ be defined by (2). Then $ƒ \in WЖ_Ƥ(ύ, ώ, ή, Ƭ)$ for all $Ƭ \neq \varphi$ if and only if

$$\sum_{ñ=1}^{\infty}(ñ+Ƥ)(ñ+Ƥ+1)(1+ήύ)a_{ñ+Ƥ} \leq ήƤ(ύ+1)(Ƥ - ώ + 1). \quad (3)$$

where $0 \leq ύ < 1, 0 \leq ώ < Ƥ, 0 < ή \leq 1$.

The result is sharp for the function ƒ given by

$$ƒ(ʑ) = ʑ^{Ƥ} - \frac{ήƤ(ύ+1)(Ƥ - ώ+1)}{(ñ+Ƥ)(ñ+Ƥ+1)(1+ήύ)}ʑ^{ñ+Ƥ}, ñ \geq 1. \quad (4)$$

Proof: Suppose that the inequality (3) holds. Then, we have

$$\left\|Ƭƒ''(Ƭ) + 2ƒ'(Ƭ) - Ƥ(1+Ƥ)Ƭ^{Ƥ-1}\right\| - ή\|ύ(Ƭƒ''(Ƭ) + 2ƒ'(Ƭ) - Ƥώ) + Ƥ(Ƥ - ώ + 1)\|$$

$$\left\|-\sum_{ñ=1}^{\infty}(ñ+Ƥ)(ñ+Ƥ+1)a_{ñ+Ƥ}Ƭ^{ñ+Ƥ-1}\right\|$$

$$-ή\left\|ύƤ(Ƥ+1)Ƭ^{Ƥ-1} - \sum_{ñ=1}^{\infty}ύ(ñ+Ƥ)(ñ+Ƥ+1)a_{ñ+Ƥ}Ƭ^{ñ+Ƥ-1} - ύƤώ + Ƥ(Ƥ - ώ + 1)\right\|$$

$$\leq \sum_{ñ=1}^{\infty}(ñ+Ƥ)(ñ+Ƥ+1)(1+ήύ)a_{ñ+Ƥ} - ήƤ(ύ+1)(Ƥ - ώ + 1) \leq 0.$$

Hence, $ƒ \in WЖ_Ƥ(ύ, ώ, ή, Ƭ)$.

To show the converse, let $ƒ \in WЖ_Ƥ(ύ, ώ, ή, Ƭ)$. Then

$\left\|Ƭƒ'(Ƭ) + 2ƒ'(Ƭ) - Ƥ(1+Ƥ)Ƭ^{Ƥ-1}\right\| < ή\|ύ(Ƭƒ''(Ƭ) + 2ƒ'(Ƭ) - Ƥώ) + Ƥ(Ƥ - ώ + 1)\|$,gives

$\left\|-\sum_{ñ=1}^{\infty}(ñ+Ƥ)(ñ+Ƥ+1)a_{ñ+Ƥ}Ƭ^{ñ+Ƥ-1}\right\|$

$< ή\left\|ύƤ(Ƥ+1)Ե^{Ƥ-1} - \sum_{ñ=1}^{\infty} ύ(ñ+Ƥ)(ñ+Ƥ+1)a_{ñ+Ƥ}Ե^{ñ+Ƥ-1} - ύƤώ + Ƥ(Ƥ-ώ+1)\right\|.$

Setting $Ե = ɍI (0 < ɍ < 1)$ in the above inequality, we get

$$\frac{\sum_{ñ=1}^{\infty}(ñ+Ƥ)(ñ+Ƥ+1)a_{ñ+Ƥ}ɍ^{ñ+Ƥ-1}}{ύƤ(Ƥ+1)ɍ^{Ƥ-1} - \sum_{ñ=1}^{\infty} ύ(ñ+Ƥ)(ñ+Ƥ+1)a_{ñ+Ƥ}ɍ^{ñ+Ƥ-1} - ύƤώ + Ƥ(Ƥ-ώ+1)} < ή. \quad (5)$$

Up0n clearing denominator in (5) and letting $ɍ \longrightarrow 1$, we obtain

$\sum_{ñ=1}^{\infty}(ñ+Ƥ)(ñ+Ƥ+1)a_{ñ+Ƥ} < ήƤ(ύ+1)(Ƥ- ώ+1) - \sum_{ñ=1}^{\infty} ήύ(ñ+Ƥ)(ñ+Ƥ+1)a_{ñ+Ƥ}.$

Thus

$$\sum_{ñ=1}^{\infty}(ñ+Ƥ)(ñ+Ƥ+1)(1+ήύ)a_{ñ+Ƥ} \leq ήƤ(ύ+1)(Ƥ- ώ+1),$$

which completes the proof.

Corollary (2.1): If $ƒ \in WЖ_Ƥ(ύ, ώ, ή, Ե)$, then

$$a_{ñ+Ƥ} \leq \frac{ήƤ(ύ+1)(Ƥ- ώ+1)}{(ñ+Ƥ)(ñ+Ƥ+1)(1+ήύ)}, ñ \geq 1.$$

Next, we obtain the growth and distortion for a function ƒ to be in the class $WЖ_Ƥ(ύ, ώ, ή, Ե)$.

Theorem (2.2): If $ƒ \in WЖ_Ƥ(ύ, ώ, ή, Ե)$, and $\|Ե\| < 1, Ե \neq \emptyset$, then

$$\|Ե\|^Ƥ - \frac{ήƤ(ύ+1)(Ƥ- ώ+1)}{(Ƥ+1)(Ƥ+2)(1+ήύ)}\|Ե\|^{Ƥ+1} \leq \|ƒ(Ե)\| \leq \|Ե\|^Ƥ + \frac{ήƤ(ύ+1)(Ƥ- ώ+1)}{(Ƥ+1)(Ƥ+2)(1+ήύ)}\|Ե\|^{Ƥ+1}$$

and

$$Ƥ\|Ե\|^{Ƥ-1} - \frac{ήƤ(ύ+1)(Ƥ- ώ+1)}{(Ƥ+2)(1+ήύ)}\|Ե\|^Ƥ \leq \|ƒ'(Ե)\| \leq Ƥ\|Ե\|^{Ƥ-1} - \frac{ήƤ(ύ+1)(Ƥ- ώ+1)}{(Ƥ+2)(1+ήύ)}\|Ե\|^Ƥ.$$

The result is sharp for the function ƒ given by

$$ƒ(ƶ) = ƶ^Ƥ - \frac{ήƤ(ύ+1)(Ƥ- ώ+1)}{(Ƥ+1)(Ƥ+2)(1+ήύ)}ƶ^{Ƥ+1}.$$

Proof: According to the Theorem (2.1), we get

$$\sum_{ñ=1}^{\infty} a_{ñ+Ƥ} \leq \frac{ήƤ(ύ+1)(Ƥ- ώ+1)}{(Ƥ+1)(Ƥ+2)(1+ήύ)}.$$

Hence

$$\|ƒ(Ե)\| \geq \|Ե\|^Ƥ - \sum_{ñ=1}^{\infty} a_{ñ+Ƥ}\|Ե\|^{ñ+Ƥ}$$

$$\geq \|Ե\|^Ƥ - \|Ե\|^{Ƥ+1}\sum_{ñ=1}^{\infty} a_{ñ+Ƥ}$$

$$\geq \|Ե\|^Ƥ - \frac{ήƤ(ύ+1)(Ƥ- ώ+1)}{(Ƥ+1)(Ƥ+2)(1+ήύ)}\|Ե\|^{Ƥ+1}.$$

Also,

$$\|ƒ(Ե)\| \leq \|Ե\|^Ƥ + \sum_{ñ=1}^{\infty} a_{ñ+Ƥ}\|Ե\|^{ñ+Ƥ}$$

$$\leq \|Ե\|^Ƥ + \frac{ήƤ(ύ+1)(Ƥ- ώ+1)}{(Ƥ+1)(Ƥ+2)(1+ήύ)}\|Ե\|^{Ƥ+1}.$$

In view of **(**Theorem 2.1**)**, we have

$$\sum_{ñ=1}^{\infty}(ñ+Ƥ)a_{ñ+Ƥ} \leq \frac{ή Ƥ(ύ+1)(Ƥ-ώ+1)}{(Ƥ+2)(1+ήύ)}.$$

Thus

$$\|f'(Ƭ)\| \geq Ƥ\|Ƭ\|^{Ƥ-1} - \sum_{ñ=1}^{\infty}(ñ+Ƥ)a_{ñ+Ƥ}\|Ƭ\|^{ñ+Ƥ-1}$$

$$\geq Ƥ\|Ƭ\|^{Ƥ-1} - \|Ƭ\|^{Ƥ}\sum_{ñ=1}^{\infty}(ñ+Ƥ)a_{ñ+Ƥ}$$

$$\geq Ƥ\|Ƭ\|^{Ƥ-1} - \frac{ήƤ(ύ+1)(Ƥ-ώ+1)}{(Ƥ+2)(1+ήύ)}\|Ƭ\|^{Ƥ}$$

and

$$\|f'(Ƭ)\| \leq Ƥ\|Ƭ\|^{Ƥ-1} - \|Ƭ\|^{Ƥ}\sum_{ñ=1}^{\infty}(ñ+Ƥ)a_{ñ+Ƥ}$$

$$\leq Ƥ\|Ƭ\|^{Ƥ-1} - \frac{ήƤ(ύ+1)(Ƥ-ώ+1)}{(Ƥ+2)(1+ήύ)}\|Ƭ\|^{Ƥ}.$$

Therefore the proof is complete.

In the following, we prove extreme points of the class $WЖ_Ƥ(ύ, ώ, ή, Ƭ)$.

The0rem (2.3): Let $ƒ_0(ƶ) = ƶ^{Ƥ}$ and

$$ƒ_ñ(ƶ) = ƶ^{Ƥ} - \frac{ήƤ(ύ+1)(Ƥ-ώ+1)}{(ñ+Ƥ)(ñ+Ƥ+1)(1+ήύ)}ƶ^{ñ+Ƥ}, \ ñ \geq 1.$$

Then $ƒ \in WЖ_Ƥ(ύ, ώ, ή, Ƭ)$ if and only if it can be expressed in the form

$$ƒ(ƶ) = \sum_{ñ=0}^{\infty} B_ñ ƒ_ñ(ƶ), \tag{6}$$

where $B_ñ \geq 0$ and $\sum_{ñ=0}^{\infty} B_ñ = 1$.

Proof: Assume that ƒ can be expressed by (6). Then, we have

$$ƒ(ƶ) = \sum_{ñ=0}^{\infty} B_ñ ƒ_ñ(ƶ) = ƶ^{Ƥ} - \sum_{ñ=0}^{\infty}\frac{ήƤ(ύ+1)(Ƥ-ώ+1)}{(ñ+Ƥ)(ñ+Ƥ+1)(1+ήύ)}B_ñ ƶ^{ñ+Ƥ}. \tag{7}$$

Thus

$$\sum_{ñ=0}^{\infty}\frac{(ñ+Ƥ)(ñ+Ƥ+1)(1+ήύ)}{ήƤ(ύ+1)(Ƥ-ώ+1)}\frac{ήƤ(ύ+1)(Ƥ-ώ+1)}{(ñ+Ƥ)(ñ+Ƥ+1)(1+ήύ)}B_ñ = \sum_{ñ=0}^{\infty} B_ñ = 1 - B_0 \leq 1, \tag{8}$$

and so $ƒ \in WЖ_Ƥ(ύ, ώ, ή, Ƭ)$.

Conversely, Suppose that ƒ given by (2) in the class $WЖ_Ƥ(ύ, ώ, ή, Ƭ)$. Then by

Corollary (2.1), we have

$$a_{ñ+Ƥ} \leq \frac{ήƤ(ύ+1)(Ƥ-ώ+1)}{(ñ+Ƥ)(ñ+Ƥ+1)(1+ήύ)}.$$

Setting

$$B_ñ = \frac{(ñ+Ƥ)(ñ+Ƥ+1)(1+ήύ)}{ήƤ(ύ+1)(Ƥ-ώ+1)}a_{ñ+Ƥ}, ñ \geq 1,$$

and $B_ñ = 1 - \sum_{ñ=1}^{\infty} B_ñ$. Then

$$ƒ(ƶ) = \sum_{ñ=0}^{\infty} B_ñ ƒ_ñ(ƶ),$$

This completes the proof of the theorem.

Next, we obtain the convex set of the class $WЖ_{Ƥ}(ύ, ώ, ή, Ѣ)$.

Theorem (2.4): The class $WЖ_{Ƥ}(ύ, ώ, ή, Ѣ)$ is a convex set.

Proof: Let f_1 and f_2 be the arbitrary element of $WЖ_{Ƥ}(ύ, ώ, ή, Ѣ)$. Then for every

$ɏ(0 \leq ɏ \leq 1)$, we show that $(1-ɏ) f_1 + ɏ f_2 \in WЖ_{Ƥ}(ύ, ώ, ή, Ѣ)$. Thus, we have

$$(1-ɏ) f_1 + ɏ f_2 = z^{Ƥ} - \sum_{ñ=1}^{\infty}((1-ɏ)a_{ñ+Ƥ} + ɏ b_{ñ+Ƥ}) z^{ñ+Ƥ}.$$

Hence,

$$\sum_{ñ=1}^{\infty}(ñ+Ƥ)(ñ+Ƥ+1)(1+ήύ)((1-ɏ)a_{ñ+Ƥ} + ɏ b_{ñ+Ƥ})$$

$$= (1-ɏ)\sum_{ñ=1}^{\infty}(ñ+Ƥ)(ñ+Ƥ+1)(1+ήύ)a_{ñ+Ƥ} + ɏ\sum_{ñ=1}^{\infty}(ñ+Ƥ)(ñ+Ƥ+1)(1+ήύ)b_{ñ+Ƥ}$$

$$\leq (1-ɏ)ήƤ(ύ+1)(Ƥ-ώ+1) + ɏήƤ(ύ+1)(Ƥ-ώ+1).$$

This completes the proof.

We will consider the function $f_{r̂}(z)$ defined, for every $r̂ = 1,2,3,\dots,ȅ$ by

$$f_{r̂}(z) = z^{Ƥ} - \sum_{ñ=1}^{\infty} a_{ñ+Ƥ,r̂} z^{ñ+Ƥ} \quad (a_{ñ+Ƥ,r̂} \geq 0). \tag{9}$$

Next, we discuss the closure theorem.

Theorem (2.5): Let the function $f_{r̂}(z)$ defined by (9) be in the class $WЖ_{Ƥ}(ύ, ώ, ή, Ѣ)$.

Then the function $Ǵ(z)$ defined by

$$Ǵ(z) = \sum_{r̂=1}^{ȅ} ê_{r̂} f_{r̂}(z) \text{ and } \sum_{r̂=1}^{ȅ} ê_{r̂} = 1, ê_{r̂} \geq 0$$

is in the class $WЖ_{Ƥ}(ύ, ώ, ή, Ѣ)$.

Proof: By using definition of $Ǵ(z)$, we get

$$Ǵ(z) = \left(\sum_{r̂=1}^{ȅ} ê_{r̂}\right) z^{Ƥ} - \sum_{ñ=1}^{\infty}\left(\sum_{r̂=1}^{ȅ} ê_{r̂} a_{ñ+Ƥ,r̂}\right) z^{ñ+Ƥ},$$

further. Since $f_{r̂}(z)$ are in the class $WЖ_{Ƥ}(ύ, ώ, ή, Ѣ)$, for every $r̂ = 1,2,3,\dots,ȅ$, we get

$$\sum_{ñ=1}^{\infty}(ñ+Ƥ)(ñ+Ƥ+1)(1+ήύ)a_{ñ+Ƥ,r̂} \leq ήƤ(ύ+1)(Ƥ-ώ+1),$$

for every $r̂ = 1,2,3,\dots,ȅ$.

Hence, we can see that

$$\sum_{ñ=1}^{\infty}(ñ+Ƥ)(ñ+Ƥ+1)(1+ήύ)\left(\sum_{r̂=1}^{ȅ} ê_{r̂} a_{ñ+Ƥ,r̂}\right)$$

$$= \sum_{r̂=1}^{ȅ} ê_{r̂}\left(\sum_{ñ=1}^{\infty}(ñ+Ƥ)(ñ+Ƥ+1)(1+ήύ)a_{ñ+Ƥ,r̂}\right) \leq \sum_{r̂=1}^{ȅ} ê_{r̂}\, ήƤ(ύ+1)(Ƥ-ώ+1),$$

which implies that $Ǵ(z)$ is in the class $WЖ_{Ƥ}(ύ, ώ, ή, Ѣ)$.

In the following, we prove the radius of close- to- convexity.

Theorem (2.6): Let the function $ƒ$ defined by (2) be in the class $WЖ_{Ƥ}(ύ, ώ, ή, Ƭ)$.

Then $ƒ$ is close-to-convex of order $(0 \leq ᾰ < 1)$ in $|ʑ| < ɍ_1$,

where

$$ɍ_1 = inf_{ñ\geq 1}\left\{\frac{(ñ+Ƥ+1)(1+ήύ)}{ή(ύ+1)(Ƥ-ώ+1)}\right\}^{\frac{1}{ñ}} \quad (10)$$

The result is sharp, the external function given (4).

Proof: It is sufficient to show that

$$\left|\frac{ƒ'(ʑ)}{ʑ^{Ƥ-1}} - Ƥ\right| \leq Ƥ \text{ f0r } |ʑ| < ɍ_1, \quad (11)$$

where $ɍ_1$ is given by (10). Indeed we find from (2) that

$$\left|\frac{ƒ'(ʑ)}{ʑ^{Ƥ-1}} - Ƥ\right| = \left|\frac{ƒ'(ʑ) - Ƥʑ^{Ƥ-1}}{ʑ^{Ƥ-1}}\right| = \left|\frac{Ƥʑ^{Ƥ-1} - \sum_{ñ=1}^{\infty}(ñ+Ƥ)a_{ñ+Ƥ}ʑ^{ñ+Ƥ-1} - Ƥʑ^{Ƥ-1}}{ʑ^{Ƥ-1}}\right|$$

$$= \left|\frac{-\sum_{ñ=1}^{\infty}(ñ+Ƥ)a_{ñ+Ƥ}ʑ^{ñ+Ƥ-1}}{ʑ^{Ƥ-1}}\right| \leq \sum_{ñ=1}^{\infty}\frac{(ñ+Ƥ)a_{ñ+Ƥ}|ʑ|^{ñ}}{Ƥ} \leq 1, \quad (12)$$

by using **Theorem (2.1)** and by (12) we get

$$\frac{(ñ+Ƥ)}{Ƥ}|ʑ|^{ñ} \leq \frac{(ñ+Ƥ)(ñ+Ƥ+1)(1+ήύ)}{ήƤ(ύ+1)(Ƥ-ώ+1)},$$

then

$$|ʑ| \leq \left\{\frac{(ñ+Ƥ+1)(1+ήύ)}{ή(ύ+1)(Ƥ-ώ+1)}\right\}^{\frac{1}{ñ}}. \quad (13)$$

The result follows easily from (13).

Definition (2.1): Let $ƒ * đ$ and $ỷ * ư$ be in the class $WЖ_{Ƥ}(ύ, ώ, ή, Ƭ)$. Then, the weighted mean $Ɏ_{ƅ}$ of $ƒ * đ$ and $ỷ * ư$ given by

$$Ɏ_{ƅ}(ʑ) = \frac{1}{2}\big((1-ƅ)(ƒ * đ)(ʑ) + (1+ƅ)(ỷ * ư)(ʑ)\big), 0 < ƅ < 1.$$

Theorem (2.7): Let $ƒ * đ$ and $ỷ * ư$ be in the class $WЖ_{Ƥ}(ύ, ώ, ή, Ƭ)$. Then, the weighted mean of $ƒ * đ$ and $ỷ * ư$ is also in the class $WЖ_{Ƥ}(ύ, ώ, ή, Ƭ)$.

Proof: By using **Definition (2.1)**, we get

$$Ɏ_{ƅ}(ʑ) = \frac{1}{2}\big((ƒ * đ)(ʑ) + (1+ƅ)(ỷ * ư)(ʑ)\big)$$

$$= \frac{1}{2}\left((1-ƅ)\left[ʑ^{Ƥ} - \sum_{ñ=1}^{\infty}a_{ñ+Ƥ}ʂ_{ñ+Ƥ}ʑ^{ñ+Ƥ}\right] + (1+ƅ)\left[ʑ^{Ƥ} - \sum_{ñ=1}^{\infty}ƨ_{ñ+Ƥ}ð_{ñ+Ƥ}ʑ^{ñ+Ƥ}\right]\right)$$

$$= ʑ^{Ƥ} - \sum_{ñ=1}^{\infty}\frac{1}{2}[(1-ƅ)a_{ñ+Ƥ}ʂ_{ñ+Ƥ} + (1+ƅ)ƨ_{ñ+Ƥ}ð_{ñ+Ƥ}]ʑ^{ñ+Ƥ}.$$

Since $ƒ * đ$ and $ỷ * ư$ are in the class $WЖ_{Ƥ}(ύ, ώ, ή, Ƭ)$ so by **Theorem (2.1)**, we have

$$\sum_{ñ=1}^{\infty}(ñ+Ƥ)(ñ+Ƥ+1)(1+ή\acute{υ})a_{ñ+Ƥ}ʂ_{ñ+Ƥ} \leq ήƤ(\acute{υ}+1)(Ƥ-\acute{ω}+1)$$

and

$$\sum_{ñ=1}^{\infty}(ñ+Ƥ)(ñ+Ƥ+1)(1+ή\acute{υ})ƨ_{ñ+Ƥ}ð_{ñ+Ƥ} \leq ήƤ(\acute{υ}+1)(Ƥ-\acute{ω}+1).$$

Hence

$$\sum_{ñ=1}^{\infty}(ñ+Ƥ)(ñ+Ƥ+1)(1+ή\acute{υ})\left[\frac{1}{2}(1-ƀ)a_{ñ+Ƥ}ʂ_{ñ+Ƥ}+\frac{1}{2}(1+ƀ)ƨ_{ñ+Ƥ}ð_{ñ+Ƥ}\right]$$

$$=\frac{1}{2}(1-ƀ)\sum_{ñ=1}^{\infty}(ñ+Ƥ)(ñ+Ƥ+1)(1+ή\acute{υ})a_{ñ+Ƥ}ʂ_{ñ+Ƥ}$$

$$=\frac{1}{2}(1+ƀ)\sum_{ñ=1}^{\infty}(ñ+Ƥ)(ñ+Ƥ+1)(1+ή\acute{υ})ƨ_{ñ+Ƥ}ð_{ñ+Ƥ}$$

$$\leq \frac{1}{2}(1-ƀ)ήƤ(\acute{υ}+1)(Ƥ-\acute{ω}+1)+\frac{1}{2}(1+ƀ)ήƤ(\acute{υ}+1)(Ƥ-\acute{ω}+1).$$

Which implies that $Ɏ_{ƀ} \in WЖ_{Ƥ}(\acute{υ},\acute{ω},ή,Ƭ)$.

Now, we obtain the inclusive properties of the class $WЖ_{Ƥ}(\acute{υ},\acute{ω},ή,Ƭ)$.

Theorem (2.8): Let $0 \leq \acute{υ} < 1, 0 \leq \acute{ω} < Ƥ, 0 < ή < 1$.

Then $WЖ_{Ƥ}(\acute{υ},\acute{ω},ή,Ƭ) \subset WЖ_{Ƥ}(ƛ,\acute{ω},\frac{1}{3},Ƭ)$, where

$$ƛ \leq \frac{1-2ή\acute{υ}-3ή}{ή-1}.$$

Proof: Let the function $ƒ(ʑ)$ given by (2) belongs to the class $WЖ_{Ƥ}(\acute{υ},\acute{ω},ή,Ƭ)$.

Then in view of **Theorem (2.1)**, we get

$$\sum_{ñ=1}^{\infty}\frac{(ñ+Ƥ)(ñ+Ƥ+1)(1+ή\acute{υ})}{ήƤ(\acute{υ}+1)(Ƥ-\acute{ω}+1)}a_{ñ+Ƥ} \leq 1. \quad (14)$$

We want to find the value ƛ such that

$$\sum_{ñ=1}^{\infty}\frac{(ñ+Ƥ)(ñ+Ƥ+1)\left(1+\frac{1}{3}ƛ\right)}{\frac{1}{3}Ƥ(ƛ+1)(Ƥ-\acute{ω}+1)}a_{ñ+Ƥ} \leq 1. \quad (15)$$

The inequality (14) would obviously imply (15) if

$$\frac{(ñ+Ƥ)(ñ+Ƥ+1)\left(1+\frac{1}{3}ƛ\right)}{\frac{1}{3}Ƥ(ƛ+1)(Ƥ-\acute{ω}+1)} \leq \frac{(ñ+Ƥ)(ñ+Ƥ+1)(1+ή\acute{υ})}{ήƤ(\acute{υ}+1)(Ƥ-\acute{ω}+1)}.$$

Rewriting the inequality, we get

$$ƛ \leq \frac{1-2ή\acute{υ}-3ή}{ή-1}.$$

This completes the proof.

CONFLICT OF INTERESTS
There are no conflicts of interest.

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بعض الخصائص الهندسية لدالة محدبة متعددة التكافؤ على فضاء هلبرت

الخلاصة

بالاستفادة من استعمال المؤثر على فضاء هلبرت، نحنُ قدمنا ودرسنا بعض الخصائص الهندسية من الدوال التحليلية المتعددة التكافؤ المحدبة ذات معاملات سالبة. $(\mathcal{X})Ж_p(ύ, ώ, ή, Ƭ)$ للصنف الجزئي كذلك حصلنا على بعض الخصائص الهندسية، مثل، متراجحة المعاملات، مبرهنة النمو والتشوية، النقاط الحرجة، المجموعة المحدبة، نظرية الانغلاق، أنصاف الاقطار المغلقة، المتوسط الوزني وخواص الاحتواء لهذا الصنف.

الكلمات المفتاحية: الدوال المتعددة التكافؤ، فضاء هلبرت، نظرية التشوية، انصاف اقطار المغلقة، النقاط الحرجة، نظرية الانغلاق والمتوسط الوزني.